

NONLINEAR LARGE DEVIATIONS: BEYOND THE HYPERCUBE

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By extending (*Adv. Math.* **299** (2016) 396–450), we present a framework to calculate large deviations for nonlinear functions of independent random variables supported on compact sets in Banach spaces. Previous research on nonlinear large deviations has only focused on random variables supported on $\{-1, +1\}^n$, and accordingly we build theory for random variables with general distributions, increasing flexibility in the applications. As examples, we compute the large deviation rate functions for monochromatic subgraph counts in edge-colored complete graphs, and for triangle counts in dense random graphs with continuous edge weights. Moreover, we verify the mean field approximation for a class of vector spin models.

1. Introduction. Large deviations theory for the linear function of i.i.d. random objects has long been studied, see [13] and references therein. Since the linear function is the simplest class of functions to analyze and only accounts for a small subset of functions people usually study, it is of natural interest to explore a corresponding theory for nonlinear functions. Recently, a nonlinear large deviations framework was built by Chatterjee and Dembo [7], where the authors deal with the large deviation principles for nonlinear functions of i.i.d. Bernoulli random variables. The main theorem in [7] gives error bounds of the mean field approximation of $\log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}]$ where μ is the uniform distribution on $\{-1, +1\}^n$. The error bounds consist of two parts: the complexity terms which involve the covering number of ∇f , and the smoothness terms which involve the first two derivatives of f . Motivated by [7], Eldan [14] comes up with a different nonlinear large deviations framework to deal with nonlinear functions of i.i.d. random variables supported on $\{-1, +1\}^n$. In [14], instead of the covering number of ∇f , a different notion of complexity called Gaussian width of the discrete gradient of f is introduced, and there f is not required to have the second derivative. In [7] many exciting applications are presented, suggesting the strong power of the new framework. Using the different method, Eldan [14] gets stronger results for the examples in [7]. However, all of the examples in [7] and [14] concern random variables with distributions supported on $\{-1, +1\}^n$, a small subset of random objects people usually study in probability theory. Therefore, it is natural to research whether a similar nonlinear large deviations regime works for random objects with more general distributions, and we can expect it since the Bernoulli random variable should not be special. Indeed, a framework similar to [7] is used by Basak and Mukherjee [4] to verify the universality of the mean field approximation on the Potts model.

In this work, we extend the framework of [7] to independent random variables compactly supported on Banach spaces. Similar to [7], our main result (Theorem 1) gives error bounds for the mean field approximation of $\log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}]$, while $\mu = \mu_1 \times \dots \times \mu_n$ could be more general than [7]. Our result has considerable flexibility in applications, because: (1) μ_i 's could be defined on general Banach spaces, and thus there is no dimension constraint on the supports of μ_i 's; (2) μ_i 's are not required to be discrete; (3) X_1, \dots, X_n are not required to be i.i.d.—only independence is needed. To show this flexibility we provide examples with

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high dimensional and continuous random variables, including an example in which the dimension of the support of μ_i 's is increasing with n ; previous methods do not work on these examples. While we take the same approach as [7] in proving our main result (Theorem 1), in [7] special calculations for the product Bernoulli distribution are used, and we find general arguments for Banach spaces. While our result works for general problems, we propose that for specific problems the error bounds in Theorem 1 could be improved by using the particular structures of the problems. As an example, we extend the result of [4] by verifying the mathematical rigor of the mean field approximation for a larger class of vector spin models. Note that it will also naturally be of interest to extend the framework in [14] for general distributions. However, when proving theorems for distributions supported on $\{-1, 1\}^n$, [14] constructs a Brownian motion running on $[-1, 1]^n$, such that whenever a facet of $[-1, 1]^n$ is hit the corresponding coordinate stops moving. In this way the Brownian motion ends up at $\{-1, 1\}^n$ uniformly, and one can change the distribution of the ending point by adding a drift to the Brownian motion. It is not clear what the corresponding objects should be for general supports.

After the first version of this paper, recently there has been some work in the area of nonlinear large deviations. In the work of Cook and Dembo [11], by providing quantitative versions of the Szemerédi's regularity lemma and the counting lemma, the authors make improvements to the large deviations of subgraph counts problem for sparse random graphs. In the independent work of Augeri [2], by applying convex analysis, the author gets new error bounds for the mean field approximation of $\log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}]$, where f is a continuous differentiable function and μ is a compactly support measure on \mathbb{R}^n . She then provides improved estimates on several applications. In the work of Austin [3], the author considers $\mu(dx) \propto e^{f(x)} \lambda(dx)$, where $\lambda(dx)$ is the product Borel probability measure on general product spaces, and f is a bounded and continuous function on that product spaces. He shows that if the covering number of ∇f is small, then μ can be approximated by a mixture of other measures, most of which are close to product measures.

1.1. *The main result.* Our goal is to find the leading term of $\log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}]$, for X_1, \dots, X_n following a product measure μ supported on a compact subset of Banach spaces and f a twice Fréchet differentiable functional (see Definition 1). As demonstrated in Section 1.2.1 and Section 1.2.2, such leading term provides us with the large deviation rate function. It further plays an important role in statistical physics, as shown in Section 1.2.3. In Theorem 1, we provide error bounds for the mean field approximation (introduced below) of $\log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}]$ (1.2), in terms of the covering number of the gradient ∇f and the norms of the first two derivatives of f . One should then show on a case by case basis that the error terms are of a smaller order than the mean field approximation. In Section 1.2 we provide three examples, demonstrating how the latter task is achieved.

For two probability measures ξ_1, ξ_2 on the same space Ω , denote by $D(\xi_1 \parallel \xi_2)$ the Kullback–Leibler divergence

$$D(\xi_1 \parallel \xi_2) := \int_\Omega \log \left(\frac{d\xi_1}{d\xi_2}(y) \right) \xi_1(dy),$$

where $\frac{d\xi_1}{d\xi_2}(\cdot)$ is the Radon–Nikodym derivative, and we set $D(\xi_1 \parallel \xi_2) \equiv \infty$ when the Radon–Nikodym derivative does not exist. From the Gibbs variational principle, we have the following identity:

$$(1.1) \quad \log \mathbb{E}_\mu[e^{f(X_1, \dots, X_n)}] = \max_{\nu \ll \mu} \{ \mathbb{E}_\nu[f(X_1, \dots, X_n)] - D(\nu \parallel \mu) \}.$$

The maximum on the right-hand side of (1.1) is taken over all measures with $\nu \ll \mu$, which is difficult to analyze. Restricting ν to be a product measure leads to the previously mentioned

mean field approximation:

$$(1.2) \quad \begin{aligned} & \log \mathbb{E}_\mu [e^{f(X_1, \dots, X_n)}] \\ & \approx \max_{\nu \ll \mu, \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n} \{ \mathbb{E}_\nu [f(X_1, \dots, X_n)] - D(\nu \parallel \mu) \}, \end{aligned}$$

which is much easier to deal with.

We next introduce some definitions needed for stating our main result. Let $[n] := \{1, \dots, n\}$. For each $i \in [n]$, we consider the probability space $(V_i, \mathcal{B}_i, \mu_i)$, where V_i is a Banach space (over the field \mathbb{R}) equipped with norm $\|\cdot\|_{V_i}$, \mathcal{B}_i is the Borel σ -algebra generated by V_i 's open sets, and μ_i is a probability measure on the measurable space (V_i, \mathcal{B}_i) . We assume that for each i , there exists a compact convex set $W_i \subset V_i$ such that $\mu_i(W_i) = 1$. Consider the product probability measure μ supported on the product space W in V where

$$\mu := \mu_1 \times \dots \times \mu_n, \quad W := W_1 \times \dots \times W_n, \quad V := V_1 \times \dots \times V_n.$$

Write the element in V as $x = (x_1, \dots, x_n)$ where $x_i \in V_i$. Set the norm $\|\cdot\|_V$ on V as

$$(1.3) \quad \|x\|_V := \max_{i \in [n]} \{\|x_i\|_{V_i}\}, \quad \forall x \in V.$$

For two Banach spaces E_1 and E_2 , and some $g : E_1 \rightarrow E_2$, we say $g(r) = o(r)$, if there exists a mapping $\varepsilon : E_1 \rightarrow E_2$ such that $\lim_{\|r\|_{E_1} \rightarrow 0} \|\varepsilon(r)\|_{E_2} = 0$, and $g(r) = \|r\|_{E_1} \varepsilon(r)$. We introduce the definition of twice Fréchet differentiability as follows.

DEFINITION 1. A functional $f(\cdot) : V \rightarrow \mathbb{R}$ is twice Fréchet differentiable on V , if

(1) For each $x \in V$ there exists a bounded linear functional $f'(x)(\cdot) : V \rightarrow \mathbb{R}$ such that

$$(1.4) \quad f(x+r) - f(x) - f'(x)(r) = o(r).$$

For each $i \in [n]$, we define the partial differential $f_i(x)(\cdot) : V_i \rightarrow \mathbb{R}$ as

$$f_i(x)(r_i) := f'(x)((0, \dots, r_i, \dots, 0)),$$

where $(0, \dots, r_i, \dots, 0) \in V$ is an element with the i th coordinate $r_i \in V_i$ and 0 otherwise.

(2) Moreover, $\forall z_i \in V_i$, $f_i(\cdot)(z_i) : V_i \rightarrow \mathbb{R}$ is Fréchet differentiable. That is, $\forall x \in V$ there exists a bounded linear functional $f'_i(x)(z_i, \cdot) : V \rightarrow \mathbb{R}$ such that

$$f_i(x+r)(z_i) - f_i(x)(z_i) - f'_i(x)(z_i, r) = o(r).$$

Similarly, $\forall i, j \in [n]$ and $z_i \in V_i$, we define the twice partial differential $f_{ij}(x)(z_i, \cdot) : V_j \rightarrow \mathbb{R}$ as

$$f_{ij}(x)(z_i, r_j) := f'_i(x)(z_i, (0, \dots, r_j, \dots, 0)).$$

For more properties about Fréchet differentials, see [10]. We define the operator norms of the first two partial derivatives of $f(x)$ as

$$\|f_i(x)\| := \sup_{\|r_i\|_{V_i} \leq 1} |f_i(x)(r_i)|,$$

$$\|f_{ij}(x)\| := \sup_{\max\{\|r_j\|_{V_j}, \|z_i\|_{V_i}\} \leq 1} |f_{ij}(x)(z_i, r_j)|, \quad \forall i, j \in [n].$$

Denote by $|f(x)|$ the absolute value of $f(x)$. We assume that there exists $a, b_i, c_{ij} > 0$ such that $\forall x \in W$,

$$|f(x)| \leq a, \quad \|f_i(x)\| \leq b_i, \quad \|f_{ij}(x)\| \leq c_{ij}, \quad \forall i, j \in [n].$$

Since W_i 's are assumed to be compact, we can find $M > 0$ such that each W_i satisfies

$$(1.5) \quad \forall z_i^{(1)}, z_i^{(2)} \in W_i, \quad \|z_i^{(1)} - z_i^{(2)}\|_{V_i} \leq M.$$

Denoting by $m(v_i) \in V_i$ the mean of v_i , namely the unique point m such that

$$(1.6) \quad \int_{V_i} h(z) dv_i(z) = h(m), \quad \forall \text{ bounded linear functional } h : V_i \rightarrow \mathbb{R}.$$

The existence of $m(v_i)$ is guaranteed by the fact that μ_i is supported on the compact set W_i ; for example, see [19], Chapter 2. Then, for any product measure $\nu = \nu_1 \times \nu_2 \times \dots \times \nu_n$ on W , let

$$(1.7) \quad m(\nu) := (m(\nu_1), \dots, m(\nu_n)).$$

Fixing some $\epsilon > 0$, assume that there exists a finite set $\mathcal{D}(\epsilon) = \{d^{(\alpha)} = (d_1^{(\alpha)}, \dots, d_n^{(\alpha)}), \alpha \in I\}$ (where I is the index set, and for each $\alpha \in I, i \in [n], d_i^{(\alpha)}$ is a bounded linear functional from V_i to \mathbb{R}) such that for any $x \in W$, there exists a $d = (d_1, \dots, d_n) \in \mathcal{D}(\epsilon)$ satisfying

$$(1.8) \quad \sum_{i=1}^n \|f_i(x) - d_i\|^2 \leq \epsilon^2 n.$$

Denote by $|\mathcal{D}(\epsilon)|$ the cardinality of $\mathcal{D}(\epsilon)$. Following is the main theorem, which gives upper and lower bounds of the mean field approximation for $\log \mathbb{E}_\mu[e^{f(X)}]$ where $X \sim \mu$.

THEOREM 1. *Under the above setting, for any $\epsilon > 0$ we have*

$$(1.9) \quad \log \int_W e^{f(x)} d\mu(x) \leq \max_{\nu \ll \mu, \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n} \left\{ f(m(\nu)) - \sum_{i=1}^n D(\nu_i \parallel \mu_i) \right\} + B_1 + B_2 + \log 2 + \log |\mathcal{D}(\epsilon)|,$$

where

$$(1.10) \quad B_1 := 4 \left(M^2 \left(a \sum_{i=1}^n c_{ii} + \sum_{i=1}^n b_i^2 \right) + M^3 \sum_{i,j=1}^n b_i c_{ij} + M^4 \left(a \sum_{i,j=1}^n c_{ij}^2 + \sum_{i,j=1}^n b_i b_j c_{ij} \right) \right)^{\frac{1}{2}},$$

$$(1.11) \quad B_2 := 4 \left(\sum_{i=1}^n b_i^2 + \epsilon^2 n \right)^{\frac{1}{2}} \left(M^3 \left(\sum_{i=1}^n c_{ii}^2 \right)^{\frac{1}{2}} + M^2 n^{\frac{1}{2}} \epsilon \right) + \sum_{i=1}^n M^2 c_{ii} + Mn\epsilon.$$

Moreover,

$$(1.12) \quad \log \int_W e^{f(x)} d\mu(x) \geq \max_{\nu \ll \mu, \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n} \left\{ f(m(\nu)) - \sum_{i=1}^n D(\nu_i \parallel \mu_i) \right\} - \frac{M^2}{2} \sum_{i=1}^n c_{ii}.$$

Theorem 1 is an extension of [7], Theorem 1.5. If μ_i 's are Bernoulli distribution with parameter $\frac{1}{2}$, Theorem 1 is merely [7], Theorem 1.5, with slight modifications. The main challenge here is to avoid the special properties of the Bernoulli distribution and the hypercube, which are used in the proof of [7], Theorem 1.5. For example, letting $\tilde{\mu}$ be a measure such that $\frac{d\tilde{\mu}}{d\mu}(x) \propto e^{f(x)}$, in [7], preceding Lemma 3.1, the authors utilize the explicit formula for $\widehat{X}_i := E_{\tilde{\mu}}[X_i \mid X_j, j \neq i]$ in case of Bernoulli $\{X_i\}$ when bounding $f(X) - f(\widehat{X})$. Lacking such a simple formula here requires a more sophisticated analysis of the error induced by approximating $f(X)$ by $f(\widehat{X})$. For another instance, in [7], for any point p in the hypercube one has a product Bernoulli measure ν^p such that $\nu^p \ll \mu$ and $m(\nu^p) = p$. Lacking such explicit description of ν^p for all $p \in W$, we instead manage to carry the proof while restricting to ν^p for p in a finite subset of W , for which the explicit description of ν^p exists. See detailed discussions on the difference from [7], important part in this extension, and the outline of the proof of Theorem 1 in Section 2. The full proof of Theorem 1 is given in Section 3.

1.2. *Applications.* We provide three applications of our framework. The first two are large deviations of subgraph counts in random graph, and the third is the mean field approximation for vector spin models.

1.2.1. *Monochromatic subgraph counts in edge-colored complete graphs.* The edge colored complete graph is an important object in combinatorics; for example, see Ramsey's theorem. People have studied this kind of graph from different perspectives; for example, see [1, 18] and [9]. On the other hand, the large deviations for subgraph counts in random graph has been studied a lot in probability; for example, see [5, 15] and [6]. In this example, we consider the large deviation for the monochromatic subgraph counts in an edge colored random graph. More precisely, we consider a complete graph G with N vertices, and assume that each edge of G has a color which is i.i.d. uniformly chosen from l different colors. Take any fixed finite simple graph H . We investigate the large deviation of the number of homomorphisms of H into G whose edges are of the same color. We formulate this problem as follows: consider a random vector $X = (X_{ij})_{1 \leq i < j \leq N}$, where X_{ij} 's are i.i.d. chosen from the set $\Lambda := \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$ (where there are l elements in Λ and the length of each element is l). Regard each element in Λ as a color, and regard X_{ij} as the color of the edge $\{i, j\}$. Then X corresponds to a coloring on G . Let m be the number of edges of H , Δ be the maximum degree of H , and k be the number of vertices of H . For convenience we let the vertex set of H be $\{1, \dots, k\}$, and denote by E the edge set of H . For $x = (x_{ij})_{1 \leq i < j \leq N}$ where $x_{ij} \in \mathbb{R}^l$, define

$$(1.13) \quad T(x) := \sum_{q_1, q_2, \dots, q_k \in [N]} \sum_{s=1}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s},$$

where $x_{q_r q_{r'} s}$ is the s th coordinate of $x_{q_r q_{r'}}$ (recall that $x_{q_r q_{r'}} \in \Lambda$ is a vector with length l), x_{ij} is interpreted as x_{ji} if $i > j$, and x_{ii} is interpreted as the 0 vector in \mathbb{R}^l for all i . It is easy to check that for coloring X , $T(X)$ is the number of homomorphisms of H in G with same color edges. Denote by $o(1)$ a quantity which goes to 0 as N goes to ∞ . We show the following large deviation result for $T(X)$.

THEOREM 2. *For $T(X)$ as above and any $u > 1$, as $N \rightarrow \infty$ we have*

$$\mathbb{P}(T(X) \geq u\mathbb{E}[T(X)]) \leq \exp(-\psi_l(u)(1 + o(1))) \quad \text{when } l \leq N^{1/(19+8m+21\Delta)}$$

and

$$\mathbb{P}(T(X) \geq u\mathbb{E}[T(X)]) \geq \exp(-\psi_l(u)(1 + o(1))),$$

$$\text{when } l \leq N^{1/(2\Delta+m+2)}(\log N)^{-1},$$

where

$$(1.14) \quad \psi_l(u) := \inf \left\{ \sum_{1 \leq i < j \leq N} \sum_{s=1}^l x_{ijs} \log \frac{x_{ijs}}{1/l} : x_{ij} \in W_0, \right. \\ \left. T((x_{ij})_{1 \leq i < j \leq N}) \geq u \mathbb{E}[T(X)] \right\},$$

and

$$(1.15) \quad W_0 := \left\{ (z_1, \dots, z_l) : \sum_{i=1}^l z_i = 1, z_i \geq 0 \forall i \in [l] \right\}.$$

Theorem 2 provides the large deviation rate function for $T(X)$ via the variational problem (1.14), in the case that the number of colors l not increasing with N faster than certain polynomial speed. We give the proof of Theorem 2 in Section 4.1.

Besides the edge-colored complete graph, one can also apply Theorem 1 to calculate the large deviations for the monochromatic subgraph counts in the edge-colored Erdős–Rényi random graph $G(N, p)$, by making X_{ij} the zero vector with probability $1 - p$. For the edge-colored random regular graph the case is different, since the edges are dependent there.

1.2.2. *Triangle counts with continuous edge weights.* The large deviation principle for the triangle counts in random graph has been studied for a long time. People study this problem for both dense Erdős–Rényi random graph $G(N, p)$, in which p is fixed [8], and sparse Erdős–Rényi random graph $G(N, p)$, in which p goes to 0 as N goes to ∞ [7, 12, 14–16, 20]. See Chatterjee [6] for more discussions and references. Here we consider the continuous version of the triangle counts problem in the dense random graph. That is, let G be a complete graph with N vertices. Let $X = (X_{ij})_{1 \leq i < j \leq N}$ where X_{ij} ’s are i.i.d. from $U(0, 1)$, the uniform distribution on $[0, 1]$. For each $1 \leq i < j \leq N$, we assign a weight X_{ij} to the edge $\{i, j\}$. For $x = (x_{ij})_{1 \leq i < j \leq N}$, we define

$$T(x) := \frac{1}{6} \sum_{i,j,k \in [N]} x_{ij} x_{jk} x_{ki},$$

where we interpret $x_{ij} = x_{ji}$ if $i > j$, and $x_{ii} = 0$ for all $i \in [N]$. Then $T(X)$ is the number of weighted triangles in G for weights X . For any $a \in (0, 1)$, we denote by ν^a the truncated exponential distribution on $[0, 1]$ with mean a , that is, the distribution whose density $p_{\nu^a}(\cdot)$ is

$$p_{\nu^a}(z) = \frac{\lambda_a e^{-\lambda_a z}}{1 - e^{-\lambda_a}} \quad \text{for } z \in (0, 1), \text{ with } \lambda_a \text{ such that } \int_0^1 p_{\nu^a}(z) dz = a.$$

By direct calculation, the KL divergence between ν^a and $U(0, 1)$ is

$$D(\nu^a || U(0, 1)) = \int_0^1 \frac{\lambda_a e^{-\lambda_a x}}{1 - e^{-\lambda_a}} \log \left(\frac{\lambda_a e^{-\lambda_a x}}{1 - e^{-\lambda_a}} \right) dx \\ = -1 + \frac{\lambda_a e^{-\lambda_a}}{1 - e^{-\lambda_a}} + \log \left(\frac{\lambda_a}{1 - e^{-\lambda_a}} \right).$$

Let $n = N(N - 1)/2$, the number of edges in G . Define

$$\psi_n(u) := \inf \left\{ \sum_{1 \leq i < j \leq N} \left(-1 + \frac{\lambda_{y_{ij}} e^{-\lambda_{y_{ij}}}}{1 - e^{-\lambda_{y_{ij}}}} + \log \left(\frac{\lambda_{y_{ij}}}{1 - e^{-\lambda_{y_{ij}}}} \right) \right) : y_{ij} \in (0, 1), \right. \\ \left. T((y_{ij})_{1 \leq i < j \leq N}) \geq u \mathbb{E}[T(X)] \right\}.$$

We show the following.

THEOREM 3. *Let $X = (X_{ij})_{1 \leq i < j \leq N}$ where X_{ij} 's are i.i.d. from $U(0, 1)$. For $T(X)$ as above and any $1 < u < 8$, we have*

$$\mathbb{P}(T(X) \geq u\mathbb{E}[T(X)]) = \exp(-\psi_n(u)(1 + o(1))) \quad \text{as } N \rightarrow \infty.$$

We give the proof of Theorem 3 in Section 4.2. One can also consider the weighted triangle counts problem for sparse random graphs, for which the challenge is to show that after approximating the upper tail probability using Theorem 1, the errors are of a smaller order than the mean field approximation term.

REMARK 1 (of Theorem 1). The bounds in Theorem 1 are not guaranteed and have no reason to be optimal; they could be improved case by case by utilizing particular structures of specific problems. We provide the following example to show this.

1.2.3. Mean field approximation on a class of vector spin models. Mean field approximation is an important method derived from Physics, and it has been applied to many different fields. See [21] or [4] for an introduction to this method. Like other methods in statistical physics, its mathematical rigor is not guaranteed and needs to be verified for specific models. In [4] the universality of the mean field approximation for a class of Potts model is verified. Our next theorem extends the result in [4] to a more general setting. We introduce some notation first. Let X_i 's be i.i.d. random variables with corresponding distributions μ_i 's supported on a compact set W_1 in \mathbb{R}^N for some $N \geq 1$. Define the product measure as $\mu := \mu_1 \times \dots \times \mu_n$. Let J be a real symmetric $N \times N$ matrix, h be a real vector with length N , and for each $n \in \mathbb{Z}^+$ let A_n be a real symmetric $n \times n$ matrix. Define the Hamiltonian $H_n^{J,h}(\cdot) : (\mathbb{R}^N)^n \rightarrow \mathbb{R}$ such that for any $x = (x_1, \dots, x_n) \in (\mathbb{R}^N)^n$

$$(1.16) \quad H_n^{J,h}(x) := \frac{1}{2} \sum_{i,j=1}^n A_n(i, j)x_i^T J x_j + \sum_{i=1}^n x_i^T h.$$

For a sequence $\{c_n\}_{n \geq 1}$ and a positive sequence $\{a_n\}$, we say $c_n = o(a_n)$ if $\lim_{n \rightarrow \infty} c_n/a_n = 0$, and $c_n = O(a_n)$ if $\limsup_{n \rightarrow \infty} |c_n|/a_n < \infty$. We have the following theorem.

THEOREM 4. *If the sequence of matrices A_n satisfies*

$$(1.17) \quad \text{tr}(A_n^2) = o(n) \quad \text{and} \quad \sup_{x \in [0,1]^n} \sum_{i \in [n]} \left| \sum_{j \in [n]} A_n(i, j)x_j \right| = O(n),$$

then

$$(1.18) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \int_{W_1^n} e^{H_n^{J,h}(x)} d\mu(x) - \max_{\nu \ll \mu, \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n, \left\{ H_n^{J,h}(m(\nu)) - \sum_{i=1}^n D(\nu_i \parallel \mu_i) \right\}} \right] = 0.$$

REMARK 2 (of Theorem 4). If we let μ_i 's be the uniform distribution on $\{(1, 0, \dots, 0), \dots, (0, 0, \dots, 1)\}$ (each element belongs to \mathbb{R}^N for $N \geq 2$ and has a unique nonzero entry), then we get the Potts model, and Theorem 4 is merely Theorem 1.1 in [4].

Theorem 4 covers a large class of models in statistical physics. In the simple case of $A_n(i, j) = 1/n$, it is easy to verify that condition (1.17) holds, and $N = 1, 2, 3$ correspond to the mean field Curie–Weiss model, XY model and Heisenberg model respectively. The validity of the mean field approximation for these mean field models has long been known; for example, see [17] and [13]. The more difficult case is when $A_n(i, j)$ are not same; see examples and discussions in [4], Section 1.3. A direct application of Theorem 4 is letting μ_i be the uniform distribution on the unit sphere S^{N-1} , which is often studied in statistical physics and is not covered by [4].

If we directly apply Theorem 1 to the setting above, we will find that (1.18) is stronger than what we can get. In order to prove Theorem 4, we need to incorporate the special properties of $H_n^{J,h}$. We give the proof of Theorem 4 in Section 4.3.

We give the proof outline of Theorem 1 in Section 2 below, including detailed discussions on the differences from [7] and important parts in our extensions. The full proof of Theorem 1 is provided in Section 3. The proofs of three applications are given in Section 4.

2. Proof outline of Theorem 1. We proceed to sketch the key part of Theorem 1, namely proving the upper bound (1.9), together with the differences from the proof in [7] (see Section 3.1 for the much easier proof of the lower bound (1.12)).

(1) We define a measure $\tilde{\mu}$ supported on W such that

$$(2.1) \quad \frac{d\tilde{\mu}}{d\mu}(x) := \frac{e^{f(x)}}{\int_W e^{f(x)} d\mu(x)} \quad \forall x \in W.$$

We define $\hat{x}_i(\cdot) : V \rightarrow W_i$ and $\hat{x}(\cdot) : V \rightarrow W$, such that for every $x = (x_1, \dots, x_n) \in V$,

$$(2.2) \quad \hat{x}_i(x) := \frac{\int_{W_i} z_i e^{f(x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_n)} d\mu_i(z_i)}{\int_{W_i} e^{f(x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_n)} d\mu_i(z_i)} \quad \text{and}$$

$$\hat{x}(x) := (\hat{x}_1(x), \dots, \hat{x}_n(x)),$$

where the integral in the numerator of $\hat{x}_i(x)$ is the Bochner integral. Another way to define $\hat{x}_i(x)$ is to let it be the expectation of the distribution of X_i conditioned on $X_j = x_j$ for $j \neq i$, where the expectation is defined at (1.6). The existence of the conditional expectation is due to the fact that W_i is compact. One can verify that indeed the two ways to define $\hat{x}_i(x)$ are same here.

For simplicity, we write \hat{x} and \hat{x}_i for $\hat{x}(x)$ and $\hat{x}_i(x)$. Obviously $\hat{x} \in W$ since W_i is convex. We first do the approximation

$$(2.3) \quad f(x) \approx f(\hat{x}).$$

In this sketch we write $L \approx R$ if under $\tilde{\mu}$ with high probability $|L - R|$ is controlled, we will not bother to make rigorous the meaning of \approx .

In [7], since each μ_i is supported on $\{0, 1\}$, \hat{x} has the good expression [7], the expression above Lemma 3.1,

$$(2.4) \quad \hat{x}_i = \frac{1}{1 + e^{-\Delta_i f(x)}},$$

where $\Delta_i f(x)$ is the discrete derivative defined as follows:

$$\Delta_i f(x) := f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n).$$

In our case we do not have a good expression as (2.4).

(2) The next step is to construct a covering set $\mathcal{D}'(\epsilon)$ of $\{\hat{x} : x \in W\}$, such that for each $x \in W$, there exists some $p^x = (p_1^x, \dots, p_n^x) \in \mathcal{D}'(\epsilon)$ which is close to \hat{x} . Consequently we have

$$(2.5) \quad f(\hat{x}) \approx f(p^x).$$

In [7], the covering set $\mathcal{D}'(\epsilon)$ is constructed by applying a function $u(x) = 1/(1 + e^{-x})$ on each point in $\mathcal{D}(\epsilon)$ ([7], 3 lines below (3.16)). This makes sense because $\mathcal{D}(\epsilon)$ is the covering set of the gradient of $f(x)$, and \hat{x}_i has the expression (2.4). Special properties of this explicit construction is used in [7], such as $|u'(x)| \leq 1/4$. In our case we construct $\mathcal{D}'(\epsilon)$ in the general setting.

(3) Next, for each i and $p = (p_1, \dots, p_n) \in \mathcal{D}'(\epsilon)$ we construct a measure v_i^p supported on W_i , such that $v_i^p \ll \mu_i$, $m(v_i^p) = p_i$, and the following approximation holds:

$$(2.6) \quad -\sum_{i=1}^n D(v_i^{p^x} \parallel \mu_i) + \sum_{i=1}^n \log\left(\frac{dv_i^{p^x}}{d\mu_i}(x)\right) \approx 0.$$

In [7], μ_i is Bernoulli($\frac{1}{2}$) (the Bernoulli distribution with parameter $\frac{1}{2}$). Therefore, for any $y = (y_1, \dots, y_n) \in [0, 1]^n$, the unique measure v_i^y with $v_i^y \ll \mu_i$ and $m(v_i^y) = y_i$ is just Bernoulli(y_i). Hence one can write down the explicit form of the KL divergence between v_i^y and μ_i as

$$D(v_i^y \parallel \mu_i) = y_i \log y_i + (1 - y_i) \log(1 - y_i) + \log 2.$$

In this way, $-\sum_{i=1}^n D(v_i^{p^x} \parallel \mu_i) + \sum_{i=1}^n \frac{dv_i^{p^x}}{d\mu_i}(x)$ becomes [7], (3.13), which has a good form to analyze. In our case, we build the measure $v_i^{p^x}$ in Section 3.2.2, and we show several general properties of this kind of measure, which help us to prove our approximation.

(4) Combining (2.3), (2.5) and (2.6), we get the following approximation

$$(2.7) \quad f(x) \approx f(p^x) - \sum_{i=1}^n D(v_i^{p^x} \parallel \mu_i) + \sum_{i=1}^n \log\left(\frac{dv_i^{p^x}}{d\mu_i}(x)\right).$$

In [7], to bound the error of the above approximation, the authors decompose the error into $f(x) - f(\hat{x})$ and [7], (3.13), which does not work in the general case here. In our proof, we find the decomposition (see (3.30) and (3.31)) that works in general.

(5) Note that if we fix $y \in W$, then by the fact that $\int_{W_i} \frac{dv_i^{p^y}}{d\mu_i}(x) d\mu_i(x) = 1$ we get

$$\int_W e^{f(p^y) - \sum_{i=1}^n D(v_i^{p^y} \parallel \mu_i) + \sum_{i=1}^n \log \frac{dv_i^{p^y}}{d\mu_i}(x)} d\mu(x) = f(p^y) - \sum_{i=1}^n D(v_i^{p^y} \parallel \mu_i).$$

Therefore, with above approximations we have that

$$\begin{aligned} \log \int_W e^{f(x)} d\mu(x) &= \log \int_W e^{f(p^x) - \sum_{i=1}^n D(v_i^{p^x} \parallel \mu_i) + \sum_{i=1}^n \log \frac{dv_i^{p^x}}{d\mu_i}(x)} d\mu(x) \\ &\quad + \text{error terms} \\ &\leq \log \sum_{p \in \mathcal{D}'(\epsilon)} \left(f(p) - \sum_{i=1}^n D(v_i^p \parallel \mu_i) \right) + \text{error terms} \\ &\leq \max_{v \ll \mu, v = v_1 \times v_2 \times \dots \times v_n} \left\{ f(m(v)) - \sum_{i=1}^n D(v_i \parallel \mu_i) \right\} \\ &\quad + \log |\mathcal{D}'(\epsilon)| + \text{error terms,} \end{aligned}$$

where in the last inequality we use the fact that $m(v_i^p) = p_i$. The above inequality leads to the desired upper bound.

3. Proof of Theorem 1.

3.1. *The lower bound part of Theorem 1.* The idea to prove the lower bound is first to use the Gibbs variational principle ((3.1) below) on any product measure ν , and then to approximate the first term on the right-hand side of (3.1) by $f(m(\nu))$ ($m(\nu)$ is defined at (1.7)), where the error is controlled by the norms of the second derivatives of f .

PROOF. For any $\nu = \nu_1 \times \nu_2 \times \dots \times \nu_n$, by the Gibbs variational principle, we have

$$(3.1) \quad \log \int_W e^{f(x)} d\mu(x) \geq \int_W f(x) d\nu(x) - D(\nu \parallel \mu).$$

Because ν and μ are both product measures, we have the following decomposition

$$(3.2) \quad D(\nu \parallel \mu) = \sum_{i=1}^n D(\nu_i \parallel \mu_i).$$

Next we approximate $\int_W f(x) d\nu(x)$ by $f(m(\nu))$. For $x \in V$, $i \in [n]$ and $z_i \in V_i$, define

$$(3.3) \quad x_{z_i}^{(i)} := (x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_n).$$

Fix $\theta = (\theta_1, \dots, \theta_n) \in W$. For $t \in [0, 1]$, by the definition of $m(\nu_i)$ (1.6) and the fact that $f_i(tx_{\theta_i}^{(i)} + (1-t)m(\nu))(\cdot)$ is linear, we have $\int_W f_i(tx_{\theta_i}^{(i)} + (1-t)m(\nu))(x_i - m(\nu_i)) d\nu(x) = 0$, which implies that

$$(3.4) \quad \begin{aligned} & \left| \int_W f_i(tx + (1-t)m(\nu))(x_i - m(\nu_i)) d\nu(x) \right| \\ &= \left| \int_W (f_i(tx + (1-t)m(\nu)) \right. \\ & \quad \left. - f_i(tx_{\theta_i}^{(i)} + (1-t)m(\nu)))(x_i - m(\nu_i)) d\nu(x) \right| \\ &\leq \int_W c_{ii} \times \|tx_i - t\theta_i\|_{V_i} \times \|x_i - m(\nu_i)\|_{V_i} d\nu(x) \leq tc_{ii} M^2. \end{aligned}$$

By (3.4) and the expression $f(x) - f(m(\nu)) = \sum_{i=1}^n \int_0^1 f_i(tx + (1-t)m(\nu))(x_i - m(\nu_i)) dt$, we further get

$$(3.5) \quad \int_W (f(x) - f(m(\nu))) d\nu(x) \geq - \sum_{i=1}^n \int_0^1 tc_{ii} M^2 dt = -\frac{M^2}{2} \sum_{i=1}^n c_{ii}.$$

Plugging (3.2) and (3.5) into (3.1), we get

$$\log \int_W e^{f(x)} d\mu(x) \geq f(m(\nu)) - \sum_{i=1}^n D(\nu_i \parallel \mu_i) - \frac{M^2}{2} \sum_{i=1}^n c_{ii}.$$

Taking the sup over $\{\nu : \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n, \nu \ll \mu\}$ completes the proof. \square

3.2. *The upper bound part of Theorem 1.* In this subsection we prove the upper bound of Theorem 1. In Section 3.2.1, we construct the covering of $\{\hat{x} : x \in W\}$, which plays an important role in our approximation. In Section 3.2.2 we show several properties of the measure ν^{p^x} , which is described in (2.6) and is defined at (3.27). We provide the error bound for the approximation (2.7) in Section 3.2.3, and we summarize and finish the proof in Section 3.2.4.

3.2.1. *The construction of $\mathcal{D}'(\epsilon)$.* In order to construct the covering of $\{\widehat{x} : x \in W\}$ (defined at (2.2)), for any $d = (d_1, d_2, \dots, d_n) \in \mathcal{D}(\epsilon)$ we construct a corresponding $p(d) = (p(d)_1, p(d)_2, \dots, p(d)_n) \in W$ in the following way: recalling that $d_i(\cdot)$ is a bounded linear functional from V_i to \mathbb{R} , let

$$(3.6) \quad p(d)_i := \frac{\int_{W_i} z_i e^{d_i(z_i)} d\mu_i(z_i)}{\int_{W_i} e^{d_i(z_i)} d\mu_i(z_i)} \quad \text{and} \quad p(d) := (p(d)_1, \dots, p(d)_n).$$

The existence of $p(d)$ is guaranteed by the fact that W_i is compact, and obviously $p(d) \in W$ since W is convex. Define

$$\mathcal{D}'(\epsilon) := \{p(d) : d \in \mathcal{D}(\epsilon)\}.$$

For each x , we choose a d^x such that

$$(3.7) \quad d^x \in \left\{ d \in \mathcal{D}(\epsilon) \text{ s.t. } \sum_{i=1}^n \|f_i(x) - d_i\|^2 \leq \epsilon^2 n \right\},$$

where if the set on the right-hand side contains more than one element, we just choose any one in it and fix the choice. Using (3.6) we can further define

$$(3.8) \quad p^x := (p_1^x, p_2^x, \dots, p_n^x), \quad \text{where } p_i^x := p(d^x)_i \quad \forall i \in [n].$$

In the following we show that $\mathcal{D}'(\epsilon)$ is a good covering of $\{\widehat{x} : x \in W\}$, by bounding the term $\sum_{i=1}^n \|\widehat{x}_i - p_i^x\|_{V_i}^2$. Recall that $d_i^x(\cdot)$ is a linear functional from W_i to \mathbb{R} . Let

$$p_i^x(t) := \frac{\int_{W_i} z_i e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)} d\mu_i(z_i)}{\int_{W_i} e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)} d\mu_i(z_i)}.$$

Then $p_i^x(t)$ is an interpolation between p_i^x and \widehat{x}_i , since it is easy to verify that

$$(3.9) \quad p_i^x(0) = p_i^x, \quad p_i^x(1) = \widehat{x}_i.$$

Let

$$(3.10) \quad e(x, i) := \frac{\widehat{x}_i - p_i^x}{\|\widehat{x}_i - p_i^x\|_{V_i}}, \quad V_{x,i} := \{ke(x, i) : k \in \mathbb{R}\}.$$

Then clearly $V_{x,i}$ is a 1-dimension subspace of V_i . Define a linear functional $g_0 : V_{x,i} \rightarrow \mathbb{R}$ as

$$(3.11) \quad g_0(ke(x, i)) = k,$$

and then obviously $\|g_0\| = 1$. By the Hahn–Banach theorem, we can extend g_0 to g , a linear functional from V_i to \mathbb{R} such that

$$(3.12) \quad g(z_i) = g_0(z_i) \quad \forall z_i \in V_{x,i}, \quad \|g\| = \|g_0\| = 1.$$

Thus for any $z_i^{(1)}, z_i^{(2)} \in W_i$ we have

$$(3.13) \quad |g(z_i^{(1)}) - g(z_i^{(2)})| \leq \|z_i^{(1)} - z_i^{(2)}\|_{V_i}.$$

Using the fact that $f(\cdot)$ is bounded and Fréchet differentiable, and W_i is compact, it is easy to see that $g(p_i^x(t))$ is differentiable with respect to t . By the definition of $p_i^x(t)$, after some algebra we arrive at

$$(3.14) \quad \begin{aligned} \frac{dg(p_i^x(t))}{dt} &= \left(\frac{\int_{W_i} g(z_i) e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)} d\mu_i(z_i)}{\int_{W_i} e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)} d\mu_i(z_i)} \right)' \\ &= \mathbb{E}_{\phi_i}[(f(x_{Z_i}^{(i)}) - d_i^x(Z_i))(g(Z_i) - \mathbb{E}_{\phi_i}[g(Z_i)])], \end{aligned}$$

where the expectation is taken with respect to Z_i , which obeys the measure $\phi_i^j \ll \mu_i$ defined as

$$\frac{d\phi_i^j}{d\mu_i}(z_i) := \frac{e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)}}{\int_{W_i} e^{tf(x_{z_i}^{(i)})+(1-t)d_i^x(z_i)} d\mu_i(z_i)}.$$

Recall that $\theta = (\theta_1, \dots, \theta_n)$ is a fixed point in W . It is easy to check that

$$\begin{aligned} & (f(tx_{z_i}^{(i)} + (1-t)x_{\theta_i}^{(i)}) - d_i^x(tx_{z_i} + (1-t)\theta_i))' \\ &= (f_i(tx_{z_i}^{(i)} + (1-t)x_{\theta_i}^{(i)}) - d_i^x)(z_i - \theta_i). \end{aligned}$$

Therefore, writing the following difference as the integral of derivative, we can see that for any $z_i \in W_i$,

$$\begin{aligned} & |f(x_{z_i}^{(i)}) - d_i^x(z_i) - (f(x_{\theta_i}^{(i)}) - d_i^x(\theta_i))| \\ &= \left| \int_0^1 (f_i(tx_{z_i}^{(i)} + (1-t)x_{\theta_i}^{(i)}) - d_i^x)(z_i - \theta_i) dt \right| \\ (3.15) \quad & \leq \int_0^1 |(f_i(tx_{z_i}^{(i)} + (1-t)x_{\theta_i}^{(i)}) - f_i(x))(z_i - \theta_i)| dt \\ & \quad + \int_0^1 |(f_i(x) - d_i^x)(z_i - \theta_i)| dt \\ & \leq c_{ii}M^2 + \|f_i(x) - d_i^x\|M. \end{aligned}$$

Noting that $\mathbb{E}_{\phi_i^j}[g(Z_i) - \mathbb{E}_{\phi_i^j}[g(Z_i)]] = 0$, we have

$$(3.16) \quad \mathbb{E}_{\phi_i^j}[(f(x_{\theta_i}^{(i)}) - d_i^x(\theta_i))(g(Z_i) - \mathbb{E}_{\phi_i^j}[g(Z_i)])] = 0.$$

From (1.5) and (3.13) it is clear that for each $z_i^{(1)}, z_i^{(2)} \in W_i$ we have $|g(z_i^{(1)}) - g(z_i^{(2)})| \leq M$, which implies that

$$(3.17) \quad \mathbb{E}_{\phi_i^j}[|g(Z_i) - \mathbb{E}_{\phi_i^j}[g(Z_i)]|] \leq M.$$

Subtracting $\mathbb{E}_{\phi_i^j}[(f(x_{\theta_i}^{(i)}) - d_i^x(\theta_i))(g(Z_i) - \mathbb{E}_{\phi_i^j}[g(Z_i)])]$ from the right-hand side of (3.14), with (3.15), (3.16) and (3.17) we have

$$(3.18) \quad \left| \frac{dg(p_i^x(t))}{dt} \right| \leq c_{ii}M^3 + \|f_i(x) - d_i^x\|M^2,$$

and consequently by (3.10), (3.11) and (3.12) we see that

$$(3.19) \quad \begin{aligned} \|\widehat{x}_i - p_i^x\|_{V_i} &= g(\widehat{x}_i - p_i^x) = g(p_i^x(1) - p_i^x(0)) \\ &\leq c_{ii}M^3 + \|f_i(x) - d_i^x\|M^2. \end{aligned}$$

Therefore, from (3.7), (3.19) and the basic inequalities $(a + b)^2 \leq 2a^2 + 2b^2$, $(a^2 + b^2)^{1/2} \leq a + b$, we have

$$(3.20) \quad \begin{aligned} \left(\sum_{i=1}^n \|\widehat{x}_i - p_i^x\|_{V_i}^2 \right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^n (c_{ii}M^3 + \|f_i(x) - d_i^x\|M^2)^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{2}M^3 \left(\sum_{i=1}^n c_{ii}^2 \right)^{\frac{1}{2}} + \sqrt{2}M^2n^{\frac{1}{2}}\epsilon. \end{aligned}$$

3.2.2. *The construction and properties of the measure ν^p .* Before constructing the measure ν^p , let us take a look at the term

$$(3.21) \quad \max_{\nu \ll \mu, \nu = \nu_1 \times \nu_2 \times \dots \times \nu_n} \left\{ f(m(\nu)) - \sum_{i=1}^n D(\nu_i \parallel \mu_i) \right\}.$$

In order to achieve the maximum, a natural question one might ask is: when $(m(\nu))$ is fixed, what is the minimum value of $\sum_{i=1}^n D(\nu_i \parallel \mu_i)$? For every $y = (y_1, \dots, y_n) \in W$, we consider the following problem:

$$(3.22) \quad \min \left\{ \sum_{i=1}^n D(\nu_i \parallel \mu_i) : \nu \text{ is a product probability measure with } \nu \ll \mu \text{ and } m(\nu) = y \right\}.$$

In this subsection, we show several properties of the minimizer of (3.22). We prove that

PROPOSITION 1. *If a measure $\nu^y = \nu_1^y \times \nu_2^y \times \dots \times \nu_n^y$ satisfies that for each $i \in [n]$,*

$$(3.23) \quad \nu_i^y \ll \mu_i, \quad m(\nu_i^y) = y_i, \quad \text{and} \quad \frac{d\nu_i^y}{d\mu_i}(z_i) = e^{R_i(z_i)}$$

for a linear functional $R_i(\cdot) : V_i \rightarrow \mathbb{R}$,

then ν^y achieves the minimum in (3.22).

PROOF. For each i , assume that ν_i^y satisfies (3.23). For any other measure $\tilde{\nu}_i^y$ with $m(\tilde{\nu}_i^y) = y_i$ and $\tilde{\nu}_i^y \ll \mu_i$, since $\log \frac{d\nu_i^y}{d\mu_i}(\cdot)$ is linear by (3.23), we have

$$(3.24) \quad \int_{W_i} \log \frac{d\nu_i^y}{d\mu_i}(z_i) d\tilde{\nu}_i^y(z_i) = \int_{W_i} \log \frac{d\nu_i^y}{d\mu_i}(z_i) d\nu_i^y(z_i) = D(\nu_i^y \parallel \mu_i).$$

Combining (3.24) and the fact that $D(\tilde{\nu}_i^y \parallel \nu_i^y) \geq 0$, we have

$$0 \leq D(\tilde{\nu}_i^y \parallel \nu_i^y) = \int_{W_i} \frac{d\tilde{\nu}_i^y}{d\mu_i}(z_i) \log \frac{\frac{d\tilde{\nu}_i^y}{d\mu_i}(z_i)}{\frac{d\nu_i^y}{d\mu_i}(z_i)} d\mu_i(z_i) = D(\tilde{\nu}_i^y \parallel \mu_i) - D(\nu_i^y \parallel \mu_i),$$

and it completes the proof. \square

Now let us consider the properties of ν^y satisfying (3.23). From (3.23) we can see that $\forall z_i \in W_i$,

$$(3.25) \quad \log \frac{d\nu_i^y}{d\mu_i}(z_i) = R_i(z_i).$$

Recalling that $\mathbb{E}_{\nu_i^y}[Z_i] = m(\nu_i^y)$, by (3.23) and (3.25), we see that

$$(3.26) \quad \begin{aligned} D(\nu_i^y \parallel \mu_i) &= \int_{W_i} \frac{d\nu_i^y}{d\mu_i}(z_i) \log \frac{d\nu_i^y}{d\mu_i}(z_i) d\mu_i(z_i) \\ &= \int_{W_i} R_i(z_i) d\nu_i^y(z_i) = R_i(m(\nu_i^y)). \end{aligned}$$

Note that we did not prove that for any $y \in W$ there exists a measure ν^y satisfying (3.23). For each $p \in \mathcal{D}'(\epsilon)$, we construct $\nu^p = (\nu_1^p, \dots, \nu_n^p)$ directly at (3.27) below, and show that it satisfies (3.23), and hence it shares the property (3.26). For each $p = p(d) \in \mathcal{D}'(\epsilon)$, recalling that d_i is a linear functional from V_i to \mathbb{R} , we can define ν_i^p , a measure on V_i , as

$$(3.27) \quad \frac{d\nu_i^p}{d\mu_i}(z_i) := \frac{e^{d_i(z_i)}}{\int_{W_i} e^{d_i(z_i)} d\mu_i(z_i)} = e^{\lambda(p_i)+d_i(z_i)},$$

where $\lambda(p_i)$ is a normalizing number satisfies that $e^{\lambda(p_i)} = (\int_{W_i} e^{d_i(z_i)} d\mu_i(z_i))^{-1}$. From the construction of $p(d)$ in (3.6), it is easy to see that

$$\int_{W_i} z_i d\nu_i^p(z_i) = \int_{W_i} z_i e^{\lambda(p_i)+d_i(z_i)} d\mu_i(z_i) = p_i.$$

The same approach we used in (3.26) can be applied here to show that

$$(3.28) \quad D(\nu_i^p \parallel \mu_i) = \lambda(p_i) + d_i(p_i),$$

and consequently

$$(3.29) \quad \sum_{i=1}^n (\lambda(p_i^x) + d_i^x(x_i)) - \sum_{i=1}^n D(\nu_i^{p_i^x} \parallel \mu_i) = \sum_{i=1}^n d_i^x(x_i - p_i^x).$$

3.2.3. *The approximation (2.7).* Due to (3.29), for the approximation (2.7) it suffices to bound

$$\left| f(p^x) + \sum_{i=1}^n d_i^x(x_i - p_i^x) - f(x) \right| \leq \Delta_1 + \Delta_2,$$

where

$$(3.30) \quad \Delta_1 := |f(\widehat{x}) - f(x)| + \left| \sum_{i=1}^n f_i(x_{\theta_i}^{(i)}) (\widehat{x}_i - x_i) \right|,$$

$$(3.31) \quad \Delta_2 := |f(\widehat{x}) - f(p^x)| + \left| \sum_{i=1}^n d_i^x(\widehat{x}_i - p_i^x) \right| + \left| \sum_{i=1}^n (d_i^x - f_i(x_{\theta_i}^{(i)})) (\widehat{x}_i - x_i) \right|.$$

So the proof of the approximation (2.7) consists of the bounds for Δ_1 and Δ_2 , which will be given separately below.

Bound for Δ_1 . Recall the definition of $\tilde{\mu}$ from (2.1). We show the following proposition.

PROPOSITION 2. *Let all notation be as in Theorem 1. We have the following bound*

$$(3.32) \quad \mathbb{E}_{\tilde{\mu}}[(f(X) - f(\widehat{X}))^2] \leq M^2 \left(a \sum_{i=1}^n c_{ii} + \sum_{i=1}^n b_i^2 \right) + M^4 \left(a \sum_{i,j=1}^n c_{ij}^2 + \sum_{i,j=1}^n b_i b_j c_{ij} \right).$$

PROOF. Let

$$h(X) := f(X) - f(\widehat{X}),$$

and then clearly

$$(3.33) \quad |h(X)| \leq 2a.$$

From the definition of \widehat{x} in (2.2), we have

$$(3.34) \quad \widehat{x}_j(x) = \frac{\int_{W_j} z_j e^{f(x_{z_j}^{(j)})} d\mu_j(z_j)}{\int_{W_j} e^{f(x_{z_j}^{(j)})} d\mu_j(z_j)}.$$

Note that $\widehat{x}_j(\cdot)$ is a functional from V to V_j . We claim that $\widehat{x}_j(\cdot)$ is Fréchet differentiable (in (1.4)) we just define the notion of Fréchet differentiability for real-valued functional. We can define it for vector-valued functional similarly; see [10], Chapter 2. For $r \in V$ we let

$$r_0^{(j)} := (r_1, \dots, r_{j-1}, 0, r_{j+1}, \dots, r_n).$$

Define $\phi_j(x)(\cdot) : V \rightarrow V_j$ as

$$\begin{aligned} \phi_j(x)(r) := & \frac{\int_{W_j} z_j f'(x_{z_j}^{(j)})(r_0^{(j)}) e^{f(x_{z_j}^{(j)})} d\mu_j(z_j)}{\int_{W_j} e^{f(x_{z_j}^{(j)})} d\mu_j(z_j)} \\ & - \frac{\int_{W_j} z_j e^{f(x_{z_j}^{(j)})} d\mu_j(z_j) \int_{W_j} f'(x_{z_j}^{(j)})(r_0^{(j)}) e^{f(x_{z_j}^{(j)})} d\mu_j(z_j)}{(\int_{W_j} e^{f(x_{z_j}^{(j)})} d\mu_j(z_j))^2}. \end{aligned}$$

By writing out $\widehat{x}_j(x+r)$ and $\widehat{x}_j(x)$ according to their definitions and calculating their difference, due to the fact that W_j is compact and $f(\cdot)$ is bounded and Fréchet differentiable, we can check that $\widehat{x}_j(x+r) - \widehat{x}_j(x) - \phi_j(x)(r) = o(r)$. We define the partial differential $\frac{d\widehat{x}_j(x)}{dx_i}(\cdot) : V_i \rightarrow V_j$ as

$$\frac{d\widehat{x}_j(x)}{dx_i}(r_i) := \phi_j((0, \dots, r_i, \dots, 0)).$$

Recall the definition of $\tilde{\mu}$ (2.1). From the definition of $\phi_j(x)(\cdot)$ we can write that for $j \neq i$,

$$\begin{aligned} (3.35) \quad \frac{d\widehat{x}_j(x)}{dx_i}(\cdot) &= \mathbb{E}_{\tilde{\mu}}[X_j f_i(X)(\cdot) - \widehat{x}_j f_i(X)(\cdot) \mid X_k = x_k \text{ for } k \neq j] \\ &= \mathbb{E}_{\tilde{\mu}}[(X_j - \widehat{x}_j)(f_i(X) - f_i(X_{\theta_j}^{(j)}))(\cdot) \mid X_k = x_k \text{ for } k \neq j] \\ &\quad + \mathbb{E}_{\tilde{\mu}}[(X_j - \widehat{x}_j) f_i(X_{\theta_j}^{(j)}) (\cdot) \mid X_k = x_k \text{ for } k \neq j]. \end{aligned}$$

By the definition of \widehat{x}_j we have that for any $r \in V$

$$(3.36) \quad \mathbb{E}_{\tilde{\mu}}[(X_j - \widehat{x}_j) f_i(X_{\theta_j}^{(j)})(r) \mid X_k = x_k \text{ for } k \neq j] = 0.$$

Due to the fact that

$$\|(f_i(X) - f_i(X_{\theta_j}^{(j)}))(\cdot)\| = \left\| \int_0^1 f_{ij}(tX + (1-t)X_{\theta_j}^{(j)}) (\cdot, X_j - \theta_j) dt \right\| \leq c_{ij}M,$$

we have

$$(3.37) \quad \|\mathbb{E}_{\tilde{\mu}}[(X_j - \widehat{x}_j)(f_i(X) - f_i(X_{\theta_j}^{(j)}))(\cdot) \mid X_k = x_k \text{ for } k \neq j]\| \leq c_{ij}M^2.$$

Combining (3.35), (3.36) and (3.37), we see that for $j \neq i$

$$(3.38) \quad \left\| \frac{d\widehat{X}_j(x)}{dx_i}(\cdot) \right\| \leq c_{ij}M^2.$$

Obviously $\frac{d\widehat{X}_i(x)}{dx_i}(\cdot) \equiv 0$. For $t \in [0, 1]$ and $x \in W$, we define a linear functional $u_i(t, x)(\cdot) : V_i \rightarrow \mathbb{R}$ as

$$(3.39) \quad u_i(t, x)(\cdot) := f_i(tx + (1-t)\widehat{x})(\cdot).$$

Then it is clear that

$$(3.40) \quad h(x) = \int_0^1 \sum_{i=1}^n u_i(t, x)(x_i - \widehat{x}_i) dt.$$

Following the same idea from [7], (3.3), to the end of the proof of [7], Lemma 3.1, we can verify that

$$(3.41) \quad \begin{aligned} & |\mathbb{E}_{\tilde{\mu}}[(u_i(t, X) - u_i(t, X_{\theta_i}^{(i)}))(X_i - \widehat{X}_i)h(X_{\theta_i}^{(i)})]| \\ & \leq 2aM \left(tMc_{ii} + (1-t)M^3 \sum_{j=1}^n c_{ij}^2 \right) \end{aligned}$$

and

$$(3.42) \quad \begin{aligned} & |\mathbb{E}_{\tilde{\mu}}[u_i(t, X)(X_i - \widehat{X}_i)(h(X) - h(X_{\theta_i}^{(i)}))]| \\ & \leq b_iM \left(Mb_i + M^3 \sum_{j=1}^n b_jc_{ij} \right). \end{aligned}$$

Due to the fact that $\mathbb{E}_{\tilde{\mu}}[u_i(t, X_{\theta_i}^{(i)})(X_i - \widehat{X}_i)h(X_{\theta_i}^{(i)})] = 0$, we have the following decomposition

$$\begin{aligned} & \mathbb{E}_{\tilde{\mu}}[u_i(t, X)(X_i - \widehat{X}_i)h(X)] \\ & = \mathbb{E}_{\tilde{\mu}}[(u_i(t, X) - u_i(t, X_{\theta_i}^{(i)}))(X_i - \widehat{X}_i)h(X_{\theta_i}^{(i)})] \\ & \quad + \mathbb{E}_{\tilde{\mu}}[u_i(t, X)(X_i - \widehat{X}_i)(h(X) - h(X_{\theta_i}^{(i)}))]. \end{aligned}$$

Thus by (3.40), (3.41) and (3.42), using the above decomposition we have

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}}[h^2(X)] & = \int_0^1 \sum_{i=1}^n \mathbb{E}_{\tilde{\mu}}[u_i(t, X)(X_i - \widehat{X}_i)h(X)] dt \\ & \leq M^2 \left(a \sum_{i=1}^n c_{ii} + \sum_{i=1}^n b_i^2 \right) + M^4 \left(a \sum_{i,j=1}^n c_{ij}^2 + \sum_{i,j=1}^n b_i b_j c_{ij} \right). \end{aligned} \quad \square$$

We provide the following proposition, which is also needed for bounding Δ_1 .

PROPOSITION 3. *If we denote*

$$G(x) := \sum_{i=1}^n f_i(x_{\theta_i}^{(i)})(x_i - \widehat{x}_i),$$

then

$$\mathbb{E}_{\tilde{\mu}}[G^2(X)] \leq M^2 \sum_{i=1}^n b_i^2 + M^3 \sum_{i,j=1}^n b_i(c_{ji} + b_jc_{ji}M).$$

PROOF. Taking derivative of G and using (3.38), we have

$$\begin{aligned}
 & \left\| \frac{\partial G(x)}{\partial x_i}(\cdot) \right\| \\
 (3.43) \quad & = \left\| f_i(x_{\theta_i}^{(i)})(\cdot) + \sum_{j \neq i}^n \left(f_{ji}(x_{\theta_j}^{(j)})(x_j - \widehat{x}_j, \cdot) + f_j(x_{\theta_j}^{(j)}) \left(-\frac{\partial \widehat{x}_j(x)}{\partial x_i}(\cdot) \right) \right) \right\| \\
 & \leq b_i + \sum_{j \neq i}^n (c_{ji}M + b_j c_{ji}M^2) \leq b_i + \sum_j^n (c_{ji}M + b_j c_{ji}M^2).
 \end{aligned}$$

Following the same idea from [7], (3.11), to the end of the proof of [7], Lemma 3.2, we finish the proof. \square

Next we combine the above two propositions. Denote

$$\begin{aligned}
 (3.44) \quad B_{1,1} & := \left(M^2 \left(a \sum_{i=1}^n c_{ii} + \sum_{i=1}^n b_i^2 \right) + M^4 \left(a \sum_{i,j=1}^n c_{ij}^2 + \sum_{i,j=1}^n b_i b_j c_{ij} \right) \right)^{\frac{1}{2}}, \\
 B_{1,2} & := \left(M^2 \sum_{i=1}^n b_i^2 + M^3 \sum_{i,j=1}^n b_i (c_{ji} + b_j c_{ji}M) \right)^{\frac{1}{2}}.
 \end{aligned}$$

And let

$$\begin{aligned}
 A_1 & := \{x \in W, |f(x) - f(\widehat{x})| \leq 2B_{1,1}\}, \\
 A_2 & := \left\{ x \in W, \left| \sum_{i=1}^n f_i(x_{\theta_i}^{(i)})(x_i - \widehat{x}_i) \right| \leq 2B_{1,2} \right\}.
 \end{aligned}$$

Define $A := A_1 \cap A_2$. Then with Proposition 2 and Proposition 3 it is easy to see that $\mathbb{P}_{\tilde{\mu}}(A) \geq \frac{1}{2}$. Therefore, with the fact that $2(B_{1,1} + B_{1,2}) < B_1$ (defined in (1.10)), we have

$$\begin{aligned}
 (3.45) \quad \log \int_W e^{f(x)} d\mu(x) & \leq \log \int_A e^{f(x)} d\mu(x) + \log 2 \\
 & \leq \log \int_A e^{f(\widehat{x}) + \sum_{i=1}^n f_i(x_{\theta_i}^{(i)})(x_i - \widehat{x}_i)} d\mu(x) \\
 & \quad + B_1 + \log 2.
 \end{aligned}$$

Bound for Δ_2 . For $|f(\widehat{x}) - f(p^x)|$, rewriting it as

$$f(\widehat{x}) - f(p^x) = \int_0^1 \sum_{i=1}^n f_i(t\widehat{x} + (1-t)(\widehat{x} - p^x))(\widehat{x}_i - p_i^x) dt,$$

by (3.20) and Cauchy's inequality we have

$$\begin{aligned}
 (3.46) \quad |f(\widehat{x}) - f(p^x)| & \leq \sum_{i=1}^n b_i \|\widehat{x}_i - p_i^x\|_{V_i} \\
 & \leq \left(\sum_{i=1}^n b_i^2 \right)^{\frac{1}{2}} \left(\sqrt{2}M^3 \left(\sum_{i=1}^n c_{ii}^2 \right)^{\frac{1}{2}} + \sqrt{2}M^2 n^{\frac{1}{2}} \epsilon \right).
 \end{aligned}$$

For $|\sum_{i=1}^n (f_i(x_{\theta_i}^{(i)}) - d_i^x)(\widehat{x}_i - x_i)|$, using (1.8) and the Cauchy-Schwarz inequality we have

$$\left| \sum_{i=1}^n (f_i(x) - d_i^x)(\widehat{x}_i - x_i) \right| \leq M \sqrt{n} \left(\sum_{i=1}^n \|f_i(x) - d_i^x\|^2 \right)^{\frac{1}{2}} \leq Mn\epsilon,$$

and thus by decomposing $f_i(x_{\theta_i}^{(i)}) - d_i^x$ as $f_i(x_{\theta_i}^{(i)}) - f_i(x)$ and $f_i(x) - d_i^x$ we get

$$\begin{aligned}
 \left| \sum_{i=1}^n (f_i(x_{\theta_i}^{(i)}) - d_i^x)(\widehat{x}_i - x_i) \right| &\leq \left| \sum_{i=1}^n (f_i(x_{\theta_i}^{(i)}) - f_i(x))(\widehat{x}_i - x_i) \right| \\
 &+ \left| \sum_{i=1}^n (f_i(x) - d_i^x)(\widehat{x}_i - x_i) \right| \\
 &\leq \sum_{i=1}^n M^2 c_{ii} + Mn\epsilon.
 \end{aligned}
 \tag{3.47}$$

For the last term $|\sum_{i=1}^n d_i^x(\widehat{x}_i - p_i^x)|$, noting that $\sum_{i=1}^n \|d_i^x\|^2 \leq 2\sum_{i=1}^n b_i^2 + 2\epsilon^2 n$, by (3.20) and Cauchy's inequality we have

$$\begin{aligned}
 \left| \sum_{i=1}^n d_i^x(\widehat{x}_i - p_i^x) \right| \\
 \leq \left(2\sum_{i=1}^n b_i^2 + 2\epsilon^2 n \right)^{\frac{1}{2}} \left(\sqrt{2}M^3 \left(\sum_{i=1}^n c_{ii}^2 \right)^{\frac{1}{2}} + \sqrt{2}M^2 n^{\frac{1}{2}} \epsilon \right).
 \end{aligned}
 \tag{3.48}$$

Recalling the definition of Δ_2 , with (3.46), (3.47), (3.48) and the definition of B_2 in (1.11), it is clear that

$$\Delta_2 \leq B_2.
 \tag{3.49}$$

3.2.4. Proof of (1.9).

PROOF. By the definition of Δ_2 (3.31) it is easy to verify that

$$\left| f(\widehat{x}) - f(p^x) + \sum_{i=1}^n f_i(x_{\theta_i}^{(i)})(x_i - \widehat{x}_i) - \sum_{i=1}^n d_i^x(x_i - p_i^x) \right| \leq \Delta_2.$$

Define

$$C(d) := \{x : x \in W, d^x = d\}.$$

Using (3.45) and (3.49) we have

$$\begin{aligned}
 \log \int_W e^{f(x)} d\mu(x) \\
 \leq \log 2 + B_1 + B_2 \\
 + \log \sum_{d \in D(\epsilon)} \int_{x \in C(d)} e^{f(p(d)) + \sum_{i=1}^n d_i(x_i - p(d)_i)} d\mu(x).
 \end{aligned}
 \tag{3.50}$$

From (3.27) it is clear that

$$\int_W e^{\sum_{i=1}^n \lambda(p(d)_i) + \sum_{i=1}^n d_i(x_i)} d\mu(x) = 1.$$

Combining the above equality and (3.28), we get the following bound

$$\begin{aligned}
 \int_{x \in C(d)} e^{f(p(d)) + \sum_{i=1}^n d_i(x_i - p(d)_i)} d\mu(x) \\
 \leq e^{f(p(d)) - \sum_{i=1}^n d_i(p(d)_i) - \sum_{i=1}^n \lambda(p(d)_i)} \\
 = e^{f(p(d)) - \sum_{i=1}^n D(v_i^{p(d)} \| \mu_i)}.
 \end{aligned}
 \tag{3.51}$$

Plugging (3.51) into (3.50) and noting the fact that for any $d \in D(\epsilon)$

$$f(p(d)) - \sum_{i=1}^n D(v_i^{p(d)} \parallel \mu_i) \leq \max_{v \ll \mu, v=v_1 \times v_2 \times \dots \times v_n} \left\{ f(m(v)) - \sum_{i=1}^n D(v_i \parallel \mu_i) \right\},$$

we finish the proof of the upper bound. \square

4. Proofs of applications. In this section we give the proofs of our examples.

4.1. *Proof of Theorem 2.* In this subsection we prove Theorem 2. Throughout the proof, C will denote any positive constant that does not depend on N . Recall the definitions in Section 1.2.1, and write $n = \binom{N}{2}$ for the total number of edges in G . Write $\tilde{T}(x)$ as the normalized version of $T(x)$, that is,

$$\tilde{T}(x) := T(x)/N^{k-2}.$$

For $u > 1$, by the above definition we see that

$$(4.1) \quad T(x) \geq u \mathbb{E}[T(X)] \iff \tilde{T}(x) \geq tn, \quad \text{with } t = \frac{\mathbb{E}[T(X)]}{nN^{k-2}}u.$$

Thanks to the choice of t we have $\psi_l(u) = \phi_l(t)$, where

$$\phi_l(t) := \inf \left\{ \sum_{1 \leq i < j \leq N} \sum_{s=1}^l x_{ijs} \log \frac{x_{ijs}}{1/l} : x_{ij} \in W_0, \tilde{T}((x_{ij})_{1 \leq i < j \leq N}) \geq tn \right\}.$$

Similar to the proof of [7], Theorem 1.1, for $K, \delta > 0$ to be determined later we define

$$g(x) := nKh((\tilde{T}(x)/n) - t)/\delta,$$

where $h(x) = -1$ if $x < -1$, $h(x) = 0$ if $x > 0$, and for $x \in [-1, 0]$

$$(4.2) \quad h(x) = 10(x + 1)^3 - 15(x + 1)^4 + 6(x + 1)^5 - 1.$$

By our choice of h we can see that it is negative on $(-1, 0)$, with bounded first and second derivatives. Denote by μ_{ij} the measure of X_{ij} for $1 \leq i < j \leq N$, and μ the measure of X . Using the definition of $g(\cdot)$ we further see that

$$(4.3) \quad \mathbb{P}(\tilde{T}(X) \geq tn) \leq \int_{W_0^n} e^{g(x)} d\mu(x).$$

For $s \in [l]$, let e_s be the length l vector with s th coordinate 1 and other coordinates 0. Recalling that μ_{ij} is the uniform distribution on $\{e_s, s \in [l]\}$, we see that for any $y_{ij} \in W_0$, the only distribution with $\nu_{ij} \ll \mu_{ij}$ and $m(\nu_{ij}) = y_{ij}$ is $\nu_{ij}(e_s) = y_{ijs}$ for all $s \in [l]$. Therefore, it is easy to see that

$$\begin{aligned} & \max_{\nu \ll \mu, \nu=v_1 \times v_2 \times \dots \times v_n} \left\{ g((m(\nu_{ij}))_{1 \leq i < j \leq N}) - \sum_{1 \leq i < j \leq N} D(\nu_{ij} \parallel \mu_{ij}) \right\} \\ &= \max_{y_{ij} \in W_0, 1 \leq i < j \leq N} \left\{ g((y_{ij})_{1 \leq i < j \leq N}) - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \right\}. \end{aligned}$$

Let $K = \phi_l(t)/n$. We claim that

$$(4.4) \quad \begin{aligned} & \max_{y_{ij} \in W_0, 1 \leq i < j \leq N} \left\{ g((y_{ij})_{1 \leq i < j \leq N}) - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \right\} \\ & \leq -\phi_l(t - \delta). \end{aligned}$$

This is because, for $y = (y_{ij})_{1 \leq i < j \leq N}$, if $\tilde{T}(y) \geq tn$, we have $g(y) = 0$, and thus

$$g(y) - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} = - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \leq -\phi_l(t) \leq -\phi_l(t - \delta).$$

If $\tilde{T}(y) \leq (t - \delta)n$, we have $g(y) = -Kn$, and then

$$g(y) - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \leq -Kn = -\phi_l(t) \leq -\phi_l(t - \delta).$$

If $\tilde{T}(y) = (t - \delta')n$ for some $\delta' \in (0, \delta)$, we have

$$g(y) - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \leq - \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \leq -\phi_l(t - \delta') \leq -\phi_l(t - \delta).$$

Observe that if we denote by $\mathcal{D}(\epsilon)$ a $\sqrt{n}\epsilon$ -covering for the gradient of $\tilde{T}(x)$ in the sense of (1.8), then $\mathcal{D}((\delta\epsilon)/(4K))$ is a $\sqrt{n}\epsilon$ -covering for the gradient of $g(x)$. Applying Theorem 1 for $g(\cdot)$, with (4.3) and (4.4) we get

$$(4.5) \quad \log \mathbb{P}(\tilde{T}(X) \geq tn) \leq -\phi_l(t - \delta) + B_1 + B_2 + \log 2 + \log |\mathcal{D}((\delta\epsilon)/(4K))|.$$

Next we analyze the right-hand side of (4.5). First we bound $\phi_l(t) - \phi_l(t - \delta)$.

4.1.1. *Upper bound of $\phi_l(t) - \phi_l(t - \delta)$.* Obviously $\phi_l(t) \geq \phi_l(t - \delta)$. If $\phi_l(t) = \phi_l(t - \delta)$, then 0 is an upper bound. Now we consider the case that $\phi_l(t) > \phi_l(t - \delta)$, and by the definition of ϕ_l , the only possibility is that $\phi_l(t - \delta)$ is achieved on some $x^* = (x_{ij}^*)_{1 \leq i < j \leq N}$ with $x_{ij}^* \in W_0$ and $\tilde{T}(x^*) \in [(t - \delta)n, tn)$. Note that in addition we can assume $x_{ij1}^* \geq x_{ij2}^* \geq \dots \geq x_{ijl}^*$ for all $1 \leq i < j \leq N$, since when $(\{x_{ij1}^*, x_{ij2}^*, \dots, x_{ijl}^*\})_{1 \leq i < j \leq N}$ is fixed, the choice $x_{ij1}^* \geq x_{ij2}^* \geq \dots \geq x_{ijl}^*$ achieves the maximum of $\tilde{T}(\cdot)$ by the rearrangement inequality. Thus if this decreasing relation is not satisfied, we can choose another x' satisfying it with $\tilde{T}(x') > \tilde{T}(x^*)$, and $\phi_l(t - \delta)$ is achieved on x' too, and we must have $\tilde{T}(x') \in [(t - \delta)n, tn)$ otherwise $\phi_l(t) = \phi_l(t - \delta)$.

For any $x = (x_{ij})_{1 \leq i < j \leq N}$ with $x_{ij} \in W_0$, $\tilde{T}(x) = (t - \delta')n$ for some $\delta' \in [0, \delta]$, and $x_{ij1} \geq x_{ij2} \geq \dots \geq x_{ijl}$ for any $1 \leq i < j \leq N$, we consider $y = (y_{ij})_{1 \leq i < j \leq N}$, where for some $\gamma > 0$ to be determined later,

$$y_{ij} := (1 - \gamma)x_{ij} + \gamma e_1, \quad e_1 = (1, 0, \dots, 0).$$

By the definition of y and \tilde{T} we have

$$(4.6) \quad \begin{aligned} & N^{k-2}(\tilde{T}(y) - \tilde{T}(x)) \\ &= \sum_{q_1, \dots, q_k \in [N]} \left(\prod_{\{r, r'\} \in E} \left(x_{q_r q_{r'} 1} + \gamma \sum_{s=2}^l x_{q_r q_{r'} s} \right) \right. \\ & \quad \left. + (1 - \gamma)^m \sum_{s=2}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} - \sum_{s=1}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \right). \end{aligned}$$

Next we show that

$$(4.7) \quad N^{k-2}(\tilde{T}(y) - \tilde{T}(x)) \geq (m-1)\gamma^2 \sum_{q_1, q_2, \dots, q_k \in [N]} \left(\sum_{\{r, r'\} \in E} \frac{1 - x_{q_r q_{r'} 1}}{x_{q_r q_{r'} 1}} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \right).$$

We fix $(q_1, q_2, \dots, q_k) \in [N]^k$ for our analysis. Denote by

$$I = \prod_{\{r, r'\} \in E} \left(x_{q_r q_{r'} 1} + \gamma \sum_{s=2}^l x_{q_r q_{r'} s} \right) - (m-1)\gamma^2 \left(\sum_{\{r, r'\} \in E} \frac{\sum_{s \geq 2} x_{q_r q_{r'} s}}{x_{q_r q_{r'} 1}} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \right).$$

For each $l' \in [l]$, we let

$$\mathcal{M}_{l'} := \left\{ \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s_{r, r'}} : (s_{r, r'})_{\{r, r'\} \in E} \in [l]^m, \max_{\{r, r'\} \in E} s_{r, r'} = l' \right\}.$$

By the decreasing assumption on x , we see that each term in $\mathcal{M}_{l'}$ is greater than or equal to $\prod_{\{r, r'\} \in E} x_{q_r q_{r'} l'}$. By direct calculation, one can check that, for $2 \leq l' \leq l$, in the expansion of $\prod_{\{r, r'\} \in E} (x_{q_r q_{r'} 1} + \gamma \sum_{s=2}^l x_{q_r q_{r'} s})$, the summation of the coefficients of those terms in $\mathcal{M}_{l'}$ is $g_0(l')$ where

$$g_0(l') := (1 + (l' - 1)\gamma)^m - (1 + (l' - 2)\gamma)^m.$$

Similarly, for $2 \leq l' \leq l$, as $\gamma < m^{-1}$, one can check that in the expansion of I , all the coefficients of terms in $\mathcal{M}_{l'}$ are positive, and the summation of them is $g_0(l') - m(m-1)\gamma^2$. From the above analysis we have that

$$(4.8) \quad I \geq \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} + \sum_{2 \leq s \leq l} \left[(g_0(s) - m(m-1)\gamma^2) \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \right].$$

It is direct to check that $g_0(\cdot)$ is increasing on \mathbb{Z}_+ . Note that we can rewrite

$$(4.9) \quad \begin{aligned} & (1 - \gamma)^m \sum_{s=2}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} - \sum_{s=1}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \\ & = - \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} - g_0(1) \sum_{s=2}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s}. \end{aligned}$$

Combining (4.8) and (4.9), and using the monotonicity of $g_0(\cdot)$, we see that

$$\begin{aligned} & I + (1 - \gamma)^m \sum_{s=2}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} - \sum_{s=1}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \\ & \geq (g_0(2) - m(m-1)\gamma^2 - g_0(1)) \sum_{2 \leq s \leq l} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \geq 0, \end{aligned}$$

where the last inequality is due to the fact that $g_0(2) - m(m-1)\gamma^2 - g_0(1) \geq 0$. Summing over all possible $(q_1, q_2, \dots, q_k) \in [N]^k$ leads to (4.7).

For any $\lambda > 1/l$, we denote by $N(\lambda)$ the number of homomorphisms of H in G whose edges all satisfy $x_{ij1} > \lambda$. Note that $x_{ij1} \geq 1/l$ always holds since $x_{ij1} \geq x_{ij s}$ for any $s \in [l]$.

Denote by $C(N, H)$ the total number of different homomorphisms of H in a N vertices complete graph. Then we have

$$N^{k-2}\tilde{T}(x) \geq \sum_{q_1, q_2, \dots, q_k \in [N]} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \geq N(\lambda)\lambda^m + (C(N, H) - N(\lambda))\frac{1}{l^m},$$

which with the fact $\tilde{T}(x) = (t - \delta')n$ implies that

$$(4.10) \quad N(\lambda) \leq ((t - \delta')N^k/2 - C(N, H)/l^m)/(\lambda^m - 1/l^m).$$

We denote by Γ_1 the set of homomorphisms of H in G who have at least an edge with $x_{ij1} \leq \lambda$. Since $x_{ij1} \leq \lambda$ implies that $(1 - x_{ij1})/x_{ij1} \geq (1 - \lambda)/\lambda$, we have

$$(4.11) \quad \begin{aligned} & \sum_{q_1, q_2, \dots, q_k \in [N]} \left(\sum_{\{r, r'\} \in E} \frac{1 - x_{q_r q_{r'} 1}}{x_{q_r q_{r'} 1}} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \right) \\ & \geq \frac{1 - \lambda}{\lambda} \sum_{(q_1, q_2, \dots, q_k) \in \Gamma_1} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1}, \end{aligned}$$

where in the right-hand side we use $(q_1, q_2, \dots, q_k) \in \Gamma_1$ to represent those (q_1, q_2, \dots, q_k) with corresponding homomorphism H (i.e., the homomorphism with vertices (q_1, q_2, \dots, q_k)) in Γ_1 . Note that

$$\begin{aligned} l \sum_{(q_1, q_2, \dots, q_k) \in \Gamma_1} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} & \geq \sum_{(q_1, q_2, \dots, q_k) \in \Gamma_1} \sum_{s=1}^l \prod_{\{r, r'\} \in E} x_{q_r q_{r'} s} \\ & \geq \tilde{T}(x)N^{k-2} - N(\lambda). \end{aligned}$$

Combining above inequality and (4.10), we get

$$\begin{aligned} & \sum_{(q_1, q_2, \dots, q_k) \in \Gamma_1} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \\ & \geq N^k[(1 - 1/N)(t - \delta')/2 \\ & \quad - ((t - \delta')/2 - C(N, H)/(N^k l^m))/(\lambda^m - 1/l^m)]/l. \end{aligned}$$

Due to the fact that $C(N, H)/N^k$ converges to a positive constant as $N \rightarrow \infty$, and that t is of order $1/l^{m-1}$ by (4.1), we can choose $\lambda = 1 - c/l$ for some constant $c > 0$ such that

$$(4.12) \quad \sum_{(q_1, q_2, \dots, q_k) \in \Gamma_1} \prod_{\{r, r'\} \in E} x_{q_r q_{r'} 1} \geq CN^k l^{-(m+1)}.$$

Combining (4.7), (4.11) and (4.12), we see that $\tilde{T}(y) - \tilde{T}(x) \geq C\gamma^2 N^2 l^{-(m+2)}$. Thus if we choose $\gamma = C_0 \delta^{1/2} l^{(m+2)/2}$ for a suitable $C_0 > 0$, we have $\tilde{T}(y) - \tilde{T}(x) \geq \delta n$ and thus $\tilde{T}(y) \geq tn$. From the convexity of $x \log x$, we have

$$(4.13) \quad \begin{aligned} \phi_l(t) & \leq \sum_{1 \leq i < j \leq N} \sum_{s=1}^l y_{ijs} \log \frac{y_{ijs}}{1/l} \\ & \leq (1 - \gamma) \sum_{1 \leq i < j \leq N} \sum_{s=1}^l x_{ijs} \log \frac{x_{ijs}}{1/l} + \gamma n \log l \\ & \leq \phi_l(t - \delta) + C_0 N^2 \delta^{1/2} l^{(m+2)/2} \log l, \end{aligned}$$

where in the last inequality we let $x = x^*$.

4.1.2. *Upper bound for $\phi_l(t)$.* Denote by $\lceil t^{1/k}N \rceil$ the smallest integer greater than $t^{1/k}N$. Choose $r = C_1 \lceil t^{1/k}N \rceil$, and let $x = (x_{ij})_{1 \leq i < j \leq N}$ where

$$x_{ij} = \begin{cases} e_1, & \text{if } 1 \leq i < j \leq r, \\ (1/l, \dots, 1/l), & \text{otherwise.} \end{cases}$$

Then it is easy to check that for a suitable $C_1 > 0$ we have $\tilde{T}(x) \geq tn$ for all N . Thus

$$(4.14) \quad \phi_l(t) \leq \sum_{1 \leq i < j \leq N} \sum_{s=1}^l x_{ijs} \log \frac{x_{ijs}}{1/l} \leq Ct^{2/k} N^2 \log l.$$

4.1.3. *Final calculation.* We give the proofs of the upper bound and lower bound of Theorem 2 separately below.

PROOF OF THE UPPER BOUND IN THEOREM 2. Recalling that $K = \phi_l(t)/n$, with (4.14) and the fact that t is of the order $l^{-(m-1)}$, we can see that

$$K \leq Cl^{-\frac{2(m-1)}{k}} \log l.$$

We work with the L_1 norm in this problem. It is easy to verify that for $g(x)$ we have

$$(4.15) \quad |g(x)| \leq nK, \quad \left\| \frac{\partial g(x)}{\partial x_{ij}} \right\| \leq \frac{CKb'_{ij}}{\delta},$$

$$(4.16) \quad \left\| \frac{\partial^2 g(x)}{\partial x_{ij} \partial x_{kl}} \right\| \leq \frac{CKc'_{ij,kl}}{\delta} + \frac{CKb'_{ij}b'_{kl}}{n\delta^2},$$

where

$$(4.17) \quad b'_i \leq C, \quad c'_{ij,kl} \leq \begin{cases} CN^{-1}, & \text{if } |\{i, j, k, l\}| = 2 \text{ or } 3, \\ CN^{-2}, & \text{otherwise.} \end{cases}$$

Denoting the $\sqrt{n}\epsilon$ -covering set in [7], Theorem 1.2, as $\tilde{D}(\epsilon)$. Since we are working with L_1 norm and each x_{ij} is l dimensional, it is not hard to observe that for any $\epsilon' > 0$, $D(\epsilon') := \tilde{D}(\epsilon'/\sqrt{l}) \times \dots \times \tilde{D}(\epsilon'/\sqrt{l})$ (the product of l sets) is a $\sqrt{n}\epsilon'$ -covering of the gradient of $\tilde{T}(x)$ in the sense of (1.8). Therefore, by [7], Lemma 5.2, we get

$$(4.18) \quad \log |D((\delta\epsilon)/(4K))| \leq C \frac{l^3 N K^4}{\delta^4 \epsilon^4} (\log N).$$

Now we bound the right-hand side of (4.5). It is clear that in this example $M \leq 2$. Using (4.15), (4.16) and (4.17), by some algebra it is easy to check that under the conditions that

$$(4.19) \quad N\delta^2 > 1, \quad k/\delta > 1, \quad N\delta\epsilon/K > 1, \quad \epsilon < 1,$$

we have

$$(4.20) \quad B_1 = CN^{\frac{3}{2}} K^{\frac{3}{2}} \delta^{-1}, \quad B_2 = CN^2 \epsilon K \delta^{-1}.$$

Thus with (4.5), (4.13), (4.18) and (4.20), we see that

$$(4.21) \quad \begin{aligned} & \log \mathbb{P}(\tilde{T}(X) \geq tn) \\ & \leq -\phi_l(t) + \log 2 + CN^2 \delta^{1/2} l^{(m+2)/2} \log l + CN^{\frac{3}{2}} K^{\frac{3}{2}} \delta^{-1} \\ & \quad + CN^2 \epsilon K \delta^{-1} + C \frac{l^3 N K^4}{\delta^4 \epsilon^4} (\log N). \end{aligned}$$

Denote by $T_i(X)$ the number of homomorphisms of H in G whose edges are all of color i , and let $\tilde{T}_i(X) := T_i(X)/N^{k-2}$. Then obviously $T_i(X)$ has the same distribution as the number of homomorphisms of H in $G(N, l^{-1})$ —the Erdős–Rényi random graph with probability l^{-1} , and thus by [15], Theorem 1.2 and Theorem 1.5, we have

$$(4.22) \quad -\log \mathbb{P}(\tilde{T}_i(X) \geq tl^{-1}n) \geq CN^2l^{-\Delta}.$$

Due to the fact that $\mathbb{P}(\tilde{T}(X) \geq tn) = \mathbb{P}(\sum_{i=1}^l \tilde{T}_i(X) \geq tn) \leq \sum_{i=1}^l \mathbb{P}(\tilde{T}_i(X) \geq tl^{-1}n)$, from (4.22) we further get

$$(4.23) \quad -\log \mathbb{P}(\tilde{T}(X) \geq tn) \geq -\log l \mathbb{P}(\tilde{T}_i(X) \geq tl^{-1}n) \geq CN^2l^{-\Delta} - \log l.$$

Choosing $\epsilon = N^{-1/5} \delta^{-3/5} K^{3/5} l^{3/5} (\log N)^{1/5}$, $\delta = N^{-(2\Delta+m+2)/(19+8m+21\Delta)} \times (\log N)^{-4}$, with (4.22) and (4.23) it is directly to derive that

$$\begin{aligned} \frac{\phi_l(t)}{-\log \mathbb{P}(\tilde{T} \geq tn)} &\leq 1 + CN^{-\frac{m/2+\Delta+1}{19+8m+21\Delta} l^{\frac{m+2\Delta+2}{2}}} (\log l)(\log N)^{-2} \\ &\quad + CN^{\frac{m+2\Delta+2}{19+8m+21\Delta} - \frac{1}{2}} l^{-\frac{3(m-1)-\Delta k}{k}} (\log l)^{\frac{3}{2}} (\log N)^4 \\ &\quad + CN^{-\frac{1}{5} + \frac{8}{5} \frac{m+2\Delta+2}{19+8m+21\Delta} l^{\frac{3}{5} - \frac{16(m-1)}{5k} + \Delta}} (\log l)^{\frac{8}{5}} (\log N)^{\frac{33}{5}} + o(1). \end{aligned}$$

Using above equation and the fact that $(m - 1)/k < \Delta/2$, we can check that if $l \leq N^{1/(19+8m+21\Delta)}$, then the right-hand side goes to 0 as $N \rightarrow \infty$, and it is directly to verify that condition (4.19) holds. Recalling that $\psi_l(u) = \phi_l(t)$, we finish the proof. \square

Next we show the lower bound.

PROOF OF THE LOWER BOUND IN THEOREM 2. Fix any $z \in W_0^n$ such that $\tilde{T}(z) \geq (t + \delta_0)n$, with δ_0 to be determined later. Recall that e_s is the l -dimension vector with 1 on the s th coordinate and 0 on others. Let Z_{ij} , $1 \leq i < j \leq N$ be independent random vectors with $\mathbb{P}(Z_{ij} = e_s) = z_{ijs}$, and denote by $\hat{\mu}$ the measure of $Z = (Z_{ij})_{1 \leq i < j \leq N}$. Let Γ be the set of $x \in W_0^n$ such that $\tilde{T}(x) \geq tn$, and let Γ' be the subset of Γ where

$$\left| \sum_{1 \leq i < j \leq N} \sum_{k=1}^l \left(x_{ijk} \log z_{ijk} - x_{ijk} \log \frac{1}{l} - z_{ijk} \log(lz_{ijk}) \right) \right| \leq \epsilon_0 n.$$

Then we have

$$(4.24) \quad \begin{aligned} \mathbb{P}(\tilde{T}(X) \geq tn) &= \int_{\Gamma} 1 d\mu(x) \\ &\geq \int_{\Gamma'} e^{\sum_{1 \leq i < j \leq N} \sum_{k=1}^l (-x_{ijk} \log z_{ijk} + x_{ijk} \log z_{ijk})} d\mu(x) \\ &\geq e^{z_{ijk} \log(lz_{ijk}) - \epsilon_0 n} \mathbb{P}_{\hat{\mu}}(Z \in \Gamma'). \end{aligned}$$

Denote

$$H(x) := \sum_{1 \leq i < j \leq N} \sum_{k=1}^l (x_{ijk} \log(lz_{ijk}) - z_{ijk} \log(lz_{ijk})).$$

Obviously $\mathbb{E}_{\hat{\mu}}[H(Z)] = 0$. By direct calculation we have

$$(4.25) \quad \text{Var}_{\hat{\mu}}(H(Z)) = \sum_{1 \leq i < j \leq N} \left[\sum_{k=1}^l z_{ijk} \log(z_{ijk}) - \left(\sum_{k=1}^l z_{ijk} \log(z_{ijk}) \right)^2 \right].$$

Noting that

$$\log\left(\frac{1}{l}\right) \leq \sum_{k=1}^l z_{ijk} \log(z_{ijk}) \leq 0,$$

with (4.25) it is clear that

$$\text{Var}_{\hat{\mu}}(H(Z)) \leq Cn(\log l)^2.$$

Thus by choosing $\epsilon_0 = C_2N^{-1}(\log l)$ for a suitable $C_2 > 0$ we have

$$(4.26) \quad \mathbb{P}_{\hat{\mu}}(|H(Z)| > \epsilon_0n) \leq \frac{C(\log l)^2}{\epsilon_0^2n} = \frac{1}{4}.$$

Let $S(x) := \tilde{T}(x) - \tilde{T}(z)$. Using the similar approach as in [7], (4.3)–(4.4), we can verify that

$$\mathbb{E}_{\hat{\mu}}[S^2] \leq CN^2.$$

Thus by choosing $\delta_0 = C_3N^{-1}$ for a suitable $C_3 > 0$, we get

$$(4.27) \quad \mathbb{P}_{\hat{\mu}}(\tilde{T}(Z) \leq tn) \leq \frac{CN^2}{\delta_0^2n^2} = \frac{1}{4}.$$

Using (4.26) and (4.27) we see that $\mathbb{P}_{\hat{\mu}}(Z \in \Gamma') \geq 1/2$, therefore, with (4.24) and by taking the sup over z we get

$$\begin{aligned} \log \mathbb{P}(\tilde{T}(X) \geq tn) &\geq -\phi_l(t + \delta_0) - \epsilon_0n - \log 2 \\ &\geq -\phi_l(t) - CN^2\delta_0^{1/2}l^{(m+2)/2} \log l - CN(\log l) - \log 2. \end{aligned}$$

Consequently with (4.23) we see that

$$\frac{-\phi_l(t)}{-\log \mathbb{P}(\tilde{T}(X) \geq tn)} \geq 1 - CN^{-\frac{1}{2}}l^{\Delta + \frac{m+2}{2}} \log N + o(1),$$

which completes the proof. \square

4.2. *Proof of Theorem 3.* In this subsection we show Theorem 3 in our second example about continuous weighted triangle counts. Throughout the proof, C will denote any positive constant that does not depend on N . We follow the routine of the above example. In the proof we use the definitions in Section 1.2.2. Define the normalized weighted triangle counts $\tilde{T}(x)$ as

$$\tilde{T}(x) := T(x)/N.$$

For any $1 < u < 8$, we let $t = u(N - 2)/(24N)$. Since $n = N(N - 1)/2$ and by calculation $\mathbb{E}[T(X)] = N(N - 1)(N - 2)/48$, we see that $\{T(x) \geq u\mathbb{E}[T(X)]\} = \{\tilde{T}(x) \geq tn\}$. Define

$$\begin{aligned} \phi_n(t) := \inf \left\{ \sum_{1 \leq i < j \leq N} \left(-1 + \frac{\lambda_{y_{ij}} e^{-\lambda_{y_{ij}}}}{1 - e^{-\lambda_{y_{ij}}}} + \log \left(\frac{\lambda_{y_{ij}}}{1 - e^{-\lambda_{y_{ij}}}} \right) \right) : \right. \\ \left. y_{ij} \in (0, 1) \text{ such that } \tilde{T}(y) \geq tn \right\}. \end{aligned}$$

Obviously $\phi_n(t) = \psi_n(u)$. Let $g(x) = nKh((\tilde{T}(x)/n) - t)/\delta$ for $h(\cdot)$ defined in (4.2), with $K = \phi_n(t)/n$ and δ to be determined later. Then same as the argument of showing (4.4), we have

$$\max_{y=(y_{ij})_{1 \leq i < j \leq N}, y_{ij} \in (0,1)} \left\{ g(y) - \sum_{i < j} D(v^{y_{ij}} \parallel \mu_{ij}) \right\} \leq -\phi_n(t - \delta).$$

Applying Theorem 1 for $g(x)$ and some ϵ to be determined later, we get

$$(4.28) \quad \begin{aligned} \log \mathbb{P}(\tilde{T}(X) \geq tn) &\leq \log \mathbb{E}[e^{g(x)}] \\ &\leq -\phi_n(t - \delta) + \log 2 + B_1 + B_2 + \log |\mathcal{D}(\epsilon)|, \end{aligned}$$

where B_1, B_2 are as defined in Theorem 1, and $\mathcal{D}(\epsilon)$ will be constructed later. Next we upper bound the rightmost side of (4.28).

4.2.1. *The upper bound for $\phi_n(t) - \phi_n(t - \delta)$.* Recall the definition of v^a in Section 1.2.2. For $\lambda^a > 0$ we define $f_1(\lambda^a) := \mathbb{E}_{v^a}[X]$. After calculation we have

$$f_1(x) = \frac{1}{x} - \frac{1}{e^x - 1},$$

and we can check that on any bounded interval $[-M_0, M_0]$, there exists $c_{M_0} > 0$ such that

$$(4.29) \quad f_1'(x) < -c_{M_0}.$$

For $\lambda^a > 0$ we define $f_2(\lambda^a) := D(v^a || U)$, which after some calculation is

$$f_2(x) = -1 + \frac{x e^{-x}}{1 - e^{-x}} + \log\left(\frac{x}{1 - e^{-x}}\right).$$

We can check that

$$(4.30) \quad \begin{aligned} f_2'(x) &< 0 \quad \text{when } x < 0; \\ f_2'(x) &> 0 \quad \text{when } x > 0; \\ |f_2'(x)| &\leq C_D \quad \text{for some } C_D < \infty. \end{aligned}$$

We assume that $t - \delta > 1/24$, since later we will choose $\delta \rightarrow 0$ as $N \rightarrow 0$, and by our choice $t > 1/24$ as $N \rightarrow 0$. In order to bound $\phi_n(t) - \phi_n(t - \delta)$, we use the same strategy as Section 4.1.1. If $\phi_n(t) \neq \phi_n(t - \delta)$, we assume that $\phi_n(t - \delta)$ is achieved on some $z = (z_{ij})_{1 \leq i < j \leq N}$ such that $\tilde{T}(z) = (t - \delta')n$ for some $\delta' \in [0, \delta]$. In addition we assume that $z_{ij} \geq 1/2$ for all $i < j$, since otherwise according to (4.30) we can change those $z_{ij} < 1/2$ to $1/2$ without increasing $\sum_{i < j} D(v^{z_{ij}} || U)$, which results in a bigger $\tilde{T}(z)$, and we can consider the new z instead. For some $s \in (1/2, 1)$ to be determined later, we define $A(s) := \{\{i, j\} : z_{ij} \geq s\}$ and $V_s(i) := |\{k \in [N] : z_{ik} \geq s\}|$ (here $|\cdot|$ refers to cardinality). Write $B(s)$ as the set of triangles whose three edges all belong to $A(s)$. Observing that for each edge $\{i, j\} \in A(s)$, the number of triangles in $B(s)$ containing $\{i, j\}$ is at least $V_s(i) + V_s(j) - N - 1$, we get that

$$(4.31) \quad \begin{aligned} |B(s)| &\geq \frac{1}{3} \sum_{\{i,j\} \in A(s)} (V_s(i) + V_s(j) - N - 1) \\ &= \frac{1}{3} \left(\sum_{i=1}^N (V_s(i))^2 - |A(s)|(N - 1) \right) \\ &\geq \frac{1}{3} \left(\frac{4|A(s)|^2}{N} - |A(s)|(N - 1) \right), \end{aligned}$$

where the second equality is by the fact that each $V_s(i)$ appears $V_s(i)$ times in the summation, and the last inequality is by Cauchy's inequality and the fact that $\sum_{i=1}^N |V_s(i)| = 2|A(s)|$. Since

$$(4.32) \quad \binom{N}{3} \geq T(z) \geq |B(s)|(s^3 - 1/8) + \mathbb{E}[T(X)],$$

with the fact that $\mathbb{E}[T(X)] = N^3/48 + o(N^2)$, substituting (4.31) into (4.32) we can verify that there exist $s \in (0, 1)$ and $c_s > 0$ independent of N such that $|A(s)| \leq (1 - c_s)n$. We find $c_s n$ number of edges in $A(s)^C$, and increase the weights on them by $\sigma > 0$ to be determined later, getting a new weight vector $\tilde{z} = (\tilde{z}_{ij})_{1 \leq i < j \leq N}$. Later we can verify that $\sigma \rightarrow 0$ as $N \rightarrow \infty$, and thus the weight-increasing operation is feasible, that is, $\tilde{z}_{ij} \leq 1$ for all $\{i, j\}$, as N is large enough. Since for each edge there are $N - 2$ triangles containing it, after the operation, with the fact that $z_{ij} > 1/2$ for all $\{i, j\}$, each edge in $A(s)^C$ at least contribute $\sigma/5$ more to $\tilde{T}(z)$. Therefore, we get

$$\tilde{T}(\tilde{z}) - \tilde{T}(z) \geq \frac{c_s n \sigma}{5},$$

which implies that we can choose $\sigma = c'_s \delta'$ for some $c'_s > 0$ to make $\tilde{T}(\tilde{z}) \geq tn$. Since for N large enough we can find $s_1 < 1$ such that $s + \sigma < s_1$, with (4.29), we see that for those $z_{ij} \in A(s)^C$, we have $|\lambda^{z_{ij}} - \lambda^{\tilde{z}_{ij}}| \leq c_{s_1} \sigma$ for some $c_{s_1} > 0$, and thus with (4.30) we have $D(v^{\tilde{z}_{ij}} \| U) - D(v^{z_{ij}} \| U) \leq C_D c_{s_1} \sigma$. Therefore, we have that

$$\begin{aligned} \phi_n(t) - \phi_n(t - \delta) &\leq \sum_{i < j} D(v^{\tilde{z}_{ij}} \| \mu_{ij}) - \sum_{i < j} D(v^{z_{ij}} \| \mu_{ij}) \\ (4.33) \qquad \qquad \qquad &\leq C_D c_{s_1} \sigma n \leq C_D c_{s_1} c'_s \delta' n \leq CN^2 \delta. \end{aligned}$$

4.2.2. *Bound for K.* In order to bound K , we just need to bound $\phi_n(t)$. Obviously we can choose $z_{ij} = s_t$ for some $s_t \in (0, 1)$ such that $\tilde{T}(z) \geq tn$ for all n , and thus $\phi_n(t) \leq CN^2$, which implies that $K \leq C$ since $K = \phi_n(t)/n$.

4.2.3. *Final calculation.* We give the proofs of the upper bound and lower bound of Theorem 3 separately below.

PROOF OF THE UPPER BOUND IN THEOREM 3. From our choice of g , it is easy to verify that

$$(4.34) \qquad B_1 = CN^{3/2} \delta^{-1} + CN \delta^{-2}, \qquad B_2 = CN \delta^{-2} + N^2 \delta^{-1} \epsilon.$$

One can check that in the sense of (1.8), the $\sqrt{n} \delta \epsilon / (4K)$ -covering of the gradient of $\tilde{T}(x)$ is a $\sqrt{n} \epsilon$ -covering of the gradient of $g(x)$, by [7], Lemma 5.2, and the fact that K is bounded by a constant, we have that for $g(x)$, $\log |D(\epsilon)| \leq CN \delta^{-4} \epsilon^{-4} \log N$. Choosing $\epsilon = N^{-1/5} \delta^{2/5}$, by (4.28), (4.33) and (4.34) we get

$$\begin{aligned} \log \mathbb{P}(\tilde{T}(X) \geq tn) \\ (4.35) \qquad \qquad \qquad &\leq -\phi_n(t) + CN^2 \delta + C \frac{N^{3/2}}{\delta} + C \frac{N}{\delta^2} + CN^{9/5} \delta^{-8/5} (\log N)^{1/5}. \end{aligned}$$

For any $s^* \in (0, 1)$, based on the graph G and weight X , we construct a graph $G'_{s^*}(X)$ by making those edges with weight $> s^*$ as connected and other edges as disconnected. Write $T_{s^*}(X)$ as the number of triangles in $G'_{s^*}(X)$. Then it is not hard to see that we can choose $0 < s_u < 1$ and $1 < u' < 8$ such that

$$\{T(X) \geq u \mathbb{E}[T(X)]\} \subset \{T_{s_u}(X) \geq u' \mathbb{E}[T_{s_u}(X)]\}.$$

Since $G'_{s_u}(X)$ is just the Erdős–Rényi random graph $G(N, 1 - s_u)$, with [15], Theorem 1.2 and Theorem 1.5, we see that

$$(4.36) \qquad -\log \mathbb{P}(\tilde{T}(X) \geq tn) \geq -\log \mathbb{P}(T_{s_u}(X) \geq u' \mathbb{E}[T_{s_u}(X)]) \geq CN^2.$$

Choosing $\delta = N^{-1/10}$ and dividing both sides of (4.35) by $-\log \mathbb{P}(\tilde{T}(X) \geq tn)$, we get the desired upper bound. \square

PROOF OF THE LOWER BOUND IN THEOREM 3. Fix any $z = (z_{ij})_{1 \leq i < j \leq N}$ with $z_{ij} \in (0, 1)$ and $\tilde{T}(z) \geq (t + \delta_0)n$ with δ_0 to be determined later. Consider $Z = (Z_{ij})_{1 \leq i < j \leq N}$ with Z_{ij} ($i < j$) independently from $\nu^{z_{ij}}$. Denote by $\hat{\mu}_z$ the distribution of Z . Denote

$$\Gamma := \{x = (x_{ij})_{1 \leq i < j \leq N} : x_{ij} \in (0, 1), \tilde{T}(x) \geq tn\}$$

and

$$\Gamma' := \Gamma \cap \left\{ x = (x_{ij})_{1 \leq i < j \leq N} : \left| \sum_{i < j} \left(-\lambda_{z_{ij}} x_{ij} - \left(-1 + \frac{\lambda_{z_{ij}} e^{-\lambda_{z_{ij}}}}{1 - e^{-\lambda_{z_{ij}}}} \right) \right) \right| < \epsilon_0 n \right\},$$

for ϵ_0 to be determined later. Noting that $\mathbb{P}(\tilde{T}(X) \geq tn) = \mathbb{E}[1_\Gamma]$ and $\Gamma' \subset \Gamma$, we have

$$\begin{aligned} & \mathbb{P}(\tilde{T}(X) \geq tn) \\ (4.37) \quad & \geq \mathbb{E} \left[1_{\Gamma'} e^{\sum_{i < j} (-\lambda_{z_{ij}} x_{ij} + \log(\frac{\lambda_{z_{ij}}}{1 - e^{-\lambda_{z_{ij}}}})) - \sum_{i < j} (-\lambda_{z_{ij}} x_{ij} + \log(\frac{\lambda_{z_{ij}}}{1 - e^{-\lambda_{z_{ij}}}}))} \right] \\ & \geq e^{-\sum_{i < j} D(\nu^{z_{ij}} \| U)} - \epsilon_0 n \mathbb{P}_{\hat{\mu}_z}(Z \in \Gamma'). \end{aligned}$$

By direct integration, we can see that for some $C_4 > 0$

$$\mathbb{E}_{\hat{\mu}_z} \left[\left(\sum_{i < j} \left(-\lambda_{z_{ij}} Z_{ij} - \left(-1 + \frac{\lambda_{z_{ij}} e^{-\lambda_{z_{ij}}}}{1 - e^{-\lambda_{z_{ij}}}} \right) \right) \right)^2 \right] \leq C_4 n.$$

Thus by choosing $\epsilon_0 = (4C_4/n)^{1/2}$ and using the Markov's inequality, we get

$$\mathbb{P}_{\hat{\mu}_z} \left(\sum_{i < j} \left(-\lambda_{z_{ij}} Z_{ij} - \left(-1 + \frac{\lambda_{z_{ij}} e^{-\lambda_{z_{ij}}}}{1 - e^{-\lambda_{z_{ij}}}} \right) \right) \geq \epsilon_0 n \right) \leq \frac{C_4}{\epsilon_0^2 n} = \frac{1}{4}.$$

Using the similar method as in [7], (4.4), by choosing $\delta_0 = CN^{-1}$, we have that $\mathbb{P}_{\hat{\mu}_z}(\tilde{T}(Z) \leq tn) \leq 1/4$. Thus $\mathbb{P}_{\hat{\mu}_z}(Z \in \Gamma') \geq 1/2$, and with (4.37) by taking sup over z we get

$$\log \mathbb{P}(\tilde{T}(X) \geq tn) \geq -\phi_n(t + \delta_0) - \epsilon_0 n - \log 2.$$

Combining above inequality and (4.33) we get

$$\log \mathbb{P}(\tilde{T}(X) \geq tn) \geq -\phi_n(t) + CN - \log 2,$$

which implies the lower bound with (4.36). \square

4.3. Proof of Theorem 4. In this section we show Theorem 4, which is an extension of [4]. Throughout the proof, C will denote any positive constant that does not depend on n . Note that in this example N is the dimension of W_1 , and it has no relation with n . Recall the definitions in Section 1.2.3. For convenience we define

$$f(x) := H_n^{J,h}(x) = \frac{1}{2} \sum_{i,j=1}^n A_n(i, j) x_i^T J x_j + \sum_{i=1}^n x_i^T h$$

and

$$\tilde{f}(x) := \frac{1}{2} \sum_{i,j=1}^n A_n(i, j) x_i^T J x_j.$$

Without the loss of generality we assume that μ_i 's are supported on the unit ball $B_{\mathbb{R}^N}(1)$ in \mathbb{R}^N . Using the similar argument to [4], Lemma 3.1, we can further assume that

$$(4.38) \quad \max_{i,j} |A_n(i, j)| = o(1) \quad \text{and} \quad A_n(i, i) = 0 \quad \text{for all } i.$$

We work with the L_1 norm, and note that

$$\sup_{x \in B_{\mathbb{R}^N}(1)} \|x\|_{L_1} = \sqrt{N}.$$

By the definition of f and (1.17), it is direct to verify that

$$(4.39) \quad \begin{aligned} a &= O(n), \\ b_i &= O\left(\sum_j |A_n(i, j)|\right) + O(1), \\ c_{ij} &= O(|A_n(i, j)|). \end{aligned}$$

Using (4.39) it is straightforward to verify that the lower bound part is implied by Theorem 1, that is,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \int_{W_1^n} e^{H_n^{J,h}(x)} d\mu(x) \right. \\ &\quad \left. - \max_{v \ll \mu, v = v_1 \times v_2 \times \dots \times v_n, \left\{ H_n^{J,h}(m(v)) - \sum_{i=1}^n D(v_i \parallel \mu_i) \right\}} \right] \geq 0. \end{aligned}$$

Next we consider the upper bound part. If we calculate B_1 and B_2 in Theorem 1, then they are of the wrong order. In order to show Theorem 4, we need to incorporate the special property of f into the proof of Theorem 1. For f and \tilde{f} we have that

$$f_i(x)(z) = \sum_{j \neq i} A_n(j, i) x_j^T J z + h^T z, \quad \tilde{f}_i(x)(z) = \sum_{j \neq i} A_n(j, i) x_j^T J z.$$

Defining $\tilde{\mu}$ same as (2.1), we claim that

$$(4.40) \quad \mathbb{E}_{\tilde{\mu}}[(\tilde{f}(X) - \tilde{f}(\hat{X}))^2] = o(n^2),$$

and

$$(4.41) \quad \mathbb{E}_{\tilde{\mu}} \left[\left(\sum_{i=1}^n \tilde{f}_i(X)(X_i - \hat{X}_i) \right)^2 \right] = o(n^2).$$

We defer their proofs to a later place, and first show how to finish the proof with them. By (4.40) and (4.41) we see that there exists $\sigma_n \rightarrow 0$ such that $\mathbb{P}_{\tilde{\mu}}(\Omega_n) \geq \frac{1}{2}$, where

$$\Omega_n := \left\{ x \in \text{supp}(\mu) : |\tilde{f}(x) - \tilde{f}(\hat{x})|, \left| \sum_{i=1}^n \tilde{f}_i(x)(x_i - \hat{x}_i) \right| \leq \sigma_n n \right\}.$$

Given any $\epsilon > 0$, by [4], Lemma 3.4, it is not hard to see that we can construct a $\mathcal{D}(\epsilon)$ such that $\log |\mathcal{D}(\epsilon)| = o(n)$. For each $d \in \mathcal{D}(\epsilon)$, we consider

$$E_d := \left\{ x \in \text{supp}(\mu) : \sum_{i=1}^n \|f_i(x) - d_i\|^2 \leq n\epsilon^2 \right\} \cap \Omega_n.$$

If E_d is not empty, we pick one element $z_d \in E_d$ and fix the choice. Consider

$$\tilde{\mathcal{D}}(\epsilon) := \{z_d : d \in \mathcal{D}(\epsilon), E_d \neq \emptyset\}.$$

Then for any $x \in \Omega_n$, recalling the definition of d^x from (3.7), we can find $y_x := z_{d^x} \in \tilde{\mathcal{D}}(\epsilon)$, such that by the triangle inequality

$$(4.42) \quad \sum_{i=1}^n \|f_i(x) - f_i(y_x)\|^2 \leq \sum_{i=1}^n \|f_i(x) - d_i^x\|^2 + \sum_{i=1}^n \|d_i^x - f_i(z_{d^x})\|^2 \leq 2n\epsilon^2.$$

Obviously $|\tilde{\mathcal{D}}(\epsilon)| \leq |\mathcal{D}(\epsilon)|$ by the construction of $\tilde{\mathcal{D}}(\epsilon)$, and thus

$$(4.43) \quad \log|\tilde{\mathcal{D}}(\epsilon)| = o(n).$$

Let $\widehat{y}_x = ((\widehat{y}_x)_1, \dots, (\widehat{y}_x)_n)$, where

$$(\widehat{y}_x)_i = \frac{\mathbb{E}_{\mu_i}[x_i e^{\sum_{j \neq i} A_n(i,j)x_i^T J(y_x)_j + x_i^T h}]}{\mathbb{E}_{\mu_i}[e^{\sum_{j \neq i} A_n(i,j)x_i^T J(y_x)_j + x_i^T h}]} = \frac{\mathbb{E}_{\mu_i}[x_i e^{f_i(y_x)(x_i)}]}{\mathbb{E}_{\mu_i}[e^{f_i(y_x)(x_i)}]}.$$

Next we do the following approximation for $x \in \Omega_n$:

$$f(x) \approx f(\widehat{y}_x) - \sum_{i=1}^n D(v_i^{\widehat{y}_x} \parallel \mu_i) + \sum_{i=1}^n \log \frac{dv_i^{\widehat{y}_x}}{d\mu_i}(x_i).$$

More precisely, we show that for any $x \in \Omega_n$

$$(4.44) \quad \left| f(x) - \left(f(\widehat{y}_x) - \sum_{i=1}^n D(v_i^{\widehat{y}_x} \parallel \mu_i) + \sum_{i=1}^n \log \frac{dv_i^{\widehat{y}_x}}{d\mu_i}(x_i) \right) \right| \leq 2\sigma_n n + 2\sqrt{2N}n\epsilon.$$

If (4.44) holds, then by (4.43) and the same method as Section 3.2.4, we see that

$$\begin{aligned} & \log \int_{W_1^n} e^{f(x)} d\mu(x) \\ & \leq \max_{v \ll \mu, v = v_1 \times v_2 \times \dots \times v_n} \left\{ f(m(v)) - \sum_{i=1}^n D(v_i \parallel \mu_i) \right\} \\ & \quad + 2\sigma_n n + 2\sqrt{2N}n\epsilon + \log 2 + \log|\mathcal{D}(\epsilon)|. \end{aligned}$$

Dividing both sides by n , and noting the fact that ϵ is arbitrary, we complete the proof by letting $\epsilon \rightarrow 0$.

Now we show (4.44). Comparing the above equality with (3.23) in Proposition 1, by (3.25) and (3.26) we see that for any $z \in \mathbb{R}^d$

$$\log \frac{dv_i^{\widehat{y}_x}}{d\mu_i}(z) - D(v_i^{\widehat{y}_x} \parallel \mu_i) = f_i(y_x)(z - (\widehat{y}_x)_i).$$

Therefore, we have

$$(4.45) \quad \begin{aligned} & \left| f(x) - \left(f(\widehat{y}_x) - \sum_{i=1}^n D(v_i^{\widehat{y}_x} \parallel \mu_i) + \sum_{i=1}^n \log \frac{dv_i^{\widehat{y}_x}}{d\mu_i}(x_i) \right) \right| \\ & \leq |\tilde{f}(x) - \tilde{f}(y_x)| + |\tilde{f}(y_x) - \tilde{f}(\widehat{y}_x)| \\ & \quad + \left| \sum_{i=1}^n \tilde{f}_i(y_x)(x_i - (y_x)_i) \right| + \left| \sum_{i=1}^n \tilde{f}_i(y_x)((y_x)_i - (\widehat{y}_x)_i) \right|, \end{aligned}$$

where in the last line we replace f by \tilde{f} since it is easy to check that all the terms involving h cancel in the first line. Recalling that $y_x \in \Omega_n$, by the definition of Ω_n we see that

$$(4.46) \quad \left| \tilde{f}(y_x) - \tilde{f}(\widehat{y}_x) \right| + \left| \sum_{i=1}^n \tilde{f}_i(y_x) ((y_x)_i - (\widehat{y}_x)_i) \right| \leq 2\sigma_n n.$$

Thus it remains to bound $|\tilde{f}(x) - \tilde{f}(y_x)|$ and $|\sum_{i=1}^n \tilde{f}_i(y_x)(x_i - (y_x)_i)|$. For $|\tilde{f}(x) - \tilde{f}(y_x)|$, we have

$$(4.47) \quad \begin{aligned} |\tilde{f}(x) - \tilde{f}(y_x)| &= \left| \frac{1}{2} \sum_{i,j=1}^n A_n(i, j) (x_i^T J(x_j - (y_x)_j) + (x_i^T - (y_x)_i^T) J(y_x)_j) \right| \\ &\leq \left| \frac{1}{2} \sum_i (\tilde{f}_i(x) - \tilde{f}_i(y_x))(x_i) \right| + \left| \frac{1}{2} \sum_j (\tilde{f}_j(x) - \tilde{f}_j(y_x))(y_x) \right| \\ &\leq \sqrt{N} \sqrt{n} \left(\sum_i \|\tilde{f}_i(x) - \tilde{f}_i(y_x)\|^2 \right)^{1/2} \leq \sqrt{2Nn} \epsilon, \end{aligned}$$

where the last inequality is by (4.42). For $|\sum_{i=1}^n \tilde{f}_i(y_x)(x_i - (y_x)_i)|$, note that

$$\begin{aligned} \left| \sum_{i=1}^n \tilde{f}_i(y_x)(x_i - (y_x)_i) \right| &= \left| \sum_{i,j=1}^n A_n(i, j) (x_i^T - (y_x)_i^T) J(y_x)_j \right| \\ &= \left| \sum_j (\tilde{f}_j(x) - \tilde{f}_j(y_x))(y_x) \right|, \end{aligned}$$

which we already bound in (4.47). Thus combining (4.45), (4.46) and (4.47), we get (4.44).

In the following we prove (4.40) and (4.41). We need the following two inequalities. By (1.17) and (4.39), there exists $\eta_n = o(n)$, such that for any $w_1, w_2, \dots, w_n \in W_1^n$

$$(4.48) \quad \begin{aligned} \sum_{i=1}^n \|\tilde{f}_i(w_i)\|^2 &\leq C \left(n + \sum_{i=1}^n \left(\sum_{j=1}^n |A_n(i, j)| \right)^2 \right) \\ &\leq C \left(n + n \sum_{i,j=1}^n |A_n(i, j)|^2 \right) \\ &\leq C(n + ntr(A_n^2)) = \eta_n n^2, \end{aligned}$$

and by (1.17) again, there exists $M_n = O(1)$ such that for all $x \in W_1^n$

$$(4.49) \quad \begin{aligned} \sum_{i=1}^n \|\tilde{f}_i(x)\| &\leq \sum_{i=1}^n \left\| \sum_{j \neq i} A_n(i, j) Jx_j \right\|_\infty \\ &\leq \sqrt{N} \|J\|_\infty \sup_{x \in [0,1]^n} \sum_{i \in [n]} \left| \sum_{j \in [n]} A_n(i, j) x_j \right| \leq M_n n. \end{aligned}$$

PROOF OF (4.40). If we directly apply Proposition 2 for \tilde{f} , then we can see that only $\sum_{i,j=1}^n b_i b_j c_{ij}$ is of wrong order, which comes from (3.42). Let $\theta = (0, 0, \dots, 0)$ in \mathbb{R}^N . Here we show

$$(4.50) \quad \sum_{i=1}^n \mathbb{E} \tilde{\mu}[u_i(t, X)(X_i - \widehat{X}_i)(h(X) - h(X_{\theta_i}^{(i)}))] = o(n^2),$$

which gives (4.40). Recall that

$$h(x) = \tilde{f}(x) - \tilde{f}(\widehat{x}), \quad u_i(t, x) = \tilde{f}_i(tx + (1-t)\widehat{x}).$$

By arrangement we have

$$\begin{aligned} h(x) - h(x_{\theta_i}^{(i)}) &= \tilde{f}(x) - \tilde{f}(x_{\theta_i}^{(i)}) + \frac{1}{2} \sum_{l,j} A_n(l, j)(\widehat{x}_l)^T J(\widehat{x}_j - x_{\theta_i}^{(i)}) \\ &\quad + \frac{1}{2} \sum_{l,j} A_n(l, j)(\widehat{x}_l - x_{\theta_i}^{(i)})^T J x_{\theta_i}^{(i)} \\ (4.51) \quad &= \tilde{f}(x) - \tilde{f}(x_{\theta_i}^{(i)}) + \frac{1}{2} \sum_j \tilde{f}_j(\widehat{x})(\widehat{x}_j - x_{\theta_i}^{(i)}) \\ &\quad + \frac{1}{2} \sum_l \tilde{f}_l(x_{\theta_i}^{(i)})(\widehat{x}_l - x_{\theta_i}^{(i)}). \end{aligned}$$

We also have

$$\begin{aligned} \tilde{f}(x) - \tilde{f}(x_{\theta_i}^{(i)}) &= \frac{1}{2} \sum_{l,j} A_n(l, j)(x_l)^T J x_j - \frac{1}{2} \sum_{l,j} A_n(l, j)((x_{\theta_i}^{(i)})_l)^T J x_j^{(i)} \\ (4.52) \quad &= \sum_{j \neq i} A_n(i, j)(x_i)^T J x_j = \tilde{f}_i(x)(x_i). \end{aligned}$$

By Cauchy’s inequality we have

$$\begin{aligned} &\left| \sum_{i=1}^n \mathbb{E}[(\|\tilde{f}_i(X)\| + \|\tilde{f}_i(\widehat{X})\|)(\|\tilde{f}_i(X)\|)] \right| \\ &\leq \left(\mathbb{E} \sum_{i=1}^n \|\tilde{f}_i(X)\|^2 \right) + \left(\mathbb{E} \sum_{i=1}^n \|\tilde{f}_i(\widehat{X})\|^2 \right)^{1/2} \left(\mathbb{E} \sum_{i=1}^n \|\tilde{f}_i(X)\|^2 \right)^{1/2} = o(n^2), \end{aligned}$$

where the last line is by (4.48). Thus with (4.52) we see that

$$\begin{aligned} &\left| \sum_{i=1}^n \mathbb{E}[u_i(t, X)(X_i - \widehat{X}_i)(\tilde{f}(X) - \tilde{f}(X^{(i)}))] \right| \\ (4.53) \quad &= \left| \sum_{i=1}^n \mathbb{E}[(t\tilde{f}_i(X) + (1-t)\tilde{f}_i(\widehat{X}))(X_i - \widehat{X}_i)(\tilde{f}_i(X)(X_i))] \right| \\ &\leq N \left| \sum_{i=1}^n \mathbb{E}[(\|\tilde{f}_i(X)\| + \|\tilde{f}_i(\widehat{X})\|)(\|\tilde{f}_i(X)\|)] \right| = o(n^2). \end{aligned}$$

Next we define $\Delta_{j,i}(x) := \widehat{x}_j - x_{\theta_i}^{(i)}$, and then by (3.38), (4.38) and (4.39) we see that for any $x \in W_1^n$,

$$(4.54) \quad \max_{i,j} |\Delta_{j,i}(x)| \leq \sqrt{N} \max_{i,j} |A_n(i, j)| = o(1).$$

By (4.49), for any $x \in W_1^n$,

$$\begin{aligned}
 & \sum_{i,j=1}^n \|\tilde{f}_j(\hat{x})\|(\|\tilde{f}_i(x)\| + \|\tilde{f}_i(\hat{x})\|) \\
 (4.55) \quad & \leq \left(\sum_{i=1}^n \|\tilde{f}_j(\hat{x})\| \right) \left(\sum_{i=1}^n \|\tilde{f}_i(x)\| + \sum_{i=1}^n \|\tilde{f}_i(\hat{x})\| \right) \\
 & \leq 2M_n^2 n^2.
 \end{aligned}$$

Noting that $|\tilde{f}_j(\hat{x})(\Delta_{j,i}(x)^T)(u_i(t, x)(x_i - \hat{x}_i))| \leq \|\tilde{f}_j(\hat{x})\|(\|\tilde{f}_i(x)\| + \|\tilde{f}_i(\hat{x})\|)$, with (4.54) and (4.55) we see that

$$\begin{aligned}
 & \left| \sum_{i=1}^n \mathbb{E} \left[u_i(t, X)(X_i - \hat{X}_i) \left(\frac{1}{2} \sum_j \tilde{f}_j(\hat{X})(\hat{X}_j - \widehat{X}^{(i)}_j) \right) \right] \right| \\
 (4.56) \quad & = \left| \frac{1}{2} \mathbb{E} \left[\sum_{i,j=1}^n \tilde{f}_j(\hat{X})(\Delta_{j,i}(X)^T)(u_i(t, X)(X_i - \hat{X}_i)) \right] \right| = o(n^2).
 \end{aligned}$$

Similarly we can show that

$$(4.57) \quad \left| \sum_{i=1}^n \mathbb{E} \left[u_i(t, X)(X_i - \hat{X}_i) \left(\frac{1}{2} \sum_l \tilde{f}_j(\widehat{X}^{(i)})(\hat{X}_l - \widehat{X}^{(i)}_l) \right) \right] \right| = o(n^2),$$

and thus with (4.53), (4.56), (4.57) and (4.51), we get (4.50) and finish the proof. \square

PROOF OF (4.41). Denote

$$G(x) := \sum_{i=1}^n \tilde{f}_i(x)(x_i - \hat{x}_i).$$

Then by the definition of \hat{X}_i we have

$$\mathbb{E}_{\tilde{\mu}}[G(X^{(i)})\tilde{f}_i(X^{(i)})(X_i - \hat{X}_i)] = 0.$$

Noting that $\tilde{f}_i(X^{(i)}) = \tilde{f}_i(X)$, it is enough to show that

$$(4.58) \quad \mathbb{E}_{\tilde{\mu}} \left[\sum_{i=1}^n \tilde{f}_i(X^{(i)})(X_i - \hat{X}_i)(G(X) - G(X^{(i)})) \right] = o(n^2).$$

After some algebra we have

$$\begin{aligned}
 & G(X) - G(X^{(i)}) \\
 (4.59) \quad & = \sum_{j=1}^n \sum_{l=1}^n A_n(l, j)(X_l^T J(X_j - \hat{X}_j) - (X_l^{(i)})^T J(X_j^{(i)} - \widehat{X}^{(i)}_j)) \\
 & = \sum_{j \neq i}^n \tilde{f}_j(X)(\widehat{X}^{(i)}_j - \hat{X}_j) + (2\tilde{f}_i(X) - \tilde{f}_i(\widehat{X}^{(i)}))(X_i).
 \end{aligned}$$

By the definition of $\Delta_{j,i}$, we have

$$\begin{aligned}
 & \mathbb{E}_{\tilde{\mu}} \left[\sum_{i=1}^n \tilde{f}_i(X^{(i)})(X_i - \hat{X}_i) \left(\sum_{j \neq i}^n \tilde{f}_j(X)(\hat{X}_j - \widehat{X}^{(i)}_j) \right) \right] \\
 (4.60) \quad & \leq \mathbb{E}_{\tilde{\mu}} \left[\max_{i,j} \|\Delta_{j,i}(X)\| \cdot \sqrt{N} \left(\sum_{i=1}^n \|\tilde{f}_i(X^{(i)})\| \right) \left(\sum_{j=1}^n \|\tilde{f}_j(X)\| \right) \right] = o(n^2).
 \end{aligned}$$

Also we have

$$(4.61) \quad \mathbb{E}_{\tilde{\mu}} \left[\left\| \sum_{i=1}^n \tilde{f}_i(X^{(i)})(X_i - \widehat{X}_i)((2\tilde{f}_i(X) - \tilde{f}_i(\widehat{X}^{(i)}))(X_i)) \right\| \right]$$

$$(4.62) \quad \leq \mathbb{E}_{\tilde{\mu}} \left[N \sum_{i=1}^n \|\tilde{f}_i(X^{(i)})\| (2\|\tilde{f}_i(X)\| + \|\tilde{f}_i(\widehat{X})\|) \right] = o(n^2),$$

where the last line is by Cauchy's inequality and (4.48). Combining (4.58), (4.59), (4.60) and (4.62), we finish the proof. \square

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