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ON THE SPREAD-OUT LIMIT FOR BOND AND CONTINUUM PERCOLATION

BY MATHEW D. PENROSE

University of California, Santa Barbara

We prove the following results on Bernoulli bond percolation on the sites of the d-dimensional lattice, $d \geq 2$, with parameters M (the maximum distance over which an open bond is allowed to form) and λ (the expected number of open bonds with one end at the origin), when the range M becomes large. If $\lambda_c(M)$ denotes the critical value of λ (for given M), then $\lambda_c(M) \to 1$ as $M \to \infty$. Also, if we make $M \to \infty$ with λ held fixed, the percolation probability approaches the survival probability for a Galton–Watson process with Poisson(λ) offspring distribution. There are analogous results for other "spread-out" percolation models, including Bernoulli bond percolation on a homogeneous Poisson process on d-dimensional Euclidean space.

1. A spread-out bond percolation model. For $M \in (0, \infty)$, let $\mathbb{Z}^d/M$ denote the set $\{z/M\colon z \in \mathbb{Z}^d\}$. In this article we consider a bond percolation model on $\mathbb{Z}^d/M$ where the range over which bonds may form is fixed and M is large. This is equivalent to bond percolation on $\mathbb{Z}^d$ with bonds being allowed to form over increasing range.

Let φ be a bounded probability density function (p.d.f.) on $\mathbb{R}^d$, symmetric in the sense that $\varphi(-x) = \varphi(x)$, $x \in \mathbb{R}^d$. Set $v(M) = \sum_{x \in (\mathbb{Z}^d/M) \setminus \{0\}} \varphi(x)$. Assume $v(M) < \infty$. Suppose $0 < \lambda \leq v(M)/\sup\{\varphi(x)\colon x \in \mathbb{Z}^d/M,\ x \neq 0\}$.

Let G be a random, undirected graph on $\mathbb{Z}^d/M$, obtained as follows: For each pair $x, y \in \mathbb{Z}^d/M$, with $x \neq y$, include $\{x, y\}$ as an edge of G with probability $\lambda\varphi(x-y)/v(M)$, independently of all other pairs. The parameter λ is the expected value of the degree of 0 in G. Let $C(0)$ denote the component of G which includes 0. As usual in percolation models, there is a critical value λ_c of λ given by $\lambda_c(M) = \inf\{\lambda\colon P[C(0) \text{ is infinite}] > 0\}$. For $\lambda > \lambda_c$, G has an infinite component almost surely.

By a branching process argument, $\lambda_c(M) \geq 1$ for all M. If $d = 1$, and φ has bounded support, $\lambda_c(M) = v(M)/\sup\{\varphi(x)\colon x \in (\mathbb{Z}^d/M) \setminus \{0\}\}$ for all M. We consider the limiting behavior of $\lambda_c(M)$ as $M \to \infty$, when φ is a fixed function on $\mathbb{R}^d$ with $d \geq 2$.

This percolation model is equivalent to the "spread-out" model of Hara and Slade (1990), who discuss critical exponents for large (fixed) M with $d > 6$; in fact, the case $d > 6$ of Theorem 1 below is implicit in their work (G. Slade, personal communication). Taking M large is a "mean-field" limit, related to

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the "Kac limit" for a potential [see e.g., Penrose, Penrose, Stell and Pemantle (1990) and references therein]. Our model is appropriate to describe the spread of disease in an orchard, if the range of infection is long. On the other hand, if we have a forest rather than an orchard, the arrangement of trees is random. In the next section we shall discuss the case where an orderly arrangement of sites on a lattice is replaced by a Poisson process in $\mathbb{R}^d$. For a general discussion of percolation models and their motivation, see for example Grimmett (1989).

We need a technical condition on φ. Let $\bar{\varphi}_M$ denote the smallest function on $\mathbb{R}^d$ which is constant on all open cubes of side $1/M$ centered on points of $\mathbb{Z}^d/M$, and which is everywhere not smaller than φ. Then φ is *directly Riemann integrable* [see Feller (1971)] if (i) φ is Riemann integrable, and (ii) $\bar{\varphi}_M$ is integrable for some M. This condition implies that $\int \bar{\varphi}_M \, dx \to \int \varphi \, dx$ and that $M^{-d} v(M) \to \int_{\mathbb{R}^d} \varphi(x) \, dx$ as $M \to \infty$. Also, if φ is bounded and has bounded support, direct Riemann integrability is immediate from Riemann integrability. We shall say a p.d.f. φ on $\mathbb{R}^d$ is *well behaved* if it is bounded, symmetric and directly Riemann integrable.

THEOREM 1. *Suppose φ is a well-behaved p.d.f. on $\mathbb{R}^d$, $d > 2$. Then*

$$\lambda_c(M) \to 1 \quad \text{as } M \to \infty.$$

We can go further than the result in Theorem 1, obtaining a limit for the percolation probability when $M \to \infty$ with λ fixed. Let $\psi(\lambda)$ denote the survival probability of a Galton–Watson branching process with a Poisson(λ) offspring distribution; that is, if $\lambda \le 1$, then $\psi(\lambda) = 0$, and if $\lambda > 1$, then $x = 1 - \psi(\lambda)$ is the solution in $0 < x < 1$ to $e^{\lambda(x-1)} = x$ [see e.g., Athreya and Ney (1972)].

THEOREM 2. *Suppose φ is a well-behaved p.d.f. on $\mathbb{R}^d$, $d \ge 2$. If $M \to \infty$ with λ fixed, then $P[C(0)$ is infinite$] \to \psi(\lambda)$.*

In Sections 3–6 we develop the machinery to prove Theorems 1 and 2. Sections 7–9 provide the probability estimates to give the proofs, and Section 10 is a discussion of related site percolation models.

Consider the special case that φ is a constant on the unit ball and is 0 elsewhere. Then if $M = 1$, G is the familiar nearest-neighbor Bernoulli bond percolation process on $\mathbb{Z}^d$; if M is large, the range over which bonds of G may form becomes large, compared to the distance between neighboring sites of $\mathbb{Z}^d/M$. Theorems 1 and 2 are analogous to the results of Kesten (1990) regarding percolation in $\mathbb{Z}^d$ when $d \to \infty$ [see also Kesten (1991), Hara and Slade (1990) and Gordon (1991)]. In fact, our methods provide another way to derive Kesten's results; see Section 10. One might expect that if there are many potential bonds at each site, each with a small probability of being open, $C(0)$ should look roughly like the graph traced out by a branching random walk with a Poisson(λ) distribution of offspring. Bramson, Durrett and

Swindle (1989) derived detailed results on the contact process under a similar limiting regime to the one here; their methods are related to those used here.

2. A continuum percolation model. The following percolation model was discussed (and described in more detail) in Penrose (1991). An integrable function $f\colon \mathbb{R}^d \to [0,1]$ with $f(-x) = f(x)$, all $x \in \mathbb{R}^d$, is prescribed beforehand. Particles are placed in $\mathbb{R}^d$ by a homogeneous rate ρ Poisson process $\mathscr{P} = \{X_1, X_2, X_3, \dots\}$. A particle is added to the system at 0 to form a random set $\mathscr{P}_0 = \mathscr{P} \cup \{0\}$. On a probability space $(\Omega, \mathscr{F}, P_\rho)$, with expectation E_ρ, a graph G on $\mathscr{P}$ (resp., a graph $\overline{G}$ on $\mathscr{P}_0$) is obtained by joining any two particles of $\mathscr{P}$ (resp., $\mathscr{P}_0$), at x and y say, with probability $f(x-y)$, independently of all other pairs of particles. Continuum models are often more realistic than discrete ones; for further discussion see Penrose (1991), Given and Stell (1990), Burton and Meester (1991), Alexander (1991) and references in these papers.

Let $C(0)$ denote the component of $\overline{G}$ which includes the vertex at 0. Let $\#(C(0))$ denote the number of vertices of $C(0)$. There is a critical value of ρ, here denoted $\rho_c(f)$, at which $P_\rho[\#(C(0)) = \infty]$ becomes positive; that is,

$$P_\rho[\#(C(0)) = \infty] = P_\rho[G \text{ has an infinite component}] = 0, \qquad \rho < \rho_c(f),$$

$$P_\rho[\#(C(0)) = \infty] > 0, \quad \rho > \rho_c(f),$$

$$P_\rho[G \text{ has an infinite component}] = 1, \qquad \rho > \rho_c(f).$$

Note that when $d = 1$, if f has bounded support then $\rho_c(f) = \infty$.

Set $\lambda = \rho\int f(x)\,dx$ (the expected value of the degree of 0 in $\overline{G}$). In view of a conditioned branching process argument [the "method of generations"; see Zuev and Sidorenko (1985) and Gilbert (1961)] one expects $C(0)$ to be finite if $\lambda < 1$; indeed, we have the following theorem.

THEOREM 3. *Suppose in the continuum percolation model that f is Riemann integrable. Then*:

$$\text{(2.1)} \qquad \begin{aligned} &\text{(i)} \qquad E_\rho[\#(C(0))] < (1-\lambda)^{-1} < \infty \quad \textit{if } \lambda < 1. \\ &\text{(ii)} \qquad P_\rho[C(0) \textit{ is infinite}] \le \psi(\lambda). \end{aligned}$$

Note that (2.1) implies that $\rho_c(f) \ge (\int f(x)\,dx)^{-1}$. In Penrose (1991), (2.1) was asserted without much proof, and we shall give a more rigorous derivation in Sections 11 and 12. While it is natural to think of $C(0)$ as a conditioned branching process, all proofs of continuum results here will be by discretization methods similar to those of Źuev and Sidorenko [(1985), Section 3].

Let φ be a fixed p.d.f. on $\mathbb{R}^d$. We consider the case when f is a small constant multiple of φ. For $h > 0$ set $f_h(x) = h\varphi(x)$, $x \in \mathbb{R}^d$ (so if $f \equiv f_h$ on $\mathbb{R}^d$, then $\lambda = \rho h$). For a given value of ρ, let $\lambda_c(\rho)$ (which also depends on the given function φ) be defined by

$$\lambda_c(\rho) = \rho \inf\left\{h \in \left[0, \|\varphi\|_\infty^{-1}\right]: \rho > \rho_c(f_h)\right\}$$

(where the infimum of the empty set is taken to be ∞). That is, for f proportional to φ, $\lambda_c(\rho)$ is the infimum of those λ, at a given Poisson density ρ, for which $C(0)$ is infinite with positive probability. In Sections 13–15 we shall prove the following continuum analogs to Theorems 1 and 2.

THEOREM 4. *Suppose $d \geq 2$ and φ is a bounded, symmetric Riemann integrable p.d.f. on $\mathbb{R}^d$. Then $\lambda_c(\rho) \to 1$ as $\rho \to \infty$.*

THEOREM 5. *Suppose $d \geq 2$ and φ is as in Theorem 4. As $\rho \to \infty$ with λ fixed (i.e., with $f \equiv f_h$ where $h = \lambda/\rho$),*

$$P_\rho[C(0) \text{ is infinite}] \to \psi(\lambda).$$

3. Preliminary definitions and estimates on branching random walk. Let $\mathscr{L}$ denote the set $\{(i, j) \in \mathbb{Z}^2: i \geq 0, |j| \leq i, (i + j)/2 \in \mathbb{Z}\}$, made into a directed graph by including all directed edges e of the form $e = e_{ij+}$ from (i, j) to $(i + 1, j + 1)$, or $e = e_{ij-}$ from (i, j) to $(i + 1, j - 1)$, $(i, j) \in \mathscr{L}$. List the edges of $\mathscr{L}$ as $e_1, e_2, e_3, \ldots,$ choosing the ordering so that $e_{ij\pm}$ comes before $e_{i'j'\pm}$ in the ordering, whenever $i < i'$ or $i = i'$ and $j < j'$, and e_{ij-} comes before e_{ij+} for all $(i, j) \in \mathscr{L}$. So $e_1 = e_{0,0,-}$, $e_2 = e_{0,0,+}$, $e_3 = e_{1,-1,-}$, $e_4 = e_{1,-1,+}$, $e_5 = e_{1,1,-}$ and so on. Also, for each $p \in \{1, 2, 3, \ldots\}$ let $i(p)$ and $j(p)$ be such that $e_p = e_{i(p), j(p), -}$ (if p is odd) or $e_p = e_{i(p), j(p), +}$ (if p is even). Here and below, p is always an integer-valued index, *not* a probability.

For $(i, j) \in \mathscr{L}$, let b_{ij} and B_{ij} be the closed hypercubes in $\mathbb{R}^d$ of side $1/2$ and 1, respectively, centered at $(i, j, 0, \ldots, 0)$. We shall show percolation can occur by comparing the high-density percolation models described above with oriented percolation on $\mathscr{L}$, where we shall say that the edge $e_{ij\pm}$ of $\mathscr{L}$ is "open" if there exist sufficiently many reasonably short paths in $C(0)$ from B_{ij} to $B_{i+1, j\pm 1}$.

Given $\varphi(\cdot)$ and λ, let $(Z_n^M,\ n = 0, 1, 2, \ldots)$ be a discrete-time branching random walk (BRW) on $\mathbb{Z}^d/M$ in which: (i) at time n, each one of the particles created at time $n - 1$ dies and is replaced by a Poisson(λ) number of offspring; and (ii) the offspring of a particle at x are independently placed in $(\mathbb{Z}^d/M) \setminus \{x\}$ according to the probability mass function $\varphi(\cdot - x)/v(M)$. Let $(Z_n,\ n = 0, 1, 2, \ldots)$ be a BRW on $\mathbb{R}^d$, defined by (i) and by (iii) the offspring of a particle at x are independently placed in $\mathbb{R}^d$ according to the p.d.f. $\varphi(\cdot - x)$.

Let $\mathscr{M}^M$ (resp., $\mathscr{M}$) denote the space of counting measures (i.e., nonnegative integer-valued measures) on $\mathbb{Z}^d/M$ (resp., $\mathbb{R}^d$). The BRW Z_n^M (resp., Z_n) is a measure-valued process taking values in $\mathscr{M}^M$ (resp., $\mathscr{M}$). If $\mu \in \mathscr{M}^M$ (resp., $\mu \in \mathscr{M}$), let P_μ be the probability measure, with corresponding expectation E_μ, under which the BRW Z_n^M (resp., Z_n) has initial position $Z_0^M = \mu$ (resp., $Z_0 = \mu$). For $x \in \mathbb{Z}^d/M$ (resp., $x \in \mathbb{R}^d$) write P_x for P_{δ_x} where δ_x is a unit mass concentrated at x.

Suppose φ has bounded support. Let $(S_n, n \geq 0)$ denote the random walk $S_n = X_1 + \cdots + X_n$, where $X_1, X_2, \dots$ are independent $\mathbb{R}^d$-valued random variables each with p.d.f. φ (set $S_0 = 0$). Let Σ denote the covariance matrix of X_1. The following multivariate local central limit theorem is from Stone (1965, 1967).

LEMMA 1. *Suppose φ is a symmetric p.d.f. on $\mathbb{R}^d$ with bounded support. If (x_n) is a sequence in $\mathbb{R}^d$ with $n^{-1/2}x_n \to x \in \mathbb{R}^d$ as $n \to \infty$, then*

$$n^{d/2}P[S_n - x_n \in b_{00}] \to (1/2)^d \varkappa(x) \quad \text{as } n \to \infty,$$

where $\varkappa(\cdot)$ is the density of a $N(0, \Sigma)$ random variable.

LEMMA 2. *Suppose φ is a well-behaved p.d.f. on $\mathbb{R}^d$ and has bounded support. Suppose $\lambda > 1$. Given $\varepsilon > 0$, there exist integers k, m and M_0, such that for any $M \geq M_0$, for any counting measure μ on $\mathbb{Z}^d/M$ supported by B_{00}, with total mass m,*

$$P_\mu\big[Z_k^M(B_{11}) < 2m\big] < \varepsilon \tag{3.1}$$

and

$$P_\mu\big[Z_k^M(B_{1,-1}) < 2m\big] < \varepsilon. \tag{3.2}$$

PROOF. Let $(S_n, n \geq 0)$ denote the random walk with density φ, as above. By Lemma 1, there is a constant $c > 0$ and a number k_0 such that for all $x \in B_{00}$ and $k \geq k_0$,

$$P[x + S_k \in b_{11}] \geq ck^{-d/2}. \tag{3.3}$$

For $x \in \mathbb{R}^d$, and measurable $A \subset \mathbb{R}^d$, we have by conditioning on the number of descendants at time k of an initial particle at x [or by Grannan and Swindle (1991), Lemma 1], that

$$E_x\big[Z_k(A)\big] = \lambda^k P[x + S_k \in A].$$

By (3.3), there exists k such that for all $x \in B_{00}$,

$$E_x\big[Z_k(b_{11})\big] \geq 3.$$

There exists M_0 such that for $M \geq M_0$ and $x \in B_{00} \cap \mathbb{Z}^d/M$,

$$E_x\big[Z_k^M(B_{11})\big] \geq 3.$$

Also, by considering the underlying Galton–Walton process, $E_x[(Z_k^M(\mathbb{R}^d))^2] \leq (\lambda^2 + \lambda)^k$ by induction on k [see Athreya and Ney (1972), page 4]. So

$$\operatorname*{Var}_x\big[Z_k^M(B_{11})\big] \leq (\lambda^2 + \lambda)^k.$$

Let μ be any counting measure on $\mathbb{Z}^d/M$ supported by B_{00}, with total mass m. Then by additivity of the branching random walk, $E_\mu[Z_k^M(B_{11})] \geq 3m$, and

$\mathrm{Var}_\mu[Z_k^M(B_{11})] \le m(\lambda^2 + \lambda)^k$. By Chebyshev's inequality,

$$P_\mu\big[Z_k^M(B_{11}) \le 2m\big] \le (\lambda^2 + \lambda)^k/m$$

and (3.1) follows by taking m large. The proof of (3.2) is similar. □

LEMMA 3. *Suppose $\lambda > 1$. Suppose φ is a well-behaved p.d.f. on $\mathbb{R}^d$ with bounded support. Given $m > 0$ and $\delta > 0$, there exists k_0 such that*

$$P_0[Z_k(b_{00}) \ge 2m] > \psi(\lambda) - \delta, \qquad k \ge k_0.$$

PROOF. Take $R \in (0, \infty)$ such that $\varphi(x) = 0$ outside $\{x\colon \|x\| \le R\}$ (here and below, $\|\cdot\|$ denotes the Euclidean norm). Recall that under the probability measure P_0, (Z_n) is a BRW with $Z_0 = \delta_0$, so $(Z_n(\mathbb{R}^d))$ is a Galton–Watson branching process with Poisson(λ) offspring distribution. By Athreya and Ney [(1972), page 9], if we set $W_n = \lambda^{-n} Z_n(\mathbb{R}^d)$, then $W_n \to W$ a.s. (P_0), where W is a random variable with $P_0[W > 0] = \psi(\lambda)$. Take $\eta > 0$ so $P_0[W \ge \eta] > \psi(\lambda) - \delta/2$. Then for some n_0,

$$P_0[W_n \ge \eta/2] \ge \psi(\lambda) - \delta/2, \qquad n \ge n_0. \tag{3.4}$$

Note that under P_0, the measure Z_n is concentrated on $\{\|x\| \le nR\}$. Also, for any n, and any x with $\|x\| \le nR$,

$$P_x[Z_{n^2}(b_{00}) \ge 1] \ge \psi(\lambda) P[x + S_{n^2} \in b_{00}],$$

where (S_n) is random walk in $\mathbb{R}^d$ with density φ as before. By Lemma 1 and a compactness argument, there are constants $c > 0$ and $n_1 \ge n_0$ such that for $n \ge n_1$,

$$\inf_{\|x\| \le nR} P_x[Z_{n^2}(b_{00}) \ge 1] \ge cn^{-d}. \tag{3.5}$$

Now if $S = I_1 + \cdots + I_r$, where I_i are independent Binomial($1, p_i$) random variables, and $p_i \ge p_0$, $1 \le i \le r$, then $\mathrm{Var}(S) \le r$, and by Chebyshev's inequality,

$$P[S < rp_0/2] \le 4/(rp_0^2). \tag{3.6}$$

Setting $p_0 = cn^{-d}$, we see from (3.5), (3.6) and the additive property of the BRW, that for any counting measure μ on $\mathbb{R}^d$ supported by $\{\|x\| \le nR\}$, with total mass exceeding $\max(4mc^{-1}n^d, 8c^{-2}\delta^{-1}n^{2d})$,

$$P_\mu p[Z_{n^2}(b_{00}) < 2m] \le \delta/2. \tag{3.7}$$

Take $n \ge n_1$ so that $\lambda^n \eta/2 \ge \max(4mc^{-1}n^d, 8c^{-2}\delta^{-1}n^{2d})$. Then by (3.4) and (3.7),

$$\begin{aligned} P_0[Z_{n+n^2}(b_{00}) \le 2m] &\le P_0[W_n < \eta/2] \\ &\quad + P_0[Z_n(\mathbb{R}^d) \ge \lambda^n \eta/2 \text{ and } Z_{n+n^2}(b_{00}) \le 2m] \\ &\le 1 - \psi(\lambda) + \delta. \end{aligned}$$

Setting $k_0 = n + n^2$, we obtain the desired result. □

4. A percolation algorithm. For each M, choose an ordering on the elements of $\mathbb{Z}^d/M$. We shall use this prechosen ordering throughout the proofs below.

We assume for now that φ has bounded support. Take R so that $\varphi(x) = 0$ for $\|x\| > R$ (remember, $\|\cdot\|$ is Euclidean norm). For $x, y \in \mathbb{R}^d$, write $x \sim y$ if $0 < \|x - y\| \leq R$. So for $x \in \mathbb{Z}^d/M$, $x \sim y$ for only finitely many $y \in \mathbb{Z}^d/M$, which prevents the following algorithm from staying at any particular x forever. Let m, k and k_1 be fixed positive integers, to be chosen later.

Let A_{00} be an arbitrary subset of $\mathbb{Z}^d/M \cap B_{00} \setminus \{0\}$, such that $|A_{00}| = 2m$ (here and below $|\cdot|$ denotes cardinality), and $\varphi(x) > 0$, $x \in A_{00}$ (such a set exists for all large enough M). Let the graph G_0 and $\mathbb{Z}^d/M$ consist of all edges of the form $\{0, x\}$, $x \in A_{00}$.

We shall argue that the following random algorithm produces a random graph G_∞ on $\mathbb{Z}^d/M$ which may be viewed as a subgraph of G. In this algorithm, the words "first" and "next" always refer to the prechosen ordering on $\mathbb{Z}^d/M$. The algorithm could lie on a probability space which generates an infinite sequence of independent random variables which are uniformly distributed on $[0, 1]$.

The algorithm will involve the construction of a set of *occupied* sites of $\mathscr{L}$, the other sites being said to be *vacant*. It will also construct a set of *open* bonds (directed edges) of the directed graph $\mathscr{L}$, the other bonds being said to be *closed*. Initially set the site $(0, 0)$ of $\mathscr{L}$ to be occupied, set all other sites (i, j) of $\mathscr{L}$ to be vacant, and set all bonds $e_{ij\pm}$ of $\mathscr{L}$ to be closed. The algorithm also constructs sets $D_{p,n}$, denoting the set of vertices which are added to the cluster at the nth generation of a part of G_∞ starting inside the hypercube $B_{i(p), j(p)}$.

ALGORITHM 1.

STEP 1. Let G_∞ be the graph G_0. Set $p = 1$.

STEP 2. Set $i = i(p)$, $j = j(p)$.

STEP 3. If the site (i, j) of $\mathscr{L}$ is occupied, go to Step 4. Otherwise go to Step 16.

STEP 4. If p is odd (resp., even), let the set A_p consist of the first m (resp., the last m) elements of A_{ij} (according to the prechosen ordering on $\mathbb{Z}^d/M$).

STEP 5. Set $D_{p,0} = A_p$. Let $D_{p,r}$ be the empty set, for all $r > 0$. Set $n = 0$.

STEP 6. Consider the first site x (according to the prechosen ordering on $\mathbb{Z}^d/M$) in $D_{p,n}$.

STEP 7. Set $h = 0$. Consider the first site y with $y \sim x$, $y \neq 0$, $y \notin D_{p,n'}$, $0 \leq n' \leq n$, $y \notin D_{q,r}$, $1 \leq q \leq p - 1$, $0 \leq r \leq k$.

STEP 8. With probability $1 - \exp\{-\lambda\varphi(x - y)/v(M)\}$, add $\{x, y\}$ to the set of edges of G_∞; and in this case, increase h by 1, and if $y \notin D_{p,n+1}$, add y to the set $D_{p,n+1}$.

STEP 9. If $h = k_1$ (i.e., the k_1th new edge from x has just been added), go to Step 11.

STEP 10. Consider the next site y with $y \sim x$, $y \neq 0$, and with y not in $D_{p,n'}$, $0 \leq n' \leq n$, or in $D_{q,r}$, $1 \leq q \leq p - 1$, $0 \leq r \leq k$ (if there is such a site y), and return to Step 8. If there is no such y, go on to Step 11.

STEP 11. Go on to the next site x in $D_{p,n}$ (if there is one), and return to Step 7. If $D_{p,n}$ has been exhausted, go to Step 12.

STEP 12. Increase n by 1.

STEP 13. If $n \leq k - 1$, go to Step 6. If $n = k$, go to Step 14.

STEP 14. Suppose p is even. Suppose that $|D_{p,k} \cap B_{i+1,j+1}| \geq 2m$. Then change the status of the bond $e_p = e_{ij+}$ of $\mathscr{L}$ to "open," and change the status of the site $(i + 1, j + 1)$ of $\mathscr{L}$ to "occupied." Also in this case, define the set $A_{i+1,j+1}$ to consist of the first $2m$ elements of $D_{p,k} \cap B_{i+1,j+1}$.

STEP 15. Suppose p is odd. Suppose that $|D_{p,k} \cap B_{i+1,j-1}| \geq 2m$. Then change the status of the bond $e_p = e_{ij-}$ of $\mathscr{L}$ to "open"; also in this case, *if* $(i + 1, j - 1)$ is vacant (which implies $A_{i+1,j-1}$ has not yet been defined), change its status to "occupied" and define the set $A_{i+1,j-1}$ to consist of the first $2m$ elements of $D_{p,k} \cap B_{+1,j-1}$.

STEP 16. Increase p by 1, and go to Step 2.

The main point about Algorithm 1 is that the randomness occurs only at Step 8. The remaining steps are just rules for choosing which edge of $\mathbb{Z}^d/M$ to look at next. Under these rules, each edge is examined either once or not at all. Let G' be a random graph on $\mathbb{Z}^d/M$ in which (i) all edges of G_0 are included in G'; (ii) all edges of the form $\{0, x\}$ not in G_0 are not included in G'; and (iii) all edges of the form $\{x, y\}$, $x \neq 0$, $y \neq 0$, are independently included as edges of G' with probability $q_M(x, y)$, where we set $q_M(x, y) = 1 - \exp\{-\lambda\varphi(x - y)/v(M)\}$. Then G_∞ may be viewed as a subgraph of G' (at Step 8, add edge $\{x, y\}$ to G_∞ iff it is an edge of G'). Since $q_M(x, y) \leq \lambda\varphi(x - y)/v(M)$, a coupling argument makes G_∞ a subgraph of a graph, also denoted G', in which (i) and (ii) hold, and (iv) all edges of the form $\{x, y\}$, $x \neq 0$, $y \neq 0$, are independently included in G' with probability $\lambda\varphi(x - y)/v(M)$.

5. A modified branching random walk algorithm. At this point our notation threatens to become overloaded with subscripts so we rewrite the BRW Z_n^M as $Z^M(n)$. On a probability space $(\Omega, \mathscr{F}, P)$, for each $x \in \mathbb{Z}^d/M$ let $(Z_x^M(n), n = 0, 1, 2, \dots, k)$ be a realization of $Z^M(\cdot)$ with $Z_x^M(0) = \delta_x$, running independently of all $Z_y^M(\cdot)$, $y \neq x$, for exactly k time units. The following random algorithm for inductively producing random graphs G_p, $p \geq 1$, and G'_∞ on $\mathbb{Z}^d/M$ can lie over the probability space $(\Omega, \mathscr{F}, P)$. We shall show that the graph G'_∞ constructed by this algorithm has the same distribution as the graph G_∞ constructed by Algorithm 1.

Again, as in Algorithm 1, all use of the words "first" and "next" refers to the prescribed ordering on $\mathbb{Z}^d/M$. Again, initially set the site $(0, 0)$ of $\mathscr{L}$ to be *occupied*, set all other sites (i, j) of $\mathscr{L}$ to be *vacant*, and set all bonds (edges) of $\mathscr{L}$ to be *closed*. Let A_{00} be as in the previous section.

ALGORITHM 2.

STEP 1. Set $p = 1$.

STEP 2. Set $i = i(p)$ and $j = j(p)$.

STEP 3. If the site (i, j) of $\mathscr{L}$ is vacant, go to Step 9. If site (i, j) is occupied, go on to Step 4.

STEP 4. If p is odd (resp., even), let the set A_p consist of the first m (resp., the last m) elements of A_{ij} (according to the prechosen ordering on $\mathbb{Z}^d/M$).

STEP 5. Let $(X_p(n), n = 1, 2, \dots, k)$ be a BRW on $\mathbb{Z}^d/M$, starting with the measure $X_p(0) = \sum_{x \in A_p} \delta_x$ at time $n = 0$, with a Poisson(λ) distribution of offspring and the position of each offspring of a particle at x chosen from $(\mathbb{Z}^d/M) \setminus \{x\}$ according to the probability mass function $\varphi(\cdot - x)/v(M)$, independently of other offspring, running for exactly k generations subject to the following modifications:

(i) If at the generation n, $1 \leq n < k$, a point x is visited simultaneously from more than one place, make the particles visiting x at that time coalesce. Similarly, if two or more particles are born at x from the same parent, let them coalesce.

(ii) If an nth generation particle $(1 \leq n < k)$ has offspring in more than k_1 positions, remove all but those in the first k_1 of these positions (using the prechosen ordering on sites of $\mathbb{Z}^d/M$) (along with their subsequent offspring).

(iii) If an nth generation particle $(0 \leq n < k)$ at x has an offspring at a site y which was already visited at an earlier generation r, $0 \leq r \leq n$, or for an earlier value of p, remove that offspring (i.e., the new particle at y), and all its subsequent offspring.

STEP 6. Let G_p be the graph traced out by the edges of the modified BRW X_p. That is, include $\{x, y\}$ as an edge of G_p if and only if for some generation n, $0 \le n < k$, there is a particle of $X_p(n)$ at an endpoint of $\{x, y\}$, which has an offspring at the other endpoint at time $n + 1$, which is not removed in the course of the modifications (ii) and (iii).

STEP 7. Suppose p is even. Suppose $X_p(k)(B_{i+1,j+1}) \ge 2m$; that is, suppose $X_p(k)$ places $2m$ or more particles in $B_{i+1,j+1}$. Then change the status of the bond $e_p = e_{ij+}$ of $\mathscr{L}$ to "open," and change the status of the site $(i+1, j+1)$ of $\mathscr{L}$ to "occupied." Also define $A_{i+1,j+1}$ by setting $A_{i+1,j+1}$ to consist of the sites of the first $2m$ of these particles (in the prechosen ordering on $\mathbb{Z}^d/M$).

STEP 8. Suppose p is odd. Suppose $X_p(k)(B_{i+1,j-1}) \ge 2m$. Then change the status of the bond $e_p = e_{ij-}$ of $\mathscr{L}$ to "open"; also, *if* $(i+1, j-1)$ is vacant (which implies $A_{i+1,j-1}$ has not yet been defined), change its status to "occupied" and define $A_{i+1,j-1}$ to consist of the first $2m$ sites of $X_p(k)(B_{i+1,j-1})$.

STEP 9. Increase p by 1, and return to Step 2.

Now let G'_∞ be the union of the graphs G_p, $p \ge 0$. By the construction, G'_∞ is connected and if infinitely many e_p are open, then G'_∞ is infinite.

By a theorem on the compound Poisson distribution [see e.g., Feller (1968), pages 291–292 or Bowers, Gerber, Hickman, Jones and Nesbitt (1986), page 330], in the BRW $Z^M(\cdot)$, for any distinct x and y, a particle at x has a Poisson$(\lambda\varphi(y-x)/v(M))$ number of offspring at y, so it has at least one offspring at y with probability $q_M(x, y) = 1 - e^{-\lambda\varphi(y-x)/v(M)}$. Also, given a particle at x, the number of offspring at different positions y are independent. It follows that the construction of G'_∞ by Algorithm 2 is equivalent to that of G_∞ by Algorithm 1, in the sense that the distribution of G'_∞ is the same as that of G_∞.

6. Comparison of modified and unmodified BRW. Recall that on our probability space $(\Omega, \mathscr{F}, P)$, $(Z_x^M(n), n = 0, 1, 2, \dots, k)$ $(x \in \mathbb{Z}^d/M)$ are independent BRW's with $Z_x^M(0) = \delta_x$, running for exactly k time units. Algorithm 2 can be performed on this probability space, with the coalescing BRW $X_p(\cdot)$ being constructed from the BRW's $Z_x^M(\cdot)$, $x \in A_p$, in a natural way. Given a realization of the BRW's $Z_x^M(\cdot)$, $x \in \mathbb{Z}^d/M$, define the events E_{pl}, $1 \le l \le 5$, $p = 1, 2, 3, \dots$ by adding the following steps to Algorithm 2:

(a) Just after Step 2 in Algorithm 2, let us say the event E_{p1} occurs if the site $(i, j) = (i(p), j(p))$ of $\mathscr{L}$ is vacant at this stage of the algorithm.

(b) Suppose E_{p1} does not occur. Then A_p is defined in Step 4, and $|A_p| = m$. Consider the BRW's $(Z_x^M(n), 0 \le n \le k)$ for $x \in A_p$. If p is odd (resp., even), let E_{p2} be the event that E_{p1} does not occur, *and* that the total mass assigned at time k by these BRW's to $B_{i(p)+1,\,j(p)-1}$ (resp., $B_{i(p)+1,\,j(p)+1}$) is less than $2m$; that is, E_{p2} is the event

$$E_{p2} = E_{p1}^c \cap \left\{ \sum_{x \in A_p} Z_x^M(k)\big(B_{i(p)+1,\,j(p)-1}\big) < 2m \right\} \qquad (p \text{ odd}),$$

$$E_{p2} = E_{p1}^c \cap \left\{ \sum_{x \in A_p} Z_x^M(k)\big(B_{i(p)+1,\,j(p)+1}\big) < 2m \right\} \qquad (p \text{ even}).$$

(c) Let us say the event E_{p3} occurs if E_{p1} does not, *and* for any $x \in A_p$, at any stage n, with $0 \le n \le k-1$, of the evolution of $Z_x^M(\cdot)$, any of the particles created at time n produces more than k_1 offspring.

(d) Define E_{p4} to be the event that E_{p1} and E_{p3} do *not* occur, *and* that there exists $y \in \mathbb{Z}^d/M$ such that y is visited by more than one of the BRW's $(Z_x^M(n))_{n=0}^k$, $x \in A_p$, or it is visited more than once by one of these BRW's.

(e) Define E_{p5} to be the event that E_{p1} and E_{p3} do *not* occur, *and* that there exists y which is assigned a mass of at least 2 at some time by one of the BRW's $(Z_x^M(n))_{n=0}^k$, $x \in A_p$. Thus,

$$E_{p4} \cup E_{p5} = E_{p1}^c \cap E_{p3}^c \cap \left[\bigcup_{y \in \mathbb{Z}^d/M} \left\{ \sum_{x \in A_p} \sum_{n=0}^{k} Z_x^M(n)(\{y\}) > 1 \right\} \right].$$

(f) Define E_{p6} to be the event that E_{p1} and E_{p3} do *not* occur, *and* that for some q, $0 \le q < p$, one of the BRWs $Z_x^M(\cdot)$, $x \in A_p$, visits one of the end-points of one of the edges of the graph G_q at some time n, $1 \le n \le k$.

(g) Define $\overline{E}_p$ to be the (good) event that none of the (bad) events E_{pl} occurs; that is, $\overline{E}_p = (\bigcup_{l=1}^6 E_{pl})^c$. If $\overline{E}_p$ occurs, then at the kth step, the coalescing BRW $X_p(k)$ places $2m$ or more particles in $B_{i+1,\,j+1}$ (resp., $B_{i+1,\,j-1}$) if p is even (resp., odd); in this case, Step 7 (resp., Step 8) of Algorithm 2 applies, and the edge e_p of $\mathscr{L}$ becomes open and the site at its end becomes (or remains) occupied.

7. Proof of Theorem 1 when φ has bounded support. Let $\lambda > 1$ be fixed. We must show that for M large, $\lambda_c(M) \le \lambda$.

Take $\varepsilon > 0$ to be so small that for oriented percolation on $\mathscr{L}$, with each bond e_p of $\mathscr{L}$ independently open with probability $1 - 5\varepsilon$, there is (with nonzero probability) an infinite path from 0 of open bonds of $\mathscr{L}$ [see Durrett (1984)]. We shall show that by suitably large fixed m, k and k_1, for large M the graph G'_∞ generated by Algorithm 2 is infinite with nonzero probability; this implies that the probability G is infinite, conditional on all bonds (edges) in G_0 being open, is nonzero. Finally the event that all bonds in G_0 are open has nonzero probability.

Let Σ_p be the σ-field generated by the outcome of the BRW's $Z_x^M(\cdot)$, $x \in (\mathbb{Z}^d/M) \cap (\bigcup_{1 \le q < p} B_q)$. Note that $E_{p1} \in \Sigma_p$. Given any outcome $B \in \Sigma_p$ of these BRWs, such that E_{p1} does not occur, we shall show that $P[\overline{E}_p | B] > 1 - 5\varepsilon$. This implies that the probability that bond e_p is closed, given that the site at the start of e_p is occupied, is at most 5ε.

First, note that by Lemma 2, for suitable k and m (which will be fixed for the remainder of the proof),

$$P[E_{p2}|B] \le \varepsilon. \tag{7.1}$$

If $N \sim \text{Poisson}(\lambda)$, then there is a constant c depending only on λ, such that for all integers $k_1 \ge 2\lambda$, $P[N \ge k_1] \le c(1/2)^{k_1}$. Hence by induction from $n = 1$ to k, the probability that there exists a particle before the kth generation of a Galton–Watson process [with a Poisson(λ) offspring distribution, starting from a single particle], having more than k_1 offspring, is at most $c(1 + k_1 + \cdots + k_1^{k-1})(1/2)^{k_1}$. Hence,

$$P[E_{p3}|B] \le mc\left(\frac{k_1^k - 1}{k_1 - 1}\right)\left(\frac{1}{2}\right)^{k_1} < \varepsilon \tag{7.2}$$

provided we take k_1 large enough (from now on k_1 will be fixed; assume $k_1 \ge 2$).

Before estimating $P[E_{p4}|B]$, let us consider a single particle at x, with a Poisson(λ) number of offspring each distributed according to the probability mass function $\varphi(\cdot - x)/v(M)$. That is, we consider $Z_x^M(1)$. For $y \ne x$, $Z_x^M(1)(\{y\})$ is a Poisson($\lambda\varphi(y - x)/v(M)$) random variable. Hence, setting $K = \sup\{\varphi(x)\colon x \in \mathbb{R}^d\}$,

$$P[Z_x^M(1)(\{y\}) \ge 1] = 1 - e^{-\lambda\varphi(y-x)/v(M)} \le \lambda K/v(M) \tag{7.3}$$

and for M sufficiently large so that $e^{\lambda K/v(M)} \le 2$, by Taylor's theorem

$$\begin{aligned} &P[Z_x^M(1)(\{y\}) \ge 2] \\ &= e^{-\lambda\varphi(y-x)/v(M)}\left[e^{\lambda\varphi(y-x)/v(M)} - 1 - \lambda\varphi(y-x)/v(M)\right] \\ &\le (\lambda K/v(M))^2. \end{aligned} \tag{7.4}$$

We can now estimate $P(E_{p4}|B)$. Recall that occurrence of the event E_{p4} implies E_{p3} does *not* occur. Let $F_{p,r}$ be the event that for $x \in A_p$, and $n \le r$, no particle in generation n of the BRW $Z_x^M(\cdot)$ has more than k_1 offspring. For $1 \le r \le k$, if $F_{p,r-1}$ occurs, then

$$\sum_{x \in A_p} \sum_{n=0}^{r-1} Z_x^M(n)(\mathbb{R}^d) \le m(1 + k_1 + k_1^2 + \cdots + k_1^{r-1}) \le mk_1^r \tag{7.5}$$

(since $k_1 \ge 2$); that is, the total number of sites visited by the BRWs $Z_x^M(\cdot)$, $x \in A_p$, at generations before the rth, is at most mk_1^r. Also, if $F_{p,r-1}$ occurs,

$$\sum_{x \in A_p} Z_x^M(r-1)(\mathbb{R}^d) \le mk_1^{r-1}. \tag{7.6}$$

Hence by (7.3), given $F_{p,r-1}$ occurs, the probability that there exists $x \in A_p$ and a particle of $Z_x^M(r-1)$ which has an offspring at a site already visited at some time before r by $Z_y^M(\cdot)$ for some $y \in A_p$, is at most $m^2k_1^{2r-1}(\lambda K/v(M))$.

Let $v_1(M)$ denote the number of sites $x \in \mathbb{Z}^d/M$ with $\|x\| \le R$ (recall, R exceeds the range of φ). Given $F_{p,r-1}$ occurs, the probability that there exists a site y, and distinct sites x_1 and x_2, such that one of the BRW's $Z_x^M(\cdot)$, $x \in A_p$, visits y from x_1 at time r, and another (possibly the same) BRW $Z_{x'}^M(\cdot)$, $x' \in A_p$, visits y from x_2, also at time r, is, by (7.6) and (7.3), at most

$$\left(mk_1^{r-1}\right)^2 v_1(M)\left(\lambda K/v(M)\right)^2$$

[since the number of (x_1, x_2) is at most $(mk_1^{r-1})^2$ by (7.6), and for each (x_1, x_2) the number of y with $\|y - x_1\| \le R$ and $\|y - x_2\| \le R$ is at most $v_1(M)$].

Combining the estimates in the last two paragraphs and summing from $r = 1$ to $r = k$, we have for large enough M that

$$(7.7)\quad P\left[E_{p,4}|B\right] \le m^2k_1^{2k-1}\left(\lambda K/v(M) + \lambda^2K^2v_1(M)/(v(M))^2\right) < \varepsilon.$$

The estimate of $P(E_{p5}|B)$ is similar. Conditional on $F_{p,r-1}$, the probability that for some $x \in A_p$, some particle in the $(r-1)$th generation of $Z_x^M(\cdot)$ has two or more offspring in the same place, is by (7.4) at most $mk_1^{r-1}\lambda^2K^2v_1(M)/(v(M))^2$, for M large. Summing from $r = 1$ to $r = k$, we obtain for large M that

$$(7.8)\qquad P\left[E_{p5}|B\right] \le mk_1^k\lambda^2K^2v_1(M)/(v(M))^2 < \varepsilon.$$

Finally, we wish to estimate $P[E_{p6}|B]$. Suppose $q < p$. By the construction of the graph G_q in Algorithm 2, the number of endpoints of edges of G_q is at most mk_1^{k+1}. Also, all these points lie within a Euclidean distance at most kR from A_q, and hence from $B_{i(q),j(q)}$ (since the points of G_q are traced by a BRW starting from A_q, and running for only k steps, with each offspring distant at most R from its parent). Similarly, all points visited by $(X_x(n))_{n=0}^k$, $x \in A_p$, are distant at most kR from A_p. It follows that the BRW's $(X_x(n))_{n=0}^k$, $x \in A_p$ cannot visit any end points of G_q unless

$$\|(i(q), j(q), 0, \dots, 0) - (i(p), j(p), 0, \dots, 0)\| \le 2kR + 2,$$

in which case we shall say edge q is *feasible*. The number of feasible q is at most $2\pi(2kR + 3)^2$.

For feasible q, for $1 \le n \le k$, given that $F_{p,n-1}$ occurs there are at most mk_1^{n-1} descendants in the $(n-1)$th generation of the parent particles at the sites of A_p, and the probability that one of these has offspring at an end point of an edge of G_q is at most $(mk_1^{n-1})(mk_1^{k+1})\lambda K/v(M)$ by (7.3). Summing over n and over feasible q, we have (for large enough M) that

$$(7.9)\qquad P\left[E_{p6}|B\right] \le 2\pi(2kR + 3)^2m^2k_1^{2k+1}\lambda K/v(M) < \varepsilon.$$

By (7.1), (7.2), (7.7), (7.8) and (7.9) we obtain $P[\overline{E}_p|B] \le 1 - 5\varepsilon$. So

$$P\left[\overline{E}_p|E_{p,1}^c\right] \ge 1 - 5\varepsilon.$$

That is, for M large, given that there is a path of open bonds of $\mathscr{L}$ to the starting point of edge e_p, the probability $\overline{E}_p$ occurs exceeds $1 - 5\varepsilon$. By the choice of ε, the probability that G'_∞ is infinite is then nonzero. □

8. Proof of Theorem 2 when φ has bounded support. Assume $\lambda > 1$ (otherwise the result is trivial). Let $\delta > 0$. Take $\varepsilon > 0$ so small that (i) $5\varepsilon < \delta$, and (ii) for oriented percolation on $\mathscr{L}$ with parameter $1 - 5\varepsilon$, with probability exceeding $1 - \delta$ there is an infinite path from 0 of open bonds. This is possible; see Durrett [(1984), page 1026].

Using Lemma 2, choose m and k so that for large M, (3.1) and (3.2) hold. Using Lemma 3, choose k_0 so that for large enough M,

$$P_0\left[Z_{k_0}^M(b_{00}) \geq 2m\right] > \psi(\lambda) - \delta. \tag{8.1}$$

Set $k_2 = \max(k, k_0)$. Let k_1 be as in the proof of Theorem 1, but now make sure k_1 is so big that $mck_1^{k_2}(1/2)^{k_1} < \varepsilon$, where c is as in (7.2). Using the same prechosen ordering on $\mathbb{Z}^d/M$ as before, let G_0 be the graph traced out by a BRW $X_0^M(\cdot)$, with $X_0^M(0) = \delta_0$, running for time k_1, subject to the same modification as in Algorithm 2.

If at time k_1, $X_0^M(B_{00}) \geq 2m$, define A_{00} to consist of the first $2m$ atoms of $X_0^M(B_{00})$, set the site (0, 0) of $\mathscr{L}$ to be occupied, and proceed with Algorithm 2. Otherwise, stop.

By (8.1), together with estimates from the proof of Theorem 1, for M large,

$$P[(0,0) \text{ is occupied}] \geq \psi(\lambda) - \delta - 5\varepsilon \geq \psi(\lambda) - 2\delta$$

and

$$P[G'_\infty \text{ is finite} | (0,0) \text{ is occupied}] \leq \delta$$

by the choice of ε at the start of this proof. As before, a coupling argument then yields that for large M,

$$P[C(0) \text{ is infinite}] \geq P[G'_\infty \text{ is infinite}] \geq \psi(\lambda) - 3\delta,$$

so by making $\delta \to 0$, we have

$$\liminf_{M\to\infty} P[C(0) \text{ is infinite}] \geq \psi(\lambda). \tag{8.2}$$

Conversely, by a branching process argument, $P[C(0) \text{ is infinite}] \leq \psi_M(\lambda)$, where $\psi_M(\lambda)$ denotes the Galton–Watson survival probability when the offspring distribution is that of the random variable $\sum_{x \in \mathbb{Z}^d/M\setminus\{0\}} I_x$, where I_x are independently Binomial$(1, \lambda\varphi(x)/v(M))$. Since $\psi_M(\lambda) \to \psi(\lambda)$ as $M \to \infty$ [see Feller (1968), page 282], we have

$$\limsup_{M\to\infty} P[C(0) \text{ is infinite}] \leq \psi(\lambda) \tag{8.3}$$

and the proof is complete. □

9. Proofs of Theorems 1 and 2 when φ has unbounded support. It suffices to prove Theorem 2 for $\lambda > 1$. For $R \geq 0$, set $\varphi_R(x) = \varphi(x)$ if $0 < \|x\| \leq R$, $\varphi_R(x) = 0$ otherwise. Set $J_R = \int_{\mathbb{R}^d} \varphi_R(x)\,dx$ and set $v_R(M) = \sum_{x \in \mathbb{Z}^d/M} \varphi_R(x)$. Also, set $\varphi'(x) = \varphi_R(x)/J_R$, so φ' is a well-behaved p.d.f. of bounded support. The percolation process by which G is constructed is specified by the function φ and the parameters λ and M. Write $P_{\varphi,\lambda,M}$ for probability for particular values of these parameters.

Since for all x we have

$$(\lambda v_R(M)/v(M))(\varphi_R(x)/v_R(M)) \le \lambda\varphi(x)/v(M),$$

we have by a coupling argument that for all M,

$$(9.1) \quad P_{\varphi',(\lambda v_R(M)/v(M)),M}[C(0) \text{ is infinite}] \le P_{\varphi,\lambda,M}[C(0) \text{ is infinite}].$$

Since φ is assumed to be well behaved, $v_R(M)/v(M) \to J_R$ as $M \to \infty$. Also, the Galton–Watson survival probability $\psi(\lambda)$ is continuous and monotone in λ. Hence, by the fact that Theorem 2 holds for φ', the left-hand side of (9.1) converges to $\psi(\lambda J_R)$ as $M \to \infty$. Hence,

$$\liminf_{M\to\infty} P_{\varphi,\lambda,M}[C(0) \text{ is infinite}] \ge \psi(\lambda J_R).$$

Making $R \to \infty$ and again utilizing the continuity of ψ, we obtain (8.2). On the other hand, the proof of (8.3) in the last section is still valid when φ has unbounded support. □

10. Discussion: Site percolation and high dimensions. Let us now consider *site* percolation on $\mathbb{Z}^d/M$. Set φ to be the indicator function of the ball of unit volume centered at 0. Let elements of $\mathbb{Z}^d/M$ ("sites") be independently occupied with probability $\lambda/v(M)$, and let G(site) be the graph on the set of occupied sites obtained by including as edges all $\{x, y\}$ with x, y occupied and $\varphi(x-y) = 1$. Let λ_c(site) be the critical λ above which G(site) has an infinite component. Then one can show

$$\lim_{M\to\infty} \lambda_c(\text{site}) > 1.$$

The limit is not 1 this time because the site model does not have mean field behavior in the limit. Instead, it converges to the "Poisson blob model" [see Grimmett (1989)] and $\lambda_c(\text{site})M^d/v(M)$ (the critical density of occupied sites) converges to the (unknown) critical Poisson density for that model; that is, to $\rho_c(\varphi)$ in the notation of Section 2. In terms of the site percolation analog of Algorithm 1, Steps 7–10, when we consider a sequence of sites y with $y \sim x$ (for a given x already determined to be occupied), there is a nonvanishing proportion of the y's whose occupancy status has already been determined (this did not happen for bonds).

On the other hand, our methods can be adapted to rederive the result of Kesten (1990) that $\lambda_c(\text{site}) \to 1$ for nearest neighbor site percolation in $\mathbb{Z}^d$ as $d \to \infty$. Here is a sketch of how to do this. Let L: $\mathbb{Z}^d \to \mathbb{Z}^2$ denote the linear map

$$L(z^{(1)}, z^{(2)}, \dots, z^{(d)}) = \left(\sum_{i=1}^{[d/2]} z^{(i)}, \sum_{i=[d/2]+1}^{d} z^{(i)} \right).$$

The place of the set B_{ij} should be taken by $L^{-1}\{(i,j)\}$. Using a lattice version of the local limit theorem, we can derive an analog to Lemma 1. Given k, m and k_1, an estimate like that on E_{p6} in Section 7 can be made because the image under L of a random walk of at most k steps on $\mathbb{Z}^d$ is a random walk of at most k steps on $\mathbb{Z}^2$; hence as before, only finitely many q are feasible.

In the equivalent to Steps 7–10 of Algorithm 1, if x is a site which has just been determined to be occupied, then one should consider only sites y which differ from x in the jth coordinate, j being such that x has the same jth coordinate as every site previously determined to be occupied during stage p or during any stage q with $q < p$, q feasible. This ensures that site y cannot have already been considered; also, the number of coordinates j prohibited by this rule is at most c, where c is a constant depending only on k, m and k_1. So for large d, one considers a proportion close to 1 of the sites neighboring x.

11. A coupling of continuum and discrete models. We now turn to the continuum model. One method [used in Penrose (1991)] to set up the random graph $\overline{G}$ on $\mathscr{P}_0$ in the continuum model is to place the Poisson process $\mathscr{P}$ and independent uniform $[0, 1]$ random variables U_{xy}, $x, y \in \mathbb{R}^d$, on a single probability space, then include as edges of $\overline{G}$ those $\{X, Y\}$, $X, Y \in \mathscr{P}_0$, for which $U_{XY} < f(X - Y)$.

For $x \in \mathbb{R}^d$ write $z_M(x)$ for the site of $\mathbb{Z}^d/M$ which is closest (in terms of Euclidean distance) to x. The function z_M is well defined except on a set of measure 0. Given a realization of the rate ρ Poisson process $\mathscr{P}$ and the independent $U[0, 1]$ random variables U_{xy}, $x, y \in \mathbb{R}^d$, let us couple the continuum percolation process to a mixed site-bond percolation process on $\mathbb{Z}^d/M$ as follows.

Let us say a site $z \in \mathbb{Z}^d/M$ is *occupied* if there is exactly one point $X \in \mathscr{P}_0$ with $z = z_M(X)$. Let the random set of occupied sites in $\mathbb{Z}^d/M$ be denoted Oc. Let us say the edge $\{z, y\}$ of $\mathbb{Z}^d/M$ is *open* if and only if (i) the sites z and y are in Oc, and (ii) the particles, Z and Y say, of $\mathscr{P}_0$ for which $Z \in C_z$ and $Y \in C_y$, satisfy $U_{Z,Y} < f_M(z - y)$, where we set

$$(11.1)\quad f_M(\zeta) = \inf\{f(u - v) : \zeta = z_M(u), 0 = z_M(v)\}, \qquad \zeta \in \mathbb{Z}^d/M$$

[and we define $\varphi_M(\zeta)$, $\zeta \in \mathbb{Z}^d/M$ similarly]. Let H denote the random graph on Oc with edges given by the open bonds.

Thus the site at 0 of $\mathbb{Z}^d/M$ is occupied with probability $\exp\{-\rho M^{-d}\}$. With probability 1, 0 is occupied for sufficiently large M. All other sites of $\mathbb{Z}^d/M$ are independently occupied with probability $\rho M^{-d} \exp\{-\rho M^{-d}\}$, and the edge between occupied sites z and y of $\mathbb{Z}^d/M$ is open, independently of other pairs of sites, with probability $f_M(z - y)$.

The random graph H on the random set Oc is a realization of the following mixed Bernoulli site-bond percolation model on $\mathbb{Z}^d/M$: (i) The site 0 of $\mathbb{Z}^d/M$ is occupied with probability $\exp(-\rho M^{-d})$; other sites are occupied with probability $\rho M^{-d} \exp(-\rho M^{-d})$; (ii) for distinct sites z and y, the bond (edge) $\{z, y\}$ is open with probability $f_M(z - y)$; otherwise the bond $\{z, y\}$ is closed; (iii) all sites and bonds are mutually independent; and (iv) having performed (i)–(iii), we remove all open bonds except those joining two occupied sites, and the remaining open bonds form the edges of H.

12. Proof of Theorem 3. Let us say a point X of $\mathscr{P}$ is *kth-generation* if it is in $C(0)$ and the shortest path (in terms of number of edges) along edges of $\overline{G}$ from 0 to X has k steps (i.e., k edges). Let N_k denote the number of kth-generation points in $\mathscr{P}$.

In the coupled site-bond percolation process, let us say a site z in Oc is *kth-generation* if there is a path along open bonds and occupied sites from 0 to z, and the shortest such path has k steps (i.e., k open bonds). Let N_k^M denote the number of kth-generation sites in $\mathbb{Z}^d/M$.

LEMMA 4. *Suppose f is continuous. Then for each positive integer k,*

$$N_k^M \to N_k \quad \textit{almost surely as } M \to \infty.$$

PROOF. Consider the case $k = 1$. For each $X \in \mathscr{P}$, if $\{0, X\} \in \overline{G}$ then $U_{0X} < f(X)$, so that for sufficiently large M, $U_{0X} < f_M(z_M(X))$. Also, with probability 1, for sufficiently large M there is no $Y \in \mathscr{P}$ distinct from X with $z_M(Y) = z_M(X)$ or $z_M(Y) = 0$. In short, for large M, for every edge from 0 in $\overline{G}$ there is a corresponding edge from 0 in H.

Conversely, if $\{0, z\}$ is an edge of H, then there is some $X \in \mathscr{P}$ with $z = z_M(X)$ and $U_{0X} \le f_M(z) \le f(X)$, so that $\{0, X\}$ is an edge of G_0. Hence, for large enough M there is a natural one-to-one correspondence between edges from 0 in G_0 and edges from 0 in H. This implies that with probability 1, $N_1^M = N_1$ for large enough M.

For a given positive integer j, let a sequence of j edges of $\overline{G}$ (resp., H) forming a path from 0 to some vertex x and not containing any loops, be denoted a *j-path in $\overline{G}$* (resp., a *j-path in H*). By a similar argument to the above, with probability 1 we have that for large M there is a natural one-to-one correspondence between j-paths in $\overline{G}$ and j-paths in H.

For each integer k, the number N_k (resp., N_k^M) is determined by the set of j-paths in G (resp., the set of j-paths in H), $1 \le j \le k$. So the remark in the previous paragraph implies that with probability 1, $N_k^M = N_k$ for large M. □

PROOF OF THEOREM 3. First suppose f is continuous. Then by dominated convergence,

$$\rho M^{-d} \sum_{z \in (\mathbb{Z}^d/M)\setminus\{0\}} f_M(z) \to \rho \int f(x)\, dx = \lambda \quad \text{as } M \to \infty. \tag{12.1}$$

Let k be a positive integer. Conditional on the value of N_{k-1}^M, the value of N_k^M is stochastically dominated by the sum of N_{k-1}^M independent random variables, each distributed as $\sum_{z \in (\mathbb{Z}^d/M)\setminus\{0\}} I_z$, where I_z are independently

Binomial(1, $f_M(z)\rho M^{-d}\exp(-\rho M^{-d})$). Setting λ_M to be the left-hand side of (12.1), we have

$$E\left[N_k^M | N_{k-1}^M\right] \le \lambda_M N_{k-1}^M,$$

so

$$E\left[N_k^M\right] \le \lambda_M^k.$$

By Fatou's lemma, $E(N_k) \le \lambda^k$, which implies (i) of Theorem 3.

By the above stochastic domination, we obtain

$$P\left[N_k^M > 0\right] \le P\left[\Xi_k^M > 0\right],$$

where Ξ_k^M, $k = 0, 1, 2, \dots,$ is a Galton–Watson branching process with offspring distribution that of $\sum_{z \in (\mathbb{Z}^d/M)\setminus\{0\}} I_z$, and with $\Xi_0^M = 1$. As $M \to \infty$ the distribution of $\sum_{z \in (\mathbb{Z}^d/M)\setminus\{0\}} I_z$ converges to a Poisson(λ) distribution; hence by Lemma 4,

$$P[N_k > 0] \le P[\Xi_k > 0],$$

where (Ξ_k) is a Galton–Watson process with Poisson(λ) offspring distribution. Now let $k \to \infty$ to obtain (ii) of Theorem 3.

Finally, we may drop the assumption that f is continuous, since the fact that f is bounded and Riemann integrable implies that for $\varepsilon > 0$ there exists a continuous function g on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} g(x)\,dx < \int_{\mathbb{R}^d} f(x)\,dx + \varepsilon$, and $g \ge f$ on $\mathbb{R}^d$. □

13. A site-bond percolation algorithm. We now turn to the proof of Theorem 4. Assume for now that $\varphi \in C_0(\mathbb{R}^d)$ (continuous with bounded support). Take R so $\varphi(x) = 0$ if $\|x\| > R - 1$. By Theorem 3, for all ρ we have $\lambda_c(\rho) \ge 1$. From now on, λ is fixed with $\lambda > 1$. The next few sections are devoted to showing that for large ρ, $P_\rho[\#(C(0)) = \infty] > 0$.

Choose $\lambda' \in (1, \lambda)$. Choose a function $(M_\rho,\ \rho > 0)$ in such a way that $\rho M_\rho^{-d} \to 0$ and $\rho^2 M_\rho^{-d} \to \infty$ as $\rho \to \infty$. From now on, we shall drop the subscript and write M for M_ρ. Note we shall now be making ρ and M approach ∞ in a linked manner, whereas in the last section we made $M \to \infty$ with ρ fixed.

Let the graph H on Oc be as constructed in Section 11; that is, H is a realization of a site-bond percolation process which is coupled to $\overline{G}$. By the construction of H, we have that if $0 \in$ Oc, and the component of H including the site at 0 is infinite, then so is $C(0)$ in the continuum model.

As in the proof of Theorem 1, choose an ordering on $\mathbb{Z}^d/M$. Let A_{00} be an arbitrary subset of $(\mathbb{Z}^d/M) \cap B_{00}$ with $|A_{00}| = 2m$. The following random algorithm generates a set of occupied sites, denoted S, and a set of vacant sites, denoted V, in $\mathbb{Z}^d/M$. It also generates a set of open bonds between sites of $\mathbb{Z}^d/M$. At any stage of the algorithm, any site which is neither in S nor in V has its status as yet undetermined.

As before, the algorithm also generates occupied and vacant sites on $\mathscr{L}$, and open and closed bonds on $\mathscr{L}$. Initially, set the site $(0, 0)$ of $\mathscr{L}$ to be occupied, other sites to be vacant and all bonds of $\mathscr{L}$ to be closed. Define the number $v_2(M)$ by $v_2(M) = \sum_{z \in (\mathbb{Z}^d/M)\setminus\{0\}} \varphi_M(z)$, with φ_M defined as in (11.1).

ALGORITHM 3.

STEP 1. Set $p = 1$. Set $S = \{0\} \cup A_{00}$. Set all bonds $\{0, x\}$, $x \in A_{00}$ to be open.

STEP 2. Set $i = i(p)$, $j = j(p)$.

STEP 3. If the site (i, j) of $\mathscr{L}$ is occupied, go to Step 4. Otherwise, go to Step 11.

STEP 4. If p is odd (resp., even), let the set A_p consist of the first m (resp., the last m) elements of A_{ij}. Set $D_{p,0} = A_p$. Set $D_{p,r}$ to be the empty set, $1 \le r \le k$. Set $n = 0$.

STEP 5. Consider the first site x in $D_{p,n}$.

STEP 6. Let $N(x) \sim \text{Poisson}(\lambda' M^d/\rho)$. Let x have $N(x)$ "potential open bonds" (POB's) attached to it, each independently placed in $(\mathbb{Z}^d/M)\setminus\{x\}$ according to the probability mass function $\varphi_M(\cdot - x)/v_2(M)$. If two or more POB's are on the same edge, remove them both. If any POB goes to a site already in S, remove that POB. If there are more than $2\lambda M^d/\rho$ POB's remaining, remove all but the first $[2\lambda M^d/\rho]$ of them. Take the remaining POB's to be actually open.

Then look at each site y at the end of an open bond from x, such that $y \notin V$. Make y a "potentially occupied site" (POS) with probability $\rho M^{-d} \exp(-\rho M^{-d})$; otherwise place y in V. If at most k_1 sites y are made POS's in this way, add them all to S and to $D_{p,n+1}$; otherwise add the first k_1 of the POS's to S and to $D_{p,n+1}$.

STEP 7. Consider the next x in $D_{p,n}$ (if there is such an x), and return to Step 6. If there is no next x in $D_{p,n}$, go to Step 8.

STEP 8. If $n < k - 1$, increase n by 1 and go to Step 5. Otherwise, increase n by 1 and go to Step 9.

STEP 9. Suppose p is even, and $|D_{p,k} \cap B_{i+1,j+1}| \ge 2m$. Then change the status of site $(i+1, j+1)$ of $\mathscr{L}$ to "occupied" and that of bond e_p of $\mathscr{L}$ to "open," and let $A_{i+1,j+1}$ consist of the first $2m$ elements of $D_{p,k} \cap B_{i+1,j+1}$.

STEP 10. Suppose p is odd, and $|D_{p,k} \cap B_{i+1,j-1}| > 2m$. Then change the status of site $(i+1, j-1)$ to "occupied" (if it was not already occupied) and that of bond e_p to "open." If $A_{i+1,j-1}$ has not yet been defined, let $A_{i+1,j-1}$ consist of the first $2m$ elements of $D_{p,k} \cap B_{i+1,j-1}$.

STEP 11. Increase p by 1, and return to Step 2.

For a particular occupied $x \in \mathbb{Z}^d/M$, and vacant $y \in \mathbb{Z}^d/M$, let N_{xy} denote the number of POB's from x assigned to the edge $\{x, y\}$ at Step 6. By the property of the compound Poisson distribution used before, $N_{xy} \sim$ Poisson$(\eta_M(x, y))$, where we set

$$(13.1) \qquad \eta_M(x,y) = (\lambda' M^d/\rho)\varphi_M(y-x)/v_2(M).$$

Also, N_{xy} is independent of all N_{xz}, $z \neq y$. We assumed $\varphi \in C_0(\mathbb{R}^d)$; hence, $M^{-d}v_2(M) \to \int \varphi(x)\,dx = 1$ as $M \to \infty$ [see (12.1)]. Since $\lambda' < \lambda$ we have for all large enough M that for all x and y,

$$(13.2) \qquad P[N_{xy} = 1] \leq \lambda\varphi_M(y-x)/\rho = f_M(y-x).$$

That is, in Algorithm 3 the probability of including a given edge is no greater than in the construction of H in Section 11. Also, the algorithm ensures that no edge gets more than one chance to be made open, and no site gets more than one chance to be placed in S.

Thus by similar reasoning to that in Section 4, the set S may be viewed as a subset of the set Oc in the mixed site-bond percolation model described in Section 11. Also, the set of open bonds on $\mathbb{Z}^d/M$ joining occupied sites produced by the algorithm may be regarded as a subset of the set of open bonds in H.

So the probability that an infinite path from 0 of sites in S and open bonds is produced by this algorithm is a lower bound for the probability that the component of H including 0 in the site-bond percolation algorithm is infinite, conditional on all sites in $\{0\} \cup A_{00}$ being occupied, and all bonds of the form $\{0, x\}$, $x \in A_{00}$, being open. As before we shall show that the algorithm can produce an infinite cluster with nonzero probability by comparison with a BRW.

14. A two-stage BRW mechanism. Let $x \in \mathbb{Z}^d/M$. Let

$$((Y_x(n), Z_x(n), V_x(n)), 1 \leq n \leq k)$$

be a two-stage branching random walk given by the rules below; here $Z_x(\cdot)$ and $V_x(\cdot)$ take values in $\mathscr{M}^M$, while $Y_x(\cdot)$ takes values in the space of counting measures on the set of edges between elements of $\mathbb{Z}^d/M$. Roughly, if $x \in A_p$ at the nth generation, $Y_x(n)$ is the set of POB's descended from x, $Z_x(n)$ is the set of POS's descended from x and $V_x(n)$ is the set of vacant sites descended from x.

(i) Set $Z_x(0) = \delta_x$.

(ii) For $1 \leq n \leq k$, create $Y_x(n)$ by making each atom of $Z_x(n-1)$, at y say, give birth to a Poisson$(\lambda' M^d/\rho)$ number of edges, each independently distributed over the set of edges $\{y, z\}$, $z \neq y$, according to the probability function assigning mass $\varphi_M(z-y)/v_2(M)$ to edge $\{y, z\}$ [recall $v_2(M) = \sum_{(\mathbb{Z}^d/M)\setminus\{0\}} \varphi_M$].

(iii) If an atom at y of $Z_x(n-1)$ creates an offspring edge of $Y_x(n)$ at $\{y, z\}$, let the site z acquire an atom of $Z_x(n)$ with probability $\rho M^{-d} \exp(-\rho M^{-d})$, and acquire an atom of $V_x(n)$ otherwise.

It should be clear that in this process, $(Z_x(n))$ is simply a BRW on $\mathbb{Z}^d/M$, with a Poisson($\lambda' \exp(-\rho M^{-d})$) offspring distribution, and offspring of a particle at y independently distributed over $(\mathbb{Z}^d/M) \setminus \{y\}$ according to the probability mass function $\varphi_M(\cdot - y)/v_2(M)$.

On a probability space let $((Y_x(n), Z_x(n), V_x(n)),\ 1 \le n \le k),\ x \in \mathbb{Z}^d/M$, be independent two-stage branching random walks defined as above. On this probability space construct a sequence of modified BRW's $(Y_p'(\cdot), Z_p'(\cdot), V_p'(\cdot))$, $p \ge 1$, by the following algorithm. Specify A_{00}, and the initial status of sites and bonds of $\mathscr{L}$, in the same way as in Algorithm 3. Also, initially set $Z_0'(0) = \delta_0$, $Y_1'(0) = \Sigma_{x \in A_{00}} \delta_{\{x,y\}}$.

ALGORITHM 4.

STEP 1. Set $p = 1$.

STEP 2. Set $i = i(p)$ and $j = j(p)$.

STEP 3. If the site (i, j) of $\mathscr{L}$ is vacant, go to Step 8. If site (i, j) is occupied, go on to Step 4.

STEP 4. If p is odd (resp., even), let A_p consist of the first (resp., the last) m elements of A_{ij}.

STEP 5. Let $(Y_p'(n), Z_p'(n), V_p'(n),\ n = 1, 2, \dots, k)$ be the multistage BRW on $\mathbb{Z}^d/M$, obtained by aggregating the multistage BRW's $(Y_x(n), Z_x(n), V_x(n))_{n=1}^k$ over $x \in A_p$ [so $Z_p'(0) = \Sigma_{x \in A_p} \delta_x$], subject to the following modifications:

(i) If an nth generation particle of $Z_p'(n)$, $0 \le n < k$, at y say, gives birth to 2 or more (edge-valued) offspring in $Y_p'(n+1)$ on the edge $\{y, z\}$ for some z, then remove these offspring from $Y_p'(n+1)$ (and remove all their descendants).

(ii) If an nth generation particle of $Z_p'(n)$, $0 \le n < k$, at y say, gives birth to an offspring in $Y_p'(n+1)$ on the edge $\{y, z\}$ for some site z which was already determined to be occupied [i.e., $z \in Z_q'(n')$, some $0 \le n' \le k$, $0 \le q < p$ or $z \in Z_p'(r)$, $0 \le r \le n$ or z is the site of a descendant in $Z_p'(n+1)$ of some particle at $y' \in Z_p'(n)$ which came before y in our ordering on $\mathbb{Z}^d/M$], then remove the edge $\{y, z\}$ from $Y_p'(n+1)$ and remove its subsequent offspring.

(iii) If a particle of $Z_p'(n)$, $0 \le n < k$, has offspring of $Y_p'(n+1)$ on more than $2\lambda M^d/\rho$ bonds [after carrying out steps (i) and (ii)], remove all but those on the first $[2\lambda M^d/\rho]$ of these bonds (using the prechosen ordering on edges of $\mathbb{Z}^d/M$) from $Y_p'(n+1)$ (and remove their subsequent offspring).

(iv) If a particle of $Z'_p(n)$, $0 \le n < k$, has offspring of $Z'_p(n+1)$ in more than k_1 positions, remove all but those in the first k_1 of these positions (along with their subsequent offspring).

(v) If a particle of $Z'_p(n)$, $0 \le n < k$, at y say, gives birth to an offspring in $Z'_p(n+1)$ at z, for some site z which was already determined to be vacant [i.e., $z \in V'_q(n')$, some $1 \le n' \le k$, $1 \le q < p$ or $z \in V'_p(r)$, $0 \le r \le n$ or z is the site of a descendant in $V'_p(n+1)$ of some particle at $y' \in Z'_p(n)$ which came before y in our ordering on $\mathbb{Z}^d/M$], then remove that site z from $Z'_p(n+1)$ and remove its subsequent offspring.

STEP 6. Suppose that p is even, and that $Z'_p(k)$ places $2m$ or more particles in $B_{i+1,j+1}$. Then change the status of the bond $e_p = e_{ij+}$ of $\mathscr{L}$ to "occupied," and change the status of the site $(i+1, j+1)$ of $\mathscr{L}$ to "open." Also define $A_{i+1,j+1}$ to consist of the sites of first $2m$ of these particles (in the prechosen ordering on $\mathbb{Z}^d/M$).

STEP 7. Suppose that p is odd, and that $Z'_p(k)$ places $2m$ or more particles in $B_{i+1,j-1}$. Then change the status of the bond $e_p = e_{ij-}$ of $\mathscr{L}$ to "open"; also, *if* $(i+1, j-1)$ is vacant (which implies $A_{i+1,j-1}$ has not yet been defined), change its status to "occupied" and define $A_{i+1,j-1}$ to consist of the first $2m$ of these particles.

STEP 8. Increase p by 1, and return to Step 2.

After running this algorithm, let S be the set of all sites which were included in $Z'_p(n)$ for some $p \ge 0$ and some $n \in \{0, 1, \dots, k\}$. On examination we find that the set S of occupied sites and the set of open bonds [those in $Y'_p(n)$ for some $p \ge 0$ and some n, $1 \le n \le k$] have the same joint distribution as the set S and the set of open bonds generated by Algorithm 3.

15. Proofs of Theorems 4 and 5.

PROOF OF THEOREM 4. First assume $\varphi \in C_0(\mathbb{R}^d)$; take R so $\varphi(x) = 0$, $\|x\| \ge R - 1$. Let $\varepsilon > 0$ be so small that for oriented Bernoulli percolation on $\mathscr{L}$ with each bond open with probability $1 - 5\varepsilon$, there is an infinite path from 0 with nonzero probability. Choose k and m so that under the hypothesis of Lemma 2, with λ replaced by λ', (3.1) and (3.2) hold for large M. Recall $N(x)$ is the number of POB's from x. For each occupied site x, $\mathrm{Var}[N(x)] \le 2\lambda M^d/\rho$, and by Chebyshev's inequality, for large enough ρ,

$$P\left[N(x) \ge 2\lambda M^d/\rho\right] \le P\left[|N(x) - EN(x)| \ge \lambda M^d/\rho\right] \le \rho/(\lambda M^d) < \varepsilon. \tag{15.1}$$

By the property of the compound Poisson distribution used earlier, the number of POS's attached to open bonds from x is dominated by a Poisson(λ)

random variable denoted $N'(x)$. As argued earlier [see (7.2)], for suitably large k_1,

$$P[N'(x) > k_1] \le \varepsilon/(mk_1^k). \tag{15.2}$$

For a particular $y \in \mathbb{Z}^d/M$, $y \notin S$, let N_{xy} denote the number of POB's from x assigned to the edge $\{x, y\}$. Then $N_{xy} \sim \text{Poisson}(\eta_M(x,y))$, $\eta_M(x,y)$ being given by (13.1), and $P[N_{xy} = 1] \le \lambda\varphi_M(y-x)/\rho$ by (13.2). So the probability that the bond $\{x, y\}$ is made open *and* y is made a POS is at most $\lambda\varphi_M(y-x)/M^d$. Also, the set of sites y with $\|y - x\| \le R$ and y already vacant is contained in the set of sites of $V_q'(r)$, for some $q \le p$, $r \le k$ and q feasible in the sense that $\|(i(q), j(q), 0, 0, \dots, 0) - (i(p), j(p), 0, 0, \dots, 0)\| \le 2kR + 2$. We have

$$\left| \sum_{\substack{q \le p \\ q \text{ feasible}}} \sum_{r \le k} V_q'(r)(\mathbb{R}^d) \right| \le 2m\pi(2kR+3)^2 k_1^{k+1}(2\lambda M^d/\rho).$$

Hence the probability of there being a POS from x at one of these sites is at most $4m\pi(2kR+3)^2 k_1^{k+1}\lambda^2 K/\rho$, where we set $K = \sup\{\varphi(x)\colon x \in \mathbb{R}^d\}$ as before.

For any fixed y, for ρ large enough, $P[N_{xy} \ge 2] \le (\lambda K/\rho)^2$, and so

$$P\left[\bigcup_{y \in \mathbb{Z}^d \setminus \text{Oc},\ \|y-x\| \le R} \{N_{xy} \ge 2\}\right] \le \text{const.}\, M^d/\rho^2 < \varepsilon$$

for ρ large (by the choice of the function M_ρ).

By similar estimates to those in the proof of Theorem 1, for the chosen values of m, k and k_1 and for large enough ρ, at stage p of the algorithm the probability that any of the mechanisms for removal of POS's occurs is at most 5ε; also, if none of these mechanisms occurs, by Lemma 2 the probability that edge e_p is not made open is at most ε. Hence, the probability of an infinite cluster is nonzero for large ρ. The case $\varphi \notin C_0(\mathbb{R}^d)$ is considered below. □

PROOF OF THEOREM 5. By Theorem 3, $P_\rho[\#(C(0)) = \infty] \le \varphi(\lambda)$, and for $\varphi \in C_0(\mathbb{R}^d)$ the opposite inequality follows from a combination of the ideas of the above section with those of the proof of Theorem 2; we omit the details.

Now suppose φ has unbounded support, but φ is Riemann integrable. Then for $\varepsilon > 0$ there exists $\varphi' \in C_0(\mathbb{R}^d)$ with $\varphi' \le \varphi$ everywhere and $\int\varphi' \ge 1 - \varepsilon$. By application of Theorem 5 to φ' we have that when $\rho \to \infty$ with λ fixed,

$$\liminf P_\rho[\#(C(0)) = \infty] \ge \psi(\lambda(1-\varepsilon)).$$

Since ε is arbitrary, the proof of Theorems 4 and 5 is now complete. □

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DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106-3110