CRITICAL PARAMETER OF RANDOM LOOP MODEL ON TREES

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We give estimates of the critical parameter for random loop models that are related to quantum spin systems. A special case of the model that we consider is the interchange- or random-stirring process. We consider here the model defined on regular trees of large degrees, which are expected to approximate high spatial dimensions. We find a critical parameter that indeed shares similarity with existing numerical results for the cubic lattice. In the case of the interchange process, our results improve on earlier work by Angel and by Hammond, in that we determine the second-order term of the critical parameter.

1. Introduction. We consider random loop models that are motivated by quantum spin systems. A special case is the random interchange model that was first introduced by Harris [12]. Tóth showed that a variant of this model, where permutations receive the weight $2^{\#\text{cycles}}$, is closely related to the quantum Heisenberg ferromagnet [17]. Another loop model was introduced by Aizenman and Nachtergaele to describe the quantum Heisenberg antiferromagnet [1]. These loop models were combined in order to describe a family of quantum systems that interpolate between the two Heisenberg models, and which contains the quantum XY model [18].

Let $G = (V, E)$ be an arbitrary finite graph with vertex set V and edge set E, and $\beta > 0$, $u \in [0, 1]$ be two parameters. To each edge $e \in E$ is assigned a time interval $[0, \beta]$, and an independent Poisson point process with two kinds of outcomes: "crosses" occur with intensity u and "double bars" occur with intensity $1 - u$. We let $\Omega(G)$ denote the set of realizations of the combined Poisson point process on $E \times [0, \beta]$.

Given a realization $\omega \in \Omega(G)$, we consider the loop passing through a point $(x, t) \in V \times [0, \beta]$ that is defined in a natural way, as follows (see Figure 1). The loop is a closed trajectory with support on $V \times [0, \beta]_{\text{per}}$ where $[0, \beta]_{\text{per}}$ is the interval $[0, \beta]$ with periodic boundary conditions, that is, the torus of length β. Starting at (x, t), move "up" until meeting the first cross or double bar with endpoint x; then jump onto the other endpoint, and continue in the same direction if a cross, in the opposite direction if a double bar; repeat until the trajectory returns to (x, t).
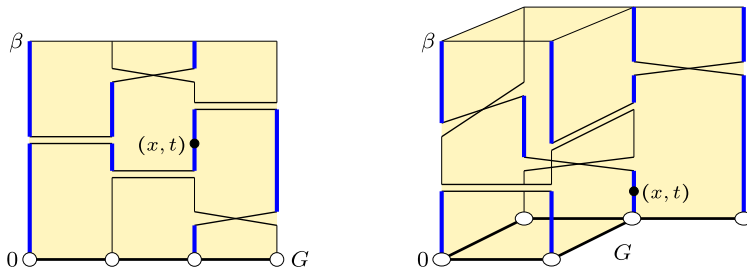
FIG. 1.   *Graphs and realizations of Poisson point processes, and the loop that contains (x, t).*

In order to represent a quantum model, one should add the weight $\theta^{\#\text{loops}}$ with $\theta = 2, 3, 4, \ldots$; quantum correlations are then given in terms of loop correlations, and magnetic long-range order is equivalent to the presence of macroscopic loops. Notice that the parameter β plays the role of the inverse temperature of the quantum spin system, hence the notation.

The random interchange model (i.e., the case $u = 1$ and $\theta = 1$) has been the object of several studies when the graph is a tree [3, 10, 11], the complete graph [5, 6, 15], the hypercube [13], and the Hamming graph [14]; a result for general graphs was also proved in [2]. In the case of arbitrary $\theta \in \{2, 3, \ldots\}$, and on the complete graph, the critical parameter has also been determined [7, 8]. Another generalization of the random interchange model is Mallows permutations, studied in [9, 16].

The occurrence of macroscopic loops can be proved using the method of reflection positivity and infrared bounds in the case where $u \in [0, \frac{1}{2}]$, $\theta = 2, 3, \ldots$, and a cubic lattice of sufficiently high dimensions (depending on θ); see [18] for precise statements.

In the case where the graph is a three-dimensional cubic lattice with edges between nearest-neighbors, and with $\theta = 1$, the critical parameter $\beta_c(u)$ has been calculated numerically in [4]. The result is depicted in Figure 2 and shows a convex curve where $\beta_c(0)$ is slightly smaller than $\beta_c(1)$ and which has a minimum at or around $u = \frac{1}{2}$. This behavior is expected to hold for all dimensions $d \geq 3$.

Trees are expected to approximate high dimensions. We consider here infinite regular trees with offspring degree d. Loops are almost surely well defined in the same way as previously (since vertices have uniformly bounded degrees) but now some loops may be unbounded. We prove that a transition takes place at the critical parameter $\beta_c = \beta_c(u, d)$ given by

$$(1.1) \qquad \beta_c(u, d) = \frac{1}{d} + \frac{1 - u(1 - u) - \frac{1}{6}(1 - u)^2}{d^2} + o(d^{-2}).$$

The second graph of Figure 2 shows β_c as function of u (with $d = 5$). The leading order of the critical parameter, $\frac{1}{d}$, is also the leading order for the percolation threshold in the associated percolation model where an edge is open if at least one
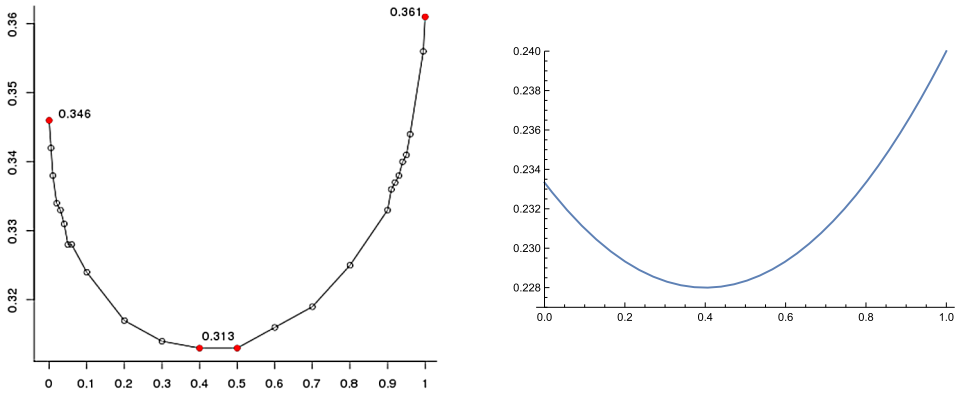
FIG. 2. *The critical parameter β_c as function of u: Left, on the three-dimensional cubic lattice (numerical results from* [4]); *right, equation* (1.1) *with $d = 5$.*

cross or double bar is present in the corresponding interval $[0, \beta]$. The next order for β_c is a nontrivial function of u and it is smallest for $u = \frac{2}{5}$. This function can be understood by looking at edges with two links and at loop connections with two crosses, two double bars or one each. As is explained below (see Figure 4), loop connections are better in the latter case, with a cross and a double bar.[2]

Let E_∞ denote the event where the root of the tree (at time 0) belongs to an infinite loop.

THEOREM 1.1. *Let $A > 0$ be arbitrary and $\beta = \frac{1}{d} + \frac{\alpha}{d^2}$ with $\alpha \leq A$. There exists d_0 (that may depend on A but not on α) such that for all $d \geq d_0$, there exists $\alpha_c(u, d)$ such that*

$$\mathbb{P}_{\beta,d,u}(E_\infty) \begin{cases} = 0 & \text{if } \alpha < \alpha_c(u, d), \\ > 0 & \text{if } \alpha > \alpha_c(u, d). \end{cases}$$

Further, we have $\alpha_c(u, d) = 1 - u(1 - u) - \frac{1}{6}(1 - u)^2 + o(1)$ as $d \to \infty$, uniformly in u.

This theorem follows from Propositions 2.1 and 3.1. Proposition 2.1 establishes the existence of $d_0(\alpha)$ such that loops occur for $\alpha > \alpha_c$ but not for $\alpha < \alpha_c$, if $d > d_0(\alpha)$. The case $u = 1$, that is, the interchange model on trees, was treated up to first order in d^{-1} by Hammond [10], following the work of Angel [3]. Proposition 3.1 implies that $d_0(\alpha)$ is uniform on bounded intervals. The corresponding result for the interchange model ($u = 1$) was proved by Hammond [11]. It turns out that his method can be adapted to $u \neq 1$ with minor modifications, as explained in Section 3.

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[2]Alan Hammond pointed out this important observation to us.

The reason we require α to be bounded by A is that our arguments only apply for β close to d^{-1}; hence d_0 depends on A. Presumably, there is some $A_0 > 0$ such that $\mathbb{P}(E_\infty) > 0$ whenever $\alpha > A_0$. For the interchange model, this was also proved by Hammond [10], but his arguments use in a crucial way a comparison with random walk, which fails for $u < 1$ due to the "time-reversal" which occurs when a double bar is traversed. [For $\alpha < 0$, we have $\mathbb{P}(E_\infty) = 0$ for all d by standard comparison with percolation.]

Of the two previous methods for proving the occurrence of infinite loops in the interchange model, due to Angel [3] and Hammond [10], respectively, our argument is thus closer to that of Angel, which also requires β to be close to d^{-1}. However, where Angel uses a comparison with a branching process, we instead directly prove recursion inequalities for the probability of long loops. These inequalities include error terms which are of higher order in d^{-1} and may be made negligible by taking d large.

The case $\theta \neq 1$ could probably be treated in a similar way, although a full study is needed in order to rule out extra obstacles. A major open problem is to establish that, in the case where the graph is a box in $\mathbb{Z}^{d'}$ with nearest-neighbor edges, the critical parameter satisfies equation (1.1) with $d = 2d' - 1$.

2. The critical parameter.

As mentioned above, we consider an infinite rooted regular tree with offspring degree d. To each edge is associated the interval $[0, 1]$, and an independent Poisson point process where "crosses" occur with intensity $u\beta \in [0, \beta]$, and "double bars" occur with intensity $(1 - u)\beta$. (This is a variation of the model discussed above, with β affecting the intensities rather than the time interval, which is obviously equivalent.)

Let us define $\bar{\alpha}(u) = 1 - u(1 - u) - \frac{1}{6}(1 - u)^2$. In what follows, we always take $0 \leq \alpha \leq A$ for some arbitrary but fixed $A > 0$, and error terms may depend on A.

PROPOSITION 2.1. *Let $\beta = \frac{1}{d} + \frac{\alpha}{d^2}$ and $\delta > 0$. There exists $d_0(\delta)$ such that the following hold for all $d > d_0$:*

(a) *For every $\alpha \leq \bar{\alpha} - \delta$, we have*

$$\mathbb{P}_{\beta,d,u}\big((\rho, 0) \leftrightarrow \infty\big) = 0.$$

(b) *For every $\alpha \geq \bar{\alpha} + \delta$, we have*

$$\mathbb{P}_{\beta,d,u}\big((\rho, 0) \leftrightarrow \infty\big) > 0.$$

Note that we prove exponential decay for (a), that is, the loop containing $(\rho, 0)$ has diameter m with probability less than $Ce^{-\eta m}$. These claims can be compared to the numerical results for three-dimensional lattices. Also, the special case $u = 1$ of our result gives a solution to Problem 10 of [3] (for large enough d).

2.1. *Preliminaries.* We let T denote an infinite tree where each vertex has $d \geq 2$ offspring, and write ρ for its root. For $m \geq 0$, let $T^{(m)}$ denote the subtree of T consisting of the first m generations.

We write σ_m for the probability that $(\rho, 0)$ belongs to a loop which reaches generation m in $T^{(m)}$, and $\zeta_m = 1 - \sigma_m$. Note that $\sigma_m \leq \sigma_{m-1}$ and that $\sigma_m \to \mathbb{P}((\rho, 0) \leftrightarrow \infty)$ as $m \to \infty$. We write $B^m_{(\rho, 0)}$ for the event that $(\rho, 0)$ does not belong to a loop which reaches generation m in $T^{(m)}$, so that $\mathbb{P}(B^m_{(\rho, 0)}) = \zeta_m$. Thus $B^m_{(\rho, 0)}$ is the event that the loop of $(\rho, 0)$ is "blocked" from generation m, and σ_m is the probability that it "survives" for m generations.

Crosses and double-bars will be referred to collectively as *links*. If $(xy, t) \in \omega$ is a link, then in general we have that the points $(x, t+)$ and $(x, t-)$ may belong to different loops [the same is true for $(y, t+)$ and $(y, t-)$]. We say that a link is a *monolink* if $(x, t+)$ and $(x, t-)$ belong to the *same* loop. The following simple observation will be useful.

PROPOSITION 2.2. *Suppose that y is a child of x in $T^{(m)}$. If there is only one link between x and y, then it is a monolink.*

PROOF. Denote the link (xy, t). In the configuration obtained by removing this link, the points (x, t) and (y, t) belong to two different loops, since we are on a tree. When the link is added back in, the loops are merged to a single loop, since the tree is finite. This proves the claim. □

Write A_1 for the event that, for each child x of ρ, there is at most one link between ρ and x. Write A_2 for the event that: (i) there is a unique child x of ρ with exactly 2 links between ρ and x, (ii) for all siblings x' of x there is at most one link between ρ and x', and (iii) for all children y of x there is at most one link between x and y. See Figure 3.
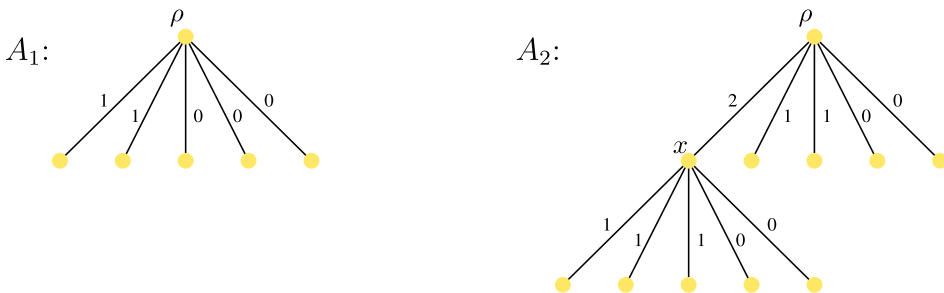

FIG. 3. *Summing over two events A_1 and A_2.*

Clearly, we have that

$$
\begin{aligned}
\zeta_m &= \mathbb{P}\big(B^m_{(\rho,0)}\big) \\
&= \mathbb{P}\big(B^m_{(\rho,0)} \cap A_1\big) + \mathbb{P}\big(B^m_{(\rho,0)} \cap A_2\big) + \mathbb{P}\big(B^m_{(\rho,0)} \setminus (A_1 \cup A_2)\big).
\end{aligned}
$$

(2.1)

In the rest of this section, we work with β of the form

$$
\beta = \frac{1}{d} + \frac{\alpha}{d^2}
$$

(2.2)

for $\alpha \in \mathbb{R}$.

2.2. *Occurrence of long loops.* We now prove part (b) of Proposition 2.1. For given $m \geq 1$ and $\varepsilon > 0$, we define

$$
\tilde{\sigma}_m = \sigma_m \wedge \sigma_{m-1} \wedge \left(\frac{\varepsilon}{d}\right) = \sigma_m \wedge \left(\frac{\varepsilon}{d}\right).
$$

(2.3)

Recall that we assume $\alpha \leq A$. In this section, we show the following.

PROPOSITION 2.3. *For all $m \geq 1$, we have*

$$
\sigma_m \geq \tilde{\sigma}_{m-1} + \frac{\tilde{\sigma}_{m-1}}{d}\big(\alpha - \bar{\alpha}(u)\big) - \frac{1}{2}\tilde{\sigma}_{m-1}^2 + O\big(d^{-3}\big),
$$

where the $O(d^{-3})$ is uniform in m (but depends on A).

Given the proposition, we can establish the occurrence of infinite loops.

PROOF OF PROPOSITION 2.1, PART (B).   We claim that if $\varepsilon < 2(\alpha - \bar{\alpha}(u))$ is small enough then, for d large enough, we have $\sigma_m \geq \tilde{\sigma}_m \geq \frac{\varepsilon}{d}$ for all m. Since $\sigma_0 = 1$ and $\sigma_1 \geq 1 - (e^{-\beta})^d$, the claim holds for $m = 1$, and Proposition 2.3 gives the claim by induction. Hence $\sigma_m \geq \frac{\varepsilon}{d}$ for all m, which gives the result.   □

PROOF OF PROPOSITION 2.3.   The starting point is the inequality

(2.4) $\zeta_m \leq \mathbb{P}\big(B^m_{(\rho,0)} \cap A_1\big) + \mathbb{P}\big(B^m_{(\rho,0)} \cap A_2\big) + 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2),$

which follows directly from (2.1). First, note that

(2.5) $\mathbb{P}(A_1) = \big(e^{-\beta}(1+\beta)\big)^d, \qquad \mathbb{P}(A_2) = \frac{1}{2}d\beta^2 e^{-\beta}\big(e^{-\beta}(1+\beta)\big)^{2d-1}.$

Next, note that

$$
\begin{aligned}
\mathbb{P}\big(B^m_{(\rho,0)} \cap A_1\big) &= \sum_{k=0}^{d} \binom{d}{k}(\beta e^{-\beta})^k (e^{-\beta})^{d-k}(\zeta_{m-1})^k \\
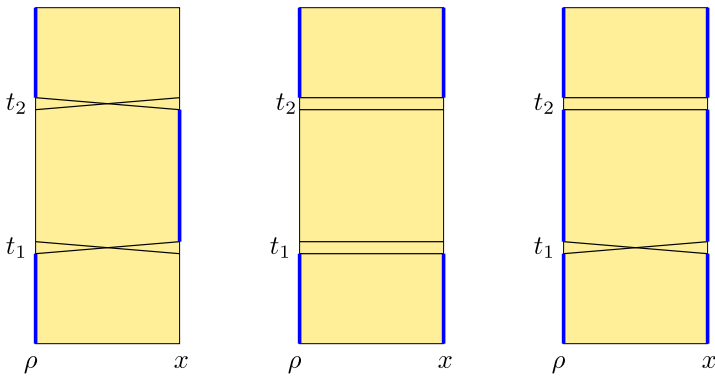&= \big(e^{-\beta}(1 + \beta\zeta_{m-1})\big)^d.
\end{aligned}
$$

(2.6)

FIG. 4.   *When there are two crosses at times t_1 and t_2, connection is only with the interval $[t_1, t_2]$ of length $X = t_2 - t_1$; when there are two double bars, connection is only with the interval $[0, t_1] \cup [t_2, 1]$ of length $1 - X$; when there is one cross and one double bar, connection is with the whole interval $[0, 1]$. The latter case is more favourable for connections.*

This relies on Proposition 2.2. Indeed, if there are k children $x_1, \ldots, x_k$ of ρ that are linked to ρ, with one link each, at times $t_1, \ldots, t_k$ say, then $(\rho, 0)$ lies in the same loop as all of $(x_1, t_1), \ldots, (x_k, t_k)$. The probability of not being connected to generation 0 is the same if one has one incoming link from a parent as if one has none, and is thus ζ_{m-1} for each of $(x_1, t_1), \ldots, (x_k, t_k)$.

In obtaining a similar expression for the case A_2, it is useful to refer to Figure 4. Let Λ_ρ and Λ_x denote the restrictions of the subset highlighted in blue to $\{\rho\} \times [0, 1]$ and $\{x\} \times [0, 1]$, respectively. Thus Λ_ρ and Λ_x have respective lengths X and $1 - X$ in the case of two crosses; X and X in the case of two double-bars; and 1 in the case of a mixture. It may look obvious that X is uniformly distributed in $[0, 1]$; this is however incorrect, since it can be written as

$$X = \min\{U_1, U_2\} + 1 - \max\{U_1, U_2\},$$

where U_1, U_2 are independent uniform random variables on $[0, 1]$; in particular, $\mathbb{E}[X] = \frac{2}{3}$.

As before, any link from ρ to a sibling x' of x, or from x to a child y, is a mono-link. Links that fall in $\Lambda_\rho \cup \Lambda_x$ have a chance of connecting $(\rho, 0)$ to generation m, the others do not. There are d choices of x and the probability of exactly two links from ρ to x is $\frac{1}{2}\beta^2 e^{-\beta}$. Conditioning on this as well as the lengths $|\Lambda_\rho|$ and $|\Lambda_x|$, and considering the probabilities for the remaining monolinks to connect $(\rho, 0)$ to generation m, one obtains

$$(2.7) \quad \mathbb{P}(B^m_{(\rho,0)} \cap A_2) = \frac{1}{2}d\beta^2 e^{-\beta}\mathbb{E}\big[\big(e^{-\beta}\big(1 + \beta\zeta_{m-1}|\Lambda_\rho| + \beta(1 - |\Lambda_\rho|)\big)\big)^{d-1}$$
$$\times \big(e^{-\beta}\big(1 + \beta\zeta_{m-2}|\Lambda_x| + \beta(1 - |\Lambda_x|)\big)\big)^d\big],$$

where the expectation is over the lengths $|\Lambda_\rho|$ and $|\Lambda_x|$. As noted above, we have

$$|\Lambda_\rho| = 1 - |\Lambda_x| = X, \qquad \text{with probability } u^2,$$
(2.8)
$$|\Lambda_\rho| = |\Lambda_x| = X, \qquad \text{with probability } (1-u)^2,$$
$$|\Lambda_\rho| = |\Lambda_x| = 1, \qquad \text{with probability } 2u(1-u).$$

We now use the inequalities

$$(2.9) \qquad \zeta_{m-1} \le 1 - \tilde\sigma_{m-1}, \qquad \zeta_{m-2} \le 1 - \tilde\sigma_{m-1}$$

to obtain from (2.6) that

$$(2.10) \qquad \mathbb{P}\big(B_{(\rho,0)}^m \cap A_1\big) \le \big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta)\big)^d$$

and from (2.7) that

$$(2.11) \qquad \mathbb{P}\big(B_{(\rho,0)}^m \cap A_2\big) \le \frac{1}{2}d\beta^2 e^{-\beta}\mathbb{E}\big[\big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta|\Lambda_\rho|)\big)^{d-1}$$
$$\times \big(e^{-\beta}(1+\beta-\tilde\sigma_{m-1}\beta|\Lambda_x|)\big)^d\big].$$

In light of (2.10), (2.11) and (2.5), we will proceed by providing estimates for terms of the form

$$(2.12) \qquad \big(e^{-\beta}(1+\beta-\sigma x\beta)\big)^d,$$

for $\sigma = O(d^{-1})$ and constant $x \in [0,1]$. Since $\beta = \frac{1}{d} + \frac{\alpha}{d^2}$, the following are easy to verify:

$$e^{-\beta} = 1 - \frac{1}{d} + \frac{1}{d^2}\Big(\frac{1}{2}-\alpha\Big) + \frac{1}{d^3}\Big(\alpha-\frac{1}{6}\Big) + O(d^{-4}),$$
(2.13)
$$1+\beta-\sigma x\beta = 1 + \frac{1}{d} + \frac{1}{d^2}(\alpha - x\sigma d) - \frac{1}{d^3}(\alpha x\sigma d).$$

Here and in what follows, the $O(\cdot)$ may depend on A (our absolute bound on α) but is uniform in the other parameters. Hence

$$e^{-\beta}(1+\beta-\sigma x\beta) = 1 + \frac{1}{d^2}\Big(-\frac{1}{2}-x\sigma d\Big)$$
(2.14)
$$+ \frac{1}{d^3}\Big(\frac{1}{3}-\alpha+x\sigma d-\alpha x\sigma d\Big) + O(d^{-4}).$$

Combining this with

$$(2.15) \qquad \Big(1+\frac{a}{n^2}+\frac{b}{n^3}+O(n^{-4})\Big)^n = 1 + \frac{a}{n} + \frac{1}{n^2}\Big(b+\frac{a^2}{2}\Big) + O(n^{-3})$$

we see that

$$\left(e^{-\beta}(1 + \beta - \sigma x\beta)\right)^d = 1 - \frac{1}{d}\left(\frac{1}{2} + x\sigma d\right)$$

(2.16)
$$+ \frac{1}{d^2}\left(\frac{1}{3} - \alpha + x\sigma d - \alpha x\sigma d + \frac{1}{2}\left(\frac{1}{2} + x\sigma d\right)^2\right)$$

$$+ O(d^{-3}).$$

Applying this to (2.10) and (2.5), we obtain

(2.17) $\quad \mathbb{P}(A_1) - \mathbb{P}\left(B_{(\rho,0)}^m \cap A_1\right) \geq \tilde{\sigma}_{m-1} - \frac{\tilde{\sigma}_{m-1}}{d}\left(\frac{3}{2} - \alpha\right) - \frac{1}{2}\tilde{\sigma}_{m-1}^2 + O(d^{-3}).$

Now consider the case of $\mathbb{P}(A_2)$ and (2.11). Since

(2.18) $$\frac{1}{2}d\beta^2 e^{-\beta} = \frac{1}{2d} + O(d^{-2})$$

it suffices in this case to use (2.16) to order $\frac{1}{d}$. We may also replace the $d-1$ in the exponent by d. We obtain that

$$\mathbb{P}(A_2) - \mathbb{P}\left(B_{(\rho,0)}^m \cap A_2\right)$$

$$\geq \left(\frac{1}{2d} + O(d^{-2})\right)\mathbb{E}\left[\left(1 - \frac{1}{d}\right)\right.$$

$$\left. - \left(1 - \frac{1}{d}\left(\frac{1}{2} + |\Lambda_\rho|\tilde{\sigma}_{m-1}d\right)\right)\left(1 - \frac{1}{d}\left(\frac{1}{2} + |\Lambda_x|\tilde{\sigma}_{m-1}d\right)\right)\right]$$

(2.19) $\qquad + O(d^{-3})$

$$= \frac{\tilde{\sigma}_{m-1}}{2d}\mathbb{E}\left[|\Lambda_\rho| + |\Lambda_x|\right] + O(d^{-3})$$

$$= \frac{\tilde{\sigma}_{m-1}}{2d}\left(u^2 + \frac{4}{3}(1-u)^2 + 4u(1-u)\right) + O(d^{-3})$$

$$= \frac{\tilde{\sigma}_{m-1}}{d}\left(\frac{1}{2} + u(1-u) + \frac{1}{6}(1-u)^2\right) + O(d^{-3}).$$

Here, we used (2.8) and $\mathbb{E}(X) = \frac{2}{3}$. Adding this to (2.17) and substituting in (2.4), we obtain the claim. $\square$

2.3. *Absence of long loops.* Interestingly, the absence of large loops for $\alpha < \bar{\alpha}(u)$ seems harder to establish than their occurrence for $\alpha > \bar{\alpha}(u)$. Intuitively, this is because for part of the range of α that we consider (namely, for $\alpha > \frac{1}{2}$) the percolation-tree (obtained by keeping only edges carrying at least one link) is infinite with positive probability, yet we still need to show that the loops are always blocked.

We will use the notation $p_0, p_1, p_2, \ldots$ and $p_{\geq 2}, p_{\geq 3}, \ldots$ for the probabilities for a Poisson(β) random variable. We also use the shorthand

$$(2.20) \qquad q = p_{\geq 3} = \sum_{j=3}^{\infty} e^{-\beta} \frac{\beta^j}{j!} = \frac{1}{6d^3} + O(d^{-4}),$$

and define for $m \geq 3$

$$(2.21) \qquad \check{\sigma}_{m-1} = \sum_{\ell=3}^{m} (dq)^{\ell-3} \sigma_{m-\ell}.$$

Here and in what follows, the $O(\cdot)$ may depend on A (our absolute bound on α) but is uniform in the other parameters. In this section, we prove the following.

PROPOSITION 2.4.   *For all $m \geq 3$, we have*

$$\sigma_m \leq \check{\sigma}_{m-1} \left(1 + \frac{\alpha - \bar{\alpha}(u)}{d} + O(d^{-2}) \right),$$

where the $O(d^{-2})$ is uniform in m (but depends on A).

The proposition implies the remaining part of Proposition 2.1.

PROOF OF PROPOSITION 2.1, PART (A).   Suppose $\alpha - \bar{\alpha}(u) \leq -2\varepsilon < 0$. For d large enough, we have $dq \leq 1/d^2$ and, by Proposition 2.4, that

$$(2.22) \qquad \sigma_m \leq \left(1 - \frac{\varepsilon}{d} \right) \check{\sigma}_{m-1} \leq \left(1 - \frac{\varepsilon}{d} \right) \sum_{\ell=3}^{m} \sigma_{m-\ell} \left(\frac{1}{d^2} \right)^{\ell-3},$$

for all $m \geq 3$. We show, by induction over m, that if d is large enough, then there are constants $C = C(d) > 0$ and $\sigma = \sigma(d) \in (0, 1)$ such that

$$(2.23) \qquad \sigma_k \leq C\sigma^k \qquad \text{for all } k \geq 0.$$

This clearly implies the result. We choose $\sigma = 1 - \frac{\varepsilon}{6d}$, and by choosing C appropriately we can assume that (2.23) holds for $k = 0, 1, 2$. Suppose that it holds for $k \leq m - 1$ for some $m \geq 3$. Then by (2.22)

$$(2.24) \qquad \begin{aligned} \sigma_m &\leq C\sigma^m \left(1 - \frac{\varepsilon}{d} \right) \left(\frac{1}{\sigma} \right)^3 \sum_{l=3}^{m} \left(\frac{1}{\sigma d^2} \right)^{\ell-3} \\ &\leq C\sigma^m \left(1 - \frac{\varepsilon}{d} \right) \left(\frac{1}{\sigma} \right)^3 \frac{1}{1 - 1/\sigma d^2}. \end{aligned}$$

But here the factor

$$
\left(1 - \frac{\varepsilon}{d}\right)\left(\frac{1}{\sigma}\right)^3 \frac{1}{1 - 1/\sigma d^2}
$$

(2.25)
$$
= \left(1 - \frac{\varepsilon}{d}\right)\left(1 + 3\frac{\varepsilon}{6d} + O(d^{-2})\right)\left(1 + O(d^{-2})\right)
$$

$$
= 1 - \frac{\varepsilon}{2d} + O(d^{-2}) \le 1,
$$

provided d is large enough. Hence (2.23) follows for $k = m$, as required. $\square$

LEMMA 2.5. *Assume that*

$$
\mathbb{P}\big(B_{(\rho,0)}^m \cap (A_1 \cup A_2)^c\big) \ge \mathbb{P}\big((A_1 \cup A_2)^c\big)[1 - c\check{\sigma}_{m-1}]
$$

for some constant $c > 0$. Then the bound of Proposition 2.4 *holds true.*

PROOF. We note that, by (2.5), (2.6) and (2.7), we have that

$$
\mathbb{P}\big(B_{(\rho,0)}^m \cap A_1\big) \ge \mathbb{P}(A_1)\left(1 - \sigma_{m-1}\left(1 + \frac{\alpha - 1}{d} + O(d^{-2})\right)\right),
$$

(2.26) $\mathbb{P}\big(B_{(\rho,0)}^m \cap A_2\big)$

$$
\ge \mathbb{P}(A_2)\left(1 - \sigma_{m-2}\left(1 + 2u(1 - u) + \frac{1}{3}(1 - u)^2 + O(d^{-1})\right)\right).
$$

This uses the inequalities $\sigma_{m-1} \le \sigma_{m-2}$ and $(1 - x)^n \ge 1 - nx$ for $x \in [0, 1]$ and $n \ge 1$, as well as the asymptotics

(2.27) $$\frac{\beta}{1 + \beta} = \frac{1}{d} + \frac{\alpha - 1}{d^2} + O(d^{-3}).$$

We have $\check{\sigma}_{m-1} \ge \sigma_{m-2} \ge \sigma_{m-1}$ and $\mathbb{P}(A_1) = 1 - \frac{1}{2d} + O(d^{-2})$ and $\mathbb{P}(A_2) = \frac{1}{2d} + O(d^{-2})$. Together with the assumption of the lemma, we have, using (2.1),

$$
\sigma_m \le 1 - \mathbb{P}(A_1)\left[1 - \sigma_{m-1}\left(1 + \frac{\alpha - 1}{d} + O(d^{-2})\right)\right]
$$

$$
- \mathbb{P}(A_2)\left[1 - \sigma_{m-2}\left(1 + 2u(1 - u) + \frac{1}{3}(1 - u)^2 + O(d^{-1})\right)\right]
$$

$$
- \mathbb{P}\big((A_1 \cup A_2)^c\big)[1 - c\check{\sigma}_{m-1}]
$$

$$
= \sigma_{m-1}\mathbb{P}(A_1)\left[1 + \frac{\alpha - 1}{d} + O(d^{-2})\right]
$$

$$
+ \sigma_{m-2}\mathbb{P}(A_2)\left[1 + 2u(1 - u) + \frac{1}{3}(1 - u)^2 + O(d^{-1})\right]
$$

(2.28) $$+ c\check{\sigma}_{m-1}\mathbb{P}\big((A_1 \cup A_2)^c\big)$$

$$\leq \check{\sigma}_{m-1}\left[1 + \frac{\alpha - 1}{d} - \frac{1}{2d} + O(d^{-2}) + \frac{1}{2d}\right.$$

$$\left. + \frac{u(1-u) + \frac{1}{6}(1-u)^2}{d} + O(d^{-2})\right].$$

This is indeed the upper bound of Proposition 2.4. $\quad\square$

The rest of this section will be devoted to the proof of the assumption of Lemma 2.5.

We write $(A_1 \cup A_2)^c$ as a union

(2.29)
$$(A_1 \cup A_2)^c = \bigcup_{k=1}^{d} (A_k' \cup A_k''),$$

of the disjoint events:

- A_1': that ρ has exactly one child with ≥ 3 links and all other children of ρ have 0 or 1 links;
- A_k' for $k \geq 2$: that ρ has exactly k children with ≥ 2 links;
- A_k'' for $k \geq 1$: that ρ has exactly one child x with exactly 2 links, all other children of ρ have 0 or 1 links, and x has exactly k children with ≥ 2 links.

The following bounds will be useful later.

LEMMA 2.6.   *For d large enough, we have*

$$\sum_{k=1}^{d} k\mathbb{P}(A_k') \leq 2\sum_{k=1}^{d} \mathbb{P}(A_k'), \quad \text{and} \quad \sum_{k=1}^{d} k\mathbb{P}(A_k'') \leq 2\sum_{k=1}^{d} \mathbb{P}(A_k'').$$

PROOF.   We start with the A_k'':s, which is actually the simpler case. For convenience, we write A_0'' for the event that ρ has exactly one child x with exactly 2 links, and that the other children of ρ have 0 or 1 links. Then

(2.30)
$$\sum_{k=1}^{d} k\mathbb{P}(A_k'') = \mathbb{P}(A_0'')\sum_{k=1}^{d} k\mathbb{P}(A_k'' \mid A_0'') = \mathbb{P}(A_0'')dp_{\geq 2},$$

since the last sum is the expected number of children of x with two links or more. Similarly,

(2.31)
$$\sum_{k=1}^{d} \mathbb{P}(A_k'') = \mathbb{P}(A_0'')\sum_{k=1}^{d} \mathbb{P}(A_k'' \mid A_0'') = \mathbb{P}(A_0'')\left(1 - (1 - p_{\geq 2})^d\right).$$

It is easy to deduce that

(2.32)
$$\frac{\sum_{k=1}^{d} k\mathbb{P}(A_k'')}{\sum_{k=1}^{d} \mathbb{P}(A_k'')} \to 1, \qquad \text{as } d \to \infty,$$

which gives the claim for the A_k''. For the A_k' a straightforward but tedious calculation gives that

$$(2.33) \qquad \sum_{k=1}^{d} k\mathbb{P}(A_k') = dp_{\geq 2} - dp_2(1 - p_{\geq 2})^{d-1} = \frac{5/12}{d^2} + O(d^{-3})$$

and

$$(2.34) \qquad \sum_{k=1}^{d} \mathbb{P}(A_k') = 1 - (1 - p_{\geq 2})^d - dp_2(1 - p_{\geq 2})^{d-1}$$
$$= \frac{7/24}{d^2} + O(d^{-3}).$$

Hence

$$(2.35) \qquad \frac{\sum_{k=1}^{d} k\mathbb{P}(A_k')}{\sum_{k=1}^{d} \mathbb{P}(A_k')} \to \frac{10}{7} < 2, \qquad \text{as } d \to \infty,$$

which gives the claim for the A_k'. $\square$

The main idea in establishing the assumption of Lemma 2.5 is to use a certain random subtree $\check{T}$ of $T^{(m)}$, which we think of as the "complex component of ρ". We will use $\check{T}$ to avoid dealing explicitly with the possibility that the loop of $(\rho, 0)$ propagates across edges carrying ≥ 3 links. Since edges carrying ≥ 3 links are rare, the connected component of ρ in the subtree spanned by such edges is typically small. This subtree is bounded by edges carrying 0, 1 or 2 links each, and we will use estimates on the probability that a loop is blocked after traversing such an edge. It will be convenient to define $\check{T}$ slightly differently than as the subtree spanned by edges with ≥ 3 links, since we want the event $(A_1 \cup A_2)^c$ be be $\check{T}$-measurable.

In order to define $\check{T}$, it helps to think that it consists of "bulk sites" and "end sites." The root ρ is a bulk site by definition. Assume that the tree has been defined up to level k, and let x be a bulk site at level k. An offspring y is:

(a) a bulk site if the number of links n_{xy} on the edge x is equal to $3, 4, \ldots$;
(b) a bulk site if $x = \rho$, $n_{xy} = 2$, and all siblings z of y satisfy $n_{xz} \in \{0, 1\}$;
(c) an end site if $n_{xy} \in \{0, 1, 2\}$, unless there is situation (b).

Note that the event $(A_1 \cup A_2)^c$ is measurable with respect to $\check{T}$.

We write $\check{\omega}$ for the configuration of crosses and double-bars within $\check{T}$. For $j = 1, 2$ and $1 \leq \ell \leq m - 1$, we write $\mathcal{E}_\ell^{(j)}$ for the set of leaves (end sites) of $\check{T}$ at distance ℓ from ρ and with j incoming links. If $x \in \mathcal{E}_\ell^{(1)}$, we write $t(x)$ for the time-coordinate of the incoming link, and if $y \in \mathcal{E}_\ell^{(2)}$ we write $t_1(y)$ and $t_2(y)$ for the time-coordinates of the two incoming links. We also let $\mathcal{E}_\ell = \mathcal{E}_\ell^{(1)} \cup \mathcal{E}_\ell^{(2)}$ ($1 \leq \ell \leq m - 1$) and we let $\mathcal{E}_m$ be the set of vertices of $\check{T}$ at distance m from ρ.

For $y \in T^{(m)}$, we let T_y be the subtree rooted at y, consisting of y and all its descendants in $T^{(m)}$. For a sub-tree T' of $T^{(m)}$, we write $\Omega(T')$ for the set of configurations of crosses and double-bars in T'. In particular, $\Omega(T_y)$ is the set of configurations in the subtree rooted at y. We write $B^k_{(y,t)} \subseteq \Omega(T_y)$ for the set of configurations in T_y such that the loop of (y, t) visits no vertex $z \in T_y$ at distance k from y (note that we do not consider any incoming links to y from its parent). And we write $B^m_{(\rho,0)}(y) \subseteq \Omega(T^{(m)})$ for the event that the loop of $(\rho, 0)$ visits no vertex $z \in T_y$ at distance m from ρ, that is, the loop does not reach distance m in the subtree rooted at y.

The next lemma concerns the probability of blocking a loop at a vertex y when there are two links between y and its parent.

LEMMA 2.7.   *Let y be a vertex of $T^{(m)}$ at distance ℓ from ρ, let $0 < t_1 < t_2 < 1$, let $\omega' \in \Omega(T^{(m)} \setminus T_y)$ be arbitrary and let $\omega'' \in B^{m-\ell}_{(y,t_1)} \cap B^{m-\ell}_{(y,t_2)}$. Consider a configuration $\omega \in \Omega(T^{(m)})$ whose restriction to $T^{(m)} \setminus T_y$ (resp., T_y) is ω' (resp., ω'') and in addition has exactly two links to y from its parent, at times t_1 and t_2. Then $\omega \in B^m_{(\rho,0)}(y)$.*

This lemma is useful since the event $B^{m-\ell}_{(y,t_1)} \cap B^{m-\ell}_{(y,t_2)}$ is defined entirely in the subgraph T_y, which is disjoint from $T^{(m)} \setminus T_y$, and due to the bound

$$(2.36) \qquad \mathbb{P}\big(B^{m-\ell}_{(y,t_1)} \cap B^{m-\ell}_{(y,t_2)}\big) \geq 1 - 2\sigma_{m-\ell}.$$

PROOF OF LEMMA 2.7.   Write x for the parent of y. In ω', the points (x, t_1) and (x, t_2) belong to some loops L'_1, L'_2, where possibly $L'_1 = L'_2$. Similarly, in ω'' the points (y, t_1) and (y, t_2) belong to some loops L''_1, L''_2, possibly equal. Note that neither L''_1 nor L''_2 reaches distance $m - \ell$ from y in T_y.

We can form ω by starting with $\omega' \cup \omega''$, and putting in the links (xy, t_1) and (xy, t_2) one at a time. When putting in (xy, t_1) we necessarily merge L'_1 and L''_1, since they were disjoint before. When we then put in (xy, t_2) we either cause another merge, involving L''_2, or we cause a loop to split. In either case, no loop of $T^{(m)} \setminus T_y$ ever merges with a loop which reaches distance m from ρ in T_y.   □

Note that, writing $\check{\mathbb{P}}(\cdot)$ for $\mathbb{P}(\cdot \mid \check{T}, \check{\omega})$,

$$(2.37) \qquad
\begin{aligned}
&\mathbb{P}\big(B^m_{(\rho,0)} \cap (A_1 \cup A_2)^c\big) \\
&\quad = \mathbb{E}\big[\mathbb{1}_{(A_1 \cup A_2)^c} \check{\mathbb{P}}\big(B^m_{(\rho,0)}\big)\big] \\
&\quad = \mathbb{E}\left[\mathbb{1}_{(A_1 \cup A_2)^c} \check{\mathbb{P}}\left(\bigcap_{\ell=1}^m \bigcap_{y \in \mathcal{E}_\ell} B^m_{(\rho,0)}(y)\right)\right].
\end{aligned}$$

But by Lemma 2.7 and (2.36) we have

$$\check{\mathbb{P}}\left(\bigcap_{\ell=1}^{m}\bigcap_{y\in\mathcal{E}_\ell} B^m_{(\rho,0)}(y)\right)$$

$$\geq \check{\mathbb{P}}\left(\bigcap_{\ell=1}^{m-1}\bigcap_{x\in\mathcal{E}_\ell^{(1)}} B^{m-\ell}_{(x,t(x))}\bigcap_{y\in\mathcal{E}_\ell^{(2)}}(B^{m-\ell}_{(y,t_1(y))}\cap B^{m-\ell}_{(y,t_2(y))})\right)\mathbb{1}\{\mathcal{E}_m=\varnothing\}$$

$$(2.38) \qquad = \prod_{\ell=1}^{m-1}\prod_{x\in\mathcal{E}_\ell^{(1)}}\check{\mathbb{P}}(B^{m-\ell}_{(x,t(x))})\prod_{y\in\mathcal{E}_\ell^{(2)}}\check{\mathbb{P}}(B^{m-\ell}_{(y,t_1(y))}\cap B^{m-\ell}_{(y,t_2(y))})\mathbb{1}\{\mathcal{E}_m=\varnothing\}$$

$$\geq \prod_{\ell=1}^{m-1}(1-\sigma_{m-\ell})^{|\mathcal{E}_\ell^{(1)}|}(1-(2\sigma_{m-\ell})\wedge 1)^{|\mathcal{E}_\ell^{(2)}|}\mathbb{1}\{\mathcal{E}_m=\varnothing\}$$

$$\geq 1 - 2\sum_{\ell=1}^{m}\sigma_{m-\ell}|\mathcal{E}_\ell|.$$

(Here, $\sigma_0 = 1$, and the last line is negative when $\mathcal{E}_m \neq \varnothing$.) Hence

$$(2.39) \qquad \mathbb{P}(B^m_{(\rho,0)}\cap(A_1\cup A_2)^c) \geq \mathbb{E}\left[\mathbb{1}_{(A_1\cup A_2)^c}\left(1-2\sum_{\ell=1}^{m}\sigma_{m-\ell}|\mathcal{E}_\ell|\right)\right],$$

and the assumption of Lemma 2.5 follows if we show that

$$(2.40) \qquad \sum_{\ell=1}^{m}\sigma_{m-\ell}\mathbb{E}[\mathbb{1}_{(A_1\cup A_2)^c}|\mathcal{E}_\ell|] \leq 48\mathbb{P}((A_1\cup A_2)^c)\check{\sigma}_{m-1}.$$

The following will let us establish (2.40) (and hence Proposition 2.4).

LEMMA 2.8.   *For d large enough, $k \geq 1$, $m \geq 3$ and $1 \leq \ell \leq m$, we have*

$$\mathbb{E}[\mathbb{1}_{A_k'}|\mathcal{E}_\ell|] \leq 4k\mathbb{P}(A_k')a_\ell' \quad and \quad \mathbb{E}[\mathbb{1}_{A_k''}|\mathcal{E}_\ell|] \leq 4k\mathbb{P}(A_k'')a_\ell'',$$

where $a_1' = 1$, $a_\ell' = (dq)^{\ell-2}$ for $\ell \geq 2$, $a_1'' = a_2'' = 1$, and $a_\ell'' = (dq)^{\ell-3}$ for $\ell \geq 3$.

PROOF.   It suffices to bound the conditional expectations

$$(2.41) \qquad \mathbb{E}[|\mathcal{E}_\ell| \,|\, A_k'] \quad and \quad \mathbb{E}[|\mathcal{E}_\ell| \,|\, A_k'']$$

by the appropriate functions. We prove the result for the A_k', the arguments for the A_k'' are similar.

There are several cases to consider, we start with $\ell = 1$. Given A_k', the number of 1's in generation $\ell = 1$ has distribution $\text{Bin}(d-k, p_1/(p_0+p_1))$, and it follows that

$$(2.42) \qquad \mathbb{E}[|\mathcal{E}_1^{(1)}| \,|\, A_k'] = (d-k)\frac{p_1}{p_0+p_1} \leq \frac{p_1 d}{p_0+p_1}$$

which is trivially bounded by $2k = 2ka_1'$. Next, we have $\mathbb{E}[|\mathcal{E}_1^{(2)}| \mid A_1'] = 0$, whereas for $k \geq 2$ the number of 2's in generation $\ell = 1$ has distribution $\mathrm{Bin}(k, p_2/p_{\geq 2})$, so that

$$(2.43) \qquad \mathbb{E}[|\mathcal{E}_1^{(2)}| \mid A_k'] = k\frac{p_2}{p_{\geq 2}} \leq k \leq 2ka_1'.$$

For $2 \leq \ell \leq m - 1$, we argue as follows. We consider the subtree of $\check{T}$ formed by edges with ≥ 3 links; the number of 1's (resp., 2's) in generation ℓ of $\check{T}$ equals the size of generation $\ell - 1$ of the subtree times an independent $\mathrm{Bin}(d, p_1)$ [resp., $\mathrm{Bin}(d, p_2)$] random variable. Each edge with ≥ 3 links from ρ is the root of a Galton–Watson tree of (≥ 3):s; these Galton–Watson trees have offspring distribution $\mathrm{Bin}(d, q)$, and hence on average $(dq)^r$ descendants after r steps. For $k = 1$, we get simply

$$(2.44) \qquad \mathbb{E}[|\mathcal{E}_\ell^{(j)}| \mid A_1'] = (dp_j)(dq)^{\ell-2} \leq 2a_\ell'.$$

For $k \geq 2$, there are $\mathrm{Bin}(k, p_{\geq 3}/p_{\geq 2})$ Galton–Watson trees to consider, hence

$$(2.45) \qquad \mathbb{E}[|\mathcal{E}_\ell^{(j)}| \mid A_k'] = \left(k\frac{p_{\geq 3}}{p_{\geq 2}}\right)(dp_j)(dq)^{\ell-2} \leq 2ka_\ell'.$$

For $\ell = m$, a similar argument gives

$$(2.46) \qquad \begin{aligned} \mathbb{E}[|\mathcal{E}_m| \mid A_1'] &= ((1-p_0)d)(dq)^{m-2} \quad \text{and} \\ \mathbb{E}[|\mathcal{E}_m| \mid A_k'] &= \left(k\frac{p_{\geq 3}}{p_{\geq 2}}\right)((1-p_0)d)(dq)^{m-2}. \end{aligned} \qquad \square$$

PROOF OF PROPOSITION 2.4.   As mentioned, it is enough to establish (2.40). Using Lemmas 2.6 and 2.8 as well as the inequalities $a_\ell' \leq a_\ell''$ and $\sigma_{m-1} \leq \sigma_{m-2} \leq \sigma_{m-3}$, we see that

$$
\begin{aligned}
(2.47) \qquad & \sum_{\ell=1}^m \sigma_{m-\ell} \mathbb{E}[\mathbb{1}_{(A_1 \cup A_2)^c}|\mathcal{E}_\ell|] \\
&= \sum_{\ell=1}^m \sigma_{m-\ell} \sum_{k=1}^d \left(\mathbb{E}[\mathbb{1}_{A_k'}|\mathcal{E}_\ell|] + \mathbb{E}[\mathbb{1}_{A_k''}|\mathcal{E}_\ell|]\right) \\
&\leq 4\sum_{\ell=1}^m \sigma_{m-\ell}(a_\ell' + a_\ell'') \sum_{k=1}^d k\left(\mathbb{P}(A_k') + \mathbb{P}(A_k'')\right) \\
&\leq 16\sum_{\ell=1}^m \sigma_{m-\ell}a_\ell'' \sum_{k=1}^d \left(\mathbb{P}(A_k') + \mathbb{P}(A_k'')\right) \\
&= 16\left(\sigma_{m-1} + \sigma_{m-2} + \sum_{\ell=3}^m \sigma_{m-\ell}(dq)^{\ell-3}\right)\mathbb{P}((A_1 \cup A_2)^c) \\
&\leq 48\mathbb{P}((A_1 \cup A_2)^c)\check{\sigma}_{m-1},
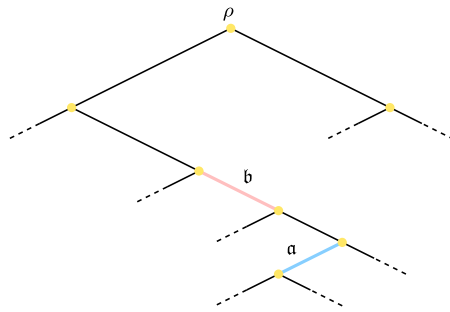\end{aligned}
$$

for d large enough, as required. $\square$

FIG. 5.   *Illustration for the random link $\mathfrak{a}$ and the bottleneck-link $\mathfrak{b}$.*

3. Sharpness of the transition.   The arguments of Hammond [11] can straightforwardly be adapted to our setting. We thus obtain the following "sharpness" result, which shows that (in the interval $\beta \in [d^{-1}, d^{-1} + 2d^{-2}]$) there is a unique β_c such that $\sigma(\beta) = \mathbb{P}((\rho, 0) \leftrightarrow \infty)$ satisfies $\sigma = 0$ for $\beta < \beta_c$ and $\sigma > 0$ for $\beta > \beta_c$:

PROPOSITION 3.1.   *For d large enough, the function $\beta \mapsto \sigma(\beta)$ is nondecreasing on the interval $\beta \in [d^{-1}, d^{-1} + 2d^{-2}]$.*

SKETCH PROOF.   Hammond's arguments [11] are written for the case $u = 1$ when there are only crosses, but they apply (almost verbatim) to the general case $u \in [0, 1]$. We provide here a synopsis of the proof, for the reader's benefit.

The starting point is a formula for the derivative $\frac{d\sigma_n}{d\beta}$, involving the concept of "the added link" (called the added *bar* by Hammond). In addition to the Poisson process ω of links (i.e., crosses and double-bars), let $\mathfrak{a}$ be an independently and uniformly placed link in $T^{(n)}$, which is a cross with probability u and otherwise a double-bar. Let P^+ and P^- denote the following *pivotality events*:

$$(3.1) \quad P^+ = \{(\rho, 0) \overset{\omega}{\nleftrightarrow} n, (\rho, 0) \overset{\omega \cup \{\mathfrak{a}\}}{\leftrightarrow} n\}, \qquad P^- = \{(\rho, 0) \overset{\omega}{\leftrightarrow} n, (\rho, 0) \overset{\omega \cup \{\mathfrak{a}\}}{\nleftrightarrow} n\}.$$

In words, P^+ is the event that $\mathfrak{a}$ creates a connection to level n that was not present in ω, and P^- is the event that $\mathfrak{a}$ breaks a connection to level n. We say that $\mathfrak{a}$ is on-pivotal if P^+ happens and off-pivotal if P^- happens. Then we have [11], Lemma 1.7:

$$(3.2) \qquad\qquad \frac{d\sigma_n}{d\beta} = |\mathcal{E}_n| \big(\mathbb{P}(P^+) - \mathbb{P}(P^-) \big).$$

Here, $\mathcal{E}_n$ denotes the set of edges of $T^{(n)}$.

Hammond shows that the difference on the right-hand side of (3.2) is positive on the interval in β considered (when d is large enough). The result then follows by letting $n \to \infty$. To show that $\mathbb{P}(P^+) - \mathbb{P}(P^-) \geq 0$, Hammond introduces the

following events. First, the *crossing-event* C that the loop $\mathcal{L}_{(\rho,0)}(\omega)$ of $(\rho,0)$ in ω visits an end-point of the added link $\mathfrak{a}$ before reaching level n. Note that $P^{\pm} \subseteq C$, since if C does not happen then the added link has no effect on whether or not $\mathcal{L}_{(\rho,0)}$ reaches level n. Second, the *bottleneck-event* B that some edge of $T^{(n)}$ on the (unique) path from ρ to $\mathfrak{a}$ supports only one link. On the event B, let the bottleneck-link $\mathfrak{b}$ be the furthest such link from ρ. And third, the *no-escape-event* $N \subseteq B$ that the loop $\mathcal{L}_{(\rho,0)}(\omega \setminus \mathfrak{b})$ of $(\rho,0)$ in $\omega \setminus \mathfrak{b}$ does not reach level n. See Figure 5.

Note that $P^{\pm}$ can be written as a disjoint union:

$$(3.3) \qquad P^{\pm} = (P^{\pm} \cap C \cap B^c) \cup (P^{\pm} \cap C \cap B \cap N).$$

Indeed, one only needs to check that $C \cap B \subseteq N$, that is, if C happens and there is a bottle-neck, then the no-escape-event happens. But if C happens and $\mathfrak{b}$ is a bottle-neck, then in $\omega \setminus \mathfrak{b}$ the loop $\mathcal{L}_{(\rho,0)}$ cannot reach level n, because if it did then it would reach level n in both ω and $\omega \cup \mathfrak{a}$ also, since $\mathfrak{b}$ is a monolink (Proposition 2.2).

Hence it suffices to provide lower bounds on the differences:

$$(3.4) \qquad \begin{aligned} \delta_1 &= \mathbb{P}(P^+ \cap C \cap B^c) - \mathbb{P}(P^- \cap C \cap B^c), \\ \delta_2 &= \mathbb{P}(P^+ \cap C \cap B \cap N) - \mathbb{P}(P^- \cap C \cap B \cap N). \end{aligned}$$

It is easy to give a lower bound on the first term in δ_1. Indeed, suppose the following happen: (i) in ω there is no link adjacent to ρ, (ii) $\mathfrak{a}$ is adjacent to ρ, (iii) the other endpoint of $\mathfrak{a}$ is connected by a loop to level n. Then $P^+ \cap C \cap B^c$ happens. It follows that

$$(3.5) \qquad \mathbb{P}(P^+ \cap C \cap B^c) \geq (e^{-\beta})^d \frac{d}{|\mathcal{E}_n|} \sigma_{n-1}.$$

It turns out that the second term in δ_1 satisfies

$$(3.6) \qquad \mathbb{P}(P^- \cap C \cap B^c) \leq c \frac{\sigma_{n-1}}{|\mathcal{E}_n|},$$

for some constant c independent of d. The detailed argument for this is more involved (see [11], Lemma 4.5), but no changes are required compared to Hammond's original argument. Very briefly, the reason that one gets a constant factor c rather than a factor which grows with d as in (3.5) is as follows. If P^- happens, then necessarily the edge supporting $\mathfrak{a}$ also supports some link of ω: if it did not then adding $\mathfrak{a}$ would necessarily merge two loops, thereby preserving any connections to level n. If also B^c happens, that is, there is no bottleneck, then necessarily $\mathfrak{a} \in \mathcal{M} \cup \mathcal{S}$ where $\mathcal{M}$ is the connected cluster of ρ consisting of edges which support ≥ 2 links in ω, and $\mathcal{S}$ is the set of edges that are adjacent to an edge of $\mathcal{M}$ and support exactly one link in ω. Now $\mathcal{M}$ is a very sub-critical Galton–Watson tree, and is therefore of at most constant (expected) size, and $\mathcal{S}$ is an approximately

constant $[\text{Bin}(d, \beta e^{-\beta})]$ multiple of the number of leaves of $\mathcal{M}$, and is thus also small. Hence there is an approximately constant number of locations for $\mathfrak{a}$ which are consistent with the event $P^- \cap C \cap B^c$, giving the factor $c/|\mathcal{E}_n|$. The factor σ_{n-1} appears in (3.6) since some link of $\mathcal{S}$ is connected to level n.

Putting together (3.5) and (3.6), we obtain that, for d large enough,

$$(3.7) \qquad \delta_1(n) \geq \frac{d}{2}(e^{-\beta})^d \frac{\sigma_{n-1}}{|\mathcal{E}_n|}.$$

Now consider the other term $\delta_2(n)$, where the bottleneck- and no-escape-events B and N happen. Since N happens, any connections to level n must occur in the subtree rooted at the bottleneck edge $\mathfrak{b}$, which is some (random) distance $n' \leq n$ from level n. Since $\mathfrak{b}$ was defined as the *furthest* bottleneck from ρ, there is no bottleneck in this subtree. We thus essentially have that $\delta_2(n) = \delta_1(n')$, so we can use the bounds on δ_1 that were already established. The only n-dependence in those bounds was in the factors $\sigma_{n-1}/|\mathcal{E}_n|$. It follows that for large enough d we certainly have $\delta_2(n) \geq 0$. Together with (3.7) and (3.2), this gives

$$(3.8) \qquad \frac{d\sigma_{n-1}}{d\beta} \geq \frac{d}{2}(e^{-\beta})^d \sigma_{n-1} \geq 0,$$

which as explained gives the result. $\square$

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