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GREEDY LATTICE ANIMALS I: UPPER BOUNDS

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Let $\{X_v: v \in \mathbb{Z}^d\}$ be an i.i.d. family of positive random variables. For each set ξ of vertices of $\mathbb{Z}^d$, its weight is defined as $S(\xi) = \sum_{v \in \xi} X_v$. A greedy lattice animal of size n is a connected subset of $\mathbb{Z}^d$ of n vertices, containing the origin, and whose weight is maximal among all such sets. Let N_n denote this maximal weight. We show that if the expectation of $X_v^d(\log^+ X_v)^{d+a}$ is finite for some $a > 0$, then w.p. 1 $N_n \le Mn$ eventually for some finite constant M. Estimates for the tail of the distribution of N_n are also derived.

1. Introduction. In this paper we investigate greedy lattice animals (GLA), as defined in the abstract, and some similar structures. Before giving formal definitions and going into the mathematics, we describe a number of different problems which motivated us to this study. We expect that the model fits a number of optimization problems and hope that, in any case, it has an appeal on its own. After these descriptions, we give the definition of GLA for a specific case which covers some of the above-mentioned problems, and in this and a companion paper [Gandolfi and Kesten (1994)], we develop some rigorous asymptotic results.

(*a*) *Vertex greedy lattice paths* (*GLP*). Let an abstract graph be given, and suppose that to each vertex v is associated a random amount X_v, measured by a positive number, which we gain during the first visit to the vertex. (Note that by "X is positive" we mean $X \ge 0$.) Suppose, further, that we start at a fixed point, and that movements on the graph can only be done through the edges and that we can visit only a fixed number n of vertices. Also, we decide never to visit a vertex twice, because we can gain the amount associated with a given vertex only once. What can we infer about the maximum total amount which can be collected, from information about the distribution of the X_v, at least asymptotically as $x \to \infty$?

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An answer can be given when the graph is the d-dimensional lattice $\mathbb{Z}^d$ and the amounts at the various vertices are measured by i.i.d. positive random variables with common distribution F. In this case we discuss the asymptotic dependence of the maximum total amount that can be collected at n vertices on the behavior of the tail of the common distribution. If for some $a > 0$, $\int x^d(\log^+ x)^{d+a}\, dF(x)$ is finite, then the maximum total amount grows at most linearly in n (see Theorem 1), and in fact, the growth is asymptotically linear [i.e., an analogue of the strong law of large numbers holds; see Gandolfi and Kesten (1994)]. This result is reasonably sharp. Indeed if the dth moment of F is infinite, then we have superlinear growth (in the weak sense of Theorem 2).

(*b*) *Vertex GLA.* The title of this paper is meant as a description of the following slightly different model. Assume again that we have a graph with i.i.d. positive quantities associated with the vertices. However, this time we are not restricted to following a self-avoiding path of n vertices. Instead we impose the restriction that *at most n distinct* vertices may be visited during the movements through the edges. In this case, it is likely that multiple visits to some vertices are convenient (even though we still can collect the amount at any given vertex at most once). The total amount we gather is thus obtained by adding up the amounts associated with n vertices belonging to some *connected* set containing the starting point. Such connected sets are often called lattice animals (of size n), and the ones which have the highest total amount are the greediest; thus, the title of our paper. Again, the same asymptotic analysis can be carried out when the graph is $\mathbb{Z}^d$, with the same qualitative results as before. As to the constants in the linear growth for vertex GLA and vertex GLP [which will be shown to exist in Gandolfi and Kesten (1994)], Lee (1993) has shown that they are different in most cases [they agree in some trivial cases, such as on $\mathbb{Z}$; see Gandolfi and Kesten (1994)].

(*c*) *Edge GLA and GLP.* Similar interpretations and results can be given when the i.i.d. positive random variables are associated with the edges of the graph, in which case we maximize the sums with the constraint of using at most a fixed number of edges. In this paper, however, we discuss the vertex case only.

(*d*) *PERT networks.* A PERT network is a model for one of the following situations. Suppose certain tasks are to be performed, or certain components of a machine are to operate, or certain computers in a network are to execute programs. For the sake of definiteness let us talk about tasks to be performed. Suppose further that the tasks have some interdependencies so that a task can be initiated only after certain other tasks are finished, and that each task requires a random amount of time to be completed. This generates an oriented graph, in which vertices represent the various tasks, and an oriented edge from a vertex v to a vertex w indicates that the task at v has to be completed before the task at w can be started. Also, there are no oriented circuits, because if tasks are to be repeated at different stages of the process, each repetition is to be counted as a new task, and to be represented by a new vertex. Oriented paths start from one of the starting vertices which have no incoming oriented edges, and end at one of the objective vertices, that is, those without outgoing edges (see Figures 1 and 2). The purpose of the activity is to begin executing the tasks associated with the starting vertices and, following the constraints indicated by the oriented edges, accomplish the tasks associated with the objective vertices. For any specification of the times of the individual tasks, it is easy to see that the total time needed to accomplish all of the objective tasks is the maximum over oriented paths leading from a starting vertex to an objective vertex of the total time associated to the path. Estimation of this time and related questions are the subject of the theory of PERT networks [see, e.g., Malcolm, Roseboom, Clark and Fazar (1959), Bigelow (1962) and Fulkerson (1962) and references therein].

The time to go from the starting to the objective vertex can sometimes also be interpreted as the time for customers to pass through a queueing network. Glynn and Whitt (1991) recently used the subadditive ergodic theorem and results about the hydrodynamic limit for a certain interacting particle system to great advantage in this context.

Results in the present paper and in Gandolfi and Kesten (1994) can be applied to some examples of PERT networks. For simplicity, consider a two-dimensional network as in Figures 1 and 2, with random times given by i.i.d. positive random variables associated with the vertices of the network [the network in Figure 2 is basically the one occurring in Glynn and Whitt (1991)]. In each example, all paths from the starting vertices to the objective vertices have the same length (9 in Figure 1 and 5 in Figure 2), and our technique can be applied to obtain asymptotic results as the size of the network, that is, the length of the paths, gets larger. As before, if, for some

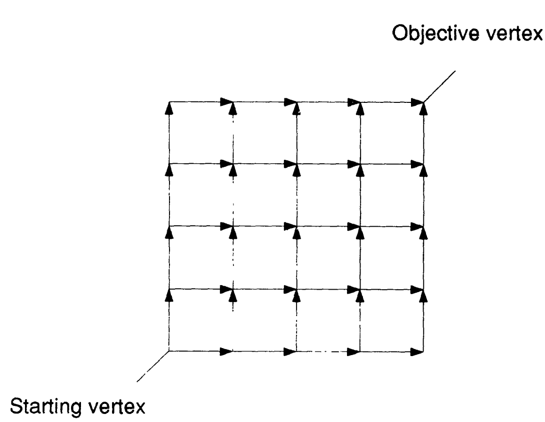

FIG. 1.

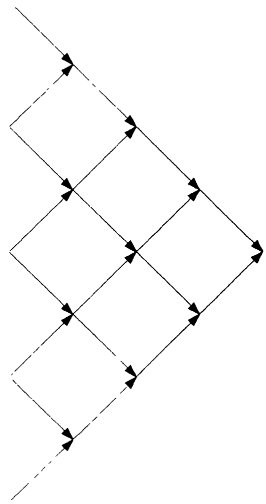

FIG. 2.

$a > 0$, $\int x^2(\log^+ x)^{2+a}\, dF(x)$ is finite, then we have asymptotic linear growth of the total time required to accomplish the objective tasks. As before, we have an analogue of the strong law of large numbers. Again, superlinear growth holds if the second moment of the X_v is infinite. These results give a theoretical framework for discussing the time behavior of large PERT networks; however, for a more careful analysis of PERT networks, a central limit theorem would be of great importance.

(*e*) *ρ-percolation.* Menshikov and Zuyev (1992) introduced ρ-percolation. Vertices of $\mathbb{Z}^d$, $d \geq 2$, are independently chosen occupied with probability p or vacant with probability $1 - p$. For $0 \leq \rho \leq 1$, they say that ρ-percolation occurs if with strictly positive probability there exists an infinite self-avoiding path $\pi = (\mathbf{0}, x_1, x_2, \dots,)$ starting at the origin, such that

$$\liminf_{n \to \infty} \frac{1}{n} [\# \text{ of occupied vertices among } \mathbf{0}, x_1,, \dots, x_{n-1}] \geq \rho.$$

It is not difficult to show that for $\rho > 0$ there is a critical probability $p_c(\rho)$, *strictly* between 0 and 1, so that for $p < p_c(\rho)$, ρ-percolation does not occur and for $p > p_c(\rho)$, ρ-percolation does occur. Menshikov and Zuyev (1992) give various estimates for $p_c(\rho)$. The critical probability $p_c(\rho)$ can also be expressed in terms of the GLP model of (a). To see this, take $X_v = 1$ or 0 with probability p and $1 - p$, respectively. Let M_n be the maximum of $\sum_{v \in \pi_n} X_v$ as π_n ranges over all self-avoiding paths of n vertices starting at the origin $\mathbf{0}$ of $\mathbb{Z}^d$. As will be shown in Gandolfi and Kesten (1994), $M^{(p)} = \lim_{n \to \infty}(1/n)M_n$

exists w.p.1. Then, as we also show in Gandolfi and Kesten [(1994), Theorem 2], ρ-percolation occurs if and only if $M^{(p)} \geq \rho$ so that

$$p_c(\rho) = \inf\{p: M^{(p)} \geq \rho\}.$$

(*f*) *Directed polymers in a random environment.* Imbrie and Spencer (1988) discuss the following model: "A directed polymer system is a statistical ensemble of walks or paths in $\mathbb{Z}^d$ parametrized by time. The graph of the walk in $\mathbb{Z}^{d+1}$ is the 'polymer' which moves at a constant rate in the time direction and so is called 'directed'." If one denotes a point of $\mathbb{Z}^{d+1}$ as (s, x), with $s \in \mathbb{Z}$ representing the time coordinate and $x \in \mathbb{Z}^d$, then the random environment is an independent family of i.i.d. positive random variables $X_{(s,x)}$. For fixed n and $x_i \in \mathbb{Z}^d$, $x_0 = \mathbf{0}$, the probability of obtaining the polymer path $(0, \mathbf{0}), (1, x_1), \dots, (n, x_n)$ is taken as $p(n, x) = (1/Z_n)\prod_0^n X_{(s, x_s)}$ if $(\mathbf{0}, x_1, \dots, x_n)$ is a nearest neighbor path on $\mathbb{Z}^d$, and 0 otherwise, where Z_n is the normalising constant

$$Z_n = \sum_{\substack{(\mathbf{0}, x_1, \dots, x_n) \\ \text{a path on } \mathbb{Z}^d}} \prod_0^n X_{(s, x_s)}.$$

Imbrie and Spencer were interested in the asymptotic behavior of $E\|x_n\|^2$, the mean square end to end distance of the polymer. Our theory has nothing to say about this [and anyway Imbrie and Spencer (1988) consider only bounded $X_{(s, x_s)}$, which is not very interesting for us]. Nevertheless, since the polymers are concentrated around paths with high values of $\prod_1^n X_{(s, x_s)}$ or of $\sum_1^n \log(X_{s, x_s})$, our question of how large the latter sums typically are is vaguely related to directed polymers. Continuous analogues of these models (without the independence for different times) have been studied in other contexts [see Sznitman (1991) and some of its references].

(*g*) *Random surfaces.* A direct stimulus of our investigation was a question by J. Bricmont and C. Newman. They investigated a variant of an Ising model in three dimensions in which a question of the following type arose. Consider "surfaces of plaquettes", where a plaquette is a face (i.e., unit square) of one of the unit cubes $[a_1, a_1 + 1] \times [a_2, a_2 + 1] \times [a_3, a_3 + 1]$ with corners on $\mathbb{Z}^3$. Assume that to each plaquette p is associated a random weight $X(p)$ with $P\{X(p) \geq x\}$ decreasing slower than exponentially [e.g., like $\exp(-k\sqrt{x})$]. How fast can the weight of surfaces of n plaquettes grow with n? If the weights of surfaces in the Bricmont–Newman model could be controlled by using i.i.d. random variables [along the lines of Fontes and Newman (1993), Theorem 5], then our result would provide a linear upper bound in n. Indeed, let us identify plaquettes with vertices of a new graph G, with two vertices of G having an edge between them if the corresponding plaquettes have at least one point in common in $\mathbb{Z}^3$. Then a surface of n plaquettes is a connected set (of a special kind) of n vertices on G. Even though we do not state our theorem for this graph G, the proof still seems to apply.

(*h*) *Random colorings.* A direct relation with GLA has been used by Fontes and Newman (1993) for several problems of the following kind. Let $\{Y_v,\ v \in \mathbb{Z}^d\}$ be an i.i.d. family of random variables concentrated on a finite set of points $\{c_1, \dots, c_l\}$, interpreted as colors. For each realization of the random variables, $\mathbb{Z}^d$ can be partitioned into clusters of equal color (clusters are connected subsets of $\mathbb{Z}^d$ with vertices all of the same color, and maximal with respect to this property). Let e_1 be the first coordinate vector of $\mathbb{Z}^d$ and, for each realization of the colors, let

$$T_n = \min\big\{k\colon \text{there is a path form the origin } \mathbf{0} \text{ of } \mathbb{Z}^d \text{ to } ne_1 \\ \text{using vertices from at most } k \text{ distinct clusters}\big\}.$$

It was conjectured that $\mu = \limsup_{n\to\infty} T_n/n > 0$ if and only if

$$P(Y_v = c_i) < p_c(d), \quad \text{for } i = 1, \dots, l, \tag{1.1}$$

with $p_c(d)$ being the critical probability for independent site percolation in $\mathbb{Z}^d$. Kesten (1986) proves that (1.1) follows from $\mu > 0$, and Fontes and Newman (1993) complete the proof of the conjecture by showing that (1.1) implies $\mu > 0$. They, in fact, show that $\mu > 0$ follows from

$$\limsup_{n\to\infty} \sup_{\mathbf{0}\in\pi, |\pi| = n} \frac{1}{n} \sum_{v\in\pi} |C_v| < \infty \quad \text{w.p.1}, \tag{1.2}$$

where π runs over paths of $\mathbb{Z}^d$ and $|C_v|$ is the number of vertices in the cluster to which v belongs. A detailed, and nonobvious, geometrical analysis shows that (1.2) itself follows from

$$\limsup_{n\to\infty} \sup_{\mathbf{0}\in\xi, |\xi| = n} \frac{1}{n} \sum_{v\in\xi} X_v^2 < \infty \quad \text{w.p.1}, \tag{1.3}$$

where ξ runs over lattice animals and $\{X_v\colon v \in \mathbb{Z}^d\}$ are i.i.d. random variables distributed like $|C_v|$. When $P(Y_v = c_i) < p_c(d)$, the distribution of the cluster size for the ith color satisfies $P(|C_v| \geq n,\ Y_v = c_i) \leq e^{-k_i n}$, for some $k_i > 0$ [see Aizenman and Barsky (1987) or Menshikov (1986)]. Therefore, (1.1) implies that the distribution of X_v satisfies $P(X_v^2 \geq n) \leq e^{-k\sqrt{n}}$, for some $k > 0$, and our Theorem 1 implies that (1.3) holds.

2. Definitions and statement of result. We introduce here a specific example of GLA and GLP based on the d-dimensional integer lattice $\mathbb{Z}^d$. We shall regard $\mathbb{Z}^d$ as a graph with vertex set $V = \{(n_1, \dots, n_d\colon n_i \in \mathbb{Z}\}$ and edge set $E = \{\{v, w\}\colon v = (v(1), \dots, v(d)),\ w = (w(1), \dots, w(d)) \in V,\ |v(i) - w(i)| = 1$ for exactly one i and $= 0$ otherwise$\}$.

Throughout this paper $c_0, c_1, \dots$ denote *constants* whose precise values are irrelevant to the calculation and whose value may change from appearance to appearance. We also use the following notation:

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$\mathbf{0}$ denotes the origin of $\mathbb{Z}^d$;
$\|v\| = \max_{1 \le i \le d} |v(i)|$;
$\Lambda(\mathbf{x}, l) = \prod_{i=1}^{d} [x(i) - l, x(i) + l]$;
$\lfloor c \rfloor$ = largest integer $\le c$ and $c^+ = \max(0, c)$ for any real number c;
$c_1 \vee c_2 = \max(c_1, c_2)$; $c_1 \wedge c_2 = \min(c_1, c_2)$;
$|A|$ = absolute value of A if A is a real number
= cardinality of A if A is a set
= size (to be defined later) of A if A a path, a lattice animal or a tree;
$\mathbf{I}(\mathscr{A})$ = indicator function of the event $\mathscr{A}$.

A self-avoiding *path* π *of size* n (on $\mathbb{Z}^d$) is a sequence $\pi = (v_1, \dots, v_n)$ of n vertices of $\mathbb{Z}^d$ such that $v_i \ne v_j$ for $i \ne j$ and such that v_i and v_{i+1} are adjacent on $\mathbb{Z}^d$ for $1 \le i \le n-1$. *Paths will always be taken self-avoiding even if not explicitly mentioned.* A *lattice animal* ξ *of size* $n \in \mathbf{N}$ is a connected set $\xi = \{v_1, \dots, v_n\}$ of n vertices of $\mathbb{Z}^d$, that is, a set such that for any two of its elements v_i and v_j it is possible to find a path π which contains v_i and v_j and which is completely contained in ξ. A (*abstract*) *tree* is a graph τ, with vertex and edge set $\mathscr{V}$ and $\mathscr{E}$ say, such that

$$\tau \text{ is conntected} \tag{2.1}$$

and

$$\tau \text{ has no circuits (or loops)}; \tag{2.2}$$

here (2.2) means that there are no sequences $(v_{i_1}, \dots, v_{i_l})$, with all v_{i_j} distinct and $l \ge 2$, $v_{i_{j-1}}$ adjacent to v_{i_j} for $2 \le j \le l$ and v_{i_l} adjacent to v_{i_1}. The *size of* τ is the cardinality of its vertex set. Here we are interested mostly in trees which are imbedded in $\mathbb{Z}^d$, that is with $\mathscr{V} \subset V$ and $\mathscr{E} \subset E$. Note that for such a tree, $\mathscr{V}$ is a lattice animal. It is easy to prove that given any lattice animal $\xi = \{v_1, \dots, v_n\}$, there exists at least one tree τ with vertex set $\{v_1, \dots, v_n\}$. Any such tree is called a *spanning tree* of ξ. [See Bollobás (1979), Chapter 12, Corollary 5.] For our purposes it will make no difference which spanning tree is selected.

We denote by $\Pi(l, n)$ and $\Xi(l, n)$ the collections of all paths π and lattice animals ξ, respectively, of size n and contained in $\Lambda(\mathbf{0}, l)$; $\Pi_0(n)$ is the collection of all paths π of size n with first vertex the origin and $\Xi_0(n)$ is the collection of lattice animals of size n which contain the origin (note that lattice animals are unordered sets, so that it is meaningless to talk about a first vertex of a lattice animal).

To complete the description of GLA and GLP we introduce a family $\{X_v: v \in V\}$ of i.i.d. positive random variables. X_v is interpreted as a quantity of some commodity located at v. The total weights of a path π or lattice animal ξ are

$$S(\pi) = \sum_{v \in \pi} X_v \quad \text{and} \quad S(\xi) = \sum_{v \in \xi} X_v. \tag{2.3}$$

The principal quantities of interest in this paper are

$$M_n^* := \max_{\pi \in \Pi(n,n)} \sum_{v \in \pi} X_v = \max_{\pi \in \Pi(n,n)} S(\pi)$$

and

$$N_n^* := \max_{\xi \in \Xi(n,n)} \sum_{v \in \xi} X_v = \max_{\xi \in \Xi(n,n)} S(\xi).$$

These are of course upper bounds for

$$M_n := \max_{\pi \in \Pi_0(n)} S(\pi)$$

and

$$N_n := \max_{\xi \in \Xi_0(n)} S(\xi),$$

respectively.

The common distribution of the X_v is denoted by F, and P is the probability measure on (the Borel σ-algebra on) $\Omega = \prod_{v \in V}[0, \infty)$, which describes the distribution of the $\{X_v\}$ (i.e., $P = \prod_v F$). Expectation with respect to P is denoted by E.

Our principal result here is the following theorem, which gives an upper bound on M_n^* and N_n^*.

THEOREM 1. *If, for some $a > 0$,*

$$E\{X_0^d (\log^+ X_0)^{d+a}\} < \infty, \tag{2.4}$$

then there exists a constant $M < \infty$ such that

$$\limsup_{n \to \infty} \frac{M_n^*}{n} \leq \limsup_{n \to \infty} \frac{N_n^*}{n} \leq M \quad w.p.1. \tag{2.5}$$

Proposition 1 in Section 3 will actually give more precise information about the tail of the distribution of N_n^*. Also, in Gandolfi and Kesten (1994) it will be shown that

$$\lim_{n \to \infty} \frac{1}{n} \max_{\pi \in \Pi_0(n)} S(\pi) = \lim_{n \to \infty} \frac{1}{n} M_n$$

and

$$\lim_{n \to \infty} \frac{1}{n} \max_{\xi \in \Xi_0(n)} S(\xi) = \lim_{n \to \infty} \frac{1}{n} N_n$$

exist and are constant w.p.1.

The next, fairly trivial, result shows that the condition (2.4) is reasonably sharp.

THEOREM 2. *If*

$$E\{X_0^d\} = \infty, \tag{2.6}$$

then

$$\limsup_{n\to\infty} \frac{M_n}{n} = \limsup_{n\to\infty} \frac{N_n}{n} = \infty \quad w.p.1. \tag{2.7}$$

If

$$\frac{x^d}{\log\log x}[1 - F(x)] \to \infty, \qquad x \to \infty, \tag{2.8}$$

then

$$\lim_{n\to\infty} \frac{M_n}{n} = \lim_{n\to\infty} \frac{N_n}{n} = \infty \quad w.p.1. \tag{2.9}$$

PROOF. Since any path can be viewed as a lattice animal, we clearly have

$$N_n \ge M_n \ge \max_{v\in\Lambda(\mathbf{0}, \lfloor n/d \rfloor)} X_v.$$

It therefore suffices to prove

$$\limsup_{n\to\infty} \frac{1}{n} \max_{v\in\Lambda(\mathbf{0}, \lfloor n/d \rfloor)} X_v = \infty \tag{2.10}$$

under (2.6) or

$$\lim_{n\to\infty} \frac{1}{n} \max_{v\in\Lambda(\mathbf{0}, \lfloor n/d \rfloor)} X_v = \infty \tag{2.11}$$

under (2.8).

These, however, are standard consequences of the Borel–Cantelli lemma. Indeed, if (2.6) holds, then for some constant $c_1 > 0$ and any A,

$$\begin{aligned}\sum_v P\{X_v \ge A\|v\|\} &\ge \sum_k \sum_{2^k \le \|v\| < 2^{k+1}} P\{X_\mathbf{0} \ge A2^{k+1}\}\\ &\ge c_1 \sum_k 2^{kd} P\{X_\mathbf{0} \ge A2^{k+1}\} = \infty,\end{aligned}$$

while under (2.8), again for some constant $c_2 > 0$ and any A,

$$\begin{aligned}\sum_k P\left\{\max_{2^k \le \|v\| < 2^{k+1}} X_v < A2^{k+1}\right\} &\le \sum_k \left[F(A2^{k+1})\right]^{c_2 2^{kd}}\\ &\le \sum_k \exp\{-c_2 2^{kd}(1 - F(A2^{k+1}))\} < \infty. \quad \square\end{aligned}$$

The following remarks should give some more feeling for Theorem 1. If

$$E\{e^{tX_0}\} = \int e^{tx}\, dF(x) < \infty \tag{2.12}$$

for some $t > 0$, then (2.5) is easily verified. Indeed, the number of paths as well as the number of lattice animals of size n in $\Lambda(\mathbf{0}, n)$ grows exponentially in n. On the other hand, one has the standard large deviation bound for any ξ of size n,

$$P\{S(\xi) \ge Mn\} \le e^{-tMn}\left[\int e^{tx}\, dF(x)\right]^n,$$

which, for any fixed C, can be made smaller than C^{-n} by taking M large, if (2.12) holds. Of course this argument breaks down when F does not have exponentially decreasing tail, so that (2.12) fails for all $t > 0$. In such a case, it is natural to consider the behavior of max $S(\xi)$ over a class of animals of size n whose cardinality grows slower than exponentially in n. For instance, Smythe (1973) considered the behavior of $S(\rho)$ as ρ varies over boxes $\prod_{i=1}^{d}[a_i, b_i]$, with size $|\rho| = \prod_{i=1}^{d}(b_i - a_i + 1)$, with $a_i \le 0 \le b_i$, $a_i, b_i \in \mathbb{Z}$ (so that $\mathbf{0} \in \rho$). He obtained the following strong law of large numbers.

THEOREM (Smythe). *If*

$$E\left\{|X_{\mathbf{0}}|\log^{+}|X_{\mathbf{0}}|)^{d-1}\right\} < \infty,$$

then

$$\lim_{\substack{|\rho|\to\infty \\ \rho \text{ a box containing } 0}} \frac{1}{|\rho|} S(\rho) = E\{X_0\} \quad w.p.1.$$

If

$$E\left\{|X_{\mathbf{0}}|\log^{+}|X_{\mathbf{0}}|)^{d-1}\right\} = \infty,$$

then

$$\lim_{|\rho|\to\infty} \sup_{\rho \text{ a box containing } 0} \frac{1}{|\rho|} S(\rho) = \infty \quad w.p.1.$$

REMARK 1. Actually Smythe considered only boxes of the form $\prod_1^d[0, b_i]$, $b_i = 1, 2, \ldots,$ but this does not influence the result.

IDEA OF THE PROOF. In order to estimate $S(\xi) = \sum_{v \in \xi} X_v$, it turns out to be useful to estimate separately the contributions from the X's with values in intervals $[0, 1)$ and $[2^k, 2^{k+1})$, $k \ge 0$. Write

$$S(L, R; \xi) = \sum_{v \in \xi} X_v \mathbf{I}(L \le X_v < R). \tag{2.13}$$

We then estimate $S(2^k, 2^{k+1}; \xi)$ by covering ξ by approximately $2|\xi|/l(|\xi|, k)$ cubes with sides of length $4l(|\xi|, k)$, and by adding up all X's with values in $[2^k, 2^{k+1})$ in these cubes. The numbers $l(|\xi|, k)$ will be chosen suitably, in most cases such that the expected number of vertices v per cube with $X_v \in [2^k, 2^{k+1})$ is of order 1. It is easy to estimate the number of vertices v with $X_v \in [2^k, 2^{k+1})$ in a union of cubes, since this number simply has a binomial distribution. We must also take into account the number of ways to choose the collection of cubes. This will be at most $[C(d)]^{|\xi|/l(|\xi|, k)}$ for some constant $C(d)$ which depends on the dimension d only. As we shall see, we can choose numbers $m(|\xi|, k)$ and a constant c such that

$$\sum_k [C(d)]^{|\xi|/l(|\xi|, k)} P\Big\{\text{a union of } 2|\xi|/l(|\xi|, k) \text{ cubes of size } [4l(|\xi|, k) + 1]^d \text{ contains more} \tag{2.14}$$

$$\text{than } cm(|\xi|, k) \text{ vertices } v \text{ with } X_v \geq 2^k\}$$

is small. This will guarantee that (with high probability) for each ξ we only have $cm(|\xi|, k)$ contributions to $S(2^k, 2^{k+1}; \xi)$. The final quantity to estimate for $S(\xi)$ is then

$$\sum_k m(|\xi|, k) 2^{k+1} \tag{2.15}$$

and our m's will be such that this is bounded by $M|\xi|$ for some $M < \infty$.

Some more technical comments on our choice of $l(|\xi|, k)$ and $m(|\xi|, k)$ can be found in the Remark at the end of Section 3.

3. Proving Theorem 1. We first show how to cover any lattice animal of size n by at most $2n/l$ cubes with edges of length $4l$. Then we choose our $m(n, k)$ and $l(n, k)$ and show that (2.15) is of order $|\xi|$. Finally we carry out the probability estimate (2.14).

We remind the reader of the notation

$$\Lambda(\mathbf{x}, 2l) = \prod_{i=1}^{d} [x(i) - 2l, x(i) + 2l]. \tag{3.1}$$

Lemma 1. *Let ξ be a lattice animal of size $|\xi| = n$ and let $1 \leq l \leq n$. Then there exists a sequence $\mathbf{x}_0, \dots, \mathbf{x}_r \in \mathbb{Z}^d$ of $r + 1 \leq 1 + (2n - 2)/l$ points such that*

$$\xi \subset \bigcup_0^r \Lambda(l\mathbf{x}_i, 2l) \tag{3.2}$$

and

$$\|\mathbf{x}_{i+1} - \mathbf{x}_i\| = \max_{1 \leq j \leq d} |x_{i+1}(j) - x_i(j)| \leq 1, \qquad 0 \leq i \leq r - 1. \tag{3.3}$$

If $\mathbf{0} \in \xi$, then we may take in addition $\mathbf{x}_0 = \mathbf{0}$.

Proof. First we observe that it suffices to prove the lemma with the lattice animal ξ replaced by a path $\pi = (v_0, \dots, v_{s-1})$ of length $s \leq 2n$ (π will not be self-avoiding, though). To see this, let τ be a spanning tree for ξ. As mentioned in the introduction, such a spanning tree always exists. Then τ has n vertices and hence $(n - 1)$ edges. There then exists a path π which contains all vertices of τ and with at most twice as many edges as τ, and hence with at most $2n - 1$ vertices. To construct π, one follows the procedure of Harris [(1965), Section 6]. It is simplest to think of τ as being imbedded in the plane and to think of π as a path which starts at an arbitrary vertex, w_0 say, and then "walks around the outside of τ" going in the clockwise direction until it returns to w_0 after having visited all vertices (see Figure 3). When π is covered by $\bigcup_i \Lambda(l\mathbf{x}_i, 2l)$, then ξ is also covered by this union, so that it

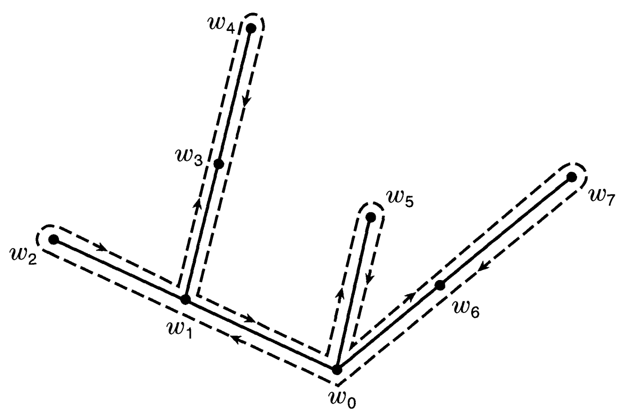

FIG. 3. *Example of a tree τ (solidly drawn) and its associated path π (dashed). π successively traverses the vertices $w_0, w_1, w_2, w_1, w_3, w_4, w_3, w_1, w_0, w_5, w_0, w_6, w_7, w_6, w_0$.*

indeed suffices to prove the lemma for π.

Now let $\pi = (v_0, v_1, \dots, v_{s-1})$ with $s \le 2n - 1$. Then choose $\mathbf{x}_i$ as the unique point of $\mathbb{Z}^d$ for which

$$x_i(j)l \le v_{il}(j) < (x_i(j) + 1)l, \qquad j = 1, \dots, d, 0 \le i \le r := \lfloor (s-1)/l \rfloor.$$

We claim that (3.2) and (3.3) hold for these $\mathbf{x}$'s. To see (3.2), note that for $il \le t \le (i+1)l$,

$$|v_t(j) - x_i(j)l| \le |v_t(j) - v_{il}(j)| + |v_{il}(j) - x_i(j)l| \le 2l,$$

so that $v_t \in \Lambda(l\mathbf{x}_i, 2l)$. (3.3) is immediate from our choice of the $\mathbf{x}_i$ and the fact that $|v_{il}(j) - v_{(i+1)l}(j)| \le l$.

As for the last statement of the lemma, this is really included in our proof, for if $\mathbf{0} \in \xi$, then we may take w_0 and v_0 equal to the origin, and then our choice of $\mathbf{x}_0$ above automatically gives $\mathbf{x}_0 = \mathbf{0}$. $\square$

Next we introduce the $l(n,k)$ and $m(n,k)$. In order to avoid dividing by $0 = \log 1$, *we take $n \ge 2$ from now on*. For convenience we shall also assume that the support of $X_{\mathbf{0}}$ is unbounded; if $X_{\mathbf{0}}$ is bounded w.p.1, then Theorem 1 is trivial. Define

$$p(k) = P\{X_{\mathbf{0}} \ge 2^k\}, \tag{3.4}$$

$$l(n,k) = \lfloor (p(k))^{-1/d} \rfloor \wedge n, \tag{3.5}$$

$$b = 1 + \frac{a}{d} \quad [\text{with } a \text{ satisfying } (2.4)]. \tag{3.6}$$

Also choose a continuous strictly increasing function $\gamma(x)$ with inverse function $\delta(x)$ such that

$$\frac{\gamma(x)(\log x)^b}{x} \to 0 \quad \text{and} \quad \frac{\gamma(x)(\log x)^b \log\log x}{x} \to \infty \qquad (x \to \infty) \tag{3.7}$$

while

$$\int_{[0,\infty)} [\delta(x)]^d \, dF(x) < \infty. \tag{3.8}$$

Such a γ exists by (2.4). [The second requirement in (3.7) will not be used here, but only in Gandolfi and Kesten (1994); we can always satisfy this requirement by replacing $\gamma(x)$ by $\gamma(x) \vee x(\log x)^{-b}(\log\log x)^{-1/2}$.] Then take

$$m(n,k) = \begin{cases} \dfrac{n}{l(n,k)}, & \text{if } l(n,k) \le \dfrac{n}{(\log n)^b} \text{ and } 2^k \le \gamma(n), \\ (\log n)^b, & \text{if } l(n,k) > \dfrac{n}{(\log n)^b} \text{ and } 2^k \le \gamma(n), \\ 0, & \text{if } 2^k > \gamma(n). \end{cases} \tag{3.9}$$

LEMMA 2. *If* (2.4) *holds, then for some* $c_0 < \infty$,

$$\sum_{k=0}^{\infty} m(n,k) 2^{k+1} \le c_0 n, \qquad n \ge 2. \tag{3.10}$$

PROOF. Note that (2.4) implies

$$p(k) \le 2^{-dk}(k \log 2)^{-d-a} E X_0^d (\log^+ X_0)^{d+a},$$

so that for $c_1 = \frac{1}{2}(\log 2)^b \{E X_0^d (\log^+ X_0)^{d+a}\}^{-1/d}$,

$$\lfloor (p(k))^{-1/d} \rfloor \ge c_1 2^k k^{1+a/d}.$$

Consequently, if Σ_1 denotes the sum over those $k \ge 0$ for which

$$l(n,k) \le \frac{n}{(\log n)^b} \quad \text{and} \quad 2^k \le \gamma(n), \tag{3.11}$$

then, by (3.9) and (3.5),

$$\begin{aligned} \textstyle\sum_1 m(n,k) 2^{k+1} &\le \sum_{k \ge 0} 2n p_k^{1/d} 2^{k+1} \\ &\le 4n \left(1 + \sum_{k \ge 1} \frac{1}{c_1} k^{-1-a/d} \right) \le c_2 n. \end{aligned} \tag{3.12}$$

Similarly, if Σ_2 is the sum over those $k \ge 0$ for which

$$l(n,k) > \frac{n}{(\log n)^b} \quad \text{and} \quad 2^k \le \gamma(n), \tag{3.13}$$

then

$$\sum_2 m(n,k)2^{k+1} \le \sum_{2^k \le \gamma(n)} 2^{k+1}(\log n)^b \tag{3.14}$$
$$\le 4\gamma(n)(\log n)^b = o(n) \quad [\text{by } (3.7)].$$

This implies (3.10) with

$$c_0 = 4\max_{n\ge 2}\frac{\gamma(n)(\log n)^b}{n} + 4 + 8(\log 2)^{-b}\{EX_0^d(\log^+ X_0)^{d+a}\}^{1/d}\sum_{k\ge 1}k^{-b}.$$

□

Next we start on the probability estimates.

LEMMA 3. *Let* $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_r$ *be arbitrary points of* $\mathbb{Z}^d$ *and* $r \le (2n-2)/l(n,k)$ *with* $l(n,k)$ *as in* (3.5). *Then there exists a* $c_3 = c_3(d) < \infty$, *which depends on d only, such that for all* $c \ge c_3$,

$$P\left\{\bigcup_{i=0}^{r}\Lambda(\mathbf{y}_i, 2l(n,k)) \text{ contains more than } cm(n,k) \text{ vertices with } X_v \ge 2^k\right\} \le \exp\left(\frac{-cn}{2l(n,k)}\right) \tag{3.15}$$

for all k which satisfy (3.11). *Moreover, the probability in* (3.15) *is at most*

$$\exp\left(-\frac{c}{2}(\log n)^b\right) \tag{3.16}$$

for all k which satisfy (3.13). *Finally*

$$P\{\text{for infinitely many vertices } v,\ X_v \ge \gamma(\|v\|)\} = 0. \tag{3.17}$$

PROOF. To obtain the estimates in (3.15) and (3.16) we use the fact that the number of vertices in

$$\bigcup_{i=0}^{r}\Lambda(\mathbf{y}_i, 2l(n,k)) \tag{3.18}$$

is at most

$$(r+1)(4l(n,k)+1)^d \le \frac{3n}{l(n,k)}5^d(l(n,k))^d = 3\cdot 5^d(l(n,k))^{d-1}n.$$

Thus, if we set

$$N = N(n,k) = 3\cdot 5^d(l(n,k))^{d-1}n,$$

then the number of v in the set (3.18) with $X_v \ge 2^k$ is stochastically smaller than a binomial variable corresponding to N trials with a success parameter $p(k)$. Consequently, by Chebychev's inequality, the probability in the left-hand side of (3.15) is at most

$$\text{(3.19)} \qquad \begin{aligned} &\exp(-cm(n,k))[1 - p(k) + p(k)e]^N \\ &\quad \le \exp(-cm(n,k) + Np(k)(e-1)). \end{aligned}$$

We now treat separately the cases where (3.11) and (3.13) hold. If (3.11) holds, then the right-hand side of (3.19) is at most

$$\text{(3.20)} \qquad \begin{aligned} &\exp\left\{-c\frac{n}{l(n,k)} + 3 \cdot 5^d \frac{n}{l(n,k)}(l(n,k))^d p(k)(e-1)\right\} \\ &\quad \le \exp\left\{-\frac{n}{l(n,k)}(c - 3 \cdot 5^d(e-1))\right\}. \end{aligned}$$

This gives (3.15) for $c \ge c_3 := 6 \cdot 5^d(e-1)$.

If (3.13) applies, then the right-hand side of (3.19) is, by our choice of $m(n,k)$, at most

$$\exp\left\{-c(\log n)^b + 3 \cdot 5^d \frac{n}{l(n,k)}(l(n,k))^d p(k)(e-1)\right\}.$$

Also, since $n(\log n)^{-b} \le l(n,k) \le (p(k))^{-1/d}$ by (3.13) and (3.5), we have

$$\frac{n}{l(n,k)}(l(n,k))^d p(k) \le (\log n)^b.$$

Again (3.16) follows from $c \ge c_3 = 6 \cdot 5^d(e-1)$.

Finally, (3.17) follows from the Borel–Cantelli lemma and the simple estimates

$$\begin{aligned} &P\{\text{there is some } v \text{ with } 2^j \le \|v\| < 2^{j+1} \text{ and } X_v \ge \gamma(\|v\|)\} \\ &\quad \le c_4 2^{dj} P\{X_0 \ge \gamma(2^j)\} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{\infty} 2^{dj} P\{X_0 \ge \gamma(2^j)\} &= \int_{(0,\infty)} dF(x) \sum_{\gamma(2^j) \le x} 2^{dj} \\ &= \int_{[0,\infty)} dF(x) \sum_{2^j \le \delta(x)} 2^{dj} \\ &\le 2\int_{[0,\infty)} dF(x)[\delta(x)]^d < \infty \quad [\text{by (3.8)}]. \qquad \square \end{aligned}$$

We shall now complete the proof by combining the three lemmas.

PROOF OF THEOREM 1. First we note that we may ignore vertices v in $[-n,n]^d$ with $X_v \ge \gamma(n)$, since these occur w.p.1 for only finitely many n [by virtue of (3.17)]. It therefore suffices to prove [see (2.13) for notation]

$$\text{(3.21)} \qquad \sum_n P\{\exists\, \xi \subset [-n,n]^d \text{ with } |\xi| = n \text{ and } S(0,\gamma(n);\xi) > Mn\} < \infty$$

for a suitable M.

Next we have by obvious translation invariance that the summand in (3.21) is at most

$$(2n+1)^d P\{\exists\, \xi \text{ such that } \mathbf{0} \in \xi, |\xi| = n \text{ and } S(0,\gamma(n);\xi) > Mn\}.$$

Furthermore, for any ξ with $|\xi| = n$,

$$\begin{aligned} S(0,\gamma(n);\xi) &\le S(0,1;\xi) + \sum_{\substack{k\ge 0\\ 2^k \le \gamma(n)}} S(2^k, 2^{k+1};\xi) \\ &\le n + \sum_{\substack{k\ge 0\\ 2^k \le \gamma(n)}} S(2^k, 2^{k+1};\xi). \end{aligned}$$

If

$$S(2^k, 2^{k+1};\xi) \le cm(n,k)2^{k+1}$$

for all $k \ge 0$ with $2^k \le \gamma(n)$, then

$$S(0,\gamma(n);\xi) \le (1 + cc_0)n$$

by Lemma 2. Thus, if we have $M > 1 + cc_0$, then it suffices to prove that for some large c,

$$\sum_n n^d \sum_{\substack{k\ge 0\\ 2^k \le \gamma(n)}} P\{\exists\, \xi \text{ such that } \mathbf{0} \in \xi, |\xi| = n \text{ and } S(2^k, 2^{k+1};\xi) > cm(n,k)2^{k+1}\} < \infty. \tag{3.22}$$

Now fix n and k for the time being and let ξ be a lattice animal with $\mathbf{0} \in \xi$, $|\xi| = n$ and $\mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_r$ such that (3.2) and (3.3) hold and

$$r \le \frac{2n-2}{l(n,k)}.$$

Such $\mathbf{x}_i$ exist by Lemma 1. Notice that we took $\mathbf{x}_0 = \mathbf{0}$ as allowed by Lemma 1. Now clearly

$$\begin{aligned} &S(2^k, 2^{k+1};\xi) \\ &\quad \le 2^{k+1}\Big(\text{number of } v \in \bigcup_{i \le r} \Lambda(l\mathbf{x}_i, 2l(n,k)) \text{ with } X_v \ge 2^k\Big). \end{aligned} \tag{3.23}$$

Therefore, if k satisfies (3.11), then the probability appearing in (3.22) is at most

$$\begin{aligned} &P\Big\{\exists\, \mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_r \text{ satisfying (3.3) and } r \le (2n-2)/l(n,k) \\ &\qquad \text{such that there are more than } cm(n,k) \text{ vertices} \\ &\qquad\qquad v \in \bigcup_{i\le r} \Lambda(l\mathbf{x}_i, 2l(n,k)) \text{ with } X_v \ge 2^k\Big\} \\ &\le [\text{number of choices for } \mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_r \text{ which satisfy (3.3)} \\ &\qquad \text{and } r \le (2n-2)/l(n,k)]\exp(-cn/(2l(n,k))) \quad [\text{by (3.15)}]. \end{aligned} \tag{3.24}$$

Finally we note that the number of choices for $\mathbf{x}_0 = \mathbf{0}, \mathbf{x}_1, \dots, \mathbf{x}_r$ which satisfy (3.3) is at most 3^{dr}, since for given $\mathbf{x}_i$ there are at most 3^d choices for $\mathbf{x}_{i+1}$ so that (3.3) holds. Therefore, (3.24) is at most

$$3^{(2dn/l(n,k))}\exp\left(-\frac{cn}{2l(n,k)}\right) \le \exp\left(-\frac{c}{4}(\log n)^b\right) \tag{3.25}$$

for k satisfying (3.11) and $c \ge$ some $c_5 = c_5(d)$.

In the same way we see from (3.16) that for any k which satisfies (3.13) and $c \ge c_5$, that the left-hand side of (3.24) is at most

$$\begin{aligned} 3^{(2dn/l(n,k))}\exp\left(-\frac{c}{2}(\log n)^b\right) &\le 3^{2d(\log n)^b}\exp\left(-\frac{c}{2}(\log n)^b\right) \\ &\le \exp\left(-\frac{c}{4}(\log n)^b\right). \end{aligned} \tag{3.26}$$

Using these bounds for (3.24), and hence for the probability in (3.22), we see that the sum in (3.22) is at most

$$\sum_n n^d \sum_{\substack{k \ge 0 \\ 2^k \le \gamma(n)}} \exp\left(-\frac{c}{4}(\log n)^b\right) \le \sum_n n^d \frac{\log n}{\log 2}\exp\left(-\frac{c}{4}(\log n)^b\right) < \infty$$

since $b > 1$ [see (3.6)]. □

For a sequel to this paper [Gandolfi and Kesten (1994)] it is useful to separate out the following explicit estimates which are contained in our proofs.

PROPOSITION 1. *Assume that* (2.4) *holds. Then there exist constants* $c_0 < \infty$ *and* $\bar{c} = \bar{c}(d) < \infty$ *such that for all* $c \ge \bar{c}$ *and* $n \ge 2$,

$$P\left\{\max_{\substack{|\xi|=n \\ \mathbf{0}\in\xi}} \sum_{v\in\xi} (X_v \wedge \gamma(n)) > (1 + cc_0)n\right\} \le \exp\left(-\frac{c}{8}(\log n)^{1+a/d}\right) \tag{3.27}$$

and for some $c_6, c_7 < \infty$,

$$\begin{aligned} P\Bigg\{ &\max_{\substack{|\xi|=n \\ \mathbf{0}\in\xi}} S\left(\frac{1}{2d}\log n, \gamma(n); \xi\right) \\ &\qquad \ge c_6 n\left[(\log\log n)^{-a/d} + \frac{\gamma(n)(\log n)^{1+a/d}}{n}\right]\Bigg\} \\ &\le \exp\left(-c_7(\log n)^{1+a/d}\right). \end{aligned} \tag{3.28}$$

PROOF. The proof of (3.27) is essentially the same as for Theorem 1. Indeed

$$\sum_{v\in\xi} (X_v \wedge \gamma(n)) > (1 + cc_0)n$$

can occur only if the event in (3.24) occurs for some $k \geq 0$ with $2^k \leq \gamma(n) = o(n(\log n)^{-b})$. There are at most $1 + (\log n)/\log 2$ such values of k and for each such k, the right-hand side of (3.24) is at most $\exp(c(\log n)^b/4)$ when $c \geq c_5$ [see (3.25) and (3.26)]. This implies (3.27).

As for (3.28), assume that for some ξ, (3.2) holds and

$$\left(\text{number of vertices in } \bigcup_0^r \Lambda(l\mathbf{x}_i, 2l(n,k)) \text{ with } X_v \geq 2^k\right) \leq c_5 m(n,k)$$

for all k satisfying

$$\frac{1}{4d}\log n \leq 2^k \leq \gamma(n). \tag{3.29}$$

Then, as in (3.12) and (3.14),

$$\begin{aligned} S\left(\frac{1}{2d}\log n, \gamma(n); \xi\right) &\leq \sum_{k \text{ satisfying } (3.29)} c_5 m(n,k) 2^{k+1} \\ &\leq 4c_5 n \sum_{2^k \geq (1/4d)\log n} \frac{1}{c_1} k^{-1-a/d} + 4c_5\gamma(n)(\log n)^b \\ &\leq c_6 n\left\{(\log\log n)^{-a/d} + \frac{\gamma(n)(\log n)^b}{n}\right\}. \end{aligned}$$

Thus, the event in the left-hand side of (3.28) can occur only if the event in the left-hand side of (3.24) with $c = c_5$ occurs for some k satisfying (3.29). Therefore, (3.28) also follows from (3.25) and (3.26). □

REMARK. In order to make (3.24) small we must take $m(n,k)$ at least of the order $n/l(n,k)$ for the k satisfying (3.11) [see (3.25), which in turn relies on (3.19)]. Since we want to make $m(n,k)$ small to get a good estimate (see Lemma 2), it appears that one wants to choose $l(n,k)$ as large as possible. However, the estimate (3.20) works only if $(l(n,k))^d p(k)$ is of order 1 so that (3.5) seems the best choice for $l(n,k)$. Then the choice of $m(n,k)$ for the k's satisfying (3.11) also seems best possible. There is somewhat more leeway, though, in the choice of $m(n,k)$ for the other k's.

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