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ON THE SPEED OF CONVERGENCE IN FIRST-PASSAGE PERCOLATION[1]

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We consider the standard first-passage percolation problem on $\mathbb{Z}^d$: $\{t(e)$: e an edge of $\mathbb{Z}^d\}$ is an i.i.d. family of random variables with common distribution F, $a_{0,n} := \inf\{\sum_1^k t(e_i)$: $(e_1, \dots, e_k)$ a path on $\mathbb{Z}^d$ from 0 to $n\xi_1\}$, where ξ_1 is the first coordinate vector. We show that $\sigma^2(a_{0,n}) \le C_1 n$ and that $P\{|a_{0,n} - Ea_{0,n}| \ge x\sqrt{n}\} \le C_2 \exp(-C_3 x)$ for $x \le C_4 n$ and for some constants $0 < C_i < \infty$. It is known that $\mu := \lim(1/n)Ea_{0,n}$ exists. We show also that $C_5 n^{-1} \le Ea_{0,n} - n\mu \le C_6 n^{5/6}(\log n)^{1/3}$. There are corresponding statements for the roughness of the boundary of the set $\tilde{B}(t) = \{v$: v a vertex of $\mathbb{Z}^d$ that can be reached from the origin by a path $(e_1, \dots, e_k)$ with $\sum t(e_i) \le t\}$.

1. Introduction. First-passage percolation was introduced in Hammersley and Welsh (1965); see Smythe and Wierman (1978), Kesten (1986) and Kesten (1987) for later surveys of the subject. The simplest setup (and this is the only one we shall discuss here) is as follows: To each edge e of $\mathbb{Z}^d$ we attach a positive random variable $t(e)$. The basic assumption is that the random variables $\{t(e)$: $e \in \mathbb{Z}^d\}$ are i.i.d. We denote the common distribution by F and assume throughout that

$$F(0-) = 0, \int_{[0,\infty)} xF(dx) < \infty, \ F \text{ is not concentrated on one point.} \tag{1.1}$$

$t(e)$ is interpreted as the passage time of e; that is, the time it takes for a particle to traverse e (in either direction). The basic question is to find asymptotic properties of the set $\tilde{B}(t)$ of vertices which a particle can reach by time t, when it starts at the origin at time 0. More formally, if r is a path on $\mathbb{Z}^d$ that successively traverses the edges $e_1, \dots, e_k$, then we define *the passage time of* r as

$$T(r) = \sum_1^k t(e_i).$$

r itself will often be denoted as $(e_1, \dots, e_k)$ when r traverses successively the edges $e_1, \dots, e_k$. For any two sets A and B of vertices of $\mathbb{Z}^d$ we define *the passage time from* A *to* B as

$$T(A, B) = \inf\{T(r): r \text{ a path from some vertex in } A \text{ to some vertex in } B\},$$

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and with $\mathbf{0}$ = the origin,

$$\tilde{B}(t) = \{v : T(\mathbf{0}, v) \le t\}.$$

We shall write ξ_i for the ith coordinate vector. The passage time from $\mathbf{0}$ to $n\xi_1$ is denoted by

$$a_{0,n} := T(\mathbf{0}, n\xi_1).$$

One of the principal results in the subjects is the following consequence of Kingman's subadditive ergodic theorem [see Smythe and Wierman (1978), Theorem 5.1].

THEOREM A. *If* (1.1) *holds, then*

$$\lim \frac{a_{0,n}}{n} = \mu \text{ w.p.1 and in } L^1 \tag{1.2}$$

for some constant $\mu = \mu(F, d)$.

μ is called the *time constant*, and it is known that

$$0 \le \mu < \int x\,dF(x) \tag{1.3}$$

when F satisfies (1.1) [cf. Hammersley and Welsh (1965), Theorem 4.1.9]. $\mu = 0$ is possible. In fact one has [Kesten (1986), Theorem 6.1]

$$\mu = 0 \text{ if and only if } F(0) \ge p_c(\mathbb{Z}^d), \tag{1.4}$$

where p_c is the critical probability of bond percolation on $\mathbb{Z}^d$.

The next fundamental result is the *shape theorem* for $\tilde{B}(t)$, or rather for the following "fattened up" version of $\tilde{B}(t)$:

$$B(t) = \left\{v + x : v \in \tilde{B}(t),\ x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^d\right\}.$$

[$B(t)$ puts a unit cube around each vertex in $\tilde{B}(t)$ and is therefore no longer made up of isolated points.] The first version of the shape theorem is due to Richardson (1973) and the following version is essentially due to Cox and Durrett (1981) [see also Kesten (1986), Section 3, and for related models, Schürger (1981), Section 3].

THEOREM B. *Assume that*

$$E \min\left\{t_1^d, \dots, t_{2d}^d\right\} < \infty, \tag{1.5}$$

where $t_1, \dots, t_{2d}$ *are independent random variables, each with distribution* F. *Then there exists a nonrandom convex set* $B_0 \subset \mathbb{R}^d$ *with nonempty interior,* *that is either compact or equals all of* $\mathbb{R}^d$ *and has the following property*:

If B_0 *is compact, then for all* $\varepsilon > 0$,

$$(1 - \varepsilon)B_0 \subset \frac{1}{t}B(t) \subset (1 + \varepsilon)B_0 \quad \textit{eventually } w.p.1; \tag{1.6}$$

If $B_0 = \mathbb{R}^d$, *then for all* $\varepsilon > 0$,

$$\{x: |x| \le \varepsilon^{-1}\} \subset \frac{B(t)}{t} \quad \textit{eventually } w.p.1. \tag{1.7}$$

If (1.5) *fails, then*

$$\limsup_{|v| \to \infty} \frac{1}{|v|} T(\mathbf{0}, v) = \infty \quad w.p.1.$$

[*Here* v *runs through the vertices of* $\mathbb{Z}^d$; $|x| = \max_{1 \le i \le d} |x(i)|$ *if* $x = (x(1), \dots, x(d))$.]

B_0 is compact if and only if $\mu > 0$ or $F(0) < p_c$ and $B_0 = \mathbb{R}^d$ if $F(0) \ge p_c$ [compare Kesten (1986), page 219]. In the former case (1.6) tells us that $(1/t)B(t)$ looks asymptotically like the nonrandom set B_0, with merely some roughness at the boundary. But $B(t)$ does not have holes or long arms with size of order t. A natural question, which appears in a slightly different form already in Smythe and Wierman [(1978), Section 10.2], is "how fast is the convergence in (1.2)", or "how rough is the boundary of the set $B(t)$"? Because of the equivalence of one version of the Eden growth model [see Eden (1961)] and first-passage percolation when F is the exponential distribution [cf. Richardson (1973)] this problem has also received much attention from statistical physicists [see Krug and Spohn (1990), Sections 4.1 and 7.1, and Family and Vicsek (1991)]. Simulations [see Zabolitzky and Stauffer (1986); Wolf and Kertész (1987)] and some scaling theory [e.g., Kardar, Parisi and Zhang (1986); Krug and Spohn (1990), Sections 3.1, 3.2 and 7.1] indicate that in two dimensions the roughness of the boundary of $B(t)$ should only be of order $t^{1/3}$. Thus, for large x, the probability

$$P\{(t - xt^{1/3})B_0 \subset B(t) \subset (t + xt^{1/3})B_0\} \tag{1.8}$$

should be close to 1 for all $t \ge 1$, at least when F is the exponential distribution. [Perhaps tB_0 should be replaced here by some other nonrandom set that plays the role of the "expectation of $B(t)$" and that may differ from tB_0 by something that is large with respect to $t^{1/3}$. This corresponds to the fact that the difference between $Ea_{0,n}$ and $n\mu$ in Theorem 1 may dominate the fluctuations in $a_{0,n} - Ea_{0,n}$.] Because

$$\{a_{0,n} \le t\} = \{n\xi_1 \in B(t)\},$$

we can also conjecture on the basis of the simulations mentioned previously that, in two dimensions, for some constants $0 < C_i < \infty$,

$$C_1 n^{2/3} \le \sigma^2(a_{0,n}) \le C_2 n^{2/3} \tag{1.9}$$

[$\sigma^2(X)$ denotes the variance of the random variable X]. (1.9) should be contrasted with classical "diffusive behavior" that would give $\sigma^2(a_{0,n})$ of order n. Indeed, if r_n is the path from $\mathbf{0}$ to $n\xi_1$ along the first coordinate axis that consists of the edges $\{j\xi_1, (j+1)\xi_1\}$, $0 \le j \le n-1$, then

$$T(r_n) = \sum_1^n t(\{j\xi_1, (j+1)\xi_1\}) \tag{1.10}$$

is the sum of n i.i.d. random variables so that

$$\frac{T(r_n) - n\int x\,dF}{\sqrt{n}} \to N(0, \sigma^2)$$

in distribution, where σ^2 is the variance of F. However, $a_{0,n}$ is the infimum of passage times over many paths, and it is intuitively plausible that its variance would be of smaller order than $\sigma^2(T(r_n))$. So far we can only prove $\sigma^2(a_{0,n}) = O(n)$, though. On the other side, we only obtain a lower bound of order 1 for $\sigma^2(a_{0,n})$. These results for $\sigma^2(a_{0,n})$ lead to tail estimates for the distribution of the deviations $a_{0,n} - Ea_{0,n}$ and $a_{0,n} - n\mu$, as well as for the deviations of $t^{-1}B(t)$ from B_0.

We now state our principal results. Throughout C_i denotes a constant with $0 < C_i < \infty$ whose precise value is of no importance; its value may change from appearance to appearance. (But C_i will always be independent of n and t; it may depend on F and d, though.)

THEOREM 1. *If* (1.1) *holds and*

$$F(0) < p_c(\mathbb{Z}^d) \tag{1.11}$$

and

$$\int x^2 F(dx) < \infty, \tag{1.12}$$

then

$$C_1 \le \sigma^2(a_{0,n}) \le C_2 n. \tag{1.13}$$

If (1.12) *is strengthened to*

$$\int e^{\gamma x} F(dx) < \infty \quad \textit{for some } \gamma > 0, \tag{1.14}$$

then

$$P\left\{\left|\frac{a_{0,n} - Ea_{0,n}}{\sqrt{n}}\right| \ge x\right\} \le C_3 e^{-C_4 x} \quad \textit{for } x \le C_5 n, \tag{1.15}$$

$$C_6 n^{-2} \le \frac{1}{n} Ea_{0,n} - \mu \le C_7 n^{-1/6} (\log n)^{1/3}, \tag{1.16}$$

$$P\{a_{0,n} - n\mu \le -x\sqrt{n}\} \le C_3 e^{-C_4 x} \tag{1.17}$$

and

$$(1.18)\qquad P\{a_{0,n} - n\mu \geq 2C_7 n^{5/6}(\log n)^{1/3}\} \leq C_3 \exp(-C_8(n \log n)^{1/3}).$$

For the set $B(t)$ one obtains the following result.

THEOREM 2. *If* (1.1), (1.11) *and* (1.14) *hold, then for* $t \geq 1$,

$$(1.19)\qquad P\left\{\frac{B(t)}{t} \subset \left(1 + \frac{x}{\sqrt{t}}\right)B_0\right\} \geq 1 - C_1 t^{2d} e^{-C_2 x} \quad \textit{if } x \leq \sqrt{t}$$

and

$$(1.20)\qquad \begin{aligned} &P\left\{\left(1 - C_3 t^{-1/(2d+4)}(\log t)^{1/(d+2)}\right)B_0 \subset \frac{B(t)}{t}\right\} \\ &\quad \geq 1 - C_4 t^d \exp\left(-C_5 t^{(d+1)/(2d+4)}(\log t)^{1/(d+2)}\right). \end{aligned}$$

Moreover,

$$(1.21)\qquad \begin{aligned} &P\left\{\left(1 - 2C_3 t^{-1/(2d+4)}(\log t)^{1/(d+2)}\right)B_0 \subset \frac{B(t)}{t}\right. \\ &\quad \left. \subset \left(1 + C_6 \frac{\log t}{\sqrt{t}}\right)B_0 \textit{ for all large } t\right\} = 1. \end{aligned}$$

REMARK 1. Theorem 1 considerably improves the estimates in Kesten [(1986), Section 5]. After completion of the present paper, Alexander (1991) showed that one can improve the right-hand side of (1.16) to $C_7 n^{-1/2}(\log n)$. Consequently (1.18) can be improved to

$$\begin{aligned} &P\{a_{0,n} - n\mu \geq C_7 n^{1/2}(\log n) + x n^{1/2}\} \\ &\quad \leq C_3 e^{-C_4 x} \quad \text{for } x \leq C_5 n. \end{aligned}$$

The next problem one should attack now is to show that $a_{0,n}$ behaves "subdiffusively"; that is, that $\sigma^2(a_{0,n}) \leq n^{1-\varepsilon}$ for some $\varepsilon > 0$. The place where one might hope to gain some power of n is in the last inequality of (2.18).

REMARK 2. Other passage times $b_{0,n}$, $s_{0,n}$, $t_{0,n}$ have been considered in the literature. For example, if H_n is the hyperplane $\{x = (x(1)), \dots, x(d)):$ $x(1) = n\}$, then $b_{0,n} = T(\mathbf{0}, H_n)$. $t_{0,n}$ and $s_{0,n}$ are the analogues of $a_{0,n}$ and $b_{0,n}$, respectively, when one only allows "cylinder paths" from $\mathbf{0}$ to $n\xi_1$ or to H_n; that is, paths that lie between the hyperplanes H_0 and H_n [see Smythe and Wierman (1978) or Kesten (1986) for more details]. All the bounds in Theorem 1 remain valid when $a_{0,n}$ is replaced by $s_{0,n}$ or $t_{0,n}$. When $a_{0,n}$ is replaced by $b_{0,n}$, then (1.13), (1.15) and (1.18) remain valid.

We also note that analogues of (1.13) and (1.15) hold for moving in directions other than along a coordinate axis. That is, one has estimates for the tails of $T(\mathbf{0}, n\xi)$ for any unit vector ξ; see (2.49).

REMARK 3. Theorems 1 and 2 deal with the "subcritical case" when $F(0) < p_c$. In the "supercritical case" when $F(0) > p_c$, then $a_{0,n}$ is of order 1 in probability. That is,

$$(1.22) \qquad P\{a_{0,n} \geq x\} \to 0 \quad \text{as } x \to \infty, \text{ uniformly in } n.$$

Indeed, there now exists an infinite connected set $\mathscr{C}$ of edges c with $t(e) = 0$. Travel along edges from $\mathscr{C}$ costs no time. The travel time from $\mathbf{0}$ to $n\xi_1$ is therefore at most the travel time from $\mathbf{0}$ to $\mathscr{C}$ plus the travel time from $n\xi_1$ to $\mathscr{C}$ [see Zhang and Zhang (1984) for details in case $d = 2$]. The so-called critical case when $F(0) = p_c$ is more complicated. The known results here all assume that F has bounded support. For $d = 2$, Chayes, Chayes and Durrett (1986) prove that $a_{0,n} = O(\log n)$ in probability, whereas for $d \geq 2$, Chayes (1991) shows that $a_{0,n} = O(n^{\varepsilon})$ in probability, for every fixed $\epsilon > 0$. His proof can be sharpened somewhat to give for some constant C,

$$(1.23) \qquad P\left\{a_{0,n} \leq \exp\left(C\sqrt{\log n}\,\right) \text{ for all large } n\right\} = 1.$$

This is probably not optimal. For the Bethe tree, which is usually regarded as the analogue of $\mathbb{Z}^d$ for $d = \infty$, Bramson (1978) shows that an analogue of $b_{0,n}$ is only $O(\log\log n)$.

Finally, we note that the exclusion of one point distributions for F is harmless, because if $t(e) = c$ with probability 1, then also $a_{0,n} = nc$ and $\tilde{B}(T) = \{v: |v| \leq n/c\}$ with probability 1.

Our proofs of (1.13) and (1.15) (which are the basic new estimates) are based on the "method of bounded differences." This method has been successfully used in a variety of combinatorial problems in the last few years [see McDiarmid (1989) for a survey]. Basically this method represents $a_{0,n} - Ea_{0,n}$ as a sum of martingale differences and, after estimating (the sum of squares of) these differences, applies standard exponential bounds for martingales with bounded differences. What is remarkable is that the martingale used to represent $a_{0,n} - Ea_{0,n}$ ignores the entire geometrical structure of our problem; it is a counterintuitive representation. [In another context, essentially the same martingale representation has been used in Aizenman and Wehr (1990).] We formulate here the abstract martingale estimate to which we reduce our problem. We believe that it is of independent interest. Its proof is given in the last section.

THEOREM 3. *Let $\{\mathscr{F}_k\}_{0 \leq k \leq N}$ be an increasing family of σ-fields of measurable sets and let $\{U_k\}_{0 \leq k \leq N}$ be a family of positive random variables that are $\mathscr{F}_N$-measurable. (We do not assume that U_k is $\mathscr{F}_k$-measurable.) Let $\{M_k\}_{0 \leq k \leq N}$ be a martingale with respect to $\{\mathscr{F}_k\}_{0 \leq k \leq N}$. (We allow $N = \infty$, in which case*

$\mathscr{F}_N = \bigvee_0^\infty \mathscr{F}_k$ *and we merely assume that* $\{M_k\}_{0 \le k < \infty}$ *is a martingale.) Assume that the increments* $\Delta_k := M_k - M_{k-1}$ *satisfy*

$$|\Delta_k| \le c \quad \text{for some constant } c \tag{1.24}$$

and

$$E\{\Delta_k^2 | \mathscr{F}_{k-1}\} \le E\{U_k | \mathscr{F}_{k-1}\}. \tag{1.25}$$

Assume further that for some constants $0 < C_1, C_2 < \infty$ *and* x_0 *with*

$$x_0 \ge e^2 c^2 \tag{1.26}$$

we have

$$P\left\{\sum_1^N U_k > x\right\} \le C_1 e^{-C_2 x} \quad \text{when } x \ge x_0. \tag{1.27}$$

Then, in the case where $N = \infty$, $M_N = \lim_{n\to\infty} M_n$ *exists w.p.*1 *Moreover, irrespective of the value of* N, *there exist universal constants* $0 < C_3, C_4 < \infty$ *that do not depend on* N, C_1, C_2, c *and* x_0, *nor on the distribution of* $\{M_k\}$ *and* $\{U_k\}$, *such that*

$$\begin{aligned} P\{M_N - M_0 \ge x\} &\le C_3\left(1 + C_1 + \frac{C_1}{C_2 x_0}\right) \\ &\quad \times \exp\left(-C_4 \frac{x}{x_0^{1/2} + C_2^{-1/3} x^{1/3}}\right). \end{aligned} \tag{1.28}$$

In particular, for $x \le C_2 x_0^{3/2}$,

$$P\{M_N - M_0 \ge x\} \le C_3\left(1 + C_1 + \frac{C_1}{C_2 x_0}\right)\exp\left(-\frac{C_4 x}{2\sqrt{x_0}}\right). \tag{1.29}$$

Notational conventions. $t_1, t_2, \ldots$ will be independent random variables, each with the distribution F.

For a vertex or vector $v = (v(1), \ldots, v(d))$ we shall use both the l_∞ and the l_2 norm. These are denoted by

$$|v| = \max_{1 \le i \le d} |v(i)| \quad \text{and} \quad \|v\| = \left\{\sum_{i=1}^d (v(i))^2\right\}^{1/2}, \tag{1.30}$$

respectively.

For a random variable X, $\sigma(X)$ denotes its standard deviation.

$\lfloor a \rfloor$ is the largest integer $\le a$, $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$.

$a \mid b$ means $b = ka$ for some $k \in \mathbb{Z}$ (i.e., a divides b).

2. Centering at $Ea_{0,n}$.

We begin with the proof of (1.13).

PROOF OF (1.13). Since we will have to consider several configurations of passage times at the same time, we need to introduce explicit notation for our probability space and its points. Order the edges of $\mathbb{Z}^d$ in some arbitrary way,

$e_1, e_2, \dots$. This ordering will remain fixed throughout. Our probability space is

$$\Omega = \prod_1^\infty \mathbb{R}_+, \qquad \mathbb{R}_+ = [0, \infty),$$

and a generic point of Ω is denoted by $\omega = (\omega_1, \omega_2, \dots)$. In the configuration ω, the passage time of e_i is

$$t(e_i) = t(e_i, \omega) = \omega_i.$$

When it is necessary to indicate the dependence on ω we write $a_{0,n}(\omega)$ instead of $a_{0,n}$, $T(r, \omega)$ instead of $T(r)$, etc. We shall use the following σ-fields of subsets of Ω:

$$\mathscr{F}_0 = \text{the trivial } \sigma\text{-field} = \{\varnothing, \Omega\},$$
$$\mathscr{F}_k = \sigma\text{-field generated by } \omega_1, \dots, \omega_k, \qquad k \geq 1.$$

The promised martingale representation of $a_{0,n} - Ea_{0,n}$ is

$$a_{0,n} - Ea_{0,n} = \sum_{k=1}^{\infty} \left(E\{a_{0,n} | \mathscr{F}_k\} - E\{a_{0,n} | \mathscr{F}_{k-1}\} \right). \tag{2.1}$$

This representation is valid because $M_0 := 0$ and

$$\begin{aligned} M_l &:= \sum_{k=1}^{l} \left(E\{a_{0,n} | \mathscr{F}_k\} - E\{a_{0,n} | \mathscr{F}_{k-1}\} \right) \\ &= E\{a_{0,n} | \mathscr{F}_l\} - Ea_{0,n}, \qquad l \geq 1, \end{aligned} \tag{2.2}$$

defines an $\{\mathscr{F}_l\}$-martingale that converges w.p.1 to $a_{0,n} - Ea_{0,n}$ [cf. Doob (1953), Corollary VII4.1). It will be seen later that this convergence also takes place in L^2. The increments of $\{M_l\}$ are denoted by

$$\Delta_k = \Delta_{k,n}(\omega) = E\{a_{0,n} | \mathscr{F}_k\} - E\{a_{0,n} | \mathscr{F}_{k-1}\}. \tag{2.3}$$

The principal step is to estimate

$$E\{\Delta_k^2 | \mathscr{F}_{k-1}\}. \tag{2.4}$$

To this end we write

$$a_{0,n}(\omega) = f(t(e_1, \omega), t(e_2, \omega), \dots) = f(\omega_1, \omega_2, \dots)$$

for some Borel function $f \colon \Omega \to \mathbb{R}_+$. Also, the following notation is useful. If $\omega = (\omega_1, \omega_2, \dots)$ and $\sigma = (\sigma_1, \sigma_2, \dots)$ are points of Ω, then

$$[\omega, \sigma]_k = (\omega_1, \dots, \omega_k, \sigma_{k+1}, \dots) \tag{2.5}$$

is the point that agrees with ω and σ on the first k coordinates and the coordinates after k, respectively. ν_{k+1} will be the product measure

$$\nu_{k+1} = \prod_{k+1}^{\infty} F_i$$

on the obvious σ-field in

$$\Omega_{k+1} = \prod_{k+1}^{\infty} R_i$$

when each $R_i = \mathbb{R}_+$ and $F_i = F$. We can think of Ω as $R_1 \times \cdots \times R_k \times \Omega_{k+1}$ and if g is a function from $\Omega \to \mathbb{R}$, then if we fix $\sigma_1, \dots, \sigma_k$, $g(\sigma)$ can be viewed as a function of $\sigma_{k+1}, \sigma_{k+2}, \dots$; that is, as a function on Ω_{k+1}. Correspondingly,

$$\int_{\Omega_{k+1}} \nu_{k+1}(d\sigma) g(\sigma) := \int \prod_{k+1}^{\infty} F(d\sigma_i) g(\sigma_1, \dots, \sigma_k, \sigma_{k+1}, \dots)$$

is the integral over all coordinates σ_i, with $i \geq k+1$, and is a function of $\sigma_1, \dots, \sigma_k$.

By the independence of the $t(e_i, \omega) = \omega_i$, $i \geq 1$, we have, in the above notation,

$$E\{a_{0,n} | \mathscr{F}_k\}(\omega) = \int_{\Omega_{k+1}} \nu_{k+1}(d\sigma) f([\omega, \sigma]_k). \tag{2.6}$$

This is a function of $t(e_i, \omega) = \omega_i$, $1 \leq i \leq k$, only. It also equals

$$\int_{\Omega_k} \nu_k(d\sigma) f([\omega, \sigma]_k), \tag{2.7}$$

because $[\omega, \sigma]_k$ does not involve σ_k and the integration over σ_k has no effect. Using (2.7) for $E\{a_{0,n} | \mathscr{F}_k\}$ and (2.6) with k replaced by $(k-1)$ for $E\{a_{0,n} | \mathscr{F}_{k-1}\}$, we find

$$\Delta_k = \int_{\Omega_k} \nu_k(d\sigma) \{ f([\omega, \sigma]_k) - f([\omega, \sigma]_{k-1}) \}. \tag{2.8}$$

Our task now is to estimate

$$g_k(\omega, \sigma) := | f([\omega, \sigma]_k) - f([\omega, \sigma]_{k-1}) |. \tag{2.9}$$

Note that

$$\begin{aligned} t(e_i, [\omega, \sigma]_k) &= t(e_i, [\omega, \sigma]_{k-1}) \\ &= \begin{cases} t(e_i, \omega), & \text{if } i \leq k-1, \\ t(e_i, \sigma), & \text{if } i \geq k+1. \end{cases} \end{aligned} \tag{2.10}$$

Only for $i = k$ do we obtain different values for the passage time of e_i in the two configurations $[\omega, \sigma]_k$ and $[\omega, \sigma]_{k-1}$:

$$t(e_k, [\omega, \sigma]_k) = t(e_k, \omega), \qquad t(e_k, [\omega, \sigma]_{k-1}) = t(e_k, \sigma). \tag{2.11}$$

We claim that this implies

$$g_k(\omega, \sigma) \leq | t(e_k, \omega) - t(e_k, \sigma) |. \tag{2.12}$$

Indeed, for any path r,

$$\begin{aligned}(2.13)\qquad |T(r,[\omega,\sigma]_k) - T(r,[\omega,\sigma]_{k-1})| &\le \sum_{\substack{e \text{ an edge}\\ \text{of } r}} |t(e,[\omega,\sigma]_k) - t(e,[\omega,\sigma]_{k-1})| \\ &\le |t(e_k,\omega) - t(e_k,\sigma)|.\end{aligned}$$

Therefore, the same estimate holds for

$$\begin{aligned}|a_{0,n}([\omega,\sigma]_k) - a_{0,n}([\omega,\sigma]_{k-1})| \\ = \left|\inf_r T(r,[\omega,\sigma]_k) - \inf_r T(r,[\omega,\sigma]_{k-1})\right|.\end{aligned}$$

This proves (2.12). We can say more, though. Write $\pi_n(\omega)$ for the optimal path from $\mathbf{0}$ to $n\xi_1$ in the configuration ω; that is, $\pi_n(\omega)$ is a path from $\mathbf{0}$ to $n\xi_1$ with

$$(2.14)\qquad a_{0,n}(\omega) = T(\pi_n(\omega),\omega).$$

It is known that such a path exists when (1.11) holds and hence $\mu > 0$ [cf. Kesten (1986), Section (9.23)]. There could be several paths with this property. To define $\pi_n(\omega)$ uniquely in case of ties, we order all paths from $\mathbf{0}$ to $n\xi_1$ in some arbitrary way, and take for $\pi_n(\omega)$ the first path in this ordering that satisfies (2.14). We write $e \in \pi$ to denote that e is an edge in the path π. Then, if

$$(2.15)\qquad e_k \notin \pi_n([\omega,\sigma]_k),$$

(2.10) and (2.13) show that

$$T(\pi_n([\omega,\sigma]_k),[\omega,\sigma]_k) = T(\pi_n([\omega,\sigma]_k),[\omega,\sigma]_{k-1}).$$

Thus, under (2.15),

$$\begin{aligned}a_{0,n}([\omega,\sigma]_{k-1}) &= \inf\{T(r,[\omega,\sigma]_{k-1}): r \text{ a path from } \mathbf{0} \text{ to } n\xi_1\} \\ &\le T(\pi_n([\omega,\sigma]_k),[\omega,\sigma]_{k-1}) \\ &= T(\pi_n([\omega,\sigma]_k,[\omega,\sigma]_k)) \\ &= a_{0,n}([\omega,\sigma]_k).\end{aligned}$$

Similarly, if

$$(2.16)\qquad e_k \notin \pi_n([\omega,\sigma]_{k-1}),$$

then

$$a_{0,n}([\omega,\sigma]_k) \le a_{0,n}([\omega,\sigma]_{k-1}).$$

It follows that $g_k(\omega,\sigma) = 0$ if (2.15) and (2.16) both hold, and by virtue of (2.12),

$$\begin{aligned}(2.17)\qquad g_k(\omega,\sigma) \le{}& |t(e_k,\omega) - t(e_k,\sigma)| \\ &\times I[e_k \in \pi_n([\omega,\sigma]_{k-1}) \text{ or } e_k \in \pi_n([\omega,\sigma]_k)].\end{aligned}$$

This is our basic estimate for g_k. Write $I_k(\omega, \sigma)$ for the indicator function in the right-hand side of (2.17). Then by (2.8) and Schwarz's inequality,

$$
\begin{aligned}
E\{\Delta_k^2|\mathscr{F}_{k-1}\} &\le E\left\{\left(\int_{\Omega_k} \nu_k(d\sigma) g_k(\omega,\sigma)\right)^2 \middle| \mathscr{F}_{k-1}\right\} \\
&\le E\left\{\left(\int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)| I_k(\omega,\sigma)\right)^2 \middle| \mathscr{F}_{k-1}\right\} \\
&\le E\left\{\int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 I_k(\omega,\sigma)\right. \\
&\qquad \left. \times \int_{\Omega_k} \nu_k(d\sigma) I_k(\omega,\sigma) \middle| \mathscr{F}_{k-1}\right\} \\
&\le E\left\{\int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 I_k(\omega,\sigma) \middle| \mathscr{F}_{k-1}\right\}.
\end{aligned} \tag{2.18}
$$

Now

$$\int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 I_k(\omega,\sigma)$$

is a function of $\omega_1, \dots, \omega_k$ only; the σ-variables all have been integrated out. Analogously to (2.6) we have

$$
\begin{aligned}
&E\left\{\int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 I_k(\omega,\sigma) \middle| \mathscr{F}_{k-1}\right\} \\
&\quad = \int F(d\omega_k) \int_{\Omega_k} \nu_k(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 \\
&\quad \times I_k(\omega,\sigma) \\
&\quad = \int F(d\omega_k) \int F(d\sigma_k) \int_{\Omega_{k+1}} \nu_{k+1}(d\sigma)|t(e_k,\omega) - t(e_k,\sigma)|^2 \\
&\quad \times I_k(\omega,\sigma).
\end{aligned} \tag{2.19}
$$

In the last step we used the fact that ν_k can be written as the product measure $F \times \nu_{k+1}$ on $R_k \times \Omega_{k+1} = \mathbb{R}_+ \times \Omega_{k+1}$. Let us write

$$J_k(\omega) = I[e_k \in \pi_n(\omega)].$$

Then

$$
\begin{aligned}
I_k(\omega,\sigma) &= J_k([\omega,\sigma]_{k-1}) \vee J_k([\omega,\sigma]_k) \\
&= J_k(\omega_1, \dots, \omega_{k-1}, \sigma_k, \sigma_{k+1}, \dots) \\
&\qquad \vee J_k(\omega_1, \dots, \omega_{k-1}, \omega_k, \sigma_{k+1}, \dots).
\end{aligned} \tag{2.20}
$$

Recall that $t(e_k, \omega) = \omega_k$, $t(e_k, \sigma) = \sigma_k$, so that

$$\begin{aligned}&|t(e_k, \omega) - t(e_k, \sigma)|^2 I_k(\omega, \sigma)\\ &\quad \le |\omega_k - \sigma_k|^2\{J_k(\omega_1, \dots, \omega_{k-1}, \sigma_k, \sigma_{k+1}, \dots)\\ &\qquad \vee J_k(\omega_1, \dots, \omega_{k-1}, \omega_k, \sigma_{k+1}, \dots)\}.\end{aligned}$$

Clearly the right-hand side is symmetric in ω_k and σ_k for fixed $\omega_1, \dots, \omega_{k-1}, \sigma_{k+1}, \sigma_{k+2}, \dots$ (as a matter of fact, this is true also for the left-hand side). It is also clear that on $\{\sigma_k \le \omega_k\}$ or on $\{t(e_k, \sigma) \le t(e_k, \omega)\}$ we have

$$|t(e_k, \omega) - t(e_k, \sigma)| \le t(e_k, \omega), \tag{2.21}$$

$$\begin{aligned}&J_k(\omega_1, \dots, \omega_{k-1}, \sigma_k, \sigma_{k+1}, \dots) \vee J_k(\omega_1, \dots, \omega_{k-1}, \omega_k, \sigma_{k+1}, \dots)\\ &\quad = J_k(\omega_1, \dots, \omega_{k-1}, \sigma_k, \sigma_{k+1}, \dots) = J_k([\omega, \sigma]_{k-1}).\end{aligned} \tag{2.22}$$

(2.22) simply says that if e_k belongs to the optimal path in configuration $[\omega, \sigma]_{k-1}$ or in configuration $[\omega, \sigma]_k$, then it will belong to the optimal path in the configuration that gives the lower value to $t(e_k)$. Substituting (2.19)–(2.22) into (2.18) we find

$$\begin{aligned}&E\{\Delta_k^2 | \mathscr{F}_{k-1}\}\\ &\quad \le 2\int_{\Omega_{k+1}} \nu_{k+1}(d\sigma) \iint_{\sigma_k \le \omega_k} F(d\omega_k) F(d\sigma_k) t^2(e_k, \omega) J_k([\omega, \sigma]_{k-1})\\ &\quad \le 2\int F(d\omega_k) t^2(e_k, \omega) \int F(d\sigma_k) \int_{\Omega_{k+1}} \nu_{k+1}(d\sigma) J_k([\omega, \sigma]_{k-1})\\ &\qquad\qquad [\text{because } J_k([\omega, \sigma]_{k-1}) \text{ is independent of } \omega_k]\\ &\quad = 2\int x^2\, dF(x) P\{e_k \in \pi_n(\omega) | \mathscr{F}_{k-1}\} \quad [\text{as in (2.6)}].\end{aligned} \tag{2.23}$$

The right-hand inequality in (1.13) now follows easily from (2.23). Indeed

$$\begin{aligned}\sigma^2(a_{0,n}) &= E\left\{\sum_1^\infty \Delta_k\right\}^2\\ &\le \sum_1^\infty E\Delta_k^2 \quad \text{(Fatou's lemma and the martingale property)}\\ &\le 2\int x^2\, dF(x) \sum_k E\{P\{e_k \in \pi_n(\omega) | \mathscr{F}_{k-1}\}\}\\ &= 2\int x^2\, dF(x) E\left\{\sum_k I[e_k \in \pi_n(\omega)]\right\}\\ &= 2\int x^2\, dF(x) E|\pi_n(\omega)|,\end{aligned} \tag{2.24}$$

where $|\pi|$ denotes the number of edges in π. So far we have not used (1.11). This is only needed to conclude that

$$E|\pi_n| \le C_1 n, \tag{2.25}$$

which, together with (2.24), will of course imply the right-hand inequality in (1.13). (2.25) itself is immediate from Kesten [(1986), Proposition 5.8]. In fact, for any $a > 0$, $y > 0$,

$$\begin{aligned} P\{|\pi_n| \ge yn\} \le P\{a_{0,n} \ge ayn\} \\ + P\{\exists \text{ self-avoiding path } r \text{ starting at } \mathbf{0} \text{ of at least} \\ yn \text{ steps but with } T(r) < ayn\}. \end{aligned} \tag{2.26}$$

With r_n as in the lines following (1.9), we have

$$P\{a_{0,n} \ge ayn\} \le P\{T(r_n) \ge ayn\} \le P\left\{\sum_1^n t_i \ge ayn\right\}. \tag{2.27}$$

Proposition 5.8 of Kesten (1986) shows that for suitable $a > 0$ the second term in the right-hand side of (2.26) is at most $C_2 \exp(-C_3 yn)$ for some $C_3 > 0$. Therefore,

$$\begin{aligned} E|\pi_n| &= n\int_0^\infty P\{|\pi_n| > yn\}\, dy \\ &\le n\int_0^\infty P\left\{\sum_1^n t_i \ge ayn\right\} dy + C_2 n\int_0^\infty e^{-C_3 yn}\, dy \\ &= \frac{1}{a}E\left\{\sum_1^n t_i\right\} + \frac{C_2}{C_3} \le C_1 n. \end{aligned}$$

Thus (2.25) and, hence, the right-hand inequality of (1.13) hold. The left-hand inequality in (1.13) will be postponed until Section 4. □

Proof of (1.15). We have set up matters such that (1.15) follows fairly easily from Theorem 3 after a simple truncation argument, which is given in the next lemma. Let n be fixed. Define

$$\hat{t}(e_i) = t(e_i) \wedge \frac{4d}{\gamma}\log n \tag{2.28}$$

with γ as in (1.14). Passage times and related quantities, when defined in terms of the $\hat{t}$ instead of the t, will be denoted by the old symbols decorated with a caret. For example, if $r = (e_1, \dots, e_k)$, then

$$\hat{T}(r) = \sum_1^k \hat{t}(e_i);$$

$$\hat{a}_{0,n} = \inf\{\hat{T}(r) : r \text{ a path from } \mathbf{0} \text{ to } n\xi_1\},$$

$$\hat{\pi}_n = \text{optimal path for } \hat{a}_{0,n} \quad [\text{compare } (2.14)].$$

LEMMA 1. *If* (1.11) *and* (1.14) *hold, then there exist constants* $0 < C_i < \infty$ *such that*

$$P\{\hat{\pi}_n \text{ is not contained in } [-C_1 n, C_1 n]^d\} \le 3e^{-C_2 n}, \tag{2.29}$$

$$P\{|a_{0,n} - \hat{a}_{0,n}| \ge x\} \le 3e^{-C_2 n} + C_3 e^{-(\gamma/2)x}, \qquad x \ge 0, \tag{2.30}$$

and

$$|Ea_{0,n} - E\hat{a}_{0,n}| \le C_4. \tag{2.31}$$

PROOF. The probability in the left-hand side of (2.29) is bounded by

$$\begin{aligned} P\{|\hat{\pi}_n| \ge C_1 n\} &\le P\{\hat{a}_{0,n} \ge aC_1 n\} \\ &\quad + P\{\exists \text{ self-avoiding path } r \text{ starting at } \mathbf{0} \text{ of at least} \\ &\qquad C_1 n \text{ steps, but with } \hat{T}(r) < aC_1 n\} \\ &\le P\left\{\sum_1^n t_i \ge aC_1 n\right\} + C_5 e^{-C_6 n}, \end{aligned} \tag{2.32}$$

exactly as in (2.26), (2.27) and succeeding lines. Note that the last inequality is true for some C_5, C_6 independent of n, even though the distribution of $\hat{t}$ does depend on n. This is so because for any constant C, $\hat{t}(e) \ge t(e) \wedge C$ for large enough n. Thus also $\hat{T}(t) \ge \sum_{e \in r}\{t(e) \wedge C\}$, and it suffices to apply Proposition 5.8 of Kesten (1986) when $X(e)$ there is replaced by $t(e) \wedge C$ for $C = 1$, say. Now we use (1.14) for the following standard large deviation estimate:

$$\begin{aligned} P\left\{\sum_1^n t_i \ge aC_1 n\right\} &\le e^{-\gamma a C_1 n}\left(Ee^{\gamma t_1}\right)^n \\ &= \left[e^{-\gamma a C_1}\int e^{\gamma x} F(dx)\right]^n. \end{aligned} \tag{2.33}$$

If we choose C_1 so that the expression in square brackets is less than e^{-1}, then we find that the probability in (2.29) is bounded by $\exp(-n) + C_5\exp(-C_6 n)$. This proves (2.29).

To prove (2.30) and (2.31) note that

$$\begin{aligned} 0 &\le a_{0,n} - \hat{a}_{0,n} \le T(\hat{\pi}_n) - \hat{T}(\hat{\pi}_n) \\ &= \sum_{e \in \hat{\pi}_n}\{t(e) - \hat{t}(e)\} \le \sum_{e \in \hat{\pi}_n} t(e) I\left[t(e) > \frac{4d}{\gamma}\log n\right]. \end{aligned} \tag{2.34}$$

[The second inequality holds because $\hat{a}_{0,n} = \hat{T}(\hat{\pi}_n)$ and $a_{0,n} \le T(r)$ for any path r from $\mathbf{0}$ in $n\xi_1$.] If $\hat{\pi}_n \subset [-C_1 n, C_1 n]^d$, then the last member of (2.34) is at most

$$\sum_{e \subset [-C_1 n, C_1 n]^d} t(e) I\left[t(e) > \frac{4d}{\gamma}\log n\right].$$

Consequently,

$$P\{|a_{0,n} - \hat{a}_{0,n}| \geq x\} \leq P\{\hat{\pi}_n \text{ is not contained in } [-C_1 n, C_1 n]^d\} \tag{2.35}$$
$$+ P\left\{\sum_1^M t_i I\left[t_i > \frac{4d}{\gamma}\log n\right] \geq x\right\},$$

where

$$M = \text{number of edges in } [-C_1 n, C_1 n]^d \sim d(2C_1 n)^d. \tag{2.36}$$

Therefore

$$P\left\{\sum_1^M t_i I\left[t_i > \frac{4d}{\gamma}\log n\right] \geq x\right\}$$
$$\leq e^{-(\gamma/2)x}\left[1 + \int_{y \geq 4(d/\gamma)\log n} (e^{(\gamma/2)y} - 1)F(dy)\right]^M$$
$$\leq e^{-(\gamma/2)x}\left[1 + e^{-2d\log n}\int e^{\gamma y}F(dy)\right]^M \tag{2.37}$$
$$\leq \exp\left\{-\frac{\gamma}{2}x + Mn^{-2d}\int e^{\gamma y}F(dy)\right\}$$
$$\leq C_3 e^{-(\gamma/2)x}.$$

(2.30) follows from (2.35), (2.29) and (2.37).

(2.31) follows from (2.30) plus the additional estimates [see the lines following (1.9) for r_n]

$$0 \leq \hat{a}_{0,n} \leq a_{0,n} \leq T(r_n)$$

and [cf. (2.27)]

$$P\{|a_{0,n} - \hat{a}_{0,n}| \geq yn\} \leq P\{T(r_n) \geq yn\}$$
$$= P\left\{\sum_1^n t_i \geq yn\right\} \leq e^{-\gamma yn}\left[\int e^{\gamma x}F(dx)\right]^n \tag{2.38}$$
$$\leq e^{-(\gamma/2)yn}$$

for $y \geq$ some y_0. □

Now we can prove (1.15). By Lemma 1 we have for $x\sqrt{n} \geq 2C_4$,

$$P\{|a_{0,n} - Ea_{0,n}| \geq x\sqrt{n}\}$$
$$\leq P\left\{|a_{0,n} - \hat{a}_{0,n}| \geq \frac{x}{4}\sqrt{n}\right\}$$
$$+ P\left\{|\hat{a}_{0,n} - E\hat{a}_{0,n}| \geq \frac{x}{4}\sqrt{n}\right\}$$
$$\leq 3\exp(-C_2 n) + C_3\exp\left(-\frac{\gamma}{8}x\sqrt{n}\right)$$
$$+ P\left\{|\hat{a}_{0,n} - E\hat{a}_{0,n}| \geq \frac{x}{4}\sqrt{n}\right\}.$$

The first term in the last member is at most $3\exp(-(C_2/C_5)x)$ for $x \le C_5 n$. (1.15) will therefore follow as soon as we prove

$$(2.39) \qquad P\left\{|\hat{a}_{0,n} - E\hat{a}_{0,n}| \ge \frac{x}{4}\sqrt{n}\right\} \le C_7 \exp(-C_8 x) \quad \text{for } x \le C_5 n.$$

The remaining part of the proof deduces (2.39) from Theorem 3.

As in (2.1)–(2.3),

$$\hat{a}_{0,n} - E\hat{a}_{0,n} = \sum_1^\infty \hat{\Delta}_k$$

with

$$\hat{\Delta}_k = E\{\hat{a}_{0,n}|\mathscr{F}_k\} - E\{\hat{a}_{0,n}|\mathscr{F}_{k-1}\}.$$

Moreover,

$$M_0 = 0, \qquad M_l = \sum_1^l \hat{\Delta}_k, \qquad l \ge 1,$$

defines a martingale. We shall now verify the hypotheses of Theorem 3 for the martingale.

Note that replacing $t(e_i), a_{0,n}$ and Δ_k by $\hat{t}(e_i)$, $\hat{a}_{0,n}$ and $\hat{\Delta}_k$ merely amounts to changing the distribution F to

$$\hat{F}(x) = F(x) \vee I\left[x \ge \frac{4d}{\gamma}\log n\right].$$

Therefore, by (2.12), (2.8) and the definition (2.28) of $\hat{t}$,

$$|\hat{\Delta}_k| \le 2\max\bigl(\operatorname{supp}\hat{F}\bigr) \le \frac{8d}{\gamma}\log n.$$

This is (1.24) for

$$c = \frac{8d}{\gamma}\log n.$$

Furthermore, by (2.23),

$$\begin{aligned} E\{\hat{\Delta}_k^2|\mathscr{F}_{k-1}\} &\le 2\int x^2 \hat{F}(dx) P\{e_k \in \hat{\pi}_n(\omega)|\mathscr{F}_{k-1}\} \\ &\le 2\int x^2 F(dx) P\{e_k \in \hat{\pi}_n(\omega)|\mathscr{F}_{k-1}\}. \end{aligned}$$

Thus (1.25) holds with

$$U_k = D\hat{J}_k,$$

where

$$D = 2\int x^2 F(dx), \qquad \hat{J}_k(\omega) = I[e_k \in \hat{\pi}_n(\omega)].$$

We next take $C > 0$ as in Proposition 5.8 of Kesten (1986) and

$$x_0 = n\frac{2D}{\gamma C}\log\left\{\int e^{\gamma u}F(du)\right\}. \tag{2.40}$$

Clearly this satisfies (1.26) for large n. Finally, we verify (1.27). We have

$$\begin{aligned}\sum_1^\infty U_k &= D\sum_1^\infty I[e_k \in \hat{\pi}_n(\omega)] \\ &= D|\hat{\pi}_n(\omega)| = D \times \text{length of } \hat{\pi}_n(\omega).\end{aligned} \tag{2.41}$$

Moreover, as in (2.32) and (2.33),

$$\begin{aligned}P\{|\hat{\pi}_n(\omega)| \geq y\} &\leq P\left\{\sum_1^n t_i \geq Cy\right\} \\ &\quad + P\{\exists \text{ self-avoiding path } r \text{ starting at } \mathbf{0} \text{ of at least} \\ &\qquad\qquad y \text{ edges, but with } \hat{T}(r) < Cy\} \\ &\leq e^{-\gamma Cy}\left[\int e^{\gamma u}F(du)\right]^n + C_9 e^{-C_{10}y}\end{aligned} \tag{2.42}$$

[by Proposition 5.8 of Kesten (1986)]. For $y \geq x_0/D$ and x_0 as in (2.40), the right-hand side of (2.42) is at most

$$e^{-(\gamma/2)Cy} + C_9 e^{-C_{10}y}.$$

Therefore, (1.27) holds with $C_1 = (1 + C_9)$ and

$$C_2 = \frac{\gamma C}{2D} \wedge \frac{C_{10}}{D}.$$

Thus, by (1.29) (applied to $\hat{a}_{0,n}$ and to $-\hat{a}_{0,n}$),

$$\begin{aligned}P\{|\hat{a}_{0,n} - E\hat{a}_{0,n}| \geq x\} &\leq 2C_3(1 + C_1)\left(1 + \frac{1}{C_2}\right)\exp\left(-\frac{C_4}{2}\frac{x}{\sqrt{x_0}}\right) \\ &\leq C_{11}\exp\left(-C_{12}\frac{x}{\sqrt{n}}\right)\end{aligned}$$

for

$$n \geq \text{some } n_0 \text{ and } x \leq C_{13}n^{3/2}.$$

This is just a reformulation of (2.39) and therefore proves (1.15) for $n \geq n_0$. For any fixed $n < n_0$ and all $x \leq C_5 n_0$, we can still obtain (1.15) by raising C_3. □

We now prove (1.17) and (1.19), basically by using subadditivity properties. First we note that from

$$Ea_{0,n+m} \leq Ea_{0,n} + Ea_{0,m},$$

it follows that $Ea_{0,n} \geq n\mu$ [cf. Hammersley and Welsh (1965), (4.3.5)]. Therefore, by (1.15),

$$P\{a_{0,n} \leq n\mu - x\sqrt{n}\} \leq P\{a_{0,n} - Ea_{0,n} \leq -x\sqrt{n}\} \leq C_3 e^{-C_4 x}$$

for $x \leq C_5 n$. This restriction on x may be dropped because $a_{0,n} \geq 0$. Thus (1.17) holds.

Next we turn to (1.19). For a general unit vector ξ [i.e., a vector with $\|\xi\| = 1$; see (1.30) for $\|\xi\|$] we pick for each n a lattice point $v = v(n, \xi)$ such that [with $|w| = \|w\|_\infty$; see (1.30)]

$$|v(n,\xi) - n\xi| = \min\{|v - n\xi| : v \in \mathbb{Z}^d\} \leq 1, \tag{2.43}$$

and define

$$a_{n,n+m}(\xi) = T(v(n,\xi), v(n+m,\xi)). \tag{2.44}$$

Then again

$$a_{0,n+m}(\xi) \leq a_{0,n}(\xi) + a_{n,n+m}(\xi) \tag{2.45}$$

and [cf. Cox and Durrett (1981), page 592; Kesten (1986), pages 158–160]

$$\frac{a_{0,n}(\xi)}{n} \text{ converges a.e. and in } L^1 \text{ to a limit, } \mu(\xi) \text{ say.} \tag{2.46}$$

Because $v(m, \xi)$ may differ from $v(n + m, \xi) - v(n, \xi)$, it is not necessarily true that $a_{n,n+m}(\xi)$ has the same distribution as $a_{0,m}(\xi)$. However, as one easily sees from (2.43), we do have

$$|v(n+m,\xi) - v(n,\xi) - v(m,\xi)| \leq 3,$$

and therefore, $a_{n,n+m}(\xi)$ is stochastically smaller than

$$a_{0,m}(\xi) + \sum{}^{(m)} t(e),$$

where $\Sigma^{(m)}$ is the sum over all edges of $\mathbb{Z}^d$ within distance 3 from $v(m, \xi)$. Combined with (2.45), this gives for some constant C_1,

$$\begin{aligned} Ea_{0,n+m}(\xi) &\leq Ea_{0,n}(\xi) + Ea_{n,n+m}(\xi) \\ &\leq Ea_{0,n}(\xi) + Ea_{0,m}(\xi) + C_1. \end{aligned} \tag{2.47}$$

Thus $Ea_{0,n}(\xi) + C_1$ is a subadditive sequence and

$$Ea_{0,n}(\xi) + C_1 \geq n\mu(\xi). \tag{2.48}$$

The proof of (1.15) did not rely on any special properties for the directions along the coordinate axes. We therefore obtain for any unit vector ξ the analogue of (1.15):

$$\begin{aligned} &P\{|a_{0,n}(\xi) - Ea_{0,n}(\xi)| \geq x\sqrt{n}\} \\ &\qquad \leq C_3 e^{-C_4 x} \quad \text{for } x \leq C_5 n, \end{aligned} \tag{2.49}$$

with $C_3 - C_5$ independent of ξ. Hence, we also have, analogously to (1.17),

$$\begin{aligned} &P\{a_{0,n}(\xi) - n\mu(\xi) \leq -x\sqrt{n}\} \\ &\qquad \leq C_3 e^{-C_4 x/2} \quad \text{for } \frac{2C_1}{\sqrt{n}} \leq x \leq C_5 n. \end{aligned} \tag{2.50}$$

By raising C_3 we may ignore the restriction $x \ge 2C_1/\sqrt{n}$.

Next note that

$$\{a_{0,n}(\xi) \le t\} = \{T(0, v(n,\xi)) \le t\} = \{v(n,\xi) \in B(t)\}. \tag{2.51}$$

This combined with (2.46) and Theorem B shows that

$$B_0 \cap \{\lambda\xi : \lambda \ge 0\} = \left\{\lambda\xi : 0 \le \lambda \le \frac{1}{\mu(\xi)}\right\}. \tag{2.52}$$

Moreover, for each fixed direction ξ,

$$\begin{aligned} P\left\{v\left(\frac{t}{\mu(\xi)}(1+\varepsilon), \xi\right) \in B(t)\right\} &= P\{a_{0,t(1+\varepsilon)/\mu(\xi)}(\xi) \le t\} \\ &\le P\left\{a_{0,t(1+\varepsilon)/\mu(\xi)}(\xi) \le \frac{t(1+\varepsilon)}{\mu(\xi)}\mu(\xi) - \varepsilon t\right\}, \end{aligned} \tag{2.53}$$

where for brevity $v(\lambda, \xi)$ and $a_{0,\lambda}(\xi)$ have been written for $v(\lfloor\lambda\rfloor, \xi)$ and $a_{0,\lfloor\lambda\rfloor}(\xi)$, respectively. (2.53), together with (2.50), gives

$$\begin{aligned} P\left\{v\left(\frac{t}{\mu(\xi)}(1+\varepsilon), \xi\right) \in B(t)\right\} \\ \le C_3 \exp\left(-\frac{C_4}{2}\varepsilon t \left(\frac{\mu(\xi)}{t(1+\varepsilon)}\right)^{1/2}\right) \end{aligned} \tag{2.54}$$

for

$$\varepsilon t \le C_5 \left\{\frac{t(1+\varepsilon)}{\mu(\xi)}\right\}^{3/2}.$$

Basically this shows that for each *fixed* direction ξ, $B(t) \cap \{\lambda\xi : \lambda \ge 0\}$ behaves as stated by (1.19) [take $\varepsilon = xt^{-1/2}$ and use (2.52)]. To obtain the estimate (1.19), simultaneously in all directions, we introduce the following set of vertices "near the boundary" of $(t + x\sqrt{t})B_0$:

$$A(t,x) = \left\{v \in \mathbb{Z}^d : \inf\{|v-y| : y \in (t + x\sqrt{t})\partial B_0\} \le 1\right\},$$

where ∂B_0 is the topological boundary of B_0. Any path on $\mathbb{Z}^d$ that starts at the origin and leaves $(t + x\sqrt{t})B_0$ must contain a vertex in $A(t, x)$. Therefore, if $B(t)$ is not contained in $(t + x\sqrt{t})B_0$, then $B(t)$ must contain at least one vertex in $A(t, x)$. Note that the number of vertices in $A(t, x)$ is at most $C_{14}t^d$ for $x \le \sqrt{t}$. For the remainder of this proof we fix a vertex v in $A(t, x)$ and estimate $P\{v \in B(t)\}$.

Fix $v \in A(t, x)$ and set $\xi = v/\|v\|$ [$\|v\|$ stands for $\|v\|_2$; see (1.30)]. Because $v \in A(t, x)$, there exists a $y \in (t + x\sqrt{t})\,\partial B_0$ with $|v - y| \le 1$ and hence

$\|v - y\| \le d^{1/2}$. Set further $\eta = y/\|y\|$. Then, (2.52) implies

(2.55) $$\|y\| = \left(t + x\sqrt{t}\right)\frac{1}{\mu(\eta)}.$$

We claim that for some $C_{15} < \infty$, independent of ξ, t, this implies

(2.56) $$\left| \|v\| - \left(t + x\sqrt{t}\right)\frac{1}{\mu(\xi)} \right| \le C_{15}.$$

To see (2.56) we need a continuity property of $\mu(\cdot)$, which is essentially given in Kesten [(1986), page 159]. To start with, note that

(2.57) $$T(\mathbf{0}, v(n,\xi)) \le T(\mathbf{0}, v(n,\eta)) + T(v(n,\eta), v(n,\xi))$$

and, by (2.43),

$$|(v(n,\xi) - v(n,\eta))| \le |v(n,\xi-\eta)| + 3 \le n|\xi - \eta| + 4.$$

Therefore,

$$\begin{aligned} ET(v(n,\eta), v(n,\xi)) &= ET(\mathbf{0}, v(n,\xi) - v(n,\eta)) \\ &\le \sum_{1}^{d(n|\xi-\eta|+4)} Et_i \le C_{16}(n|\xi-\eta|+1). \end{aligned}$$

By virtue of (2.46) and (2.57) we therefore have

$$\mu(\xi) \le \mu(\eta) + C_{16}|\xi - \eta|.$$

By interchanging the role of η and ξ we obtain

(2.58) $$|\mu(\xi) - \mu(\eta)| \le C_{16}|\xi - \eta|.$$

Note also that for some $C_{17} > 0$,

(2.59) $$\inf_{\|\eta\|=1} \mu(\eta) \ge C_{17};$$

this follows from convexity of B_0 as in the lines following (6.10) in Kesten (1986) or by (3.16) in that reference. Also by (2.58) or (3.12) in Kesten (1986),

(2.60) $$\sup_{\|\eta\|=1} \mu(\eta) \le C_{18}.$$

Finally, $|v - y| \le 1$, $\|v - y\| \le d^{1/2}$, (2.55) and (2.58)–(2.60) imply

$$\begin{aligned} \left| \|v\| - \left(t + x\sqrt{t}\right)\frac{1}{\mu(\xi)} \right| &= \left| \|v\| - \|y\|\frac{\mu(\eta)}{\mu(\xi)} \right| \\ &\le d^{1/2} + \frac{\|y\|}{\mu(\xi)}|\mu(\xi) - \mu(\eta)| \\ &\le d^{1/2} + C_{19}\|y\|\,|\xi - \eta| = d^{1/2} + C_{19}\|y\|\left|\frac{v}{\|v\|} - \frac{y}{\|y\|}\right| \\ &= d^{1/2} + \frac{C_{19}}{\|v\|}\left|\, \|v\|(v - y) + (\|y\| - \|v\|)v\right| \\ &\le C_{15}. \end{aligned}$$

This completes the proof of (2.56).

Now take

$$n = \left\lfloor t\left(1 + \frac{x}{\sqrt{t}}\right)\frac{1}{\mu(\xi)}\right\rfloor. \tag{2.61}$$

By (2.56),

$$|n - \|v\|\,| \le C_{15} + 1$$

and therefore

$$\begin{aligned}|v - v(n,\xi)| &\le |v - n\xi| + |n\xi - v(n,\xi)|\\ &\le \|v - n\xi\| + 1 \le (C_{15} + 2).\end{aligned}$$

In particular, there exists a path $\pi = \pi(v)$ of at most $C_{20} := (C_{15} + 2)$ edges that connects v to $v(n, \xi)$. This implies that

$$\begin{aligned}&|T(\mathbf{0}, v) - T(\mathbf{0}, v(n,\xi))|\\ &\quad\le \sum_{e\in\pi} t(e)\\ &\quad\le C_{20}\max\{t(e)\colon e \text{ within distance } (C_{15} + 3) \text{ from } (t + x\sqrt{t})\partial B_0\}.\end{aligned}$$

Again the max in the right-hand side is over at most $C_{14}t^d$ edges, so that

$$\begin{aligned}&P\left\{|T(\mathbf{0}, v) - T(\mathbf{0}, v(n,\xi))| \ge \frac{x}{3}\sqrt{t}\right\}\\ &\quad\le C_{14}t^d P\left\{t_1 \ge \frac{x}{3C_{20}}\sqrt{t}\right\}\\ &\quad\le C_{14}t^d \exp\left(-\frac{\gamma}{3C_{20}}x\sqrt{t}\right)\int e^{\gamma u}F(du).\end{aligned} \tag{2.62}$$

Finally, if $v \in B(t)$ and $|T(\mathbf{0}, v) - T(\mathbf{0}, v(n, \xi))| < (x/3)\sqrt{t}$, then $T(\mathbf{0}, v) \le t$ and (by the choice of n)

$$\begin{aligned}a_{0,n}(\xi) = T(\mathbf{0}, v(n,\xi)) &\le t + \frac{x}{3}\sqrt{t}\\ &\le n\mu(\xi) + \mu(\xi) - \frac{2x}{3}\sqrt{t}\\ &\le n\mu(\xi) - \frac{x}{2}\sqrt{t}\end{aligned} \tag{2.63}$$

provided $x\sqrt{t} \ge 6C_{18}$ [cf. (2.60)].

Finally, we obtain from (2.63), (2.50) and (2.62) for $x \le t^{1/2}$,

$$\begin{aligned}P\{v \in B(t)\} &\le P\left\{|T(\mathbf{0}, v) - T(\mathbf{0}, v(n,\xi))| \ge \frac{x}{3}\sqrt{t}\right\}\\ &\quad + P\left\{a_{0,n}(\xi) \le n\mu(\xi) - \frac{x}{2}\sqrt{t}\right\}\\ &\le C_{14}t^d \exp\left(-C_{22}x\sqrt{t}\right) + C_3\exp(-C_{23}x).\end{aligned}$$

Because there are only $C_{14}t^d$ vertices v in $A(t, x)$ and, as we saw before,

$$\{B(t) \not\subset (t + x\sqrt{t})B_0\} \subset \bigcup_{v \in A(t, x)} \{v \in B(t)\},$$

this implies (1.19).

3. Centering at $n\mu$. We already proved the one-sided bound (1.17) for deviations from $n\mu$. The bound (1.18) for deviations in the other direction will follow immediately from (1.15) and

$$Ea_{0,n} - n\mu \le C_7 n^{5/6}(\log n)^{1/3}, \tag{3.1}$$

which is the right-hand inequality in (1.16). We do not prove (3.1), but instead concentrate on its analogue for arbitrary directions ξ, to wit

$$Ea_{0,n}(\xi) - n\mu(\xi) \le C_7 n^{1-1/(2d+4)}(\log n)^{1/(d+2)} \tag{3.2}$$

[cf. (2.44) and (2.46) for notation]. An improved version of (3.1) is given in Alexander (1991). Once we have (3.2), (1.20) can be deduced from the same "duality relation" (2.51) that was used in the last section to prove (1.19) from (2.50).

Proof of (3.2). This proof is very similar to the proof of Lemma (5.68) and (5.18) in Kesten (1986). However, the argument on pages 212–214 of this reference uses special symmetry properties with respect to the coordinate axes. We therefore must modify (in the following Step 2) the argument for Lemma (5.68) of Kesten (1986). Analogously to pages 210 and 211, the idea of the proof is to show that if for some fixed n and unit vector ξ, $Ea_{0,n}(\xi) - n\mu(\xi) \ge \Lambda$ for a certain (comparatively) large Λ, then $P\{a_{0,ln}(\xi) \le ln\mu(\xi) + l\Lambda/4\}$ decreases exponentially fast in l. This contradicts $(ln)^{-1}a_{0,ln} \to \mu(\xi)$ and will allow us to conclude $Ea_{0,n}(\xi) \le n\mu(\xi) + \Lambda$.

We break the proof into three steps. Throughout, ξ will be a fixed unit vector, but the estimates will be uniform in ξ.

Step 1. For some integer M, let $U_1, \dots, U_\nu$ be all the vectors with integer components and $|U| = M$. For $n \ge M$ define

$$\Lambda(M, n) = \min\left\{\sum_1^\nu p(k)ET(\mathbf{0}, U_k)\right\} - n\mu(\xi), \tag{3.3}$$

where the minimum is over all choices of $p(k) \in \mathbb{Z}_+ = \{0, 1, \dots\}$ such that

$$\left|\sum_1^\nu p(k)U_k - n\xi\right| \le M. \tag{3.4}$$

In this step we show that for $l = 1, 2, \dots$ and some constant C_1 (which is independent of M, n, l and ξ),

$$\Lambda(M, ln) \ge l\Lambda(M, n) - C_1 lM^{1/d}n^{(d-1)/d} \quad \text{for } n \ge M \ge n^{1/(d+1)}. \tag{3.5}$$

To prove (3.5) we also introduce a subset $V_1, \dots, V_\sigma$ of the U's, such that for each U_i there exists a $j(i)$ with

$$|U_i - V_{j(i)}| \le M^{(d+1)/d} n^{-1/d}. \tag{3.6}$$

For instance we can take for the V's the vectors $V = (V(1), \dots, V(d))$ with

$$|V(r)| = M \quad \text{for some } r,$$
$$|V(s)| \le M \quad \text{and} \quad \lfloor M^{(d+1)/d} n^{-1/d} \rfloor | V(s) \quad \text{for } s \ne r$$

(note that for our choice of n and M, $M^{(d+1)/d} n^{-1/d} \ge 1$). Thus, the number of V's that we need satisfies

$$\sigma \le C_2 \left(\frac{M}{M^{(d+1)/d} n^{-1/d}} \right)^{d-1} = C_2 \left(\frac{n}{M} \right)^{(d-1)/d}. \tag{3.7}$$

Now let $p(k,l) \in \mathbb{Z}^+$ be such that $\Lambda(M, ln)$ is achieved with these coefficients; that is,

$$\Lambda(M, ln) = \sum_{k=1}^{\nu} p(k,l) E\{T(\mathbf{0}, U_k)\} - ln\mu(\xi), \tag{3.8}$$

while

$$\left| \sum_{k=1}^{\nu} p(k,l) U_k - ln\xi \right| \le M. \tag{3.9}$$

First we derive the rather crude bound

$$\sum_{k=1}^{\nu} p(k,l) \le C_3 \frac{ln}{M} \tag{3.10}$$

for some constant $C_3 < \infty$ that depends on F and d only. To see (3.10), let $\xi(i)$ be the ith component of ξ (this should be distinguished from the ith coordinate vector, ξ_i) and consider the sum

$$\sum_{i=1}^{d} \left\lfloor \frac{ln|\xi(i)|}{M} \right\rfloor (\operatorname{sgn}(\xi(i))M)\xi_i. \tag{3.11}$$

Because

$$|(\operatorname{sgn}(\xi(i))M)\xi_i| = M,$$

$\operatorname{sgn}(\xi(i))M\xi_i$ is one of the U's and the sum in (3.11) is therefore of the form $\sum_1^d q(i) U_{\alpha(i)}$ for some $\alpha(i)$ and $q(i) = \lfloor ln|\xi(i)|/M \rfloor$. These satisfy

$$\left| \sum_1^d q(i) U_{\alpha(i)} - ln\xi \right| \le M. \tag{3.12}$$

Therefore,

$$\Lambda(M, ln) \le \sum_1^d q(i) ET(\mathbf{0}, U_{\alpha(i)}) - ln\mu(\xi)$$
$$\le \sum_1^d q(i) ET(\mathbf{0}, U_{\alpha(i)}) \tag{3.13}$$
$$\le \sum_1^d q(i)\, dMEt_1 \le C_4 ln.$$

In the third inequality we used that $T(\mathbf{0}, U_{\alpha(i)})$ is smaller than $T(r)$ for any path r from $\mathbf{0}$ to $U_{\alpha(i)}$ of $\Sigma_{j=1}^d |U_{\alpha(i)}(j)|$ steps; in particular, $T(\mathbf{0}, U_{\alpha(i)})$ is stochastically smaller than $\Sigma_1^{dM} t_j$.

On the other hand, there exists a constant $C_5 > 0$ such that for $M \ge 1$,

$$ET(\mathbf{0}, U_i) \ge C_5 M \quad \text{for all } i. \tag{3.14}$$

To see this, note that if, for instance, the first coordinate of U_i equals M, then [$v(1)$ is the first coordinate of v]

$$T(\mathbf{0}, U_i) \ge b_{0,M} := \inf\{T(\mathbf{0}, v) : v(1) = M\}. \tag{3.15}$$

But it is well known [cf. Smythe and Wierman (1978), Section 5.3, or Cox and Durrett (1981), Theorem 6, or Kesten (1986), pages 166, 167] that

$$\frac{b_{0,M}}{M} \to \mu \quad \text{a.e.} \tag{3.16}$$

Thus (3.14) holds for large M. However, for any fixed M, (3.14) is clear, so that we can obtain (3.14) for all $M \ge 1$ by lowering C_5.

It follows from (3.13), (3.8) and (3.14) that

$$C_4 ln \ge \Lambda(M, ln) \ge \sum_{k=1}^{\nu} p(k,l) C_5 M - ln\mu(\xi)$$

and hence

$$\sum_{k=1}^{n} p(k,l) \le \frac{1}{C_5}\Big(C_4 + \sup_{\|\eta\|=1} \mu(\eta)\Big)\frac{ln}{M},$$

which is (3.10) [recall (2.60)].

Now that (3.10) is established, we replace $\Sigma p(k,l)U_k$ by $\Sigma p(k,l)V_{j(k)}$. Then, by (3.6) and (3.10),

$$\Big|\sum_k p(k,l)U_k - \sum_k p(k,l)V_{j(k)}\Big|$$
$$\le C_3 \frac{ln}{M} M^{(d+1)/d} n^{-1/d}. \tag{3.17}$$

Moreover,

$$\Big|\sum_k p(k,l)ET(\mathbf{0},U_k) - \sum_k p(k,l)ET(\mathbf{0},V_{j(k)})\Big| \\ \le C_3\frac{ln}{M}\max_k\big|ET(\mathbf{0},U_k) - ET(\mathbf{0},V_{j(k)})\big|.$$

As in (2.57),

$$\big|ET(\mathbf{0},U_k) - ET(\mathbf{0},V_{j(k)})\big| \le ET(U_k,V_{j(k)}).$$

In addition, there exists a path of at most $d|U_k - V_{j(k)}| \le dM^{(d+1)/d}n^{-1/d}$ edges from U_k to $V_{j(k)}$, so that

$$ET(U_k,V_{j(k)}) \le dM^{(d+1)/d}n^{-1/d}Et_1. \tag{3.18}$$

Thus

$$\Big|\sum_k p(k,l)ET(\mathbf{0},U_k) - \sum_k p(k,l)ET(\mathbf{0},V_{j(k)})\Big| \\ \le C_6 lM^{1/d}n^{(d-1)/d}. \tag{3.19}$$

Now set

$$r(j,l) = \sum_{k \text{ with } j(k)=j} p(k,l)$$

and

$$s(j) = \left\lfloor \frac{r(j,l)}{l} \right\rfloor. \tag{3.20}$$

Then

$$\sum_k p(k,l)V_{j(k)} = \sum_{j=1}^{\sigma} r(j,l)V_j \tag{3.21}$$

and

$$\begin{aligned}\Big|\sum s(j)V_j - n\xi\Big| &\le \frac{1}{l}\sum_j |ls(j) - r(j,l)||V_j| + \frac{1}{l}\Big|\sum_j r(j,l)V_j - ln\xi\Big| \\ &\le \frac{1}{l}\sum_{j=1}^{\sigma} lM + \frac{1}{l}\Big|\sum_k p(k,l)V_{j(k)} - \sum_k p(k,l)U_k\Big| \\ &\quad + \frac{1}{l}\Big|\sum_k p(k,l)U_k - ln\xi\Big| \\ &\le M\sigma + C_3M^{1/d}n^{(d-1)/d} + \frac{M}{l} \quad \text{[by (3.17) and (3.9)]} \\ &\le C_7M^{1/d}n^{(d-1)/d} \quad \text{[by (3.7)]}.\end{aligned}$$

These estimates quickly imply (3.5). Indeed, as in (3.11) and (3.12) we can now find some $t(i) \in \mathbb{Z}^+$ with

$$\sum_1^d t(i) \le C_8 M^{-(d-1)/d} n^{(d-1)/d}, \tag{3.22}$$

$$\left| \sum_1^d t(i) U_{\alpha(i)} - \left(\sum_j s(j) V_j - n\xi \right) \right| \le M.$$

Then

$$\left| \sum_j s(j) V_j - \sum_i t(i) U_{\alpha(i)} - n\xi \right| \le M.$$

Thus $\Sigma s(j)V_j - \Sigma t(i)U_{\alpha(i)}$ is a sum of the form $\Sigma p(k)U_k$, which satisfies (3.4). We may even take the $p(k)$ positive because $-U_k$ equals U_l for some l. Thus, by definition

$$\begin{aligned} \Lambda(M, n) &\le \sum s(j) ET(\mathbf{0}, V_j) - \sum t(i) ET(\mathbf{0}, U_{\alpha(i)}) - n\mu(\xi) \\ &\le \sum t(i) M E t_1 + \frac{1}{l}\left\{ \sum r(j, l) ET(\mathbf{0}, V_j) - ln\mu(\xi) \right\} \end{aligned} \tag{3.23}$$

[cf. the third inequality in (3.13) and (3.20)]. Furthermore, as in (3.21),

$$\begin{aligned} \sum_j r(j, l) ET(\mathbf{0}, V_j) &= \sum_k p(k, l) ET(\mathbf{0}, V_{j(k)}) \\ &\le \sum_k p(k, l) ET(\mathbf{0}, U_k) + C_6 l M^{1/d} n^{(d-1)/d} \quad \text{[by (3.19)]} \\ &= \Lambda(M, ln) + ln\mu(\xi) + C_6 l M^{1/d} n^{(d-1)/d} \quad \text{[by (3.8)]}. \end{aligned}$$

Therefore, by (3.22) and (3.23),

$$\Lambda(M, n) \le C_1 M^{1/d} n^{(d-1)/d} + \frac{1}{l}\Lambda(M, ln).$$

This proves (3.5).

STEP 2. Here we prove that for suitable constants $0 < C_i < \infty$ (independent of n, M, l and ξ) we have for all large M and $n^{1/(d+1)} \le M \le n$ and $l \ge 2$,

$$\begin{aligned} P&\left\{ a_{0, ln}(\xi) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M, n) \right\} \\ &\le C_9 \exp(-ln) \\ &\quad + C_9 \exp\left\{ C_{10} \frac{ln}{M} \log M + C_{10} l M^{(2-d)/(2d)} n^{(d-1)/d} \right. \\ &\qquad\qquad \left. - C_{11} \frac{l\Lambda^2(M, n)}{nM^{1/2}} \right\}. \end{aligned} \tag{3.24}$$

The proof of (3.24) is a minor modification of pages 210 and 211 in Kesten (1986). An attempt has been made to keep the notation the same as in this reference. [The proof here corresponds to taking $N = 1$ in Kesten (1986); the contribution of the form

$$N^{Q(d-1)}P\{t_1 + \cdots + t_{Q\lfloor dN/2\rfloor} \geq y\}$$

can be dropped in this situation and will therefore not appear in the argument that follows.] Let $r = (v_0 = \mathbf{0}, v_1, \dots, v_p = v(ln, \xi))$ be any self-avoiding path from $\mathbf{0}$ to $v(ln, \xi)$ with passage time $T(r) \leq ln\mu(\xi) + (l/2)\Lambda(M, n)$. Then we successively define the indices

$$\tau_0 = 0, \qquad \tau_{i+1} = \min\{t > \tau_i : |v_t - v_{\tau_i}| = M\}$$

(if such a t exists; if no such t exists, $\tau_{i+1} = \infty$). We define

$$Q = \max\{i : \tau_i < \infty\}$$

and

$$a_i = v_{\tau_i}, \qquad i \leq Q.$$

Then by definition of Q,

$$|v_j - v_{\tau_Q}| < M \quad \text{for } \tau_Q < j \leq p$$

and, in particular,

$$|v_{\tau_Q} - ln\xi| \leq |v_{\tau_Q} - v(ln, \xi)| + |v(ln, \xi) - ln\xi| \leq M. \tag{3.25}$$

Moreover,

$$|a_i - a_{i-1}| = |v_{\tau_i} - v_{\tau_{i-1}}| = M \tag{3.26}$$

so that

$$a_i - a_{i-1} = \text{one of the } U_k \text{ of Step 1.}$$

First we take care of the paths with

$$Q \geq C_{12}\frac{ln}{M} \tag{3.27}$$

for a suitably chosen (large) constant C_{12}. To do this, note that by (3.26), r must have at least M edges between a_{i-1} and a_i. Thus, if (3.27) holds, then r has at least $C_{12}ln$ edges. On the other hand, by (3.13) with $l = 1$ and by (2.60),

$$ln\mu(\xi) + \frac{l}{2}\Lambda(M, n) \leq C_{13}ln.$$

But, by Proposition 5.8 of Kesten (1986), we know that we can choose C_{12} so large that

$$P\{\exists \text{ self-avoiding path } r \text{ of at least } C_{12}ln \text{ edges but with } T(r) < C_{13}ln\} \leq C_9 e^{-ln}.$$

Thus the contribution to (3.24) from all paths satisfying (3.27) is at most $C_9 \exp(-ln)$ and these paths can be ignored from now on.

For the time being we now *fix* $Q < C_{12} ln/M$ *and* $a_1, \dots, a_Q$ such that (3.25) and (3.26) are satisfied. Later on we shall sum over all possible values of $Q, a_1, \dots, a_Q$. We denote the number of $i \in [1, Q]$ with $a_i - a_{i-1} = U_k$ by $p(k)$. The $p(k)$ are therefore also fixed at the moment. Then we have

$$(3.28)\quad \begin{aligned} P\Big\{&\text{there exists a self-avoiding path } r \text{ with } v_{\tau_i} = a_i,\ 1 \le i \le Q, \\ &\text{and satisfying (3.25) and } T(r) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M, n)\Big\} \\ &\le P\left\{\sum_1^Q T(a_{i-1}, a_i) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M, n)\right\}. \end{aligned}$$

By Theorem 4.8 in Kesten (1986) and the illustration following it, the last probability is at most

$$(3.29)\quad P\left\{\sum_1^Q T'(a_{i-1}, a_i) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M, n)\right\},$$

where the $T'(a_{i-1}, a_i)$ are *mutually independent* copies of the $T(a_{i-1}, a_i)$. In particular, if $a_i - a_{i-1} = U_k$ then $T'(a_{i-1}, a_i)$ has the same distribution as $T(\mathbf{0}, U_k)$. Therefore, for any $\delta \ge 0$, (3.29) is at most

$$(3.30)\quad \begin{aligned} &\exp\left(\delta ln\mu(\xi) + \frac{\delta l}{2}\Lambda(M, n)\right)\prod_1^Q Ee^{-\delta T'(a_{i-1}, a_i)} \\ &\quad = \exp\left(\delta ln\mu(\xi) + \frac{\delta l}{2}\Lambda(M, n)\right)\prod_k \left[Ee^{-\delta T(\mathbf{0}, U_k)}\right]^{p(k)}. \end{aligned}$$

In addition (3.25) says

$$(3.31)\quad \left|\sum_1^Q (v_{\tau_i} - v_{\tau_{i-1}}) - ln\xi\right| = \left|\sum_k p(k)U_k - ln\xi\right| \le M,$$

so that by definition of Λ,

$$\begin{aligned} \sum p(k) ET(\mathbf{0}, U_k) &\ge \Lambda(M, ln) + ln\mu(\xi) \\ &\ge l\Lambda(M, n) + ln\mu(\xi) - C_1 l M^{1/d} n^{(d-1)/d} \quad [\text{by } (3.5)]. \end{aligned}$$

Substitution of this estimate into (3.30) shows that the right-hand side of (3.28) is bounded by

$$(3.32)\quad \begin{aligned} &\exp\left(-\frac{\delta l}{2}\Lambda(M, n) + \delta C_1 l M^{1/d} n^{(d-1)/d}\right) \\ &\quad \times \prod_k \left[E\exp(-\delta\{T(\mathbf{0}, U_k) - ET(\mathbf{0}, U_k)\})\right]^{p(k)}. \end{aligned}$$

It remains to estimate the product in (3.32). Note that $\Sigma p(k) = Q$, the number of $a_i - a_{i-1}$ appearing above. We now write

$$(3.33)\quad \begin{aligned} &E\exp(-\delta\{T(\mathbf{0},U_k) - ET(\mathbf{0},U_k)\}) \\ &\quad\le \exp\left(C_{14}\frac{\delta l}{Q}\Lambda(M,n)\right) + \exp(\delta ET(\mathbf{0},U_k)) \\ &\qquad\times P\left\{T(\mathbf{0},U_k) - ET(\mathbf{0},U_k) \le -\frac{C_{14}}{Q}l\Lambda(M,n)\right\}. \end{aligned}$$

C_{14} will be chosen such that for large enough M, $n \ge M$ and $l \ge 2$ we have

$$(3.34)\quad \frac{C_{14}}{Q}l\Lambda(M,n) \le \frac{1}{2}MEt_1 \quad\text{and}\quad C_{14} \le \frac{1}{4}.$$

Such a $C_{14} > 0$ exists, because by (3.13) (with $l = 1$), $l\Lambda(M,n) \le C_4 ln$, while by (3.31),

$$QM = \sum_1^Q |v_{\tau_i} - v_{\tau_{i-1}}| \ge ln - M,$$

whence, for $n \ge M$ and $l \ge 2$,

$$(3.35)\quad Q \ge \frac{ln}{M} - 1 \ge \frac{ln}{2M}.$$

We now estimate the last probability in (3.33) by means of (2.49) with ξ replaced by $\bar{\xi} := U_k/\|U_k\|$ and n by $\bar{n} := \lfloor \|U_k\| \rfloor \in [M, dM]$. Note that $a_{0,\bar{n}}(\bar{\xi}) = T(\mathbf{0}, v(\bar{n},\bar{\xi}))$ and that

$$|v(\bar{n},\bar{\xi}) - U_k| \le |v(\bar{n},\bar{\xi}) - \bar{n}\bar{\xi}| + |\bar{n}\bar{\xi} - \|U_k\|\bar{\xi}| \le 2.$$

Therefore, U_k and $v(\bar{n},\bar{\xi})$ can be connected by a path of at most $2d$ edges and $T(\mathbf{0}, v(\bar{n},\bar{\xi})) - T(\mathbf{0},U_k)$ is stochastically smaller than $t_1 + \cdots + t_{2d}$. In particular,

$$(3.36)\quad \begin{aligned} &P\left\{\left|T(\mathbf{0},v(\bar{n},\bar{\xi})) - ET(\mathbf{0},v(\bar{n},\bar{\xi})) - T(\mathbf{0},U_k) + ET(\mathbf{0},U_k)\right| \ge y\right\} \\ &\quad\le P\left\{\sum_1^{2d}(t_i + Et_i) \ge y\right\} \\ &\quad\le e^{-\gamma y}e^{2d\gamma Et_1}\left\{\int e^{\gamma u}F(du)\right\}^{2d}. \end{aligned}$$

We thus obtain from (2.49) that the last probability in (3.33) is at most

$$
\begin{aligned}
&P\left\{\left|T(\mathbf{0}, v(\bar{n}, \bar{\xi})) - ET(\mathbf{0}, v(\bar{n}, \bar{\xi}))\right| \geq \frac{C_{14}}{2Q} l\Lambda(M,n)\right\} \\
&\quad + \left(\text{l.h.s. of (3.36) with } y = \frac{C_{14}}{2Q} l\Lambda(M,n)\right) \\
&\quad \leq C_3 \exp\left(-\frac{C_4 C_{14}}{2Q\bar{n}^{1/2}} l\Lambda(M,n)\right) + C_{15}\exp\left(-\frac{\gamma C_{14}}{2Q} l\Lambda(M,n)\right) \\
&\quad \leq 2C_3 \exp\left(-\frac{C_{16}}{QM^{1/2}} l\Lambda(M,n)\right),
\end{aligned}
\tag{3.37}
$$

provided

$$\frac{l\Lambda(M,n)}{QM^{1/2}} \leq C_5 M.$$

This proviso holds automatically for large M, $n \geq M$ and $l \geq 2$, by virtue of (3.34).

We now use (3.37) to obtain that the right-hand side of (3.33) is at most

$$\exp\left(C_{14}\frac{\delta l}{Q}\Lambda(M,n)\right) + 2C_3 \exp\left(\delta\, dMEt_1 - \frac{C_{16}}{QM^{1/2}} l\Lambda(M,n)\right)$$

[compare the third inequality in (3.13) for the estimate of $ET(\mathbf{0}, U_k)$]. Finally, we choose δ such that the two exponents here become equal; that is,

$$
\begin{aligned}
\delta &= \frac{C_{16}}{QM^{1/2}} l\Lambda(M,n)\left[dMEt_1 - \frac{C_{14}}{Q} l\Lambda(M,n)\right]^{-1} \\
&\leq \frac{2C_{16}}{QM^{1/2}} l\Lambda(M,n)[MEt_1]^{-1} \quad [\text{by (3.34)}] \\
&\leq \frac{C_{17}}{M^{1/2}} \quad [\text{by (3.35) and (3.13) with } l = 1].
\end{aligned}
$$

Using that

$$\exp\left(-\frac{\delta l}{2}\Lambda(M,n)\right) = \prod_k \left[\exp\left(-\frac{\delta l}{2Q}\Lambda(M,n)\right)\right]^{p(k)},$$

we conclude that (3.32), and hence (3.28), is bounded by

$$
\begin{aligned}
&\exp\left(C_1 C_{17} l M^{(2-d)/(2d)} n^{(d-1)/d}\right) \prod_k \left[(1+2C_3)\exp\left(\left(C_{14} - \frac{1}{2}\right)\frac{\delta l}{Q}\Lambda(M,n)\right)\right]^{p(k)} \\
&\quad \leq \exp\left(C_1 C_{17} l M^{(2-d)/(2d)} n^{(d-1)/d}\right)(1+2C_3)^Q \exp\left(-C_{18}\frac{l^2\Lambda^2(M,n)}{QM^{3/2}}\right)
\end{aligned}
$$

(recall that $C_{14} \leq 1/4$).

Finally, to estimate (3.24) we have to sum our estimate for (3.28) over all possible values of Q and $a_1, \dots, a_Q$. Because we already estimated the contribution to (3.24) from all paths satisfying (3.27) we may restrict ourselves to $Q < C_{12} ln/M$. For given a_i, a_{i+1} must satisfy $|a_{i+1} - a_i| = M$, so that there are at most $C_{19} M^{d-1}$ choices for a_{i+1} when a_i is given. Hence, for given Q, there are at most

$$\left(C_{19} M^{d-1}\right)^Q$$

choices for $a_1, \dots, a_Q$. It follows that the left-hand side of (3.24) is at most

$$\begin{aligned} C_9 \exp(-ln) &+ \sum_{Q \le C_{12} ln/M} \left[C_{19} M^{d-1}(1 + 2C_3)\right]^Q \\ &\times \exp\left\{C_1 C_{17} l M^{(2-d)/(2d)} n^{(d-1)/d} - C_{18} \frac{l^2 \Lambda^2(M,n)}{QM^{3/2}}\right\} \\ \le C_9 \exp(-ln) &+ C_9 \exp\left\{C_{10} \frac{ln}{M} \log M + C_{10} l M^{(2-d)/(2d)} n^{(d-1)/d} \right. \\ &\left. - \frac{C_{18}}{C_{12}} \frac{l \Lambda^2(M,n)}{nM^{1/2}}\right\}. \end{aligned}$$

This proves (3.24) with $C_{11} = C_{18}/C_{12}$.

STEP 3. It is now easy to complete the proof of (3.2). If $n^{1/(d+1)} \le M \le n$ and

$$C_{11} \frac{\Lambda^2(M,n)}{nM^{1/2}} > C_{10}\left(\frac{n \log M}{M} + M^{(2-d)/(2d)} n^{(d-1)/d}\right), \tag{3.38}$$

then (3.24) shows that

$$P\left\{a_{0,ln}(\xi) \le ln\mu(\xi) + \frac{l}{2}\Lambda(M,n)\right\} \to 0 \text{ as } l \to \infty.$$

This contradicts $(ln)^{-1} a_{0,ln}(\xi) \to \mu(\xi)$ w.p.1. Therefore, (3.38) fails for large M, which gives

$$\Lambda(M,n) \le \left(\frac{C_{10}}{C_{11}}\right)^{1/2} n \left\{\frac{(\log M)^{1/2}}{M^{1/4}} + \left(\frac{M}{n}\right)^{1/(2d)}\right\}. \tag{3.39}$$

For given n, we may take any M in $[n^{1/(d+1)}, n]$. We more or less minimize the right-hand side of (3.39) by taking

$$M = \left\lfloor n^{2/(d+2)} (\log n)^{2d/(d+2)} \right\rfloor. \tag{3.40}$$

With this M we find

$$\Lambda(M,n) \le C_{20} n^{(2d+3)/(2d+4)} (\log n)^{1/(d+2)}. \tag{3.41}$$

The remainder is just the subadditivity relation

$$ET(u,w) \le ET(u,v) + ET(v,w) \tag{3.42}$$

for any $u, v, w \in \mathbb{Z}^d$. By (3.41) and the definition of $\Lambda(M,n)$ there exist $p(k) \ge 0$ that satisfy (3.4) and for which

$$\sum_1^{\nu} p(k) ET(\mathbf{0}, U_k) \le n\mu(\xi) + C_{20} n^{(2d+3)/(2d+4)} (\log n)^{1/(d+2)}.$$

Now let $u_1, u_2, \dots, u_{\Sigma p(k)}$ be the vertices defined by $u_i - u_{i-1} = U_k$ for

$$\sum_1^{k-1} p(l) < i \le \sum_1^{k} p(l).$$

Thus exactly $p(k)$ of the differences $u_i - u_{i-1}$ equal U_k. Then (3.42) gives

$$Ea_{0,n}(\xi) \le \sum_1^{\gamma} ET(u_{i-1}, u_i) + ET\big(u_\gamma, v(n,\xi)\big),$$

where

$$\gamma = \sum_1^{\nu} p(k).$$

Thus

$$u_\gamma = \sum_1^{\nu} p(k) U_k$$

and by (3.4) and (2.43) [compare (3.18)],

$$ET\big(u_\gamma, v(n,\xi)\big) \le d|v(n,\xi) - u_\gamma| Et_1 \le d(M+1)\, Et_1,$$

while

$$\sum_1^{\gamma} ET(u_{i-1}, u_i) = \sum p(k) ET(\mathbf{0}, U_k)$$
$$\le n\mu(\xi) + C_{20} n^{(2d+3)/(2d+4)} (\log n)^{1/(d+2)}.$$

For the choice (3.40) of M, (3.2) follows. □

(1.18) is immediate from the right-hand inequality in (1.16) and (1.15). Also, from (3.2) and (2.49) we conclude that

$$P\Big\{a_{0,n}(\xi) \ge n\mu(\xi) + C_7 n^{1-1/(2d+4)} (\log n)^{1/(d+2)} + x\sqrt{n}\Big\} \le C_3 e^{-C_4 x} \quad \text{for } x \le C_5 n. \tag{3.43}$$

As remarked before, (1.20) can now be obtained from (3.43) via the relation (2.51). We merely have to estimate

$$\sum_{v \in (t - C_3 t^{(2d+3)/(2d+4)} (\log t)^{1/(d+2)}) B_0} P\{T(\mathbf{0}, v) > t\}.$$

For $|v|$ small with respect to t, use [as in (3.18)]

$$P\{T(\mathbf{0},v) > t\} \le P\left\{\sum_{1}^{d|v|} t_i > t\right\}$$

and the standard large deviation bound (2.33). For the other v, write again $\xi = v/\|v\|$, $n = \lfloor \|v\| \rfloor$ and use that as in (3.36),

$$P\{|T(\mathbf{0},v) - a_{0,n}(\xi)| \ge y\} \le P\{t_1 + \cdots + t_{2d} \ge y\} \le e^{-\gamma y}\left\{\int e^{\gamma u} F(du)\right\}^{2d}.$$

This, together with (3.43), will give a good upper bound for $P\{T(\mathbf{0},v) > t\}$ for all

$$v \in \left(t - C_3 t^{(2d+3)/(2d+4)}(\log t)^{1/(d+2)}\right)B_0,$$

when C_3 is large enough. We skip further details.

Finally, to prove (1.21), it suffices to restrict t to the integers, because $B(t)$ is increasing in t. But when t is restricted to the integers, then (1.21) is obvious from (1.19), (1.20) and the Borel–Cantelli lemma.

4. A lower bound for $Ea_{0,n} - n\mu$. In this section we give the proof of the left-hand inequalities in (1.13) and (1.16), which are the last remaining inequalities. For (1.16) the idea is to construct many paths from $\mathbf{0}$ to $kn\xi_1$ and to show that with high probability at least one of these has a passage time not exceeding $kEa_{0,n} - k^{1/2}\sigma(a_{0,n})$. This will lead to an inequality of the form

$$nk\mu \le Ea_{0,kn} \le kEa_{0,n} - k^{1/2}\sigma(a_{0,n}), \tag{4.1}$$

which quickly leads to the left-hand inequality in (1.16). Actually, we shall replace $a_{0,n}$ by the more restricted passage time

$$a_{0,n}^* := \inf\left\{T(r): r \text{ is a path from } \mathbf{0} \text{ to } n\xi_1 \text{ inside } (-2n,2n)^d\right\}.$$

For some k to be chosen later [see (4.10)] we now consider the minimal passage time, $T(j)$, of all paths r from $4jn\xi_2$ to $kn\xi_1 + 4jn\xi_2$ that lie entirely in the "tube" $(-2n,(k+2)n) \times ((4j-2)n,(4j+2)n) \times (-2n,2n)^{d-2}$. One way to construct such a path r is to connect successively $ln\xi_1 + 4jn\xi_2$ to $(l+1)n\xi_1 + 4jn\xi_2$ inside the cube

$$((l-2)n,(l+2)n) \times ((4j-2)n,(4j+2)n) \times (-2n,2n)^{d-2},$$

for $l = 0,\dots,k-1$. Denote the minimal passage time of all such connections for a given j and l by $\tau(l) = \tau(l,j)$. Then $\tau(l)$ has the same distribution as $a_{0,n}^*$. Even though, for fixed j, $\tau(0,j),\dots,\tau(k-1,j)$ are not independent, the variables $\tau(i+4l,j)$, $l = 0,1,\dots$ *are* independent because they depend on pairwise disjoint sets of edges. Therefore, by the Berry–Esseen theorem [cf.

Feller (1971), Theorem XVI.5.1] there exist two constants $C_1, C_2 > 0$ such that

$$(4.2)\qquad \frac{kE|a^*_{0,n} - Ea^*_{0,n}|^3}{k^{3/2}\sigma^3(a^*_{0,n})} \le C_1$$

implies that for fixed $i = 0, 1, 2, 3$ and k,

$$P\Biggl\{ \sum_{0 \le l \le (k-1-i)/4} [\tau(i+4l, j) - Ea^*_{0,n}] \le -k^{1/2}\sigma(a^*_{0,n}) \Biggr\} \ge C_2 > 0.$$

By the Harris–FKG inequality [cf. Durrett (1988), page 130] we then also have

$$(4.3)\qquad \begin{aligned} P\Biggl\{ &\sum_{0 \le l \le (k-1-i)/4} [\tau(i+4l, j) - Ea^*_{0,n}] \\ &\le -k^{1/2}\sigma(a^*_{0,n}) \text{ for } i = 0, 1, 2, 3 \Biggr\} \ge C_2^4. \end{aligned}$$

If the event in braces in (4.3) occurs, then

$$T(j) = \sum_{0 \le l \le k-1} \tau(l, j) \le kEa^*_{0,n} - 4k^{1/2}\sigma(a^*_{0,n}).$$

Thus (4.2) implies

$$(4.4)\qquad P\{T(j) \le kEa^*_{0,n} - 4k^{1/2}\sigma(a^*_{0,n})\} \ge C_2^4.$$

Next we shall prove the very crude lower bound (4.7) for $\sigma(a^*_{0,n})$. Despite the fact that we believe this bound to be very poor, we have been unable to improve (4.7).

Note added in proof: C. Newman and M. S. T. Piza, as well as R. Pemantle and Y. Peres, have now shown that $\sigma^2(a_{0,n}) \ge C_3 \log n$.

LEMMA 2. *Let $X, X_1, \dots, X_{2d}$ be i.i.d. random variables whose distribution is not concentrated on one point. Then*

$$(4.5)\qquad \inf_{c_1, \dots, c_{2d}} \sigma^2\Bigl\{ \min_{1 \le i \le 2d} (X_i + c_i) \Bigr\} > 0.$$

PROOF. Without loss of generality we restrict ourselves to $-\infty < c_1 \le c_2 \le \cdots \le c_{2d} < \infty$. Now find an x_0 such that

$$(4.6)\qquad P\{X \le x_0\} > 0, \qquad P\{X \ge x_0\} > 0 \quad \text{and} \quad \sigma^2(X | X \le x_0) > 0.$$

Here $\sigma^2(X|A)$ denotes the conditional variance of X, given that A occurs. There exists an x_0 that satisfies (4.6), because the distribution of X is not a one-point distribution. Now let E be the event

$$E = \{X_1 \le x_0, X_2 \ge x_0, \dots, X_{2d} \ge x_0\}.$$

Then $P\{E\} > 0$ and because the c_i are increasing,

$$\min_{1 \le i \le 2d} (X_i + c_i) = X_1 + c_1 \quad \text{on } E.$$

Finally, by the independence of the X_i,

$$\sigma^2\Big(\min_i(X_i + c_i)\Big) \geq P\{E\}\sigma^2\Big(\min_i(X_i + c_i)|E\Big)$$
$$= P\{E\}\sigma^2\{(X_1 + c_1)|E\}$$
$$= P\{E\}\sigma^2\{X_1|X_1 \leq x_0\} > 0. \qquad \square$$

We now show that this lemma implies for some $C_3 > 0$,

$$\sigma^2(a^*_{0,n}) \geq C_3. \tag{4.7}$$

The same argument gives a proof of the left-hand inequality of (1.13), but we leave this to the reader. To see (4.7) we condition on all the $t(e)$ with e not incident to the origin. For X_i we take the passage time of the edge from $\mathbf{0}$ to ξ_i if $i \leq d$ and of the edge from $\mathbf{0}$ to $-\xi_{i-d}$ if $d < i \leq 2d$. Also, for $i \leq d$ (for $d < i \leq 2d$), we take for c_i the infimum of $T(r)$ over all self-avoiding paths from ξ_i (from $-\xi_{i-d}$, respectively) to $n\xi_1$ inside $(-2n, 2n)^d$ and not passing through any of the $2d$ edges incident to the origin. Then

$$a^*_{0,n} = \min_{1 \leq i \leq 2d}\{X_i + c_i\},$$

because any self-avoiding path from $\mathbf{0}$ to $n\xi_1$ has to go first from $\mathbf{0}$ to some $\pm\xi_i$ and to continue from there to $n\xi_1$ without using any edge incident to $\mathbf{0}$ (otherwise it is not self-avoiding). If all edges not incident to $\mathbf{0}$ are fixed, then the c_i are also fixed, and the X_i are still i.i.d. Thus, by Lemma 2,

$$\sigma^2(a^*_{0,n}) \geq E\big\{\sigma^2(a^*_{0,n}|\,\text{all } t(e) \text{ for } e \text{ not incident to } \mathbf{0})\big\} \geq C_3,$$

as claimed.

Next we derive the upper bound

$$E|a^*_{0,n} - Ea^*_{0,n}|^3 \leq C_4 n\sigma^2(a^*_{0,n}). \tag{4.8}$$

This is easy, because [with r_n the path along the first coordinate axis from $\mathbf{0}$ to $n\xi_1$ as in the lines below (1.9)] for $C_5 > Et_1$, we have $a^*_{0,n} \leq T(r_n)$, $Ea^*_{0,n} \leq ET(r_n) < C_5 n$ and

$$E|a^*_{0,n} - Ea^*_{0,n}|^3 \leq C_5 nE|a^*_{0,n} - Ea^*_{0,n}|^2 + E\Big\{[T(r_n) + ET(r_n)]^3; |a^*_{0,n} - Ea^*_{0,n}| \geq C_5 n\Big\}$$
$$\leq C_5 n\sigma^2(a^*_{0,n}) + 8E\Big\{[T(r_n)]^3 + [ET(r_n)]^3; T(r_n) \geq C_5 n\Big\}.$$

(4.8) for a suitable C_4 now follows from the fact that the last term decreases exponentially in n for $C_5 > Et_1$ and from (4.7).

From (4.8) we see that (4.2), and hence (4.4) hold as soon as

$$k \geq \left(\frac{C_4}{C_1}\right)^2 \frac{n^2}{\sigma^2(a^*_{0,n})}. \tag{4.9}$$

After these preparations we are already to construct a path from $\mathbf{0}$ to $kn\xi_1$ with a "small" passage time. Take

$$k = \left\lfloor C_6 \frac{n^2}{\sigma^2(a^*_{0,n})} \right\rfloor + 1 \tag{4.10}$$

for a $C_6 \geq (C_4/C_1)^2$ to be chosen below [see (4.15)] and define

$$J = \min\{j \geq 0 \colon T(j) \leq kEa^*_{0,n} - 4k^{1/2}\sigma(a^*_{0,n})\}.$$

Then $T(J)$ is the passage time of some path, $r(J)$ say, from $4Jn\xi_2$ to $kn\xi_1 + 4Jn\xi_2$ inside

$$(-2n, (k+2)n) \times ((4J-2)n, (4J+2)n) \times (-2n, 2n)^{d-2}.$$

We make this into a path from $\mathbf{0}$ to $kn\xi_1$ by adding paths π' and π'' from $\mathbf{0}$ to $4Jn\xi_2$ and from $kn\xi_1 + 4Jn\xi_2$ to $kn\xi_1$, respectively. π' is the path from $\mathbf{0}$ to $4Jn\xi_2$ along the second coordinate axis, and π'' is a path, similarly chosen parallel to the second coordinate axis in the obvious way. Then

$$\begin{aligned} a^*_{0,kn} &\leq T(\pi') + T(J) + T(\pi'') \\ &\leq T(\pi') + T(\pi'') + kEa^*_{0,n} - 4k^{1/2}\sigma(a^*_{0,n}). \end{aligned} \tag{4.11}$$

Now let s_j be the path from $\mathbf{0}$ to $4jn\xi_2$ along the second coordinate axis. Then $\pi' = s_J$ and

$$\begin{aligned} T(\pi') &\leq 8(Et_1)Jn + [T(\pi') - 8(Et_1)Jn]^+ \\ &\leq 8(Et_1)Jn + \sum_{j=0}^{\infty} [T(s_j) - 8(Et_1)jn]^+. \end{aligned}$$

Because $T(s_j)$ has the distribution of $\Sigma_1^{4jn} t_i$, we obtain

$$ET(\pi') \leq 8n(Et_1)EJ + \sum_{j=0}^{\infty} E\left[\sum_1^{4jn} t_i - 8(Et_1)jn\right]^+. \tag{4.12}$$

The $T(j)$, $j = 0, 1, \dots,$ are independent, because they depend on disjoint sets of edges. Therefore by (4.4),

$$P\{J \geq m\} \leq (1 - C_2^4)^m$$

and

$$EJ \leq C_2^{-4} < \infty. \tag{4.13}$$

In addition, for any $\delta \geq 0$,

$$\begin{aligned} E\left[\sum_1^{4jn} t_i - 8(Et_1)jn\right]^+ &= jn\int_0^{\infty} P\left\{\sum_1^{4jn} t_i \geq (8(Et_1) + x)jn\right\} dx \\ &\leq jn\int_0^{\infty} \left[e^{-\delta x/4} E\{e^{\delta t_1 - 2\delta Et_1}\}\right]^{4jn} dx. \end{aligned}$$

If we choose $\delta > 0$ so small that

$$E\{e^{\delta t_1 - 2\delta E t_1}\} \le 1 - C_7$$

for some $C_7 > 0$, then we obtain

$$\begin{aligned} &\sum_{j=0}^{\infty} E\left[\sum_1^{4jn} t_i - 8(Et_1)jn\right]^+ \\ (4.14)\qquad &\le \sum_{j=0}^{\infty} jn(1-C_7)^{4jn}\int_0^{\infty} e^{-\delta jnx}\,dx \\ &= \left[\delta\{1-(1-C_7)^{4n}\}\right]^{-1}. \end{aligned}$$

Combining (4.12)–(4.14) yields

$$ET(\pi') \le C_8 n.$$

Of course, by translation invariance, $ET(\pi'') = ET(\pi')$ and taking expectations in (4.11) finally gives

$$Ea^*_{0,kn} \le 2C_8 n + kEa^*_{0,n} - 4k^{1/2}\sigma(a^*_{0,n}).$$

By definition, $a^*_{0,kn} \ge a_{0,kn}$ and by subadditivity,

$$Ea_{0,kn} \ge kn\mu = kn \inf_m E\frac{a_{0,m}}{m}.$$

Thus

$$Ea^*_{0,n} \ge n\mu + 4k^{-1/2}\sigma(a^*_{0,n}) - 2C_8\frac{n}{k}.$$

Note that C_8 depends only on C_2 and C_7, but not on C_6 [as long as $C_6 \ge (C_4/C_1)^2$]. Thus, with k as in (4.10) with C_6 so large that

$$C_6^{1/2} > C_8, \tag{4.15}$$

we find that for large n,

$$\begin{aligned} Ea^*_{0,n} - n\mu &\ge 2k^{-1/2}\sigma(a^*_{0,n}) \\ (4.16)\qquad &\ge C_6^{-1/2}\frac{\sigma^2(a^*_{0,n})}{n} \ge \frac{C_9}{n}. \end{aligned}$$

Last, we show that

$$Ea^*_{0,n} - Ea_{0,n} \le C_{10}\exp(-C_{11}n). \tag{4.17}$$

Clearly (4.16) and (4.17) will imply the left-hand inequality of (1.16). To obtain (4.17) we note that

$$0 \le a^*_{0,n} - a_{0,n} \le T(r_n)I[a_{0,n} < a^*_{0,n}].$$

Therefore,

$$Ea_{0,n}^* - Ea_{0,n} \le \left(E\{T^2(r_n)\}P\{a_{0,n} < a_{0,n}^*\}\right)^{1/2}$$
$$(4.18)\qquad = \left(E\left\{\sum_1^n t_i\right\}^2\right)^{1/2} P^{1/2}\{a_{0,n} < a_{0,n}^*\}$$
$$\le C_{12} n P^{1/2}\{a_{0,n} < a_{0,n}^*\}.$$

Moreover, if π_n is the optimal path for $a_{0,n}$, as in (2.14), then

$$\{a_{0,n} < a_{0,n}^*\} \subset \left\{\pi_n \text{ is not contained in } (-2n, 2n)^d\right\}$$
$$(4.19)\qquad \subset \left\{a_{0,n} > \tfrac{3}{2} n\mu\right\}$$
$$\cup \Big\{\exists \text{ path } v \text{ from } \mathbf{0} \text{ to } n\xi_1 \text{ that is not contained in } (-2n, 2n)^d \text{ but has } T(r) \le \tfrac{3}{2} n\mu\Big\}.$$

If there exists a path r such that the second event in the right-hand side of (4.19) occurs, then its piece from $\mathbf{0}$ to the boundary of $(-2n, 2n)^d$ or its piece from $n\xi_1$ to the boundary of $(-2n, 2n)^d$ has passage time at most $\frac{3}{4} n\mu$. Thus

$$(4.20)\qquad P\{a_{0,n} < a_{0,n}^*\} \le P\left\{a_{0,n} > \tfrac{3}{2} n\mu\right\} + 4dP\left\{b_{0,n} \le \tfrac{3}{4} n\mu\right\},$$

where

$$b_{0,n} = \inf\{T(r): r \text{ a path from } \mathbf{0} \text{ to } H_n\}$$

and H_n is the hyperplane $\{x = (x(1), \dots, x(d)) \colon x(1) = n\}$. (4.17) now follows from (4.18), (4.20) and the exponential bounds for the right-hand side of (4.20) in Grimmett and Kesten [(1984), Theorems 3.1 and 3.2] or Kesten [(1986), Theorems 5.2 and 5.9].

5. Some martingale estimates. In this section we prove Theorem 3. The proof will be broken down into several steps again. Most of these steps have been used repeatedly in the martingale literature.

STEP 1. In this step we appeal to exponential bounds to martingales with bounded increments. Let

$$(5.1)\qquad A = \sum_1^N E\{\Delta_k^2 | \mathscr{F}_{k-1}\}.$$

For the time being, assume that M_∞ exists in the case $N = \infty$. This will be proven to be the case in Steps 2 and 3, which do not depend on this step. Then, by Neveu [(1972), pages 154 and 155], for $x, y > 0$,

$$P\{M_N - M_0 \ge x\} \le P\{A \ge y\} + P\left\{M_N - M_0 > \frac{x}{2} + \frac{x}{2y}A\right\}$$
$$\le P\{A \ge y\} + \exp\left(-\frac{x\lambda}{2}\right),$$

where λ is the solution of

$$\frac{\phi_c(\lambda)}{\lambda} := \frac{e^{c\lambda} - 1 - c\lambda}{c^2\lambda} = \frac{x}{2y}. \tag{5.2}$$

In our application we will choose

$$y \geq \frac{\sqrt{x_0}}{e} x. \tag{5.3}$$

For this y we have [by (1.26)]

$$\frac{xc}{y} \leq 1, \tag{5.4}$$

and therefore the solution λ of (5.2) satisfies

$$\frac{1}{2}c\lambda \leq \frac{e^{c\lambda} - 1 - c\lambda}{c\lambda} = \frac{xc}{2y} \leq \frac{1}{2}.$$

But for $c\lambda \leq 1$ we also have

$$\frac{e^{c\lambda} - 1 - c\lambda}{c\lambda} \leq \frac{1}{2}c\lambda e^{c\lambda} \leq \frac{1}{2}c\lambda e.$$

Consequently, under (5.4),

$$c\lambda e \geq \frac{xc}{y} \quad \text{or equivalently} \quad \lambda \geq \frac{x}{ey}$$

and

$$P\{M_N - M_0 \geq x\} \leq P\{A \geq y\} + \exp\left(-\frac{x^2}{2ey}\right). \tag{5.5}$$

STEP 2. In order to apply (5.5) we need an estimate for $P\{A \geq y\}$. We shall derive such an estimate by bounding the moments of a truncated version of A. The argument is reminiscent of Dellacherie and Meyer [(1980), Chapter VI.105].

Set

$$Z_l = \sum_{k=l+1}^{N} E\{U_k | \mathscr{F}_l\}, \qquad 0 \leq l \leq N,$$

and for some $z > 0$, set

$$\nu = \inf\{l\colon Z_l > z\} \qquad (\nu = \infty \text{ if no such } l \text{ exists}).$$

We shall later take $z = 2x_0$, but for the time being the value of z is not important. Finally define

$$\tilde{A} = \sum_{1}^{\nu \wedge N} E\{\Delta_k^2 | \mathscr{F}_{k-1}\} = \sum_{1}^{N} I_k E\{\Delta_k^2 | \mathscr{F}_{k-1}\},$$

where

$$I_k = I[\nu \geq k].$$

Clearly

$$P\{A \geq y\} \leq P\{\nu < N\} + P\{\tilde{A} \geq y\}. \tag{5.6}$$

In this step we estimate the second term on the right-hand side:

$$\begin{aligned} E\{\tilde{A}^r\} &= E\left[\sum_{1}^{N} I_k E\{\Delta_k^2 | \mathscr{F}_{k-1}\}\right]^r \\ &\leq \sum_{1 \leq k_1, \dots, k_r \leq N} E\left\{\prod_{i=1}^{r} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right]\right\} \\ &\leq r! \sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq N} E\left\{\prod_{i=1}^{r} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right]\right\}. \end{aligned} \tag{5.7}$$

Introduce the abbreviation

$$\Gamma_r = \sum_{1 \leq k_1 \leq \cdots \leq k_r \leq N} E\left\{\prod_{i=1}^{r} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right]\right\}.$$

Then

$$\begin{aligned} \Gamma_r = \sum_{1 \leq k_1 \leq \cdots \leq k_{r-1} \leq N} & E\left\{\prod_{i=1}^{r-1} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right]\right. \\ & \left. \times E\left\{\sum_{k_r \geq k_{r-1}} I_{k_r} E\{U_{k_r} | \mathscr{F}_{k_r - 1}\} | \mathscr{F}_{k_{r-1} - 1}\right\}\right\} \\ \leq \sum_{1 \leq k_1 \leq \cdots \leq k_{r-1} \leq N} & E\left\{\prod_{i=1}^{r-1} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right] Z_{k_{r-1} - 1}\right\}. \end{aligned}$$

(In the last step we estimated I_{k_r} by 1 and used a standard property of repeated conditional expectations.) Now observe that $I_k = 1$ implies that $Z_{k-1} \leq z$, so that

$$I_k Z_{k-1} \leq I_k z.$$

Applying this for $k = k_{r-1}$ (in case $r \geq 2$), we obtain

$$\begin{aligned}\Gamma_r &\leq \sum_{1 \leq k_1 \leq \cdots \leq k_{r-1} \leq N} E\left\{\prod_{i=1}^{r-1} \left[I_{k_i} E\{U_{k_i} | \mathscr{F}_{k_i - 1}\}\right] z\right\} \\ &= z\Gamma_{r-1} \leq z^2 \Gamma_{r-2} \leq \cdots \leq z^{r-1}\Gamma_1 \\ &\leq z^{r-1} E\left\{\sum_1^N U_k\right\} = z^{r-1} \int_0^\infty P\left\{\sum_1^N U_k > x\right\} dx \\ &\leq z^{r-1}\left\{x_0 + C_1 \int_{x_0}^\infty e^{-C_2 x}\, dx\right\} \quad [\text{by } (1.27)] \\ &\leq z^{r-1}(x_0 + C_1/C_2).\end{aligned}$$

Substituting the last estimate into (5.7) gives

$$E\{\tilde{A}^r\} \leq r! z^{r-1}(x_0 + C_1/C_2)$$

and, by taking $r = \lfloor y/z \rfloor$ for $y \geq z$,

$$\begin{aligned}P\{\tilde{A} \geq y\} &\leq \frac{1}{y^r} E\{\tilde{A}^r\} \\ &\leq C_5\left(\frac{x_0 + C_1/C_2}{z}\right)\left\lfloor \frac{y}{z} \right\rfloor^{r + \frac{1}{2}} \left(\frac{z}{y}\right)^r e^{-\lfloor y/z \rfloor} \qquad (5.8) \\ &\leq C_6\left(x_0 + \frac{C_1}{C_2}\right)\frac{1}{z}\exp\left(-\frac{y}{2z}\right), \qquad y \geq z.\end{aligned}$$

STEP 3. Here we prove the following estimate for ν:

$$P\{\nu < N\} \leq (1 + C_1)\exp\left(\tfrac{1}{2}C_2(x_0 - z)\right). \qquad (5.9)$$

To see (5.9) we note first that $\nu < N$ means

$$\sum_{l+1}^N E\{U_k | \mathscr{F}_l\} > z \quad \text{for some } l < N,$$

and hence

$$\sum_1^N E\{U_k | \mathscr{F}_l\} > z \quad \text{for some } 0 \leq l \leq N. \qquad (5.10)$$

Thus

$$P\{\nu < N\} \leq P\{(5.10) \text{ occurs}\}.$$

However, for any $t \geq 0$, $m > 0$,

$$\exp\left(tE\left\{\sum_1^{N \wedge m} U_k \middle| \mathscr{F}_l\right\}\right), \qquad l = 0, 1, \ldots, N,$$

is an $\{\mathscr{F}_l\}$-submartingale, by Jensen's inequality. Therefore, by Doob [(1953), Theorem VII.3.2], for $0 \le t < C_2$,

$$\begin{aligned}
P\{(5.10) \text{ occurs}\} &\le \lim_{m\to\infty} P\left\{\sup_l \exp\left(t \sum_1^{N\wedge m} E\{U_k|\mathscr{F}_l\}\right) > e^{tz}\right\} \\
&\le \lim_{m\to\infty} e^{-tz} E\left\{\exp\left(t \sum_1^{N\wedge m} E\{U_k|\mathscr{F}_N\}\right)\right\} \\
&= e^{-tz} E\left\{\exp\left(t \sum_1^{N} U_k\right)\right\} \quad (\text{because } U_k \text{ is } \mathscr{F}_N\text{-measurable}) \\
&= e^{-tz}\left[1 + t\int_0^\infty e^{tx} P\left\{\sum_1^N U_k > x\right\} dx\right] \\
&\le e^{-tz}\left[1 + t\int_0^{x_0} e^{tx}\, dx + t\int_{x_0}^\infty C_1 e^{(t-C_2)x}\, dx\right] \quad [\text{by } (1.27)] \\
&\le e^{-tz}\left[e^{tx_0} + \frac{C_1 t}{C_2 - t} e^{(t-C_2)x_0}\right].
\end{aligned}$$

For $t = \frac{1}{2}C_2$ we obtain (5.9).

We also obtain from (5.6), (5.8) and (5.9), by first letting $y \to \infty$ and then $z \to \infty$, that $P\{A = \infty\} = 0$. It then follows from Neveu [(1972), Proposition VII.2.3C] that $M_\infty = \lim_{l\to\infty} M_l$ exists w.p.1, in case $N = \infty$.

STEP 4. Finally we combine (5.5), (5.6), (5.8) and (5.9). This gives for $z \ge 0$ and $y \ge z \vee (\sqrt{x_0}/e)x$,

$$\begin{aligned}
P\{M_N - M_0 \ge x\} \le \exp\left(-\frac{x^2}{2ey}\right) &+ (1 + C_1)\exp\left(\frac{1}{2}C_2(x_0 - z)\right) \\
&+ C_6\left(x_0 + \frac{C_1}{C_2}\right)\frac{1}{z}\exp\left(-\frac{y}{2z}\right).
\end{aligned}$$

(1.28) now follows by choosing

$$y = \frac{x}{\sqrt{e}}\sqrt{z}, \qquad z = x_0 + C_2^{-2/3}x^{2/3}$$

(this choice makes the three exponents of the same order). Note that this choice satisfies (5.3). Also $y \ge z$, provided $\sqrt{z} \le x/\sqrt{e}$. However, for $\sqrt{z} > x/\sqrt{e}$, the ratio

$$\frac{x}{x_0^{1/2} + C_2^{-1/3}x^{1/3}}$$

in the exponent in (1.28) is at most $\sqrt{e}$, so that (1.28) is trivial in that case.

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