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KNUDSEN GAS IN FLAT TIRE

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We consider random reflections (according to the Lambertian distribution) of a light ray in a thin variable width (but almost circular) tube. As the width of the tube goes to zero, properly rescaled angular component of the light ray position converges in distribution to a diffusion whose parameters (diffusivity and drift) are given explicitly in terms of the tube width.

1. Introduction. We will prove an invariance principle for a light ray reflecting inside a very thin variable width (but almost circular) planar domain. The reflections are random and have the Lambertian distribution introduced in [16]. An alternative physical representation of the process is that of a gas molecule with a velocity so high that the effect of the gravitation is negligible. In this alternative context, the Lambertian distribution is known as Knudsen's law, introduced in [15].

We will now present a (very) informal version of our main result. Consider a smooth function $h : \mathbb{R} \to [1, 3]$ with period 2π. For each $\varepsilon \in (0, 1/100)$, consider a planar domain $\mathcal{D}_\varepsilon$ that is very close to a thin annulus with the center $(0, 0)$ and radii close to 1, except that its width is $\varepsilon h(\alpha)$, where α measures the angle along the tube in radians. Suppose that a light ray travels inside $\mathcal{D}_\varepsilon$ and reflects randomly according to the Lambertian distribution, that is, the direction of the reflected trajectory forms an angle Θ with the inner normal to the boundary of $\mathcal{D}_\varepsilon$ and the density of Θ is proportional to $\cos\theta$. The directions of reflections are independent. If $\boldsymbol{\beta}^\varepsilon(t)$ denotes the angular coordinate of the light ray in the polar coordinates at time t, then properly rescaled process $\{\boldsymbol{\beta}^\varepsilon(t), t \geq 0\}$ converges in the Skorokhod topology, as ε goes to 0, to the solution of

$$dX_t = h'(X_t)\,dt + \sqrt{h(X_t)}\,dW_t, \tag{1.1}$$

where W is standard Brownian motion.

We will now discuss related results and motivation for this research.

The idea of multidimensional processes converging in distribution to a process on a lower dimensional manifold goes back at least to Katzenberger [14]. Roughly

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speaking, such convergence can be induced by a strong drift keeping multidimensional processes close to the manifold.

The reflection problem in thin domains was investigated in [13, 20]. More specifically, this research was devoted to eigenfunctions of the Laplacian with Neumann boundary conditions. It was proved that when the width of the domain goes to 0, the eigenfunctions converge to those of a one-dimensional Sturm–Liouville operator. In our notation, the limiting operator could be expressed as $\Delta + \frac{h'(x)}{h(x)} \frac{d}{dx}$. This is strikingly close to (1.1) in the following sense. We could time change the diffusion in (1.1) so that it has the quadratic variation equal to 1. Then the time-changed process would correspond to the operator $\frac{1}{2}\Delta + \frac{h'(x)}{h(x)} \frac{d}{dx}$. Whether the usual probabilistic factor $\frac{1}{2}$ in front of the Laplacian is a real difference between the two operators or whether the two operators are actually equal under proper scaling, we are not able to determine due to considerable differences in the presentations of the models in [13, 20] and in our paper. Either way, we consider it remarkable that significantly different families of processes (reflected Brownian motions and Knudsen random walks) have limits that are so closely related.

There has been recently interest in billiards in fractal domains. The authors of [17, 18] take the "classical" approach in which the reflection is specular, that is, the angle of reflection is the same as the angle of incidence. This idea can be applied in "prefractals" approximating, for example, the von Koch snowflake, and then one can hope to pass to the limit, in some sense. Another approach, based on Lambertian reflections, was taken in [6, 7]. It was proved in [1] that Lambertian reflections are the only physically realistic reflections if the distribution of the reflected path does not depend on the location of reflection and the incidence angle. As a prelude to the study of fractal domains, the authors of [3, 4] investigated Lambertian reflections in thin tubes; this shed a light on the distribution of light rays leaving crevices in fractal domains. The present project may be considered as a continuation of [3, 4] although no invariance principle was proved in those papers.

The present article is focused on two-dimensional domains only, as a result of research results in [3, 4]. It was shown in [3] that Knudsen's random walk in a two-dimensional tube has steps with infinite variance but the step distribution is nevertheless in the domain of attraction of the normal law—a rare occurrence in probability literature. The variance of the steps is finite in dimensions 3 and higher, so less interesting (see [4]). Moreover, formulas become cumbersome in higher dimensions. The same remarks explain why we put our process inside a circular tube rather than a straight tube with variable width. In the latter case, steps could have infinite variance or, in some cases, the light ray could escape to infinity in one go.

The invariance principle or, at least, the central limit theorem, for billiards has received some attention when the reflection is random (see [7], Theorems 2.1 and 2.2, or [5], Theorem 3, for the case when the reflecting angle is chosen among finitely many) or deterministic when the domain has cusps (see [2]). In those invariance principles, the domain is fixed and time is accelerated.

Interest in stochastic billiards arose when researchers started to investigate deterministic billiards with microscopic irregularities at the boundary (see, e.g., [8, 11, 12]). Instead of zooming in on these irregularities to do deterministic analysis, the idea was to consider irregularities as points of random refections. It turns out that the Lambertian distribution is the invariant and ergodic probability measure for such random processes, in an appropriate sense (see, e.g., [8, 9]).

On the technical side, we will use two classical versions of the invariance principle, available in [10]. The main effort will be in verifying the assumptions of those theorems. The ballistic character of our process and the smoothness of the boundary make the calculations harder than in the Brownian case—a situation that seems paradoxical but it is well known in other contexts.

1.1. *Organization of the paper.* Sections 2–3 are devoted to the simplified model, in which the domain is a true annulus, that is, its two parts of the boundary are concentric circles. This may be helpful to the reader as our general result, presented and proved in Sections 4–5, has a proof that contains many details which obscure the basic strategy.

We would like to point out Proposition 1, a result that may have a separate interest. It holds only in the case when the domain is a true annulus.

2. Reflections in an annulus: Model and results. Given $r > 0$ and $\varepsilon \in (0, 1)$, let

$$\mathcal{D}(\varepsilon, r) = \{(x, y) \in \mathbb{R}^2 : (r - \varepsilon)^2 \leq x^2 + y^2 \leq r^2\}. \tag{2.1}$$

We will use $\mathcal{C}((x, y), r)$ to denote the circle with center (x, y) and radius r. We will refer to $\mathcal{C}_{\text{int}} := \mathcal{C}((0, 0), r - \varepsilon)$ as the inner boundary of $\mathcal{D}(\varepsilon, r)$ and to $\mathcal{C}_{\text{out}} := \mathcal{C}((0, 0), r)$ as the outer boundary of $\mathcal{D}(\varepsilon, r)$.

We will consider a ray of light traveling inside $\mathcal{D}(\varepsilon, r)$ and reflecting from the boundary. Its position at time $t \geq 0$ will be denoted

$$Q(t) = \mathbf{r}(t)\big(\cos \boldsymbol{\beta}(t), \sin \boldsymbol{\beta}(t)\big). \tag{2.2}$$

We give a label to the following assumption for later reference:

(**A**) We will assume that the light ray always travels with speed 1. Every time the light ray is reflected, the reflection angle is independent from the past trajectory and has the Lambertian distribution, that is, the reflection angle Θ with respect to the inner normal vector at the point of reflection has the probability density given by

$$\mathbb{P}(\Theta \in d\theta) = \frac{1}{2} \cos(\theta)\, d\theta \qquad \text{for } \theta \in (-\pi/2, \pi/2). \tag{2.3}$$

It is easy to see that the light ray process is invariant under scaling, that is, if the process in $\mathcal{D}(\varepsilon, r)$ is denoted $\{\mathbf{r}(t)(\cos\boldsymbol{\beta}(t), \sin\boldsymbol{\beta}(t)), t \geq 0\}$ then for $c > 0$,

$$\{c\mathbf{r}(t/c)(\cos\boldsymbol{\beta}(t/c), \sin\boldsymbol{\beta}(t/c)), t \geq 0\}$$

is the analogous process in $\mathcal{D}(c\varepsilon, cr)$. For this reason, we will assume that the light ray travels inside $\mathcal{D}(\varepsilon, 1)$ in Sections 2–3. Since $\varepsilon > 0$ remains the only parameter, we will incorporate it in the notation by writing

$$\{\mathbf{r}^\varepsilon(t)(\cos\boldsymbol{\beta}^\varepsilon(t), \sin\boldsymbol{\beta}^\varepsilon(t)), t \geq 0\}.$$

We now state our main result on reflections in an annulus.

THEOREM 1. *Processes $\{\boldsymbol{\beta}^\varepsilon(\frac{\pi}{\varepsilon\log(1/\varepsilon)}t), t \geq 0\}$ converge in law to Brownian motion in the Skorokhod topology as ε goes to* 0.

The proof will be given at the end of Section 3.

3. Reflections in an annulus: Proofs. We start with some notation. We will write $\mathbf{1}_a(b) = 1$ if $a = b$ and $\mathbf{1}_a(b) = 0$ otherwise. Similarly, for a set A, we will say $\mathbf{1}_A(b) = 1$ if $b \in A$ and $\mathbf{1}_A(b) = 0$ otherwise.

We will define a number of objects needed in the proofs. We will assume that the light ray is on the boundary of $\mathcal{D}(\varepsilon, 1)$ at time $t = 0$, as it clearly does not affect the validity of Theorem 1.

We will encode the nth reflection point as

$$(1 - \mathbf{s}_n^\varepsilon\varepsilon)(\cos(\boldsymbol{\alpha}_n^\varepsilon), \sin(\boldsymbol{\alpha}_n^\varepsilon)), \tag{3.1}$$

where $\mathbf{s}_n^\varepsilon$ can be 0 or 1, and $\boldsymbol{\alpha}_n^\varepsilon \in \mathbb{R}$ is chosen for $n \geq 0$ so that $|\boldsymbol{\alpha}_{n+1}^\varepsilon - \boldsymbol{\alpha}_n^\varepsilon| < \pi$. By convention, the first reflection occurs at time $t = 0$.

It is clear that $\{(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon), n \geq 0\}$ is a time homogeneous discrete time Markov chain.

Since the light ray travels with speed 1, the time between the kth and $(k-1)$st reflections can be calculated as

$$\begin{aligned}\Delta\mathcal{T}_k^\varepsilon := \big|&(1 - \mathbf{s}_k^\varepsilon\varepsilon)(\cos(\boldsymbol{\alpha}_k^\varepsilon), \sin(\boldsymbol{\alpha}_k^\varepsilon)) \\ &- (1 - \mathbf{s}_{k-1}^\varepsilon\varepsilon)(\cos(\boldsymbol{\alpha}_{k-1}^\varepsilon), \sin(\boldsymbol{\alpha}_{k-1}^\varepsilon))\big|.\end{aligned} \tag{3.2}$$

Set $\mathcal{T}_0^\varepsilon = 0$ and, for $n \geq 1$,

$$\mathcal{T}_n^\varepsilon = \sum_{k=1}^n \Delta\mathcal{T}_k^\varepsilon. \tag{3.3}$$

Given $t > 0$ and $\varepsilon \in (0, 1)$, let

$$N^\varepsilon(t) = \inf\{n \geq 0 : \mathcal{T}_{n+1}^\varepsilon > t\} = \sup\{n \geq 0 : \mathcal{T}_n^\varepsilon \leq t\}. \tag{3.4}$$

Then $N^\varepsilon(t)$ is the number of reflections made by the light ray before time t, while $\mathcal{T}_n^\varepsilon$ represents the time of the nth reflection. With this notation, using (2.2) we can rewrite (3.1) as

$$Q(\mathcal{T}_n^\varepsilon) = (1 - \mathbf{s}_n^\varepsilon \varepsilon)(\cos(\boldsymbol{\alpha}_n^\varepsilon), \sin(\boldsymbol{\alpha}_n^\varepsilon)). \tag{3.5}$$

We will derive formulas linking the angle of reflection Θ with the increment of angle $\boldsymbol{\beta}$ between reflections. Since $\{(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon), n \geq 0\}$ is a time homogeneous Markov chain, it will suffice to analyze $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$. By rotation invariance of the process, we may and will assume without loss of generality that $\boldsymbol{\alpha}_0^\varepsilon = \pi/2$. Set

$$a = a(\theta) = \tan\left(\frac{\pi}{2} - \theta\right) = 1/\tan(\theta) = \cot(\theta). \tag{3.6}$$

Recall that Θ denotes the reflection angle at the first reflection point sampled from the cosine law.

Suppose that $\mathbf{s}_0^\varepsilon = 1$, that is, the light ray starts at the inner circle. Then the next reflection must be on the outer circle. If $\Theta = 0$, then $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon = 0$.

We will denote the coordinates of the second reflection point $(x, y) = (x(\Theta), y(\Theta))$, that is,

$$(x(\Theta), y(\Theta)) = (1 - \mathbf{s}_1^\varepsilon \varepsilon)(\cos(\boldsymbol{\alpha}_1^\varepsilon), \sin(\boldsymbol{\alpha}_1^\varepsilon)). \tag{3.7}$$

Then

$$\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon = \arctan\left(\frac{x(\Theta)}{y(\Theta)}\right) = \arctan\left(\frac{x(\Theta)}{a(\Theta)x(\Theta) + 1 - \varepsilon}\right). \tag{3.8}$$

If $\Theta \neq 0$, then $x = x(\Theta)$ is the solution of

$$x^2 + y^2 = x^2 + (ax + 1 - \varepsilon)^2 = 1 \tag{3.9}$$

such that $xa > 0$. Elementary computations yield

$$x = \frac{-a(1-\varepsilon) + \operatorname{sgn}(a)\sqrt{a^2 - \varepsilon^2 + 2\varepsilon}}{1 + a^2}. \tag{3.10}$$

For $a > 0$, we obtain the following formula using (3.6):

$$\begin{aligned} x(\Theta) &= \frac{-a(\Theta)(1-\varepsilon) + \sqrt{a(\Theta)^2 - \varepsilon^2 + 2\varepsilon}}{1 + a(\Theta)^2} \\ &= \frac{-\cot(\Theta)(1-\varepsilon) + \sqrt{\cot(\Theta)^2 - \varepsilon^2 + 2\varepsilon}}{1 + \cot(\Theta)^2} \qquad (3.11) \\ &= \sin(\Theta)\cos(\Theta)\left(-1 + \varepsilon + \sqrt{1 + (2-\varepsilon)\varepsilon \tan^2(\Theta)}\right). \end{aligned}$$

Suppose that $\mathbf{s}_0^\varepsilon = 0$, that is, the light ray starts at the outer boundary. Then the next reflection may occur at the outer or inner boundary.

LEMMA 1. *If* $\mathbf{s}_0^\varepsilon = 0$, *then* $\mathbf{s}_1^\varepsilon = 0$ *if and only if*

$$a(\Theta)^2 < \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}.$$

PROOF. The light ray hits the inner boundary if and only if there is a solution to

$$x^2 + (ax+1)^2 = (1-\varepsilon)^2. \tag{3.12}$$

This equation has a solution if and only if

$$a^2 - 2\varepsilon - 2a^2\varepsilon + \varepsilon^2 + a^2\varepsilon^2 \geq 0, \tag{3.13}$$

that is, if and only if

$$a^2 \geq \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}. \tag{3.14}$$

□

LEMMA 2. *We have* $\mathbb{P}(\mathbf{s}_1^\varepsilon = 0 \mid \mathbf{s}_0^\varepsilon = 0) = \varepsilon$.

PROOF. Set $\gamma(\varepsilon) = \arctan(\sqrt{(2\varepsilon - \varepsilon^2)/((1-\varepsilon)^2)})$. Then, by Lemma 1 and using the fact that $\cos(\arctan(x)) = (1+x^2)^{-1/2}$, we have

$$\begin{aligned}
\mathbb{P}(\mathbf{s}_1^\varepsilon = 0 \mid \mathbf{s}_0^\varepsilon = 0) &= \mathbb{P}\left(|a(\Theta)| < \sqrt{\frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}}\right) = \mathbb{P}\left(|\Theta| \in \left[\frac{\pi}{2} - \gamma(\varepsilon), \frac{\pi}{2}\right]\right) \\
&= 2\int_{\frac{\pi}{2}-\gamma(\varepsilon)}^{\frac{\pi}{2}} \frac{1}{2}\cos(\theta)\, d\theta = 1 - \cos\gamma(\varepsilon) \\
&= 1 - \left(1 + \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}\right)^{-1/2} \\
&= \varepsilon.
\end{aligned}$$

□

The following representation of the process $\{(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon), n \geq 0\}$ will be useful.

DEFINITION 1. Let $T_n^\varepsilon, n \geq 1$, be i.i.d. random variables with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 1\}$, that is, on the event that the light ray starts from the inner boundary.

Let $R_n^\varepsilon, n \geq 1$, be i.i.d. random variables with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 0, \mathbf{s}_1^\varepsilon = 1\}$, that is, on the event that the light ray starts from the outer boundary and the next reflection is on the inner boundary.

Let $S_n^\varepsilon, n \geq 1$, be i.i.d. random variables with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 0, \mathbf{s}_1^\varepsilon = 0\}$, that is, on the event that the light ray starts from the outer boundary and the next reflection is also on the outer boundary.

Let Λ_n^ε, $n \geq 1$, be i.i.d. random variables ("Bernoulli sequence") with the distribution given by $\mathbb{P}(\Lambda_n^\varepsilon = 1) = 1 - \mathbb{P}(\Lambda_n^\varepsilon = 0) = \varepsilon$.

We assume that all random variables defined above, for all $n \geq 1$ and $\varepsilon \in (0, 1)$, are jointly independent.

We can represent the process $\{(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon), n \geq 0\}$ as follows. For $n \geq 0$,

$$\begin{cases} \mathbf{s}_{n+1}^\varepsilon = (1 - \mathbf{s}_n^\varepsilon)(1 - \Lambda_{n+1}^\varepsilon), \\ \boldsymbol{\alpha}_{n+1}^\varepsilon = \boldsymbol{\alpha}_n^\varepsilon + T_{n+1}^\varepsilon \mathbf{s}_n^\varepsilon + (1 - \mathbf{s}_n^\varepsilon)(\Lambda_{n+1}^\varepsilon S_{n+1}^\varepsilon + (1 - \Lambda_{n+1}^\varepsilon) R_{n+1}^\varepsilon). \end{cases} \tag{3.15}$$

We record the following property of random variables T_n^ε and R_n^ε because it is useful in our arguments but we also find the property interesting on its own.

PROPOSITION 1. *Random variables T_n^ε and R_n^ε have the same distribution.*

PROOF. Recall the notation from (3.5). The following claims follow from [6], Theorem 2.1. The discrete Markov chain $\{Q(\mathcal{T}_n^\varepsilon), n \geq 0\}$ representing consecutive reflection locations has a stationary distribution. [The stationary distribution is uniform on the boundary of $\mathcal{D}(\varepsilon, 1)$ but this is not relevant in this proof.] The Markov chain is symmetric (see the first displayed formula on page 507 of [6]) and its time reversal has the same distribution as the process itself. Consider any $-\infty < b_1 < b_2 < \infty$ and let $N_+(b_1, b_2, t)$ be the number of n such that $\mathcal{T}_{n+1}^\varepsilon \leq t$, $Q(\mathcal{T}_n^\varepsilon) \in \mathcal{C}_{\text{int}}$, $Q(\mathcal{T}_{n+1}^\varepsilon) \in \mathcal{C}_{\text{out}}$ and $\boldsymbol{\alpha}_{n+1}^\varepsilon - \boldsymbol{\alpha}_n^\varepsilon \in (b_1, b_2)$. By the ergodic theorem, $\lim_{t\to\infty} N_+(b_1, b_2, t)/t \to \ell_+ \in [0, \infty)$.

We will apply the same argument to the "reversed events." Let $N_-(b_1, b_2, t)$ be the number of n such that $\mathcal{T}_{n+1}^\varepsilon \leq t$, $Q(\mathcal{T}_n^\varepsilon) \in \mathcal{C}_{\text{out}}$, $Q(\mathcal{T}_{n+1}^\varepsilon) \in \mathcal{C}_{\text{int}}$ and $\boldsymbol{\alpha}_{n+1}^\varepsilon - \boldsymbol{\alpha}_n^\varepsilon \in (-b_2, -b_1)$. By the ergodic theorem, $\lim_{t\to\infty} N_-(b_1, b_2, t)/t \to \ell_- \in [0, \infty)$. Since the time reversed process has the same distribution as the original one, $\ell_- = \ell_+$.

The above observations, the symmetry of the reflection angle and the rotation invariance of the model easily imply the lemma. □

The proof of Proposition 1 is based on the symmetry of the process of Lambertian reflections, that is, the fact that the time reversed process has the same distribution as the original one. This symmetry is not obvious so we will present a physical heuristic argument which makes this symmetry plausible. It has been proved in [1] that (random) Lambertian reflections can be approximated by (deterministic) specular reflections from a collection of finite number of mirrors (a specular reflection occurs when the angle of reflection is equal to the angle of incidence). Time reversibility of the classical optics implies time reversibility of the process of Lambertian reflections.

Proposition 1 allows us to rewrite (3.15) as follows:

$$\begin{cases} \mathbf{s}_{n+1}^\varepsilon = (1 - \mathbf{s}_n^\varepsilon)(1 - \Lambda_{n+1}^\varepsilon), \\ \boldsymbol{\alpha}_{n+1}^\varepsilon = \boldsymbol{\alpha}_n^\varepsilon + T_{n+1}^\varepsilon \mathbf{s}_n^\varepsilon + (1 - \mathbf{s}_n^\varepsilon)(\Lambda_{n+1}^\varepsilon S_{n+1}^\varepsilon + (1 - \Lambda_{n+1}^\varepsilon) T_{n+1}^\varepsilon). \end{cases} \tag{3.16}$$

Since the evolution of $\{\mathbf{s}_n^\varepsilon, n \ge 0\}$ does not depend on $\{\boldsymbol{\alpha}_n^\varepsilon, n \ge 0\}$, it is a Markov chain in its own right. The chain $\{\mathbf{s}_n^\varepsilon, n \ge 0\}$ is irreducible and aperiodic because the transition from 0 to 0 is possible. The unique invariant probability measure μ^ε is given by

$$\mu^\varepsilon(0) = \frac{1}{2-\varepsilon}, \qquad \mu^\varepsilon(1) = \frac{1-\varepsilon}{2-\varepsilon}.$$

From now on, we will assume that $\mathbf{s}_0^\varepsilon$ (and, therefore, $\mathbf{s}_n^\varepsilon$ for all $n \ge 0$) is distributed according to μ^ε. It is easy to see that this assumption does not affect the validity of our main results.

LEMMA 3. *Set* $b_\varepsilon = \arctan(\sqrt{2\varepsilon - \varepsilon^2})$. *Then the support of the distribution of* T_n^ε *is* $[-b_\varepsilon, b_\varepsilon]$, *while the support of the distribution of* S_n^ε *is* $[-2b_\varepsilon, 2b_\varepsilon]$.

PROOF. Suppose that $\mathbf{s}_0^\varepsilon = 1$ and $\boldsymbol{\alpha}_0^\varepsilon = \pi/2$, that is, the light ray starts from the top point on the inner boundary. Recall the representation of the jumps and the notation introduced in (3.8). The absolute value of the angular component of the jump $|\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon|$ is maximized when $|a(\Theta)|$ is minimized; in other words, if $\Theta = \pm\pi/2$. It is easy to check that when $\Theta = \pm\pi/2$ then $|\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon| = b_\varepsilon$. This proves our claim about the support of the distribution of T_n^ε.

Next, suppose that $\mathbf{s}_0^\varepsilon = 0$ and $\boldsymbol{\alpha}_0^\varepsilon = \pi/2$, that is, the light ray starts from the top point on the outer boundary. Assume that $|\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon|$ corresponds to a jump from the outer boundary to outer boundary. Then this quantity is maximal when the light ray is almost tangent to the inner boundary. Simple geometry shows that the length of such a light ray segment is bounded by twice the maximum length of a light ray starting from the inner boundary and ending at the outer boundary. By the first part of the proof, $|\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon|$ corresponding to a jump from the outer boundary to outer boundary is bounded by $2b_\varepsilon$. This proves the second claim of the lemma. $\square$

LEMMA 4. *We have*

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}((T_1^\varepsilon)^2)}{(1/2)\varepsilon^2 \log(1/\varepsilon)} = 1. \tag{3.17}$$

PROOF. We will use formula (3.11), that is,

$$x(\theta) = \sin(\theta)\cos(\theta)\bigl(-1 + \varepsilon + \sqrt{1 + (2-\varepsilon)\varepsilon \tan^2(\theta)}\bigr). \tag{3.18}$$

We will use the notation introduced in (3.6)–(3.8). Hence we can and will identify T_1^ε with a function of Θ, that is, $T_1^\varepsilon(\Theta) = \arctan(x(\Theta)/y(\Theta))$. Assume that $\mathbf{s}_0^\varepsilon = 1$ and $\boldsymbol{\alpha}_0^\varepsilon = \pi/2$. Then $1 - \varepsilon \le y(\Theta) \le 1$. Recall from Lemma 3 that

$|T_1^\varepsilon(\Theta)| \le \arctan(\sqrt{2\varepsilon - \varepsilon^2})$. These observations imply that

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\mathbb{E}((T_1^\varepsilon)^2)}{(1/2)\varepsilon^2 \log(1/\varepsilon)} &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}(\arctan^2(x(\Theta)/y(\Theta)))}{(1/2)\varepsilon^2 \log(1/\varepsilon)} \\ &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}(x(\Theta)^2)}{(1/2)\varepsilon^2 \log(1/\varepsilon)},\end{aligned} \tag{3.19}$$

assuming that at least one of these limits exists. It follows from (3.18) that

$$\begin{aligned}\mathbb{E}\big(x(\Theta)^2\big) &= 2\int_0^{\pi/2} x(\theta)^2 \frac{1}{2}\cos(\theta)\, d\theta \\ &= \int_0^{\pi/2} \sin^2(\theta)\cos^2(\theta)\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2 \\ &\quad \times \cos(\theta)\, d\theta \\ &= \int_0^{\pi/2} \sin^2(\theta)\cos^3(\theta)\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2\, d\theta.\end{aligned} \tag{3.20}$$

Set

$$\theta_0 = \theta_0(\varepsilon) = \frac{\pi}{2} - \arctan\big(2\varepsilon^{1/2}\big). \tag{3.21}$$

Then for $\varepsilon \in (0, 1)$ and $\theta \in (\theta_0, \frac{\pi}{2})$,

$$\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)} \le \sqrt{2\cdot 1} + \sqrt{2(2-\varepsilon)\varepsilon\tan^2(\theta)} \le 2 + 2\sqrt{\varepsilon}\tan(\theta).$$

This implies that, for $\varepsilon \in (0, 1)$ and $\theta \in (\theta_0, \frac{\pi}{2})$,

$$\begin{aligned}&\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2 \\ &\quad \le (1-\varepsilon)^2 + \big(\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2 \\ &\quad \le 1 + \big(2+2\sqrt{\varepsilon}\tan(\theta)\big)^2 \le 1 + \big(2\cdot 2^2 + 2\cdot\big(2\sqrt{\varepsilon}\tan(\theta)\big)^2\big) \\ &\quad = 9 + 8\varepsilon\tan^2(\theta).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\theta_0}^{\pi/2} \sin^2(\theta)\cos^3(\theta)\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2\, d\theta \\ &\quad \le \int_{\theta_0}^{\pi/2} \sin^2(\theta)\cos^3(\theta)\big(9+8\varepsilon\tan^2(\theta)\big)\, d\theta \\ &\quad = \int_{\theta_0}^{\pi/2} 9\sin^2(\theta)\cos^3(\theta)\, d\theta + \int_{\theta_0}^{\pi/2} 8\varepsilon\sin^4(\theta)\cos(\theta)\, d\theta\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\theta_0}^{\pi/2} 9\cos^3(\theta)\,d\theta + \int_{\theta_0}^{\pi/2} 8\varepsilon\cos(\theta)\,d\theta \\
&\le \int_{\theta_0}^{\pi/2} 9(\pi/2-\theta)^3\,d\theta + \int_{\theta_0}^{\pi/2} 8\varepsilon(\pi/2-\theta)\,d\theta \\
&= 9\cdot\frac{1}{4}(\pi/2-\theta)^4 + 8\varepsilon\frac{1}{2}(\pi/2-\theta)^2 \\
&= 9\cdot\frac{1}{4}\arctan^4(2\varepsilon^{1/2}) + 8\varepsilon\frac{1}{2}\arctan^2(2\varepsilon^{1/2}) \\
&= O(\varepsilon^2).
\end{aligned}
\tag{3.22}
$$

For $\varepsilon \in (0,1)$ and $\theta \in [0,\theta_0]$, $\varepsilon\tan^2(\theta) \le 1/4$, so

$$
\begin{aligned}
\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)} &= 1 + \frac{1}{2}(2-\varepsilon)\varepsilon\tan^2(\theta) + O\big(((2-\varepsilon)\varepsilon\tan^2(\theta))^2\big) \\
&= 1 + \varepsilon\tan^2(\theta) - \frac{1}{2}\varepsilon^2\tan^2(\theta) + O\big(\varepsilon^2\tan^4(\theta)\big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2 \\
&\quad= \Big(-1+\varepsilon+1+\varepsilon\tan^2(\theta) - \frac{1}{2}\varepsilon^2\tan^2(\theta) + O\big(\varepsilon^2\tan^4(\theta)\big)\Big)^2 \\
&\quad= \Big(\varepsilon\frac{1}{\cos^2(\theta)} - \frac{1}{2}\varepsilon^2\tan^2(\theta) + O\big(\varepsilon^2\tan^4(\theta)\big)\Big)^2 \\
&\quad= \varepsilon^2\frac{1}{\cos^4(\theta)} + \frac{1}{4}\varepsilon^4\tan^4(\theta) + O\big(\varepsilon^4\tan^8(\theta)\big) \\
&\qquad + O\Big(\varepsilon^3\frac{\tan^2(\theta)}{\cos^2(\theta)}\Big) + O\Big(\varepsilon^3\frac{\tan^4(\theta)}{\cos^2(\theta)}\Big) + O\big(\varepsilon^4\tan^6(\theta)\big) \\
&\quad= \varepsilon^2\frac{1}{\cos^4(\theta)} + \frac{1}{4}\varepsilon^4\frac{\sin^4(\theta)}{\cos^4(\theta)} + O\Big(\varepsilon^4\frac{\sin^8(\theta)}{\cos^8(\theta)}\Big) \\
&\qquad + O\Big(\varepsilon^3\frac{\sin^2(\theta)}{\cos^4(\theta)}\Big) + O\Big(\varepsilon^3\frac{\sin^4(\theta)}{\cos^6(\theta)}\Big) + O\Big(\varepsilon^4\frac{\sin^6(\theta)}{\cos^6(\theta)}\Big) \\
&\quad= \varepsilon^2\frac{1}{\cos^4(\theta)} + O\Big(\varepsilon^4\frac{1}{\cos^4(\theta)}\Big) + O\Big(\varepsilon^4\frac{1}{\cos^8(\theta)}\Big) \\
&\qquad + O\Big(\varepsilon^3\frac{1}{\cos^4(\theta)}\Big) + O\Big(\varepsilon^3\frac{1}{\cos^6(\theta)}\Big) + O\Big(\varepsilon^4\frac{1}{\cos^6(\theta)}\Big) \\
&\quad= \varepsilon^2\frac{1}{\cos^4(\theta)} + O\Big(\varepsilon^3\frac{1}{\cos^6(\theta)}\Big) + O\Big(\varepsilon^4\frac{1}{\cos^8(\theta)}\Big).
\end{aligned}
\tag{3.23}
$$

We have

$$\int_0^{\theta_0} \sin^2(\theta)\cos^3(\theta)\frac{1}{\cos^6(\theta)}\,d\theta \le \int_0^{\theta_0} \frac{1}{\cos^3(\theta)}\,d\theta \le \int_0^{\theta_0} \frac{1}{(1-2\theta/\pi)^3}\,d\theta = \frac{\pi(\pi-\theta_0)\theta_0}{(\pi-2\theta_0)^2} = O(\varepsilon^{-1}), \tag{3.24}$$

and

$$\int_0^{\theta_0} \sin^2(\theta)\cos^3(\theta)\frac{1}{\cos^8(\theta)}\,d\theta \le \int_0^{\theta_0} \frac{1}{\cos^5(\theta)}\,d\theta \le \int_0^{\theta_0} \frac{1}{(1-2\theta/\pi)^5}\,d\theta = \frac{1}{8}\pi\left(\frac{\pi^4}{(\pi-2\theta_0)^4}-1\right) = O(\varepsilon^{-2}). \tag{3.25}$$

It follows from (3.21), (3.24) and (3.25) that

$$\begin{aligned}
&\int_0^{\theta_0} \sin^2(\theta)\cos^3(\theta)\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2\,d\theta\\
&\quad= \int_0^{\theta_0} \sin^2(\theta)\cos^3(\theta)\\
&\qquad\times\left(\varepsilon^2\frac{1}{\cos^4(\theta)} + O\left(\varepsilon^3\frac{1}{\cos^6(\theta)}\right) + O\left(\varepsilon^4\frac{1}{\cos^8(\theta)}\right)\right)d\theta\\
&\quad= \varepsilon^2\int_0^{\theta_0}\frac{\sin^2(\theta)}{\cos(\theta)}\,d\theta + O(\varepsilon^2) \qquad (3.26)\\
&\quad= \frac{\varepsilon^2}{2}\big(\log(1+\sin(\theta_0)) - \log(1-\sin(\theta_0)) - \sin(\theta_0)\big) + O(\varepsilon^2)\\
&\quad= -\frac{\varepsilon^2}{2}\log\big(1-\cos(\pi/2-\theta_0)\big) + O(\varepsilon^2) \qquad (3.27)\\
&\quad= -\frac{\varepsilon^2}{2}\log\big(1-\cos\big(\arctan\big(2\varepsilon^{1/2}\big)\big)\big) + O(\varepsilon^2)\\
&\quad= -\frac{\varepsilon^2}{2}\log(2\varepsilon) + O(\varepsilon^2)\\
&\quad= \frac{\varepsilon^2}{2}|\log\varepsilon| + O(\varepsilon^2).
\end{aligned}$$

This estimate and (3.22) imply that

$$\begin{aligned}
&\int_0^{\pi/2} \sin^2(\theta)\cos^3(\theta)\big(-1+\varepsilon+\sqrt{1+(2-\varepsilon)\varepsilon\tan^2(\theta)}\big)^2\,d\theta\\
&\quad= \frac{\varepsilon^2}{2}|\log\varepsilon| + O(\varepsilon^2).
\end{aligned}$$

The lemma follows from this, (3.19) and (3.20). $\square$

LEMMA 5. *Recall the definition of $\Delta\mathcal{T}_k^\varepsilon$ stated in (3.2). For every $k \geq 1$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\big(\Delta\mathcal{T}_k^\varepsilon \mid \mathbf{s}_{k-1}^\varepsilon = 1\big) = \frac{\pi}{2}, \tag{3.28}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\big(\Delta\mathcal{T}_k^\varepsilon \mid \mathbf{s}_{k-1}^\varepsilon = 0\big) = \frac{\pi}{2}. \tag{3.29}$$

PROOF. It will suffice to prove the lemma for $k = 1$. By rotation invariance, we can and will assume that $\boldsymbol{\alpha}_0^\varepsilon = \pi/2$. Then (3.2), (3.16) and Definition (1) yield

$$\begin{aligned}\mathbb{E}\big(\Delta\mathcal{T}_1^\varepsilon \mid \mathbf{s}_0^\varepsilon = 1\big) &= \mathbb{E}\sqrt{\sin^2(T_1^\varepsilon) + \big(1-\varepsilon-\cos(T_1^\varepsilon)\big)^2}\\ &= \mathbb{E}\sqrt{2(1-\varepsilon)\big(1-\cos(T_1^\varepsilon)\big)+\varepsilon^2}.\end{aligned} \tag{3.30}$$

Let

$$G(\varepsilon) = \frac{1}{\varepsilon}\mathbb{E}\Big(\sqrt{2(1-\varepsilon)\big(1-\cos(T_1^\varepsilon)\big)+\varepsilon^2}\Big).$$

Since $|T_1^\varepsilon| \leq \arctan(\sqrt{2\varepsilon+\varepsilon^2})$ by Lemma 3, the Taylor expansion for the cosine function at 0 yields

$$1-\cos\big(T_1^\varepsilon\big) = \frac{1}{2}\big(1+O(\varepsilon)\big)\big(T_1^\varepsilon\big)^2. \tag{3.31}$$

Therefore, using notation from (3.8),

$$\begin{aligned}G(\varepsilon) &= \mathbb{E}\Big(\sqrt{1+(1-\varepsilon)\big(1+O(\varepsilon)\big)\big(T_1^\varepsilon/\varepsilon\big)^2}\Big)\\ &= \mathbb{E}\bigg(\sqrt{1+(1-\varepsilon)\big(1+O(\varepsilon)\big)\Big(\frac{1}{\varepsilon}\arctan\big(x(\Theta)/y(\Theta)\big)\Big)^2}\ \bigg|\ \mathbf{s}_0^\varepsilon = 1\bigg).\end{aligned} \tag{3.32}$$

We will estimate $\frac{1}{\varepsilon}\arctan(x(\Theta)/y(\Theta))$. The following geometric interpretation of the quantity $\frac{1}{\varepsilon}\arctan(x(\theta)/y(\theta))$ follows from (3.8), rescaling (enlarging) the annulus $\mathcal{D}(\varepsilon,1)$ by the factor of $1/\varepsilon$, and then shifting it down by $1/\varepsilon$ so that its outer boundary passes through the origin. Consider the half-line L starting at $(0,-1)$ at an angle $\theta \in [0,\pi/2)$ with the vertical line. Let $A_1(\varepsilon)$ be the intersection point of L with the circle $\mathcal{C}((0,-1/\varepsilon),1/\varepsilon)$ (i.e., the outer boundary of the transformed domain) and let A_2 be the intersection point of L with the horizontal axis. Then $\frac{1}{\varepsilon}\arctan(x(\theta)/y(\theta))$ is the angle between the vertical line and the line passing through points $A_1(\varepsilon)$ and $(0,-1/\varepsilon)$. Let $\alpha(\varepsilon)$ be the angle between the vertical line and the line passing through points A_2 and $(0,-1/\varepsilon)$. For every fixed $\theta \in [0,\pi/2)$, $A_1(\varepsilon) \to A_2$ as $\varepsilon \to 0$. This implies that

$$\lim_{\varepsilon\downarrow 0}\frac{\arctan(x(\theta)/y(\theta))}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{\alpha(\varepsilon)}{\varepsilon} = \operatorname{dist}\big(A_2,(0,0)\big) = \tan\theta. \tag{3.33}$$

Moreover, we have $\arctan(x(\Theta)/y(\Theta)) \le \alpha(\varepsilon)$ and $\alpha(\varepsilon) \le \tan\alpha(\varepsilon) = \operatorname{dist}(A_2, (0,0))/(1/\varepsilon)$ so for all $\varepsilon > 0$ and $\theta \in [0, \pi/2)$,

$$\frac{\arctan(x(\theta)/y(\theta))}{\varepsilon} \le \frac{\alpha(\varepsilon)}{\varepsilon} \le \operatorname{dist}\bigl(A_2, (0,0)\bigr) = \tan\theta. \tag{3.34}$$

A similar analysis applies to $\theta \in (-\pi/2, 0]$. By the dominated convergence theorem and (3.33)–(3.34),

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}\bigl(\Delta\mathcal{T}_k^\varepsilon \mid \mathbf{s}_{k-1}^\varepsilon = 1\bigr) \\ &\quad= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}\Bigl(\sqrt{2(1-\varepsilon)\bigl(1-\cos(T_1^\varepsilon)\bigr)+\varepsilon^2}\Bigr) \\ &\quad= \lim_{\varepsilon\to 0} G(\varepsilon) = \mathbb{E}\bigl(\sqrt{1+\tan^2(\Theta)}\bigr) = \mathbb{E}\biggl(\frac{1}{\cos(\Theta)}\biggr) = \frac{\pi}{2}. \end{aligned} \tag{3.35}$$

This proves (3.28).

By (3.2), (3.16) and Definition 1,

$$\begin{aligned} &\mathbb{E}\bigl(\Delta\mathcal{T}_1^\varepsilon \mid \mathbf{s}_0^\varepsilon = 0\bigr) \\ &\quad= \mathbb{E}\Bigl(\Lambda_1^\varepsilon\sqrt{2\bigl(1-\cos(S_1^\varepsilon)\bigr)} + \bigl(1-\Lambda_1^\varepsilon\bigr)\sqrt{2(1-\varepsilon)\bigl(1-\cos(T_1^\varepsilon)\bigr)+\varepsilon^2}\Bigr). \end{aligned} \tag{3.36}$$

By Lemma 2 and Lemma 3,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\Bigl|\Lambda_1^\varepsilon\sqrt{2\bigl(1-\cos(S_1^\varepsilon)\bigr)}\Bigr| = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\varepsilon\mathbb{E}\Bigl|\sqrt{2\bigl(1-\cos(S_1^\varepsilon)\bigr)}\Bigr| = 0 \tag{3.37}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\Bigl|-\Lambda_1^\varepsilon\sqrt{2(1-\varepsilon)\bigl(1-\cos(T_1^\varepsilon)\bigr)+\varepsilon^2}\Bigr| \\ &\qquad= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\varepsilon\mathbb{E}\Bigl(\sqrt{2(1-\varepsilon)\bigl(1-\cos(T_1^\varepsilon)\bigr)+\varepsilon^2}\Bigr) = 0. \end{aligned} \tag{3.38}$$

By (3.35),

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\Bigl(\sqrt{2(1-\varepsilon)\bigl(1-\cos(T_1^\varepsilon)\bigr)+\varepsilon^2}\Bigr) = \frac{\pi}{2}. \tag{3.39}$$

The combination of (3.36), (3.37), (3.38) and (3.39) implies (3.29). □

LEMMA 6. *For $t \ge 0$,*

$$\mathbb{E}\bigl(N^\varepsilon(t)\bigr) \le 2(t+2\varepsilon)/\varepsilon. \tag{3.40}$$

PROOF. The light ray travels at speed 1 so it takes at least ε units of time between any two consecutive reflections that do not take place on the same piece of the boundary. Thus n crossings from the inner to the outer boundary and n

crossings from the outer to the inner boundary must take at least $2n\varepsilon$ units of time. Let $U(n)$ be the total number of reflections (including consecutive reflections from the outer boundary) that have occurred by the time when n crossings from the inner to the outer boundary and n crossings from the outer to the inner boundary have happened. Then $N^\varepsilon(t) \le U(\lceil t/(2\varepsilon)\rceil)$. We can represent $U(n)$ as

$$U(n) = n + \sum_{k=1}^{n} X_k^\varepsilon,$$

where X_k^ε are i.i.d. random variables with the geometric distribution (taking values $1, 2, \dots$) with parameter $1-\varepsilon$ (see Lemma 2). Therefore, for $\varepsilon < 1/2$,

$$\begin{aligned}
\mathbb{E}(N^\varepsilon(t)) &\le \mathbb{E}(U(\lceil t/(2\varepsilon)\rceil)) \\
&= \lceil t/(2\varepsilon)\rceil + \lceil t/(2\varepsilon)\rceil \frac{1}{1-\varepsilon} = \lceil t/(2\varepsilon)\rceil \frac{2-\varepsilon}{1-\varepsilon} \\
&\le \left(\frac{t}{2\varepsilon} + 1\right)\frac{2-\varepsilon}{1-\varepsilon} \le 2(t+2\varepsilon)/\varepsilon. \qquad \square
\end{aligned}$$

LEMMA 7. *Processes* $\{\frac{\pi}{2}\varepsilon^2 \log(1/\varepsilon) N^\varepsilon(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon)}) - t, t \ge 0\}$ *converge in probability toward* 0 *in the uniform topology on compact sets when* $\varepsilon \to 0$.

PROOF. Computations similar to those in (3.30) and (3.31) yield

$$\begin{aligned}
\mathbb{E}((\Delta \mathcal{T}_1^\varepsilon)^2 \mid \mathbf{s}_0^\varepsilon = 1) &= \mathbb{E}(\sin^2(T_1^\varepsilon) + (1-\varepsilon-\cos(T_1^\varepsilon))^2) \\
&= \mathbb{E}(2(1-\varepsilon)(1-\cos(T_1^\varepsilon)) + \varepsilon^2) \\
&= \mathbb{E}((1-\varepsilon)(1+O(\varepsilon))(T_1^\varepsilon)^2 + \varepsilon^2).
\end{aligned}$$

This and Lemma 4 imply that, for small $\varepsilon > 0$,

$$\mathbb{E}((\Delta \mathcal{T}_1^\varepsilon)^2 \mid \mathbf{s}_0^\varepsilon = 1) < \varepsilon^2 \log(1/\varepsilon). \tag{3.41}$$

By (3.36),

$$\begin{aligned}
&\mathbb{E}((\Delta \mathcal{T}_1^\varepsilon)^2 \mid \mathbf{s}_0^\varepsilon = 0) \\
&\quad = \mathbb{E}\Big(\Big(\Lambda_1^\varepsilon \sqrt{2(1-\cos(S_1^\varepsilon))} \\
&\qquad + (1-\Lambda_1^\varepsilon)\sqrt{2(1-\varepsilon)(1-\cos(R_1^\varepsilon)) + \varepsilon^2}\Big)^2\Big) \qquad (3.42) \\
&\quad \le 2\mathbb{E}((\Lambda_1^\varepsilon)^2(2(1-\cos(S_1^\varepsilon)))) \\
&\qquad + 2\mathbb{E}((1-\Lambda_1^\varepsilon)^2(2(1-\varepsilon)(1-\cos(R_1^\varepsilon)) + \varepsilon^2)).
\end{aligned}$$

By Lemma 4 and Proposition 1, for small $\varepsilon > 0$,

$$2\mathbb{E}((1-\Lambda_1^\varepsilon)^2(2(1-\varepsilon)(1-\cos(R_1^\varepsilon)) + \varepsilon^2)) < 2\varepsilon^2 \log(1/\varepsilon). \tag{3.43}$$

It follows from the definition of Λ_1^ε and Lemma 3 that, for small $\varepsilon > 0$,

$$2\mathbb{E}\big((\Lambda_1^\varepsilon)^2\big(2\big(1-\cos\big(S_1^\varepsilon\big)\big)\big)\big) \le 5\varepsilon^2.$$

This, (3.42) and (3.43) imply that, for small $\varepsilon > 0$,

$$\mathbb{E}\big((\Delta\mathcal{T}_1^\varepsilon)^2 \mid \mathbf{s}_0^\varepsilon = 0\big) \le 2\varepsilon^2\log(1/\varepsilon). \tag{3.44}$$

Recall definition (3.3) and set

$$M_n^\varepsilon = \frac{1}{2}\varepsilon\log(1/\varepsilon)\left(\mathcal{T}_n^\varepsilon - \sum_{k=1}^n \mathbb{E}\big(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon\big)\right).$$

Then $(M_n^\varepsilon)_{n\ge 0}$ is a martingale starting at 0 and its quadratic variation is

$$\langle M^\varepsilon\rangle_n = \frac{1}{4}\varepsilon^2\log^2(1/\varepsilon)\sum_{k=1}^n \operatorname{Var}\big(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon\big).$$

From (3.41) and (3.44), we obtain for small $\varepsilon > 0$,

$$\langle M^\varepsilon\rangle_n \le \frac{1}{4}\varepsilon^2\log^2(1/\varepsilon)\sum_{k=1}^n \mathbb{E}\big((\Delta\mathcal{T}_k^\varepsilon)^2 \mid \mathcal{F}_{k-1}^\varepsilon\big) \le \frac{1}{2}n\varepsilon^4\log^3(1/\varepsilon). \tag{3.45}$$

In this proof, we will use the notation $W(\varepsilon,t) = N^\varepsilon(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon))})$. By Lemma 6, $W(\varepsilon,t)$ is a stopping time with a finite expectation so by the optional stopping theorem, (3.40) and (3.45), for small $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\big((M_{W(\varepsilon,t)}^\varepsilon)^2\big) &= \mathbb{E}\langle M^\varepsilon\rangle_{W(\varepsilon,t)} \le \frac{1}{2}\varepsilon^4\log^3(1/\varepsilon)\mathbb{E}W(\varepsilon,t)\\ &\le \frac{1}{2}\varepsilon^4\log^3(1/\varepsilon)\frac{2}{\varepsilon}\left(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon)} + 2\varepsilon\right)\\ &= \varepsilon^3\log^3(1/\varepsilon)\left(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon)} + 2\varepsilon\right).\end{aligned} \tag{3.46}$$

For a fixed t, the right-hand side goes to 0 as $\varepsilon \to 0$.

The definition of $N^\varepsilon(t)$ implies that

$$\mathcal{T}_{W(\varepsilon,t)}^\varepsilon \le \frac{t}{(1/2)\varepsilon\log(1/\varepsilon))} \le \mathcal{T}_{W(\varepsilon,t)+1}^\varepsilon = \mathcal{T}_{W(\varepsilon,t)}^\varepsilon + \Delta\mathcal{T}_{W(\varepsilon,t)+1}^\varepsilon. \tag{3.47}$$

It follows easily from (3.2) and Lemma 3 that $\lim_{\varepsilon\to 0}\sup_{k\ge 1}\Delta\mathcal{T}_k^\varepsilon = 0$ almost surely. Hence, a.s.,

$$\lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\left|\mathcal{T}_{W(\varepsilon,t)}^\varepsilon - \frac{t}{(1/2)\varepsilon\log(1/\varepsilon))}\right| = 0. \tag{3.48}$$

It follows from the definition of M_n^ε and (3.46) that

$$\lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\left|\mathcal{T}_{W(\varepsilon,t)}^\varepsilon - \sum_{k=1}^{W(\varepsilon,t)}\mathbb{E}\big(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon\big)\right| = 0, \tag{3.49}$$

in probability.

Lemma 5 implies that, a.s.,

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\sup_{k\geq 0}\left|\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)-\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\right)=0. \tag{3.50}$$

By Lemmas 5 and 6, and (3.50), for $t>0$,

$$\begin{aligned}
0 &\leq \lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\mathbb{E}\left|\frac{\pi}{2}\varepsilon W(\varepsilon,t)-\sum_{k=1}^{W(\varepsilon,t)}\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)\right|\\
&\leq \lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\mathbb{E}\left|\frac{\pi}{2}\varepsilon W(\varepsilon,t)-\sum_{k=1}^{W(\varepsilon,t)}\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\\
&\quad+\lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\mathbb{E}\left|\sum_{k=1}^{W(\varepsilon,t)}\bigl(\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)-\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\bigr)\right|\\
&\leq \lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\mathbb{E}\bigl(W(\varepsilon,t)\bigr)\left|\frac{\pi}{2}\varepsilon-\mathbb{E}(\Delta\mathcal{T}_1^\varepsilon)\right|\\
&\quad+\lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\mathbb{E}\bigl(W(\varepsilon,t)\bigr)\sup_{k\geq 0}\left|\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)-\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\\
&\leq \lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\frac{2}{\varepsilon}\left(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon))}+2\varepsilon\right)\varepsilon\left|\frac{\pi}{2}-\frac{1}{\varepsilon}\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\\
&\quad+\lim_{\varepsilon\to 0}\varepsilon\log(1/\varepsilon)\frac{2}{\varepsilon}\left(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon))}+2\varepsilon\right)\sup_{k\geq 0}\left|\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)-\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\\
&\leq \lim_{\varepsilon\to 0}4t\left|\frac{\pi}{2}-\frac{1}{\varepsilon}\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|+\lim_{\varepsilon\to 0}4t\frac{1}{\varepsilon}\sup_{k\geq 0}\left|\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon \mid \mathcal{F}_{k-1}^\varepsilon)-\mathbb{E}(\Delta\mathcal{T}_k^\varepsilon)\right|\\
&=0.
\end{aligned}$$

This, (3.48) and (3.49) imply that for any fixed $t\geq 0$,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\left|\frac{\pi}{2}\varepsilon^2\log(1/\varepsilon)N^\varepsilon\left(\frac{t}{(1/2)\varepsilon\log(1/\varepsilon))}\right)-t\right| &= \lim_{\varepsilon\to 0}\left|\frac{\pi}{2}\varepsilon^2\log(1/\varepsilon)W(\varepsilon,t)-t\right|\\
&=0,
\end{aligned}$$

in probability. The stronger statement given in the lemma follows from this and the fact that the process $t\to N^\varepsilon(t)$ is nondecreasing. □

PROOF OF THEOREM 1. First, we are going to apply [10], Theorem 1.4, Chapter 7, to a time change of $\boldsymbol{\alpha}_k^\varepsilon$. We extend the time parameter for this process from integers to reals by letting $\boldsymbol{\alpha}_t^\varepsilon := \boldsymbol{\alpha}_{\lfloor t\rfloor}^\varepsilon$ for $t\geq 0$. Next, we rescale, that is, we let

$$\begin{aligned}
\sigma_\varepsilon^2 &= \frac{1}{2}\varepsilon^2\log(1/\varepsilon),\\
\widetilde{\boldsymbol{\alpha}}_t^\varepsilon &= \boldsymbol{\alpha}_{t/\sigma_\varepsilon^2}^\varepsilon \qquad \text{for } t\geq 0.
\end{aligned}$$

We will prove that processes $\{\widetilde{\boldsymbol{\alpha}}^{\varepsilon}_t, t \geq 0\}$ converge weakly to Brownian motion as $\varepsilon \to 0$.

It follows easily from the symmetry of jumps of $\boldsymbol{\alpha}^{\varepsilon}_k$ and from Lemma 3 that the process $\{\widetilde{\boldsymbol{\alpha}}^{\varepsilon}_t, t \geq 0\}$ is a martingale.

We will use assumption (a) of [10], Theorem 1.4, Chapter 7. According to Definition 1, Proposition 1 and Lemma 3, $|\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_n| \leq 2\arctan(\sqrt{2\varepsilon - \varepsilon^2})$, a.s., for all n. Hence, for all $t_0 > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big(\sup_{t\leq t_0} |\widetilde{\boldsymbol{\alpha}}^{\varepsilon}_t - \widetilde{\boldsymbol{\alpha}}^{\varepsilon}_{t-}|\Big) \leq \lim_{\varepsilon\to 0} 2\arctan\big(\sqrt{2\varepsilon - \varepsilon^2}\big) = 0.$$

This means that condition (1.14) of [10], Theorem 1.4, Chapter 7, is satisfied. It remains to show that the quadratic variation $\langle \widetilde{\boldsymbol{\alpha}}^{\varepsilon}\rangle_t$ of $\widetilde{\boldsymbol{\alpha}}^{\varepsilon}$ converges to t. More precisely, we have to show that for each $t \geq 0$, $\langle \widetilde{\boldsymbol{\alpha}}^{\varepsilon}\rangle_t \to t$ in probability. We will compute the quadratic variation $\langle \boldsymbol{\alpha}^{\varepsilon}\rangle_n$ of $\boldsymbol{\alpha}^{\varepsilon}_n$ first.

Recall from (3.16) that

$$\boldsymbol{\alpha}^{\varepsilon}_{n+1} = \boldsymbol{\alpha}^{\varepsilon}_n + T^{\varepsilon}_{n+1}\mathbf{s}^{\varepsilon}_n + (1 - \mathbf{s}^{\varepsilon}_n)\big(\Lambda^{\varepsilon}_{n+1} S^{\varepsilon}_{n+1} + (1 - \Lambda^{\varepsilon}_{n+1})T^{\varepsilon}_{n+1}\big).$$

We have assumed that $\{\mathbf{s}^{\varepsilon}_n, n \geq 0\}$ is in the stationary regime, that is, for all $n \geq 0$, $\mathbf{s}^{\varepsilon}_n$ is distributed according to the stationary distribution μ^{ε}, where

$$\mu^{\varepsilon}(0) = \frac{1}{2-\varepsilon}, \qquad \mu^{\varepsilon}(1) = \frac{1-\varepsilon}{2-\varepsilon}. \tag{3.51}$$

Let $\mathcal{F}^{\varepsilon}_n = \sigma(\boldsymbol{\alpha}^{\varepsilon}_k, \mathbf{s}^{\varepsilon}_k, k = 1, \dots, n)$. Then

$$\begin{aligned}
&\mathbb{E}\big((\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_n)^2 \mid \mathcal{F}^{\varepsilon}_n\big)\\
&\quad = \mathbb{E}\big((T^{\varepsilon}_{n+1})^2\big)\mathbf{s}^{\varepsilon}_n + \big(\varepsilon\mathbb{E}\big((S^{\varepsilon}_{n+1})^2\big) + (1-\varepsilon)\mathbb{E}\big((T^{\varepsilon}_{n+1})^2\big)\big)(1 - \mathbf{s}^{\varepsilon}_n),
\end{aligned}$$

so

$$\begin{aligned}
\langle \boldsymbol{\alpha}^{\varepsilon}\rangle_n &= \sum_{k=0}^{n} \mathbb{E}\big((\boldsymbol{\alpha}^{\varepsilon}_{k+1} - \boldsymbol{\alpha}^{\varepsilon}_k)^2 \mid \mathcal{F}^{\varepsilon}_k\big)\\
&= \mathbb{E}\big((T^{\varepsilon}_1)^2\big)\sum_{k=0}^{n}\mathbf{s}^{\varepsilon}_k + \big(\varepsilon\mathbb{E}\big((S^{\varepsilon}_1)^2\big) + (1-\varepsilon)\mathbb{E}\big((T^{\varepsilon}_1)^2\big)\big)\sum_{k=0}^{n}(1 - \mathbf{s}^{\varepsilon}_k) \qquad (3.52)\\
&= n(1-\varepsilon)\mathbb{E}\big((T^{\varepsilon}_1)^2\big) + \varepsilon\mathbb{E}\big((T^{\varepsilon}_1)^2\big)\sum_{k=0}^{n}\mathbf{s}^{\varepsilon}_k + \varepsilon\mathbb{E}\big((S^{\varepsilon}_1)^2\big)\sum_{k=0}^{n}(1 - \mathbf{s}^{\varepsilon}_k).
\end{aligned}$$

It follows from Lemmas 3 and 4 that

$$\lim_{\varepsilon\to 0}\frac{1}{\sigma^2_{\varepsilon}}(1-\varepsilon)\mathbb{E}\big((T^{\varepsilon}_1)^2\big) = 1,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\sigma^2_{\varepsilon}}\big(\varepsilon\mathbb{E}\big((S^{\varepsilon}_1)^2\big) + \varepsilon\mathbb{E}\big((T^{\varepsilon}_1)^2\big)\big) = 0.$$

This and (3.52) imply that, for each $t \geq 0$,

$$\lim_{\varepsilon\to 0}\langle\widetilde{\boldsymbol{\alpha}}^{\varepsilon}\rangle_t = \lim_{\varepsilon\to 0}\langle\boldsymbol{\alpha}^{\varepsilon}\rangle_{t/\sigma_\varepsilon^2} = \lim_{\varepsilon\to 0}\langle\boldsymbol{\alpha}^{\varepsilon}\rangle_{\lfloor t/\sigma_\varepsilon^2\rfloor} = t,$$

almost surely. This completes the proof that processes $\{\widetilde{\boldsymbol{\alpha}}_t^{\varepsilon}, t \geq 0\}$ converge weakly to Brownian motion as $\varepsilon \to 0$.

To complete the proof, we need to time change the process $\{\widetilde{\boldsymbol{\alpha}}_t^{\varepsilon}, t \geq 0\}$. More precisely, we note that

$$\boldsymbol{\beta}^{\varepsilon}(t) = \boldsymbol{\alpha}^{\varepsilon}_{N^{\varepsilon}(t)} = \widetilde{\boldsymbol{\alpha}}^{\varepsilon}_{\sigma_\varepsilon^2 N^{\varepsilon}(t)} = \widetilde{\boldsymbol{\alpha}}^{\varepsilon}_{(1/2)\varepsilon^2\log(1/\varepsilon)N^{\varepsilon}(t)} \tag{3.53}$$

for t at which $N^{\varepsilon}(t)$ jumps. We will apply the last formula with $t = \frac{\pi s}{\varepsilon\log(1/\varepsilon)}$. We have

$$\begin{aligned}&\frac{1}{2}\varepsilon^2\log(1/\varepsilon)N^{\varepsilon}\left(\frac{\pi s}{\varepsilon\log(1/\varepsilon)}\right)\\ &\quad = s + \frac{2}{\pi}\left(\frac{\pi}{2}\varepsilon^2\log(1/\varepsilon)N^{\varepsilon}\left(\frac{\pi s}{\varepsilon\log(1/\varepsilon)}\right) - \frac{\pi s}{2}\right).\end{aligned} \tag{3.54}$$

The jumps of $\boldsymbol{\alpha}^{\varepsilon}$ are uniformly bounded by a quantity going to 0 when $\varepsilon \to 0$, by Lemma 3. This observation, Lemma 7, (3.53) and (3.54) imply that for a fixed $s \geq 0$,

$$\lim_{\varepsilon\to 0}\left|\boldsymbol{\beta}^{\varepsilon}\left(\frac{\pi s}{\varepsilon\log(1/\varepsilon)}\right) - \widetilde{\boldsymbol{\alpha}}_s^{\varepsilon}\right| = 0$$

in probability. This formula, the uniform bound for the jumps of $\boldsymbol{\alpha}^{\varepsilon}$ and weak convergence of processes $\{\widetilde{\boldsymbol{\alpha}}_t^{\varepsilon}, t \geq 0\}$ to Brownian motion imply weak convergence of processes $\{\boldsymbol{\beta}^{\varepsilon}(\pi t/(\varepsilon\log(1/\varepsilon))), t \geq 0\}$ to Brownian motion as $\varepsilon \to 0$. □

4. Reflections in a perturbed annulus: Model and results. We will generalize Theorem 1 to "perturbed annuli" whose boundaries are smooth curves close to circles. The precise definition follows.

For any function $f : \mathbb{R} \to \mathbb{R}$, $\|f\|_\infty$ will denote its supremum norm, that is, $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$.

Let $(f_\varepsilon)_{0<\varepsilon<1/2}$ and $(g_\varepsilon)_{0<\varepsilon<1/2}$ be families of 2π-periodic C^3 functions from $\mathbb{R}$ to $\mathbb{R}$, satisfying the following assumptions:

H1: For all $\alpha \in [0, 2\pi]$, $\varepsilon \leq f_\varepsilon(\alpha) \leq 2\varepsilon$ and $0 \leq g_\varepsilon(\alpha) \leq \varepsilon$,
H2: $f_\varepsilon/\varepsilon$ and $g_\varepsilon/\varepsilon$ converge uniformly to f and g, respectively.
H3: $f_\varepsilon'/\varepsilon$ and $g_\varepsilon'/\varepsilon$ converge uniformly to f' and g', respectively.
H4: $f_\varepsilon''/\varepsilon$ and $g_\varepsilon''/\varepsilon$ converge uniformly to f'' and g'', respesctively.
H5: For some $c < \infty$ and all $\varepsilon \in (0, 1/2)$, $\|f_\varepsilon'''\|_\infty < c$ and $\|g_\varepsilon'''\|_\infty < c$.

REMARK 1. (i) A good example to keep in mind is $f_\varepsilon(\alpha) = \varepsilon f(\alpha)$ and $g_\varepsilon(\alpha) = \varepsilon g(\alpha)$, where $f : \mathbb{R} \to [1, 2]$ and $g : \mathbb{R} \to [0, 1]$ are 2π-periodic C^3 function.

(ii) Assuming **H2**, if $f'_\varepsilon/\varepsilon$ and $g'_\varepsilon/\varepsilon$ converge uniformly then they must converge to f' and g', resp. (see [19], Theorem 7.17).

(iii) Assumption **H1** could have been $c_1\varepsilon \le f_\varepsilon(\alpha) \le c_2\varepsilon$ and $c_3\varepsilon \le g_\varepsilon(\alpha) \le c_4\varepsilon$, for some constants $0 < c_1 < c_2 < \infty$ and $0 \le c_3 < c_4 < \infty$. We gave **H1** its present form to avoid adding further complexity to the already highly complex notation.

It will be convenient to use complex notation occasionally. For example, we will write $e^{i\alpha} = \exp(i\alpha) = (\cos\alpha, \sin\alpha)$.

Given $\varepsilon \in (0, 1/2)$, let $\Gamma^0_\varepsilon, \Gamma^1_\varepsilon$ be closed simple curves parametrized as follows:

$$\Gamma^0_\varepsilon(\alpha) = \big(1 + g_\varepsilon(\alpha)\big)e^{i\alpha}, \qquad \Gamma^1_\varepsilon(\alpha) = \big(1 - f_\varepsilon(\alpha)\big)e^{i\alpha}, \tag{4.1}$$

for $\alpha \in [0, 2\pi)$; the formulas are valid for $\alpha \in \mathbb{R}$ because of the periodicity of f_ε and g_ε. Let $\mathcal{U}^j_\varepsilon$ denote the bounded connected component of $\mathbb{R}^2 \setminus \Gamma^j_\varepsilon$ for $j = 0, 1$.

We consider a ray of light traveling inside $\mathcal{D}_\varepsilon := \overline{\mathcal{U}^0_\varepsilon \setminus \mathcal{U}^1_\varepsilon}$. Its position at time $t \ge 0$ will be denoted by

$$Q^\varepsilon(t) = \mathbf{r}^\varepsilon(t)\exp\big(i\boldsymbol{\beta}^\varepsilon(t)\big).$$

We assume that the trajectory $Q^\varepsilon(t)$ conforms to (**A**) and (2.3) in Section 2.

Our main result on reflections in a perturbed annulus is the following.

THEOREM 2. *Let $h = f + g$. Processes $\{\boldsymbol{\beta}^\varepsilon(\frac{\pi}{\varepsilon\log(1/\varepsilon)}t), t \ge 0\}$ converge in law to X in the Skorokhod topology as ε goes to* 0, *where X solves the stochastic differential equation*

$$dX_t = h'(X_t)\,dt + \sqrt{h(X_t)}\,dW_t, \tag{4.2}$$

and W is standard Brownian motion.

5. Reflections in a perturbed annulus: Proofs. We will encode the nth reflection point as

$$\mathbf{p}_\varepsilon(\boldsymbol{\alpha}^\varepsilon_n, \mathbf{s}^\varepsilon_n)\exp\big(i\boldsymbol{\alpha}^\varepsilon_n\big) = \Gamma^{\mathbf{s}^\varepsilon_n}_\varepsilon(\boldsymbol{\alpha}^\varepsilon_n), \tag{5.1}$$

where $\mathbf{s}^\varepsilon_n$ can be 0 or 1, $\boldsymbol{\alpha}^\varepsilon_n \in \mathbb{R}$ is chosen for $n \ge 0$ so that $|\boldsymbol{\alpha}^\varepsilon_{n+1} - \boldsymbol{\alpha}^\varepsilon_n| < \pi$ and

$$\mathbf{p}_\varepsilon(\alpha, s) = 1 + (1 - s)g_\varepsilon(\alpha) - sf_\varepsilon(\alpha). \tag{5.2}$$

By convention, the first reflection occurs at time $t = 0$.

We will sometimes write $\boldsymbol{\alpha}^\varepsilon(n)$ instead of $\boldsymbol{\alpha}^\varepsilon_n$, for typographical convenience.

It is clear that $\{(\boldsymbol{\alpha}^\varepsilon_n, \mathbf{s}^\varepsilon_n), n \ge 0\}$ is a time homogeneous discrete time Markov chain. Since the light ray travels with speed 1, the time between the kth and $(k-1)$st reflections can be calculated as

$$\Delta\mathcal{T}^\varepsilon_k := \big|\mathbf{p}_\varepsilon(\boldsymbol{\alpha}^\varepsilon_k, \mathbf{s}^\varepsilon_k)\exp\big(i\boldsymbol{\alpha}^\varepsilon_k\big) - \mathbf{p}_\varepsilon(\boldsymbol{\alpha}^\varepsilon_{k-1}, \mathbf{s}^\varepsilon_{k-1})\exp\big(i\boldsymbol{\alpha}^\varepsilon_{k-1}\big)\big|. \tag{5.3}$$

Set $\mathcal{T}_0^\varepsilon = 0$ and for $n \geq 1$,

$$\mathcal{T}_n^\varepsilon = \sum_{k=1}^{n} \Delta \mathcal{T}_k^\varepsilon. \tag{5.4}$$

Given $t > 0$ and $\varepsilon \in (0, 1/2)$, let

$$N^\varepsilon(t) = \inf\{n \geq 0 : \mathcal{T}_{n+1}^\varepsilon > t\} = \sup\{n \geq 0 : \mathcal{T}_n^\varepsilon \leq t\}. \tag{5.5}$$

Recall that $N^\varepsilon(t)$ is the number of reflections made by the light ray before time t, while $\mathcal{T}_n^\varepsilon$ represents the time of the nth reflection. We have

$$Q(\mathcal{T}_n^\varepsilon) = \mathbf{p}_\varepsilon(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon)\exp(i\boldsymbol{\alpha}_n^\varepsilon) = \Gamma_\varepsilon^{\mathbf{s}_n^\varepsilon}(\boldsymbol{\alpha}_n^\varepsilon). \tag{5.6}$$

LEMMA 8. *There exists $\varepsilon_0 \in (0, 1/4)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{U}_\varepsilon^0$ and $\mathcal{U}_\varepsilon^1$ are strictly convex.*

PROOF. For $j = 0, 1$, Γ_ε^j is a closed simple curve, so it suffices to show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, its curvature is strictly positive at every point.

Standard calculations show that the curvature of Γ_ε^0 at $\Gamma_\varepsilon^0(\alpha)$ is given by

$$\kappa_\varepsilon^0(\alpha) := \frac{(1 + g_\varepsilon(\alpha))(1 + g_\varepsilon(\alpha) - g_\varepsilon''(\alpha)) + 2g_\varepsilon'(\alpha)^2}{((1 + g_\varepsilon(\alpha))^2 + g_\varepsilon'(\alpha)^2)^{3/2}}, \tag{5.7}$$

while the curvature of Γ_ε^1 at $\Gamma_\varepsilon^1(\alpha)$ is given by

$$\kappa_\varepsilon^1(\alpha) := \frac{(1 - f_\varepsilon(\alpha))(1 - f_\varepsilon(\alpha) + f_\varepsilon''(\alpha)) + 2f_\varepsilon'(\alpha)^2}{((1 - f_\varepsilon(\alpha))^2 + f_\varepsilon'(\alpha)^2)^{3/2}}. \tag{5.8}$$

By assumptions **H1**–**H4**, it is clear that κ_ε^s converges uniformly to 1 as ε goes to 0, for $s = 0, 1$. $\square$

From now on, we will assume that $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is given in Lemma 8, so that $\mathcal{U}_\varepsilon^0$ and $\mathcal{U}_\varepsilon^1$ are strictly convex for all values of the parameter ε. This implies that when the light ray reflects from Γ_ε^1, the next reflection is from Γ_ε^0. On the other hand, when the light ray reflects from Γ_ε^0 at $\Gamma_\varepsilon^0(\alpha)$, there is a strictly positive probability $p_\varepsilon(\alpha)$ that the next reflection point is again on Γ_ε^0. In other words,

$$p_\varepsilon(\alpha) = \mathbb{P}(\mathbf{s}_{n+1}^\varepsilon = 0 \mid \mathbf{s}_n^\varepsilon = 0, \boldsymbol{\alpha}_n^\varepsilon = \alpha) > 0.$$

Since Γ_ε^0 and Γ_ε^1 are not necessary circles, $p_\varepsilon(\alpha)$ may depend on the reflection point $Q(\mathcal{T}_n^\varepsilon)$.

DEFINITION 2. We will define some random variables for $n \geq 1$, $\alpha \in [0, 2\pi)$ and $\varepsilon \in (0, \varepsilon_0)$.

Let $T_n^\varepsilon(\alpha)$ be a random variable with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 1, \boldsymbol{\alpha}_0^\varepsilon = \alpha\}$, that is, on the event that the light ray starts from $\Gamma_\varepsilon^1(\alpha)$.

Let $R_n^\varepsilon(\alpha)$ be a random variable with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 0, \mathbf{s}_1^\varepsilon = 1, \boldsymbol{\alpha}_0^\varepsilon = \alpha\}$, that is, on the event that the light ray starts from $\Gamma_\varepsilon^0(\alpha)$ and the next reflection is on the inner boundary.

Let $S_n^\varepsilon(\alpha)$ be a random variable with the distribution of $\boldsymbol{\alpha}_1^\varepsilon - \boldsymbol{\alpha}_0^\varepsilon$ conditioned on $\{\mathbf{s}_0^\varepsilon = 0, \mathbf{s}_1^\varepsilon = 0, \boldsymbol{\alpha}_0^\varepsilon = \alpha\}$, that is, on the event that the light ray starts from $\Gamma_\varepsilon^0(\alpha)$ and the next reflection is also on the outer boundary.

Let $\Lambda_n^\varepsilon(\alpha)$ be a random variable with the distribution given by $\mathbb{P}(\Lambda_n^\varepsilon(\alpha) = 1) = 1 - \mathbb{P}(\Lambda_n^\varepsilon(\alpha) = 0) = p_\varepsilon(\alpha)$.

We assume that all random variables listed above, for all $n \geq 1$, $\alpha \in [0, 2\pi)$ and $\varepsilon \in (0, \varepsilon_0)$, are jointly independent.

The process $\{(\boldsymbol{\alpha}_n^\varepsilon, \mathbf{s}_n^\varepsilon), n \geq 0\}$ can be represented as follows. For $n \geq 0$,

$$
\begin{aligned}
\mathbf{s}_{n+1}^\varepsilon &= (1 - \mathbf{s}_n^\varepsilon)(1 - \Lambda_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)),\\
\boldsymbol{\alpha}_{n+1}^\varepsilon &= \boldsymbol{\alpha}_n^\varepsilon + T_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)\mathbf{s}_n^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)\\
&\qquad + (1 - \mathbf{s}_n^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon))(\Lambda_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon) S_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)\\
&\qquad + (1 - \Lambda_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)) R_{n+1}^\varepsilon(\boldsymbol{\alpha}_n^\varepsilon)).
\end{aligned}
\tag{5.9}
$$

LEMMA 9. *For all $n \geq 0$ and $0 < \varepsilon < 1/2$, a.s.*, $|\boldsymbol{\alpha}_{n+1}^\varepsilon - \boldsymbol{\alpha}_n^\varepsilon| \leq 12\sqrt{\varepsilon}$.

PROOF. Let $\mathcal{B}((x, y), r)$ denote the open disc with center (x, y) and radius r. The estimate follows easily from the argument in the proof of Lemma 3 and the fact that $\mathcal{B}((0,0), 1 - 2\varepsilon) \subset \mathcal{D}_\varepsilon \subset \overline{\mathcal{B}}((0,0), 1 + \varepsilon)$. □

For $\alpha \in \mathbb{R}$ and $s \in \{0, 1\}$, let $\gamma_\varepsilon^s(\alpha)$ denote the angle between the inner normal vector in $\mathcal{D}_\varepsilon$ at $\Gamma_\varepsilon^s(\alpha)$ and the vector $(2s - 1)\mathbf{p}_\varepsilon(\alpha, s)e^{i\alpha}$. The latter vector goes from $\mathbf{p}_\varepsilon(\alpha, s)e^{i\alpha}$ to $(0, 0)$, so it has the same direction as $-e^{i\alpha}$. By convention, we choose the sign of $\gamma_\varepsilon^s(\alpha)$ so that it is positive if $s = 0$ and $g_\varepsilon'(\alpha) > 0$, or $s = 1$ and $f_\varepsilon'(\alpha) > 0$. This means that if $\gamma_\varepsilon^s(\alpha) > 0$ for both $s = 0$ and $s = 1$ then $\mathcal{D}_\varepsilon$ is locally widening in the direction of increasing α.

LEMMA 10. *We have*

$$
\gamma_\varepsilon^0(\alpha) = \arcsin\left(\frac{g_\varepsilon'(\alpha)}{\sqrt{(1 + g_\varepsilon(\alpha))^2 + g_\varepsilon'(\alpha)^2}}\right),
\tag{5.10}
$$

$$
\gamma_\varepsilon^1(\alpha) = \arcsin\left(\frac{f_\varepsilon'(\alpha)}{\sqrt{(1 - f_\varepsilon(\alpha))^2 + f_\varepsilon'(\alpha)^2}}\right).
\tag{5.11}
$$

PROOF. Let $\langle\cdot,\cdot\rangle$ denote the scalar product. Then

$$\gamma_\varepsilon^0(\alpha) = \arcsin\left\langle \frac{(\Gamma_\varepsilon^0)'(\alpha)}{|(\Gamma_\varepsilon^0)'(\alpha)|}, e^{i\alpha}\right\rangle.$$

We have

$$(\Gamma_\varepsilon^0)'(\alpha) = \frac{d}{d\alpha}\big((1+g_\varepsilon(\alpha))e^{i\alpha}\big) = \big(1+g_\varepsilon(\alpha)\big)ie^{i\alpha} + g_\varepsilon'(\alpha)e^{i\alpha},$$

so that

$$\big\langle(\Gamma_\varepsilon^0)'(\alpha), e^{i\alpha}\big\rangle = g_\varepsilon'(\alpha). \tag{5.12}$$

Thus

$$\gamma_\varepsilon^0(\alpha) = \arcsin\left\langle \frac{(\Gamma_\varepsilon^0)'(\alpha)}{|(\Gamma_\varepsilon^0)'(\alpha)|}, e^{i\alpha}\right\rangle = \arcsin\left(\frac{g_\varepsilon'(\alpha)}{\sqrt{(1+g_\varepsilon(\alpha))^2 + g_\varepsilon'(\alpha)^2}}\right).$$

This proves (5.10). The proof of (5.11) is analogous. $\square$

LEMMA 11. *For some $\varepsilon_1 > 0$, all $\varepsilon \in (0,\varepsilon_1)$ and all $\alpha \in \mathbb{R}$,*

$$\varepsilon/2 \le p_\varepsilon(\alpha) \le \varepsilon\big(4 + 6\|g'\|_\infty\big). \tag{5.13}$$

PROOF. Let L be the straight line passing through $\Gamma_\varepsilon^0(\alpha)$ and orthogonal to Γ_ε^0 at this point. It follows from **H3** and Lemma 10 that for some $\varepsilon_2 > 0$ and all $\varepsilon \in (0,\varepsilon_2)$, the angle θ_ε between L and the line segment with endpoints $\Gamma_\varepsilon^0(\alpha)$ and $(0,0)$ is less than $2\varepsilon\|g'\|_\infty$. Hence, for some $\varepsilon_3 > 0$ and all $\varepsilon \in (0,\varepsilon_3)$, the distance between L and $(0,0)$ is less than $3\varepsilon\|g'\|_\infty$. Let x be the point in L closest to $(0,0)$. Thus, $\mathrm{dist}(x,(0,0)) < 3\varepsilon\|g'\|_\infty$ for small ε. Assumption **H1** and (4.1) imply that the circle $\mathcal{C}(x, 1-\varepsilon(2+3\|g'\|_\infty))$ lies inside $\mathcal{U}_\varepsilon^1$. Hence, a light ray starting from $\Gamma_\varepsilon^0(\alpha)$ will hit Γ_ε^1 before hitting the circle $\mathcal{C}(x, 1-\varepsilon(2+3\|g'\|_\infty))$. The Lambertian direction of the light ray starting from $\Gamma_\varepsilon^0(\alpha)$, defined as in (2.3), is the same whether it is defined relative to $\mathcal{D}_\varepsilon$ or the interior of $\mathcal{C}(x, |\Gamma_\varepsilon^0(\alpha) - x|)$ because the boundaries of the two domains are tangent at $\Gamma_\varepsilon^0(\alpha)$. We can apply Lemma 2 to the domain between the circles $\mathcal{C}(x, 1-\varepsilon(2+3\|g'\|_\infty))$ and $\mathcal{C}(x, |\Gamma_\varepsilon^0(\alpha) - x|)$. By **H1**, for small $\varepsilon > 0$,

$$\begin{aligned} 1 - 3\varepsilon\|g'\|_\infty \le \big|\Gamma_\varepsilon^0(\alpha) - x\big| &\le \big|\Gamma_\varepsilon^0(\alpha) - (0,0)\big| + \big|(0,0) - x\big| \\ &\le 1 + \varepsilon + 3\varepsilon\|g'\|_\infty. \end{aligned} \tag{5.14}$$

Lemma 2, (5.14) and rescaling by the factor of $|\Gamma_\varepsilon^0(\alpha) - x|$ imply that, for sufficiently small $\varepsilon > 0$,

$$p_\varepsilon(\alpha) \le \frac{|\Gamma_\varepsilon^0(\alpha) - x| - (1 - \varepsilon(2 + 3\|g'\|_\infty))}{|\Gamma_\varepsilon^0(\alpha) - x|}$$

$$\le \frac{1+\varepsilon+3\varepsilon\|g'\|_\infty-(1-\varepsilon(2+3\|g'\|_\infty))}{1-3\varepsilon\|g'\|_\infty}$$
$$= \frac{\varepsilon(3+6\|g'\|_\infty))}{1-3\varepsilon\|g'\|_\infty} \le \varepsilon(4+6\|g'\|_\infty)).$$

Assumption **H1** and (4.1) imply that the circle $\mathcal{U}_\varepsilon^1$ lies inside the circle $\mathcal{C}((0,0),1-\varepsilon)$, and $|\Gamma_\varepsilon^0(\alpha)-(0,0)|\ge 1$. Since the line L does not have to pass through $(0,0)$, we can use only one half of the estimate in Lemma 2 to conclude that for a light ray starting from $\Gamma_\varepsilon^0(\alpha)$ with the Lambertian direction, the probability of avoiding of $\mathcal{C}((0,0),1-\varepsilon)$ is bounded below by $\varepsilon/2$. Hence, the probability of avoiding $\mathcal{U}_\varepsilon^1$ is also bounded below by $\varepsilon/2$. This proves the lower bound. $\square$

LEMMA 12. *The following assertions hold uniformly in $\alpha \in [0,2\pi)$:*

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}(T_1^\varepsilon(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h'(\alpha)h(\alpha), \tag{5.15}$$

$$\lim_{\varepsilon\to 0}\frac{\operatorname{Var}(T_1^\varepsilon(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\alpha). \tag{5.16}$$

PROOF. (i) We will prove the lemma for $\alpha=\pi/2$. This will cause no loss of generality because the constants in our estimates do not dependent on α.

Let $L(\alpha)$ be the straight line passing through $\Gamma_\varepsilon^0(\alpha)$ in the direction of the normal vector to Γ_ε^0 at $\Gamma_\varepsilon^0(\alpha)$, for $\alpha\in[0,2\pi)$.

It follows from **H3** that for some $c_1<\infty$ and all $\varepsilon\in(0,1/2)$, we have $\|g_\varepsilon'\|_\infty < c_1\varepsilon$. Let $c_2=30c_1$ and $\alpha_1=\pi/2-c_2\varepsilon^2$. The unsigned angle between $L(\alpha_1)$ and the vertical axis is $\rho_0:=\pi/2-\alpha_1+\arctan(\frac{g_\varepsilon'(\alpha_1)}{1+g_\varepsilon(\alpha_1)})$. For small $\varepsilon>0$, $\rho_0\le 2c_1\varepsilon$, so $\tan\rho_0\le 3c_1\varepsilon$. It is possible that $L(\alpha_1)$ does not cross the vertical axis below $\Gamma_\varepsilon^0(\alpha_1)$. Suppose that it does and denote by $(0,u_1)$ the intersection point with the vertical axis. Then

$$\bigl(1+g_\varepsilon(\alpha_1)\bigr)\cos\bigl(c_2\varepsilon^2\bigr)-u_1=\frac{(1+g_\varepsilon(\alpha_1))\sin(c_2\varepsilon^2)}{\tan(\rho_0)}.$$

Therefore, by **H1**, for small $\varepsilon>0$,

$$1-u_1=\frac{(1+g_\varepsilon(\alpha_1))\sin(c_2\varepsilon^2)}{\tan(\rho_0)}-g_\varepsilon(\alpha_1)\cos\bigl(c_2\varepsilon^2\bigr)+1-\cos\bigl(c_2\varepsilon^2\bigr)$$
$$\ge\frac{c_2\varepsilon^2/2}{\tan(\rho_0)}-\|g_\varepsilon\|_\infty\ge\left(\frac{c_2}{6c_1}-2\right)\varepsilon=3\varepsilon.$$

Since $f_\varepsilon(\pi/2)\le 2\varepsilon$, $L(\alpha_1)$ crosses the vertical axis below $\Gamma_\varepsilon^1(\pi/2)$, and stays to the right of $\Gamma_\varepsilon^1(\pi/2)$ above $(0,u_1)$. We have shown that no matter whether $L(\alpha_1)$ crosses the vertical axis below $\Gamma_\varepsilon^0(\alpha_1)$ or not, it stays to the right of $\Gamma_\varepsilon^1(\pi/2)$.

An analogous argument shows that if $\alpha_2 = \pi/2 + c_2\varepsilon^2$ then $L(\alpha_2)$ stays to the left of $\Gamma_\varepsilon^1(\pi/2)$. Since g_ε is C^3, when α varies continuously from $\pi/2 - c_2\varepsilon^2$ to $\pi/2 + c_2\varepsilon^2$, we must encounter α' such that $L(\alpha')$ passes through $\Gamma_\varepsilon^1(\pi/2)$. We have

$$|\pi/2 - \alpha'| = O(\varepsilon^2). \tag{5.17}$$

Let $\mathcal{R}$ be the rotation about $\Gamma_\varepsilon^1(\pi/2)$, with the angle of rotation ρ chosen so that $\mathcal{R}(\Gamma_\varepsilon^0(\alpha')) = (0, z)$ with $z \ge 1$. We will estimate ρ. The angle between $L(\alpha')$ and the line segment between 0 and $e^{i\alpha'}$ is equal to $\gamma_\varepsilon^0(\alpha')$, by definition. Hence, the angle between $L(\alpha')$ and the vertical line (i.e., ρ) is $\gamma_\varepsilon^0(\alpha') + (\pi/2 - \alpha')$. Recall that $\|g_\varepsilon'\|_\infty < c_1\varepsilon$ and a similar bound holds for g_ε. This, (5.10), **H1**–**H4**, and the Taylor expansion imply that

$$\begin{aligned}
\rho &= \gamma_\varepsilon^0(\alpha') + (\pi/2 - \alpha') = g_\varepsilon'(\alpha')(1 + O(\varepsilon)) + O(\varepsilon^2) \\
&= g_\varepsilon'(\alpha') + O(\varepsilon^2) = g_\varepsilon'(\pi/2)(1 + O(\varepsilon(\pi/2 - \alpha'))) + O(\varepsilon^2) \\
&= g_\varepsilon'(\pi/2) + O(\varepsilon^2) = O(\varepsilon).
\end{aligned} \tag{5.18}$$

Let $\mathcal{C}$ be the osculating circle of $\mathcal{R}(\Gamma_\varepsilon^0)$ at $(0, z)$. We have chosen ρ so that the topmost point of the circle is on the vertical axis. The radius of $\mathcal{C}$ is $1/|\kappa_\varepsilon^0(\alpha')|$ [see (5.7) for a formula for the curvature κ_ε^0], so the center of $\mathcal{C}$ is at $(0, z_1) := (0, z - 1/|\kappa_\varepsilon^0(\alpha')|)$. We will now estimate z and z_1. The definition of z and the formula for the image of $\Gamma_\varepsilon^0(\alpha') = ((1 + g_\varepsilon(\alpha'))\cos(\alpha'), (1 + g_\varepsilon(\alpha'))\sin(\alpha'))$ under rotation about $(0, 1 - f_\varepsilon(\pi/2))$ by angle ρ yield

$$\begin{aligned}
z &= (1 - f_\varepsilon(\pi/2))(1 - \cos(\rho)) + \sin(\rho)(1 + g_\varepsilon(\alpha'))\cos(\alpha') \\
&\quad + \cos(\rho)(1 + g_\varepsilon(\alpha'))\sin(\alpha') \\
&= (1 - f_\varepsilon(\pi/2))(1 - \cos(\rho)) + (1 + g_\varepsilon(\alpha'))\sin(\rho + \alpha').
\end{aligned}$$

This, **H1**, **H3**, (5.17) and (5.18) show that

$$\begin{aligned}
&|z - (1 + g_\varepsilon(\pi/2))| \\
&\quad = |(1 - f_\varepsilon(\pi/2))(1 - \cos(\rho)) + (1 + g_\varepsilon(\alpha'))\sin(\rho + \alpha') \\
&\qquad - (1 + g_\varepsilon(\pi/2))| \\
&\quad = |(1 - f_\varepsilon(\pi/2))(1 - \cos(\rho)) + (1 + g_\varepsilon(\alpha')) \\
&\qquad + (1 + g_\varepsilon(\alpha'))(\sin(\rho + \alpha') - 1) - (1 + g_\varepsilon(\pi/2))| \\
&\quad = |(1 - f_\varepsilon(\pi/2))(1 - \cos(\rho)) + (g_\varepsilon(\alpha') - g_\varepsilon(\pi/2)) \\
&\qquad + (1 + g_\varepsilon(\alpha'))(\sin(\rho + \alpha') - 1)| \\
&\quad = |(1 - f_\varepsilon(\pi/2))O(\rho^2) + (g_\varepsilon(\alpha') - g_\varepsilon(\pi/2))
\end{aligned} \tag{5.19}$$

$$+\big(1+g_\varepsilon(\alpha')\big)O\big((\pi/2-\alpha'-\rho)^2\big)\big|$$
$$=O(\varepsilon^2)+O(\varepsilon^2)+O(\varepsilon^2)=O(\varepsilon^2).$$

Assumptions **H1**–**H4**, (5.7) and the Taylor expansion imply that $|\kappa_\varepsilon^0(\alpha)|=1+O(\varepsilon)$ uniformly in α. We combine this with **H2** and (5.19) to see that

$$|z_1|=\big|z-1/|\kappa_\varepsilon^0(\alpha')|\big|=O(\varepsilon). \tag{5.20}$$

Suppose that a light ray leaves $\Gamma_\varepsilon^1(\pi/2)$ at an angle θ, relative to the normal vector to Γ_ε^1 at $\Gamma_\varepsilon^1(\pi/2)$. The light ray will intersect $\mathcal{R}(\Gamma_\varepsilon^0)$ at a point that we will denote $r'(\theta)\exp(i(\pi/2+T'(\theta)))$. In other words, $T'(\theta)$ denotes the angular distance between $\Gamma_\varepsilon^1(\pi/2)$ and the intersection of the light ray with $\mathcal{R}(\Gamma_\varepsilon^0)$. The same light ray will intersect the circle $\mathcal{C}$ at a point $\widehat{r}(\theta)\exp(i(\pi/2+\widehat{T}(\theta)))$.

For later reference, we record the following estimates valid for all θ. They follow from an argument similar to the one used in the proof of Lemma 9:

$$|T_1^\varepsilon|=O(\varepsilon^{1/2}),\qquad |T'(\theta)|=O(\varepsilon^{1/2}),\qquad |\widehat{T}(\theta)|=O(\varepsilon^{1/2}). \tag{5.21}$$

If we recall the notation from (2.3) and write $T_1^\varepsilon(\pi/2,\theta)=T_1^\varepsilon(\pi/2)$ to emphasize the dependence on θ, then

$$T'(\theta+\rho)=T_1^\varepsilon(\pi/2,\theta). \tag{5.22}$$

The curvature of $\mathcal{C}$ matches that of $\mathcal{R}(\Gamma_\varepsilon^0)$ at $(0,z)$, by the definition of the osculating circle. Hence, if $v\in\mathcal{C}$ and $\operatorname{dist}(v,(0,z))=b<10\sqrt{\varepsilon}$ then for some $c_1<\infty$ (not depending on our choice of $\boldsymbol{\alpha}_0^\varepsilon=\pi/2$, in view of **H5**), the distance from v to $\mathcal{R}(\Gamma_\varepsilon^0)$ is less than c_1b^3. This and an elementary analysis of the triangle with vertices $\widehat{T}(\theta)$, $T'(\theta)$ and $\Gamma_\varepsilon^0(\widehat{T}(\theta))$ shows that

$$\big|\widehat{T}(\theta)-T'(\theta)\big|=O\big(\widehat{T}(\theta)^3\tan\theta\big). \tag{5.23}$$

We will need a stronger version of this estimate for $\theta\le-\pi/2+c_3\varepsilon^{1/2}$ and $\theta\ge\pi/2-c_3\varepsilon^{1/2}$. If θ is in this range, $|\widehat{T}(\theta)|\ge c_4\varepsilon^{1/2}$. It follows that, for some $c_5>0$, the slope of the osculating circle $\mathcal{C}$ at $\widehat{r}(\theta)\exp(i(\pi/2+\widehat{T}(\theta)))$, considered to be the graph of a function in the usual coordinate system, is greater than $c_5\varepsilon^{1/2}$ for $\theta\le-\pi/2+c_3\varepsilon^{1/2}$ and smaller than $-c_5\varepsilon^{1/2}$ for $\theta\ge\pi/2-c_3\varepsilon^{1/2}$. The same remark applies to the slope of $\mathcal{R}(\Gamma_\varepsilon^0)$ at $r'(\theta)\exp(i(\pi/2+T'(\theta)))$. Hence, for $\theta\le-\pi/2+c_3\varepsilon^{1/2}$ and $\theta\ge\pi/2-c_3\varepsilon^{1/2}$,

$$\big|\widehat{T}(\theta)-T'(\theta)\big|=O\big(\widehat{T}(\theta)^3\varepsilon^{-1/2}\big). \tag{5.24}$$

We will write

$$(x,y)=\big(x(\theta),y(\theta)\big)=\widehat{r}(\theta)\exp\big(i\widehat{T}(\theta)\big) \tag{5.25}$$

and we will find a formula for x in terms of θ, f_ε and g_ε. If we let $a=1/\tan\theta$, then $y(\theta)=ax(\theta)+1-f_\varepsilon(\pi/2)$. Since $(x,y)\in\mathcal{C}((0,z_1),1/|\kappa_\varepsilon^0(\alpha')|)$,

$$x^2+y^2=x^2+\big(ax+1-f_\varepsilon(\pi/2)-z_1\big)^2=\kappa_\varepsilon^0(\alpha')^{-2}.$$

This and $a = 1/\tan\theta$ yield, for $a > 0$,

$$
\begin{aligned}
x(\theta) &= \frac{-a(1 - f_\varepsilon(\pi/2) - z_1) + \sqrt{(a^2+1)\kappa_\varepsilon^0(\alpha')^{-2} - (1 - f_\varepsilon(\pi/2) - z_1)^2}}{1 + a^2} \\
&= \sin\theta\cos\theta\big(f_\varepsilon(\pi/2) + z_1 - 1\big) \\
&\quad + \sin\theta\cos\theta\sqrt{\kappa_\varepsilon^0(\alpha')^{-2} + \tan^2\theta\big(\kappa_\varepsilon^0(\alpha')^{-2} - \big(1 - f_\varepsilon(\pi/2) - z_1\big)^2\big)}.
\end{aligned}
\tag{5.26}
$$

We have

$$
\widehat{T}(\theta) = \arctan\big(x(\theta)/y(\theta)\big) = \arctan\bigg(\frac{x(\theta)}{x(\theta)/\tan\theta + 1 - f_\varepsilon(\pi/2)}\bigg). \tag{5.27}
$$

The density of the angle of reflection given in (2.3) is relative to the normal vector at the boundary of the domain, which is tilted by $\gamma_\varepsilon^1(\pi/2)$ relative to the vertical if the reflection takes place at $\Gamma_\varepsilon^1(\pi/2)$, so

$$
\mathbb{E}\widehat{T}(\Theta + \rho) = \int_{-\pi/2}^{\pi/2} \widehat{T}\big(\theta + \rho + \gamma_\varepsilon^1(\pi/2)\big)\frac{1}{2}\cos\theta\, d\theta. \tag{5.28}
$$

Let $\rho_1 = \rho + \gamma_\varepsilon^1(\pi/2)$. We will assume that $\rho_1 \geq 0$. The argument is analogous in the opposite case. We have

$$
\begin{aligned}
\mathbb{E}\widehat{T}(\Theta + \rho) &= \int_{-\pi/2}^{\pi/2} \widehat{T}(\theta + \rho_1)\frac{1}{2}\cos\theta\, d\theta \\
&= \int_{-\pi/2}^{\pi/2 - 2\rho_1} \widehat{T}(\theta + \rho_1)\frac{1}{2}\cos\theta\, d\theta \\
&\quad + \int_{\pi/2 - 2\rho_1}^{\pi/2} \widehat{T}(\theta + \rho_1)\frac{1}{2}\cos\theta\, d\theta.
\end{aligned}
\tag{5.29}
$$

We will estimate the two integrals separately. We start with the first integral:

$$
\int_{-\pi/2}^{\pi/2 - 2\rho_1} \widehat{T}(\theta + \rho_1)\frac{1}{2}\cos\theta\, d\theta = \int_{-\pi/2 + \rho_1}^{\pi/2 - \rho_1} \widehat{T}(\theta)\frac{1}{2}\cos(\theta - \rho_1)\, d\theta. \tag{5.30}
$$

Recall that $\cos(\theta - \rho_1) = \cos\theta\cos\rho_1 + \sin\theta\sin\rho_1$. Note that $\theta \to \widehat{T}(\theta)$ is an odd function. Thus

$$
\begin{aligned}
\int_{-\pi/2}^{\pi/2 - 2\rho_1} \widehat{T}(\theta + \rho_1)\frac{1}{2}\cos\theta\, d\theta &= \frac{1}{2}\sin\rho_1 \int_{-\pi/2 + \rho_1}^{\pi/2 - \rho_1} \widehat{T}(\theta)\sin\theta\, d\theta \\
&= \sin\rho_1 \int_0^{\pi/2 - \rho_1} \widehat{T}(\theta)\sin\theta\, d\theta.
\end{aligned}
\tag{5.31}
$$

Once again, we will analyze the factors on the right-hand side separately. First, by (5.18) and Lemma 10,

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\sin\rho_1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\sin\big(\rho + \gamma_\varepsilon^1(\pi/2)\big)
$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sin\bigl(\gamma_\varepsilon^0(\pi/2)+O(\varepsilon^2)+\gamma_\varepsilon^1(\pi/2)\bigr) \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sin\bigl(\gamma_\varepsilon^0(\pi/2)+\gamma_\varepsilon^1(\pi/2)\bigr)=h'(\pi/2).
\end{aligned}
\tag{5.32}
$$

Next, we tackle the integral on the right-hand side of (5.31). It is easy to see the $y(\theta)$ converges to 1 and $x(\theta)$ converges to 0, both uniformly in θ, as $\varepsilon\to 0$. This observation and (5.27) imply that

$$
\lim_{\varepsilon\to 0}\widehat{T}(\theta)/x(\theta)=1, \tag{5.33}
$$

uniformly in θ, so

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\int_0^{\pi/2-\rho_1}\widehat{T}(\theta)\sin\theta\,d\theta \\
&\quad= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\int_0^{\pi/2-\rho_1}x(\theta)\sin\theta\,d\theta,
\end{aligned}
\tag{5.34}
$$

assuming that at least one of these limits exists.

In order to estimate the integral on the right-hand side of (5.34), we will split the interval of integration into two parts. Set $h_\varepsilon=f_\varepsilon+g_\varepsilon$ and

$$
\theta_0=\frac{\pi}{2}-\sqrt{\varepsilon}.
$$

An easy argument, similar to the one in the the proof of Lemma 3, shows that for some c_3, all $\varepsilon\in(0,1/2)$ and all θ, $|x(\theta)|\le c_3\sqrt{\varepsilon}$. Hence

$$
\left|\int_{\theta_0}^{\pi/2-\rho_1}x(\theta)\sin\theta\,d\theta\right|\le\int_{\theta_0}^{\pi/2}\bigl|x(\theta)\bigr|\,d\theta\le c_3\sqrt{\varepsilon}(\pi/2-\theta_0)=O(\varepsilon),
$$

and, therefore,

$$
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\left|\int_{\theta_0}^{\pi/2-\rho_1}x(\theta)\sin\theta\,d\theta\right|=0. \tag{5.35}
$$

Recall from (5.26) that

$$
\begin{aligned}
x(\theta)&=\sin\theta\cos\theta\bigl(f_\varepsilon(\pi/2)+z_1-1\bigr) \\
&\quad+\sin\theta\cos\theta \\
&\quad\times\sqrt{\kappa_\varepsilon^0(\alpha')^{-2}+\tan^2\theta\bigl(\kappa_\varepsilon^0(\alpha')^{-2}-\bigl(1-f_\varepsilon(\pi/2)-z_1\bigr)^2\bigr)}.
\end{aligned}
\tag{5.36}
$$

It follows from (5.19) that

$$
\bigl|\kappa_\varepsilon^0(\alpha')\bigr|^{-1}=z-z_1=1+g_\varepsilon(\pi/2)-z_1+O(\varepsilon^2), \tag{5.37}
$$

so

$$
\kappa_\varepsilon^0(\alpha')^{-2}=\bigl(1+g_\varepsilon(\pi/2)-z_1\bigr)^2+O(\varepsilon^2). \tag{5.38}
$$

This implies the following representation for the expression under square root in (5.36):

$$\begin{aligned}
&\sqrt{\kappa_\varepsilon^0(\alpha')^{-2} + \tan^2\theta\big(\kappa_\varepsilon^0(\alpha')^{-2} - (1 - f_\varepsilon(\pi/2) - z_1)^2\big)}\\
&\qquad = \big((1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2)\\
&\qquad\quad + \tan^2\theta\big((1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2) - (1 - f_\varepsilon(\pi/2) - z_1)^2\big)\big)^{1/2}.
\end{aligned} \tag{5.39}$$

It follows from **H2** and (5.20) that

$$\begin{aligned}
&(1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2) - (1 - f_\varepsilon(\pi/2) - z_1)^2 \qquad (5.40)\\
&\qquad = g_\varepsilon(\pi/2)^2 - f_\varepsilon(\pi/2)^2 + 2(1 - z_1)\big(g_\varepsilon(\pi/2) + f_\varepsilon(\pi/2)\big) + O(\varepsilon^2)\\
&\qquad = 2(1 - z_1)h_\varepsilon(\pi/2) + O(\varepsilon^2) = O(\varepsilon). \qquad (5.41)
\end{aligned}$$

Since $\tan\theta = O(\varepsilon^{1/2})$ for $0 \le \theta \le \theta_0$, the above estimate implies that

$$\tan^2\theta\big((1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2) - (1 - f_\varepsilon(\pi/2) - z_1)^2\big) = O(\varepsilon^2).$$

This and (5.40)–(5.41) imply that we can apply the Taylor expansion to the right-hand side of (5.39) as follows:

$$\begin{aligned}
&\big((1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2)\\
&\quad + \tan^2\theta\big((1 + g_\varepsilon(\pi/2) - z_1)^2 + O(\varepsilon^2) - (1 - f_\varepsilon(\pi/2) - z_1)^2\big)\big)^{1/2}\\
&\quad = 1 + g_\varepsilon(\pi/2) - z_1 + \frac{\tan^2\theta(2(1 + g_\varepsilon(\pi/2) - z_1)h_\varepsilon(\pi/2) + O(\varepsilon^2))}{2(1 + g_\varepsilon(\pi/2) - z_1)}\\
&\qquad + O\big(\tan^4\theta h_\varepsilon(\pi/2)^2\big)\\
&\quad = 1 + g_\varepsilon(\pi/2) - z_1 + \tan^2\theta h_\varepsilon(\pi/2) + \tan^2\theta O(\varepsilon^2) + O\big(\tan^4\theta h_\varepsilon(\pi/2)^2\big)\\
&\quad = 1 + g_\varepsilon(\pi/2) - z_1 + \tan^2\theta h_\varepsilon(\pi/2) + \big(\tan^2\theta + \tan^4\theta\big)O(\varepsilon^2).
\end{aligned}$$

We combine this with (5.36), (5.39) and (5.41) to obtain

$$x(\theta) = \sin\theta\cos\theta\big(h_\varepsilon(\pi/2)\big(1 + \tan^2\theta\big) + \big(\tan^2\theta + \tan^4\theta\big)O(\varepsilon^2)\big). \tag{5.42}$$

Hence,

$$\begin{aligned}
&\int_0^{\theta_0} x(\theta)\sin\theta\,d\theta\\
&\qquad = \int_0^{\theta_0} \sin\theta\cos\theta\big(h_\varepsilon(\pi/2)\big(1 + \tan^2\theta\big) + \big(\tan^2\theta + \tan^4\theta\big)O(\varepsilon^2)\big)\\
&\qquad\quad \times \sin\theta\,d\theta\\
&\qquad = h_\varepsilon(\pi/2)\int_0^{\theta_0} \frac{\sin^2\theta}{\cos\theta}\,d\theta + O(\varepsilon^2)\int_0^{\theta_0}\left(\frac{\sin^4\theta}{\cos\theta} + \frac{\sin^6\theta}{\cos^3\theta}\right)d\theta.
\end{aligned} \tag{5.43}$$

We use (3.26)–(3.27) in the following calculation:

$$h_\varepsilon(\pi/2)\int_0^{\theta_0}\frac{\sin^2\theta}{\cos\theta}\,d\theta = -\frac{h_\varepsilon(\pi/2)}{2}\log\bigl(1-\cos(\pi/2-\theta_0)\bigr)$$
$$= -\frac{h_\varepsilon(\pi/2)}{2}\log\bigl(1-\cos(\sqrt{\varepsilon})\bigr)$$
$$= \frac{h_\varepsilon(\pi/2)}{2}\log\Bigl(\frac{1+o(1)}{\varepsilon}\Bigr).$$

Thus, in view of **H2**,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}h_\varepsilon(\pi/2)\int_0^{\theta_0}\frac{\sin^2\theta}{\cos\theta}\,d\theta\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\frac{h_\varepsilon(\pi/2)}{2}\log\Bigl(\frac{1+o(1)}{\varepsilon}\Bigr)\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{2}\frac{h_\varepsilon(\pi/2)}{\varepsilon}=h(\pi/2)/2.
\end{aligned}\tag{5.44}$$

We use (3.24) and **H2** as follows:

$$O(\varepsilon^2)\int_0^{\theta_0}\Bigl(\frac{\sin^4\theta}{\cos\theta}+\frac{\sin^6\theta}{\cos^3\theta}\Bigr)d\theta \le O(\varepsilon^2)2\int_0^{\theta_0}\frac{1}{\cos^3\theta}\,d\theta\le O(\varepsilon^2)2\frac{\pi(\pi-\theta_0)\theta_0}{(\pi-2\theta_0)^2}$$
$$=O(\varepsilon),$$

from which we conclude that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}O(\varepsilon^2)\int_0^{\theta_0}\Bigl(\frac{\sin^4\theta}{\cos\theta}+\frac{\sin^6\theta}{\cos^3\theta}\Bigr)d\theta=0.$$

We combine this with (5.43) and (5.44) to conclude that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\int_0^{\theta_0}x(\theta)\sin\theta\,d\theta=h(\pi/2)/2.$$

Thus, in view of (5.34) and (5.35),

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log(1/\varepsilon)}\int_0^{\pi/2-\rho_1}\widehat{T}(\theta)\sin\theta\,d\theta=h(\pi/2)/2.$$

Combining the formula with (5.31) and (5.32) yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2\log(1/\varepsilon)}\int_{-\pi/2}^{\pi/2-2\rho_1}\widehat{T}(\theta+\rho_1)\frac{1}{2}\cos\theta\,d\theta=h'(\pi/2)h(\pi/2)/2.\tag{5.45}$$

We next estimate the second integral on the right-hand side of (5.29). Recall that $\rho_1=\rho+\gamma_\varepsilon^1(\pi/2)$. We have

$$|\rho_1|=O(\varepsilon)\tag{5.46}$$

in view of (5.11) and (5.18). Hence (5.21) gives

$$\int_{\pi/2-2\rho_1}^{\pi/2} \widehat{T}(\theta+\rho_1)\frac{1}{2}\cos\theta\,d\theta = O(\varepsilon^{1/2})\int_{\pi/2-2\rho_1}^{\pi/2}\cos\theta\,d\theta$$
$$= O(\varepsilon^{1/2})O(\rho_1^2) = O(\varepsilon^{5/2}).$$

This, (5.29) and (5.45) yield

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}\widehat{T}(\Theta+\rho)}{\varepsilon^2\log(1/\varepsilon)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2\log(1/\varepsilon)}\int_{-\pi/2}^{\pi/2}\widehat{T}(\theta+\rho_1)\frac{1}{2}\cos\theta\,d\theta \tag{5.47}$$
$$= h'(\pi/2)h(\pi/2)/2.$$

We will now estimate $\mathbb{E}|\widehat{T}(\theta+\rho) - T_1^\varepsilon(\pi/2,\theta)|$. By (5.22), (5.23) and (5.24), for some c_4,

$$\begin{aligned}
&\mathbb{E}\big|\widehat{T}(\Theta+\rho) - T_1^\varepsilon(\pi/2,\Theta)\big| \\
&\quad = \mathbb{E}\big|\widehat{T}(\Theta+\rho) - T'(\Theta+\rho)\big| \\
&\quad \le c_4\mathbb{E}\big(\big|\widehat{T}(\Theta+\rho)^3\tan(\Theta+\rho)\big|\mathbf{1}_{(-\pi/2+\varepsilon^{1/2},\pi/2-\varepsilon^{1/2}-2\rho_1)}(\Theta)\big) \\
&\qquad + c_4\mathbb{E}\big(\big|\widehat{T}(\Theta+\rho)^3\varepsilon^{-1/2}\big|\mathbf{1}_{(-\pi/2,-\pi/2+\varepsilon^{1/2})\cup(\pi/2-\varepsilon^{1/2}-2\rho_1,\pi/2)}(\Theta)\big).
\end{aligned} \tag{5.48}$$

By (5.21),

$$\begin{aligned}
&\mathbb{E}\big(\big|\widehat{T}(\Theta+\rho)^3\varepsilon^{-1/2}\big|\mathbf{1}_{(-\pi/2,-\pi/2+\varepsilon^{1/2})\cup(\pi/2-\varepsilon^{1/2}-2\rho_1,\pi/2)}(\Theta)\big) \\
&\qquad \le O(\varepsilon)\bigg(\int_{-\pi/2}^{-\pi/2+\varepsilon^{1/2}} + \int_{\pi/2-\varepsilon^{1/2}-2\rho_1}^{\pi/2}\bigg)\frac{1}{2}\cos\theta\,d\theta = O(\varepsilon^2).
\end{aligned} \tag{5.49}$$

We calculate as in (5.29) and (5.30), use the fact that $\theta\to|\widehat{T}(\theta)^3\tan(\theta)|$ is even, and then apply (5.33),

$$\begin{aligned}
&\mathbb{E}\big(\big|\widehat{T}(\Theta+\rho)^3\tan(\Theta+\rho)\big|\mathbf{1}_{(-\pi/2+\varepsilon^{1/2},\pi/2-\varepsilon^{1/2}-2\rho_1)}(\Theta)\big) \\
&\quad = \int_{-\pi/2+\varepsilon^{1/2}}^{\pi/2-\varepsilon^{1/2}-2\rho_1}\big|\widehat{T}(\theta+\rho_1)^3\tan(\theta+\rho_1)\big|\frac{1}{2}\cos\theta\,d\theta \\
&\quad = \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1}\big|\widehat{T}(\theta)^3\tan(\theta)\big|\frac{1}{2}\cos(\theta-\rho_1)\,d\theta \\
&\quad = \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1}\big|\widehat{T}(\theta)^3\tan(\theta)\big|\frac{1}{2}(\cos\theta\cos\rho_1+\sin\theta\sin\rho_1)\,d\theta \\
&\quad \le \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1}\big|\widehat{T}(\theta)^3\tan(\theta)\big|\frac{1}{2}\cos\theta\,d\theta \\
&\quad \le \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1}\big|\widehat{T}(\theta)^3\big|\,d\theta \le \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1}\big|x(\theta)^3\big|\,d\theta.
\end{aligned} \tag{5.50}$$

It follows from (5.42) that

$$|x(\theta)| = O(\varepsilon)(\cos\theta)^{-1} + O(\varepsilon^2)(\cos\theta)^{-3},$$

so

$$|x(\theta)|^3 = O(\varepsilon^3)(\cos\theta)^{-3} + O(\varepsilon^6)(\cos\theta)^{-9}.$$

These bounds and (5.50) yield

$$\begin{aligned}
&\mathbb{E}\big(\big|\widehat{T}(\Theta+\rho)^3 \tan(\Theta+\rho)\big|\mathbf{1}_{(-\pi/2+\varepsilon^{1/2},\pi/2-\varepsilon^{1/2}-2\rho_1)}(\Theta)\big)\\
&\quad\le \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1} |x(\theta)^3|\,d\theta\\
&\quad\le \int_{-\pi/2+\varepsilon^{1/2}+\rho_1}^{\pi/2-\varepsilon^{1/2}-\rho_1} \big(O(\varepsilon^3)(\cos\theta)^{-3} + O(\varepsilon^6)(\cos\theta)^{-9}\big)\,d\theta\\
&\quad\le O(\varepsilon^3)O(\varepsilon^{-1}) + O(\varepsilon^6)O(\varepsilon^{-4}) = O(\varepsilon^2).
\end{aligned} \tag{5.51}$$

The inequality, (5.48) and (5.49) imply that

$$\mathbb{E}\big|\widehat{T}(\Theta+\rho) - T_1^\varepsilon(\pi/2,\Theta)\big| = O(\varepsilon^2). \tag{5.52}$$

This estimate and (5.47) give

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}T_1^\varepsilon(\pi/2,\Theta)}{\varepsilon^2 \log(1/\varepsilon)} = h'(\pi/2)h(\pi/2)/2,$$

and, therefore, complete the proof of (5.15).

(ii) Recall definitions and notation from the first part of the proof. We have

$$\begin{aligned}
\operatorname{Var}\big(T_1^\varepsilon(\pi/2,\Theta)\big) &= \mathbb{E}\big(T_1^\varepsilon(\pi/2,\Theta)^2\big) - \big(\mathbb{E}T_1^\varepsilon(\pi/2,\Theta)\big)^2\\
&= \mathbb{E}\big(\big(T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big)^2\big) + \mathbb{E}\big(\widehat{T}(\Theta+\rho)^2\big)\\
&\quad + 2\mathbb{E}\big(\big(T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big)\widehat{T}(\Theta+\rho)\big)\\
&\quad - \big(\mathbb{E}T_1^\varepsilon(\pi/2,\Theta)\big)^2.
\end{aligned} \tag{5.53}$$

We use (5.21) and (5.52) in the following two estimates:

$$\begin{aligned}
\mathbb{E}\big(\big(T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big)^2\big) &\le O(\varepsilon^{1/2})\mathbb{E}\big|T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big|\\
&= O(\varepsilon^{5/2}),
\end{aligned} \tag{5.54}$$

$$\begin{aligned}
&\mathbb{E}\big(\big(T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big)\widehat{T}(\Theta+\rho)\big)\\
&\quad\le O(\varepsilon^{1/2})\mathbb{E}\big|T_1^\varepsilon(\pi/2,\Theta) - \widehat{T}(\Theta+\rho)\big|\\
&\quad= O(\varepsilon^{5/2}).
\end{aligned} \tag{5.55}$$

From (5.15), we obtain

$$(\mathbb{E}T_1^\varepsilon(\pi/2,\Theta))^2 = O(\varepsilon^4 \log^2(1/\varepsilon)) = o(\varepsilon^2 \log(1/\varepsilon)). \tag{5.56}$$

We now use the same strategy as in (5.29),

$$\begin{aligned}\mathbb{E}(\widehat{T}(\Theta+\rho)^2) &= \int_{-\pi/2}^{\pi/2} (\widehat{T}(\Theta+\rho_1)^2)\frac{1}{2}\cos\theta\, d\theta \\ &= \int_{-\pi/2}^{\pi/2-2\rho_1} (\widehat{T}(\Theta+\rho_1)^2)\frac{1}{2}\cos\theta\, d\theta \\ &\quad + \int_{\pi/2-2\rho_1}^{\pi/2} (\widehat{T}(\Theta+\rho_1)^2)\frac{1}{2}\cos\theta\, d\theta.\end{aligned}\tag{5.57}$$

The second integral can be estimated as follows, using (5.21) and (5.46),

$$\begin{aligned}\int_{\pi/2-2\rho_1}^{\pi/2} (\widehat{T}(\Theta+\rho_1)^2)\frac{1}{2}\cos\theta\, d\theta &= O(\varepsilon)\int_{\pi/2-2\rho_1}^{\pi/2} \cos\theta\, d\theta \\ &= O(\varepsilon)O(\rho_1^2) = O(\varepsilon^3).\end{aligned}\tag{5.58}$$

For the first integral on the right-hand side of (5.57), we use the formula $\cos(\theta - \rho_1) = \cos\theta\cos\rho_1 + \sin\theta\sin\rho_1$. and the fact that $\widehat{T}(\theta)^2$ is an even function,

$$\begin{aligned}&\int_{-\pi/2}^{\pi/2-2\rho_1} (\widehat{T}(\Theta+\rho_1)^2)\frac{1}{2}\cos\theta\, d\theta \\ &\quad= \frac{1}{2}\int_{-\pi/2+\rho_1}^{\pi/2-\rho_1} \widehat{T}(\theta)^2\cos(\theta-\rho_1)\, d\theta \\ &\quad= \cos\rho_1 \int_{-\pi/2+\rho_1}^{\pi/2-\rho_1} \widehat{T}(\theta)^2 \frac{1}{2}\cos\theta\, d\theta \\ &\quad= \cos\rho_1 \int_{-\pi/2}^{\pi/2} \widehat{T}(\theta)^2 \frac{1}{2}\cos\theta\, d\theta + O(\varepsilon^3).\end{aligned}\tag{5.59}$$

The last equality above follows from an estimate similar to the one in (5.58). We combine (5.59) with (5.57) and (5.58), and also use (5.46), to obtain

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}(\widehat{T}(\Theta+\rho)^2)}{(\varepsilon^2/2)\log(1/\varepsilon)} = \lim_{\varepsilon\to 0} \frac{1}{(\varepsilon^2/2)\log(1/\varepsilon)} \int_{-\pi/2}^{\pi/2} \widehat{T}(\theta)^2 \frac{1}{2}\cos\theta\, d\theta. \tag{5.60}$$

The last formula matches (3.17) except that ε in (3.17) has to be replaced with $|\Gamma_\varepsilon^1(\pi/2) - z|$, which is $h_\varepsilon(\pi/2) + O(\varepsilon^2)$, in view of (5.19). It follows from **H2**, (3.17) and (5.60) that

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}(\widehat{T}(\Theta+\rho)^2)}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\pi/2).$$

Combining this with (5.53)–(5.56) yields

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Var}(T_1^\varepsilon(\pi/2,\Theta))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\pi/2). \qquad \square$$

LEMMA 13. *The following assertions hold uniformly in $\alpha \in [0, 2\pi)$:*

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}(R_1^\varepsilon(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h'(\alpha)h(\alpha), \tag{5.61}$$

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Var}(R_1^\varepsilon(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\alpha). \tag{5.62}$$

PROOF. The proof proceeds along the same lines as the proof of Lemma 12. We will discuss only the changes to that proof that need to be made to accommodate it to the current setting.

(i) The roles of the following objects need to be interchanged:

1. Γ_ε^0 and Γ_ε^1,
2. f_ε and g_ε; the sign in front of the function needs to be adjusted, for example, we typically need $1 + g_\varepsilon$ and $1 - f_\varepsilon$, to match the definitions of Γ_ε^0 and Γ_ε^0,
3. $|\kappa_\varepsilon^0|$ and $|\kappa_\varepsilon^1|$.

We define $\widehat{R}$ and R' in the way analogous to $\widehat{T}$ and T'.

(ii) The equation for $x(\theta)$ is analogous to (5.26),

$$\begin{aligned} x(\theta) &= \Big(-a\big(1 + g_\varepsilon(\pi/2) - z_1\big) \\ &\quad + \sqrt{(a^2+1)\kappa_\varepsilon^1(\alpha')^{-2} - \big(1 + g_\varepsilon(\pi/2) - z_1\big)^2}\Big)/(1 + a^2). \end{aligned} \tag{5.63}$$

In view of (5.38), the expression under the square root sign in (5.26) is equal to

$$\begin{aligned} &(a^2+1)\kappa_\varepsilon^0(\alpha')^{-2} - \big(1 - f_\varepsilon(\pi/2) - z_1\big)^2 \\ &\quad = (\sin\theta)^{-2}\big(\big(1 + g_\varepsilon(\pi/2) - z_1\big)^2 + O(\varepsilon^2)\big) - \big(1 - f_\varepsilon(\pi/2) - z_1\big)^2. \end{aligned}$$

It is easy to see that this quantity is always nonnegative for small $\varepsilon > 0$. The analogous expression in (5.63) is

$$\begin{aligned} &(a^2+1)\kappa_\varepsilon^1(\alpha')^{-2} - \big(1 + g_\varepsilon(\pi/2) - z_1\big)^2 \\ &\quad = (\sin\theta)^{-2}\big(\big(1 - f_\varepsilon(\pi/2) - z_1\big)^2 + O(\varepsilon^2)\big) - \big(1 + g_\varepsilon(\pi/2) - z_1\big)^2. \end{aligned}$$

This quantity is equal to 0 if

$$|\sin\theta| = 1 - h_\varepsilon(\pi/2) + O(\varepsilon^2). \tag{5.64}$$

Let θ_- and θ_+ be the two solutions to (5.64) in $(-\pi/2, \pi/2)$. Since

$$\theta_- - (-\pi/2) = O\big(\sqrt{h_\varepsilon(\pi/2)}\big) = O\big(\varepsilon^{1/2}\big)$$

and

$$\pi/2 - \theta_+ = O\big(\sqrt{h_\varepsilon(\pi/2)}\big) = O\big(\varepsilon^{1/2}\big),$$

we have

$$c_* := \int_{\theta_-}^{\theta_+} \frac{1}{2} \cos\theta \, d\theta = 1 - O(\varepsilon).$$

It follows that all integrals of the form $\int_{-\pi/2}^{\pi/2} (\dots) \frac{1}{2} \cos\theta \, d\theta$ that appear in the proof of Lemma 12 should be replaced with the integrals of the form $\int_{\theta_-}^{\theta_+} (\dots) \frac{1}{2c_*} \cos\theta \, d\theta$ in the present proof. Since $c_* = 1 - O(\varepsilon)$, the extra factor $1/c_*$ will not affect the normalizing constant in (5.61)–(5.62) relative to (5.15)–(5.16).

(iii) The last element of the proof of Lemma 12 that needs to be modified is the estimate of $|\widehat{T}(\Theta + \rho) - T_1^\varepsilon(\pi/2, \Theta)|$. We start by modifying (5.23) and (5.24). We divide the interval (θ_-, θ_+) into two subsets,

$$A_1 := \big(-\pi/2 + \varepsilon^{1/2} \log^{1/4}(1/\varepsilon), \pi/2 - \varepsilon^{1/2} \log^{1/4}(1/\varepsilon)\big)$$

and

$$A_2 := \big(\theta_-, -\pi/2 + \varepsilon^{1/2} \log^{1/4}(1/\varepsilon)\big) \cup \big(\pi/2 - \varepsilon^{1/2} \log^{1/4}(1/\varepsilon), \theta_+\big).$$

The same geometric analysis as in the proof of Lemma 12 yields

$$\big|\widehat{R}(\theta) - R'(\theta)\big| = O\big(\widehat{R}(\theta)^3 \tan\theta\big)$$

for $\theta \in A_1$, and

$$\big|\widehat{R}(\theta) - R'(\theta)\big| = O\big(\widehat{R}(\theta)^3 \varepsilon^{-1/2}\big)$$

for $\theta \in A_2$. The analogue of (5.48) is

$$\begin{aligned} \mathbb{E}\big|\widehat{R}(\Theta + \rho) - R_1^\varepsilon(\pi/2, \Theta)\big| &= \mathbb{E}\big|\widehat{R}(\Theta + \rho) - R'(\Theta + \rho)\big| \\ &\le c_4 \mathbb{E}\big(\big|\widehat{R}(\Theta + \rho)^3 \tan(\Theta + \rho)\big| \mathbf{1}_{A_1}(\Theta)\big) \\ &\quad + c_4 \mathbb{E}\big(\big|\widehat{R}(\Theta + \rho)^3 \varepsilon^{-1/2}\big| \mathbf{1}_{A_2}(\Theta)\big). \end{aligned} \tag{5.65}$$

The analogue of (5.49) is

$$\begin{aligned} \mathbb{E}\big(\big|\widehat{R}(\Theta + \rho)^3 \varepsilon^{-1/2}\big| \mathbf{1}_{A_2}(\Theta)\big) &\le O(\varepsilon) \int_{A_2} \frac{1}{2} \cos\theta \, d\theta \\ &= O(\varepsilon) O\big(\varepsilon \log^{1/2}(1/\varepsilon)\big) \\ &= O\big(\varepsilon^2 \log^{1/2}(1/\varepsilon)\big). \end{aligned} \tag{5.66}$$

The analogue of (5.51) is

$$\begin{aligned}
&\mathbb{E}\big(\big|\widehat{R}(\Theta+\rho)^3\tan(\Theta+\rho)\big|\mathbf{1}_{A_1}(\Theta)\big)\\
&\qquad\le \int_{-\pi/2+\varepsilon^{1/2}\log^{1/4}(1/\varepsilon)+\rho_1}^{\pi/2-\varepsilon^{1/2}\log^{1/4}(1/\varepsilon)-\rho_1} \big|x(\theta)^3\big|\,d\theta\\
&\qquad\le \int_{-\pi/2+\varepsilon^{1/2}\log^{1/4}(1/\varepsilon)+\rho_1}^{\pi/2-\varepsilon^{1/2}\log^{1/4}(1/\varepsilon)-\rho_1} \big(O(\varepsilon^3)(\cos\theta)^{-3}+O(\varepsilon^6)(\cos\theta)^{-9}\big)\,d\theta \qquad (5.67)\\
&\qquad\le O(\varepsilon^3)O\big(\varepsilon^{-1}\log^{-1/2}(1/\varepsilon)\big)+O(\varepsilon^6)O\big(\varepsilon^{-4}\log^{-1}(1/\varepsilon)\big)\\
&\qquad= O(\varepsilon^2).
\end{aligned}$$

The estimates (5.65), (5.66) and (5.67) are accurate enough to yield an analogue of (5.52).

With these changes, the other steps in the proof of Lemma 12 can be easily adjusted to generate a proof of (5.61)–(5.62). □

LEMMA 14. *We have uniformly in* $\alpha \in [0, 2\pi)$,

$$\big|\mathbb{E}\big(S_1^\varepsilon(\alpha)\big)\big| = O(\varepsilon).$$

PROOF. Let $\mathcal{C}$ be the osculating circle of Γ_ε^0 at $\Gamma_\varepsilon^0(\pi/2)$. Note that the osculating circle is defined relative to Γ_ε^0 and not relative to a rotation of Γ_ε^0, unlike in the proofs of Lemmas 12 and 13. Let $\mathcal{R}$ be the rotation about the point $\Gamma_\varepsilon^0(\pi/2)$ such that the center of the circle $\mathcal{C}_* := \mathcal{R}(\mathcal{C})$ is at a point $(0, z_1)$, with $z_1 < 1$.

Suppose that a light ray leaves $\Gamma_\varepsilon^0(\pi/2)$ at an angle θ, relative to the normal vector to Γ_ε^0 at $\Gamma_\varepsilon^0(\pi/2)$. It follows from Lemma 8 that there exist $\theta_- \in (-\pi/2, 0)$ and $\theta_+ \in (0, \pi/2)$ such that (i) the light ray does not intersect Γ_ε^1 and it intersects Γ_ε^0 for $\theta \in A_1 := (-\pi/2, \theta_-) \cup (\theta_+, \pi/2)$, and (ii) the light ray intersects Γ_ε^1 before intersecting Γ_ε^0 for $\theta \in (\theta_-, \theta_+)$.

For $\theta \in A_1$, the light ray intersects Γ_ε^0 at a point $\mathbf{p}_\varepsilon(\pi/2, 0, \theta)\exp(i(\pi/2 + S_1^\varepsilon(\theta)))$, in the notation of (5.1); we added θ to the notation to make dependence on θ explicit. The same light ray will intersect the circle $\mathcal{C}$ at a point $\widehat{r}(\theta)\exp(i(\pi/2 + \widehat{S}(\theta)))$.

Let $r_*(\theta)\exp(i(\pi/2 + S_*(\theta))) = \mathcal{R}(\widehat{r}(\theta)\exp(i(\pi/2 + \widehat{S}(\theta))))$. In other words, $r_*(\theta)\exp(i(\pi/2 + S_*(\theta)))$ represents the point of intersection with the rotated circle $\mathcal{C}_*$.

Let $\theta_0 = \max(-\theta_-, \theta_+)$ and $A_2 = (-\pi/2, \theta_0) \cup (\theta_0, \pi/2)$. A calculation similar to that in Lemma 1 gives $\theta_- - (-\pi/2) = O(\varepsilon^{1/2})$ and $\pi/2 - \theta_+ = O(\varepsilon^{1/2})$, so

$$\pi/2 - \theta_0 = O(\varepsilon^{1/2}). \qquad (5.68)$$

By symmetry,

$$\mathbb{E}\big(S_*(\Theta)\mathbf{1}_{A_2}(\Theta)\big) = 0. \qquad (5.69)$$

Elementary geometry shows that

$$\lim_{\varepsilon\to 0}\sup_{\theta\in A_1}\left|\frac{\widehat{S}(\theta)}{2|\pi/2-\theta|}-1\right|=0. \tag{5.70}$$

The angle of rotation for $\mathcal{R}$ is equal to $|\gamma_\varepsilon^0(\pi/2)|$ and this is of order $O(\varepsilon)$, by (5.10). The radius of $\mathcal{C}$ is $|\kappa_\varepsilon^0(\pi/2)|^{-1}$ and this is $1+O(\varepsilon)$, by the same reasoning that gave (5.37). These observations easily imply that $|\widehat{S}(\theta)-S_*(\theta)|=O(|\pi/2-\theta|\varepsilon)$, uniformly in $\theta\in A_1$. Hence, using (5.13), (5.68) and (5.69),

$$\begin{aligned}
\bigl|\mathbb{E}\bigl(\widehat{S}(\Theta)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr| &= \bigl|\mathbb{E}\bigl(S_*(\Theta)\mathbf{1}_{A_2}(\Theta)\bigr)+\mathbb{E}\bigl(\bigl(\widehat{S}(\Theta)-S_*(\Theta)\bigr)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr|\\
&= \bigl|\mathbb{E}\bigl(\bigl(\widehat{S}(\Theta)-S_*(\Theta)\bigr)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr|\\
&\le \left|\frac{2}{\varepsilon}\int_{(-\pi/2,\theta_0)\cup(\theta_0,\pi/2)} O\bigl(|\pi/2-\theta|\varepsilon\bigr)\frac{1}{2}\cos\theta\,d\theta\right|\\
&= O\bigl(|\pi/2-\theta_0|^3\bigr)=O\bigl(\varepsilon^{3/2}\bigr).
\end{aligned}\tag{5.71}$$

We have the following analogue of (5.23),

$$\bigl|\widehat{S}(\theta)-S_1^\varepsilon(\theta)\bigr|=O\bigl(\widehat{S}(\theta)^3\tan\theta\bigr),$$

which, combined with (5.68), (5.70) and (5.71), implies

$$\begin{aligned}
\bigl|\mathbb{E}\bigl(S_1^\varepsilon(\Theta)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr| &\le \bigl|\mathbb{E}\bigl(\bigl(\widehat{S}(\Theta)-S_1^\varepsilon(\Theta)\bigr)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr|+\bigl|\mathbb{E}\bigl(\widehat{S}(\Theta)\mathbf{1}_{A_2}(\Theta)\bigr)\bigr|\\
&\le \left|\frac{2}{\varepsilon}\int_{(-\pi/2,\theta_0)\cup(\theta_0,\pi/2)} O\bigl(|\pi/2-\theta|^3\bigr)\tan\theta\frac{1}{2}\cos\theta\,d\theta\right|\\
&\quad + O\bigl(\varepsilon^{3/2}\bigr)\\
&= O\bigl(|\pi/2-\theta_0|^4/\varepsilon\bigr)+O\bigl(\varepsilon^{3/2}\bigr)=O(\varepsilon).
\end{aligned}\tag{5.72}$$

It remains to estimate $|\mathbb{E}(S_1^\varepsilon(\Theta)\mathbf{1}_{A_1\setminus A_2}(\Theta))|$.

Assume without loss of generality that $\theta_0=-\theta_-$ so $A_3:=A_1\setminus A_2=(\theta_+,\theta_0)=(\theta_+,-\theta_-)$. Lemma 9 implies that if a light ray starting from $\Gamma_\varepsilon^0(\pi/2)$ intersects Γ_ε^1 at a point $\Gamma_\varepsilon^1(\alpha)$ then $\pi/2-12\sqrt{\varepsilon}\le\alpha\le\pi/2+12\sqrt{\varepsilon}$. Let

$$\begin{aligned}
f_\varepsilon^+ &= \sup\bigl(f_\varepsilon(\alpha):\pi/2-12\sqrt{\varepsilon}\le\alpha\le\pi/2+12\sqrt{\varepsilon}\bigr),\\
f_\varepsilon^- &= \inf\bigl(f_\varepsilon(\alpha):\pi/2-12\sqrt{\varepsilon}\le\alpha\le\pi/2+12\sqrt{\varepsilon}\bigr).
\end{aligned}$$

It follows from **H3** that $f_\varepsilon^+-f_\varepsilon^-\le 24\sqrt{\varepsilon}\|f_\varepsilon'\|=O(\varepsilon^{3/2})$. If a light ray starting from $\Gamma_\varepsilon^0(\pi/2)$ intersects Γ_ε^1 then it must intersect the circle $\mathcal{C}((0,0),1-f_\varepsilon^-)$ but it cannot intersect $\mathcal{C}((0,0),1-f_\varepsilon^+)$. We will rescale the circles so that we can apply Lemma 1. We define ε_- and ε_+ by

$$1-\varepsilon_-=\frac{1-f_\varepsilon^-}{1+g_\varepsilon(\pi/2)},\qquad 1-\varepsilon_+=\frac{1-f_\varepsilon^+}{1+g_\varepsilon(\pi/2)},$$

and note that $\varepsilon_+ - \varepsilon_- = O(\varepsilon^{3/2})$ because $f_\varepsilon^+ - f_\varepsilon^- = O(\varepsilon^{3/2})$. Then a light ray starting from $\Gamma_\varepsilon^0(\pi/2)$ at an angle θ relative to vertical intersects $\mathcal{C}((0,0), 1-f_\varepsilon^-)$ and does not intersect $\mathcal{C}((0,0), 1-f_\varepsilon^+)$ if and only if a light ray starting from $(0,1)$ at an angle θ relative to vertical intersects $\mathcal{C}((0,0), 1-\varepsilon_-)$ and does not intersect $\mathcal{C}((0,0), 1-\varepsilon_+)$. According to Lemma 1, the angle must be in the range

$$A_4 := \left(\operatorname{arccot}\left(\left(\frac{2\varepsilon_- - \varepsilon_-^2}{(1-\varepsilon_-)^2}\right)^{1/2}\right), \operatorname{arccot}\left(\left(\frac{2\varepsilon_+ - \varepsilon_+^2}{(1-\varepsilon_+)^2}\right)^{1/2}\right)\right).$$

Since $\varepsilon_+ - \varepsilon_- = O(\varepsilon^{3/2})$, the length of A_4 is $O(\varepsilon)$. This implies that $\theta_+ - (-\theta_-) = O(\varepsilon)$. By (5.68), $\cos\theta = O(\varepsilon^{1/2})$ for $\theta \in (-\theta_-, \theta_+)$. We combine these observations with Lemma 9 to obtain

$$\left|\mathbb{E}\big(S_1^\varepsilon(\Theta)\mathbf{1}_{A_1\setminus A_2}(\Theta)\big)\right| \le \left|\frac{2}{\varepsilon}\int_{\theta_+}^{-\theta_-} 12\sqrt{\varepsilon}\frac{1}{2}\cos\theta\, d\theta\right| = O(\varepsilon).$$

The lemma follows from this and (5.72). □

Recall definition (5.5) of $N^\varepsilon(t)$ and for $n \ge 0$, let

$$\begin{aligned}
\mathcal{F}_n^\varepsilon &= \sigma\big(\boldsymbol{\alpha}_k^\varepsilon, \mathbf{s}_k^\varepsilon, k = 1, \dots, n\big),\\
\Delta B_{n+1}^\varepsilon &= \mathbb{E}\big(\boldsymbol{\alpha}_{n+1}^\varepsilon - \boldsymbol{\alpha}_n^\varepsilon \mid \mathcal{F}_n^\varepsilon\big),\\
B^\varepsilon(n) &= \sum_{k=1}^n \Delta B_k^\varepsilon,\\
M^\varepsilon(n) &= \boldsymbol{\alpha}_n^\varepsilon - B^\varepsilon(n),\\
\Delta A_{n+1}^\varepsilon &= \mathbb{E}\big(\big(M_{n+1}^\varepsilon - M_n^\varepsilon\big)^2 \mid \mathcal{F}_n^\varepsilon\big),\\
A^\varepsilon(n) &= \sum_{k=1}^n \Delta A_k^\varepsilon,\\
\chi_{n+1}^\varepsilon &= \mathbb{E}\big(\Delta\mathcal{T}_{n+1}^\varepsilon \mid \mathcal{F}_n^\varepsilon\big),\\
\zeta(\varepsilon, t) &= \frac{\pi t}{\varepsilon\log(1/\varepsilon)},\\
\mathbf{M}_t^\varepsilon &= M^\varepsilon\big(N^\varepsilon\big(\zeta(\varepsilon,t)\big)\big) \qquad \text{for } t \ge 0,\\
\mathbf{B}_t^\varepsilon &= B^\varepsilon\big(N^\varepsilon\big(\zeta(\varepsilon,t)\big)\big) \qquad \text{for } t \ge 0,\\
\mathbf{A}_t^\varepsilon &= A^\varepsilon\big(N^\varepsilon\big(\zeta(\varepsilon,t)\big)\big) \qquad \text{for } t \ge 0.
\end{aligned} \tag{5.73}$$

The expectations in the above definitions exist and are finite because of the estimate given in Lemma 9. For $\alpha \in [0, 2\pi)$ and $s = 0, 1$, let

$$\chi^\varepsilon(\alpha, s) = \mathbb{E}\big(\Delta\mathcal{T}_{n+1}^\varepsilon \mid \boldsymbol{\alpha}_n^\varepsilon = \alpha, \mathbf{s}_n^\varepsilon = s\big).$$

LEMMA 15. *We have uniformly in α,*

$$\lim_{\varepsilon\to 0}\frac{\chi^{\varepsilon}(\alpha,1)}{\varepsilon}=\frac{\pi}{2}h(\alpha),\tag{5.74}$$

$$\lim_{\varepsilon\to 0}\frac{\chi^{\varepsilon}(\alpha,0)}{\varepsilon}=\frac{\pi}{2}h(\alpha).\tag{5.75}$$

PROOF. Let $\beta=2/3$ and assume that $\varepsilon\in(0,\varepsilon_0)$. Since the curvature of the unit circle is 1, for every $c_1>0$ there exists $c_2>0$ such that the arc of the circle

$$\{(1+g_\varepsilon(\pi/2))e^{i(\pi/2+\alpha)}:-c_1\varepsilon^\beta\le\alpha\le c_1\varepsilon^\beta\}$$

lies below the line $\{(z_1,z_2):z_2=1+g_\varepsilon(\pi/2)\}$ and above the line $\{(z_1,z_2):z_2=1+g_\varepsilon(\pi/2)-c_2\varepsilon^{2\beta}\}$. We have assumed that $\|g'_\varepsilon\|_\infty=O(\varepsilon)$ so for some $c_3>0$ and all $-c_1\varepsilon^\beta\le\alpha\le c_1\varepsilon^\beta$ we have $|g_\varepsilon(\pi/2+\alpha)-g_\varepsilon(\pi/2)|\le c_3\varepsilon^{1+\beta}$. These two observations imply that for some $c_4>0$, the set

$$\{\Gamma^0_\varepsilon(\pi/2+\alpha):-c_1\varepsilon^\beta\le\alpha\le c_1\varepsilon^\beta\}$$

lies below the line $L:=\{(z_1,z_2):z_2=1+g_\varepsilon(\pi/2)+c_4\varepsilon^{2\beta}\}$ and above the line $\{(z_1,z_2):z_2=1+g_\varepsilon(\pi/2)-c_4\varepsilon^{2\beta}\}$. This implies that if the light ray starts from $\Gamma^1_\varepsilon(\pi/2)$ at time $t=0$, at an angle θ relative to the vector $(0,1)$ and $-\pi/2+c_5\varepsilon^{1-\beta}\le\theta\le\pi/2-c_5\varepsilon^{1-\beta}$ then the light ray crosses L at a point (z_1,z_2) with $|z_1|\le c_6\varepsilon^\beta$. This implies that

$$\frac{h_\varepsilon(\pi/2)-c_4\varepsilon^{2\beta}}{\cos\theta}\le\Delta\mathcal{T}^\varepsilon_1\le\frac{h_\varepsilon(\pi/2)+c_4\varepsilon^{2\beta}}{\cos\theta}.\tag{5.76}$$

Recall the definition of $\gamma^1_\varepsilon(\pi/2)$ stated before Lemma 10. In the following formula, we have to shift the angle θ by $\gamma^1_\varepsilon(\pi/2)$ because θ is the angle relative to $(0,1)$ in (5.76). It follows from (5.11) that $\gamma^1_\varepsilon(\pi/2)=O(\varepsilon)$, so if $-\pi/2+c_5\varepsilon^{1-\beta}\le\theta\le\pi/2-c_5\varepsilon^{1-\beta}$ then $-\pi/2+2c_5\varepsilon^{1-\beta}\le\theta+\gamma^1_\varepsilon(\pi/2)\le\pi/2-2c_5\varepsilon^{1-\beta}$, for small $\varepsilon>0$. Let $\theta_-=-\pi/2+\varepsilon^{1-\beta}-\gamma^1_\varepsilon(\pi/2)$ and $\theta_+=\pi/2-\varepsilon^{1-\beta}-\gamma^1_\varepsilon(\pi/2)$. We use these observations and the estimate from Lemma 9 to derive the following:

$$\begin{aligned}
\mathbb{E}(\Delta\mathcal{T}^\varepsilon_1)&=\mathbb{E}(\Delta\mathcal{T}^\varepsilon_1\mathbf{1}_{(\theta_-,\theta_+)}(\Theta))+\mathbb{E}(\Delta\mathcal{T}^\varepsilon_1\mathbf{1}_{(-\pi/2,\theta_-)\cup(\theta_+,\pi/2)}(\Theta))\\
&=\int_{\theta_-}^{\theta_+}\frac{h_\varepsilon(\pi/2)+O(\varepsilon^{2\beta})}{\cos(\theta+\gamma^1_\varepsilon(\pi/2))}\frac{1}{2}\cos\theta\,d\theta\\
&\quad+\int_{(-\pi/2,\theta_-)\cup(\theta_+,\pi/2)}O(\varepsilon^{1/2})\frac{1}{2}\cos\theta\,d\theta\\
&=\int_{-\pi/2+\varepsilon^{1-\beta}}^{\pi/2-\varepsilon^{1-\beta}}\frac{h_\varepsilon(\pi/2)+O(\varepsilon^{2\beta})}{\cos(\theta)}\frac{1}{2}\cos(\theta-\gamma^1_\varepsilon(\pi/2))\,d\theta\\
&\quad+O(\varepsilon^{2(1-\beta)+1/2}).
\end{aligned}$$

Recall that $\cos(\theta - \gamma_\varepsilon^1(\pi/2)) = \cos\theta \cos\gamma_\varepsilon^1(\pi/2) + \sin\theta \sin\gamma_\varepsilon^1(\pi/2)$. Thus

$$\begin{aligned}
\mathbb{E}(\Delta\mathcal{T}_1^\varepsilon) &= \int_{-\pi/2+\varepsilon^{1-\beta}}^{\pi/2-\varepsilon^{1-\beta}} h_\varepsilon(\pi/2)\cos\gamma_\varepsilon^1(\pi/2)\frac{1}{2}\,d\theta \\
&\quad + O(\varepsilon^{2\beta})\int_{-\pi/2+\varepsilon^{1-\beta}}^{\pi/2-\varepsilon^{1-\beta}} \frac{|\sin\theta\sin\gamma_\varepsilon^1(\pi/2)|}{\cos(\theta)}\frac{1}{2}\,d\theta \\
&\quad + O(\varepsilon^{5/2-2\beta}) \\
&= \frac{\pi}{2}h_\varepsilon(\pi/2)(1+O(\varepsilon^{1-\beta})) + O(\varepsilon^{2\beta})O(\varepsilon)O(\varepsilon^{\beta-1}) + O(\varepsilon^{7/6}) \\
&= \frac{\pi}{2}h_\varepsilon(\pi/2)(1+O(\varepsilon^{1/3})) + O(\varepsilon^2) + O(\varepsilon^{7/6}).
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0}\frac{\chi^\varepsilon(\pi/2,1)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\mathbb{E}(\Delta\mathcal{T}_1^\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\frac{\pi}{2}h_\varepsilon(\pi/2)}{\varepsilon} = \frac{\pi}{2}h(\pi/2).$$

Our estimates are uniform in α so (5.74) follows. The proof of (5.75) proceeds along similar lines, with only minor modifications, so it is left to the reader. $\square$

LEMMA 16. *For any* $T > 0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T}|\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))} - \boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t-))}|^2\Big) = 0,$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T}|\mathbf{B}_t^\varepsilon - \mathbf{B}_{t-}^\varepsilon|^2\Big) = 0,$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T}|\mathbf{A}_t^\varepsilon - \mathbf{A}_{t-}^\varepsilon|\Big) = 0.$$

PROOF. The quantities $|\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))} - \boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t-))}|^2$, $|\mathbf{B}_t^\varepsilon - \mathbf{B}_{t-}^\varepsilon|^2$ and $|\mathbf{A}_t^\varepsilon - \mathbf{A}_{t-}^\varepsilon|$ can be nonzero only if $\zeta(\varepsilon,t) = \mathcal{T}_k^\varepsilon$ for some $k \ge 1$. It follows from Lemma 9 and definitions of these quantities that they are bounded by 144ε. Since this bound is deterministic, the lemma follows. $\square$

LEMMA 17. *There exists a constant* c, *depending only on* $\|g'\|_\infty$, *such that for any* $\varepsilon \le \varepsilon_0$,

$$\mathbb{E}(N^\varepsilon(t)) \le c(t+2\varepsilon)/\varepsilon. \tag{5.77}$$

In particular, $N^\varepsilon(t)$ *is finite almost surely.*

PROOF. Assumption **H1** implies that the distance between Γ_ε^0 and Γ_ε^1 is at least ε. The light ray travels at speed 1 so it takes at least ε units of time between any two consecutive reflections that do not take place on the same piece of the boundary. Thus n crossings from the inner to the outer boundary and n crossings from the outer to the inner boundary must take at least $2n\varepsilon$ units of time. It follows that $N^\varepsilon(t)$ is stochastically majorized by $U(\lceil t/(2\varepsilon)\rceil)$, where

$$U(n) = n + \sum_{k=1}^{n} X_k^\varepsilon,$$

and X_k^ε are i.i.d. random variables with the geometric distribution (taking values $1, 2, \dots$) with parameter $1 - \varepsilon(4 + 6\|g'\|_\infty)$ (see Lemma 11). Therefore,

$$\begin{aligned}\mathbb{E}(N^\varepsilon(t)) &\le \mathbb{E}(U(\lceil t/(2\varepsilon)\rceil)) = \lceil t/(2\varepsilon)\rceil + \lceil t/(2\varepsilon)\rceil \frac{1}{1-\varepsilon(4+6\|g'\|_\infty)} \\ &= \lceil t/(2\varepsilon)\rceil \frac{2-\varepsilon(4+6\|g'\|_\infty)}{1-\varepsilon(4+6\|g'\|_\infty)} \le \left(\frac{t}{2\varepsilon}+1\right)\cdot c_1 \le c(t+2\varepsilon)/\varepsilon. \qquad \square\end{aligned}$$

LEMMA 18. *Suppose that $\{X_t\}_{t\ge 0}$ is a martingale and let τ be a stopping time such that $\mathbb{E}\tau < \infty$. Then for $a > 0$,*

$$\mathbb{P}\Big(\sup_{0\le t\le\tau} |X_t| \ge a\Big) \le \sup_{s\ge 0} \frac{2}{a^2}\mathbb{E}(X_{s\wedge\tau}^2). \tag{5.78}$$

PROOF. Let $M_t = X_{t\wedge\tau}$. By the optional stopping theorem, $\{M_t\}_{t\ge 0}$ is a martingale and $\{|M_t|\}_{t\ge0}$ is a positive submartingale. By Doob's inequality, for any fixed $s > 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le s} |X_{t\wedge\tau}| \ge a\Big) \le \frac{2}{a^2}\mathbb{E}(X_{s\wedge\tau}^2). \tag{5.79}$$

Events $\{\sup_{0\le n\le k} |X_{n\wedge\tau}| \ge a\}$ converge monotonically to $\{\sup_{0\le n\le\tau} |X_n| \ge a\}$ when $k \to \infty$ so the left-hand side of (5.79) converges to the left-hand side of (5.78). $\square$

LEMMA 19. *For any $T > 0$,*

$$\sup_{0\le t\le T}\left|\mathbf{B}_t^\varepsilon - \int_0^t h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))}\big)\,ds\right|$$

converges to 0 *in probability when $\varepsilon \to 0$.*

PROOF. Recall that $\zeta(\varepsilon,t) = \frac{\pi t}{\varepsilon\log(1/\varepsilon)}$. Since $N^\varepsilon(\mathcal{T}_k^\varepsilon) = k$, we get by a change of variable,

$$\begin{aligned}&\int_0^t h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))}\big)\,ds \\ &\quad= \frac{\varepsilon\log(1/\varepsilon)}{\pi}\left(\int_0^{\mathcal{T}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))}} h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(s)}\big)\,ds + \int_{\mathcal{T}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))}}^{\zeta(\varepsilon,t)} h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(s)}\big)\,ds\right)\end{aligned} \tag{5.80}$$

$$= \frac{\varepsilon \log(1/\varepsilon)}{\pi} \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} h'(\boldsymbol{\alpha}_k^\varepsilon) \Delta \mathcal{T}_{k+1}^\varepsilon + \frac{\varepsilon \log(1/\varepsilon)}{\pi} \big(\zeta(\varepsilon,t) - \mathcal{T}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))}\big) h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))}\big).$$

From (5.4), we have for any $t > 0$,

$$\mathcal{T}^\varepsilon_{N^\varepsilon(t)} \le t \le \mathcal{T}^\varepsilon_{N^\varepsilon(t)+1} = \mathcal{T}^\varepsilon_{N^\varepsilon(t)} + \Delta \mathcal{T}^\varepsilon_{N^\varepsilon(t)+1}. \tag{5.81}$$

Assumption **H1** and Lemma 9 imply that, a.s., for all k and $\varepsilon \le \varepsilon_0$,

$$\begin{aligned} \Delta \mathcal{T}_k^\varepsilon &= \big| Q^\varepsilon(\mathcal{T}_{k-1}^\varepsilon) - Q^\varepsilon(\mathcal{T}_k^\varepsilon) \big| \le 2 \big| \boldsymbol{\alpha}_{k-1}^\varepsilon - \boldsymbol{\alpha}_k^\varepsilon \big| + 6\varepsilon \\ &\le 24 \varepsilon^{1/2} + 6\varepsilon = O\big(\varepsilon^{1/2}\big). \end{aligned} \tag{5.82}$$

This and (5.81) imply that $\zeta(\varepsilon,t) - \mathcal{T}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))} = O(\varepsilon^{1/2})$. Since $\|h'\|_\infty < \infty$, we have uniformly in $\alpha \in [0, 2\pi)$ and $t \ge 0$,

$$\frac{\varepsilon \log(1/\varepsilon)}{\pi} \big(\zeta(\varepsilon,t) - \mathcal{T}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,t))}\big) \big| h'(\alpha) \big| = O\big(\varepsilon^{3/2} \log(1/\varepsilon)\big). \tag{5.83}$$

We combine this with (5.80) to see that

$$\begin{aligned} &\left| \mathbf{B}_t^\varepsilon - \int_0^t h'\big(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))}\big)\, ds \right| \\ &\quad \le \left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Delta B^\varepsilon\big(\boldsymbol{\alpha}_{k+1}^\varepsilon, \mathbf{s}_{k+1}^\varepsilon\big) - \frac{\varepsilon \log(1/\varepsilon)}{\pi} h'\big(\boldsymbol{\alpha}_k^\varepsilon\big) \Delta \mathcal{T}_{k+1}^\varepsilon \right| \\ &\qquad + O\big(\varepsilon^{\frac{3}{2}} \log(1/\varepsilon)\big) \\ &\quad \le \left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Delta B^\varepsilon\big(\boldsymbol{\alpha}_{k+1}^\varepsilon, \mathbf{s}_{k+1}^\varepsilon\big) - \frac{\varepsilon \log(1/\varepsilon)}{\pi} h'\big(\boldsymbol{\alpha}_k^\varepsilon\big) \chi_{k+1}^\varepsilon \right| \\ &\qquad + \frac{\varepsilon \log(1/\varepsilon)}{\pi} \left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} h'\big(\boldsymbol{\alpha}_k^\varepsilon\big)\big(\chi_{k+1}^\varepsilon - \Delta \mathcal{T}_{k+1}^\varepsilon\big) \right| \\ &\qquad + O\big(\varepsilon^{\frac{3}{2}} \log(1/\varepsilon)\big). \end{aligned} \tag{5.84}$$

We have

$$\begin{aligned} &\left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Delta B^\varepsilon\big(\boldsymbol{\alpha}_{k+1}^\varepsilon, \mathbf{s}_{k+1}^\varepsilon\big) - \frac{\varepsilon \log(1/\varepsilon)}{\pi} h'\big(\boldsymbol{\alpha}_k^\varepsilon\big) \chi_{k+1}^\varepsilon \right| \\ &\quad \le \frac{\varepsilon^2 \log(1/\varepsilon)}{2} \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Big| h'\big(\boldsymbol{\alpha}_k^\varepsilon\big) h\big(\boldsymbol{\alpha}_k^\varepsilon\big) \end{aligned}$$

$$\times \left(\frac{\Delta B^\varepsilon(\boldsymbol{\alpha}_k^\varepsilon, \mathbf{s}_k^\varepsilon)}{(1/2)\varepsilon^2 \log(1/\varepsilon) h'(\boldsymbol{\alpha}_k^\varepsilon) h(\boldsymbol{\alpha}_k^\varepsilon)} - \frac{2\chi_{k+1}^\varepsilon}{\pi \varepsilon h(\boldsymbol{\alpha}_k^\varepsilon)} \right) \Bigg| \tag{5.85}$$

$$\le \frac{\varepsilon^2 \log(1/\varepsilon)}{2} N^\varepsilon(\zeta(\varepsilon,t)) \sup_{\alpha \in \mathbb{R}, s \in \{0,1\}} \Bigg| h'(\alpha) h(\alpha) \times \left(\frac{\Delta B^\varepsilon(\alpha, s)}{(1/2)\varepsilon^2 \log(1/\varepsilon) h'(\alpha) h(\alpha)} - \frac{2\chi^\varepsilon(\alpha, s)}{\pi \varepsilon h(\alpha)} \right) \Bigg|.$$

Lemmas 11, 12, 13 and 14 imply that the following limit holds uniformly in α and s:

$$\lim_{\varepsilon \to 0} \frac{\Delta B^\varepsilon(\alpha, s)}{(1/2)\varepsilon^2 \log(1/\varepsilon) h'(\alpha) h(\alpha)} = 1. \tag{5.86}$$

Lemma 15 implies that the following limit holds uniformly in α and s:

$$\lim_{\varepsilon \to 0} \frac{2\chi^\varepsilon(\alpha, s)}{\pi \varepsilon h(\alpha)} = 1.$$

This and (5.85)–(5.86) imply that

$$\left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Delta B^\varepsilon(\boldsymbol{\alpha}_{k+1}^\varepsilon, \mathbf{s}_{k+1}^\varepsilon) - \frac{\varepsilon \log(1/\varepsilon)}{\pi} h'(\boldsymbol{\alpha}_k^\varepsilon) \chi_{k+1}^\varepsilon \right| \le \frac{\varepsilon^2 \log(1/\varepsilon)}{2} N^\varepsilon(\zeta(\varepsilon,t)) o(1). \tag{5.87}$$

Since $t \mapsto N^\varepsilon(t)$ is a nondecreasing function, Lemma 17 implies that for some c_1 and all $t \le T$,

$$\mathbb{E} N^\varepsilon(\zeta(\varepsilon,t)) \le \mathbb{E} N^\varepsilon(\zeta(\varepsilon,T)) \le c_1 \zeta(\varepsilon,T)/\varepsilon = c_1 \frac{\pi T}{\varepsilon^2 \log(1/\varepsilon)}.$$

It follows from this and (5.87) that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left(\sup_{0 \le t \le T} \left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} \Delta B^\varepsilon(\boldsymbol{\alpha}_{k+1}^\varepsilon, \mathbf{s}_{k+1}^\varepsilon) - \frac{\varepsilon \log(1/\varepsilon)}{\pi} h'(\boldsymbol{\alpha}_k^\varepsilon) \chi_{k+1}^\varepsilon \right| \right) = 0. \tag{5.88}$$

This and (5.84) imply that it will suffice to show that

$$\sup_{0 \le t \le T} \frac{\varepsilon \log(1/\varepsilon)}{\pi} \left| \sum_{k=0}^{N^\varepsilon(\zeta(\varepsilon,t))-1} h'(\boldsymbol{\alpha}_k^\varepsilon)(\chi_{k+1}^\varepsilon - \Delta \mathcal{T}_{k+1}^\varepsilon) \right| \to 0 \tag{5.89}$$

in probability as $\varepsilon \to 0$.

Recall that $\chi_{k+1}^{\varepsilon} = \mathbb{E}(\Delta \mathcal{T}_{k+1}^{\varepsilon} \mid \mathcal{F}_k^{\varepsilon})$. Let $(\mathcal{M}_{\varepsilon}(n))_{n\geq 0}$ be defined by $\mathcal{M}_{\varepsilon}(0) = 0$ and for $n \geq 1$ by

$$\mathcal{M}_{\varepsilon}(n) = \frac{\varepsilon \log(1/\varepsilon)}{\pi} \sum_{k=0}^{n-1} h'(\boldsymbol{\alpha}_k^{\varepsilon})\big(\Delta \mathcal{T}_{k+1}^{\varepsilon} - \mathbb{E}(\Delta \mathcal{T}_{k+1}^{\varepsilon} \mid \mathcal{F}_k^{\varepsilon})\big). \tag{5.90}$$

Then $(\mathcal{M}_{\varepsilon}(n))_{n\geq 0}$ is a martingale and its quadratic variation is given by

$$\begin{aligned} \langle \mathcal{M}_{\varepsilon}\rangle_n &= \frac{\varepsilon^2 \log^2(1/\varepsilon)}{\pi^2} \sum_{k=0}^{n-1} \big(h'(\boldsymbol{\alpha}_k^{\varepsilon})\big)^2 \operatorname{Var}\big(\Delta \mathcal{T}_{k+1}^{\varepsilon} \mid \mathcal{F}_k^{\varepsilon}\big) \\ &\leq \frac{\varepsilon^2 \log^2(1/\varepsilon)\|h'\|_{\infty}^2}{\pi^2} \sum_{k=0}^{n-1} \mathbb{E}\big((\Delta \mathcal{T}_{k+1}^{\varepsilon})^2 \mid \mathcal{F}_k^{\varepsilon}\big). \end{aligned}$$

By (5.82), a.s., for some c_2 and all k, $\mathbb{E}((\Delta \mathcal{T}_{k+1}^{\varepsilon})^2 \mid \mathcal{F}_k^{\varepsilon}) \leq c_2 \varepsilon$. This implies that for some c_3,

$$\langle \mathcal{M}_{\varepsilon}\rangle_n \leq c_3 \varepsilon^3 \log^2(1/\varepsilon) n. \tag{5.91}$$

By Lemma 17, $N^{\varepsilon}(\zeta(\varepsilon, t))$ is a stopping time with a finite expectation so by the optional stopping theorem and estimates (5.77) and (5.91), for any $s > 0$,

$$\begin{aligned} \mathbb{E}\big(\mathcal{M}_{\varepsilon}^2\big(s \wedge N^{\varepsilon}\big(\zeta(\varepsilon, t)\big)\big)\big) &= \mathbb{E}\big(\langle \mathcal{M}_{\varepsilon}\rangle_{s\wedge N^{\varepsilon}(\zeta(\varepsilon,t))}\big) \\ &\leq c_3 \varepsilon^3 \log^2(1/\varepsilon) \mathbb{E}\big(s \wedge N^{\varepsilon}\big(\zeta(\varepsilon, t)\big)\big) \\ &\leq c_4 \varepsilon^3 \log^2(1/\varepsilon) \bigg(\frac{\pi t}{\varepsilon \log(1/\varepsilon)} \frac{1}{\varepsilon} + 2\bigg) \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned}$$

We see that the assumptions of Lemma 18 are satisfied and we can use that lemma as follows. For any $a > 0$,

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} \big|\mathcal{M}_{\varepsilon}\big(N^{\varepsilon}\big(\zeta(\varepsilon, t)\big)\big)\big| \geq a\Big) \leq \sup_{s>0} \frac{2}{a^2} \mathbb{E}\big(\mathcal{M}_{\varepsilon}^2\big(N^{\varepsilon}\big(\zeta(\varepsilon, t)\big)\big)\big) \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.92}$$

The claim (5.89) is proved and, therefore, so is the lemma. □

Recall from (5.73) that $\Delta A_{n+1}^{\varepsilon} = \mathbb{E}((M_{n+1}^{\varepsilon} - M_n^{\varepsilon})^2 \mid \mathcal{F}_n^{\varepsilon})$. We will write

$$\Delta A^{\varepsilon}(\alpha, s) = \mathbb{E}\big(\big(M_{n+1}^{\varepsilon} - M_n^{\varepsilon}\big)^2 \mid \boldsymbol{\alpha}_n^{\varepsilon} = \alpha, \mathbf{s}_n^{\varepsilon} = s\big)$$

to emphasize the dependence on $\boldsymbol{\alpha}_n^{\varepsilon}$ and $\mathbf{s}_n^{\varepsilon}$. We will also use the self-explanatory notation $\Delta A^{\varepsilon}(\boldsymbol{\alpha}_n^{\varepsilon}, \mathbf{s}_n^{\varepsilon})$.

LEMMA 20. *The following limit holds uniformly in α and s*:

$$\lim_{\varepsilon\to 0} \frac{\Delta A^{\varepsilon}(\alpha, s)}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\alpha). \tag{5.93}$$

PROOF. Definition (5.73) yields

$$\begin{aligned}\Delta A^{\varepsilon}(\alpha, s) &= \mathbb{E}\big((M^{\varepsilon}_{n+1} - M^{\varepsilon}_{n})^2 \mid \boldsymbol{\alpha}^{\varepsilon}_{n} = \alpha, \mathbf{s}^{\varepsilon}_{n} = s\big) \\ &= \mathbb{E}\big((\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_{n} - \Delta B^{\varepsilon}(n+1))^2 \mid \boldsymbol{\alpha}^{\varepsilon}_{n} = \alpha, \mathbf{s}^{\varepsilon}_{n} = s\big) \\ &= \operatorname{Var}\big(\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_{n} \mid \boldsymbol{\alpha}^{\varepsilon}_{n} = \alpha, \mathbf{s}^{\varepsilon}_{n} = s\big).\end{aligned}$$

For $s = 1$, (5.16) implies that

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\Delta A^{\varepsilon}(\alpha, 1)}{(\varepsilon^2/2)\log(1/\varepsilon)} &= \lim_{\varepsilon\to 0} \frac{\operatorname{Var}(\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_{n} \mid \boldsymbol{\alpha}^{\varepsilon}_{n} = \alpha, \mathbf{s}^{\varepsilon}_{n} = 1)}{(\varepsilon^2/2)\log(1/\varepsilon)} \\ &= \lim_{\varepsilon\to 0} \frac{\operatorname{Var}(T^{\varepsilon}_1(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\alpha).\end{aligned} \tag{5.94}$$

Next, we consider the case $s = 0$. Recall (5.9). By Lemmas 11 and 9, for all α, $\mathbb{E}((\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha))^2) = O(\varepsilon^2)$. By Lemmas 11 and 14, for all α, $|\mathbb{E}(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha))| = O(\varepsilon^2)$. Hence, in view of Lemma 13,

$$\begin{aligned}&\big|\mathbb{E}\big(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha) + \big(1 - \Lambda^{\varepsilon}_1(\alpha)\big) R^{\varepsilon}_1(\alpha)\big)\big| \\ &\quad= \big(1 + o(1)\big)\big(\varepsilon^2/2\big)\log(1/\varepsilon)\big|h'(\alpha)h(\alpha)\big|\big(1 - O(\varepsilon)\big) + O\big(\varepsilon^2\big).\end{aligned} \tag{5.95}$$

By Lemma 13 and the Cauchy–Schwarz inequality,

$$\begin{aligned}&\mathbb{E}\big(\big(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha) + \big(1 - \Lambda^{\varepsilon}_1(\alpha)\big) R^{\varepsilon}_1(\alpha)\big)^2\big) \\ &\quad= \mathbb{E}\big(\big(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha)\big)^2\big) + \mathbb{E}\big(\big(\big(1 - \Lambda^{\varepsilon}_1(\alpha)\big) R^{\varepsilon}_1(\alpha)\big)^2\big) \\ &\qquad+ 2\mathbb{E}\big(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha)\big(1 - \Lambda^{\varepsilon}_1(\alpha)\big) R^{\varepsilon}_1(\alpha)\big) \\ &\quad= O\big(\varepsilon^2\big) + \big(1 + O(\varepsilon)\big)\big(\varepsilon^2/2\big)\log(1/\varepsilon)h^2(\alpha) + 0 \\ &\quad= \big(1 + O(\varepsilon)\big)\big(\varepsilon^2/2\big)\log(1/\varepsilon)h^2(\alpha).\end{aligned}$$

This and (5.95) imply that

$$\begin{aligned}&\operatorname{Var}\big(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha) + \big(1 - \Lambda^{\varepsilon}_1(\alpha)\big) R^{\varepsilon}_1(\alpha)\big) \\ &\quad= \big(1 + O(\varepsilon)\big)\big(\varepsilon^2/2\big)\log(1/\varepsilon)h^2(\alpha) + o\big(\varepsilon^2\log(1/\varepsilon)\big).\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{\Delta A^{\varepsilon}(\alpha, 0)}{(\varepsilon^2/2)\log(1/\varepsilon)} &= \lim_{\varepsilon\to 0} \frac{\operatorname{Var}(\boldsymbol{\alpha}^{\varepsilon}_{n+1} - \boldsymbol{\alpha}^{\varepsilon}_{n} \mid \boldsymbol{\alpha}^{\varepsilon}_{n} = \alpha, \mathbf{s}^{\varepsilon}_{n} = 0)}{(\varepsilon^2/2)\log(1/\varepsilon)} \\ &= \lim_{\varepsilon\to 0} \frac{\operatorname{Var}(\Lambda^{\varepsilon}_1(\alpha) S^{\varepsilon}_1(\alpha) + (1 - \Lambda^{\varepsilon}_1(\alpha)) R^{\varepsilon}_1(\alpha))}{(\varepsilon^2/2)\log(1/\varepsilon)} = h^2(\alpha).\end{aligned}$$

In view of (5.94), the proof is complete. $\square$

LEMMA 21. *For any* $T > 0$, $\sup_{t \le T} |\mathbf{A}_t^\varepsilon - \int_0^t h(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))})\, ds|$ *converges to* 0 *in probability when* $\varepsilon \to 0$.

PROOF. The proof is the same as that for Lemma 19, except for the following changes:

(i) $\Delta B^\varepsilon(\alpha, s)$ should be replaced with $\Delta A^\varepsilon(\alpha, s)$.

(ii) $h'(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))})$ should be replaced with $h(\boldsymbol{\alpha}^\varepsilon_{N^\varepsilon(\zeta(\varepsilon,s))})$.

(iii) Formula (5.86) should be replaced with the following consequence of Lemma 20:

$$\lim_{\varepsilon \to 0} \frac{\Delta A^\varepsilon(\alpha, s)}{(1/2)\varepsilon^2 \log(1/\varepsilon) h^2(\alpha)} = 1. \qquad \square$$

PROOF OF THEOREM 2. We will prove that processes

$$\{\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon, t))), t \ge 0\}$$

converge in law to X in the Skorokhod topology as ε goes to 0, where X solves the stochastic differential equation (4.2).

The above claim implies easily Theorem 2 because $\boldsymbol{\beta}^\varepsilon(\mathcal{T}_k^\varepsilon) = \boldsymbol{\alpha}_k^\varepsilon$ and the jumps of $\boldsymbol{\alpha}^\varepsilon$ are uniformly bounded by a quantity going to 0 when $\varepsilon \to 0$, by Lemma 9.

To prove the claim stated at the beginning of the proof, we will apply [10], Theorem 4.1, Chapter 7. We start with a dictionary translating our notation to that in [10]. In the following list, our symbol is written to the left of the arrow and the corresponding symbol used in [10] is written to the right of the arrow. Note that our family of processes is indexed by a continuous parameter $\varepsilon \in (0, 1/2)$ and the corresponding family of processes in [10] is indexed by a discrete parameter n. Standard arguments show that nevertheless [10], Theorem 4.1, Chapter 7, applies in our setting:

$$\begin{aligned}
\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon, t))) &\Longrightarrow X_n, \\
\mathbf{B}_t^\varepsilon &\Longrightarrow B_n, \\
\mathbf{M}_t^\varepsilon &\Longrightarrow M_n, \\
\mathbf{A}_t^\varepsilon &\Longrightarrow A_n, \\
h'(\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon, s)))) &\Longrightarrow b(X_n(s)), \\
h^2(\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon, s)))) &\Longrightarrow a(X_n(s)).
\end{aligned}$$

Many of the assumptions of [10], Theorem 4.1, Chapter 7, are clearly satisfied and, therefore, we will not discuss them explicitly. For example, our assumptions on the smoothness of h are so strong that the martingale problem corresponding to (4.2) is well posed.

We will now review the crucial assumptions of [10], Theorem 4.1, Chapter 7. Recall notation introduced in (5.73).

Let $\mathcal{G}_t^\varepsilon = \sigma((\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon,s))), \mathbf{B}_s^\varepsilon, \mathbf{A}_s^\varepsilon), s \le t)$. It was proved in Lemma 9 that the absolute value of a jump of $\boldsymbol{\alpha}_n^\varepsilon$ is bounded by $12\sqrt{\varepsilon}$. The same bound applies to jumps of $B^\varepsilon(n)$ and, therefore, the absolute value of a jump of $M^\varepsilon(n)$ is bounded by $24\sqrt{\varepsilon}$, a.s. Hence, it is easy to see from the definition (5.73) that $M^\varepsilon(n)$ is a martingale. Let $\tau_r^\varepsilon = \inf\{n : |M^\varepsilon(n)| \ge r\}$ and note that $|M^\varepsilon(n)| \le r + 24\sqrt{\varepsilon}$ for $n \le \tau_r^\varepsilon$. This easily implies that the optional stopping theorem applies to the martingale $M^\varepsilon(n)$ at the stopping time $N^\varepsilon(\zeta(\varepsilon,t)) \wedge \tau_r^\varepsilon$, for every t and r. This in turn implies that $\mathbf{M}_t^\varepsilon$ is a $\mathcal{G}_t^\varepsilon$-local martingale. A similar argument shows that $(\mathbf{M}_t^\varepsilon)^2 - \mathbf{A}_t^\varepsilon$ is a $\mathcal{G}_t^\varepsilon$-local martingale. We have verified the assumption that processes defined in (4.1) and (4.2) in [10], Chapter 7, are local martingales.

Assumptions (4.3)–(4.5) in [10], Chapter 7, are satisfied due to Lemma 16. Assumptions (4.6) and (4.7) in [10], Chapter 7, are satisfied due to Lemmas 19 and 21.

We have shown that the assumptions of [10], Theorem 4.1, Chapter 7, are satisfied. Therefore, we may conclude that $\{\boldsymbol{\alpha}^\varepsilon(N^\varepsilon(\zeta(\varepsilon,t))), t \ge 0\}$ converge in law to X in the Skorokhod topology as ε goes to 0, where X solves the stochastic differential equation (4.2). We have already pointed out that this implies Theorem 2. □

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