

A PHASE TRANSITION REGARDING THE EVOLUTION OF BOOTSTRAP PROCESSES IN INHOMOGENEOUS RANDOM GRAPHS

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A bootstrap percolation process on a graph with infection threshold $r \geq 1$ is a dissemination process that evolves in time steps. The process begins with a subset of infected vertices and in each subsequent step every uninfected vertex that has at least r infected neighbours becomes infected and remains so forever.

Critical phenomena in bootstrap percolation processes were originally observed by Aizenman and Lebowitz in the late 1980s as finite-volume phase transitions in \mathbb{Z}^d that are caused by the accumulation of small local islands of infected vertices. They were also observed in the case of dense (homogeneous) random graphs by Janson et al. [*Ann. Appl. Probab.* **22** (2012) 1989–2047]. In this paper, we consider the class of inhomogeneous random graphs known as the *Chung-Lu model*: each vertex is equipped with a positive weight and each pair of vertices appears as an edge with probability proportional to the product of the weights. In particular, we focus on the *sparse* regime, where the number of edges is proportional to the number of vertices.

The main results of this paper determine those weight sequences for which a critical phenomenon occurs: there is a critical density of vertices that are infected at the beginning of the process, above which a small (sublinear) set of infected vertices creates an avalanche of infections that in turn leads to an outbreak. We show that this occurs essentially only when the tail of the weight distribution dominates a power law with exponent 3 and we determine the critical density in this case.

Received September 2016; revised February 2017.

¹Supported by EPSRC Grant No. EP/K019749/1. Part of this research was carried out while this author was visiting TU Graz, supported by TU Graz and Austrian Science Fund (FWF) P26826.

²Supported by Austrian Science Fund (FWF): P26826.

³Supported by European Research Council (ERC): No. 639046. Part of this research was carried out while this author was a Ph.D. student at the Institute of Discrete Mathematics, TU Graz, supported by the Austrian Science Fund (FWF): W1230, P26826.

⁴Supported by Austrian Science Fund (FWF): P26826. Part of this research was carried out while this author was a postdoctoral research associate at the School of Mathematics, University of Birmingham, supported by the Marie Curie CIG PCIG09-GA2011-293619.

MSC2010 subject classifications. Primary 05C80, 60K37; secondary 82B26, 82B27.

Key words and phrases. Bootstrap percolation, inhomogeneous random graphs, critical phenomena.

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1. Introduction. A *bootstrap percolation process* with *infection threshold* $r \geq 1$ is a dissemination process on a graph G which evolves in steps. Each vertex may have one of two possible states: it is either *infected* or *uninfected*. At the beginning, there is a subset \mathcal{A}_0 of *initially infected* vertices, and all remaining vertices are uninfected. In each subsequent step, any uninfected vertex with at least r infected neighbours also becomes infected, and never changes its state again. The set of vertices infected until step $t \geq 0$ is denoted by \mathcal{A}_t . The process will stop once there is a step in which no vertices became infected. In particular, if G is a finite graph then the process always stops, and we denote by \mathcal{A}_F the set of all vertices which became infected throughout the entire process.

The bootstrap percolation process was introduced in the context of magnetic disordered systems by Chalupa, Leath and Reich (1979). Since then it has been used as a model for several phenomena in various areas, from jamming transitions [Toninelli, Biroli and Fisher (2006)] and magnetic systems [Sabhapandit, Dhar and Shukla (2002)] to neuronal activity [Amini (2010a), Tlusty and Eckmann (2009)]. Certain variations of this process are related to the dynamics of the Ising model at zero temperature [Fontes, Schonmann and Sidoravicius (2002), Morris (2009)]. A short survey regarding applications can be found in Adler and Lev (2003).

Several qualitative characteristics of bootstrap percolation, for instance the dependence of the final set of infected vertices \mathcal{A}_F on the set \mathcal{A}_0 of initially infected vertices, have been studied on a variety of families of graphs, such as trees [Balogh, Peres and Pete (2006), Bollobás et al. (2014), Fontes and Schonmann (2008)], grids [Balogh and Pete (1998), Balogh et al. (2012), Cerf and Manzo (2002), Holroyd (2003)], lattices on the hyperbolic plane [Sausset et al. (2010)] and hypercubes [Balogh and Bollobás (2006)], as well as on many models of random graphs [Amini (2010b), Balogh and Pittel (2007), Janson et al. (2012)].

The most well-studied quantity is the probability that all vertices of the underlying graph are eventually infected. In particular, this quantity has been considered as a function of the density p_0 of initially infected vertices. More specifically, assuming that before the process begins each vertex of the graph is independently infected with probability p_0 , what is the probability that the final set contains every vertex? In other words, what is the probability that the process *percolates*?

In several families of infinite graphs, it turns out that there is a critical value for p_0 above which the probability of percolation is positive. This is the case for the family of infinite regular trees with degree $d + 1$ and $d \geq r$, as it was proved by Balogh, Peres and Pete (2006), as well as for (infinite) Galton–Watson trees [this was shown by Bollobás et al. (2014)]. Fontes and Schonmann (2008) showed that infinite regular trees also exhibit two thresholds: a critical density p_f [that was proved in Balogh, Peres and Pete (2006)] above which percolation occurs almost surely and a critical density $p_c < p_f$ above which infinite infected clusters exist almost surely.

A large part of the literature on bootstrap percolation processes has been devoted to the d -dimensional integer lattice \mathbb{Z}^d . Schonmann (1992) showed that if the elements of \mathcal{A}_0 are selected independently with probability p_0 , then the evolution of the process is in some sense “trivial”: if $r \leq d$, then for every $p_0 > 0$ all vertices of the lattice become infected with probability 1, whereas if $r > d$, then this does not happen unless $p_0 = 1$. The former had already been shown by Enter (1987) for $d = r = 2$. Moreover, for this case, Aizenman and Lebowitz (1988) identified a phase-transition phenomenon when the process is restricted to a box of \mathbb{Z}^2 of side-length $n \rightarrow \infty$, which Holroyd (2003) later made precise. Let $r = 2$ and G be the 2-dimensional grid with vertex set $V = \{1, \dots, n\}^2$, and let $\mathcal{A}_0 \subseteq V$ be a random subset containing every element independently with probability $p_0 = p_0(n)$.

Holroyd (2003) showed that the probability $I(n, p_0)$ that the entire square is eventually infected satisfies $I(n, p_0) \rightarrow 1$ if $\liminf_{n \rightarrow \infty} p_0(n) \log n > \pi^2/18$, and $I(n, p_0) \rightarrow 0$ if $\limsup_{n \rightarrow \infty} p_0(n) \log n < \pi^2/18$. This result has been generalised to higher dimensions by Balogh, Bollobás and Morris (2009) (when G is the 3-dimensional grid on $\{1, \dots, n\}^3$ and $r = 3$) and Balogh et al. (2012) (in general). This is an instance of the so-called *metastability phenomenon*.

Similar thresholds have been identified in the case of the binomial random graph $G(n, p)$, where every edge on a set of n vertices is present independently with probability p . Janson et al. (2012) presented a complete analysis of the bootstrap percolation process for various ranges of p . Among other results they showed that when $1/n \ll p \ll n^{-1/r}$, there is a critical function $a_c = a_c(n)$ such that with high probability⁵ the following occurs: if $p_0 \ll a_c/n$, then there is very little evolution of the process, whereas if $p_0 \gg a_c/n$, then eventually almost every vertex becomes infected. For sparser graphs, this is not the case. When $p = c/n$ (i.e., the average degree is approximately constant) and if $p_0 = o(1)$, then only a sub-linear number of vertices will ever be infected with high probability. In fact, no evolution occurs with high probability. This had been observed previously by Balogh and Bollobás [see Balogh and Pittel (2007)].

However, this is no longer the case if one considers sparse random graphs which are *inhomogeneous*. We focus on random graphs which are defined through a sequence of weights assigned to the vertices: these weights determine the probability that two vertices are adjacent. More specifically, we are interested in the case where this probability is proportional to the product of the weights of these vertices. Hence, pairs of vertices where at least one of them has high weight are more likely to appear as edges. Of course, $G(n, p)$ is a special case of such a random graph, in which all vertices have the same weight. Amini and Fountoulakis (2014) showed that such a threshold does exist when the sequence of weights follows a power law distribution with exponent in the interval $(2, 3)$. They showed that there is a function $a_c = a_c(n) = o(n)$ such that if $p_0 \ll a_c/n$, then with high probability no evolution occurs, but if $p_0 \gg a_c/n$, then even if $p_0 = o(1)$, with high probability a constant fraction of all vertices become infected eventually. In addition, Amini, Fountoulakis and Panagiotou (2014) determined the value of this constant. More general results which include those in Amini and Fountoulakis (2014) were obtained by Karbasi, Lengler and Steger (2015). Similar behaviour was also observed in geometric random graph models that exhibit a power law degree distribution with such exponent [Candellero and Fountoulakis (2016), Koch and Lengler (2016)] as well as in several versions of the preferential attachment model [Abdullah and Fountoulakis (2014), Ebrahimi et al. (2014)].

The aim of this paper is to determine the conditions on the sequence of weights which characterise the existence of such a threshold function. We show that a *crit-*

⁵With probability tending to one as $n \rightarrow \infty$.

ical droplet [in the terminology of Aizenman and Lebowitz (1988)] is formed by a certain set of vertices of high weight, which we call the *nucleus* of the infection. Informally, this is a set consisting of vertices of high weight which become infected at some stage and from this set the infection spreads to a positive fraction of the vertices of the random graph. Effectively, we show that such a nucleus is formed, if the tail of the empirical distribution function of the weight sequence dominates the tail of a power law distribution with exponent equal to 3. Furthermore, we determine the critical density of the initially infected vertices below which this phenomenon does not occur.

2. Model and notation.

2.1. *Inhomogeneous random graphs.* The random graph model that we consider is a special yet general enough version of an *inhomogeneous random graph* introduced by Söderberg (2002) and studied in its full generality by Bollobás, Janson and Riordan (2007). The model is asymptotically equivalent to a model considered by Chung and Lu (2002, 2003) and Chung, Lu and Vu (2004). They analysed several typical properties of the resulting graphs, including the average distance between two randomly chosen vertices that belong to the same component and the distribution of the component sizes. Let $n \in \mathbb{N}$ and $[n] := \{1, \dots, n\}$. We consider a graph $G = (V, E)$ with vertex set $V := [n]$ and a random edge set E defined as follows: each vertex v is assigned a positive weight $w_v = w_v(n) \in \mathbb{R}^+$, and without loss of generality we will assume throughout the paper that $w_1 \leq w_2 \leq \dots \leq w_n$. We denote this *weight sequence* by $\mathbf{w} = \mathbf{w}(n) := (w_1, \dots, w_n)$ and the *total weight* by $W := \sum_{v \in V} w_v$. Any two distinct vertices $u, v \in V$ form an edge, that is, $\{u, v\} \in E$, independently with probability

$$(1) \quad p_{u,v} = p_{u,v}(\mathbf{w}) := \min \left\{ \frac{w_u w_v}{W}, 1 \right\}.$$

We refer to this model as the *Chung–Lu random graph* $CL(\mathbf{w})$. A fundamental observation is that the weights (essentially) determine the *expected* degrees of all vertices: if we ignore the minimum in (1), and also allow for a loop at vertex u , then the expected degree of that vertex is $\sum_{v \in V} w_u w_v / W = w_u$.

For the sake of a more concise exposition of the results and proofs, we assume that the minimal weight w_1 is at least 1 and the total weight satisfies

$$(2) \quad W = \lambda n \quad \text{for some } \lambda \geq 1.$$

Central in our results will be the distribution of the weight of a vertex selected randomly with probability proportional to its weight. More formally, let X denote a V -valued random variable whose distribution is given by $\mathbb{P}[X = u] = \frac{w_u}{W}$ for any vertex $u \in V$ of weight w_u . Then the weight w_X of this randomly chosen vertex X is a \mathbb{R}^+ -valued random variable w_X whose distribution function is given by

$$\mathbb{P}[w_X \leq a] = \sum_{u:w_u \leq a} \frac{w_u}{W},$$

for any $a \in \mathbb{R}$. This distribution (of w_X) is called the *size-biased distribution* and we denote a weight chosen randomly according to this distribution by Z_w .

Let $\mathbf{w} = \mathbf{w}(n)$ be a sequence of weight sequences. We will consider events on the probability space that is the product of the space induced by $\text{CL}(\mathbf{w})$ and the one representing the set of initially infected vertices $\mathcal{A}_0 \subset V$, where each vertex is initially infected independently with probability $p_0 = p_0(n)$. We let Ω_n denote the sequence of these spaces. If $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$ is a sequence of events with $\mathcal{E}_n \subseteq \Omega_n$, we say that they occur *with high probability* (whp) if the probability of \mathcal{E}_n tends to 1 as $n \rightarrow \infty$. Moreover, any unspecified limits and asymptotics will be as $n \rightarrow \infty$.

We will also be using a probabilistic version of the standard Landau notation. Let X_n be a sequence of nonnegative random variables (where for each n the variable X_n is defined on Ω_n) and y_n be a sequence of nonnegative real numbers. We say that whp $X_n = O(y_n)$ if there exists $C > 0$ such that $X_n \leq C y_n$ whp, and whp $X_n = \Omega(y_n)$ if there exists $C > 0$ such that $X_n \geq C y_n$ whp. If both hold, we say that whp $X_n = \Theta(y_n)$. Furthermore, we say that whp $X_n = o(y_n)$ if for any $\varepsilon > 0$ whp $X_n/y_n < \varepsilon$. In other words, X_n/y_n converges to 0 *in probability*.

We sometimes also write $x_n \ll y_n$ to denote $x_n = o(y_n)$ and $x_n \gg y_n$ to denote $x_n = \omega(y_n)$ (as in the standard Landau notation), for two sequences of nonnegative real numbers. Also, the meaning of “whp $X_n \ll y_n$ ”, where $\{X_n\}$ is a sequence of nonnegative random variables on Ω_n , is now obvious from the above.

2.2. Bootstrap processes. Consider an integer $r \geq 2$ and a weight sequence $\mathbf{w} = (w_1, \dots, w_n)$. Let $G \sim \text{CL}(\mathbf{w})$ be a Chung–Lu random graph with weight sequence \mathbf{w} . Given an *initial infection rate* $p_0 = p_0(n) \in [0, 1]$, we *initially* infect a random subset $\mathcal{A}_0 \subseteq V$ which contains each element with probability p_0 independently. As we already mentioned, a bootstrap process is a process evolving in discrete time steps. At any time $t \geq 0$, there is a set \mathcal{A}_t of *infected* vertices, which is defined iteratively by

$$\mathcal{A}_{t+1} := \mathcal{A}_t \cup \{v \in V \mid v \text{ has at least } r \text{ neighbours in } \mathcal{A}_t\}$$

for all $t \geq 0$. Furthermore, we set $\mathcal{A}_F := \bigcup_{t \geq 0} \mathcal{A}_t$.

Critical function. A function $p_c = p_c(n)$, $0 \leq p_c \leq 1$, is called a *critical function* (with respect to the initial infection rate p_0) if the following three conditions are satisfied:

1. $p_c = o(1)$;
2. if $p_0 \ll p_c$, then whp $|\mathcal{A}_F| = o(n)$;
3. if $p_0 \gg p_c$, then whp $|\mathcal{A}_F| = \Theta(n)$.

We refer to the two latter cases as the *subcritical* and the *supercritical* regime, respectively. Of course, the above definition yields a class of functions that have the same order of magnitude. With slight abuse of notation, we will be referring to *the* threshold p_c , and treat it as if it was uniquely defined.

2.3. *Some more notation.* Given an interval $I \subseteq \mathbb{R}^+$, we denote the set of vertices having weight in I by

$$V_I := \{v \in V: w_v \in I\},$$

and in particular if $I = [f, \infty)$ for some $f \in \mathbb{R}^+$ we write “ $\geq f$ ” instead of “[$f, \infty)$ ”, and likewise for intervals (f, ∞) , $(0, f]$ and $(0, f)$.

For a set $U \subseteq V$ of vertices, we write $W[U]$ for the sum of the weights of the vertices in U and $w_{\max}[U]$, $w_{\min}[U]$ for their maximal and minimal weight, respectively. Moreover, we denote by

$$\widehat{U} := U \cap \mathcal{A}_F$$

the subset of all those vertices in U which eventually become infected. In case $U = V_I$ for some interval $I \subseteq \mathbb{R}^+$, we also use the abbreviations

$$W_I := W[V_I] \quad \text{and} \quad \widehat{W}_I := W[\widehat{V}_I].$$

We denote by $\binom{U}{\ell}$ the set of all ℓ -element subsets of a set U and for a vertex $u \in V$ we denote by $N(u)$ the set of neighbours of u in G . Furthermore, we write $X \sim \mathcal{D}$ for a random variable with distribution \mathcal{D} .

3. Main results and proof outline.

3.1. *Main results.* Our main result is to characterise two classes of weight sequences \mathbf{w} : for the first class bootstrap percolation on the Chung–Lu random graph, $\text{CL}(\mathbf{w})$ exhibits a critical phenomenon, while for the second it does not. Roughly speaking, the first class stochastically dominates the size-biased distribution associated to a power law of exponent 3, with a suitable large constant. In contrast, the second class is stochastically dominated by the size-biased distribution associated to a power law of exponent 3, with a suitable small constant. In this sense, the characterisation only has a constant “gap”.

Interestingly, there can be different types of nuclei, each leading to an outbreak once infected, depending on some property of the weight sequence \mathbf{w} , each providing its own *candidate threshold*, which if exceeded [by an $\omega(1)$ -factor] guarantees an outbreak whp. This behaviour depends sensitively on the following weight bound $\phi_H = \phi_H(n)$ defined (pointwise) by

$$(3) \quad \phi_H(n) := \min \left\{ x \in \mathbb{R}^+ : |V_{\geq x}| \geq \left(\frac{W}{4x^2} \right)^r \right\},$$

where we use the convention that $\phi_H(n) := w_n + 1$ if this set is empty. Vertices whose weight is at least ϕ_H are called *heavy*. The subgraph spanned by the heavy vertices will be called the *dense* subgraph, while the subgraph spanned by all non-heavy vertices will be called the *sparse* subgraph.

If we consider the bootstrap process restricted to the sparse subgraph, we obtain the first candidate threshold

$$(4) \quad p_s = p_s(n) := \left(\frac{W}{\sum_{u \in V_{<\phi_H}} w_u^{r+1}} \right)^{1/(r-1)},$$

which always exists. On the other hand, studying the process restricted to the dense subgraph yields the second candidate threshold

$$(5) \quad p_d = p_d(n) := \left(\frac{1}{\sum_{u \in V_{\geq\phi_H}} w_u^r} \right)^{1/r},$$

provided this quantity is finite. This happens if and only if $\phi_H \leq w_n$. Should both candidate thresholds exist, the threshold will always be the smallest of the two.

In fact, in Example 9.1, we show that which of the candidates is smaller actually depends on the weight sequence. Even though it is a priori not obvious why, considering these two candidate thresholds turns out to be sufficient, due to a matching lower bound on the threshold p_c .

THEOREM 3.1. *Let $r \geq 2$ be an integer, let $\alpha > 0$, $C \geq 64r(\min\{\alpha, 1/2\})^{-3}$, and $C_1 > 0$. Furthermore, let $\mathbf{w} = \mathbf{w}(n)$ be a sequence of ordered weight sequences satisfying $1 \leq w_1 \leq \dots \leq w_n \rightarrow \infty$, $w_{n-r+1} \geq \alpha w_n$, and*

$$(6) \quad \mathbb{P}[Z_{\mathbf{w}} \geq x] \geq \frac{C}{x},$$

for all $C_1 \leq x \leq w_n$. Consider the Chung–Lu random graph $G \sim \text{CL}(\mathbf{w})$. Then the bootstrap process on G with infection threshold r and initial infection rate $p_0 = p_0(n)$ has a critical function p_c .

Furthermore, if $\phi_H \leq w_n$, then $\min\{p_s, p_d\} = o(1)$ and $p_c = \Theta(\min\{p_s, p_d\})$; otherwise $p_s = o(1)$ and $p_c = \Theta(p_s)$.

Condition (6) on $Z_{\mathbf{w}}$ essentially states that the distribution of the weights in the weight sequence stochastically dominates a distribution that has power law tail with exponent equal to 3. Indeed, recall that a distribution function $F(x)$ has a power law tail with exponent equal to $\tau > 0$ if there exist constants $\gamma > 0$ and $x_0 > 0$ such that $1 - F(x) \geq \frac{\gamma}{x^{\tau-1}}$, for any $x > x_0$. Assume that $\tau > 2$ (otherwise the distribution has infinite expected value). If F^* denotes the distribution function of the size-biased version of a random variable whose distribution is F , then for any $x > x_0$

$$1 - F^*(x) \geq \gamma \int_x^\infty \frac{1}{z^{\tau-1}} dz = \Omega\left(\frac{1}{x^{\tau-2}}\right).$$

Hence, our claim is verified if $\tau = 3$.

However, such a critical function does not always exist. In particular, if the size-biased distribution associated to the weight sequence is dominated by a size-biased distribution associated to a power law with exponent 3 with a suitably small constant, then there is no critical phenomenon.

THEOREM 3.2. *Let $r \geq 2$ be an integer and let $\mathbf{w} = \mathbf{w}(n)$ be a sequence of ordered weight sequences satisfying $1 \leq w_1 \leq \dots \leq w_n \rightarrow \infty$. Consider a bootstrap process on the random graph $\text{CL}(\mathbf{w})$ with infection threshold r and initial infection rate $p_0 = p_0(n)$. If there exist constants $0 < c < 1/30$ and $c_1 > 0$ and a function $h = h(n) \rightarrow \infty$ such that the size-biased distribution associated with \mathbf{w} satisfies*

$$\mathbb{P}[Z_{\mathbf{w}} \geq f] \leq \frac{c}{f}$$

for every $c_1 \leq f \leq h$, then no critical function exists.

3.2. Proof outline. Theorem 3.1 states that the following holds whp: if $p_0 \gg p_c$, then a constant fraction of all vertices become infected by the end of the process, and if $p_0 \ll p_c$, then only few additional vertices become infected.

In the supercritical regime, that is, $p_0 \gg p_c$, there are two phases. In the first phase, we show that if $p_0 \gg p_c$, then there exists a weight-bound $\phi_K = \phi_K(n)$ such that the subset of vertices $V_{\geq \phi_K}$ has the property that whp “almost all”⁶ of its vertices become infected. The set $V_{\geq \phi_K}$ is called a *nucleus* of the process. For the construction of a nucleus $V_{\geq \phi_K}$, we first observe that the behaviour of the infection process on the subgraph spanned by the nonheavy vertices [i.e., the vertices with weight less than ϕ_H , defined in (3)] is quite different from that on the subgraph spanned by the heavy vertices. This is due to the fact that the vertices of weight less than ϕ_H span a “sparse” subgraph, while the vertices of weight at least ϕ_H span a “dense” subgraph. We show that either of these restricted processes creates a nucleus on its own under some suitable condition on p_0 . Therefore, the actual threshold cannot be (substantially) larger than the minimum of these two candidate thresholds. However, this is already sufficient because we will prove a matching lower bound for the threshold in Section 7.

In Section 5.2, we analyse the *sparse process*. We show that if $p_0 \gg p_s$, then the sum of the weights of the infected vertices with weight less than ϕ_H increases until whp “almost all” vertices of weight at least ϕ_K become infected.

The *dense process* provides a candidate threshold if $\phi_H \leq w_n$, that is, p_d exists, and $p_0 \gg p_d$, then we use one of the following two approaches. In the first case, we use the important observation that the subgraph spanned by $V_{\geq \phi_H}$ stochastically dominates the binomial random graph $G(|V_{\geq \phi_H}|, \phi_H^2/W)$. For binomial random graphs, the evolution of bootstrap percolation is well understood [see Janson et al. (2012)]. In particular, it follows (cf. Theorem 4.6 below) that if we can find $\omega(1)$ infected vertices with weight at least ϕ_H , then whp every vertex of weight ϕ_H becomes infected. However, in the second case, when the number of vertices of weight at least ϕ_H is small, then this result is not applicable. The condition that $w_{n-r+1} \geq \alpha w_n$ essentially states that the r highest weights are the same up to a

⁶With respect to the total weight.

multiplicative constant. This is a technical condition which ensures that there are r vertices of approximately the same weight which will become infected whp. Since every vertex with weight at least W/w_{n-r+1} is connected to each of these vertices with probability one, they also become infected. The proof appears in Section 5.3.

Thereafter, in the second phase, we show that the condition on the size-biased distribution ensures that whp a set of linear size becomes infected. This is achieved by partitioning the vertices according to their weights and showing that the infection spreads from one part to the next, that is, the one containing vertices of slightly smaller weight. In particular, we show that most vertices of a given part have at least r neighbours in the previous part. The structure we discover there is very reminiscent of the construction of a giant r -core, that is, a subgraph of minimum degree at least r that has linear order. This concept has been studied extensively in the random graph literature; see, for example, Pittel, Spencer and Wormald (1996). The details can be found in Section 6.

For the subcritical regime, that is, when $p_0 \ll p_c$, we show in Section 7 that the number of infected vertices can be approximated by the total progeny of a subcritical branching process. Considering the current generation of newly infected vertices, we expose sequentially their uninfected neighbours. If an uninfected vertex is adjacent to a newly infected vertex and in addition, it has $r - 1$ neighbours within the infected set, then we declare this vertex to be an offspring of the newly infected vertex. The event that a given vertex becomes infected in a certain step, conditional on the history of the process is the intersection of nondecreasing and nonincreasing events. At this point, we make use of the FKG inequality (cf. Theorem 4.7 below) to deduce that these events are negatively correlated, whereby we can obtain a simple upper bound on the probability of infection at a certain step. We show that if the initial density of infected vertices is asymptotically below the threshold function, the process has expected progeny per vertex that is less than 1, and thus the infection spreads only to a few additional vertices (Section 7.2).

The proof of Theorem 3.2 follows a similar argument. In this case, its assumption on the distribution of the weights implies that the process we just described is again subcritical. With little more work, we show that this implies that the bootstrap process terminates after a small number of steps and ends with a set of infected vertices that is sublinear whp. The details can be found in Section 8.2.

4. Fundamental properties and tools. We first perform some fundamental calculations, which we will be using throughout the paper. We continue with a collection of concentration inequalities (from the literature) that we will make use of at some point in our arguments.

4.1. *Fundamental properties of the weight sequence.* We start out by observing that the weight-bound ϕ_H defined in (3), which characterises heavy vertices tends to infinity.

CLAIM 4.1. *Let ϕ_H be defined as in (3). Then we have $\phi_H \rightarrow \infty$ and $\phi_H + 1 \leq \frac{2}{3}\sqrt{W}$ for any sufficiently large n .*

PROOF. Assume that $\phi_H \leq w_n$, then we have

$$\phi_H^{2r} \geq \frac{W^r}{4^r |V_{\geq \phi_H}|} \stackrel{(2)}{\geq} \frac{\lambda^r n^{r-1}}{4^r} \rightarrow \infty,$$

since there are only n vertices in total; otherwise we have $\phi_H > w_n \rightarrow \infty$.

Now for the second statement, note that if $w_n \leq \frac{2}{3}\sqrt{W} - 2$, then we obtain $\phi_H + 1 \leq w_n + 2 \leq \frac{2}{3}\sqrt{W}$; on the other hand, if $w_n > \frac{2}{3}\sqrt{W} - 2$ then $|V_{\geq \frac{2}{3}\sqrt{W} - 2}| \geq 1$, and thus

$$\left(\frac{W}{4(\frac{2}{3}\sqrt{W} - 2)^2} \right)^r \leq (1 + o(1)) \left(\frac{9}{16} \right)^r \leq 1,$$

where the last inequality holds for any sufficiently large n . Consequently, by the definition of ϕ_H we have $\phi_H + 1 \leq \frac{2}{3}\sqrt{W} - 1 < \frac{2}{3}\sqrt{W}$, and thus the claim follows. □

REMARK 4.2. Claim 4.1 implies that for any nonheavy vertices $u, v \in V_{< \phi_H}$ we may drop the minimum in (1), that is, they form an edge with probability $p_{u,v} = w_u w_v / W$.

Next, we relate sums of powers of vertex weights to the size-biased distribution. Recall the following standard formula for the moments of a nonnegative random variable.

LEMMA 4.3 [e.g., Williams (1991)]. *Let $k \geq 1$ be an integer and X be a non-negative random variable. Then*

$$\mathbb{E}[X^k] = k \int_0^\infty x^{k-1} \mathbb{P}[X \geq x] dx.$$

PROPOSITION 4.4. *Let $0 < a < b$ and let $\vartheta \geq 2$ be an integer. Then*

$$(7) \quad \begin{aligned} W^{-1} \sum_{v \in V_{(a,b)}} w_v^\vartheta &= \mathbb{P}[a \leq Z_{\mathbf{w}} < b] a^{\vartheta-1} \\ &+ (\vartheta - 1) \int_a^b x^{\vartheta-2} \mathbb{P}[x \leq Z_{\mathbf{w}} < b] dx. \end{aligned}$$

In particular, for $b = \phi_H$ defined in (3) and $r \leq \vartheta \leq 2r - 1$, we obtain the upper bound

$$\sum_{v \in V_{[a, \phi_H)}} w_v^\vartheta \leq 4r W^r a^{\vartheta-2r}.$$

PROOF. We write

$$\sum_{v \in V_{[a,b]}} w_v^\vartheta = W \sum_{v \in V_{[a,b]}} w_v^{\vartheta-1} \frac{w_v}{W} = W \mathbb{E}[Z_{\mathbf{w}}^{\vartheta-1} \mathbb{1}_{\{a \leq Z_{\mathbf{w}} < b\}}].$$

Hence, using Lemma 4.3 with $k = \vartheta - 1$ for the random variable $Z_{\mathbf{w}} \mathbb{1}_{\{a \leq Z_{\mathbf{w}} < b\}}$ the first claim (7) follows.

For the upper bound, we will apply (7) with $b = \phi_H$. As a first step, we prove the following bound on the size-biased distribution for any $0 < y < \phi_H$:

$$(8) \quad \mathbb{P}[y \leq Z_{\mathbf{w}} < \phi_H] < 2W^{r-1}/y^{2r-1}.$$

To prove this, we observe that for any $0 < y_1 < y_2$ we have

$$(9) \quad \mathbb{P}[y_1 \leq Z_{\mathbf{w}} \leq y_2] = W^{-1} \sum_{u \in V_{[y_1, y_2]}} w_u \leq W^{-1} |V_{\geq y_1}| y_2.$$

First, note that if $\phi_H/2 \leq y < \phi_H$, then we have something stronger than (8):

$$(10) \quad \mathbb{P}[y \leq Z_{\mathbf{w}} < \phi_H] \stackrel{(9)}{\leq} \frac{|V_{\geq y}| \phi_H}{W} \stackrel{y < \phi_H}{<} \frac{W^{r-1} \phi_H}{2y^{2r}} \stackrel{y \geq \phi_H/2}{\leq} \frac{W^{r-1}}{y^{2r-1}}.$$

Now let $\mathcal{S} = \{y < \phi_H : \mathbb{P}[y \leq Z_{\mathbf{w}} < \phi_H] \geq 2W^{r-1}/y^{2r-1}\}$ and assume for contradiction that \mathcal{S} is not empty. Note that (10) implies that for any element $y' \in \mathcal{S}$ we have $y' < \phi_H/2$. Therefore, there exists a $y' \in \mathcal{S}$ with $y' < \phi_H/2$ such that $2y' \notin \mathcal{S}$ and $2y' < \phi_H$.

Since $2y' \notin \mathcal{S}$ and $2y' < \phi_H$, we have

$$\mathbb{P}[2y' \leq Z_{\mathbf{w}} < \phi_H] \leq \frac{2W^{r-1}}{(2y')^{2r-1}}.$$

Furthermore,

$$\mathbb{P}[y' \leq Z_{\mathbf{w}} < 2y'] \stackrel{(9)}{<} \frac{2y'|V_{\geq y'}|}{W} \stackrel{y' < \phi_H}{<} \frac{W^{r-1}}{(y')^{2r-1}}.$$

Consequently, we obtain

$$\mathbb{P}[y' \leq Z_{\mathbf{w}} < \phi_H] < \frac{W^{r-1}}{(y')^{2r-1}} + \frac{2W^{r-1}}{(2y')^{2r-1}} < \frac{2W^{r-1}}{(y')^{2r-1}},$$

resulting in a contradiction, and hence (8) holds for all $0 < y < \phi_H$.

Next, note that

$$\int_a^{\phi_H} \frac{2W^{r-1}}{x^{2r-1}} x^{\vartheta-2} dx \stackrel{\vartheta < 2r}{\leq} \frac{2W^{r-1} a^{\vartheta-2r}}{2r - \vartheta},$$

and thus we obtain the upper bound

$$\begin{aligned} \sum_{v \in V_{[a, \phi_H]}} w_v^\vartheta &\stackrel{(7),(8)}{\leq} W \frac{2W^{r-1}}{a^{2r-1}} a^{\vartheta-1} + W(\vartheta - 1) \int_a^{\phi_H} x^{\vartheta-2} \frac{2W^{r-1}}{x^{2r-1}} dx \\ &\leq 2W^r a^{\vartheta-2r} (1 + (\vartheta - 1)/(2r - \vartheta)) \stackrel{\vartheta < 2r}{\leq} 4r W^r a^{\vartheta-2r}. \quad \square \end{aligned}$$

Note that a priori it is not clear by the definition of the candidate thresholds p_s and p_d defined in (4) and (5), respectively, that the claimed critical function p_c in Theorem 3.1 satisfies $p_c = o(1)$. However, this is an almost direct consequence of Proposition 4.4.

COROLLARY 4.5. *Under the assumptions of Theorem 3.1 if in addition $\phi_H > w_n$, then*

$$p_s = o(1).$$

If instead of $\phi_H > w_n$, we have $\phi_H \leq w_n$ then

$$\min\{p_s, p_d\} = o(1).$$

PROOF. Assume that $\phi_H > w_n$. Then we have that $\mathbb{P}[a \leq Z_{\mathbf{w}} < \phi_H] = \mathbb{P}[a \leq Z_{\mathbf{w}}]$. Proposition 4.4 implies that under the assumptions of Theorem 3.1 we have

$$\sum_{u \in V_{<\phi_H}} w_u^{r+1} \geq rW \int_{C_1}^{w_n} Cx^{r-2} dx = \Omega(Ww_n^{r-1}),$$

and since $w_n \rightarrow \infty$ we have

$$p_s = O(w_n^{-1}) \stackrel{C.4.1}{=} o(1).$$

Now, if $w_n \geq \phi_H$, by the definition in (5) we have

$$\min\{p_s, p_d\} \leq p_d = O(\phi_H^{-1}) \stackrel{C.4.1}{=} o(1). \quad \square$$

4.2. Tools. We will need the following result due to Janson et al. (2012) for bootstrap percolation on the binomial random graph $G(n, p)$. The result holds for any choice (both random and deterministic) of the set \mathcal{A}_0 of initially infected vertices.

THEOREM 4.6 [Theorems 5.6 and 5.8 in Janson et al. (2012)]. *Consider a bootstrap percolation process with threshold $r \geq 2$ on $G(n, p)$, where $p \geq bn^{-1/r}$ for some constant $b > 0$, and the number a of initially infected vertices satisfies $a \rightarrow \infty$. Then whp all n vertices become infected eventually.*

We also apply the FKG inequality several times in our proofs. We consider the following setting. Let $G = (V, E)$ be a random graph where every pair of distinct vertices $u, v \in V$ appears as an edge in E independently with some probability $p_{u,v} \in [0, 1]$.

A graph property is called *nondecreasing* if it is preserved under the addition of edges, and it is *nonincreasing* if it is preserved under the removal of edges.

THEOREM 4.7 [FKG inequality, see, e.g., Janson, Łuczak and Ruciński (2000)]. *Let A be a nonincreasing graph property and B be a nondecreasing graph property. Consider the random graph G and denote by \mathcal{A} and \mathcal{B} the events that G has property A or B , respectively. Then we have*

$$\mathbb{P}[\mathcal{A} \cap \mathcal{B}] \leq \mathbb{P}[\mathcal{A}]\mathbb{P}[\mathcal{B}].$$

Throughout the paper, the proofs will rely on the following collection of concentration inequalities. Often we deal with sums of independent random variables which guarantee exponentially small bounds on the probability of nonconcentration.

THEOREM 4.8 [Chernoff inequality, e.g., Janson, Łuczak and Ruciński (2000)]. *Let X_i , for $1 \leq i \leq m$, be independent Bernoulli random variables with mean $0 \leq p_i \leq 1$, and let $X = \sum_{i=1}^m X_i$ denote their sum. Then for any $s > 0$ we have*

$$\mathbb{P}[X \leq \mathbb{E}[X] - s] \leq \exp\left(-\frac{s^2}{2\mathbb{E}[X]}\right)$$

and

$$\mathbb{P}[X \geq \mathbb{E}[X] + s] \leq \exp\left(-\frac{s^2}{2(\mathbb{E}[X] + s/3)}\right).$$

A similar bound due to McDiarmid (1998) applies also in a more general setting.

THEOREM 4.9 [McDiarmid (1998)]. *Let X_i , for $1 \leq i \leq m$, be independent random variables satisfying $X_i \leq \mathbb{E}[X_i] + M$ for some constant M , and let $X = \sum_{i=1}^m X_i$ denote their sum. Then, for any $s > 0$ we have*

$$\mathbb{P}[X \geq \mathbb{E}[X] + s] \leq \exp\left(-\frac{s^2}{2(\text{Var}[X] + Ms/3)}\right).$$

When the order of magnitude of each individual random variable varies strongly the following bound may prove to be stronger.

THEOREM 4.10 [Azuma–Hoeffding inequality, e.g., Janson, Łuczak and Ruciński (2000)]. *Let X_i , for $1 \leq i \leq m$, be independent random variables satisfying $a_i \leq X_i \leq b_i$, and let $X = \sum_{i=1}^m X_i$ denote their sum. Then, for any $s > 0$ we have*

$$\mathbb{P}[X \leq \mathbb{E}[X] - s] \leq \exp\left(-\frac{s^2}{2\sum_{i=1}^m (b_i - a_i)^2}\right).$$

5. Nuclei. As we described in Section 3.2, the first step towards proving that an outbreak occurs whp is to show that a subset of vertices of high weight becomes almost completely infected. We shall call this set a nucleus (see Definition 5.1 below). The total weight of this set is high enough, so that it functions as a source of the infection of a large part of the random graph. Moreover, the density within the set is high enough, so that the bootstrap process restricted to the subgraph induced by these vertices results in its almost complete infection.

DEFINITION 5.1. Let $\phi_K = \phi_K(n)$ satisfy $\phi_K \leq \min\{w_{n-r+1}, W/w_{n-r+1}\}$ and $\phi_K = o(\sqrt{W})$. We call the quantity ϕ_K the *weight-bound* of the nucleus if there exists a function $\varepsilon = \varepsilon(n) : \mathbb{N} \rightarrow [0, 1)$, tending to 0 as $n \rightarrow \infty$, such that whp

$$\widehat{W}_{\geq \phi_K} \geq (1 - \varepsilon)W_{\geq \phi_K}.$$

In addition, the set $V_{\geq \phi_K}$ is called a *nucleus*.

The goal of this section is to prove the existence of a nucleus in the supercritical regime.

THEOREM 5.2. *Suppose that the assumptions of Theorem 3.1 hold. If additionally*

$$\phi_H > w_n \quad \text{and} \quad p_0 \gg p_s,$$

or

$$\phi_H \leq w_n \quad \text{and} \quad p_0 \gg \min\{p_s, p_d\},$$

then there exists a nucleus with weight-bound ϕ_K .

The proof of this theorem is rather sophisticated and requires some preparation. Hence, we defer it to Section 5.4. Its proof consists of two theorems (Theorems 5.12 and 5.16) depending on certain properties of the weight sequence. In fact, it is these two theorems that we use towards the proof of Theorem 3.1. However, we also prove Theorem 5.2, as we believe it is of independent interest.

5.1. *Modified process.* The existence of a nucleus will be shown in several stages. The first stage is a restriction of the bootstrap process to a carefully selected set of vertices and its analysis therein. We modify the percolation process by restricting it to some *breeding ground* S for some number of steps. The steps when this restriction holds is called the *breeding phase*. The actual definitions of the breeding ground S used in the sparse and dense process will differ: in the former it consists of vertices of intermediate weight, while in the latter it is the set of all heavy vertices.

More formally, let $S \subseteq V$ be a subset of vertices, and let $\phi_0 = \phi_0(n)$ be a lower bound on their weights, that is, $\phi_0 \leq w_{\min}[S]$. Then we initially infect any vertex with weight less than ϕ_0 independently with probability p_0 (but no vertices of larger weight). They form the set \mathcal{B}_0 and for convenience of notation we define $\mathcal{B}_{-1} := \emptyset$. Now, for any $t \geq 0$ we denote by \mathcal{B}_t the set of all vertices which became infected either initially or in some step $1 \leq t' \leq t$ in the following process: in the t th step, $t \geq 1$, we infect all (uninfected) vertices in S having at least r infected neighbours in $\mathcal{B}_{t-1} \setminus \mathcal{B}_{t-2}$ (but no other vertices are infected). Note that apart from the vertices in \mathcal{B}_0 which were infected initially, only vertices within the breeding ground S ever become infected in this restricted process. Furthermore, whether a vertex $u \in V \setminus \mathcal{B}_{t-1}$ becomes infected at time t depends only on the edges connecting u to any of the vertices in $\mathcal{B}_{t-1} \setminus \mathcal{B}_{t-2}$ (none of which has been revealed so far). Thus all these events are independent.

Because this restriction of the process can only delay the time at which any particular vertex becomes infected (if it becomes infected at all), we have $\mathcal{B}_t \cap V_I \subseteq \mathcal{A}_t \cap V_I$, for any $t \geq 0$ and any interval $I \subseteq \mathbb{R}^+$. In particular, this holds at the end of the breeding phase. At this point, it will be necessary to return to the original process instead (or another subprocess). Considering these modifications is sufficient for showing that a candidate nucleus is indeed a nucleus, since this only depends on a *lower* bound on the total weight of the vertices within the candidate nucleus which become infected eventually.

We will give a few results that will allow us to control the evolution of the modified process. We will start with a concentration result on the weight of \mathcal{B}_0 . Recall that we consider a breeding ground S of minimum weight that is bounded from below by ϕ_0 .

LEMMA 5.3. *If $\phi_0 \rightarrow \infty$ and $np_0 \rightarrow \infty$, then whp $W[\mathcal{B}_0] \geq np_0/2$.*

PROOF. Since $\phi_0 \gg 1$, we have $|V_{\geq \phi_0}| \leq W/\phi_0 = o(n)$. Therefore, $(1 + o(1))n$ vertices have weight less than ϕ_0 and each of these vertices is infected independently with probability p_0 . Since $np_0 \gg 1$, the Chernoff bound (Theorem 4.8) implies that whp at least $np_0/2$ of these vertices become infected. Since the weight of every vertex is at least 1, the result follows. \square

Next, we show that the probability of a vertex u becoming infected in step $t + 1$ is essentially determined by its weight w_u and the total weight of the vertices which became infected in the previous steps, that is, $W[\mathcal{B}_t]$. The upper bound is almost immediate.

LEMMA 5.4. *For any $t \geq 0$ and any vertex $u \in S \setminus \mathcal{B}_t$, we have*

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t] \leq \frac{w_u^r W[\mathcal{B}_t]^r}{r! W^r}.$$

PROOF. By a union bound, we have

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t] \leq \sum_{\mathcal{R} \in \binom{\mathcal{B}_t}{r}} \prod_{v \in \mathcal{R}} \frac{w_u w_v}{W} \leq \frac{w_u^r}{W^r} \sum_{\mathcal{R} \in \binom{\mathcal{B}_t}{r}} \prod_{v \in \mathcal{R}} w_v \leq \frac{w_u^r}{W^r} \frac{W[\mathcal{B}_t]^r}{r!}. \quad \square$$

For the lower bound, we use the following intuition which turns out to be justified: the total weight of infected vertices should increase substantially in each step, and it should not be dominated by a few vertices of very large weight. So assume for now that this heuristic applies. In this case, we prove that if the total weight of infected vertices is already large enough, then u becomes infected whp. Otherwise, the total weight of infected vertices is still sufficiently large and the upper bound in Lemma 5.4 is actually tight.

LEMMA 5.5. *Let $\xi = \xi(n)$, $\xi_1 = \xi_1(n)$, $\xi_2 = \xi_2(n)$ be three functions such that $\xi, \xi_1, \xi_2 \rightarrow \infty$. Let $t \geq 0$ and assume that*

$$(11) \quad W[\mathcal{B}_t] \geq \xi_1 W[\mathcal{B}_{t-1}] \quad \text{and}$$

$$(12) \quad w_{\max}[\mathcal{B}_t] \leq \frac{W[\mathcal{B}_t]}{\xi_2}.$$

Furthermore, let $u \in S \setminus \mathcal{B}_t$ be such that

$$(13) \quad W[\mathcal{B}_t] \leq (W/w_u) \sqrt{\log \xi} \quad \text{and}$$

$$(14) \quad w_{\max}[\mathcal{B}_t] \leq \frac{4}{9} \frac{W}{w_u}.$$

Then we have

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq \frac{w_u^r W[\mathcal{B}_t]^r}{r! W^r} \exp(-1.9 \sqrt{\log \xi}).$$

If $u \in V \setminus \mathcal{B}_t$ is such that

$$(15) \quad W[\mathcal{B}_t] \geq (W/w_u) \sqrt[4]{\log \xi},$$

instead of (13), then we have

$$\mathbb{P}[|N(u) \cap (\mathcal{B}_t \setminus \mathcal{B}_{t-1})| \geq r \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq 1 - o(1/\sqrt{\log \xi}),$$

and consequently, if $u \in S$, then

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq 1 - o(1/\sqrt{\log \xi}).$$

PROOF. In order to prove the first statement, consider a vertex u of weight w_u . Next, observe that for any vertex $v \in \mathcal{B}_t$ we have $w_u w_v / W < 1$ by (14), and thus we may drop the minimum in (1), that is, u and v form an edge with probability $p_{u,v} = w_u w_v / W$, independently. Furthermore, since $W[\mathcal{B}_t \setminus \mathcal{B}_{t-1}] =$

$(1 + o(1))W[\mathcal{B}_t]$ by (11), and $W[\mathcal{B}_t] \gg w_{\max}[\mathcal{B}_t]$ by (12), there are at least r vertices in $\mathcal{B}_t \setminus \mathcal{B}_{t-1}$. We bound the probability that u has exactly r infected neighbours in $\mathcal{B}_t \setminus \mathcal{B}_{t-1}$:

$$(16) \quad \mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq \mathbb{P}[|N(u) \cap (\mathcal{B}_t \setminus \mathcal{B}_{t-1})| = r \mid \mathcal{B}_t, \mathcal{B}_{t-1}].$$

To compute the right-hand side of (16), we sum over all sets \mathcal{R} of r distinct vertices in $\mathcal{B}_t \setminus \mathcal{B}_{t-1}$ the probability that the event $\{N(u) \cap (\mathcal{B}_t \setminus \mathcal{B}_{t-1}) = \mathcal{R}\}$ holds, because these events are mutually exclusive for different r -tuples of vertices. This provides a lower bound on (16)

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq \sum_{\mathcal{R} \in \binom{\mathcal{B}_t \setminus \mathcal{B}_{t-1}}{r}} \prod_{v \in \mathcal{R}} \frac{w_u w_v}{W} \prod_{v' \in \mathcal{B}_t \setminus (\mathcal{B}_{t-1} \cup \mathcal{R})} \left(1 - \frac{w_u w_{v'}}{W}\right).$$

Furthermore, by (14) we have $w_u w_{v'}/W \leq 4/9$. Because $1 - x \geq \exp(-x/(1 - x))$, for any $x < 1$, the innermost product is therefore bounded by

$$\begin{aligned} & \prod_{v' \in \mathcal{B}_t \setminus (\mathcal{B}_{t-1} \cup \mathcal{R})} \left(1 - \frac{w_u w_{v'}}{W}\right) \\ & \geq \prod_{v' \in \mathcal{B}_t \setminus (\mathcal{B}_{t-1} \cup \mathcal{R})} \exp\left(-\frac{w_u w_{v'}}{W} / \left(1 - \frac{w_u w_{\max}[\mathcal{B}_t]}{W}\right)\right) \\ & \stackrel{(14)}{\geq} \exp\left(-\frac{9}{5} w_u \sum_{v' \in \mathcal{B}_t} \frac{w_{v'}}{W}\right) \\ & = \exp\left(-\frac{9}{5} w_u \frac{W[\mathcal{B}_t]}{W}\right) \\ & \stackrel{(13)}{\geq} \exp\left(-\frac{9}{5} \sqrt{\log \xi}\right). \end{aligned}$$

Note that this lower bound holds uniformly over all choices of \mathcal{R} . Hence, it suffices to bound

$$\begin{aligned} \sum_{\mathcal{R} \in \binom{\mathcal{B}_t \setminus \mathcal{B}_{t-1}}{r}} \prod_{v \in \mathcal{R}} \frac{w_u w_v}{W} &= \frac{w_u^r}{W^r} \sum_{\mathcal{R} \in \binom{\mathcal{B}_t \setminus \mathcal{B}_{t-1}}{r}} \prod_{v \in \mathcal{R}} w_v \\ &\geq \frac{w_u^r}{r! W^r} \left(W[\mathcal{B}_t \setminus \mathcal{B}_{t-1}]^r - W[\mathcal{B}_t]^{r-2} \sum_{w_v \in \mathcal{B}_t} w_v^2 \right) \\ &\stackrel{(11)}{=} \frac{w_u^r}{r! W^r} \left(W[\mathcal{B}_t]^r (1 - o(1)) - W[\mathcal{B}_t]^{r-2} \sum_{w_v \in \mathcal{B}_t} w_v^2 \right). \end{aligned}$$

The first claim follows from the fact that

$$W[\mathcal{B}_t]^2 \stackrel{(12)}{\gg} w_{\max}[\mathcal{B}_t] W[\mathcal{B}_t] = w_{\max}[\mathcal{B}_t] \sum_{u \in \mathcal{B}_t} w_u \geq \sum_{u \in \mathcal{B}_t} w_u^2.$$

For the second statement, let u be a vertex of weight w_u and note that if there are at least r vertices in $\mathcal{B}_t \setminus \mathcal{B}_{t-1}$ with weight at least W/w_u , then all of them will be neighbours of u with probability 1. Otherwise, consider $\mathcal{B}'_t \subseteq \mathcal{B}_t \setminus \mathcal{B}_{t-1}$ where all such vertices have been removed. We have

$$W[\mathcal{B}'_t] \geq W[\mathcal{B}_t] - W[\mathcal{B}_{t-1}] - (r - 1)w_{\max}[\mathcal{B}_t] \stackrel{(11),(12)}{=} (1 - o(1))W[\mathcal{B}_t].$$

We have

$$\begin{aligned} \mathbb{E}[|N(u) \cap \mathcal{B}'_t|] &= \sum_{v \in \mathcal{B}'_t} \frac{w_u w_v}{W} = \frac{w_u W[\mathcal{B}'_t]}{W} = (1 + o(1)) \frac{w_u W[\mathcal{B}_t]}{W} \\ &\stackrel{(15)}{=} \Omega((\log \xi)^{1/4}). \end{aligned}$$

Since $|N(u) \cap \mathcal{B}'_t|$ is the sum of independent indicator random variables, the Chernoff bound (Theorem 4.8) implies

$$\mathbb{P}[|N(u) \cap \mathcal{B}'_t| < r \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \leq \exp\left(-\frac{(\mathbb{E}[X] - r)^2}{2\mathbb{E}[X]}\right) = o(1/\sqrt{\log \xi}),$$

and the second claim follows. \square

In the analysis of the sparse process, the breeding ground is chosen to be a specific subset $S \subseteq V_{(\phi_s, \phi_H)}$ for some suitably chosen function $\phi_s = \phi_s(n) \rightarrow \infty$. We show that whp the total weight of infected vertices (in S) increases significantly in every step until it is large enough so that in the following step “almost all” vertices in the (candidate) nucleus $V_{\geq \phi_K}$ become infected. We first show that this holds in expectation, and then using Chebyshev’s inequality we deduce concentration around the expected value. However, since heavy vertices, that is, the ones with weight at least ϕ_H , increase the variance significantly we need to exclude them. The details can be found in Section 5.2.

Our approach for the dense process is different. Here, the breeding ground S is formed by all heavy vertices, that is, the ones with weight at least ϕ_H . Because these induce a dense graph only the number of infected vertices matters, while their total weight becomes irrelevant. By Theorem 4.6, we have that if at some point there are $\omega(1)$ infected heavy vertices, then every heavy vertex becomes infected eventually, hence the set of all heavy vertices forms a nucleus $V_{\geq \phi_K}$. We show that whp there are $\omega(1)$ infected heavy vertices as long as there are $\omega(1)$ heavy vertices in total. However, this is not guaranteed. If there are only $O(1)$ heavy vertices, then we can still ensure that the r vertices of largest weight, $n - r + 1, \dots, n$, become infected because their weights differ by at most a constant factor. These will in turn automatically infect any vertex u of weight $w_u \geq W/w_{n-r+1}$, because u is connected to $n - r + 1, \dots, n$ with probability 1 by (1), providing a nucleus $V_{\geq \phi_K}$ also in this case.

5.2. *Sparse process.* Ultimately, we want to show that if the initial infection rate $0 \leq p_0 \leq 1$ satisfies

$$(17) \quad p_0 \gg p_s,$$

where p_s is the candidate threshold defined in (4), then there is an outbreak. As a first step, we show that (17) implies the existence of a nucleus $V_{\geq \phi_K}$. Since the details of the proof are very delicate, we split it into three parts: the first contains all relevant definitions, the second addresses some preliminary calculations which allow for a more concise exposition of the main argument proving Theorem 5.12 in the final part. How this nucleus triggers an outbreak will be shown in Section 6.

5.2.1. *Setup.* First of all, we assume that (17) holds, and introduce the parameter

$$(18) \quad \bar{v}_s := p_0/p_s \rightarrow \infty.$$

Furthermore, in order to define the breeding ground S we observe that roughly speaking Lemmas 5.4 and 5.5 imply that for some time the probability that a vertex u of some intermediate weight w_u becomes infected is proportional to w_u^r and, therefore, the expected total weight of infected vertices of this type is essentially given by the sum of the $(r + 1)$ st powers of their weights.

With this in mind, we define an auxiliary weight bound $\phi_s = \phi_s(n)$ as the following (pointwise) maximum:

$$(19) \quad \phi_s(n) := \max \left\{ x \in \mathbb{R}^+ : \sum_{V_{[x, \phi_H)}} w_u^{r+1} \geq \frac{1}{2} \sum_{V_{< \phi_H}} w_u^{r+1} \right\},$$

and note that we have $\phi_s < \phi_H$ and $V_{[\phi_s, \phi_H]} \neq \emptyset$. Now we define S to be a nonempty subset of $V_{[\phi_s, \phi_H]}$. In particular, assume that the vertices in $V_{[\phi_s, \phi_H]}$ are $v_{j_s}, v_{j_s+1}, \dots, v_{j_H}$ arranged in nondecreasing order of weight. Let

$$(20) \quad S' := \{v_{j_s}, v_{j_s+2}, \dots\} \cup \{v_{j_H}\},$$

that is, we include in S' every other vertex of $V_{[\phi_s, \phi_H]}$ starting from the first one (the lightest), but we always include the last one. For $r \geq 3$, we take $S = S'$.

Recall that the infection of the vertices in a given step is independent and that the probability that a vertex u of some intermediate weight w_u becomes infected is proportional to w_u^r . Therefore, the variance is proportional to the $(r + 2)$ nd powers. Since vertices with large weight give rise to large variance we have to exclude some heavy vertices. In the $r \geq 3$ case, we show that it is enough to exclude the vertices of weight at least ϕ_H . On the other hand, for $r = 2$, we need to be a bit more careful and choose S according to the following procedure. We include in S each vertex in S' following the ordering as it appears in (20), so that S is maximal with respect to

$$(21) \quad \sum_{u \in S} w_u^4 \leq 9W^2.$$

Note that since $\phi_H^4 \leq W^2$ by Claim 4.1, this greedily chosen breeding ground S contains at least one vertex. Furthermore, by construction we have for any $v \in S' \setminus S$

$$(22) \quad w_v^4 + \sum_{u \in S} w_u^4 > 9W^2.$$

Now we use \bar{v}_s to define the weight-bound $\phi_0 = \phi_0(n)$, which provides an upper bound on the weights of initially infected vertices in the modified process, by

$$(23) \quad \phi_0 := \min\{\phi_s, np_0 \bar{v}_s^{-1/2}\}.$$

5.2.2. *Preliminaries.* First of all, we want to guarantee that we can apply Lemma 5.5 for $t = 0$.

CLAIM 5.6. *We have $np_0 \rightarrow \infty$ as $n \rightarrow \infty$. Furthermore, whp $W[\mathcal{B}_0] \geq np_0/2$ and $w_{\max}[\mathcal{B}_0] \leq 2W[\mathcal{B}_0] \bar{v}_s^{-1/2} = o(W[\mathcal{B}_0])$.*

PROOF. We first show that $\phi_s \rightarrow \infty$. For this, note that for $x > 0$, we have

$$(24) \quad \sum_{u \in V_{<x}} w_u^{r+1} \leq x^r \sum_{u \in V_{<x}} w_u \leq x^r \sum_{u \in V} w_u \leq x^r W.$$

Setting $m_0 := (p_s^{1-r}/2)^{1/r}$, we thus obtain

$$\sum_{u \in V_{<m_0}} w_u^{r+1} \stackrel{(24)}{\leq} m_0^r W = \frac{1}{2} p_s^{1-r} W \stackrel{(4)}{=} \frac{1}{2} \sum_{u \in V_{<\phi_H}} w_u^{r+1}$$

and thus

$$\sum_{u \in V_{(m_0, \phi_H)}} w_u^{r+1} = \sum_{u \in V_{<\phi_H}} w_u^{r+1} - \sum_{u \in V_{<m_0}} w_u^{r+1} \geq \frac{1}{2} \sum_{u \in V_{<\phi_H}} w_u^{r+1}.$$

Therefore, by the definition (19) of ϕ_s , we have $\phi_s \geq m_0$. Due to Corollary 4.5, we have $p_s = o(1)$, and thus $m_0 = \Theta(p_s^{(1-r)/r}) \gg 1$.

Using this, we prove that $\phi_0 \gg 1$. By Proposition 4.4, for $\vartheta = r + 1$ and $a = 1$, we have

$$\sum_{u \in V_{<\phi_H}} w_u^{r+1} \leq 4r W^r,$$

and, therefore, by (18) and (2) we also have

$$np_0 \bar{v}_s^{-1/2} \geq \frac{W}{\lambda} \bar{v}_s^{1/2} p_s \stackrel{(4)}{=} \frac{\bar{v}_s^{1/2}}{\lambda} \left(\frac{W^r}{\sum_{u \in V_{<\phi_H}} w_u^{r+1}} \right)^{1/(r-1)} \geq \frac{\bar{v}_s^{1/2}}{4r\lambda} \rightarrow \infty,$$

implying $\phi_0 \rightarrow \infty$ and $np_0 \rightarrow \infty$.

Now, we apply Lemma 5.3, which implies that whp

$$W[\mathcal{B}_0] \stackrel{L.5.3}{\geq} np_0/2.$$

Conditional on $W[\mathcal{B}_0] \geq np_0/2$ by (23), we have

$$w_{\max}[\mathcal{B}_0] \leq \phi_0 \leq np_0 \bar{v}_s^{-1/2} \leq 2W[\mathcal{B}_0] \bar{v}_s^{-1/2}. \quad \square$$

However, we need some more preparation for the inductive argument. The following lower bound of the sums of the $(r + 1)$ st powers of the vertex weights in S is reminiscent of the definition of ϕ_s in (19). In fact, for $r \geq 3$ the bound follows from this. However, the proof for $r = 2$ is nontrivial due to the slightly different construction of the breeding ground S .

LEMMA 5.7. *For any integer $r \geq 2$, we have*

$$\sum_{u \in S} w_u^{r+1} \geq \frac{1}{8} \sum_{u \in V_{<\phi_H}} w_u^{r+1}.$$

PROOF. Let $r \geq 3$. The definition of ϕ_s in (19) implies that

$$\sum_{u \in V_{[\phi_s, \phi_H)}} w_u^{r+1} \geq \frac{1}{2} \sum_{u \in V_{<\phi_H}} w_u^{r+1}.$$

Now, note that the set S in (20) contains every other vertex (ordered by weight), and moreover, the vertex of largest weight within $V_{[\phi_s, \phi_H)}$, hence this implies that

$$(25) \quad \sum_{u \in S} w_u^{r+1} \geq \frac{1}{2} \sum_{u \in V_{[\phi_s, \phi_H)}} w_u^{r+1}.$$

The same is true if $r = 2$ and $S = S'$.

Hence, consider $r = 2$ and assume that $S \subsetneq S'$, that is, there exists a vertex $v \in S' \setminus S$. For any such vertex, we have that $w_v^4 < \phi_H^4 \leq W^2$ by Claim 4.1 and by (22) we have $\sum_{v \in S} w_v^4 \geq 8W^2$. Therefore,

$$\sum_{u \in S} w_u^3 \geq \sum_{u \in S} w_u^4 / w_{\max}[S] \geq 8W^2 / w_{\max}[S].$$

On the other hand, since we picked the elements of S in a nondecreasing order of the weights, we have for any $v \in S' \setminus S$ that $w_v \geq w_{\max}[S]$, and thus

$$\sum_{v \in S' \setminus S} w_v^3 \leq \sum_{v \in V_{[w_{\max}[S], \phi_H)}} w_v^3 \stackrel{P.4.4, \vartheta=3}{\leq} 8W^2 / w_{\max}[S].$$

But this implies by the pigeon-hole principle that

$$\sum_{u \in S} w_u^3 \geq \frac{1}{2} \sum_{u \in S'} w_u^3 \stackrel{(25)}{\geq} \frac{1}{4} \sum_{u \in V_{[\phi_s, \phi_H)}} w_u^3 \stackrel{(19)}{\geq} \frac{1}{8} \sum_{u \in V_{<\phi_H}} w_u^3$$

as desired. \square

The following lemma will be used later in the proof of Lemma 5.9 as an upper bound on the second moment of the total weight of the newly infected vertices during the modified process. The bound will be used along with Chebyshev’s inequality in order to show that whp the weight of these vertices is at least a certain multiple of the current generation.

LEMMA 5.8. *For any integer $r \geq 2$, we have*

$$\sum_{u \in S} w_u^{r+2} \leq 2^{r+7} W^2 p_s^{2-r}.$$

PROOF. If $r = 2$, then by (21) we have $\sum_{u \in S} w_u^4 \leq 9W^2 < 2^9 W^2$, and thus the claim is immediate.

If $r \geq 3$, then recall that $S = S'$. We abbreviate

$$(26) \quad \Upsilon := p_s^{r-1} W^{-1} \sum_{u \in S} w_u^{r+2}.$$

Note that $\Upsilon = \sum_{u \in S} w_u^{r+2} / \sum_{u \in V_{<\phi_H}} w_u^{r+1}$ by the definition of p_s [cf. (4)], and hence $\Upsilon \leq \phi_H$.

Moreover, since $r + 2 \leq 2r - 1$ for any $r \geq 3$, applying Proposition 4.4 with $a = \Upsilon/2$, $\vartheta = r + 1$ and $\vartheta = r + 2$, respectively, implies that

$$(27) \quad \sum_{u \in V_{[\Upsilon/2, \phi_H]}} w_u^{r+1} \leq 4r W^r \Upsilon^{1-r} \leq 2^{r+2} W^r \Upsilon^{1-r}$$

and

$$(28) \quad \sum_{u \in V_{[\Upsilon/2, \phi_H]}} w_u^{r+2} \leq 4r W^r \Upsilon^{2-r} \leq 2^{r+2} W^r \Upsilon^{2-r}.$$

Now if

$$\sum_{u \in V_{<\Upsilon/2}} w_u^{r+1} \leq 2^{r+6} W^r \Upsilon^{1-r},$$

then (27) implies

$$\sum_{u \in V_{<\phi_H}} w_u^{r+1} = \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1} + \sum_{u \in V_{[\Upsilon/2, \phi_H]}} w_u^{r+1} \leq 2^{r+7} W^r \Upsilon^{1-r}.$$

Moreover, expressing the left-hand side using p_s [cf. (4)] and the right-hand side using (26) we obtain

$$p_s^{1-r} W \leq 2^{r+7} p_s^{-(r-1)^2} W^{2r-1} \left(\sum_{u \in S} w_u^{r+2} \right)^{1-r}$$

and the claim follows by taking the $(r - 1)$ st root and solving for the sum.

Otherwise, we may assume that

$$(29) \quad \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1} > 2^{r+6} W^r \Upsilon^{1-r}.$$

However, we will show that this leads to a contradiction. First, note that this implies

$$\sum_{u \in V_{[\Upsilon/2, \phi_H)}} w_u^{r+2} \stackrel{(28)}{\leq} 2^{r+4} W^r \Upsilon^{2-r} \stackrel{(29)}{<} \frac{1}{4} \Upsilon \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1}$$

and we also have

$$\sum_{u \in V_{<\phi_H}} w_u^{r+2} \geq \sum_{u \in S} w_u^{r+2} \stackrel{(26)}{=} \Upsilon \sum_{u \in V_{<\phi_H}} w_u^{r+1} \geq \Upsilon \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1}.$$

Therefore, we obtain a lower bound on the difference

$$\sum_{u \in V_{<\phi_H}} w_u^{r+2} - \sum_{u \in V_{[\Upsilon/2, \phi_H)}} w_u^{r+2} \geq \frac{3}{4} \Upsilon \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1}.$$

However, at the same time this difference satisfies

$$\sum_{u \in V_{<\phi_H}} w_u^{r+2} - \sum_{u \in V_{[\Upsilon/2, \phi_H)}} w_u^{r+2} \stackrel{\Upsilon \leq \phi_H}{\leq} \sum_{u \in V_{<\Upsilon/2}} w_u^{r+2} \leq \frac{1}{2} \Upsilon \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1},$$

a clear contradiction, since $\Upsilon \sum_{u \in V_{<\Upsilon/2}} w_u^{r+1} > 0$ by (26) and (29), and since $S \neq \emptyset$. \square

5.2.3. *Main argument.* Now we show that until almost every vertex with weight at least $\phi_H / (\log \bar{v}_s)^{1/4}$ becomes infected whp in every step of the process, the weight of the infected set increases significantly. For simplicity of notation, we define the *scaled total weight* v_t by

$$(30) \quad v_t = v_t(n) := \frac{2W[\mathcal{B}_t]}{p_s n}.$$

Note that by Claim 5.6 and (18) we have whp $v_0 = 2W[\mathcal{B}_0] / (p_s n) \geq \bar{v}_s$ (this is why there is a factor 2 in the definition) and, therefore, these scaled total weights satisfy

$$(31) \quad 1 \ll p_0 / p_s = \bar{v}_s \leq v_0 \leq v_1 \leq \dots$$

LEMMA 5.9. *Let $t \geq 0, r \geq 2$ be integers and assume that*

$$(32) \quad W[\mathcal{B}_t] \geq \frac{1}{2} \bar{v}_s^{r-1} \exp(-2\sqrt{\log \bar{v}_s}) W[\mathcal{B}_{t-1}],$$

$$(33) \quad w_{\max}[\mathcal{B}_t] \leq \frac{1}{2 \log \bar{v}_s} W[\mathcal{B}_t],$$

and

$$(34) \quad W[\mathcal{B}_t] \leq (W/\phi_H)\sqrt{\log \bar{v}_s}.$$

Then conditional on the random sets \mathcal{B}_t and \mathcal{B}_{t-1} with probability at least $1 - v_t^{1-r}$ we have

$$(35) \quad W[\mathcal{B}_{t+1}] \geq \frac{1}{2}v_t^{r-1}W[\mathcal{B}_t]\exp(-2\sqrt{\log \bar{v}_s})$$

and

$$(36) \quad w_{\max}[\mathcal{B}_{t+1}] \leq v_t^{r-1}W[\mathcal{B}_t]\exp(-2\sqrt{\log \bar{v}_s})/\log \bar{v}_s.$$

PROOF. We first calculate the total weight of vertices that become infected in step $t + 1$. Afterwards, we use a second moment argument to show that it is actually concentrated around its expectation, thereby proving the first statement. Then we use a first moment argument to guarantee that no vertices of too large weight became infected whp.

Let $t \geq 0$ be an integer and assume that \mathcal{B}_t and \mathcal{B}_{t-1} have been realised. Furthermore, recall that we consider the process in which only vertices within the breeding ground S become infected (apart from those being infected initially). Moreover, observe that Conditions (32), (33) and (34) imply that their counterparts, (11), (12) and (13), respectively, are satisfied. Also, Condition (14) holds since any two vertices $u, u' \in S$ have weight at most ϕ_H and by Claim 4.1, the product of their weights satisfies $w_u w_{u'} \leq \frac{4}{9}W$. Thereby, Lemma 5.5 is applicable for all vertices $u \in S$, that is, we have

$$\mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq \exp(-1.9\sqrt{\log \bar{v}_s})\frac{w_u^r W[\mathcal{B}_t]^r}{r!W^r}.$$

Consequently,

$$\mathbb{E}[W[\mathcal{B}_{t+1}] \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \geq \exp(-1.9\sqrt{\log \bar{v}_s}) \sum_{u \in S} \frac{w_u^{r+1} W[\mathcal{B}_t]^r}{r!W^r} - W[\mathcal{B}_t],$$

where the last term is a (crude) upper bound on the contribution of the vertices in S which are already infected at time t . Now observe that

$$(37) \quad \begin{aligned} \frac{W[\mathcal{B}_t]^{r-1}}{W^r} \sum_{u \in S} w_u^{r+1} &\stackrel{L.5.7}{\geq} \frac{W[\mathcal{B}_t]^{r-1}}{8W^r} \sum_{u \in V_{<\phi_H}} w_u^{r+1} \stackrel{(4)}{=} \frac{W[\mathcal{B}_t]^{r-1}}{8(p_s W)^{r-1}} \\ &\stackrel{(30)}{=} \frac{v_t^{r-1}}{8} \left(\frac{1}{2\lambda}\right)^{r-1}, \end{aligned}$$

where in particular the right-hand side grows polynomially in ν_t . Hence, because $\bar{\nu}_s \leq \nu_t$, and thus also $\exp(-1.9\sqrt{\log \bar{\nu}_s})\nu_t^{r-1} \rightarrow \infty$, we obtain

$$\begin{aligned}
 \mathbb{E}[W[\mathcal{B}_{t+1}] \mid \mathcal{B}_t, \mathcal{B}_{t-1}] &\geq \exp(-1.9\sqrt{\log \bar{\nu}_s} + O(1)) \frac{W[\mathcal{B}_t]^r}{W^r} \sum_{u \in S} w_u^{r+1} \\
 (38) \qquad \qquad \qquad &\stackrel{(37)}{\geq} \exp(-1.9\sqrt{\log \bar{\nu}_s} + O(1)) \nu_t^{r-1} W[\mathcal{B}_t] \\
 &\geq \exp(-2\sqrt{\log \bar{\nu}_s}) \nu_t^{r-1} W[\mathcal{B}_t]
 \end{aligned}$$

for any sufficiently large n , because $\bar{\nu}_s \rightarrow \infty$.

Next, we want to apply Chebyshev’s inequality, so we need to provide an upper bound on the variance of $W[\mathcal{B}_{t+1}]$. Because infections (at time $t + 1$) take place independently for all vertices in $S \setminus W[\mathcal{B}_t]$, we have

$$\begin{aligned}
 \text{Var}[W[\mathcal{B}_{t+1}] \mid \mathcal{B}_t, \mathcal{B}_{t-1}] &\leq \sum_{u \in S \setminus \mathcal{B}_t} w_u^2 \mathbb{P}[u \in \mathcal{B}_{t+1} \mid \mathcal{B}_t, \mathcal{B}_{t-1}] \\
 (39) \qquad \qquad \qquad &\leq \stackrel{L.5.4}{\sum_{u \in S} w_u^{r+2} (W[\mathcal{B}_t]/W)^r} \\
 &\leq \stackrel{L.5.8}{2^{r+7} p_s^{2-r} W^2 (W[\mathcal{B}_t]/W)^r} \\
 &\leq \stackrel{(30), W \geq n}{2^{r+7} W[\mathcal{B}_t]^2 \nu_t^{r-2}}.
 \end{aligned}$$

By Chebyshev’s inequality, we thus obtain

$$\begin{aligned}
 \mathbb{P}[W[\mathcal{B}_{t+1}] \leq \mathbb{E}[W[\mathcal{B}_{t+1}]]/2] &\stackrel{(38),(39)}{\leq} \frac{2^{r+9} W[\mathcal{B}_t]^2 \nu_t^{r-2}}{(\exp(-2\sqrt{\log \bar{\nu}_s}) \nu_t^{r-1} W[\mathcal{B}_t])^2} \\
 &\leq \nu_t^{-r} \exp(5\sqrt{\log \bar{\nu}_s}),
 \end{aligned}$$

and the first statement follows since $\bar{\nu}_s \leq \nu_t$.

For the second part of the statement, we define

$$\zeta := \nu_t^{r-1} W[\mathcal{B}_t] \exp(-2\sqrt{\log \bar{\nu}_s}) / \log \bar{\nu}_s$$

and note that by Lemma 5.4 the expected number of vertices in S of weight at least ζ becoming infected at time $t + 1$ is at most

$$\sum_{u \in V_{[\zeta, \phi_H]}} w_u^r \frac{W[\mathcal{B}_t]^r}{W^r} \stackrel{P.4.4, \vartheta=r}{\leq} 4r W[\mathcal{B}_t]^r \zeta^{-r} \leq (\nu_t^{1-r} \exp(3\sqrt{\log \bar{\nu}_s}) \log \bar{\nu}_s)^r.$$

Because $\bar{\nu}_s \leq \nu_t$ the second statement follows from Markov’s inequality.

In fact, both error terms are sufficiently small so that by a union bound, both statements hold simultaneously with probability at least $1 - \nu_t^{1-r}$. \square

Note that (35) and (36) imply that Conditions (32), (33) hold for $t + 1$. We apply Lemma 5.9 repeatedly in order to show that the total weight of the infected vertices becomes large enough so that every vertex of large enough weight becomes infected in the next step whp.

LEMMA 5.10. *For any integer $r \geq 2$, there exists an integer $T \geq 0$ such that whp*

$$W[\mathcal{B}_T] > (W/\phi_H)\sqrt{\log \bar{v}_s}.$$

PROOF. We show that in every step $\tau \geq 0$ there are two possibilities: either this condition is satisfied, that is, we can *stop* and use $T = \tau$, or else we can apply Lemma 5.9 once more; note that this ensures that $W[\mathcal{B}_{\tau+1}] = \omega(W[\mathcal{B}_\tau])$ and $w_{\max}[\mathcal{B}_{\tau+1}] = o(W[\mathcal{B}_{\tau+1}])$ for the next step. In this spirit, we denote the event that we stopped by time $\tau \geq 0$ by \mathcal{T}_τ (and by $\neg\mathcal{T}_\tau$ its complement), and define a family of *good events*

$$\mathcal{G}_\tau := \{\forall 0 \leq t \leq \tau : (35) \text{ and } (36) \text{ hold}\}$$

for $\tau \geq 0$. The core of the proof is formed by the following recursive argument, whose proof we will postpone for a moment.

CLAIM 5.11. *If $\neg\mathcal{T}_0$, then*

$$(40) \quad \mathbb{P}[\mathcal{G}_0 \mid \mathcal{B}_0] \geq 1 - o(1) - v_0^{1-r}.$$

Similarly, for any $\tau \geq 1$,

$$(41) \quad \mathbb{P}[\mathcal{G}_\tau \mid \mathcal{G}_{\tau-1}, \neg\mathcal{T}_\tau, \mathcal{B}_\tau, \mathcal{B}_{\tau-1}] \geq 1 - v_\tau^{1-r}.$$

Now assume that Claim 5.11 holds. Then we observe that by definition of the scaled total weights

$$\frac{v_t}{v_{t-1}} \stackrel{(30)}{=} \frac{W[\mathcal{B}_t]}{W[\mathcal{B}_{t-1}]} \stackrel{\mathcal{G}_{t-1}}{\geq} \frac{1}{2} v_{t-1}^{r-1} \exp(-2\sqrt{\log \bar{v}_s}) \geq \frac{1}{2} \bar{v}_s^{r-1} \exp(-2\sqrt{\log \bar{v}_s}) \rightarrow \infty$$

for any $t \geq 1$, and thus by a union bound

$$\mathbb{P}[\mathcal{G}_\tau \mid \neg\mathcal{T}_\tau, \mathcal{B}_\tau, \dots, \mathcal{B}_0] \geq 1 - o(1) - \sum_{t=0}^{\infty} v_t^{1-r} = 1 - o(1).$$

Since we have $W[\mathcal{B}_t] \leq W$ for any time $t \geq 0$, the recursion must end in finite time; however, this can only happen because there exists a $T \geq 0$ such that \mathcal{T}_T holds, proving the statement of the lemma.

It remains to prove Claim 5.11. By Claim 5.6, we initially have $W[\mathcal{B}_0] \geq np_0/2 \rightarrow \infty$ and $w_{\max}[\mathcal{B}_0] \leq 2W[\mathcal{B}_0]\bar{v}_s^{-1/2}$.

If in addition $W[\mathcal{B}_0] \leq (W/\phi_H)\sqrt{\log \bar{v}_s}$, then Lemma 5.9 is applicable for $t = 0$, and thus \mathcal{G}_0 holds with probability at least $1 - o(1) - v_0^{1-r}$. Otherwise, we must have $W[\mathcal{B}_0] > (W/\phi_H)\sqrt{\log \bar{v}_s}$, and, therefore, \mathcal{T}_0 holds. Both cases together prove (40).

The induction step is proven analogously: let $\tau \geq 1$ and assume $\mathcal{G}_{\tau-1}$ holds, then if additionally $W[\mathcal{B}_\tau] \leq (W/\phi_H)\sqrt{\log \bar{v}_s}$, then Lemma 5.9 with $t = \tau$ implies that with sufficiently high probability \mathcal{G}_τ holds. Otherwise, we have $W[\mathcal{B}_\tau] > (W/\phi_H)\sqrt{\log \bar{v}_s}$ and, therefore, \mathcal{T}_τ holds. Both cases together prove the induction step (41). \square

Next, we construct a nucleus (cf. Definition 5.1) for the sparse process. We consider the *candidate nucleus* $V_{\geq \phi_K}$ given by the weight-bound

$$(42) \quad \phi_K := \begin{cases} W/w_{n-r+1} & \text{if } \phi_H(\log \bar{v}_s)^{-1/4} \eta > W/w_{n-r+1}, \\ \phi_H(\log \bar{v}_s)^{-1/4} & \text{if } \phi_H(\log \bar{v}_s)^{-1/4} \eta \leq W/w_{n-r+1}, \end{cases}$$

where $\eta = \eta(n)$ is an arbitrarily slowly growing function, satisfying $\eta \rightarrow \infty$ but $(\log \bar{v}_s)^{-1/4} \eta = o(1)$.⁷ We do so by determining a subset $\mathcal{U}_K \subseteq V_{\geq \phi_K} \cap \mathcal{A}_F$ of weight

$$(43) \quad W[\mathcal{U}_K] \geq (1 - o(1))W_{\geq \phi_K}.$$

It will be crucial that this construction requires us to only expose a specific subset of edge indicator variables.

THEOREM 5.12. *Under the assumptions of Theorem 3.1, suppose that $p_0 \gg p_s$. Then whp the following two statements hold:*

- *There exists a set $\mathcal{U}_K \subseteq (V_{\geq \phi_K} \cap \mathcal{A}_F)$ [where $V_{\geq \phi_K}$ is defined as in (42)] which satisfies (43), contains $\{n - r + 1, \dots, n\}$, and is constructed by only exposing edge-indicator random variables corresponding to edges in $S' \times (\mathcal{B}_0 \cup S' \cup V_{\geq \phi_K})$.*
- *The weight-bound ϕ_K satisfies $\phi_K \leq \min\{w_{n-r+1}, W/w_{n-r+1}\}$ and $\phi_K = o(\sqrt{W})$.*

In particular, this means that whp $V_{\geq \phi_K}$ is a nucleus.

PROOF. For this proof, we abbreviate $\phi'_H := \phi_H(\log \bar{v}_s)^{-1/4}$ for convenience of notation. We first show that the two conditions on the weight-bound ϕ_K [defined in (42)] are satisfied. Because $w_{n-r+1} = \alpha w_n$ and $\phi_H \leq w_n + 1$ [cf. (3)] we have

$$(44) \quad \phi'_H \eta = \phi_H(\log \bar{v}_s)^{-1/4} \eta = o(\phi_H) = o(w_{n-r+1}),$$

⁷Such an η exists since $\bar{v}_s \rightarrow \infty$ by (31).

since $(\log \bar{v}_s)^{-1/4} \eta = o(1)$. Now if $\phi_K = \phi'_H$, then we have

$$\phi_K = \phi'_H \stackrel{(42)}{\leq} W/(\eta w_{n-r+1}) = o(W/w_{n-r+1}) \quad \text{and} \quad \phi_K = \phi'_H \stackrel{(44)}{=} o(w_{n-r+1}).$$

Otherwise, if $\phi_K = W/w_{n-r+1}$, then by (42) we obtain

$$\phi_K = W/w_{n-r+1} < \phi'_H \eta \stackrel{(44)}{=} o(w_{n-r+1}).$$

Thus in both cases the first property of ϕ_K is satisfied. Furthermore, the previous argument also showed $\phi_K = o(\phi_H)$ and since $\phi_H = O(\sqrt{W})$ by Claim 4.1 the second property follows.

It remains to construct the set \mathcal{U}_K , or, in other words, show that a significant proportion of the candidate nucleus becomes infected eventually. For this, let $T \geq 0$ be an integer satisfying $W[\mathcal{B}_T] \geq (W/\phi_H)\sqrt{\log \bar{v}_s}$ as in Lemma 5.10 and condition on this (high probability) event. In particular, we have

$$(45) \quad W[\mathcal{B}_T] \geq (W/\phi_H)\sqrt{\log \bar{v}_s} \geq (W/w_u)\sqrt[4]{\log \bar{v}_s},$$

for any $w_u \geq \phi'_H$.

First, assume that we have $\phi_K = W/w_{n-r+1}$. We consider the r vertices of largest weight and observe that

$$w_n \geq \dots \geq w_{n-r+1} \stackrel{(44)}{\gg} \phi'_H \eta \gg \phi'_H$$

and thus the assertion in (45) is true for each of them. Therefore, the second statement of Lemma 5.5 is applicable (with $t = T$ and $\xi = \bar{v}_s \rightarrow \infty$) showing that whp $\{n - r + 1, \dots, n\} \subseteq \mathcal{A}_{T+1}$ [using only edges between $\{n - r + 1, \dots, n\}$ and $(\mathcal{B}_T \setminus \mathcal{B}_{T-1}) \subseteq S'$]. Moreover, we certainly have $V_{\geq \phi_K} = V_{\geq W/w_{n-r+1}} \subseteq \mathcal{A}_{T+2}$, because every vertex in this set is connected to each vertex in $\{n - r + 1, \dots, n\}$ with probability 1. Thus we set $\mathcal{U}_K := V_{\geq \phi_K}$, which obviously satisfies (43). Since the process $\{\mathcal{B}_t\}$ exposes only edges in $S' \times (\mathcal{B}_0 \cup S')$ and the last step only depends on edges in $S' \times V_{\geq \phi_K}$ the statement follows in this case.

Otherwise, we have $\phi_K = \phi'_H$ and we need to consider the vertices in $V_{\geq \phi'_H} \setminus \mathcal{B}_T$. We observe that (45) shows that the weight of any vertex $u \in V_{\geq \phi'_H} \setminus \mathcal{B}_T$ is sufficiently large to apply the second statement of Lemma 5.5 (with $t = T$ and $\xi = \bar{v}_s \rightarrow \infty$). Hence, whp each of them has at least r neighbours in $\mathcal{B}_T \setminus \mathcal{B}_{T-1}$, and this events depend only on mutually disjoint sets of edges. So now define $\mathcal{U}_K := \{u \in V_{\geq \phi'_H} \mid u \in \mathcal{B}_T \text{ or } |N(u) \cap (\mathcal{B}_T \setminus \mathcal{B}_{T-1})| \geq r\}$, and note that $\mathbb{E}[W[\mathcal{U}_K]] = (1 - o(1))W_{\geq \phi'_H}$.

Consequently, since the weight of each vertex is at most w_n , McDiarmid's inequality (Theorem 4.9) is applicable (with $X_u = \mathbb{1}_{\{u \in \mathcal{U}_K\}}$, $M = w_n$, and $s = \sqrt{w_n W_{\geq \phi'_H}}$) yielding

$$(46) \quad \begin{aligned} & \mathbb{P}[W[\mathcal{U}_K] \leq (1 - o(1))W_{\geq \phi'_H} - \sqrt{w_n W_{\geq \phi'_H}}] \\ & \leq \exp\left(-\frac{w_n W_{\geq \phi'_H}}{2(W_{\geq \phi'_H}(1 - o(1)) + w_n \sqrt{w_n W_{\geq \phi'_H}}/3)}\right). \end{aligned}$$

Observe that by the assumption on the size-biased distribution in Theorem 3.1 we have

$$(47) \quad W_{\geq \phi'_H} = W \sum_{u \in V_{\geq \phi'_H}} \frac{w_u}{W} = W \mathbb{P}[Z_{\mathbf{w}} \geq \phi'_H] \geq \frac{CW}{\phi'_H}.$$

Furthermore, since $\phi_K = \phi'_H$ we have $\phi'_H \eta \leq W/w_{n-r+1}$ by (42), and this in turn shows that $W/\phi'_H = \omega(w_n)$, because $w_{n-r+1} = \alpha w_n$ and $\eta \rightarrow \infty$. Combined with (47) this provides $W_{\geq \phi'_H} = \omega(w_n)$, and consequently the right-hand side of (46) is $o(1)$. Because $\mathcal{U}_K \subseteq (V_{\geq \phi'_H} \cap \mathcal{A}_F)$, we thus obtain whp $\widehat{W}_{\geq \phi_K} = (1 - o(1))W_{\geq \phi_K}$ also in this case. Because the process $\{\mathcal{B}_t\}$ exposes only edges in $S' \times (\mathcal{B}_0 \cup S')$ and the last step only depends on edges in $S' \times V_{\geq \phi_K}$. This completes the proof. \square

5.3. *Dense process.* In this section, we want to show that if the candidate threshold p_d for the dense process—as defined in (4)—exists, in other words,

$$(48) \quad \phi_H \leq w_n,$$

and the initial infection rate $0 \leq p_0 = p_0(n) \leq 1$ satisfies

$$p_0 \gg p_d,$$

then there exists a nucleus $V_{\geq \phi_K}$. Throughout this section, we assume that these two conditions hold, and introduce the parameter

$$(49) \quad \bar{v}_d := p_0/p_d \rightarrow \infty.$$

Furthermore, we observe that the candidate threshold for the dense process satisfies

$$p_d W \stackrel{(5)}{=} \left(\frac{W^r}{\sum_{u \in V_{\geq \phi_H}} w_u^r} \right)^{1/r} \geq 1,$$

and thus, since $W = \lambda n$ for some constant $\lambda > 0$, we have

$$(50) \quad np_0 = (W/\lambda)\bar{v}_d p_d \geq \bar{v}_d/\lambda \rightarrow \infty.$$

The setup for the dense process is much simpler than that for the sparse process. We use the set

$$(51) \quad S := V_{\geq \phi_H}$$

as breeding ground and define the weight-bound for the initial infection by

$$(52) \quad \phi_0 := \min\{\phi_H, \sqrt{np_0}\}.$$

Similarly, as in Section 5.2, we establish some basic properties of $W[\mathcal{B}_0]$, which for instance guarantee that Lemma 5.5 is applicable at time $t = 0$.

CLAIM 5.13. *Whp $np_0/2 \leq W[\mathcal{B}_0] < (\lambda + 1)np_0$ and $w_{\max}[\mathcal{B}_0] = o(W[\mathcal{B}_0])$.*

PROOF. In order to apply Lemma 5.3, it suffices to show that $\phi_H \rightarrow \infty$, since we already showed in (50) that $np_0 \rightarrow \infty$. To do so, we observe that $\phi_H = O(1)$ (over a subsequence) would imply

$$|V_{\geq \phi_H}| \stackrel{(3)}{\geq} \left(\frac{W}{4\phi_H^2}\right)^r = \Omega(W^r) = \omega(n),$$

yielding a contradiction since there are only n vertices in total. Hence, Lemma 5.3 is applicable and we have whp

$$W[\mathcal{B}_0] \stackrel{L.5.3}{\geq} np_0/2 \gg \sqrt{np_0} \stackrel{(52)}{\geq} \phi_0 \geq w_{\max}[\mathcal{B}_0],$$

since only vertices of weight at most ϕ_0 are infected initially. On the other hand, we have

$$\text{Var}[W[\mathcal{B}_0]] = \sum_{u \in V_{< \phi_0}} (w_u^2 p_0 - (w_u p_0)^2) \leq \sum_{u \in V_{< \phi_0}} w_u^2 p_0 \leq \phi_0 p_0 W.$$

Theorem 4.9 implies

$$\mathbb{P}[W[\mathcal{B}_0] \geq (\lambda + 1)np_0] \leq \exp\left(-\frac{(np_0)^2}{3\phi_0 p_0 W}\right) \stackrel{(52)}{\leq} \exp(-\sqrt{np_0}/(3\lambda)) \stackrel{(50)}{=} o(1),$$

that is, the claimed upper bound on $W[\mathcal{B}_0]$ holds whp. \square

The crucial observation for the dense process is the following consequence of Theorem 4.6, asserting that once a substantial number of heavy vertices become infected, all of them will be infected eventually.

LEMMA 5.14. *Assume that there is a function $a = a(n)$ such that $a \rightarrow \infty$ for which $|\mathcal{B}_1 \cap V_{\geq \phi_H}| \geq a$. Conditional on this, whp there exists a time $T \geq 1$ such that $V_{\geq \phi_H} \subseteq \mathcal{A}_T$.*

PROOF. Consider the random graph induced by the vertex set $V_{\geq \phi_H}$, and note that it stochastically dominates the binomial random graph $G(n', p')$, with

$$n' := |V_{\geq \phi_H}| \geq |\mathcal{B}_1 \cap V_{\geq \phi_H}| \geq a \rightarrow \infty$$

and

$$p' := \phi_H^2 / W \stackrel{(3)}{\geq} |V_{\geq \phi_H}|^{-1/r} / 4 = (n')^{-1/r} / 4.$$

Now we consider bootstrap process with parameter r on $G(n', p')$ where the set of initially infected vertices is $\mathcal{B}_1 \cap V_{\geq \phi_H}$. So $|\mathcal{B}_1 \cap V_{\geq \phi_H}| \geq a \rightarrow \infty$. Hence, Theorem 4.6 is applicable. Since all vertices which become infected by this process within $\tau \geq 0$ steps are contained in $\mathcal{A}_{1+\tau}$, we obtain

$$\mathbb{P}[\exists T \geq 1: V_{\geq \phi_H} \subseteq \mathcal{A}_T] \rightarrow 1,$$

as $n' \rightarrow \infty$ and the claim follows since $n' \rightarrow \infty$ as $n \rightarrow \infty$. \square

Next, we show that a significant number of heavy vertices become infected whp if either one of the following additional assumptions holds:

$$(53) \quad (\lambda + 1)p_0w_n \leq (\log \bar{v}_d)^{1/2};$$

$$(54) \quad w_{n-r+1} \leq W^{1/2}(\log \bar{v}_d)^{1/16} \quad \wedge \quad \phi_H \leq W^{1/2}(\log \bar{v}_d)^{-1/8}.$$

Before we make this statement precise, we perform some calculations motivating the first assumption. It allows us to distinguish the two regimes of Lemma 5.5 (which is applicable at time $t = 0$ by Claim 5.13): if (53) holds, then since $W = \lambda n$, for some $\lambda \geq 1$ [cf. (2)] we have

$$(55) \quad W[\mathcal{B}_0] \stackrel{C.5.13}{\leq} (\lambda + 1)p_0W \stackrel{(53)}{\leq} (W/w_n)(\log \bar{v}_d)^{1/2},$$

and thus we also obtain

$$\begin{aligned} w_{\max}[\mathcal{B}_0] &\stackrel{(52)}{\leq} \sqrt{np_0} \stackrel{C.5.13}{\leq} \frac{2W[\mathcal{B}_0]}{\sqrt{np_0}} \stackrel{(50)}{\leq} \frac{2W[\mathcal{B}_0]\lambda^{1/2}}{\bar{v}_d^{1/2}} \stackrel{(55)}{\leq} \frac{2\lambda^{1/2}W(\log \bar{v}_d)^{1/2}}{w_n\bar{v}_d^{1/2}} \\ &\stackrel{(49)}{\leq} \frac{W}{4w_n}, \end{aligned}$$

for any n large enough. This together with Claim 5.13 implies that the first assertion of Lemma 5.5 holds (with $\xi = \bar{v}_d$) for all vertices $u \in V_{\geq \phi_H}$ (other vertices are not considered in the restricted process), that is, we have

$$\mathbb{P}[u \in \mathcal{B}_1 \mid \mathcal{B}_0] \geq \frac{w_u^r W[\mathcal{B}_0]^r}{r!W^r} \exp(-1.9\sqrt{\log \bar{v}_d}).$$

Because $W[\mathcal{B}_0]/W \geq p_0/(2\lambda) = \bar{v}_d p_d/(2\lambda)$, this implies

$$(56) \quad \mathbb{P}[u \in \mathcal{B}_1 \mid \mathcal{B}_0] \gg w_u^r p_d^r \stackrel{(49)}{=} \frac{w_u^r}{\sum_{v \in V_{\geq \phi_H}} w_v^r} \stackrel{(5)}{=} \dots$$

On the other hand, if (53) does not hold, that is, we have $(\lambda + 1)p_0w_n > (\log \bar{v}_d)^{1/2}$, then we obtain

$$W[\mathcal{B}_0] \stackrel{C.5.13}{\geq} np_0/2 > \frac{W(\log \bar{v}_d)^{1/2}}{2(\lambda + 1)\lambda w_n} \gg (W/w_n)(\log \bar{v}_d)^{1/4},$$

for any vertex u whose weight satisfies $w_u \gg w_n(\log \bar{v}_d)^{-1/4}$. Thus the second assertion of Lemma 5.5 (with $\xi = \bar{v}_d$) holds for any such vertex showing that

$$(57) \quad \mathbb{P}[u \in \mathcal{B}_1 \mid \mathcal{B}_0] \geq 1 - o(1/\sqrt{\log \bar{v}_d}).$$

Using these two observations, we prove the following result.

LEMMA 5.15. *If either (53) or (54) holds, then there exists a function $a = a(n)$ such that $a \rightarrow \infty$ for which whp we have $|\mathcal{B}_1 \cap V_{\geq \phi_H}| \geq a$.*

PROOF. The idea of this proof is to show that whp a significant number of heavy vertices becomes infected and, therefore, by Lemma 5.14 all heavy vertices become infected. The proof is split into two cases.

Case I. Assume that (53) holds and, therefore, also (56). Summing (56) over all $u \in V_{\geq \phi_H}$ we have that $\mathbb{E}[|\mathcal{B}_1 \cap V_{\geq \phi_H}| \mid \mathcal{B}_0] \gg 1$. Consequently, as $|\mathcal{B}_1 \cap V_{\geq \phi_H}|$ is the sum of independent Bernoulli random variables, the Chernoff bound (Theorem 4.8) yields that whp $|\mathcal{B}_1 \cap V_{\geq \phi_H}| \gg 1$, completing Case I.

Case II. Now assume that (54) holds, but (53) does not hold which implies that (57) holds. We consider vertices of weight at least $w' := \sqrt{W}(\log \bar{v}_d)^{-1/8}$, and observe that by (54) all of them are heavy, that is, $w' \geq \phi_H$. Moreover by the assumptions of Theorem 3.1 the number of these vertices satisfies

$$|V_{\geq w'}| \geq \frac{W\mathbb{P}[Z_{\mathbf{w}} \geq w']}{w_n} \geq \frac{\alpha CW}{w_{n-r+1}w'} \stackrel{(54)}{=} \Omega((\log \bar{v}_d)^{1/16}) \rightarrow \infty.$$

Now note that any such vertex u satisfies

$$w_u \geq w' \stackrel{(54)}{\geq} \alpha w_n (\log \bar{v}_d)^{-3/16} \gg w_n (\log \bar{v}_d)^{-1/4},$$

and thus by (57) it becomes infected at time $t = 1$ with probability at least $1 - o(1/\sqrt{\log \bar{v}_d})$. Applying a union bound to a sufficiently small but growing number of these vertices implies that there is a function $a \rightarrow \infty$ such that whp

$$|\mathcal{B}_1 \cap V_{\geq \phi_H}| \geq |\mathcal{B}_1 \cap V_{\geq w'}| \geq a,$$

completing the proof of the lemma. \square

With this preparation, we will now construct a nucleus (cf. Definition 5.1) for the dense process. We consider the *candidate nucleus* $V_{\geq \phi_K}$ given by the weight-bound

$$(58) \quad \phi_K := \begin{cases} W/w_{n-r+1} & \text{if } w_{n-r+1} > W^{1/2}(\log \bar{v}_d)^{1/16}, \\ \phi_H & \text{if (54) holds.} \end{cases}$$

Note that this does not always define a weight-bound ϕ_K ; however, we will demonstrate in Section 5.4 that this suffices to guarantee the existence of a nucleus.

THEOREM 5.16. *Under the assumptions of Theorem 3.1, suppose also that $p_0 \gg p_d$. Then whp $V_{\geq \phi_K}$ as defined in (58) is a nucleus with weight-bound ϕ_K such that $\phi_K \leq \min\{w_{n-r+1}, W/w_{n-r+1}\}$ and $\phi_K = o(\sqrt{W})$. Moreover, the nucleus is completely infected whp and no edges in $G[V_{\leq \phi_0}]$ have been exposed yet.*

PROOF. We start by showing that the weight-bound ϕ_K is sufficiently small. Recall that $\bar{v}_d \rightarrow \infty$ by (49). Now if $\phi_K = W/w_{n-r+1}$, then

$$\phi_K = W/w_{n-r+1} < W^{1/2}(\log \bar{v}_d)^{-1/16} = o(\sqrt{W}) = o(w_{n-r+1}),$$

and both conditions are satisfied. On the other hand, if $\phi_K = \phi_H$, then we have

$$\phi_K = \phi_H \leq W^{1/2}(\log \bar{v}_d)^{-1/8} \leq W^{1/2}(\log \bar{v}_d)^{-1/16} \leq W/w_{n-r+1}.$$

Furthermore, by (3) the number of heavy vertices satisfies

$$|V_{\geq \phi_H}| \geq \left(\frac{W}{4\phi_H^2}\right)^r \geq 4^{-r}(\log \bar{v}_d)^{r/4} \rightarrow \infty,$$

and thus, in particular, we also obtain

$$\phi_K = \phi_H \leq w_{n-r+1}.$$

Therefore, the weight-bound ϕ_K is sufficiently small in this case as well.

We will prove that whp all vertices of the candidate nucleus becomes infected eventually. Again, we first assume that $\phi_K = W/w_{n-r+1}$. In this case, we start by showing that there is a $T \geq 0$ at which whp the r vertices of largest weight become infected, that is,

$$(59) \quad \{n - r + 1, \dots, n\} \subseteq \mathcal{A}_T.$$

If (53) holds, then Lemma 5.15 implies in particular that the number of heavy vertices is $\omega(1)$ and by Lemma 5.14 they all become infected by some time $T \geq 0$, hence (59) holds. On the other hand, if (53) does not hold, then (57) holds for each of the r vertices of largest weight (since $w_{n-r+1} = \alpha w_n$, for some constant $\alpha > 0$), and thus a union bound shows that whp $\{n - r + 1, \dots, n\} \subseteq \mathcal{B}_1 \subseteq \mathcal{A}_1$, that is, (59) holds.

Now consider any vertex $u \in V_{\geq W/w_{n-r+1}}$ and note that u is connected to each vertex in $\{n - r + 1, \dots, n\}$ with probability 1 [cf. (1)]. Consequently, we have $V_{\geq \phi_K} = V_{\geq W/w_{n-r+1}} \subseteq \mathcal{A}_{T+1}$ implying $\widehat{W}_{\geq \phi_K} = W_{\geq \phi_K}$, in other words $V_{\geq \phi_K}$ is a nucleus.

Now suppose that $\phi_K = \phi_H$, that is, (54) holds. Then Lemmas 5.15 and 5.14 imply that whp all heavy vertices become infected eventually, and thus $\widehat{W}_{\geq \phi_K} = W_{\geq \phi_K}$, completing the proof. \square

We conclude this section by proving that if we cannot apply Theorem 5.16, then the premises of Theorem 5.12 are met, justifying the previous case distinction.

CLAIM 5.17. *Assume that we have $w_{n-r+1} \leq \sqrt{W}(\log \bar{v}_d)^{1/16}$ and additionally $\phi_H > \sqrt{W}(\log \bar{v}_d)^{-1/8}$. Then we have*

$$p_d(\log \bar{v}_d)^{2r} \geq p_s,$$

and in particular

$$p_0 = \omega(p_s).$$

PROOF. First, set $w' := \sqrt{W}(\log \bar{v}_d)^{-1/8} < \phi_H$ and note that

$$(60) \quad |V_{\geq \phi_H}| \leq |V_{\geq w'}| \stackrel{(3)}{<} \left(\frac{W}{4(w')^2} \right)^r = 4^{-r} (\log \bar{v}_d)^{r/4},$$

and consequently

$$(61) \quad p_d \stackrel{(5)}{=} \left(\frac{1}{\sum_{u \in V_{\geq \phi_H}} w'_u} \right)^{1/r} \geq \left(\frac{1}{|V_{\geq \phi_H}| w'_n} \right)^{1/r} \stackrel{(60)}{=} \Omega(W^{-1/2} (\log \bar{v}_d)^{-5/16}),$$

since $w_{n-r+1} = \alpha w_n$, for some constant $\alpha > 0$ by the assumptions of Theorem 3.1. Therefore,

$$\begin{aligned} W_{\geq \phi_H} &\stackrel{(60)}{\leq} 4^{-r} (\log \bar{v}_d)^{r/4} w_n = O(\sqrt{W} (\log \bar{v}_d)^{r/4} (\log \bar{v}_d)^{1/16}) \\ &= o(\sqrt{W} (\log \bar{v}_d)^{r/2}) \end{aligned}$$

and thus $\mathbb{P}[Z_{\mathbf{w}} \geq \phi_H] = o((\log \bar{v}_d)^{r/2} / \sqrt{W})$. By Proposition 4.4, we have

$$\begin{aligned} \sum_{u \in V_{< \phi_H}} w_u^{r+1} &\geq rW \left(\int_{C_1}^{\phi_H} \mathbb{P}[\phi_H > Z_{\mathbf{w}} \geq \mu] \mu^{r-1} d\mu \right) \\ &\geq rW \left(\int_{C_1}^{w' / (\log \bar{v}_d)^{r/2}} \left(\frac{C}{\mu} - o\left(\frac{(\log \bar{v}_d)^{r/2}}{\sqrt{W}} \right) \right) \mu^{r-1} d\mu \right) \\ &\stackrel{w' = o(\sqrt{W})}{=} \Omega\left(\frac{W(w')^{r-1}}{(\log \bar{v}_d)^{(r-1)r/2}} \right). \end{aligned}$$

Combining (61) with this, we obtain the first statement. Moreover, this then implies

$$p_0 \stackrel{(49)}{=} \bar{v}_d p_d \geq \frac{\bar{v}_d}{(\log \bar{v}_d)^{2r}} p_s = \omega(p_s),$$

that is, the second statement holds as well. \square

5.4. *The existence of a nucleus: Proof of Theorem 5.2.* In the preceding sections, we gave several constructions for a nucleus, and we will now demonstrate how these results can be combined into the proof of Theorem 5.2. It is structured in three steps: at the beginning of each step, we check whether some condition is satisfied. If it is, then whp one of the preceding results guarantees the existence of a nucleus; if not, we gained some information and proceed to the next step.

PROOF OF THEOREM 5.2. We intend to use the subsubsequence principle [cf., e.g., Janson, Łuczak and Ruciński (2000)], which states that in order to show that a certain property holds whp, it suffices to show that for every sequence of natural numbers there exists a subsequence along which the property holds whp.

So let $N \subseteq \mathbb{N}$ be an arbitrary (infinite) sequence of natural numbers satisfying the conditions of Theorem 3.1. Then it must contain an (infinite) subsequence $N_0 \subseteq N$ satisfying either

$$(62) \quad \phi_H > w_n \quad \text{and} \quad p_0 \gg p_s,$$

or

$$(63) \quad \phi_H \leq w_n \quad \text{and} \quad p_0 \gg \min\{p_s, p_d\}.$$

We thus distinguish two cases.

Case I. If along N_0 , we have

$$\phi_H > w_n,$$

then by (62) we have $p_0 \gg p_s$, and furthermore Corollary 4.5 implies $p_s = o(1)$. Therefore, we consider the sparse process and note that Theorem 5.12 is applicable, proving that whp there exists a nucleus $V_{\geq \phi_K}$. In this case, we call N_0 *sparse*.

Case II. Assume now that $\phi_H \leq w_n$ along N_0 and, therefore, also $p_0 \gg \min\{p_s, p_d\}$ by (63). Similarly as before, Corollary 4.5 provides $\min\{p_s, p_d\} = o(1)$. We split N_0 into two subsequences: let $N_1 = \{n \in N_0 : p_d \geq p_s\}$ and $N_2 = \{n \in N_0 : p_d < p_s\} = N_0 \setminus N_1$.

If N_1 is infinite, then we have

$$p_0 \gg p_s$$

along N_1 . In this case, we consider the sparse process and observe that since $p_s = o(1)$ Theorem 5.12 is applicable, proving that whp there exists a nucleus $V_{\geq \phi_K}$. Once again, we call N_0 *sparse*.

Otherwise, N_2 must be infinite, and we have

$$(64) \quad p_0 \gg p_d$$

along N_2 . Now, if

$$w_{n-r+1} > \sqrt{W}(\log \bar{v}_d)^{1/16},$$

along an infinite subsequence of N_2 , then we consider the dense process. It follows from Theorem 5.16 that along this subsequence whp there exists a nucleus $V_{\geq \phi_K}$. In this case, we call N_0 *dense*.

Thus, we may assume that there exists an infinite subsequence N_3 of N_2 along which we have $w_{n-r+1} \leq \sqrt{W}(\log \bar{v}_d)^{1/16}$. We consider two more cases. If there exists an infinite subsequence of N_3 along which we have

$$\phi_H \leq \sqrt{W}(\log \bar{v}_d)^{-1/8},$$

then we also consider the dense process and apply Theorem 5.16, thus showing that whp there exists a nucleus $V_{\geq \phi_K}$. Again we call N_0 *dense*.

Therefore, we finally assume that N_3 contains an infinite subsequence N_4 along which $\phi_H > \sqrt{W}(\log \bar{v}_d)^{-1/8}$ is satisfied. But this tells us that the requirements of Claim 5.17 are met along N_4 . Hence, it would imply that over N_4 we also have $p_0 \gg p_s$. Thereby, Theorem 5.12 is again applicable, proving that whp there exists a nucleus $V_{\geq \phi_K}$. In this final case, we call N_0 *sparse*.

By the subsubsequence principle, the statement of Theorem 5.2 holds along \mathbb{N} . □

Inspired by the case distinction in the above proof of Theorem 5.2, we define another partition of the set all infinite sequences of natural numbers, which we will use in Section 6.

DEFINITION 5.18. Let $\mathcal{N}_s := \{N \subseteq \mathbb{N} : |N| = \infty \text{ and } N \text{ is sparse}\}$ and $\mathcal{N}_d := \{N \subseteq \mathbb{N} : |N| = \infty \text{ and } N \text{ is dense}\}$.

6. Outbreak. In the previous section, we proved that in the supercritical regime there exists a nucleus which gets infected almost entirely. In this section, we show that once this happens, whp it also causes an outbreak.

THEOREM 6.1. *Suppose that the premises of Theorem 3.1 hold. If $p_0 \gg p_c$, then whp there is an outbreak, that is, we have*

$$|\mathcal{A}_F| = \Theta(n).$$

Again we use the subsubsequence principle to prove Theorem 6.1. So we fix an arbitrary infinite sequence of natural numbers N_0 and show that it contains a subsequence along which the probability of an outbreak is $1 - o(1)$. To this end, we use the distinction between N_0 being sparse or dense provided by Definition 5.18.

In both cases, the proof is based on partitioning the vertex set $V_{\geq C_1}$, where the constant $C_1 > 0$ is the one from Theorem 3.1, into several *layers*, each corresponding to a subinterval of $[C_1, \infty)$, and then showing that the infection spreads from layer to layer, starting from a completely or an almost completely infected nucleus.

More precisely, assume that most of the vertices in each of the, say i , heaviest layers are infected eventually. We prove that also most of the vertices in the $(i + 1)$ st layer must become infected eventually. Some of the details depend sensitively on the relation of various error-terms. We begin with the definitions of these in the next subsection.

6.1. Setup. Recall that by the assumptions of Theorem 3.1 there are constants $\alpha > 0$ and C satisfying

$$(65) \quad C \geq 64r(\min\{\alpha, 1/2\})^{-3}$$

and $C_1 > 0$ such that $w_{n-r+1} = \alpha w_n$ and for any $C_1 \leq x \leq w_n$ we have

$$(66) \quad \mathbb{P}[Z_{\mathbf{w}} \geq x] \geq \frac{C}{x}.$$

Next, abbreviate

$$(67) \quad C' := C \min\{\alpha, 1/2\}/2,$$

and use this constant to recursively define a nonincreasing sequence of layer weight-bounds.

More specifically, we set $\psi_1 := \min\{\phi_K, W/w_n\}$ and

$$(68) \quad \psi_{i+1} := \frac{C'}{\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i]}$$

for any $i \geq 1$ as long as

$$(69) \quad \psi_i \geq 2 \max\{C_1, \lambda\},$$

where $\lambda \geq 1$ is the constant in (2). We denote the maximal such i by i^* and observe that for each $1 \leq i \leq i^*$ we obtain

$$\psi_{i+1} \leq \psi_i \frac{C'}{C} < \psi_i.$$

This implies, in particular, that

$$(70) \quad \psi_{i+1} \leq \psi_1 \left(\frac{C'}{C}\right)^i.$$

Moreover, for convenience of notation we also set $\psi_0 := \phi_K$ and note that we have

$$(71) \quad \psi_0 \geq \psi_1 > \psi_2 > \dots > \psi_{i^*+1} > 0.$$

Furthermore, recall that the weight-bound ψ_0 satisfies

$$(72) \quad \psi_0 \leq \min\{w_{n-r+1}, W/w_{n-r+1}\} \quad \text{and} \quad \psi_0 = o(\sqrt{W}).$$

If N_0 is sparse, then we consider a subsequence N_s which witnesses this fact, that is, Theorem 5.12 is applicable along N_s . In this case, we set S' as in Section 5.2 (and recall that $S \subseteq S'$). On the other hand, if N_0 is dense, then we consider a subsequence N_d which witnesses this fact, that is, Theorem 5.16 is applicable along N_d , and define $S' := \emptyset$. For the remainder of this section, all asymptotic statements will be with respect to N_s or N_d , respectively.

Then for any $1 \leq i \leq i^*$ such that $\psi_i < \psi_{i-1}$ we call the set

$$(73) \quad \mathcal{S}_i := V_{[\psi_i, \psi_{i-1})} \setminus S'$$

the i th layer, and in case $\psi_i = \psi_{i-1}$ the i th layer is the empty set. We also set $\mathcal{S}_0 := V_{\geq \phi_K}$. We let $\mathcal{K}_i := \bigcup_{0 \leq j \leq i} \mathcal{S}_j$ be the union of the first $i + 1$ layers and let $W^{(i)} := W[\mathcal{K}_i]$ be its total weight.

The following claim will be useful later in our argument.

CLAIM 6.2. For any $0 \leq i \leq i^*$, we have

$$W^{(i)} \geq \frac{1}{4}W[V_{\geq \psi_i}].$$

PROOF. For $N_0 \in \mathcal{N}_d$, we have $S' = \emptyset$, whereby $W^{(i)} = W[V_{\geq \psi_i}]$, and hence the statement holds trivially.

So now assume $N_0 \in \mathcal{N}_s$. Recall that we consider all vertices in V to be ordered by weight. By the definition of S' , the set \mathcal{K}_i contains at least every other vertex of $V_{\geq \psi_i}$ together with all vertices in $V_{\geq \phi_K}$. Therefore, for every vertex in $\mathcal{K}_i \setminus \mathcal{K}_0$ its preceding vertex could belong to S' . Thus, roughly speaking, $W^{(i)}$ is approximately at least half of $W[V_{\geq \psi_i}]$. More precisely, by excluding either the last vertex or the last two vertices of $V_{[\psi_i, \phi_K)}$ (if the penultimate vertex is not contained in \mathcal{K}_i), we obtain the lower bound

$$W^{(i)} \geq \frac{1}{2}(W[V_{\geq \psi_i}] - 2\phi_K) = \frac{1}{2}(W\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i] - 2\phi_K),$$

as the excluded vertices have weight at most ϕ_K . Now, we would like to show that

$$W\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i] - 2\phi_K \geq \frac{1}{2}W\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i]$$

or, equivalently,

$$(74) \quad \frac{1}{2}W\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i] \geq 2\phi_K.$$

Recall that by (66) we have

$$\frac{1}{2}W\mathbb{P}[Z_{\mathbf{w}} \geq \psi_i] \geq \frac{1}{2} \frac{CW}{\psi_i}.$$

Furthermore, (71) implies $\psi_i \leq \phi_K$ and by Theorem 5.12 we have $\phi_K = o(\sqrt{W})$. Hence, $\phi_K = o(W/\psi_i)$ and inequality (74) follows. \square

Now, for some subset $U \subset V_{\geq \phi_K}$, we consider the restricted process \mathcal{A}_t^U on $G[\mathcal{K}_{i^*}]$ with initially infected set $\mathcal{A}_0^U := U$. In fact, we will use either $U = V_{\geq \phi_K}$ (for $N_0 \in \mathcal{N}_d$) or $U = \mathcal{U}_K$ (for $N_0 \in \mathcal{N}_s$).

Finally, for any $i \geq 0$ we introduce two error-terms: we set

$$\delta_i := \frac{1}{4} \left(\frac{C'}{C}\right)^i \quad \text{and} \quad \varepsilon_i := \sum_{j=0}^i \delta_j$$

and note that this implies

$$(75) \quad \begin{aligned} \frac{1}{4} = \varepsilon_0 &\leq \varepsilon_1 \leq \dots \leq \varepsilon_{i^*+1} \leq \sum_{j=0}^{\infty} \delta_j \\ &= \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{C'}{C}\right)^j = \frac{1}{4} \cdot \frac{1}{1 - C'/C} \stackrel{(67)}{\leq} 1/2, \end{aligned}$$

and also, for any $1 \leq i < i^*$, the following estimate (which will be used later):

$$(76) \quad \delta_{i+1} \psi_i^{-2} \stackrel{(70)}{\geq} \frac{1}{4} \psi_1^{-2} \left(\frac{C'}{C}\right)^{3-i} \stackrel{(67)}{\geq} 2^{i-7} \psi_1^{-2} \alpha^2.$$

Using these we define, for any $0 \leq i \leq i^*$, the event

$$\mathcal{W}_i := \{W[\mathcal{A}_i^U \cap \mathcal{K}_i] \geq (1 - \varepsilon_i) W^{(i)}\},$$

that is, \mathcal{W}_i asserts that the union of the nucleus and the first i layers got infected almost entirely in the first i steps. Hence, the goal will be to prove that \mathcal{W}_{i^*} holds whp. We do so inductively in Section 6.2. Depending on the total weight of a layer (in comparison to the union of the nucleus and the previous layers) the induction steps can be straightforward or rather complicated.

More precisely, for any $1 \leq i \leq i^*$ we say that the i th layer has property \mathcal{L}_i if

$$W^{(i)} \geq (1 + \delta_i) W^{(i-1)}.$$

We will prove in Section 6.2 that if this property holds, then with sufficiently high probability a large enough fraction of the total weight in the i th layer becomes infected eventually.

6.2. *Infecting large layers.* First, we show that if the $(i + 1)$ st layer has property \mathcal{L}_{i+1} , then conditional on \mathcal{W}_i the expected number of eventually infected neighbours of a vertex in the i th layer is not too small. More precisely, for $0 \leq i < i^*$ and a vertex $v \in \mathcal{S}_{i+1}$ we denote the number of its neighbours in $\mathcal{A}_i^U \cap \mathcal{K}_i$ by

$$\hat{d}^{(i)}(v) := \sum_{u \in \mathcal{A}_i^U \cap \mathcal{K}_i} \mathbb{1}_{\{u,v\} \in E}.$$

CLAIM 6.3. *Let $0 \leq i < i^*$. If \mathcal{L}_{i+1} holds, then for every $v \in \mathcal{S}_{i+1} \setminus \mathcal{B}_0$ we have $\mathbb{E}[\hat{d}^{(i)}(v) \mid \mathcal{W}_i] \geq (1 - \varepsilon_i) C' / 4$.*

PROOF. Let $0 \leq i < i^*$ and let $v \in \mathcal{S}_{i+1} \setminus \mathcal{B}_0$. Assume that the event \mathcal{W}_i has been realised. Observe that each one of the indicator random variables satisfies

$$\mathbb{P}[\mathbb{1}_{\{u,v\} \in E} = 1 \mid \mathcal{W}_i] = \min\left\{\frac{w_v w_u}{W}, 1\right\} \geq \min\left\{\frac{\psi_{i+1} w_u}{W}, 1\right\} = \frac{\psi_{i+1} w_u}{W}$$

since $\psi_{i+1} \leq \psi_1 \leq W/w_n$. Thus, we have

$$(77) \quad \begin{aligned} \mathbb{E}[\hat{d}^{(i)}(v) \mid \mathcal{W}_i] &\geq \sum_{u \in \mathcal{A}_i^U \cap \mathcal{K}_i} \frac{\psi_{i+1} w_u}{W} = \frac{\psi_{i+1} W[\mathcal{A}_i^U \cap \mathcal{K}_i]}{W} \\ &\stackrel{\mathcal{W}_i}{\geq} \frac{(1 - \varepsilon_i) \psi_{i+1} W^{(i)}}{W}. \end{aligned}$$

By Claim 6.2, we have

$$(78) \quad W^{(i)} \geq \frac{1}{4} W \mathbb{P}[Z_{\mathbf{w}} \geq \psi_i].$$

This implies that

$$(79) \quad \mathbb{E}[\hat{d}^{(i)}(v) \mid \mathcal{W}_i] \geq (1 - \varepsilon_i) \frac{1}{4} \psi_{i+1} \mathbb{P}[Z_{\mathbf{w}} \geq \psi_i].$$

Now, for any $i \geq 1$ the claim follows directly by plugging (68) into the right-hand side of (79).

Thus, it remains to consider the case $i = 0$. First, observe that $W/w_n \geq \psi_0$ would imply $\psi_1 = \psi_0$, and thus the first layer would be empty contradicting property \mathcal{L}_1 . Hence, we may also assume that $\psi_1 = W/w_n$, and consequently

$$\mathbb{E}[\hat{d}^{(0)}(v) \mid \mathcal{W}_0] \geq (1 - \varepsilon_0) W \mathbb{P}[Z_{\mathbf{w}} \geq \psi_0]/w_n \stackrel{(66)}{\geq} (1 - \varepsilon_0) \frac{CW}{\psi_0 w_n}.$$

But recall that $\psi_0 = \phi_K \leq W/w_{n-r+1} = W/(\alpha w_n)$, which, in turn, implies that

$$\mathbb{E}[\hat{d}^{(0)}(v) \mid \mathcal{W}_0] \geq (1 - \varepsilon_0) \alpha C \stackrel{(67)}{\geq} (1 - \varepsilon_0) C',$$

proving the claim also in this case, completing the proof. \square

Claim 6.3 allows us to compute a suitable lower bound on the expected total infected weight within the $(i + 1)$ st layer. More precisely, for $0 \leq i < i^*$ and a vertex $v \in \mathcal{S}_{i+1}$ we consider the random variables

$$\hat{X}_{i,v} := w_v \mathbb{1}_{\{\hat{d}^{(i)}(v) \geq r\}}$$

and note that these satisfy $0 \leq \hat{X}_{i,v} \leq w_v$. Note that

$$W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] = \sum_{v \in \mathcal{S}_{i+1} \setminus \mathcal{B}_0} \hat{X}_{i,v} + \sum_{v \in \mathcal{S}_{i+1} \cap \mathcal{B}_0} w_v.$$

For each $v \in \mathcal{S}_{i+1} \cap \mathcal{B}_0$ we define the random variable $\hat{X}_{i,v} = w_v I_v$, where I_v is the indicator random variable such that $\mathbb{P}[I_v = 1] = \mathbb{P}[\sum_{u \in \mathcal{A}_i^U \cap \mathcal{K}_i} I_{u,v} \geq r]$, where $\{I_{u,v}\}_{v \in \mathcal{S}_{i+1}, u \in \mathcal{K}_i}$ is a collection of independent indicator random variables with $I_{u,v} = 1$ with probability given by (1).

Hence, setting

$$\hat{X}_{i+1} := \sum_{v \in \mathcal{S}_{i+1}} \hat{X}_{i,v}$$

conditional on \mathcal{W}_i the random variable $W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}]$ stochastically dominates \hat{X}_{i+1} . In the next two lemmas, we will use the stochastic domination in order to deduce that whp $W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}]$ is a significant proportion of $W[\mathcal{S}_{i+1}]$.

CLAIM 6.4. For any $0 \leq i < i^*$, we have $\mathbb{E}[\widehat{X}_{i+1} \mid \mathcal{W}_i] \geq (1 - \varepsilon_i^2)W[\mathcal{S}_{i+1}]$.

PROOF. First, consider a vertex $v \in \mathcal{S}_{i+1} \setminus \mathcal{B}_0$. Since $\widehat{d}^{(i)}(v)$ is a sum of independent indicator random variables, we have

$$\text{Var}[\widehat{d}^{(i)}(v) \mid \mathcal{W}_i] \leq \mathbb{E}[\widehat{d}^{(i)}(v) \mid \mathcal{W}_i].$$

Consequently, we obtain from Chebyshev’s inequality that

$$\begin{aligned} \mathbb{P}[\widehat{d}^{(i)}(v) < r \mid \mathcal{W}_i] &\leq \frac{\mathbb{E}[\widehat{d}^{(i)}(v) \mid \mathcal{W}_i]}{(\mathbb{E}[\widehat{d}^{(i)}(v) \mid \mathcal{W}_i] - r)^2} \\ (80) \qquad &\stackrel{\text{c.6.3}}{\leq} \frac{4}{(1 - \varepsilon_i)C'} \frac{1}{(1 - 4r/((1 - \varepsilon_i)C'))^2} \\ &\stackrel{(75)}{\leq} \frac{8}{C'} \frac{1}{(1 - 8r/C')^2}. \end{aligned}$$

Now observe that from (67) and (65) we obtain the following lower bounds:

$$C' \geq 64r(\min\{\alpha, 1/2\})^{-2} \geq 2^7r.$$

The second inequality implies $(1 - (8r/C'))^{-2} \leq 2$ and also the following upper bound on the right-hand side in (80)

$$\frac{16}{C'} \leq \frac{1}{8r} \stackrel{r \geq 2}{\leq} \varepsilon_0^2.$$

Hence, since $\varepsilon_0 < \varepsilon_i$, for $i > 0$, we have $\frac{16}{C'} \leq \varepsilon_i^2$. We therefore obtain

$$\mathbb{P}[\widehat{d}^{(i)}(v) \geq r \mid \mathcal{W}_i] \geq 1 - \varepsilon_i^2,$$

for any $0 \leq i < i^*$. The random variable $\sum_{u \in \mathcal{A}_i^U \cap \mathcal{K}_i} I_{u,v}$ also satisfies Claim 6.3 and, therefore, the above argument also holds there. By summing up over all vertices $v \in \mathcal{S}_{i+1}$, the statement follows. \square

Next, we extend Claim 6.4 and show that the probability that the total infected weight in the $(i + 1)$ st layer is not large enough is sufficiently small.

LEMMA 6.5. For any $0 \leq i < i^*$ for which property \mathcal{L}_{i+1} holds, we have

$$\mathbb{P}[W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] \leq (1 - \varepsilon_i)W[\mathcal{S}_{i+1}] \mid \mathcal{W}_i] \leq \exp(-2^{i-6}W\psi_1^{-2}).$$

PROOF. Let $0 \leq i < i^*$ and recall that \widehat{X}_{i+1} is a sum of independent random variables $\widehat{X}_{i,v}$ satisfying $0 \leq \widehat{X}_{i,v} \leq w_v$ for $v \in \mathcal{S}_{i+1}$. Moreover, note that Claim 6.4 implies

$$(1 - \varepsilon_i)W[\mathcal{S}_{i+1}] \leq \mathbb{E}[\widehat{X}_{i+1} \mid \mathcal{W}_i] - \varepsilon_i(1 - \varepsilon_i)W[\mathcal{S}_{i+1}].$$

Therefore, using the above stochastic domination and applying the Azuma–Hoeffding inequality (Theorem 4.10) to \widehat{X}_{i+1} we obtain

$$(81) \quad \begin{aligned} \mathbb{P}[W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] \leq (1 - \varepsilon_i)W[\mathcal{S}_{i+1}] \mid \mathcal{W}_i] \\ \leq \mathbb{P}[\widehat{X}_{i+1} \leq (1 - \varepsilon_i)W[\mathcal{S}_{i+1}] \mid \mathcal{W}_i] \leq \exp\left(-\frac{\varepsilon_i^2(1 - \varepsilon_i)^2}{2} \frac{W[\mathcal{S}_{i+1}]^2}{\sum_{u \in \mathcal{S}_{i+1}} w_u^2}\right). \end{aligned}$$

We proceed by bounding the argument of the exponential function on the right-hand side from below by splitting it into three factors. First, note that we have

$$\frac{W[\mathcal{S}_{i+1}]}{\sum_{u \in \mathcal{S}_{i+1}} w_u^2} \geq \psi_i^{-1}$$

and also

$$W[\mathcal{S}_{i+1}] = W^{(i+1)} - W^{(i)} \stackrel{\mathcal{L}_{i+1}}{\geq} \delta_{i+1}W^{(i)} \stackrel{C.6.2, (66)}{\geq} \frac{\delta_{i+1}}{4}\psi_i^{-1}CW.$$

Multiplying these two factors we obtain

$$(82) \quad \frac{W[\mathcal{S}_{i+1}]^2}{\sum_{u \in \mathcal{S}_{i+1}} w_u^2} \geq \frac{\delta_{i+1}}{4}\psi_i^{-2}CW \stackrel{(76)}{\geq} 2^{i-9}\psi_1^{-2}CW\alpha^2 \geq 2^{i+1}\psi_1^{-2}W\alpha^2,$$

because $C \geq 2^{10}$ by (65). Consequently, it remains to bound the last factor

$$\varepsilon_i^2(1 - \varepsilon_i)^2/2 \stackrel{(75)}{\geq} 2^{-7},$$

and since $W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] \geq \widehat{X}_{i+1}$ this yields

$$\mathbb{P}[W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] \leq (1 - \varepsilon_i)W[\mathcal{S}_{i+1}] \mid \mathcal{W}_i] \stackrel{(81), (82)}{\leq} \exp(-2^{i-6}\alpha^2W\psi_1^{-2}),$$

as desired. \square

We will use the above construction to show inductively the following.

LEMMA 6.6. *If $U \subseteq V_{\geq \phi_K}$ satisfies*

$$(83) \quad W[U] \geq (1 - o(1))W_{\geq \psi_0},$$

then \mathcal{W}_{i^} holds whp.*

PROOF. We proceed by induction on i and show that \mathcal{W}_i holds for all $0 \leq i \leq i^*$ with sufficiently high probability. The statement for the base case $i = 0$ holds since $V_{\geq \phi_K}$ is a nucleus, and thus \mathcal{W}_0 holds (with probability 1) by (83).

So now assume that $0 \leq i < i^*$ and \mathcal{W}_i holds. Additionally, suppose that

$$W^{(i+1)} < (1 + \delta_{i+1})W^{(i)},$$

that is, the property \mathcal{L}_{i+1} does not hold. Then deterministically we have

$$\begin{aligned} W[\mathcal{A}_{i+1}^U \cap \mathcal{K}_{i+1}] &\stackrel{(71)}{\geq} W[\mathcal{A}_i^U \cap \mathcal{K}_i] \stackrel{\mathcal{W}_i}{\geq} (1 - \varepsilon_i)W^{(i)} \stackrel{\neg \mathcal{L}_{i+1}}{>} \frac{1 - \varepsilon_i}{1 + \delta_{i+1}} W^{(i+1)} \\ &> (1 - \varepsilon_{i+1})W^{(i+1)}, \end{aligned}$$

implying \mathcal{W}_{i+1} .

Otherwise, property \mathcal{L}_{i+1} holds, and so Lemma 6.5 is applicable, showing that

$$W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] > (1 - \varepsilon_i)W[\mathcal{S}_{i+1}]$$

with (conditional) probability at least $1 - \exp(-2^{i-6}\alpha^2 W \psi_1^{-2})$. However, this implies

$$\begin{aligned} W[\mathcal{A}_{i+1}^U \cap \mathcal{K}_{i+1}] &= W[\mathcal{A}_{i+1}^U \cap \mathcal{S}_{i+1}] + W[\mathcal{A}_i^U \cap \mathcal{K}_i] > (1 - \varepsilon_i)(W[\mathcal{S}_{i+1}] + W^{(i)}) \\ &\geq (1 - \varepsilon_{i+1})W^{(i+1)}, \end{aligned}$$

and \mathcal{W}_{i+1} follows since $\varepsilon_{i+1} \geq \varepsilon_i$.

Therefore, \mathcal{W}_{i^*} holds with probability at least

$$(1 - o(1)) \prod_{i=0}^{i^*-1} (1 - \exp(-2^{i-6}\alpha^2 W \psi_1^{-2})) \geq 1 - o(1) - \sum_{i=1}^{\infty} \exp(-W \psi_1^{-2} 2^{i-6}\alpha^2),$$

by a union bound. Because $\psi_1 \leq \psi_0 = \phi_K = o(\sqrt{W})$, by (72), the right-hand side is $1 - o(1)$, that is, whp we have $W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}] \geq (1 - \varepsilon_{i^*})W^{(i^*)}$. \square

REMARK 6.7. Observe that in the proofs of Claims 6.3 and 6.4 and Lemmas 6.5 and 6.6 we only exposed edge-indicator random variables corresponding to edges in the set $\mathcal{K}_{i^*} \times \bigcup_{i=1}^{i^*} (\mathcal{S}_i \setminus \mathcal{B}_0)$.

Now, observe that we have the (deterministic) lower bound

$$\begin{aligned} (84) \quad \frac{1}{2} W^{(i^*)} &\stackrel{C.6.2}{\geq} \frac{1}{8} W[V_{\geq \psi_{i^*}}] = \frac{W}{8} \mathbb{P}[Z_{\mathbf{w}} \geq \psi_{i^*}] = \frac{W}{8} \frac{C'}{\psi_{i^*+1}} \\ &\stackrel{(68)}{\geq} \frac{C'W}{16 \max\{C_1, \lambda\}}. \end{aligned}$$

In other words, we have already proven that the total weight of all eventually infected vertices is at least a constant fraction of the total weight W . It remains to show that this guarantees that whp a constant fraction of all vertices become infected eventually, that is, there is an outbreak.

6.3. *Witnessing the outbreak.* To witness the outbreak, we consider the vertices in $V_{<\psi_{i^*}} \setminus \mathcal{B}_0$ of which there are at least

$$(85) \quad n - |V_{\geq\psi_{i^*}}| - |\mathcal{B}_0| \geq n - \frac{W}{\psi_{i^*}} - o(n) \stackrel{(69),(2)}{\geq} \left(1 - \frac{\lambda}{2 \max\{C_1, \lambda, 1\}} - o(1)\right)n \geq \frac{1}{3}n.$$

For any vertex $u \in V_{<\psi_{i^*}} \setminus \mathcal{B}_0$, we will consider the random variables

$$\widehat{Y}_u := \mathbb{1}_{\{\widehat{d}^{(i^*)}(u) \geq r\}}$$

and denote their sum by

$$(86) \quad \widehat{Y} := \sum_{u \in V_{<\psi_{i^*}} \setminus \mathcal{B}_0} \widehat{Y}_u.$$

LEMMA 6.8. *There exists a constant $\gamma > 0$ such that for any $U \subseteq V_{\geq\phi_K}$ satisfying (83) and such that $\{n - r + 1, \dots, n\} \subseteq U$, conditional on \mathcal{W}_{i^*} , whp we have*

$$\widehat{Y} > \gamma n.$$

PROOF. As the \widehat{Y}_u s are independent, we will deduce this by applying the Chernoff bound (Theorem 4.8).

First of all, observe that if we replace u by a vertex u_0 of weight $w_{u_0} := 1 \leq w_u$, then we have

$$(87) \quad \mathbb{P}[\widehat{Y}_u = 1 \mid \mathcal{W}_{i^*}] \geq \mathbb{P}[\widehat{d}^{(i^*)}(u) \geq r \mid \mathcal{W}_{i^*}] \geq \mathbb{P}[\widehat{d}^{(i^*)}(u_0) = r \mid \mathcal{W}_{i^*}].$$

Because $w_{u_0} = 1$ and also

$$(88) \quad w_n \stackrel{(85), w_1 \geq 1}{\leq} W - n/3 \stackrel{(2)}{=} (1 - 1/(3\lambda))W,$$

we can drop the minimum in (1), and thus the above probability can be computed as

$$(89) \quad \mathbb{P}[\widehat{d}^{(i^*)}(u_0) = r \mid \mathcal{W}_{i^*}] = \sum_{\mathcal{R} \in (\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*})_r} \prod_{v \in \mathcal{R}} \frac{w_v}{W} \prod_{v' \in (\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}) \setminus \mathcal{R}} \left(1 - \frac{w_{v'}}{W}\right).$$

Because $1 - x \geq \exp(-x/(1 - x))$, for any $x < 1$, for the innermost product we obtain

$$(90) \quad \prod_{v' \in (\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}) \setminus \mathcal{R}} \left(1 - \frac{w_{v'}}{W}\right) \geq \exp\left(-\frac{\sum_{v' \in \mathcal{K}_{i^*}} w_{v'}}{W(1 - \frac{w_n}{W})}\right) \geq \exp\left(-1/\left(1 - \frac{w_n}{W}\right)\right) \stackrel{(88)}{\geq} \exp(-3\lambda),$$

independently of u . Moreover, we have

$$(91) \quad \sum_{\mathcal{R} \in \binom{\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}}{r}} \prod_{v \in \mathcal{R}} w_v \geq \left((\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*})^r - (\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*})^{r-2} \sum_{v \in \mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}} w_v^2 \right) / r!$$

since the sum on the left-hand side ranges only over monomials corresponding to r distinct vertices. Furthermore, we have

$$(92) \quad \sum_{v \in \mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}} w_v^2 \leq w_n W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}],$$

and thus in order for the right-hand side of (89) to be at least a (small) positive constant, it suffices to show that

$$(93) \quad w_n \leq (1 - \eta) W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}],$$

for $\eta := 1 - 1/(1 + (r - 1)\alpha)$ (and note that $0 < \eta < 1$).

We distinguish two cases, first assume that

$$W^{(0)} - W[\{n - r + 1, \dots, n\}] < w_{n-r+1}/2.$$

Then we observe that $w_{n-r+1} \geq \psi_0 \geq \psi_{i^*}$ by (71) and (72). Since $w_n \geq \dots \geq w_{n-r+1} \geq \psi_0$ and the r vertices of largest weight are infected, that is, $\{n - r + 1, \dots, n\} \subseteq U$, we obtain

$$W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}] \geq W[U] \geq W[\{n - r + 1, \dots, n\}] \geq (1 + (r - 1)\alpha)w_n,$$

implying (93).

Otherwise, we have

$$W^{(0)} \geq W[\{n - r + 1, \dots, n\}] + w_{n-r+1}/2 \geq (1 + (r - 1/2)\alpha)w_n,$$

and (83) implies

$$W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}] \geq W[U] \geq (1 - o(1))(1 + (r - 1/2)\alpha)w_n \geq (1 + (r - 1)\alpha)w_n,$$

for any sufficiently large n . Hence, (93) also holds in this case.

Combining the bounds (90), (91), (92) and (93), it follows from (87) and (89) that

$$\begin{aligned} \mathbb{P}[\widehat{Y}_u = 1 \mid \mathcal{W}_{i^*}] &\geq \exp(-3\lambda)\eta \frac{(W[\mathcal{A}_{i^*}^U \cap \mathcal{K}_{i^*}])^r}{r!W^r} \\ &\stackrel{(84)}{\geq} \exp(-3\lambda)\eta \left(\frac{C'}{16 \max\{C_1, \lambda\}} \right)^r / r!, \end{aligned}$$

where we define the right-hand side to be 4γ , which is positive and independent of U . Thus, by (85) and (86) we have

$$\frac{4\gamma n}{3} \leq \mathbb{E}[\widehat{Y} \mid \mathcal{W}_{i^*}] \leq n,$$

and hence applying the Chernoff bound (Theorem 4.8) on \widehat{Y} yields

$$\mathbb{P}[\widehat{Y} \leq \gamma n \mid \mathcal{W}_{i^*}] \leq \exp\left(-\frac{(\gamma n/3)^2}{2n}\right) = o(1). \quad \square$$

REMARK 6.9. Observe that in the proof of Lemma 6.8 we only exposed edge-indicator random variables corresponding to edges in $(V_{<\psi_{i^*}} \setminus \mathcal{B}_0) \times \mathcal{K}_{i^*}$.

Now we combine the above two lemmas and prove Theorem 6.1

PROOF OF THEOREM 6.1. Let \mathcal{E}_d be the event of Theorem 5.16, and \mathcal{E}_s denote the event of Theorem 5.12. For a given set $U \subseteq V_{\geq\phi_K}$ satisfying (83), let \mathcal{E}_U be the event that the random variable $\widehat{Y} > \gamma n$, where $\gamma > 0$ is as in Lemma 6.8. Now Lemmas 6.6 and 6.8 imply that if U satisfies (83) and $\{n-r+1, \dots, n\} \subseteq U$, then $\mathbb{P}[\mathcal{E}_U] = 1 - o(1)$.

Recall that we are aiming to use the subsubsequence principle. Hence, consider an infinite sequence $N_0 \subseteq \mathbb{N}$.

If $N_0 \in \mathcal{N}_d$, then whp (over a subsequence of N_0) the event \mathcal{E}_d is realised, that is, we have complete infection of the nucleus $V_{\geq\phi_K}$. Using $U = V_{\geq\phi_K}$ and $\{w_{n-r+1}, \dots, w_n\} \subseteq U$, a union bound implies

$$\mathbb{P}[\mathcal{E}_d \cap \mathcal{E}_{V_{\geq\phi_K}}] = 1 - o(1).$$

The definition of \widehat{Y} implies that if the event $\mathcal{E}_d \cap \mathcal{E}_{V_{\geq\phi_K}}$ is realised, then every vertex contributing to \widehat{Y} will eventually be infected, and thus we have $|\mathcal{A}_F| \geq \widehat{Y} > \gamma n$, in other words, there is an outbreak.

On the other hand, let $N_0 \in \mathcal{N}_s$. Then whp (over a subsequence of N_0) the event \mathcal{E}_s is realised, that is, we have shown almost complete infection of the nucleus $V_{\geq\phi_K}$ through the breeding ground $S \subseteq S'$. Let $\mathcal{U}_K \subseteq V_{\geq\phi_K}$ be the random subset of $V_{\geq\phi_K}$ as in Theorem 5.12.

Next, we observe that the sets $\mathcal{B}_0 \cup S' \cup \mathcal{K}_0$ and $(\bigcup_{i=1}^{i^*} \mathcal{S}_i) \cup V_{<\psi_{i^*}} \setminus \mathcal{B}_0$ are disjoint. Moreover, note that $\{\mathcal{U}_K = U\}$ depends only on edges in $S' \times (\mathcal{B}_0 \cup S' \cup \mathcal{K}_0)$, whereas \mathcal{E}_U depends on edges in $\mathcal{K}_{i^*} \times ((\bigcup_{i=1}^{i^*} \mathcal{S}_i) \cup V_{<\psi_{i^*}}) \setminus \mathcal{B}_0$ (cf. Remarks 6.7 and 6.9). Therefore, the two events are independent. Thus, we have

$$\begin{aligned} \mathbb{P}[\mathcal{E}_s \cap \mathcal{E}_{\mathcal{U}_K}] &\geq \sum_{\substack{U \subseteq V_{\geq\phi_K}: \\ W[U] \geq \frac{1}{2} W_{\geq\psi_0}}} \mathbb{P}[\mathcal{E}_U \cap \{\mathcal{U}_K = U\}] \\ &= \sum_{\substack{U \subseteq V_{\geq\phi_K}: \\ W[U] \geq \frac{1}{2} W_{\geq\psi_0}}} \mathbb{P}[\mathcal{E}_U] \mathbb{P}[\mathcal{U}_K = U] \\ &= (1 - o(1)) \sum_{\substack{U \subseteq V_{\geq\phi_K}: \\ W[U] \geq \frac{1}{2} W_{\geq\psi_0}}} \mathbb{P}[\mathcal{U}_K = U] = 1 - o(1). \end{aligned}$$

Similar to the previous case, if the event $\mathcal{E}_s \cap \mathcal{E}_{\mathcal{U}_K}$ is realised, then every vertex contributing to \widehat{Y} will eventually be infected, and thus $|\mathcal{A}_F| \geq \widehat{Y} > \gamma n$, that is, there is an outbreak completing the proof of Theorem 6.1. \square

7. Subcritical regime: No linear outbreak. The goal of this section is to show that in the subcritical regime whp there is no outbreak.

THEOREM 7.1. *Under the assumptions of Theorem 3.1, if additionally either*

$$\phi_H \leq w_n \quad \text{and} \quad p_0 \ll \min\{p_s, p_d\}$$

or

$$\phi_H > w_n \quad \text{and} \quad p_0 \ll p_s,$$

then there is no outbreak, that is, whp we have

$$|\mathcal{A}_F| = o(n).$$

We will use the parameter

$$(94) \quad \mu := \begin{cases} \min\{p_s, p_d\}/p_0 & \text{if } \phi_H \leq w_n, \\ p_s/p_0 & \text{if } \phi_H > w_n. \end{cases}$$

Since we consider the subcritical regime, we have

$$(95) \quad \mu \rightarrow \infty.$$

Instead of tracking the infection process, we relate it to a branching process, motivated by the following observation: if vertex v becomes infected at time $t > 0$ it must have at least one neighbour u which became infected at time $t - 1$; actually one can show that typically it has exactly one such neighbour. Hence, we may consider the vertex v a child of this unique neighbour u (and in case there are several, choose the smallest among them). Then for each vertex the number of its children is a random variable, however, they may have different distributions and be dependent on each other.

We will show that the condition on p_0 implies that this process is subcritical and whp it dies out quickly, thereby proving that the total infected population remains small.

Some of these arguments require us to work on the subgraph spanned by all non-heavy vertices, and then argue separately for the heavy vertices. We will show that whp no heavy vertex becomes infected during the process, and thus the relevant part of the proof is to analyse the behaviour of bootstrap process on the subgraph spanned by the nonheavy vertices.

We run the bootstrap process on $G[V_{<\phi_H}]$ in the usual way (cf. Section 2.2) and we denote the set of vertices that have become infected by time $t \geq 0$ by $\overset{\circ}{C}_t \subseteq V_{<\phi_H}$. Let $\overset{\circ}{C}_F$ denote the set of infected vertices at the end of the process.

7.1. *A branching process approximation.* In this section, we will prove that the total infected weight of the \mathring{C} -process will not be significantly larger than the total weight of the initially infected vertices.

LEMMA 7.2. *Whp we have $W[\mathring{C}_F] = o(\mu W p_0)$.*

We will prove this lemma by coupling the evolution of the bootstrap process with a stochastic process that is reminiscent of a branching process, where the offspring distribution depends on the current state.

For $t \geq 0$, let $\mathcal{I}_t := \mathring{C}_t \setminus \mathring{C}_{t-1}$ denote the set of vertices that belong to the t th generation and let $I_t = W[\mathcal{I}_t]$. In other words, the set \mathcal{I}_t consists of the vertices that become infected in step t . Thus, $\mathcal{I}_0 = \mathring{C}_0$ and $\mathring{C}_t = \bigcup_{s=0}^t \mathcal{I}_s$.

Now, let us condition on \mathring{C}_{t-1} . For every vertex $y \in \mathcal{I}_{t-1}$, following the ordering of the vertices, we expose its neighbours in $V_{<\phi_H} \setminus \mathring{C}_{t-1}$. For every $v \in V_{<\phi_H} \setminus \mathring{C}_{t-1}$ adjacent to y , we expose whether or not there are at least $r - 1$ other edges between v and \mathring{C}_{t-1} . If this is the case, then we include v into \mathcal{I}_t and, in particular, we include the vertex v among the offspring of y – we write $v \in \mathcal{X}_y$, where \mathcal{X}_y denotes the set of offspring of y . This leads to a partition of \mathcal{I}_t into sets \mathcal{X}_y of *children* for $y \in \mathcal{I}_{t-1}$, that is, we have

$$\mathcal{I}_t = \bigcup_{y \in \mathring{C}_{t-1} \setminus \mathring{C}_{t-2}} \mathcal{X}_y.$$

Using the FKG inequality (Theorem 4.7), we will show the following lemma, which bounds the probability that $v \in \mathcal{X}_y$.

LEMMA 7.3. *For any $t \geq 0$, let $v \in V_{\leq\phi_H} \setminus \mathring{C}_{t-1}$. We have*

$$\mathbb{P}[v \in \mathcal{X}_y \mid \mathring{C}_{t-1}] \leq w_y W[\mathring{C}_{t-1}]^{r-1} \left(\frac{w_v}{W}\right)^r.$$

PROOF. We will condition on a realisation of \mathring{C}_{t-1} . Consider the conditional space where $\mathcal{I}_i = S_i$, for $i = 0, \dots, t - 1$ and $v \notin \bigcup_{i=0}^{t-1} S_i = \mathring{C}_{t-1}$. In order for the event $v \in \mathcal{X}_y$ to hold, we need the following three conditions to hold:

- (i) y is a neighbour of v ;
- (ii) v has $r - 1$ additional infected neighbours;
- (iii) there exists no $z \in \mathring{C}_{t-1} \setminus \mathring{C}_{t-2}$ with $z < y$ such that $v \in \mathcal{X}_z$.

We write $\mathcal{D}(v)$ for the event that $\sum_{j=0}^{t-2} d_{S_j}(v) < r$, where $d_{S_j}(v)$ is the degree of v in S_j . So, if we ignore the third condition, the conditional probability of the event

$v \in \mathcal{X}_y$ can be bounded as

$$\begin{aligned} & \mathbb{P}\left[v \in \mathcal{X}_y \mid \bigcap_{i=0}^{t-1} \{\mathcal{I}_i = S_i\}\right] \\ & \leq \mathbb{P}\left[\{y, v\} \in E, \exists y_1, \dots, y_{r-1} \in \bigcup_{i=0}^{t-1} S_i : \bigcap_{i=0}^{r-1} \{\{y_i, v\} \in E\} \mid \mathcal{D}(v), \Gamma\right], \end{aligned}$$

where Γ is some event which does not depend on edges incident to v . Thus, we can omit Γ from the conditioning.

We bound the latter probability from above using the FKG inequality (Theorem 4.7). Note that the event $\{y, v\} \in E, \exists y_1, \dots, y_{r-1} \in \bigcup_{i=0}^{t-1} S_i : \bigcap_{i=0}^{r-1} \{\{y_i, v\} \in E\} \in E$ is nondecreasing, whereas the event $\mathcal{D}(v)$ is nonincreasing. Therefore, Theorem 4.7 implies that

$$\begin{aligned} & \mathbb{P}\left[\{y, v\} \in E, \exists y_1, \dots, y_{r-1} \in \bigcup_{i=0}^{t-1} S_i : \bigcap_{i=0}^{r-1} \{\{y_i, v\} \in E\} \mid \mathcal{D}(v)\right] \\ & \leq \mathbb{P}\left[\{y, v\} \in E, \exists y_1, \dots, y_{r-1} \in \bigcup_{i=0}^{t-1} S_i : \bigcap_{i=0}^{r-1} \{\{y_i, v\} \in E\}\right] \\ & \leq w_y w_v^r \left(\sum_{i=0}^{t-1} W[S_i]\right)^{r-1} \left(\frac{1}{W}\right)^r. \end{aligned}$$

In other words, we have

$$\mathbb{P}\left[v \in \mathcal{X}_y \mid \bigcap_{i=0}^{t-1} \{\mathcal{I}_i = S_i\}\right] \leq w_y \left(\sum_{i=0}^{t-1} W[S_i]\right)^{r-1} \left(\frac{w_v}{W}\right)^r$$

and the lemma follows. \square

PROOF OF LEMMA 7.2. We now provide a stochastic upper bound on I_t using a process that is very similar to a branching process except the offspring distribution depends on the history of the process. Moreover, the number of offspring of each individual in each generation are not independent.

Consider the family of Bernoulli random variables $I_{y,v}(t)$, where $y \in \mathcal{I}_{t-1}$ and v is any vertex, which satisfies $I_{y,v}(t) = 1$ if and only if $v \in \mathcal{X}_y$. Hence, by Lemma 7.3

$$\mathbb{E}[I_{y,v}(t) \mid \mathring{\mathcal{C}}_{t-1}] \leq w_y W[\mathring{\mathcal{C}}_{t-1}]^{r-1} \left(\frac{w_v}{W}\right)^r.$$

Given $\mathring{\mathcal{C}}_{t-1}$, we write

$$I_t = \sum_{y \in \mathcal{I}_{t-1}} \sum_{v \in V_{<\phi_H} \setminus \mathring{\mathcal{C}}_{t-1}} I_{y,v}(t) w_v.$$

This implies that

$$(96) \quad \mathbb{E}[I_t \mid \mathring{C}_{t-1}] \leq I_{t-1} W[\mathring{C}_{t-1}]^{r-1} W^{-r} \sum_{v \in V_{<\phi_H}} w_v^{r+1}.$$

Now, we introduce the stopping time T which is the first step t where either $I_t = 0$ or $W[\mathring{C}_t] > \mu^{1/2} W p_0$. Note that $T < \mu^{1/2} W p_0$, since if $I_t > 0$, then in fact $I_t \geq 1$.

Now, let $\hat{I}(t)$ be equal to I_t , if $t \leq T$ and equal to 0 otherwise. In other words, $\hat{I}(t) = I_t \mathbb{1}_{\{t \leq T\}}$. Let

$$\chi(t) := W[\mathring{C}_{t-1}]^{r-1} W^{-r} \sum_{v \in V_{<\phi_H}} w_v^{r+1}.$$

If $t \leq T$, then

$$(97) \quad \chi(t) \leq (\mu^{1/2} W p_0)^{r-1} W^{-r} \sum_{v \in V_{<\phi_H}} w_v^{r+1} =: \hat{\chi},$$

and furthermore this implies

$$(98) \quad \mathbb{E}[\hat{I}(t) \mid \mathring{C}_{t-1}] \leq \hat{\chi} \hat{I}(t-1).$$

Next, we show that

$$(99) \quad \hat{\chi} \leq \mu^{-1/2}.$$

Indeed, since $p_0 \leq \mu^{-1} p_s$ we have that

$$\begin{aligned} (W p_0)^{r-1} &\leq \mu^{-(r-1)} W^{r-1} p_s^{r-1} \\ &= \mu^{-(r-1)} W^{r-1} \frac{W}{\sum_{v \in V_{<\phi_H}} w_v^{r+1}} \\ &= \mu^{-(r-1)} \frac{W^r}{\sum_{v \in V_{<\phi_H}} w_v^{r+1}}. \end{aligned}$$

This together with (97) imply

$$\hat{\chi} \leq \mu^{-(r-1)/2},$$

and (99) follows since $r \geq 2$.

Therefore, taking expectations on both sides of (98) we deduce that

$$\mathbb{E}[\hat{I}(t)] \leq \hat{\chi} \mathbb{E}[\hat{I}(t-1)].$$

Repeating this inequality, we obtain

$$\mathbb{E}[\hat{I}(t)] \leq \hat{\chi}^t \mathbb{E}[\hat{I}(0)] = \hat{\chi}^t \mathbb{E}[W[\mathring{C}_0]].$$

Note that $\mathbb{E}[W[\mathring{C}_0]] \leq W p_0$, and thus

$$\mathbb{E}[\hat{I}(t)] \leq \hat{\chi}^t W p_0.$$

implying

$$(100) \quad \mathbb{E}[W[\mathring{C}_T]] = \mathbb{E}\left[\sum_{t=0}^T \hat{I}(t)\right] \leq \left(\sum_{t=0}^T \hat{\chi}^t\right) Wp_0 \leq \frac{1}{1 - \hat{\chi}} Wp_0.$$

Now, let \mathcal{B} be the event $\sum_{t=0}^T \hat{I}(t) \geq \mu^{1/2} Wp_0$. We have

$$\mathbb{E}\left[\mathbb{1}_{\{\mathcal{B}\}} \sum_{t=0}^T \hat{I}(t)\right] \geq \mu^{1/2} Wp_0 \mathbb{E}[\mathbb{1}_{\{\mathcal{B}\}}].$$

This together with (100) imply that

$$\mathbb{E}[\mathbb{1}_{\{\mathcal{B}\}}] = O(\mu^{-1/2}) = o(1).$$

Therefore, Markov’s inequality implies that whp the process stops before the total weight of the infected vertices reaches $\mu^{1/2} Wp_0$, as desired. \square

7.2. *No outbreak: Proof of Theorem 7.1.* Now we consider the process on the whole vertex set V . Until this point, we showed that if we restrict ourselves to the nonheavy vertices, then no linear outbreak occurs. We now have to take care of the heavy vertices. The first observation is that initially whp none of them are infected.

CLAIM 7.4. *Whp $\mathcal{A}_0 \cap V_{\geq \phi_H} = \emptyset$.*

PROOF. In the case $\phi_H > w_n$, there is nothing to prove, hence assume $\phi_H \leq w_n$. Recall that in this case p_d is well defined and by its definition (5) we have

$$(101) \quad p_d^{-r} = \sum_{u \in V_{\geq \phi_H}} w_u^r \geq |V_{\geq \phi_H}| \phi_H^r \stackrel{(3)}{\geq} \left(\frac{W}{4\phi_H}\right)^r.$$

Thus, since $|V_{\geq \phi_H}| \leq W/\phi_H$, we obtain

$$\mathbb{E}[|\mathcal{A}_0 \cap V_{\geq \phi_H}|] = |V_{\geq \phi_H}| p_0 \stackrel{(94)}{\leq} \frac{p_d W}{\mu \phi_H} \stackrel{(101)}{\leq} \frac{4}{\mu} \stackrel{(95)}{=} o(1).$$

Therefore, the claim follows from Markov’s inequality. \square

PROOF OF THEOREM 7.1. Now we consider the unrestricted process as described in Section 2.2, but starting with $\mathcal{A}'_0 := \mathring{C}_F$ as the initial set of infected vertices. This defines a sequence of sets \mathcal{A}'_τ for $\tau \geq 0$ and a final set \mathcal{A}'_F . Now Claim 7.4 implies that whp $\mathcal{A}_0 \subseteq \mathring{C}_F = \mathcal{A}'_0$, and thus

$$(102) \quad \mathcal{A}_F \subseteq \mathcal{A}'_F.$$

Lemma 7.2 implies that whp the total weight of \mathring{C}_F satisfies

$$(103) \quad W[\mathcal{A}'_0] = W[\mathring{C}_F] = o(\mu Wp_0).$$

Moreover, this shows that \mathcal{C}_F contains only few vertices

$$(104) \quad |\mathcal{A}'_0| \leq W[\mathcal{A}'_0]/w_1 \stackrel{w_1 \geq 1, (94), Cor. 4.5}{=} o(W) \stackrel{(2)}{=} o(n).$$

The last step is to show that whp

$$(105) \quad \mathcal{A}'_1 = \mathcal{A}'_0,$$

because due to (102) and (104) this will imply

$$|\mathcal{A}_F| \leq |\mathcal{A}'_F| = |\mathcal{A}'_0| = o(n).$$

It remains to prove (105). In other words, we have to show that whp in the next step none of the heavy vertices become infected. Once again, in case $\phi_H > w_n$ there is nothing to be shown, so assume $\phi_H \leq w_n$. For any vertex $v \in V_{\geq \phi_H}$ we have

$$\mathbb{P}[v \in \mathcal{A}'_1 | \mathcal{A}'_0] \stackrel{L.5.4}{\leq} \frac{w_v^r W[\mathcal{A}'_0]^r}{r! W^r}$$

and note that only heavy vertices can become infected in this step. Thus, summing over all heavy vertices and using (103), we obtain

$$\mathbb{E}[\mathcal{A}'_1 \setminus \mathcal{A}'_0 | \mathcal{A}'_0] = o\left((\mu p_0)^r \sum_{v \in V_{\geq \phi_H}} w_v^r\right).$$

Moreover, we have

$$\mu p_0 \stackrel{(94)}{\leq} p d \stackrel{(5)}{=} \frac{1}{\sum_{u \in V_{\geq \phi_H}} w_u^r},$$

implying

$$\mathbb{E}[\mathcal{A}'_1 \setminus \mathcal{A}'_0 | \mathcal{A}'_0] = o(1).$$

Therefore, Markov's inequality implies that whp (105) holds, and as argued previously this completes the proof of Theorem 7.1. \square

8. Proof of main results.

8.1. *Proof of Theorem 3.1.* Theorem 3.1 follows directly from Theorems 5.2 and 6.1.

8.2. *Proof of Theorem 3.2.* Recall that the assumption on the weight sequence in Theorem 3.2 is that there exist constants $c < 1/30$, c_1 and a function $h = \omega(1)$ such that for $c_1 \leq f \leq h$ we have $\mathbb{P}[Z_{\mathbf{w}} \geq f] \leq c/f$.

For the remainder of the section, assume that $p_0 \geq h^{-1}$ and $p_0 = \omega(W^{-1/2})$. Clearly, if whp there is no outbreak for such a p_0 , then this is the case for any

$p_0 = o(1)$. Moreover, note that if there is no linear outbreak for $r = 2$, then there is no linear outbreak for $r > 2$. So we may restrict ourselves to the $r = 2$ case.

We aim to apply the branching process argument from Section 7 here as well. Similarly, as before we could construct the branching process, but unlike the previous case the typical vertex would have multiple parents during the early stages of the process. We use Theorem 4.9 instead to track the total weight of the infected vertices during the early stages of the process. After a point, due to the rapid decrease in the sum of the weight of the vertices which become infected in a given step, a typical vertex which becomes infected in step $t - 1$ has on average less than one child in the next generation (in the sense we discussed in Section 7), giving rise to a subcritical process which dies out quickly. In turn, this implies that the bootstrap process stops quickly.

We modify the initial step of the process slightly which leads to a stochastic upper bound: in the initial step, we infect every vertex with weight at least p_0^{-1} and in addition we infect every vertex with weight less than p_0^{-1} with probability p_0 independently. We denote this set by $\tilde{\mathcal{A}}_0$. More generally, for any $t \geq 0$ we let $\tilde{\mathcal{A}}_t$ be the set of infected vertices after the t th step. As usual, we set $\tilde{\mathcal{A}}_{-1} := \emptyset$.

Let

$$\kappa_k := W^{-1} \sum_{v \in V_{<p_0^{-1}}} w_v^{k+1}.$$

Using Proposition 4.4, we bound κ_k from above by

$$\begin{aligned} \kappa_k &\leq k \int_0^{p_0^{-1}} \mu^{k-1} \mathbb{P}[Z_{\mathbf{w}} \geq \mu] d\mu \\ (106) \quad &\leq k \int_{c_1}^{p_0^{-1}} \mu^{k-1} \mathbb{P}[Z_{\mathbf{w}} \geq \mu] d\mu + k \int_0^{c_1} \mu^{k-1} d\mu \\ &\leq \begin{cases} c \ln(p_0^{-1}) + c_1, & k = 1, \\ c \frac{k}{k-1} p_0^{-k+1} + c_1^k, & k \geq 2. \end{cases} \end{aligned}$$

LEMMA 8.1. *Whp $W[\tilde{\mathcal{A}}_0] \leq (1 + 2c)Wp_0$.*

PROOF. Since $p_0^{-1} \leq h$ and $p_0^{-1} \rightarrow \infty$, we have $\sum_{u \in V_{\geq p_0^{-1}}} w_u \leq cW/p_0^{-1} = cWp_0$. Clearly, $\mathbb{E}[W[\tilde{\mathcal{A}}_0 \cap V_{<p_0^{-1}}]] \leq Wp_0$. Theorem 4.9 implies that for large n we have

$$\begin{aligned} &\mathbb{P}[W[\tilde{\mathcal{A}}_0 \cap V_{<p_0^{-1}}] \geq Wp_0 + cWp_0] \\ &\leq \exp\left(-\frac{(cWp_0)^2}{2(\sum_{u \in V_{\leq p_0^{-1}}} w_u^2 p_0 + p_0^{-1} cWp_0/3)}\right) \end{aligned}$$

$$\begin{aligned} &\stackrel{(106)}{\leq} \exp\left(-\frac{(cWp_0)^2}{2(c(1+o(1))Wp_0 \ln p_0^{-1} + p_0^{-1}cWp_0/3)}\right) \\ &\leq \exp\left(-\frac{(cWp_0)^2}{cW}\right) \\ &\leq \exp(-cWp_0^2) = o(1), \end{aligned}$$

because $p_0^2 = \omega(W^{-1})$. \square

LEMMA 8.2. *If $v \notin \tilde{\mathcal{A}}_t$, then*

$$\mathbb{P}[v \in \tilde{\mathcal{A}}_{t+1} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \leq \frac{w_v^2}{W^2} \frac{W[\tilde{\mathcal{A}}_t]^2 - W[\tilde{\mathcal{A}}_{t-1}]^2}{2}.$$

PROOF. Given a vertex v , in order for v to become infected at step $t + 1$ it must neighbour at least 2 vertices in $\tilde{\mathcal{A}}_t$ and at least one of these vertices must be contained in $\tilde{\mathcal{A}}_t \setminus \tilde{\mathcal{A}}_{t-1}$. For a set of vertices U , let \mathcal{E}_U be the event that for every $u \in U$ we have that the pair (u, v) is an edge. If $v \notin \tilde{\mathcal{A}}_t$, then we have

$$\mathbb{P}[v \in \tilde{\mathcal{A}}_{t+1} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \leq \sum_{k=1}^2 \sum_{U \in \binom{\tilde{\mathcal{A}}_t \setminus \tilde{\mathcal{A}}_{t-1}}{k}} \sum_{U' \in \binom{\tilde{\mathcal{A}}_{t-1}}{2-k}} \mathbb{P}[\mathcal{E}_U, \mathcal{E}_{U'} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}].$$

The event \mathcal{E}_U is independent of $\mathcal{E}_{U'}$, $\tilde{\mathcal{A}}_t$, $\tilde{\mathcal{A}}_{t-1}$, because it depends on edges that have not been exposed up to time t . Now, the event $\mathcal{E}_{U'}$ is nondecreasing, whereas $\tilde{\mathcal{A}}_t$, $\tilde{\mathcal{A}}_{t-1}$'s only connection to $\mathcal{E}_{U'}$ is the event that v has at most one neighbour in \mathcal{A}_{t-1} . But the latter is a nonincreasing event. Hence, the FKG inequality (Theorem 4.7) implies that $\mathbb{P}[\mathcal{E}_{U'} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \leq \mathbb{P}[\mathcal{E}_{U'}]$. Thus, we have

$$\mathbb{P}[\mathcal{E}_U, \mathcal{E}_{U'} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \leq \mathbb{P}[\mathcal{E}_U] \mathbb{P}[\mathcal{E}_{U'}] = \frac{w_v^2}{W^2} \prod_{u \in U} w_u \prod_{u' \in U'} w_{u'}.$$

Therefore,

$$\begin{aligned} \mathbb{P}[v \in \tilde{\mathcal{A}}_{t+1} \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] &\leq \sum_{k=1}^2 \sum_{U \in \binom{\tilde{\mathcal{A}}_t \setminus \tilde{\mathcal{A}}_{t-1}}{k}} \sum_{U' \in \binom{\tilde{\mathcal{A}}_{t-1}}{2-k}} \frac{w_v^2}{W^2} \prod_{u \in U} w_u \prod_{u' \in U'} w_{u'} \\ &\leq \frac{w_v^2}{W^2} \sum_{k=1}^2 \frac{W[\tilde{\mathcal{A}}_t \setminus \tilde{\mathcal{A}}_{t-1}]^k}{k!} \frac{W[\tilde{\mathcal{A}}_{t-1}]^{2-k}}{(2-k)!} \\ &= \frac{w_v^2}{W^2} \frac{W[\tilde{\mathcal{A}}_t]^2 - W[\tilde{\mathcal{A}}_{t-1}]^2}{2}, \end{aligned}$$

where the last equality follows, by writing $W[\tilde{\mathcal{A}}_t] = W[\tilde{\mathcal{A}}_t \setminus \tilde{\mathcal{A}}_{t-1}] + W[\tilde{\mathcal{A}}_{t-1}]$. \square

Since $c < 1/30$, we have that $3c(1 + 2c) < 1/8$. Set $\beta = 1/4$ and note that

$$(107) \quad (1 - 2\beta)\beta = 1/8 > 3c(1 + 2c).$$

LEMMA 8.3. *Assume that $W[\tilde{\mathcal{A}}_\tau \setminus \tilde{\mathcal{A}}_{\tau-1}] \leq (1 + 2c)(2\beta)^\tau Wp_0$ for every $\tau \leq t$. Conditional on $\tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}$, with probability at least*

$$1 - \exp\left(-\frac{\beta(2\beta)^t(1 + 2c)Wp_0^2}{3}\right),$$

we have

$$W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t] \leq (2\beta)^{t+1}(1 + 2c)Wp_0.$$

PROOF. From Lemma 8.2, we have

$$\mathbb{E}[W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t] \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \leq \sum_{v \in V_{<p_0^{-1}}} \frac{w_v^3}{W^2} \frac{W[\tilde{\mathcal{A}}_t]^2 - W[\tilde{\mathcal{A}}_{t-1}]^2}{2}.$$

Clearly, for n sufficiently large, by (106) we have

$$\sum_{v \in V_{<p_0^{-1}}} w_v^3 \leq (1 + o(1))2cWp_0^{-1}$$

and thus

$$\begin{aligned} \mathbb{E}[W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t] \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] &\leq (1 + o(1))2c(Wp_0)^{-1} \frac{W[\tilde{\mathcal{A}}_t]^2 - W[\tilde{\mathcal{A}}_{t-1}]^2}{2} \\ &\leq (1 + o(1))2c(Wp_0)^{-1}(W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}])W[\tilde{\mathcal{A}}_t], \end{aligned}$$

because $W[\tilde{\mathcal{A}}_{t-1}] + W[\tilde{\mathcal{A}}_t] \leq 2W[\tilde{\mathcal{A}}_t]$.

From the assumptions, we have

$$\begin{aligned} &\mathbb{E}[W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t] \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \\ &\leq (1 + o(1))2c(Wp_0)^{-1}(W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}])W[\tilde{\mathcal{A}}_t] \\ &\stackrel{\tilde{\mathcal{A}}_{-1}=\emptyset}{=} (1 + o(1))2c(Wp_0)^{-1}(W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}]) \sum_{\tau=0}^t W[\tilde{\mathcal{A}}_\tau \setminus \tilde{\mathcal{A}}_{\tau-1}] \\ &\leq (1 + o(1))2c(Wp_0)^{-1}(W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}]) \left((1 + 2c)Wp_0 \sum_{\tau=0}^t (2\beta)^\tau \right) \\ &\leq (W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}])3c(1 + 2c)/(1 - 2\beta) \\ &\stackrel{(107)}{\leq} \beta(W[\tilde{\mathcal{A}}_t] - W[\tilde{\mathcal{A}}_{t-1}]) \\ &\leq \beta(2\beta)^t(1 + 2c)Wp_0. \end{aligned}$$

A similar argument provides

$$\text{Var}[W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t]] \leq \sum_{v \in V_{<p_0^{-1}}} \frac{w_v^4}{W^2} \frac{W[\tilde{\mathcal{A}}_t]^2 - W[\tilde{\mathcal{A}}_{t-1}]^2}{2} \leq \beta(2\beta)^t(1+2c)W.$$

Now all we have to show is concentration. We will use Theorem 4.9 for the sum of the weighted Bernoulli-distributed random variables, which gives $W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t]$. Each summand is bounded from above by p_0^{-1} . Hence, we can take $M = p_0^{-1}$ in Theorem 4.9 and deduce that

$$\begin{aligned} \mathbb{P}[W[\tilde{\mathcal{A}}_{t+1} \setminus \tilde{\mathcal{A}}_t] \geq (2\beta)^{t+1}(1+2c)Wp_0 \mid \tilde{\mathcal{A}}_t, \tilde{\mathcal{A}}_{t-1}] \\ \leq \exp\left(-\frac{(\beta(2\beta)^t(1+2c)Wp_0)^2}{2(\beta(2\beta)^t(1+2c)W + p_0^{-1}\beta(2\beta)^t(1+2c)Wp_0/3)}\right) \\ \leq \exp\left(-\frac{(\beta(2\beta)^t(1+2c)Wp_0)^2}{3\beta(2\beta)^t(1+2c)W}\right) \\ \leq \exp\left(-\frac{\beta(2\beta)^t(1+2c)Wp_0^2}{3}\right). \end{aligned} \quad \square$$

PROOF OF THEOREM 3.2. Fix a function $T := T(n)$ satisfying $T \rightarrow \infty$, but $(2\beta)^T Wp_0^2 = \omega(1)$. Note that such a function exists, because $Wp_0^2 = \omega(1)$. Let \mathcal{E} denote the event $W[\tilde{\mathcal{A}}_{T-1}] \leq (1+2c)Wp_0/(1-2\beta)$ and $W[\tilde{\mathcal{A}}_T \setminus \tilde{\mathcal{A}}_{T-1}] = o(Wp_0)$.

Now, by Lemma 8.3, with probability

$$1 - O\left(\exp\left(-\frac{\beta(2\beta)^T(1+2c)Wp_0^2}{3}\right)\right) = 1 - o(1)$$

we have $W[\tilde{\mathcal{A}}_\tau \setminus \tilde{\mathcal{A}}_{\tau-1}] \leq (2\beta)^\tau(1+2c)Wp_0$ for every $\tau \leq T$, and thus \mathcal{E} holds whp as well. From this point on, we will condition on the event \mathcal{E} .

We will give a stochastic upper bound on the evolution of the bootstrap process after step T , using the branching process framework we introduced in Section 7.

Recall that \mathcal{X}_v denotes the children of v . Consider the family of Bernoulli random variables $I_{v,u}(t)$, where $v \in \mathcal{I}_{t-1}$ and u is any vertex, which satisfies $I_{v,u}(t) = 1$ if and only if $u \in \mathcal{X}_v$.

Following the steps of Lemma 7.3, one can show that

$$\mathbb{P}[u \in \mathcal{X}_v \mid \tilde{\mathcal{A}}_{t-1}, \mathcal{E}] \leq w_v W[\tilde{\mathcal{A}}_{t-1}] \left(\frac{w_u}{W}\right)^2.$$

Hence,

$$\mathbb{E}[I_{v,u}(t) \mid \tilde{\mathcal{A}}_{t-1}, \mathcal{E}] \leq w_v W[\tilde{\mathcal{A}}_{t-1}] \left(\frac{w_u}{W}\right)^2.$$

Let S be the stopping time which is the first step t after T where either $\tilde{\mathcal{A}}_t = \tilde{\mathcal{A}}_{t-1}$ or $W[\tilde{\mathcal{A}}_t] > 4Wp_0$. In other words, S is the first time where either the process dies out or the total weight of the set $\tilde{\mathcal{A}}_t$ exceeds $4Wp_0$.

Denote by $I_v(t)$ the total weight of the offspring of v , that is,

$$I_v(t) = \sum_{u \in V_{<p_0^{-1}} \setminus \tilde{\mathcal{A}}_{t-1}} I_{v,u}(t)w_u.$$

Now, we show that if $t \leq S$, then provided that c is small enough, the expected value of $I_v(t)$ is smaller than 1. We have conditional on \mathcal{E}

$$\mathbb{E}[I_v(t) \mid \tilde{\mathcal{A}}_{t-1}] \leq w_v \frac{W[\tilde{\mathcal{A}}_{t-1}]}{W^2} \sum_{u \in V_{<p_0^{-1}}} w_u^3 \stackrel{(106)}{\leq} 3w_v c W[\tilde{\mathcal{A}}_{t-1}] (Wp_0)^{-1}.$$

Now, as long as $t \leq S$, we have $W[\tilde{\mathcal{A}}_{t-1}] \leq 4Wp_0$. We deduce that

$$\mathbb{E}[I_v(t) \mathbb{1}_{\{t \leq S\}} \mid \tilde{\mathcal{A}}_{t-1}, \mathcal{E}] \leq 12cw_v.$$

Set $z = 12c$. Since $12c < 1$ (from our assumption that $c < 1/30$), we have $z < 1$, for any n sufficiently large. Denote by $I(t)$ the total weight of the vertices in the t th generation. Then this is equal to the sum of the random variables $I_v(t)$ over all v which belong to the $(t - 1)$ th generation. Therefore, we have

$$\mathbb{E}[I(t) \mathbb{1}_{\{t \leq S\}} \mid \tilde{\mathcal{A}}_{t-1}, \mathcal{E}] \leq zI(t - 1) \mathbb{1}_{\{t \leq S\}}.$$

Now, we set $\hat{I}(t) = I(t) \mathbb{1}_{\{t \leq S\}}$. Taking expectations on both sides of the above inequality implies that for $t > T$

$$\mathbb{E}[\hat{I}(t) \mid \mathcal{E}] \leq z \mathbb{E}[\hat{I}(t - 1) \mid \mathcal{E}] \leq \dots \leq z^{t-T} \mathbb{E}[\hat{I}(T) \mid \mathcal{E}].$$

The random variable

$$X_S = \sum_{t=T}^{\infty} \hat{I}(t)$$

is the total progeny (but without the first T generations) until the stopping time S . We deduce that

$$\mathbb{E}[X_S \mid \mathcal{E}] \leq \mathbb{E}[\hat{I}(T) \mid \mathcal{E}] \sum_{t=T}^{\infty} z^{t-T} = \frac{1}{1-z} \mathbb{E}[\hat{I}(T) \mid \mathcal{E}] = o(Wp_0),$$

as $\hat{I}(T) \leq I(T) = W[\tilde{\mathcal{A}}_T \setminus \tilde{\mathcal{A}}_{T-1}] = o(Wp_0)$ when \mathcal{E} holds.

Markov's inequality implies $\mathbb{P}[X_S > Wp_0 \mid \mathcal{E}] = o(1)$. Recall that \mathcal{E} also implies that $W[\tilde{\mathcal{A}}_{T-1}] \leq 3Wp_0$. The result follows as \mathcal{E} holds whp. \square

9. Concluding remarks. To sum up, we have considered the evolution of the classical bootstrap percolation processes on a general class of inhomogeneous random graphs, which is known as the Chung–Lu model. In this model, the vertices are equipped with positive weights and each potential edge is present with probability proportional to the product of the weights. Essentially, the typical properties of the resulting random graph are determined by the sequence of the weights and its asymptotic properties.

We gave an approximate characterisation of those weight sequences for which the evolution of a bootstrap process exhibits a critical phenomenon. This is connected to the existence of a critical density of the initial set of infected vertices such that when the initial density crosses this value then an outbreak occurs whp even if the initial set is small. Our main finding has to do with the existence of constants c and C such that if $\mathbb{P}[Z_w \geq f] < c/f$, then no such critical value exists whereas if $\mathbb{P}[Z_w \geq f] > C/f$, then it does exist.

The results we have shown assume that $W = \lambda n$ and the minimum weight is at least 1. However, the results also hold under the more general assumption that $W = (1 + o(1))\lambda n$ and the minimal weight is bounded away from 0. However, if we remove the condition that the minimum weight is at least 1, then the lower bound on C in Theorem 3.1 should also depend on this quantity.

As mentioned earlier, the smaller of the two candidate thresholds gives the critical threshold. In the following example, we demonstrate that either of the candidate thresholds can be the minimum.

EXAMPLE 9.1. Fix $r = 2$ and take a weight sequence such that it contains $W^{1/9}$ vertices of weight $W^{7/12}$, $W^{2/3}/20$ vertices of weight $W^{1/3}$ and each of the remaining vertices has weight 1. Note that $|V_{\geq W^{1/3}}| = (1 + o(1))W^{2/3}/20 < W^{2/3}/2^4 = W^2/(2W^{1/3})^4$ and $|V_{\geq 1}| = n = o(W^2)$. In addition, $|V_{\geq \sqrt{W}}| = W^{1/9} \geq 2^{-4} = W^2/(2\sqrt{W})^4$. Therefore, $W^{1/2} > \phi_H > W^{1/3}$. Thus,

$$\sum_{u \in V_{< \phi_H}} w_u^3 = \Theta(W^{5/3}).$$

Clearly, $\sum_{u \in V_{\geq \phi_H}} w_u^2 = W^{23/18}$. This gives us

$$\frac{W}{\sum_{u \in V_{< \phi_H}} w_u^3} = \Theta(W^{-2/3}) = o(W^{-23/36}) = o\left(\left(\frac{1}{\sum_{u \in V_{\geq \phi_H}} w_u^2}\right)^{1/2}\right).$$

Thus, the minimum in Theorem 3.1 is achieved by the vertices of weight less than ϕ_H .

Now replace the $W^{1/9}$ vertices of weight $W^{7/12}$ by $W^{1/9}$ vertices of weight $W^{3/4}$. Note that this has no effect on the value of ϕ_H . However, we have

$$\left(\frac{1}{\sum_{u \in V_{\geq \phi_H}} w_u^2}\right)^{1/2} = \Theta(W^{-29/36}) = o(W^{-2/3}),$$

so this time the minimum is achieved by the vertices of weight at least ϕ_H .

Several open questions still remain. The first concerns the exact values of c and C . In particular, is $c = C$? If not, then what happens in the case where the weight sequences satisfies $\mathbb{P}[Z_{\mathbf{w}} \geq f] = c'/f$ for $c < c' < C$.

Furthermore, our analysis does not consider the case where the initial density has the same order of magnitude as the critical density. We believe that in this case an outbreak occurs with probability that is asymptotically bounded away from 0 and 1. If this is the case, it would be interesting to know whether a limiting value exists for this probability and how it depends upon the parameters of the model.

Acknowledgement. We would like to thank an anonymous referee for making helpful comments on the presentation of the paper.

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