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VOLATILITY AND ARBITRAGE

BY E. ROBERT FERNHOLZ[*], IOANNIS KARATZAS[†,*,1] AND JOHANNES RUF[‡,2]

INTECH Investment Management[*], Columbia University[†] and London School of Economics and Political Science[‡]

The capitalization-weighted cumulative variation

$$\sum_{i=1}^{d} \int_0^{\cdot} \mu_i(t)\, \mathrm{d}\langle \log \mu_i \rangle(t)$$

in an equity market consisting of a fixed number d of assets with capitalization weights $\mu_i(\cdot)$, is an observable and a nondecreasing function of time. If this observable of the market is not just nondecreasing but actually grows at a rate bounded away from zero, then strong arbitrage can be constructed relative to the market over sufficiently long time horizons. It has been an open issue for more than ten years, whether such strong outperformance of the market is possible also over arbitrary time horizons under the stated condition. We show that this is not possible in general, thus settling this long-open question. We also show that, under appropriate additional conditions, outperformance over any time horizon indeed becomes possible, and exhibit investment strategies that effect it.

1. Introduction and summary. It has been known since Fernholz (2002) that volatility in an equity market can generate arbitrage, or at least *relative arbitrage* between a specified portfolio and the market portfolio, under idealized conditions on market structure and on trading. However, the questions of exactly what level of volatility is required and of how long it might take for this arbitrage to be realized, have never been fully answered. Here, we hope to shed some light on these questions, and come to an understanding about what might represent adequate volatility and over which time-frame relative arbitrage might be achieved.

A common condition regarding market volatility, sometimes known as *strict nondegeneracy*, is the requirement that the eigenvalues of the market covariation matrix be bounded away from zero. It was shown in Fernholz (2002) that strict nondegeneracy coupled with market *diversity*, the condition that the largest relative market weight be bounded away from one, will produce relative arbitrage with respect to the market over sufficiently long time horizons. Later,

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Fernholz, Karatzas and Kardaras (2005) showed that these two conditions lead to relative arbitrage over arbitrarily short time horizons (or "short-term relative arbitrage", for brevity). Market diversity is actually a rather mild condition, very likely to be satisfied in any market with even a semblance of anti-trust regulation. However, strict nondegeneracy is a much stronger condition, and probably not amenable to statistical verification in any realistic market setting. While it might be reasonable to assume that the market covariation matrix is *nonsingular*, it would seem rather courageous to make strong assumptions regarding the behavior over time of the smallest eigenvalue of a random $d \times d$ matrix, where $d \in \mathbb{N}$ is usually a large integer standing for the number of stocks in an equity market.

Accordingly, it is preferable to avoid the use of strict nondegeneracy as a characterization of adequate volatility and consider instead measures based on aggregated relative variations. An important such measure is the so-called *cumulative internal variation* of the market:

$$\Gamma(\cdot) := \frac{1}{2} \sum_{i=1}^{d} \int_0^{\cdot} \mu_i(t)\, \mathrm{d}\langle \log \mu_i \rangle(t). \tag{1.1}$$

This is based on the cumulative weighted average of the variations of the logarithmic market weights, and will be discussed at some length below. Here, $\mu_i(t)$ represents the market weight of the ith stock at time $t \geq 0$, for each $i = 1, \ldots, d$. Fernholz and Karatzas (2005) show that if the slope of $\Gamma(\cdot)$ is bounded away from zero, then relative arbitrage with respect to the market will exist over a long enough time horizon. We shall see in Section 6 that this condition does not necessarily lead to relative arbitrage over an arbitrary time horizon. However, not all is lost: in Section 5, we shall show that under *additional* assumptions, such as when there are only two stocks ($d = 2$) or when the market weights satisfy appropriate time-homogeneity properties, relative arbitrage *does* exist over arbitrary time horizons when the slope of $\Gamma(\cdot)$ in (1.1) is bounded away from zero. Other sufficient conditions are also provided. We remark also that Pal (2016) recently derived sufficient conditions for large markets that yield asymptotic short-term arbitrage.

Preview: The structure of the paper is as follows: Sections 2, 3 and 4 set up the basic definitions, including the concept of functional generation of trading strategies. Introduced by Fernholz (1999, 2002) and developed in Fernholz and Karatzas (2009) and Karatzas and Ruf (2017), functionally-generated strategies are very useful for constructing arbitrages relative to the market. Section 5 establishes conditions under which such relative arbitrage can be shown to exist over arbitrary time horizons. Section 6 constructs examples of markets which have adequate volatility but admit no such relative arbitrage—indeed, the price processes in these examples are all martingales. Section 7 summarizes the results of this paper and discusses some open questions.

2. The market setup and trading strategies. We fix a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ endowed with a right-continuous filtration $\mathfrak{F} = (\mathscr{F}(t))_{t\geq 0}$. For simplicity, we take $\mathscr{F}(0) = \{\varnothing, \Omega\}$, mod. P. All processes to be encountered will be adapted to this filtration. On this filtered probability space and for some integer $d \geq 2$, we consider a continuous d-dimensional semimartingale $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_d(\cdot))'$ taking values in the lateral face

$$\mathbf{\Delta}^d := \left\{ (x_1, \ldots, x_d)' \in [0,1]^d : \sum_{i=1}^d x_i = 1 \right\} \subset \mathbb{H}^d$$

of the unit simplex, where $\mathbb{H}^d$ denotes the hyperplane

$$\mathbb{H}^d := \left\{ (x_1, \ldots, x_d)' \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \right\}. \tag{2.1}$$

We assume that $\mu(0) \in \mathbf{\Delta}^d_+$, where we set

$$\mathbf{\Delta}^d_+ := \mathbf{\Delta}^d \cap (0,1)^d. \tag{2.2}$$

We interpret $\mu_i(t)$ as the relative *market weight*, in terms of capitalization in the market, of company $i = 1, \ldots, d$ at time $t \geq 0$. An individual company's market weight is allowed to become zero, but we insist that $\sum_{i=1}^d \mu_i(t) = 1$ must hold for all $t \geq 0$. In this spirit, it is useful to think of the generic market weight process $\mu_i(\cdot)$ as the ratio

$$\mu_i(\cdot) := \frac{S_i(\cdot)}{\Sigma(\cdot)}, \qquad i = 1, \ldots, d; \qquad \Sigma(\cdot) := S_1(\cdot) + \cdots + S_d(\cdot) > 0. \tag{2.3}$$

Here, $S_i(\cdot)$ is a continuous nonnegative semimartingale for each $i = 1, \ldots, d$, representing the capitalization (stock price, multiplied by the number of shares outstanding) of the ith company; whereas the process $\Sigma(\cdot)$, assumed to be strictly positive, stands for the total capitalization of the entire market.

For later reference, we introduce the stopping times

$$\mathscr{D} := \mathscr{D}_1 \wedge \cdots \wedge \mathscr{D}_d, \qquad \mathscr{D}_i := \inf\{t \geq 0 : \mu_i(t) = 0\}. \tag{2.4}$$

To avoid notational inconvenience, we assume that $\mu_i(\mathscr{D}_i + t) = 0$ holds for all $i = 1, \ldots, d$ and $t \geq 0$; in other words, the origin is an absorbing state for any of the market weights.

One of our results, Theorem 5.10 below, needs the following notion.

DEFINITION 2.1 (Deflator). A strictly positive process $Z(\cdot)$ is called *deflator* for the vector semimartingale $\mu(\cdot)$ of relative market weights, if the product $Z(\cdot)\mu_i(\cdot)$ is a local martingale for every $i = 1, \ldots, d$.

Except when explicitly stated otherwise, the results below will hold independently of whether the vector semimartingale $\mu(\cdot)$ of relative market weights admits a deflator, or not.

In order to fix ideas and simplify the presentation, *we shall assume from now onwards that each stock in the market has exactly one share outstanding.* With this normalization, capitalizations and prices coincide.

We consider now a predictable process $\vartheta(\cdot) = (\vartheta_1(\cdot), \ldots, \vartheta_d(\cdot))'$ with values in $\mathbb{R}^d$, and interpret $\vartheta_i(t)$ as the number of shares held at time $t \geq 0$ in the stock of company $i = 1, \ldots, d$. Then the total *value*, or "wealth", of this investment, measured in terms of the total market capitalization, is

$$V^{\vartheta}(\cdot) := \sum_{i=1}^{d} \vartheta_i(\cdot)\mu_i(\cdot).$$

DEFINITION 2.2 (Trading strategies). Suppose that the $\mathbb{R}^d$-valued, predictable process $\vartheta(\cdot)$ is integrable with respect to the continuous $\mathbb{R}^d$-valued semimartingale $\mu(\cdot)$. We shall say that $\vartheta(\cdot)$ is a *trading strategy* if it satisfies the so-called "self-financibility" condition

$$V^{\vartheta}(T) = V^{\vartheta}(0) + \int_0^T \sum_{i=1}^{d} \vartheta_i(t)\, \mathrm{d}\mu_i(t), \qquad T \geq 0. \tag{2.5}$$

We call a trading strategy $\vartheta(\cdot)$ *long-only*, if it never sells any stock short: i.e., if $\vartheta_i(t) \geq 0$ holds for all stocks $i = 1, \ldots, d$ and all times $t \geq 0$.

The vector stochastic integral in (2.5) gives the *gains-from-trade* realized over $[0, T]$. The self-financibility requirement of (2.5) posits that these "gains" account for the entire change in the value generated by the trading strategy $\vartheta(\cdot)$ between the start $t = 0$ and the end $t = T$ of the interval $[0, T]$.

Instead of defining trading strategies in terms of the market weights $\mu(\cdot)$, we could have defined them in terms of the capitalizations $S(\cdot)$ that appear in (2.3). This approach would have led to exactly the same class of trading strategies, thanks to the insights on "changes of numéraires" by Geman, El Karoui and Rochet (1995); see also Proposition 2.3 in Karatzas and Ruf (2017).

The special trading strategy $\vartheta^*(\cdot) = (1, \ldots, 1)'$ buys an equal number of shares in each one of the assets at time $t = 0$, then holds on to these shares indefinitely (equivalently, invests in each asset in proportion to its relative capitalization at all times). This strategy implements the so-called *market portfolio*.

The following observation, a result of straightforward computation, will be needed in the proof of Theorem 5.7.

REMARK 2.3 (Concatenation of trading strategies). Suppose we are given a real number $b \in \mathbb{R}$, a stopping time τ of the underlying filtration $\mathfrak{F}$, and a trading strategy $\varphi(\cdot)$. We then form a new process $\psi(\cdot) = (\psi_1(\cdot), \ldots, \psi_d(\cdot))'$, again

integrable with respect to $\mu(\cdot)$ and with components

$$\psi_i(\cdot) := b + \big(\varphi_i(\cdot) - V^{\varphi}(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}(\cdot), \qquad i = 1, \ldots, d. \tag{2.6}$$

Then the process $\psi(\cdot)$ is a trading strategy itself, and its associated wealth process $V^{\psi}(\cdot)$ is given by

$$V^{\psi}(\cdot) = b + \big(V^{\varphi}(\cdot) - V^{\varphi}(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}(\cdot).$$

3. Functional generation of trading strategies. There is a special class of trading strategies for which the representation of (2.5) takes an exceptionally simple and explicit form; in particular, one in which stochastic integrals disappear entirely from the right-hand side of (2.5). In order to present this class of trading strategies, we start with the definition of regular functions.

DEFINITION 3.1 (Regular function). A continuous mapping $\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}$ is called *regular function* for the vector process $\mu(\cdot)$ of relative market weights, if the process $\boldsymbol{G}(\mu(\cdot))$ is a semimartingale of the form

$$\boldsymbol{G}\big(\mu(\cdot)\big) = \boldsymbol{G}\big(\mu(0)\big) + \int_0^{\cdot} \sum_{i=1}^{d} D_i \boldsymbol{G}\big(\mu(t)\big)\, \mathrm{d}\mu_i(t) - \Gamma^{\boldsymbol{G}}(\cdot) \tag{3.1}$$

for some measurable function $D\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}^d$ and a process $\Gamma^{\boldsymbol{G}}(\cdot)$ of finite variation on compact time intervals.

The notion comes from Karatzas and Ruf (2017). We stress that it pertains to a specific market weight process $\mu(\cdot)$; a function $\boldsymbol{G}$ might be regular with respect to some such process, but not with respect to another.

In this paper, we shall only deal with regular functions $\boldsymbol{G}$ that can be extended to twice continuously differentiable functions in a neighborhood of the set $\boldsymbol{\Delta}^d_+$ in (2.2). *From now on, every regular function $\boldsymbol{G}$ we encounter will be supposed to have this smoothness property.* Then we may assume that $D\boldsymbol{G}(x)$ is the gradient of $\boldsymbol{G}$ evaluated at x, at least for all $x \in \boldsymbol{\Delta}^d_+$. Moreover, the finite-variation process $\Gamma^{\boldsymbol{G}}(\cdot)$ of (3.1) will then be given, on the stochastic interval $[\![0, \mathscr{D}[\![$ and in the notation of (2.4), by the expression

$$\Gamma^{\boldsymbol{G}}(\cdot) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^{\cdot} D^2_{i,j} \boldsymbol{G}\big(\mu(t)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(t), \tag{3.2}$$

a $\boldsymbol{G}$-aggregated cumulative measure of the market's internal variation.

There are two ways in which a regular function $\boldsymbol{G}$ can generate a trading strategy:

1. *Additive functional generation*: The vector process $\boldsymbol{\varphi}^{\boldsymbol{G}}(\cdot) = (\boldsymbol{\varphi}_1^{\boldsymbol{G}}(\cdot), \ldots, \boldsymbol{\varphi}_d^{\boldsymbol{G}}(\cdot))'$ with components

$$\boldsymbol{\varphi}_i^{\boldsymbol{G}}(\cdot) := D_i \boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot) + \boldsymbol{G}\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j \boldsymbol{G}\big(\mu(\cdot)\big), \tag{3.3}$$
$$i = 1, \ldots, d$$

is a trading strategy, and is said to be *additively generated by* $\boldsymbol{G}$. The wealth process $V^{\boldsymbol{\varphi}^{\boldsymbol{G}}}(\cdot)$ associated with this trading strategy has the extremely simple form

$$V^{\boldsymbol{\varphi}^{\boldsymbol{G}}}(\cdot) = \boldsymbol{G}\big(\mu(\cdot)\big) + \Gamma^{\boldsymbol{G}}(\cdot). \tag{3.4}$$

That is, $V^{\boldsymbol{\varphi}^{\boldsymbol{G}}}(\cdot)$ can be represented as the sum of a completely observed and "controlled" term $\boldsymbol{G}(\mu(\cdot))$, plus a "cumulative earnings" term $\Gamma^{\boldsymbol{G}}(\cdot)$.

2. *Multiplicative functional generation*: The second way to generate a trading strategy requires the process $1/\boldsymbol{G}(\mu(\cdot))$ to be locally bounded [equivalently, the continuous, adapted process $\boldsymbol{G}(\mu(\cdot))$ in the denominator not to vanish]. This assumption allows us to define the process

$$Z^{\boldsymbol{G}}(\cdot) := \boldsymbol{G}\big(\mu(\cdot)\big) \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^{\boldsymbol{G}}(t)}{\boldsymbol{G}(\mu(t))} \right) > 0. \tag{3.5}$$

Then the vector process $\boldsymbol{\psi}^{\boldsymbol{G}}(\cdot) = (\boldsymbol{\psi}_1^{\boldsymbol{G}}(\cdot), \ldots, \boldsymbol{\psi}_d^{\boldsymbol{G}}(\cdot))'$ with components

$$\boldsymbol{\psi}_i^{\boldsymbol{G}}(\cdot) := Z^{\boldsymbol{G}}(\cdot) \left(1 + \frac{1}{\boldsymbol{G}(\mu(\cdot))} \left(D_i \boldsymbol{G}\big(\mu(\cdot)\big) - \sum_{j=1}^{d} D_j \boldsymbol{G}\big(\mu(\cdot)\big) \mu_j(\cdot) \right) \right), \tag{3.6}$$
$$i = 1, \ldots, d$$

is a trading strategy, and is said to be *multiplicatively generated by* $\boldsymbol{G}$. The wealth $V^{\boldsymbol{\psi}^{\boldsymbol{G}}}(\cdot)$ associated with this strategy is given by the process of (3.5); namely, $V^{\boldsymbol{\psi}^{\boldsymbol{G}}}(\cdot) = Z^{\boldsymbol{G}}(\cdot)$.

It is important to note that the expressions in (3.3), (3.4) and (3.5), (3.6) are completely devoid of stochastic integrals.

If the regular function $\boldsymbol{G}$ is nonnegative and concave, then it can be shown that both strategies $\boldsymbol{\varphi}^{\boldsymbol{G}}(\cdot)$ and $\boldsymbol{\psi}^{\boldsymbol{G}}(\cdot)$ are *long-only*: to wit, all quantities in (3.3), (3.6) are nonnegative, and in this case the process $\Gamma^{\boldsymbol{G}}(\cdot)$ in (3.1) is actually nondecreasing. More generally, we introduce the following notion.

DEFINITION 3.2 (Lyapunov function). A regular function $\boldsymbol{G}$ is called *Lyapunov function*, if the process $\Gamma^{\boldsymbol{G}}(\cdot)$ in (3.1) is nondecreasing.

For a Lyapunov function $\boldsymbol{G}$, the process $\Gamma^{\boldsymbol{G}}(\cdot)$ has the significance of an aggregated *measure of cumulative internal variation* in the market. The Hessian matrix-valued process $-D^2\boldsymbol{G}(\mu(\cdot))$ which quantifies the aggregation, acts then as a sort of "local curvature" on the covariation matrix of the semimartingale $\mu(\cdot)$ to give us this cumulative measure of variation or "volatility". A concave mapping $\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}$ that can be extended to a twice continuously differentiable function in a neighborhood of the set $\boldsymbol{\Delta}^d_+$ in (2.2) is a Lyapunov function for *any* continuous semimartingale process $\mu(\cdot)$ representing relative market weights.

The theory and applications of functionally-generated trading strategies were developed by Fernholz (1999, 2002); see also Karatzas and Ruf (2017). All claims made in this section are proved in these references, and several examples of functionally-generated trading strategies are discussed. We introduce now four regular functions which will be important in our present context.

1. The *entropy function* of statistical mechanics and information theory

$$\boldsymbol{H} := -\sum_{i=1}^{d} x_i \log x_i, \qquad x \in \boldsymbol{\Delta}^d, \tag{3.7}$$

with the convention $0 \times \log \infty = 0$, is a particularly important regular function [provided that $\mu(\cdot)$ allows for a deflator]. Note that $\boldsymbol{H}$ is concave, thus also a Lyapunov function, and takes values in $[0, \log d]$. It generates additively the long-only *entropy-weighted trading strategy*

$$\boldsymbol{\varphi}^{\boldsymbol{H}}_i(\cdot) = \left(\log\left(\frac{1}{\mu_i(\cdot)}\right) + \Gamma^{\boldsymbol{H}}(\cdot)\right)\mathbf{1}_{\{\mu_i(\cdot)>0\}}, \qquad i = 1, \ldots, d. \tag{3.8}$$

Here,

$$\begin{aligned} \Gamma^{\boldsymbol{H}}(\cdot) &= \frac{1}{2}\sum_{i=1}^{d} \int_0^{\cdot} \frac{\mathrm{d}\langle \mu_i \rangle(t)}{\mu_i(t)} \\ &= \frac{1}{2}\sum_{i=1}^{d} \int_0^{\cdot} \mu_i(t)\, \mathrm{d}\langle \log \mu_i \rangle(t) = \Gamma(\cdot) \qquad \text{on } [\![0, \mathscr{D}[\![\end{aligned} \tag{3.9}$$

coincides with the process $\Gamma(\cdot)$ in (1.1); it denotes both the cumulative earnings of the strategy $\boldsymbol{\varphi}^{\boldsymbol{H}}(\cdot)$ in the manner of (3.4), as well as the $\boldsymbol{H}$-aggregated measure of the market's cumulative internal volatility as in (3.2).

This nondecreasing, trace-like process $\Gamma^{\boldsymbol{H}}(\cdot)$ coincides also with the *cumulative excess growth* of the market in the context of stochastic portfolio theory, where it plays an important role. The process $\Gamma^{\boldsymbol{H}}(\cdot)$ measures the market's cumulative internal variation—stock-by-stock, then averaged according to each stock's weight. As such, it offers a gauge of the market's "intrinsic volatility". Figure 1 uses the monthly stock database of the Center for Research in Securities Prices (CRSP) at the University of Chicago to plot the quantity of (3.9) over the 80-year period 1926–2005.

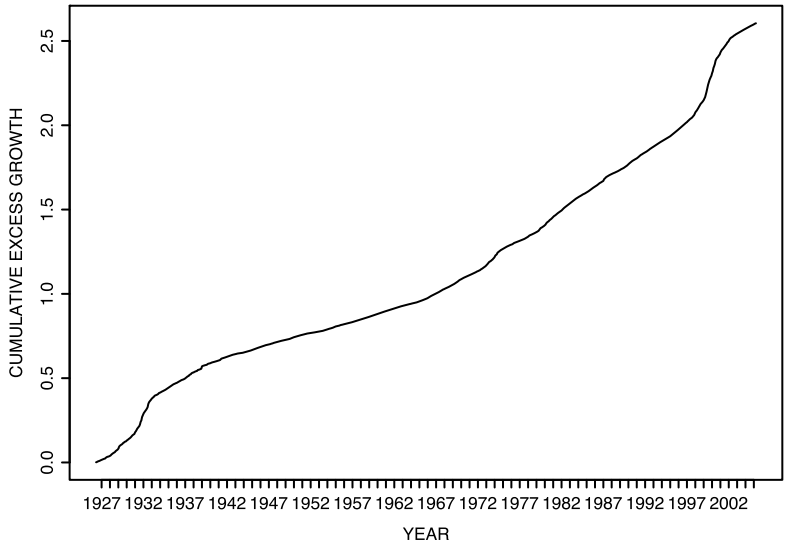

FIG. 1. *Cumulative internal variation* (*excess growth*) $\Gamma^{\boldsymbol{H}}(\cdot)$ *for the U.S. Market*, 1926–2005.

2. The *quadratic function*

$$\boldsymbol{Q}(x) := 1 - \sum_{i=1}^{d} x_i^2, \qquad x \in \boldsymbol{\Delta}^d, \tag{3.10}$$

takes values in $[0, 1 - 1/d]$, and is also a concave regular function. It is mathematically very convenient to work with $\boldsymbol{Q}$, so this function will play a major role when constructing specific counterexamples in Section 6. The corresponding aggregated measure of cumulative variation, or volatility, is given by the nondecreasing trace process

$$\Gamma^{\boldsymbol{Q}}(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle(\cdot). \tag{3.11}$$

We note also that the difference

$$2\Gamma^{\boldsymbol{H}}(\cdot) - \Gamma^{\boldsymbol{Q}}(\cdot)$$

$$= \sum_{i=1}^{d} \int_0^{\cdot} \mathbf{1}_{\{\mu_i(t)>0\}} \left(\frac{1}{\mu_i(t)} - 1 \right) \mathrm{d}\langle \mu_i \rangle(t) \qquad \text{is nondecreasing,} \tag{3.12}$$

where $\Gamma^{\boldsymbol{H}}(\cdot)$ is given in (3.9), thanks to the property $\int_0^{\cdot} \mathbf{1}_{\{\mu_i(t)=0\}} \mathrm{d}\langle \mu_i \rangle(t) = 0$ from Exercise 3.7.10 in Karatzas and Shreve (1991).

3. We shall also have a close look at the concave, *geometric mean function*

$$\boldsymbol{R}(x) := \left(\prod_{i=1}^{d} x_i \right)^{1/d}, \qquad x \in \boldsymbol{\Delta}^d \tag{3.13}$$

with corresponding aggregated cumulative variation process given on $[\![0, \mathscr{D}[\![$ by

$$\Gamma^{\boldsymbol{R}}(\cdot) = \frac{1}{2d}\sum_{i=1}^{d}\int_0^{\cdot} \boldsymbol{R}\big(\mu(t)\big)\,\mathrm{d}\langle \log \mu_i\rangle(t) - \frac{1}{2d^2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^{\cdot} \boldsymbol{R}\big(\mu(t)\big)\,\mathrm{d}\langle \log \mu_i, \log \mu_j\rangle(t).$$

4. Finally, we shall use in one of the proofs the power function

$$\boldsymbol{F}(x) := x_1^q, \qquad x \in \boldsymbol{\Delta}^d \tag{3.14}$$

for $q \geq 1$. Note that $\boldsymbol{F}$ is, for $q > 1$, a *convex* rather than concave function, as was the case in the other three examples. It is nevertheless regular, so it can still be used to generate a trading strategy. Indeed, if $\mu_1(\cdot)$ does not vanish, the multiplicatively generated strategy $\boldsymbol{\psi}(\cdot)$ of (3.6) exists. More precisely, with the process of (3.5) given now by

$$Z^{\mathbf{F}}(\cdot) = \big(\mu_1(\cdot)\big)^q \exp\Big(-\frac{1}{2}q(q-1)\int_0^{\cdot}\big(\mu_1(t)\big)^{-2}\,\mathrm{d}\langle \mu_1\rangle(t)\Big), \tag{3.15}$$

the expression in (3.6) can be written here as

$$\begin{aligned} \boldsymbol{\psi}_1^{\mathbf{F}}(\cdot) &= \left(\frac{q}{\mu_1(\cdot)} + 1 - q\right) Z^{\mathbf{F}}(\cdot); \\ \boldsymbol{\psi}_i^{\mathbf{F}}(\cdot) &= (1-q)Z^{\mathbf{F}}(\cdot) < 0, \qquad i = 2, \ldots, d. \end{aligned} \tag{3.16}$$

The resulting strategy $\boldsymbol{\psi}^{\mathbf{F}}(\cdot)$ takes short positions in all stocks but the first; whereas in this first stock it invests heavily and in a leveraged manner, that is, $\boldsymbol{\psi}_1^{\mathbf{F}}(\cdot) > Z^{\mathbf{F}}(\cdot) \equiv V^{\boldsymbol{\psi}^{\mathbf{F}}}(\cdot)$.

4. Relative arbitrage and an old question. We introduce now the important notion of relative arbitrage with respect to the market.

DEFINITION 4.1 (Relative arbitrage). Given a real constant $T > 0$, we say that a trading strategy $\vartheta(\cdot)$ is *relative arbitrage with respect to the market* over the time horizon $[0, T]$ if $V^{\vartheta}(0) = 1$, $V^{\vartheta}(\cdot) \geq 0$, and

$$\mathsf{P}\big(V^{\vartheta}(T) \geq 1\big) = 1, \qquad \mathsf{P}\big(V^{\vartheta}(T) > 1\big) > 0.$$

If in fact $\mathsf{P}(V^{\vartheta}(T) > 1) = 1$ holds, this relative arbitrage is called *strong*.

REMARK 4.2 (Equivalent martingale measure). Let us fix a real number $T > 0$. Then, over the time horizon $[0, T]$, no relative arbitrage is possible with respect to a market whose relative weight processes $\mu_1(\cdot \wedge T), \ldots, \mu_d(\cdot \wedge T)$ are martingales under some equivalent probability measure $\mathsf{Q}_T \sim \mathsf{P}$ defined on $\mathscr{F}(T)$.

Suppose now that no relative arbitrage is possible over the time horizon $[0, T]$, with respect to a market with relative weights $\mu(\cdot)$. Provided that $\mu(\cdot)$ admits a deflator, there exists then on $\mathscr{F}(T)$ an equivalent probability measure $\mathsf{Q}_T \sim \mathsf{P}$ under which the relative weights $\mu_1(\cdot \wedge T), \ldots, \mu_d(\cdot \wedge T)$ are martingales; see Delbaen and Schachermayer (1994) or Karatzas and Kardaras (2007).

Since the process $\mu(\cdot)$ expresses the market portfolio, the arbitrage of Definition 4.1 is interpreted as relative arbitrage with respect to the market. The question of whether a given market portfolio can be "outperformed" as in Definition 4.1, is of great theoretical and practical importance—particularly given the proliferation of index-type mutual funds[3] which try to track and possibly outperform a specific market portfolio or "index" or "benchmark".

To wit: *Under what conditions is there relative arbitrage with respect to the market? over which time horizons? if it exists, can such relative arbitrage be strong?*

Functionally-generated trading strategies are ideal for answering such questions, thanks to the representations of (3.4) and (3.5) which describe their performance relative to the market in a pathwise manner, devoid of stochastic integration. The following result is taken from Karatzas and Ruf (2017); its lineage goes back to Fernholz (2002) and to Fernholz and Karatzas (2005). In our present context, it is a straightforward consequence of the representation (3.4).

THEOREM 4.3 (Strong relative arbitrage over sufficiently long time horizons). *Suppose that $\boldsymbol{G} : \boldsymbol{\Delta}^d \to [0, \infty)$ is a Lyapunov function with $\boldsymbol{G}(\mu(0)) > 0$. Suppose, moreover, that there is a real number $T_* > 0$ with the property*

$$\mathsf{P}\big(\Gamma^{\boldsymbol{G}}(T_*) > \boldsymbol{G}(\mu(0))\big) = 1. \tag{4.1}$$

Then the trading strategy $\boldsymbol{\varphi}^{\boldsymbol{G}_*}(\cdot) = (\boldsymbol{\varphi}_1^{\boldsymbol{G}_*}(\cdot), \ldots, \boldsymbol{\varphi}_d^{\boldsymbol{G}_*}(\cdot))'$, generated additively in the manner of (3.3) by the function $\boldsymbol{G}_* := \boldsymbol{G}/\boldsymbol{G}(\mu(0))$, is strong relative arbitrage with respect to the market over any time horizon $[0, T]$ with $T \in [T_*, \infty)$.

The following result is also a direct corollary of the representation (3.4). It pertains to regular functions $\boldsymbol{G} : \boldsymbol{\Delta}^d \to [0, \infty)$ that satisfy

$$\begin{aligned} &\mathsf{P}\big(\text{the mapping } [0, \infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\big) = 1 \\ &\qquad \text{for some constant } \eta > 0. \end{aligned} \tag{4.2}$$

This condition will be very important from now onwards.

[3] As Jason Zweig writes for *The Wall Street Journal* in his August 31, 2016 article *"Birth of the Index Mutual Fund: Bogle's Folly turns 40"*: "Today [...] index mutual funds and exchange-traded funds invest nearly $5 trillion in combined assets".

PROPOSITION 4.4 (Slope bounded from below). *Suppose that* $\boldsymbol{G} : \boldsymbol{\Delta}^d \to [0, \infty)$ *is a regular function with* $\boldsymbol{G}(\mu(0)) > 0$ *satisfying the condition* (4.2). *Then the trading strategy* $\boldsymbol{\varphi}^{\boldsymbol{G}_*}(\cdot)$ *of Theorem* 4.3 *is strong relative arbitrage with respect to the market, over any time horizon* $[0, T]$ *with*

$$\frac{\boldsymbol{G}(\mu(0))}{\eta} < T < \infty. \tag{4.3}$$

The assertion of Proposition 4.4 appears already in Fernholz and Karatzas (2005) for the entropy function $\boldsymbol{H}$ of (3.7). In this case, the condition of (4.2) posits that the cumulative internal variation $\Gamma^{\boldsymbol{H}}(\cdot)$ as in (3.9) or (1.1) is not just increasing, but actually dominates a straight line with positive slope. This assumption can be read most instructively in conjunction with the plot of Figure 1. Under it, Proposition 4.4 guarantees the existence of relative arbitrage with respect to the market over any time horizon $[0, T]$ of finite length $T > \boldsymbol{H}(\mu(0))/\eta$.

REMARK 4.5. The following question was raised in Fernholz and Karatzas (2005), and was posed again in Banner and Fernholz (2008).

Assume that (4.2) *holds with* $\boldsymbol{G} = \boldsymbol{H}$, *the entropy function of* (3.7). *Is then relative arbitrage with respect to the market possible over every time horizon* $[0, T]$ *of finite length* $T > 0$*?*

In Section 5, we shall present results which guarantee, under appropriate *additional* conditions, affirmative answers to this question. Then in Section 6 we shall construct market models illustrating that, in general, the answer to the above question is negative. This settles an issue which had remained open for more than 10 years.

5. Existence of short-term relative arbitrage. Given a Lyapunov function $\boldsymbol{G} : \boldsymbol{\Delta}^d \to [0, \infty)$ with $\boldsymbol{G}(\mu(0)) > 0$, Theorem 4.3 provides the condition $\mathsf{P}(\Gamma^{\boldsymbol{G}}(T) > \boldsymbol{G}(\mu(0))) = 1$ on the length $T > 0$ of the time horizon $[0, T]$ as sufficient for the existence of strong relative arbitrage with respect to the market over $[0, T]$.

In this section, we study conditions under which relative arbitrage exists on the time horizon $[0, T]$, for any given real number $T > 0$. Section 5.1 discusses conditions that guarantee the existence of short-term relative arbitrage which is strong. The conditions of Section 5.2 guarantee only the existence of short-term relative arbitrage, not necessarily strong.

5.1. *Existence of strong short-term relative arbitrage.* The following theorem greatly extends and simplifies the results in Section 8 of Fernholz, Karatzas and Kardaras (2005) and in Section 8 of Fernholz and Karatzas (2009).

THEOREM 5.1 (One asset with sufficient variation). *In a market as in Section 2 with relative weight processes $\mu_1(\cdot), \ldots, \mu_d(\cdot)$, suppose there exists a constant $\eta > 0$ such that $\langle \mu_1 \rangle(t) \geq \eta t$ holds on the stochastic interval $[\![0, \mathscr{D}^*[\![$ with*

$$\mathscr{D}^* := \inf\left\{ t \geq 0 : \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.$$

Then, given any real number $T > 0$, there exists a long-only trading strategy $\varphi(\cdot)$ which is strong relative arbitrage with respect to the market over the time horizon $[0, T]$.

PROOF. Let us fix a real number $T > 0$ and consider the market with weights $\nu(\cdot) := \mu(\cdot \wedge \mathscr{D}^*)$. It suffices to prove the existence of a long-only trading strategy $\varphi(\cdot)$ which is strong relative arbitrage with respect to the new market with weights $\nu(\cdot)$ over the time horizon $[0, T]$. For then the long-only trading strategy $\varphi(\cdot \wedge \mathscr{D}^*)$ is strong relative arbitrage with respect to the original market with weights $\mu(\cdot)$ over the time horizon $[0, T]$.

For some number $q > 1$ to be determined presently, we recall the regular function $\boldsymbol{F}$ of (3.14). Since $1/\nu_1(\cdot)$ is locally bounded, $\boldsymbol{F}$ generates multiplicatively, for the market with weight process $\nu(\cdot)$, the strategy $\psi^{\boldsymbol{F}}(\cdot)$ given by (3.16), with $\mu(\cdot)$ replaced by $\nu(\cdot)$. We note that $\boldsymbol{\psi}_i^{\boldsymbol{F}}(\cdot) < 0$ holds for $i = 2, \ldots, d$, that $\boldsymbol{\psi}_1^{\boldsymbol{F}}(\cdot) \leq 1$, and that $V^{\boldsymbol{\psi}^{\boldsymbol{F}}}(\cdot) = Z^{\boldsymbol{F}}(\cdot)$ is given as in (3.15). We introduce now the long-only trading strategy

$$\varphi_i(\cdot) = 1 + \big(\nu_1(0)\big)^q - \boldsymbol{\psi}_i^{\boldsymbol{F}}(\cdot), \qquad i = 1, \ldots, d$$

with associated wealth process

$$V^{\varphi}(\cdot) = 1 + \big(\nu_1(0)\big)^q - Z^{\boldsymbol{F}}(\cdot).$$

In particular, we note $V^{\varphi}(0) = 1$ and $V^{\varphi}(\cdot) \geq 0$. On the event $\{\mathscr{D}^* \leq T\}$, we have

$$V^{\varphi}(T) \geq 1 + \big(\nu_1(0)\big)^q - \big(\nu_1(T)\big)^q = 1 + \big(\nu_1(0)\big)^q - \left(\frac{\nu_1(0)}{2}\right)^q > 1;$$

whereas, on the event $\{\mathscr{D}^* > T\}$ we have

$$\begin{aligned} V^{\varphi}(T) &\geq 1 + \big(\nu_1(0)\big)^q - \exp\left(-\frac{1}{2} q(q-1)\langle \nu_1 \rangle(T)\right) \\ &\geq 1 + \big(\nu_1(0)\big)^q - \left(\exp\left(-\frac{\eta}{2}(q-1)T\right)\right)^q > 1, \end{aligned}$$

provided we choose $q = q(T)$ large enough so that

$$\exp\left(-\frac{\eta}{2}(q-1)T\right) < \nu_1(0) = \mu_1(0). \tag{5.1}$$

This shows $\mathsf{P}(V^{\varphi}(T) > 1) = 1$, as claimed. □

REMARK 5.2 [Dependence of $\varphi(\cdot)$ on the length of the time horizon]. The trading strategy $\varphi(\cdot)$ of Theorem 5.1 is constructed in a very explicit and "model-free" manner, but *does* depend on the length of the time horizon over which it effects arbitrage with respect to the market. The shorter the length of the time-horizon $T > 0$, the higher the "leverage parameter" $q = q(T)$ needs to be in (5.1).

The following two remarks recall two alternative ways for obtaining the existence of strong relative arbitrage opportunities over arbitrary time horizons.

REMARK 5.3 (Smallest asset with sufficient variation). In the spirit of Theorem 5.1, Banner and Fernholz (2008) also prove the existence of a strong relative arbitrage over arbitrary time horizons. However, they do not assume that one fixed asset contributes to the overall market volatility—but rather that it is always the smallest stock that has sufficient variation. The strategy they construct is again "model-free" but does depend on the length of the time horizon.

REMARK 5.4 (Completeness and arbitrage imply strong arbitrage). If the underlying market model allows for a deflator (recall Definition 2.1) and is complete (any contingent claim can be replicated), then the existence of an arbitrage opportunity implies the existence of a strong one; see Theorem 8 in Ruf (2011). However, this strong arbitrage usually will depend on the model and on the length of the time horizon.

Diversity and strict nondegeneracy.

DEFINITION 5.5 (Diversity). We say that a market with weight processes $\mu_1(\cdot), \ldots, \mu_d(\cdot)$ is *diverse* if

$$\mathbb{P}\Big(\sup_{t\in[0,\infty)} \max_{1\le i\le n} \mu_i(t) \le 1-\delta \Big) = 1 \qquad \text{holds for some constant } \delta \in (0,1). \tag{5.2}$$

Diversity posits that no company can come close to dominating the entire market capitalization. It is a typical characteristic of large, deep and liquid equity markets.

Let us assume now that the continuous semimartingales $S_1(\cdot), \ldots, S_d(\cdot)$ in (2.3), to wit, the capitalization processes of the various assets in the market, have covariations of the form

$$\langle S_i, S_j \rangle(\cdot) = \int_0^{\cdot} S_i(t) S_j(t) A_{i,j}(t)\, \mathrm{d}t, \qquad i, j = 1, \ldots, d \tag{5.3}$$

for a suitable symmetric, nonnegative definite matrix-valued and progressively measurable process $A(\cdot) = (A_{i,j}(\cdot))_{1\le i,j\le d}$.

COROLLARY 5.6 (Diversity and strict nondegeneracy). *Suppose that a market as in* (2.3) *is diverse, that* (5.3) *holds, and that the asset covariation matrix-valued process* $A(\cdot) = (A_{i,j}(\cdot))_{1\le i,j\le d}$ *of* (5.3) *satisfies the strict nondegeneracy condition*

$$\xi' A(t,\omega)\xi \ge \lambda\|\xi\|^2 \qquad \text{for all } \xi \in \mathbb{R}^d,\ (t,\omega) \in [0,\infty)\times\Omega \tag{5.4}$$

for some $\lambda > 0$. *Then, given any real number* $T > 0$, *there exists a long-only trading strategy* $\varphi(\cdot)$ *which is strong relative arbitrage with respect to the market over the time horizon* $[0, T]$.

PROOF. With the help of (2.6) and (3.11) in Fernholz and Karatzas (2009), the conditions in (5.2) and (5.4)—namely, diversity and strict nondegeneracy—lead to the lower bound

$$\langle \mu_1 \rangle\big(t\wedge\mathscr{D}^*\big) \ge \lambda \int_0^{t\wedge\mathscr{D}^*} \big(\mu_1(s)\big)^2\Big(1-\max_{1\le i\le n}\mu_i(s)\Big)^2\,\mathrm{d}s \ge \eta\big(t\wedge\mathscr{D}^*\big), \qquad t\ge 0$$

for $\eta := \lambda(\delta\mu_1(0)/2)^2$. The claim is now a direct consequence of Theorem 5.1. □

5.2. *Existence of short-term relative arbitrage, not necessarily strong.* In this subsection, we provide three more criteria that guarantee the existence of relative arbitrage with respect to the market over arbitrary time horizons. The first criterion is a condition on the time-homogeneity of the support of the market weight vector process $\mu(\cdot)$. The second criterion uses failure of diversity. The third criterion is a condition on the nondegeneracy of the covariation matrix-valued process of $\mu(\cdot)$.

5.2.1. *Time-homogeneous support.* Let us recall the condition in (4.1). There, the threshold $\boldsymbol{G}(\mu(0))$ "is at its lowest", when the initial market-weight configuration $\mu(0)$ is at a site where $\boldsymbol{G}$ attains, or is very close to, its smallest value on $\boldsymbol{\Delta}^d$. These are the most propitious sites from which relative arbitrage can be launched: starting at them the accumulated volatility is large relative to the overall displacement of $\boldsymbol{G}$ evaluated along the market weight vector process $\mu(\cdot)$.

The following result assumes that the essential infimum of the continuous semimartingale $\boldsymbol{G}(\mu(\cdot))$ remains constant over time. In particular, that sites in the state space which are close to this infimum—and thus "propitious" for launching relative arbitrage—can be visited "quickly" (i.e., over any given time horizon) with positive probability. The result shows that relative arbitrage with respect to such a market can then be realized over *any* time horizon $[0, T]$ of arbitrary length $T > 0$.

THEOREM 5.7 (A time-homogeneity condition on the support). *Suppose that for a given regular function* $\boldsymbol{G}$ *and appropriate real constants* $\eta > 0$ *and* $h \ge 0$, *the condition in* (4.2) *is satisfied along with the lower bound*

$$\mathsf{P}\big(\boldsymbol{G}\big(\mu(t)\big) \ge h,\ t \ge 0\big) = 1 \tag{5.5}$$

and the "time homogeneous support" property

$$\text{(5.6)}\quad \mathsf{P}\big(\boldsymbol{G}(\mu(t)) \in [h, h+\varepsilon) \text{ for some } t \in [0,T]\big) > 0 \qquad \text{for all } T > 0, \varepsilon > 0.$$

Then arbitrage relative to the market exists over the time horizon $[0,T]$, *for every real number* $T > 0$.

The basic argument in the proof of Theorem 5.7 is quite simple to describe: Given a time horizon $[0,T]$, the condition in (5.6) declares that the vector process $\mu(\cdot)$ of relative market weights will visit, before time $T/2$ and with positive probability, sites which are "very propitious" for arbitrage relative to the market. The moment this happens it makes good sense to switch from the market portfolio to the trading strategy $\boldsymbol{\varphi}^\star(\cdot)$ generated by the function $\boldsymbol{G}^\star = c(\boldsymbol{G} - h)$ in the manner of (3.3), for some appropriately chosen constant $c > 0$. The challenge then is to show that such a strategy does not underperform the market, and has a positive probability of outperforming it strictly.

PROOF OF THEOREM 5.7. For an arbitrary but fixed real number $T > 0$, we introduce the regular function

$$\boldsymbol{G}^\star := (\boldsymbol{G} - h)\frac{3}{\eta T} \quad \text{and denote} \quad \Gamma^\star(\cdot) := \Gamma^{\boldsymbol{G}^\star}(\cdot) = \frac{3}{\eta T}\Gamma^{\boldsymbol{G}}(\cdot).$$

We also introduce the stopping time

$$\tau := \inf\left\{t \in \left[0, \frac{T}{2}\right] : \boldsymbol{G}(\mu(t)) < h + \frac{\eta T}{3}\right\}$$

with the usual convention that the infimum of the empty set is equal to infinity, and note that (5.6) implies

$$\text{(5.7)}\qquad \mathsf{P}\left(\tau \le \frac{T}{2}\right) > 0.$$

We let $\boldsymbol{\varphi}^\star(\cdot) := \boldsymbol{\varphi}^{\boldsymbol{G}^\star}(\cdot)$ denote the trading strategy generated by the function $\boldsymbol{G}^\star$ in the manner of (3.3). We also introduce the trading strategy $\boldsymbol{\varphi}(\cdot)$ which follows the market portfolio up to the stopping time τ, then switches for the remainder of the time horizon $[0,T]$ to the trading strategy $\boldsymbol{\varphi}^\star(\cdot)$; namely,

$$\boldsymbol{\varphi}_i(\cdot) := 1 + \big(\boldsymbol{\varphi}_i^\star(\cdot) - \boldsymbol{G}^\star(\mu(\tau)) - \Gamma^\star(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}, \qquad i = 1, \ldots, d$$

in the "self-financed" manner of (2.6). According to (5.7), this switching occurs with positive probability. As we saw in Remark 2.3, the value that results from this concatenation is given by

$$\begin{aligned} V^{\boldsymbol{\varphi}}(t) &= 1 + \big(\boldsymbol{G}^\star(\mu(t)) + \Gamma^\star(t) - \boldsymbol{G}^\star(\mu(\tau)) - \Gamma^\star(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}(t) \\ &\ge \mathbf{1}_{[\![0,\tau[\![}(t) + \frac{3}{\eta T}\big(\Gamma^{\boldsymbol{G}}(t) - \Gamma^{\boldsymbol{G}}(\tau)\big)\mathbf{1}_{[\![\tau,\infty[\![}(t) \\ &\ge \mathbf{1}_{[\![0,\tau[\![}(t) + \frac{3(t-\tau)}{T}\mathbf{1}_{[\![\tau,\infty[\![}(t), \qquad t \ge 0. \end{aligned}$$

Here, we have used the comparisons $\boldsymbol{G}^\star(\mu(\cdot)) \geq 0$ and $\boldsymbol{G}^\star(\mu(\tau)) \leq 1$ in the first inequality, and (4.2) in the second inequality.

Now it clear from this last display that $V^{\boldsymbol{\varphi}}(\cdot) \geq 0$ holds, and that $V^{\boldsymbol{\varphi}}(T) \geq 3/2$ holds on the event $\{\tau \leq T/2\}$; it is also clear that $V^{\boldsymbol{\varphi}}(T) = 1$ holds on $\{\tau > T/2\} = \{\tau = \infty\}$. Since the event $\{\tau \leq T/2\}$ has positive probability on account of (5.7), it follows that the trading strategy $\boldsymbol{\varphi}(\cdot)$ is relative arbitrage with respect to the market over the time horizon $[0, T]$. □

REMARK 5.8 (On the type of arbitrage). There is nothing in the above argument to suggest that the probability in (5.7), which is argued there to be positive, is actually equal to 1. Thus, the relative arbitrage constructed in Theorem 5.7 need not be strong. It should also be noted that the trading strategy which implements this arbitrage depends on the length T of the time horizon $[0, T]$—in marked contrast to the strategy of Theorem 4.3 which does not, as long as $T \geq T_*$.

5.2.2. *Failure of diversity.* Theorem 5.7 has the following corollary. Taken together, Corollaries 5.6 and 5.9 illustrate that both diversity, *and* its failure, can lead to arbitrage over arbitrary time horizons—under appropriate additional conditions in each case. For the statement, we let $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ denote the extremal points (unit vectors) of the lateral face $\boldsymbol{\Delta}^d$ of the unit simplex.

COROLLARY 5.9 (Failure of diversity). *Suppose that diversity fails for a market with relative weights $\mu(\cdot)$, in the sense that*

$$\mathsf{P}\Big(\sup_{t\in[0,T)} \max_{1\leq i\leq d} \mu_i(t) > 1-\delta\Big) > 0 \qquad \text{holds for every } (T,\delta) \in (0,\infty)\times(0,1).$$

Suppose also that, for some nonnegative regular function $\boldsymbol{G}$ which satisfies

$$\boldsymbol{G}(\mathfrak{e}_i) = \min_{x\in\boldsymbol{\Delta}^d} \boldsymbol{G}(x) \qquad \text{for each } i = 1, \ldots, d,$$

the condition in (4.2) holds for some real constant $\eta > 0$. Relative arbitrage with respect to the market exists then over every time horizon $[0, T]$ of finite length $T > 0$.

5.2.3. *Strict nondegeneracy.* Theorem 5.7 has yet another important consequence, Theorem 5.10 below. This establishes the existence of relative arbitrage with respect to the market, under the "sufficient intrinsic volatility" condition of (4.2) and under additional nondegeneracy conditions. Theorem 5.10 is proved at the end of this subsection.

In order to prepare the ground for the result, let us recall the trace process $\Gamma^{\boldsymbol{Q}}(\cdot)$ of (3.11). From Proposition II.2.9 of Jacod and Shiryaev (2003), we have the representation

$$\langle \mu_i, \mu_j \rangle(\cdot) = \int_0^\cdot \alpha_{i,j}(t)\, \mathrm{d}\Gamma^{\boldsymbol{Q}}(t), \qquad 1 \leq i, j \leq d \tag{5.8}$$

for some symmetric and nonnegative-definite matrix-valued process $\alpha(\cdot) = (\alpha_{i,j}(\cdot))_{i,j=1,\dots,d}$ whose entries are progressively measurable and satisfy $\sum_{j=1}^{d} \alpha_{i,j}(\cdot) \equiv 0$ for every $i = 1, \dots, d$. Furthermore, thanks to the Kunita–Watanabe inequality [Proposition 3.2.14 in Karatzas and Shreve (1991)], the process $\alpha_{i,j}(\cdot)$ takes values in $[-1, 1]$ for every $1 \le i, j \le d$. Since

$$\Gamma^{Q}(\cdot) = \sum_{i=1}^{d} \langle \mu_i \rangle(\cdot) = \int_0^{\cdot} \left(\sum_{i=1}^{d} \alpha_{i,i}(t) \right) \mathrm{d}\Gamma^{Q}(t),$$

we may also assume that

$$\sum_{i=1}^{d} \alpha_{i,i}(\cdot) = 1. \tag{5.9}$$

We shall also consider the sequence $(\mathscr{D}^n)_{n\in\mathbb{N}}$ of stopping times

$$\mathscr{D}^n := \inf\left\{ t \ge 0 : \min_{1\le i\le d} \mu_i(t) < \frac{1}{n} \right\}, \qquad n \in \mathbb{N}. \tag{5.10}$$

THEOREM 5.10 (A strict nondegeneracy condition). *Let us suppose that for the process $\mu(\cdot) = (\mu_1(\cdot), \dots, \mu_d(\cdot))'$ of market weights there exists a deflator, as well as a regular function $\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}$ which satisfies the condition of (4.2) for some real constant $\eta > 0$. Moreover, we assume that the $d-1$ largest eigenvalues of the matrix-valued process $\alpha(\cdot)$ in (5.8) are bounded away from zero on $[\![0, \mathscr{D}^n[\![$ uniformly in (t, ω), for each $n \in \mathbb{N}$ and with the notation of (5.10). Then relative arbitrage with respect to the market exists over $[0, T]$, for every real number $T > 0$.*

The proof of Theorem 5.10 is at the end of this subsection.

For the conclusion of this result to hold, it is *not* sufficient that the $d-1$ largest eigenvalues of the matrix-valued process $\alpha(\cdot)$ be strictly positive. If these eigenvalues are not additionally bounded away from zero, an example is constructed in the proof of Proposition 6.11 where relative arbitrage with respect to the market does not exist over the time horizon $[0, T]$ for some suitable real number $T > 0$.

It is important to stress that Theorem 5.10 establishes only the *existence* of a trading strategy, which effects the claimed relative arbitrage. Moreover, unlike the trading strategy of Theorem 4.3 which is strong relative arbitrage, explicit, model-free, and independent of the time horizon, the trading strategy whose existence is claimed in Theorem 5.10 may be none of these things.

REMARK 5.11 (Itô-process covariation structure). If $\langle \mu_i, \mu_j \rangle(\cdot) = \int_0^\cdot \beta_{i,j}(t)\,\mathrm{d}t$ holds for all $i, j = 1, \ldots, d$, then

$$\Gamma^{\boldsymbol{Q}}(\cdot) = \sum_{j=1}^{d} \int_0^\cdot \beta_{j,j}(t)\,\mathrm{d}t;$$

$$\alpha_{i,j}(\cdot)\mathbf{1}_{\{\sum_{k=1}^d \beta_{k,k}(\cdot)>0\}} = \frac{\beta_{i,j}(\cdot)}{\sum_{k=1}^d \beta_{k,k}(\cdot)}\mathbf{1}_{\{\sum_{k=1}^d \beta_{k,k}(\cdot)>0\}}, \qquad i, j = 1, \ldots, d.$$

Hence, in this case, a sufficient (though not necessary) condition for the nondegeneracy assumption in Theorem 5.10 to hold, is that the $d-1$ largest eigenvalues of the matrix-valued process $\beta(\cdot)$ be bounded away from zero and from infinity on $[\![0, \mathscr{D}^n[\![$, for each $n \in \mathbb{N}$.

The proof of Theorem 5.10 uses the following lemma.

LEMMA 5.12 (Sum of quadratic variations bounded from below). *Assume that there exist a regular function $\boldsymbol{G}$ and a constant $\eta > 0$ such that (4.2) is satisfied. Then, for each $n \in \mathbb{N}$, there exists a real constant $C_n = C(n, \eta, d, \boldsymbol{G})$ such that the mapping $t \mapsto \Gamma^{\boldsymbol{Q}}(t) - C_n t$ is nondecreasing on $[\![0, \mathscr{D}^n[\![$.*

PROOF. Let us fix $n \in \mathbb{N}$. Thanks to (3.2), (5.8) we get

$$\Gamma^{\boldsymbol{G}}(\cdot) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^\cdot D^2_{i,j}\boldsymbol{G}\big(\mu(t)\big)\alpha_{i,j}(t)\,\mathrm{d}\Gamma^{\boldsymbol{Q}}(t) \qquad \text{on } [\![0, \mathscr{D}^n[\![.$$

Next, we observe that $\sum_{i=1}^d \sum_{j=1}^d |D^2_{i,j}\boldsymbol{G}(\mu(\cdot))|$ is bounded from above by a real constant $K_n > 0$ on the stochastic interval $[\![0, \mathscr{D}^n[\![$. The inequality $|\alpha_{i,j}(\cdot)| \le 1$ for all $1 \le i, j \le d$ implies that the difference

$$K_n\Gamma^{\boldsymbol{Q}}(\cdot) - 2\Gamma^{\boldsymbol{G}}(\cdot) \qquad \text{is nondecreasing on } [\![0, \mathscr{D}^n[\![. \tag{5.11}$$

Hence, (4.2) yields the statement with $C_n = 2\eta/K_n$. □

Proposition 5.13 right below shows that in the case $d = 2$ of two assets, the condition in (4.2) yields, for *every* given time horizon, the existence of a long-only trading strategy which is strong relative arbitrage with respect to the market over this time horizon. This result is a consequence of Theorem 5.1; its proof requires, however, the technical observation made in Lemma 5.12.

PROPOSITION 5.13 (The case of two assets). *Assume that $d = 2$ and that there exist a regular function $\boldsymbol{G}$ and a real constant $\eta > 0$ such that (4.2) is satisfied. Then strong arbitrage relative to the market can be realized by a long-only trading strategy over the time horizon $[0, T]$, for any given real number $T > 0$.*

PROOF. This follows directly from Lemma 5.12 and Theorem 5.1, as in this case $\mu_1(\cdot) + \mu_2(\cdot) = 1$ and $\Gamma^{\mathcal{Q}}(\cdot) = 2\langle \mu_1 \rangle(\cdot)$. $\square$

Alternatively, a weaker formulation of Proposition 5.13, which guarantees the existence of relative arbitrage over any given time horizon but not the fact that this relative arbitrage is strong, can be obtained from Theorem 5.10. To apply this result, it suffices to check that the largest eigenvalue of $\alpha(\cdot)$ equals 1. However, this is easy to see here: we have $\mu_2(\cdot) = 1 - \mu_1(\cdot)$; hence $\alpha_{1,1}(\cdot) = \alpha_{2,2}(\cdot) = 1/2$ and $\alpha_{1,2}(\cdot) = \alpha_{2,1}(\cdot) = -1/2$, so the eigenvalues of the matrix $\alpha(\cdot)$ are then indeed 0 and 1.

PROOF OF THEOREM 5.10. We may assume without loss of generality that, in addition to satisfying (4.2), the function $\boldsymbol{G}$ is nonnegative. We shall argue by contradiction, assuming that for some real number $T_* > 0$ no relative arbitrage is possible with respect to the market on the time horizon $[0, T_*]$. Remark 4.2 gives then the existence of an equivalent probability measure $\mathsf{Q}_* \sim \mathsf{P}$ on $\mathscr{F}(T_*)$, under which the relative market weights $\mu_1(\cdot \wedge T_*), \ldots, \mu_d(\cdot \wedge T_*)$ are martingales.

The plan is to show that this leads to the *time-homogeneous support* property (5.6) with $h = \min_{x \in \boldsymbol{\Delta}^d} \boldsymbol{G}(x)$, and hence, on the strength of Theorem 5.7, to the desired contradiction. In order to make headway with this approach, we fix real numbers $\varepsilon > 0$ and $T \in (0, T_*]$, define

$$\mathcal{U} := G^{-1}\big([h, h+\varepsilon)\big) \cap (0,1)^d \subset \boldsymbol{\Delta}_+^d, \tag{5.12}$$

choose a point $x \in \mathcal{U}$, and fix an integer $N \in \mathbb{N}$ large enough so that

$$\min_{1 \le i \le d} x_i > \frac{2}{N}; \qquad \min_{1 \le i \le d} \mu_i(0) > \frac{2}{N}.$$

We recall the constant $C_N = C(N, \eta, d, \boldsymbol{G})$ from Lemma 5.12, define the stopping time

$$\varrho := \inf\big\{t \ge 0 : \Gamma^{\mathcal{Q}}(t) > C_N T\big\}, \tag{5.13}$$

and note that Lemma 5.12 yields the set inclusion

$$\big\{\mathscr{D}^N \ge T\big\} \subset \{\varrho \le T\}. \tag{5.14}$$

Now in order to obtain the property (5.6) of Theorem 5.7 it suffices, on account of (5.12) and of the equivalence of Q_* and P, to show that the stopped process

$$\nu(\cdot) := \mu(\cdot \wedge \varrho) \tag{5.15}$$

satisfies $\mathsf{Q}_*(\nu(T) \in \mathcal{U}) > 0$. This, in turn, will follow as soon as we have shown

$$\mathsf{Q}_*\left(\sum_{i=1}^d \big(\nu_i(T) - x_i\big)^2 < \delta\right) > 0. \tag{5.16}$$

Here, the constant $\delta \in (0, 1/N)$ is sufficiently small so that, for all $y \in \mathbf{\Delta}^d$, we have

$$\sum_{i=1}^{d}(y_i - x_i)^2 < \delta \quad \text{implies} \quad y \in \mathcal{U} \quad \text{and} \quad \min_{1\le i\le d} y_i > \frac{1}{N};$$

$$\sum_{i=1}^{d}\big(y_i - \mu_i(0)\big)^2 < \delta \quad \text{implies} \quad \min_{1\le i\le d} y_i > \frac{1}{N}.$$

Clearly, on account of (5.13), we have for the stopped process $\nu(\cdot)$ in (5.15) the upper bound

$$\sum_{i=1}^{d}\langle \nu_i\rangle(\cdot) = \sum_{i=1}^{d}\langle \mu_i\rangle(\cdot \wedge \boldsymbol{\varrho}) = \Gamma^{\mathbf{Q}}(\cdot \wedge \boldsymbol{\varrho}) \le C_N T. \tag{5.17}$$

In order to establish (5.16), we modify the arguments in Stroock and Varadhan (1972) and Stroock (1971). We fix the vector

$$\zeta := \frac{x - \mu(0)}{C_N T} \tag{5.18}$$

and let $\alpha^\flat(\cdot)$ denote the Moore–Penrose pseudo-inverse of the matrix-valued process $\alpha(\cdot)$ in (5.8). Next, we note that the vector ζ is in the range of $\alpha(\cdot)$ on $[\![0, \mathscr{D}^N[\![$, since the matrix-valued process $\alpha(\cdot)$ has rank $d-1$ and satisfies $\alpha(\cdot)\mathfrak{e} = 0$, where $\mathfrak{e} = (1, \ldots, 1)'$. Thus, we have $\alpha(\cdot)\alpha^\flat(\cdot)\zeta = \zeta$ on $[\![0, \mathscr{D}^N[\![$.

We introduce now the continuous Q_*-local martingale

$$M(\cdot) := \int_0^{\cdot\wedge\mathscr{D}^N} \big\langle \alpha^\flat(t)\zeta, \mathrm{d}\nu(t)\big\rangle = \int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \big\langle \alpha^\flat(t)\zeta, \mathrm{d}\mu(t)\big\rangle.$$

The quadratic variation of this local martingale is dominated by a real constant, namely

$$\begin{aligned}\langle M\rangle(\cdot) &= \int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \big(\zeta'\alpha^\flat(t)\big)\alpha(t)\big(\alpha^\flat(t)\zeta\big)\,\mathrm{d}\Gamma^{\mathbf{Q}}(t) = \int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \zeta'\alpha^\flat(t)\zeta\,\mathrm{d}\Gamma^{\mathbf{Q}}(t)\\ &\le \frac{\zeta'\zeta}{c_N}\Gamma^{\mathbf{Q}}\big(\cdot \wedge \mathscr{D}^N \wedge \boldsymbol{\varrho}\big) \le \frac{1}{C_N T c_N}\sum_{i=1}^{d}\big(x_i - \mu_i(0)\big)^2,\end{aligned}$$

on account of (5.17) and (5.18); here the real constant $c_N > 0$ stands for a lower bound on the smallest positive eigenvalue of $\alpha(\cdot)$ on the stochastic interval $[\![0, \mathscr{D}^N[\![$. Likewise, we have

$$\begin{aligned}\langle \nu_i, M\rangle(\cdot) &= \sum_{j=1}^{d}\sum_{k=1}^{d}\int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \zeta_k\alpha^\flat_{k,j}(t)\alpha_{i,j}(t)\,\mathrm{d}\Gamma^{\mathbf{Q}}(t)\\ &= \zeta_i\Gamma^{\mathbf{Q}}\big(\cdot \wedge \mathscr{D}^N \wedge \boldsymbol{\varrho}\big)\\ &= \frac{x_i - \mu_i(0)}{C_N T}\Gamma^{\mathbf{Q}}\big(\cdot \wedge \mathscr{D}^N \wedge \boldsymbol{\varrho}\big), \qquad i = 1, \ldots, d.\end{aligned} \tag{5.19}$$

The Novikov theorem [Proposition 3.5.12 in Karatzas and Shreve (1991)] now implies that the stochastic exponential $\mathcal{E}(M(\cdot))$ is a uniformly integrable Q_*-martingale. Thus, this exponential martingale generates a new probability measure Q on $\mathscr{F}(T_*)$, which is equivalent to Q_*. According to the van Schuppen–Wong extension of the Girsanov theorem (ibid., Exercise 3.5.20), the processes

$$X_i(\cdot) := \nu_i\big(\cdot \wedge \mathscr{D}^N\big) - \mu_i(0) - \langle \nu_i, M\rangle\big(\cdot \wedge \mathscr{D}^N\big), \qquad i = 1, \ldots, d \tag{5.20}$$

are then Q-local martingales with $X_i(0) = 0$.

We consider next the event

$$A := \left\{ \max_{0\le t\le T} \sum_{i=1}^{d} X_i^2(t) < \delta \right\}. \tag{5.21}$$

Thanks to (5.19), any vector process with components $\mu_i(0) + \langle \nu_i, M\rangle(\cdot)$, $i = 1, \ldots, d$, is a random, convex combination of the vectors $\mu(0)$ and x; in particular, this is true of the vector process $(X_1(\cdot), \ldots, X_d(\cdot))'$ with components as in (5.20), and leads to the set-inclusion

$$A \subset \big\{\mathscr{D}^N \ge T\big\}.$$

Now, in conjunction with (5.19), this set inclusion implies $\Gamma^{\varrho}(T \wedge \mathscr{D}^N \wedge \varrho) = C_N T$ thanks to (5.14), thus

$$\langle \nu_i, M\rangle(T) = x_i - \mu_i(0), \qquad i = 1, \ldots, d$$

on the event A of (5.21) and, therefore,

$$\begin{aligned}\sum_{i=1}^{d} \big(\nu_i(T) - x_i\big)^2 &= \sum_{i=1}^{d} \big(\nu_i(T) - \mu_i(0) - \langle \nu_i, M\rangle(T)\big)^2 \\ &= \sum_{i=1}^{d} X_i^2(T) < \delta \qquad \text{on } A.\end{aligned}$$

Consequently, the claim (5.16) will follow from the equivalence $\mathsf{Q}_* \sim \mathsf{Q}$, as soon as we have established that the event in (5.21) satisfies

$$\mathsf{Q}(A) > 0. \tag{5.22}$$

In order to argue this positivity, we introduce the processes

$$R(\cdot) := \sum_{i=1}^{d} X_i^2(\cdot) \quad \text{and} \quad Y(\cdot) := \int_0^{\cdot} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}}\, \mathrm{d}R(t) \tag{5.23}$$

and the Q-local martingale

$$\begin{aligned} N(\cdot) &:= Y(\cdot) - \int_0^{\cdot \wedge \mathscr{D}^N \wedge \varrho} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}}\, \mathrm{d}\Gamma^{\varrho}(t) \\ &= 2\int_0^{\cdot} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}} \big\langle X(t), \mathrm{d}X(t)\big\rangle, \end{aligned} \tag{5.24}$$

where we used (5.9). We now perform another change of measure. To this end, we define the Q-local martingale

$$\widehat{M}(\cdot) = -\frac{1}{2}\int_0^{\cdot} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}} \frac{1}{X'(t)\alpha(t)X(t)} \big\langle X(t), \mathrm{d}X(t)\big\rangle,$$

whose quadratic variation is dominated again by a real constant, namely

$$\begin{aligned}\langle \widehat{M}\rangle(\cdot) &= \frac{1}{4}\int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}} \frac{1}{X'(t)\alpha(t)X(t)}\, \mathrm{d}\Gamma^{\boldsymbol{\varrho}}(t) \\ &\le \frac{4}{c_N\delta^2}\Gamma^{\boldsymbol{\varrho}}(\cdot\wedge\boldsymbol{\varrho}) \le \frac{4C_N T}{c_N\delta^2}\end{aligned}$$

on account of (5.17). Here, the real constant $c_N > 0$ stands again for a lower bound on the smallest positive eigenvalue of $\alpha(\cdot)$ on the stochastic interval $[\![0, \mathscr{D}^N[\![$. Another application of Novikov's theorem yields now that the stochastic exponential $\mathcal{E}(\widehat{M}(\cdot))$ is a uniformly integrable Q-martingale and generates via the Girsanov theorem a probability measure $\widehat{\mathsf{Q}}$, equivalent to Q on $\mathscr{F}(T_*)$. Thus, in order to show (5.22) it suffices to argue

$$\widehat{\mathsf{Q}}\Big(\max_{0\le t\le T} R(t) < \delta\Big) > 0. \tag{5.25}$$

To make headway, we start by noting that $R(0) = 0$. Moreover, with the stopping times

$$\tau_1 := \inf\Big\{t \ge 0 : R(t) > \frac{\delta}{2}\Big\}, \qquad \tau_2 := \inf\Big\{t \ge \tau_1 : R(t) \notin \Big(\frac{\delta}{4}, \delta\Big)\Big\},$$

we note that (5.25) is implied by the assertion $\widehat{\mathsf{Q}}(\tau_2 > T) > 0$, which we argue now. To this end, we observe that, from the Girsanov theorem and in the notation of (5.23) and (5.24), the process

$$Y(\cdot) = N(\cdot) + \int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}}\, \mathrm{d}\Gamma^{\boldsymbol{\varrho}}(t) = N(\cdot) - \langle N, \widehat{M}\rangle(\cdot)$$

is a $\widehat{\mathsf{Q}}$-local martingale, and has bounded quadratic variation

$$\begin{aligned}\langle Y\rangle(\cdot) &= 4\int_0^{\cdot\wedge\mathscr{D}^N\wedge\boldsymbol{\varrho}} \mathbf{1}_{\{\delta/4 \le R(t) \le \delta\}} X'(t)\alpha(t)X(t)\, \mathrm{d}\Gamma^{\boldsymbol{\varrho}}(t) \\ &\le 4\int_0^{\cdot\wedge\boldsymbol{\varrho}} \mathbf{1}_{\{R(t)\le\delta\}} R(t)\, \mathrm{d}\Gamma^{\boldsymbol{\varrho}}(t) \le 4\delta C_N T\end{aligned}$$

by virtue of (5.9) and (5.13). Therefore, $Y(\cdot)$ can be expressed in its Dambis–Dubins–Schwarz representation $Y(\cdot) = \boldsymbol{\beta}(\langle Y\rangle(\cdot))$ for a suitable scalar $\widehat{\mathsf{Q}}$-Brownian motion $\boldsymbol{\beta}(\cdot)$ starting at the origin [e.g., Theorem 3.4.6 and Problem 3.4.7 in

Karatzas and Shreve (1991)]. Moreover, we have $R(\cdot) - R(\tau_1) = Y(\cdot) - Y(\tau_1)$ on the stochastic interval $[\![\tau_1, \tau_2[\![$, and thus

$$\{\tau_2 > T\} \supset \left\{ \max_{0 \le t \le T} |Y(t \vee \tau_1) - Y(\tau_1)| \le \frac{\delta}{4} \right\} \supset \left\{ \max_{0 \le t \le 4\delta C_N T} |\widetilde{\boldsymbol{\beta}}(u)| \le \frac{\delta}{4} \right\},$$

where $\widetilde{\boldsymbol{\beta}}(\cdot)$ is another scalar $\widehat{\mathbb{Q}}$-Brownian motion starting at the origin. The event on the right-hand side has positive $\widehat{\mathbb{Q}}$-probability, so the same is true for the event on the left-hand side. This then yields (5.25), concluding the proof. □

In the absence of strict nondegeneracy conditions as in Theorem 5.10, the controllability approach of Kunita (1974, 1978) yields conditions which guarantee that the assumptions of Theorem 5.7 hold in the special case when the vector market weight process $\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_d(\cdot))'$ is an Itô diffusion. In the same spirit, suitable Hörmander-type hypoellipticity conditions on the covariations of the components of such a diffusion, along with additional technical conditions on the drifts, yield good "tube estimates" which then again avoid the need to impose strict nondegeneracy conditions; see Bally, Caramellino and Pigato (2016) and the literature cited there.

6. Lack of short-term relative arbitrage opportunities. In Remark 4.5, we raised the question, whether the condition of (4.2) yields the existence of relative arbitrage over sufficiently short time horizons. In Section 5, we saw that, under appropriate *additional conditions* on the covariation structure of the market weights, the answer to this question is affirmative. In general, however, the answer to the question of Remark 4.5 is negative, as we shall see in the present section. Specific counterexamples of market models will be constructed here in a fairly systematic way, to illustrate that arbitrage opportunities over arbitrarily short time horizons do not necessarily exist in models which satisfy (4.2).

In Sections 6.1, 6.2 and 6.3, we shall focus on the quadratic function $\boldsymbol{Q}$ of (3.10). More precisely, we construct there variations of market models $\mu(\cdot)$ that satisfy (4.2) with $\boldsymbol{G} = \boldsymbol{Q}$, but do not admit relative arbitrage over any time horizon. We recall from (3.12) that $2\Gamma^{\boldsymbol{H}}(\cdot) - \Gamma^{\boldsymbol{Q}}(\cdot)$ is nondecreasing for the cumulative internal variation $\Gamma^{\boldsymbol{H}}(\cdot)$ of (3.9); thus, if (4.2) is satisfied by the quadratic function $\boldsymbol{Q}$, it is automatically also satisfied by the entropy function $\boldsymbol{H}$ of (3.7). This then also yields a negative answer to the question posed in Remark 4.5. In Section 6.4, market weight processes $\mu(\cdot)$ are constructed, such that $\boldsymbol{G}(\mu(\cdot))$ moves along the level sets of a general Lyapunov function $\boldsymbol{G}$ at unit speed, namely, with $\boldsymbol{G}(\mu(t)) = \boldsymbol{G}(\mu(0)) - t$ and $\Gamma^{\boldsymbol{G}}(t) = t$, but which do not admit relative arbitrage over any time horizon.

REMARK 6.1 (Some simplifications for notational convenience). Throughout this section, we shall make certain assumptions, mostly for notational convenience:

- We shall focus on the case $d = 3$. Indeed, Proposition 5.13 shows that it would be impossible to find a counterexample to the question of Remark 4.5 when $d = 2$. The counterexamples of this section can be generalized to more than three assets, but at the cost of additional notation and without any major additional insights.
- We shall construct market models that satisfy for a certain nonnegative Lyapunov function $\boldsymbol{G}$ the condition

$$\mathsf{P}\big(\text{the mapping } [0, T^\flat] \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\big) = 1 \quad \text{for some } \eta > 0 \text{ and } T^\flat > 0. \tag{6.1}$$

We may do this without loss of generality. Indeed, let us assume that the relative market weight process $\mu(\cdot)$ satisfies (6.1) and does not allow relative arbitrage over any time horizon. By appropriately adjusting the dynamics of $\mu(\cdot)$, say after time $T^\flat/2$, it is then always possible to construct a market model $\widehat{\mu}(\cdot)$ that satisfies (4.2), and also does not allow for arbitrage over short time horizons (as it displays the same dynamics up to time $T^\flat/2$).

6.1. *A first step for the quadratic function.* Here is a first result on absence of relative arbitrage under the condition of (6.1).

PROPOSITION 6.2 (Counterexample with Lipschitz-continuous dispersion matrix). *Assume that the filtered probability space* $(\Omega, \mathscr{F}, \mathsf{P})$, $\mathfrak{F} = (\mathscr{F}(t))_{t\geq 0}$ *supports a scalar Brownian motion* $W(\cdot)$. *Then there exists an Itô diffusion* $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot))'$ *with values in* $\boldsymbol{\Delta}^3$ *and with a time-homogeneous and Lipschitz-continuous dispersion matrix in* $\boldsymbol{\Delta}^3_+$, *for which the following properties hold*:

(i) *No relative arbitrage exists with respect to the market with weight process* $\mu(\cdot)$, *over any time horizon* $[0, T]$ *with* $T > 0$.

(ii) *The condition of* (6.1) *is satisfied by the quadratic* $\boldsymbol{G} = \boldsymbol{Q}$ *of* (3.10) *with* $\eta = 2/3 - \boldsymbol{Q}(\mu(0))$ *and with* $T^\flat = T^\flat(\mu(0))$ *a strictly positive real number, provided that* $\boldsymbol{Q}(\mu(0)) \in (1/2, 2/3)$.

We provide the proof of Proposition 6.2 at the end of this subsection. The following system of stochastic differential equations will play a fundamental role when deriving the dynamics of the relative market weight process $\mu(\cdot)$ in this proof:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}\big(v_2(t) - v_3(t)\big)\,\mathrm{d}W(t), \qquad t \geq 0, \tag{6.2}$$

$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}\big(v_3(t) - v_1(t)\big)\,\mathrm{d}W(t), \qquad t \geq 0, \tag{6.3}$$

$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}}\big(v_1(t) - v_2(t)\big)\,\mathrm{d}W(t), \qquad t \geq 0, \tag{6.4}$$

where $W(\cdot)$ denotes a scalar Brownian motion. The Lipschitz continuity of the coefficients guarantees that this system has a pathwise unique, strong solution for any initial point $v(0) \in \mathbb{R}^3$. If, moreover, the point $v(0)$ lies on the hyperplane $\mathbb{H}^3$ of (2.1), then we also have $v(t) \in \mathbb{H}^3$ for all $t \geq 0$.

REMARK 6.3 (Explicit solution). Let us give an explicit solution $v(\cdot)$ of the system (6.2)–(6.4) assuming that $v(0) \in \mathbb{H}^3$. If $v(0) = (1/3, 1/3, 1/3)'$, then we have also $v(t) = (1/3, 1/3, 1/3)'$ for all $t \geq 0$.

More generally, some determined but fairly basic stochastic calculus shows that the solution of the system (6.2)–(6.4) is given by

$$\begin{aligned}
v_1(t) &= \frac{1}{3} + \frac{e^{t/2}}{3}\big[2v_1(0)\cos\big(W(t)\big) + v_2(0)\big(-\cos\big(W(t)\big) + \sqrt{3}\sin\big(W(t)\big)\big) \\
&\quad + v_3(0)\big(-\cos\big(W(t)\big) - \sqrt{3}\sin\big(W(t)\big)\big)\big]; \\
v_2(t) &= \frac{1}{3} + \frac{e^{t/2}}{3}\big[v_1(0)\big(-\cos\big(W(t)\big) - \sqrt{3}\sin\big(W(t)\big)\big) + 2v_2(0)\cos\big(W(t)\big) \\
&\quad + v_3(0)\big(-\cos\big(W(t)\big) + \sqrt{3}\sin\big(W(t)\big)\big)\big]; \\
v_3(t) &= \frac{1}{3} + \frac{e^{t/2}}{3}\big[v_1(0)\big(-\cos\big(W(t)\big) + \sqrt{3}\sin\big(W(t)\big)\big) \\
&\quad + v_2(0)\big(-\cos\big(W(t)\big) - \sqrt{3}\sin\big(W(t)\big)\big) + 2v_3(0)\cos\big(W(t)\big)\big].
\end{aligned}$$

REMARK 6.4 (Representation in a special case). With the initial condition

$$v_i(0) = \frac{1}{3} + \delta\cos\left(2\pi\left(u + \frac{i-1}{3}\right)\right), \qquad i = 1, 2, 3$$

for some constants $\delta \in [0, 1/3]$ and $u \in \mathbb{R}$, we find another useful representation of the solution in Remark 6.3. Indeed, a computation shows

$$\sum_{i=1}^{3} v_i(0) = 1 + \delta\left(\cos(2\pi u) + \cos\left(2\pi u + \frac{2\pi}{3}\right) + \cos\left(2\pi u + \frac{4\pi}{3}\right)\right) = 1, \tag{6.5}$$

hence $v(0) \in \mathbb{H}^3$. We now claim that

$$v_i(t) = \frac{1}{3} + \delta e^{t/2}\cos\left(W(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right), \qquad i = 1, 2, 3,\ t \geq 0 \tag{6.6}$$

solves the system (6.2)–(6.4). To this end, we note that Itô's rule yields the dynamics

$$\mathrm{d}v_i(t) = -\delta e^{t/2}\sin\left(W(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right)\mathrm{d}W(t), \qquad i = 1, 2, 3,\ t \geq 0.$$

Moreover, since $\sin(\pi/3) = \sqrt{3}/2$, it suffices to argue that

$$2\sin\left(\frac{\pi}{3}\right)\sin\left(x + \frac{2\pi(i-1)}{3}\right) = \cos\left(x + \frac{2\pi(i+1)}{3}\right) - \cos\left(x + \frac{2\pi i}{3}\right),$$
$$i = 1, 2, 3,\ x \in \mathbb{R}.$$

This is a basic trigonometric identity, from which the claim (6.6) follows.

To study the dynamics of the solution $v(\cdot)$ for the system in (6.2)–(6.4) further, we introduce the function

$$\boldsymbol{r} : \mathbb{R}^3 \to [0, \infty),$$
$$x \mapsto \boldsymbol{r}(x) := \frac{1}{3}\big((x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2\big). \tag{6.7}$$

The following result shows that $\sqrt{\boldsymbol{r}(x)}$ is the distance of x from the "node" $(1/3, 1/3, 1/3)'$ on the lateral face of the unit simplex.

LEMMA 6.5 (Another representation for the function $\boldsymbol{r}$). *We have the representation*

$$\boldsymbol{r}(x) = \sum_{i=1}^{3}\left(x_i - \frac{1}{3}\right)^2 = \sum_{i=1}^{3} x_i^2 - \frac{1}{3} = \frac{2}{3} - \boldsymbol{Q}(x), \qquad x \in \mathbb{H}^3 \tag{6.8}$$

in the notation of (2.1), (3.10). *Moreover, if $v(\cdot)$ denotes a solution to the system of stochastic differential equations* (6.2)–(6.4) *with $v(0) \in \mathbb{H}^3$, then*

$$\boldsymbol{r}\big(v(t)\big) = \boldsymbol{r}\big(v(0)\big)e^t, \qquad t \geq 0. \tag{6.9}$$

PROOF. Fix $x \in \mathbb{H}^3$ and define $y_i := x_i - 1/3$ for each $i = 1, 2, 3$. Then we get

$$\begin{aligned}
\boldsymbol{r}(x) &= \frac{1}{3}\big((y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_2 - y_3)^2\big) \\
&= \frac{2}{3}\sum_{i=1}^{3} y_i^2 - \frac{2}{3}(y_1y_2 + y_1y_3 + y_2y_3) \\
&= \frac{2}{3}\sum_{i=1}^{3} y_i^2 + \frac{2}{3}\big(y_1^2 + y_2^2 + y_1y_2\big) \\
&= \frac{2}{3}\sum_{i=1}^{3} y_i^2 + \frac{1}{3}\sum_{i=1}^{3} y_i^2 = \sum_{i=1}^{3} y_i^2 = \sum_{i=1}^{3}\left(x_i - \frac{1}{3}\right)^2,
\end{aligned}$$

using $y_3 = -y_1 - y_2$ repeatedly. Basic stochastic calculus and (6.8) yield now, in conjunction with (6.2)–(6.4), the very simple deterministic dynamics $\mathrm{d}\boldsymbol{r}(v(t)) = \boldsymbol{r}(v(t))\,\mathrm{d}t$ for all $t \geq 0$, provided that $v(0) \in \mathbb{H}^3$; and (6.9) follows. □

PROOF OF PROPOSITION 6.2. We let $v(\cdot)$ denote the solution of the system of stochastic equations described in (6.2)–(6.4) for some $v(0) \in \mathbf{\Delta}_+^3$. Next, we define the stopping time

$$\tau := \inf\{t \geq 0 : v(t) \notin \mathbf{\Delta}_+^3\} = \inf\{t \geq 0 : v_i(t) = 0 \text{ for some } i \in \{1,2,3\}\} \tag{6.10}$$

and the stopped process $\mu(\cdot) := v(\cdot \wedge \tau)$. This is a vector of martingales, so relative arbitrage with respect to a market with the components of $\mu(\cdot)$ as its relative weights is impossible, over any given time horizon $[0, T]$ of finite length $T > 0$; see also Remark 4.2.

Now, the definition (6.10) of the stopping time τ implies $\boldsymbol{Q}(\mu(\tau)) \leq 1/2$, thus also $\boldsymbol{r}(\mu(\tau)) \geq 1/6$. In conjunction with Lemma 6.5, this yields that τ is bounded away from zero, namely that

$$T^\flat(\mu(0)) := \log\left(\frac{1}{6\boldsymbol{r}(\mu(0))}\right) \leq \tau$$

holds, since $\boldsymbol{Q}(\mu(0)) > 1/2$ holds by assumption. Moreover, with (6.7) and (6.9) we have

$$\frac{\partial}{\partial t}\Gamma^{\boldsymbol{Q}}(t) = \sum_{i=1}^{3} \frac{\partial}{\partial t}\langle \mu_i \rangle(t) = \boldsymbol{r}\big(\mu(t)\big) \geq \boldsymbol{r}\big(\mu(0)\big), \qquad t \in \big[0, T^\flat(\mu(0))\big]$$

on account of (3.11) and (6.2)–(6.4). Hence, the requirement (6.1) is satisfied with $T^\flat = T^\flat(\mu(0))$, $\boldsymbol{G} = \boldsymbol{Q}$ and $\eta = \boldsymbol{r}(\mu(0)) = 2/3 - \boldsymbol{Q}(\mu(0)) \in (0, 1/6)$, thanks to (6.8). □

REMARK 6.6 (A sanity check). We can verify that $T^\flat(\mu(0)) < \boldsymbol{Q}(\mu(0))/\eta$ holds with the notation of the above proof, and in accordance with Proposition 4.4.

REMARK 6.7 (Expanding circle). Let us observe from (6.9) that the market weights constructed in Proposition 6.2 live on an expanding circle. More specifically, from (6.8) and (6.9) we have

$$\mu_1^2(t) + \mu_2^2(t) + \mu_3^2(t) = \frac{1}{3} + \boldsymbol{r}\big(\mu(t)\big) = \frac{1}{3} + \boldsymbol{r}\big(\mu(0)\big)e^t, \qquad t \in \big[0, T^\flat\big];$$

hence the vector $\mu(\cdot)$ of relative market weights lies on the intersection of the hyperplane $\mathbb{H}^3$ with the sphere of radius $\sqrt{(1/3) + \boldsymbol{r}(\mu(0))e^t}$ centered at the origin. This intersection is a circle of radius $\sqrt{\boldsymbol{r}(\mu(0))e^t}$ centered at the node $(1/3, 1/3, 1/3)'$.

6.2. *Starting away from the node, and "moving slowly".* As (6.9) shows, in the context of Proposition 6.2, the process $\mu(\cdot)$ starts out away from the node on the lateral face of the unit simplex, then spins outward *very fast* (namely,

exponentially fast), until it reaches the boundary of the simplex at some time $\tau \geq \log(1/(6\boldsymbol{r}(\mu(0))))$; this time is bounded away from zero.

We construct here another market model, similar to the one in Section 6.1, but in which the spinning motion of $\mu(\cdot)$ is "slowed down" quite considerably. More precisely, the diffusion process of Theorem 6.8 starts away from the node $(1/3, 1/3, 1/3)'$ at some point $\mu(0)$ with $\boldsymbol{Q}(\mu(0)) \in (1/2, 2/3)$, then moves outwards along level sets of the quadratic function $\boldsymbol{Q}$. This takes time at least $T = \boldsymbol{Q}(\mu_0) - 1/2$; on the interval $[0, T]$ the condition in (6.1) is satisfied with $\boldsymbol{G} = \boldsymbol{Q}$ and $\eta = 1$, but no arbitrage with respect to the market can exist.

THEOREM 6.8 (Lack of short term relative arbitrage opportunities). *Assume that the filtered probability space* $(\Omega, \mathcal{F}, \mathsf{P})$, $\mathfrak{F} = (\mathcal{F}(t))_{t \geq 0}$ *supports a Brownian motion* $W(\cdot)$. *Fix* $\mu_0 \in \boldsymbol{\Delta}^3_+$ *with* $\boldsymbol{Q}(\mu_0) \in (1/2, 2/3)$. *Then there exists an Itô diffusion* $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot))'$ *with values in* $\boldsymbol{\Delta}^3$, *time-homogeneous dispersion matrix, starting point* $\mu(0) = \mu_0$, *and the following properties*:

(i) *No relative arbitrage exists with respect to the market with weight process* $\mu(\cdot)$, *over any time horizon* $[0, T]$ *of finite length* $T > 0$.

(ii) *The condition of* (6.1) *is satisfied for the quadratic function* $\boldsymbol{G} = \boldsymbol{Q}$ *with* $\eta = 1$ *and with* $T^\flat = \boldsymbol{Q}(\mu_0) - 1/2$.

We prove this theorem at the end of the subsection, after a remark and a preliminary result.

REMARK 6.9 (An open question). Suppose that the condition in (4.2) is satisfied by a market model with relative weight process $\mu(\cdot)$, for the quadratic function $\boldsymbol{Q}$ with $\eta = 1$. Theorem 4.3 yields then the existence of a strong relative arbitrage with respect to this market, over any time horizon $[0, T]$ with $T > \boldsymbol{Q}(\mu(0))$. On the other hand, Theorem 6.8 shows that, for time horizons $[0, T]$ with $T \leq \boldsymbol{Q}(\mu(0)) - 1/2$, there exist market models with respect to which no relative arbitrage is possible, even if (4.2) holds for them. We do not know what happens for time horizons $[0, T]$ with $T \in (\boldsymbol{Q}(\mu(0)) - 1/2, \boldsymbol{Q}(\mu(0)]$.

For the next result, we recall the function $\boldsymbol{r}$ from (6.7), (6.8).

PROPOSITION 6.10 [Time-changed, slowed-down version of (6.2)–(6.4)]. *Assume that the filtered probability space* $(\Omega, \mathcal{F}, \mathsf{P})$, $\mathfrak{F} = (\mathcal{F}(t))_{t \geq 0}$ *supports a Brownian motion* $W(\cdot)$. *Then, for any initial condition* $w(0) \in \mathbb{H}^3$ *with* $w(0) \neq (1/3, 1/3, 1/3)'$, *the following system of stochastic differential equations has a*

pathwise unique strong solution $w(\cdot)$, *taking values in* $\mathbb{H}^3$:

$$\mathrm{d}w_1(t) = \frac{1}{\sqrt{3\boldsymbol{r}(w(t))}}\big(w_2(t) - w_3(t)\big)\,\mathrm{d}W(t); \tag{6.11}$$

$$\mathrm{d}w_2(t) = \frac{1}{\sqrt{3\boldsymbol{r}(w(t))}}\big(w_3(t) - w_1(t)\big)\,\mathrm{d}W(t); \tag{6.12}$$

$$\mathrm{d}w_3(t) = \frac{1}{\sqrt{3\boldsymbol{r}(w(t))}}\big(w_1(t) - w_2(t)\big)\,\mathrm{d}W(t). \tag{6.13}$$

PROOF. Let $w(\cdot)$ denote any solution to the system (6.11)–(6.13) with $w(0) \in \mathbb{H}^3 \setminus \{(1/3, 1/3, 1/3)'\}$. Then it is clear that $w(t) \in \mathbb{H}^3$ holds for all $t \geq 0$. Next, we define the stopping time

$$\sigma = \inf\left\{t \geq 0 : \boldsymbol{r}\big(w(t)\big) < \frac{\boldsymbol{r}(w(0))}{2}\right\}$$

and note as in Lemma 6.5, via an application of Itô's rule, that

$$\mathrm{d}\boldsymbol{r}\big(w(\sigma \wedge t)\big) = \mathbf{1}_{\{\sigma > t\}}\,\mathrm{d}t, \qquad t \geq 0.$$

Hence, if $\sigma > 0$, the process $\boldsymbol{r}(w(\cdot))$ is nondecreasing and deterministic; indeed, we then have $\boldsymbol{r}(w(t)) = \boldsymbol{r}(w(0)) + t$ for all $t \geq 0$, and $\sigma = \infty$. For this reason, given any $\varepsilon \in (0, 3\boldsymbol{r}(w(0))$, any solution to the system (6.11)–(6.13) solves also the system

$$\mathrm{d}w_1^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon \vee 3\boldsymbol{r}(w^\varepsilon(t))}}\big(w_2^\varepsilon(t) - w_3^\varepsilon(t)\big)\,\mathrm{d}W(t), \tag{6.14}$$

$$\mathrm{d}w_2^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon \vee 3\boldsymbol{r}(w^\varepsilon(t))}}\big(w_3^\varepsilon(t) - w_1^\varepsilon(t)\big)\,\mathrm{d}W(t), \tag{6.15}$$

$$\mathrm{d}w_3^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon \vee 3\boldsymbol{r}(w^\varepsilon(t))}}\big(w_1^\varepsilon(t) - w_2^\varepsilon(t)\big)\,\mathrm{d}W(t). \tag{6.16}$$

Since the system (6.14)–(6.16) has Lipschitz-continuous coefficients, its solution is unique. This yields uniqueness of the solution to the system (6.11)–(6.13). Existence of a solution to the system (6.11)–(6.13) follows by checking that any solution to (6.14)–(6.16) is also a solution to (6.11)–(6.13). □

PROOF OF THEOREM 6.8. We proceed as in the proof of Proposition 6.2. With $w(0) = \mu_0$, we recall the process $w(\cdot)$ of Proposition 6.10, define the stopping time $\tau := \inf\{t \geq 0 : w(t) \notin \boldsymbol{\Delta}_+^3\}$ as in (6.10), and set $\mu(\cdot) := w(\cdot \wedge \tau)$. This process $\mu(\cdot)$ is a vector of continuous martingales; hence, no relative arbitrage is possible with respect to this market, over *any* given time horizon. Finally, we note that $\Gamma^{\boldsymbol{Q}}(t) = t \wedge \tau$ holds for all $t \geq 0$, and that $\boldsymbol{Q}(\mu_0) - \tau = \boldsymbol{Q}(\mu(\tau)) \leq 1/2$. This then yields the statement. □

6.3. *Modifications*. We provide here modifications of the examples in the previous subsections. More precisely, we recall the first system of stochastic differential equations (6.2)–(6.4) along with the solution given in (6.6). That is, we consider, for some $\delta \in (0, 1/3)$ and $u \in \mathbb{R}$, the corresponding market model with weight processes

$$\mu_i(t) = \frac{1}{3} + \delta e^{t/2} \cos\left(W(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right),$$
$$i = 1, 2, 3,\ t \in \left[0, -2\log(3\delta)\right],$$

where $W(\cdot)$ is Brownian motion and u a real number.

The first modification perturbs this model radially in the plane of $\mathbf{\Delta}^3$, so that the resulting new model has a covariation matrix-valued process $\alpha(\cdot)$ as in (5.8) with two strictly positive eigenvalues.

PROPOSITION 6.11 (Nondegeneracy and absence of arbitrage). *Assume that the filtered probability space* $(\Omega, \mathscr{F}, \mathsf{P})$, $\mathfrak{F} = (\mathscr{F}(t))_{t\geq 0}$ *supports two independent Brownian motions* $W(\cdot)$ *and* $B(\cdot)$. *Then there exist a real number* $T^\flat > 0$ *and an Itô diffusion* $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot))'$, *with the following properties*:

(i) *No relative arbitrage with respect to the market with weights* $\mu(\cdot)$ *exists, over any time horizon* $[0, T]$ *of finite length* $T > 0$.

(ii) *The condition of* (6.1) *is satisfied by the quadratic function* $\boldsymbol{G} = \boldsymbol{Q}$ *of* (3.10), *with* $\eta = \boldsymbol{r}(\mu(0))/4$.

(iii) *The covariation matrix of* $\mu(\cdot)$ *has two strictly positive eigenvalues on* $[0, T^\flat]$; *that is, for each* $t \in [0, T^\flat]$ *the matrix* $\alpha(t)$ *of* (5.8) *has two strictly positive eigenvalues*.

PROOF. To describe the model, let us fix a real constant $\delta \in (0, 1/9)$. We consider also a process $\Phi(\cdot)$ which is both a diffusion and a martingale of the filtration generated by the Brownian motion $B(\cdot)$; the process $\Phi(\cdot)$ takes values in the interval $(\delta, 3\delta)$ and starts at $\Phi(0) = 2\delta$.

More precisely, we introduce the Itô diffusion

$$\Psi(\cdot) := \int_0^\cdot \big(\Psi(t) - \delta\big)\big(\Psi(t) + \delta\big)\, \mathrm{d}B(t)$$

with state space $(-\delta, \delta)$ and inaccessible boundaries, thanks to Feller's test of explosions. We note that the quadratic variation

$$\langle \Psi \rangle(\cdot) = \int_0^\cdot \big(\Psi(t) - \delta\big)^2\big(\Psi(t) + \delta\big)^2\, \mathrm{d}t$$

of this martingale is strictly increasing. Next, we define the martingale $\Phi(\cdot) := 2\delta + \Psi(\cdot)$ with takes values in the interval $(\delta, 3\delta)$.

We now define $T^\flat := -2\log(9\delta) > 0$ and the market weights as positive Itô processes

$$\mu_i(t) := \frac{1}{3} + \Phi\big(t \wedge T^\flat\big) e^{(t\wedge T^\flat)/2} \cos\Big(W\big(t \wedge T^\flat\big) + \frac{2\pi}{3}(i-1)\Big), \quad i = 1, 2, 3,\ t \geq 0. \tag{6.17}$$

Since $\Phi(\cdot)$ and $W(\cdot)$ are independent, $\mu_i(\cdot)$ is a martingale for each $i = 1, \ldots, 3$. Indeed, it has the dynamics

$$\begin{aligned} d\mu_i(t) = &-\Phi(t) e^{t/2} \sin\Big(W(t) + \frac{2\pi}{3}(i-1)\Big)\, dW(t) \\ &+ e^{t/2} \cos\Big(W(t) + \frac{2\pi}{3}(i-1)\Big)\, d\Phi(t) \end{aligned}$$

for all $i = 1, 2, 3$ and $t \in [0, T^\flat]$. As a result, no relative arbitrage can exist with respect to this market. Moreover, we note

$$\begin{aligned} \langle \mu_i \rangle(\cdot) &\geq \int_0^{\cdot \wedge T^\flat} \Phi^2(t) e^t \sin^2\Big(W(t) + \frac{2\pi}{3}(i-1)\Big)\, dt \\ &\geq \delta^2 \int_0^{\cdot \wedge T^\flat} e^t \sin^2\Big(W(t) + \frac{2\pi}{3}(i-1)\Big)\, dt. \end{aligned}$$

Since $\Gamma^{\boldsymbol{Q}}(\cdot) = \sum_{i=1}^3 \langle \mu_i \rangle(\cdot)$, we get

$$\frac{\partial}{\partial t} \Gamma^{\boldsymbol{Q}}(t) \geq \delta^2 \sum_{i=1}^3 \sin^2\Big(W(t) + \frac{2\pi}{3}(i-1)\Big) = \boldsymbol{r}\big(\mu(0)\big), \qquad t \in [0, T^\flat]$$

and (ii) follows. Here, the last equality follows from the same type of computations as the ones in Remark 6.4.

Of the two Brownian motions that drive this market model, $W(\cdot)$ generates circular motion on the plane of $\boldsymbol{\Delta}^3$ about the point $(1/3, 1/3, 1/3)'$; while $B(\cdot)$ generates the martingale $\Phi(\cdot)$, whose quadratic variation has strictly positive time-derivative. Thus, these two independent random motions span the two-dimensional space, and the covariation process $\alpha(\cdot)$ of the market weight process $\mu(\cdot)$ has rank 2. □

REMARK 6.12 (Contrasting Theorem 5.10 to Proposition 6.11). Theorem 5.10 yields the existence of short-term relative arbitrage, if the $(d-1)$ largest eigenvalues of the covariance matrix $\alpha(\cdot)$ are uniformly bounded away from zero. According to Proposition 6.11 the weaker requirement, that all these $(d-1)$ largest eigenvalues be strictly positive, is *not* sufficient to guarantee short-term relative arbitrage. Indeed, the slope of the quadratic variation $\langle \Psi \rangle(\cdot)$ in the proof of Proposition 6.11 can get arbitrarily close to zero over any given time horizon, with

positive probability; see, for example, Bruggeman and Ruf (2016). This prevents the second-largest eigenvalue of the covariance matrix $\alpha(\cdot)$ from being uniformly bounded from below, away from zero.

REMARK 6.13 (Spiral expansion). In the market model of (6.17), the relative weights take values on a circle whose radius is allowed to expand at the rate $\Phi(t)e^{t/2}$; more precisely, we have

$$\mu_1^2(t) + \mu_2^2(t) + \mu_3^2(t) = \frac{1}{3} + \frac{3\Phi^2(t)e^t}{2} < \frac{1}{3} + \frac{3(3\delta)^2 e^{T^\flat}}{2} = \frac{1}{2}, \qquad t \in [0, T^\flat].$$

Therefore, at any given time $t \in [0, T^\flat]$ the vector $\mu(t)$ of relative market weights lies on the intersection of the hyperplane $\mathbb{H}^3$ with the sphere of radius $\sqrt{(1/3) + (3/2)\Phi^2(t)e^t}$ centered at the origin. This intersection is a circle of radius $\sqrt{3/2}\Phi(t)e^{t/2} < \sqrt{1/6}$, contained in $\mathbf{\Delta}^3_+$ and centered at its node $(1/3, 1/3, 1/3)'$; see also Remark 6.7. Thus, a more precise description of the current situation might be that the market weights live on a spiral. The rate of this "spiral expansion" is exactly that one, for which the market weights become martingales.

This remark raises the following question: *What happens if the market weights are confined to a stationary circle in* $\mathbf{\Delta}^3$*?* Such a diffusion, confined to a circle, turns out to be incompatible with a martingale structure and to open the door to "egregious" forms of arbitrage. This is the subject of the example that follows.

EXAMPLE 6.14 (Immediate arbitrage). Let $W(\cdot)$ denote a scalar Brownian motion, fix a real constant $\delta \in (0, 1/3)$, and define the positive market weight processes

$$\mu_i(\cdot) := \frac{1}{3} + \delta \cos\left(W(\cdot) + \frac{2\pi}{3}(i-1)\right), \qquad i = 1, 2, 3.$$

These market weights take values in the interval $(0, 2/3)$; in fact they live on a circle, namely

$$\mu_1^2(t) + \mu_2^2(t) + \mu_3^2(t) = \frac{1}{3} + \frac{3\delta^2}{2} < \frac{1}{2} \qquad \text{so } \boldsymbol{r}\big(\mu(\cdot)\big) \equiv \frac{3\delta^2}{2}$$

in the notation of (6.7), and have dynamics

$$\begin{aligned} \mathrm{d}\mu_i(t) = &-\delta \sin\left(W(t) + \frac{2\pi}{3}(i-1)\right) \mathrm{d}W(t) \\ &- \frac{\delta}{2} \cos\left(W(t) + \frac{2\pi}{3}(i-1)\right) \mathrm{d}t, \qquad t \geq 0. \end{aligned}$$

As above, we have

$$\Gamma^{\boldsymbol{Q}}(t) = \int_0^t \boldsymbol{r}\big(\mu(s)\big)\,\mathrm{d}s = \boldsymbol{r}\big(\mu(0)\big)t = \frac{3\delta^2 t}{2}, \qquad t \geq 0.$$

Let us introduce now the normalized quadratic function $\boldsymbol{Q}^\star := \boldsymbol{Q}/\boldsymbol{Q}(\mu(0))$, where $\boldsymbol{Q}(\mu(0)) = (2/3) - (3\delta^2/2) > 0$. For the trading strategy $\boldsymbol{\varphi}^{\boldsymbol{Q}^\star}(\cdot)$ generated additively by this function $\boldsymbol{Q}^\star$ as in (3.3), and the associated wealth process $V^{\boldsymbol{\varphi}^{\boldsymbol{Q}^\star}}(\cdot)$ of (3.4), we get

$$V^{\boldsymbol{\varphi}^{\boldsymbol{Q}^\star}}(t) = \boldsymbol{Q}^\star\big(\mu(t)\big) + \Gamma^{\boldsymbol{Q}^\star}(t) = 1 + \frac{3\delta^2}{2\boldsymbol{Q}(\mu(0))}t > 1, \qquad t > 0.$$

Hence, the additively-generated strategy $\boldsymbol{\varphi}^{\boldsymbol{Q}^\star}(\cdot)$ yields a strong relative arbitrage over any given time horizon $[0, T]$. Investing according to this strategy $\boldsymbol{\varphi}^{\boldsymbol{Q}^\star}(\cdot)$ is a sure way to do better than the market right away, and to keep doing better and better as time goes on.

Indeed, the existence of such "egregious" or "immediate" arbitrage, should not come here as a surprise, since the so-called "structure equation" is not satisfied. In particular, no deflator as in Definition 2.1 exists for $\mu(\cdot)$; see, for example, Schweizer (1992), or Theorem 1.4.2 in Karatzas and Shreve (1998).

REMARK 6.15 (General submanifolds). The diffusion constructed in Example 6.14 lives on a submanifold of $\mathbb{R}^3$, which is incompatible with a martingale structure. By contrast, in Section 6.1 the submanifold was allowed to evolve as an expanding circle. The diffusions in these examples could probably be generalized to diffusions with support on an arbitrary submanifold of $\mathbb{R}^d$, for $d \geq 2$, and then the submanifold could be allowed to evolve through $\mathbb{R}^d$ in the manner of our expanding circle in $\mathbb{R}^3$.

In this case, a natural question would be to characterize the evolution that would cause the diffusion on the evolving submanifold to become a martingale. How would this martingale-compatible evolution depend on the diffusion? What would become of this evolving submanifold over time? Would the evolving submanifold develop singularities? (Et cetera.) We provide some partial answers to such questions in the following Section 6.4, but the picture that emerges seems to be far from complete.

6.4. *General Lyapunov functions*. We extend now the construction of market models, for which (6.1) is satisfied but no short-term relative arbitrage is possible, to a general class of regular functions. Throughout this section, we fix a Lyapunov function $\boldsymbol{G} : \boldsymbol{\Delta}^3 \to [0, \infty)$; this function is assumed to be strictly concave and twice continuously differentiable in a neighborhood of $\boldsymbol{\Delta}^3_+$. We assume moreover that its Hessian $D^2\boldsymbol{G}$ is locally Lipschitz continuous. Next, we introduce the nonnegative number

$$\mathfrak{g} := \sup_{x \in \boldsymbol{\Delta}^3 \setminus \boldsymbol{\Delta}^3_+} \boldsymbol{G}(x).$$

If $\boldsymbol{G}$ attains an interior local (hence also global) maximum at a point $c \in \boldsymbol{\Delta}^3_+$, we call this c the "navel", or "umbilical point", of the function $\boldsymbol{G}$.

THEOREM 6.16 (General Lyapunov functions, lack of short-term relative arbitrage). *Assume that the filtered probability space* $(\Omega, \mathscr{F}, \mathsf{P})$, $\mathfrak{F} = (\mathscr{F}(t))_{t\geq 0}$ *supports a Brownian motion* $W(\cdot)$. *Suppose also that we are given a Lyapunov function* $\boldsymbol{G} : \boldsymbol{\Delta}^3 \to [0, \infty)$ *with the properties and notation just stated, along with a vector* $\mu_0 \in \boldsymbol{\Delta}^3_+$ *such that* $\boldsymbol{G}(\mu_0) \in (\mathfrak{g}, \max_{x\in\boldsymbol{\Delta}^3} \boldsymbol{G}(x))$. *Then there exists an Itô diffusion* $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot))'$ *with starting point* $\mu(0) = \mu_0$, *values in* $\boldsymbol{\Delta}^3$, *and the following properties*:

(i) *No relative arbitrage is possible with respect to* $\mu(\cdot)$ *over any time horizon* $[0, T]$ *with* $T > 0$.

(ii) *The condition of* (6.1) *is satisfied with* $\eta = 1$ *and* $T^\flat = \boldsymbol{G}(\mu_0) - \mathfrak{g}$.

Moreover, we have

$$\boldsymbol{G}\big(\mu(t)\big) = \boldsymbol{G}(\mu_0) - t, \qquad \Gamma^{\boldsymbol{G}}(t) = t; \qquad t \in \big[0, T^\flat\big].$$

PROOF. We introduce the vector function $\sigma = (\sigma_1, \sigma_2, \sigma_3)'$ with components

$$\sigma_1(x) := D_3\boldsymbol{G}(x) - D_2\boldsymbol{G}(x);$$

$$\sigma_2(x) := D_1\boldsymbol{G}(x) - D_3\boldsymbol{G}(x);$$

$$\sigma_3(x) := D_2\boldsymbol{G}(x) - D_1\boldsymbol{G}(x)$$

for $x \in \boldsymbol{\Delta}^3_+$. If $\sigma_1(x) = \sigma_2(x) = \sigma_3(x) = 0$, then $x = c$ is the umbilical point. Indeed, if for some $x \in \boldsymbol{\Delta}^3_+$ we have $D_1\boldsymbol{G}(x) = D_2\boldsymbol{G}(x) = D_3\boldsymbol{G}(x)$, then the strict concavity of $\boldsymbol{G}$ yields

$$0 = \sum_{i=1}^{3} D_i\boldsymbol{G}(x)(x_i - y_i) > \boldsymbol{G}(y) - \boldsymbol{G}(x), \qquad y \in \boldsymbol{\Delta}^3_+ \setminus \{x\}.$$

Next, we introduce the function

$$L(x) := -\frac{1}{2}\sigma'(x)D^2\boldsymbol{G}(x)\sigma(x), \qquad x \in \boldsymbol{\Delta}^3_+ \tag{6.18}$$

and note that $L(x) > 0$ holds, as long as x is not the umbilical point c.

Let us now consider the Itô diffusion process $\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot))'$ with initial condition $\mu(0) = \mu_0$ and dynamics

$$\mathrm{d}\mu_i(t) = \frac{\sigma_i(\mu(t))}{\sqrt{L(\mu(t))}}\,\mathrm{d}W(t), \qquad i = 1, 2, 3. \tag{6.19}$$

It is clear from this construction and the property $\sum_{i=1}^3 \sigma_i(x) = 0$ for all $x \in \boldsymbol{\Delta}^n_+$ that, as long as the process $\mu(\cdot)$ is well-defined by the above system, it satisfies $\mu_1(\cdot) + \mu_2(\cdot) + \mu_3(\cdot) = 1$. Furthermore, as long as the process $\mu(\cdot)$ is well-defined, elementary stochastic calculus and the property

$$\sum_{i=1}^{3} D_i\boldsymbol{G}(x)\sigma_i(x) = 0, \qquad x \in \boldsymbol{\Delta}^n_+$$

lead to the dynamics

$$d\boldsymbol{G}(\mu(t)) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} D_{ij}^2 \boldsymbol{G}(\mu(t))\, d\langle \mu_i, \mu_j\rangle(t) = -d\Gamma^{\boldsymbol{G}}(t).$$

This double summation is identically equal to $-dt$, by virtue of (6.18) and (6.19). Thus, $\boldsymbol{G}(\mu(\cdot))$ is strictly decreasing at the constant rate -1, and stays clear of the navel c (when this exists); by the same token, $\Gamma^{\boldsymbol{G}}(\cdot)$ is strictly increasing at the constant rate $+1$. Hence, by analogy with the proof of Proposition 6.10, we deduce that the process $\mu(\cdot)$ is well defined, since the system of stochastic differential equations in (6.19) has a pathwise unique and strong solution up until the time $\mathscr{D}$. This stopping time was defined in (2.4) and describes the first time when $\mu(\cdot)$ hits the boundary of $\boldsymbol{\Delta}^3$; in particular, we have for it

$$\boldsymbol{G}(\mu_0) - \mathfrak{g} \le \mathscr{D} = \boldsymbol{G}(\mu_0) - \boldsymbol{G}\big(\mu(\mathscr{D})\big) \le \boldsymbol{G}(\mu_0). \tag{6.20}$$

The market weight processes $\mu_i(\cdot)$ for $i = 1, 2, 3$ are continuous martingales with values in the unit interval $[0, 1]$. As a result, no relative arbitrage can exist with respect to the resulting market, over any given time horizon. Since $\mathscr{D} \ge \boldsymbol{G}(\mu_0) - \mathfrak{g}$, we can conclude the proof. □

Theorem 6.8 is a special case of Theorem 6.16. We do not know whether the assumption $\boldsymbol{G}(\mu_0) \ne \max_{x\in\boldsymbol{\Delta}^3_+} \boldsymbol{G}(x)$ can be removed from Theorem 6.16, but conjecture that this should be possible.

REMARK 6.17 (Presence of a gap). We continue here the discussion started in Remark 6.9. It is very instructive to compare the interval $(\boldsymbol{G}(\mu_0), \infty)$ of (4.3), which provides the lengths of time horizons $[0, T]$ over which strong arbitrage is possible with respect to *any* market that satisfies the condition in (4.2) with $\eta = 1$; and the interval $[0, \boldsymbol{G}(\mu_0) - \mathfrak{g}]$ of Theorem 6.16, giving the lengths of time horizons $[0, T]$ over which examples of markets can be constructed that do not admit relative arbitrage but satisfy the condition in (4.2).

There is a gap in these two intervals, when $\mathfrak{g}$ is positive. For instance, in the case of the entropy function $\boldsymbol{G} = \boldsymbol{H}$ of (3.7), the gap is very much there, as we have $\mathfrak{g} = 2\log 2$ but $\max_{x\in\boldsymbol{\Delta}^3} \boldsymbol{H}(x) = 3\log 3$. In this "entropic" case, the dynamics of (6.19) and (6.18) take the form

$$d\mu_i(t) = \log\left(\frac{\mu_{i+1}(t)}{\mu_{i-1}(t)}\right)\frac{dW(t)}{\sqrt{L(\mu(t))}},$$

$$i = 1, 2, 3,\ t \ge 0 \text{ with } \mu_0(\cdot) := \mu_3(\cdot),\ \mu_4(\cdot) := \mu_1(\cdot);$$

$$L(x) = \sum_{i=1}^{3} \frac{1}{2x_i}\left(\log\left(\frac{x_{i+1}}{x_{i-1}}\right)\right)^2, \qquad x \in \boldsymbol{\Delta}^3_+ \text{ with } x_0 := x_3,\ x_4 := x_1.$$

REMARK 6.18 (Absence of a gap). The gap in Remark 6.17 disappears, of course, when $\mathfrak{g} = 0$; most eminently, when the function $\boldsymbol{G}$ vanishes on the boundary $\boldsymbol{\Delta}^d \setminus \boldsymbol{\Delta}^d_+$, but is strictly positive on $\boldsymbol{\Delta}^d_+$.

For such a function $\boldsymbol{G}$ and for the market model constructed in the proof of Theorem 6.16, we are then led from (6.20) and with the notation of (2.4), to the rather remarkable identity

$$\mathscr{D} = \mathscr{D}_1 \wedge \mathscr{D}_2 \wedge \mathscr{D}_3 = \boldsymbol{G}(\mu_0)$$

for the stopping time of (2.4). That is, at *exactly* the time $T^\flat = \boldsymbol{G}(\mu_0)$, one of the components of the diffusion $\mu(\cdot)$ vanishes for the first time.

Assuming now that $\mathbb{P}(\Gamma^{\boldsymbol{G}}(t) \geq t, 0 \leq t < \infty) = 1$ holds, we conclude then from Theorems 4.3 and 6.16 that strong arbitrage with respect to the market is possible over any horizon $[0, T]$ with $T > \boldsymbol{G}(\mu_0)$; and that there exists an Itô diffusion $\mu(\cdot)$ for the vector process of market weights, with respect to which no arbitrage is possible over any time-horizon $[0, T]$ with $0 < T \leq \boldsymbol{G}(\mu_0)$.

A prominent example of such a function is the geometric mean $\boldsymbol{R}$ from (3.13); in this case, and with $\mu_0(\cdot)$, $\mu_4(\cdot)$, x_0, and x_4 as above, the dynamics corresponding to (6.19) and (6.18) are given as

$$\mathrm{d}\mu_i(t) = \left(\frac{1}{\mu_{i-1}(t)} - \frac{1}{\mu_{i+1}(t)}\right)\frac{\mathrm{d}W(t)}{\sqrt{L_*(\mu(t))}}, \qquad i = 1, 2, 3, t \geq 0.$$

Here, we have set

$$\begin{aligned} L_*(x) &= \frac{\boldsymbol{R}(x)}{18}\left(\sum_{i=1}^{3}\frac{3}{x_i^2}\left(\frac{1}{x_{i-1}} - \frac{1}{x_{i+1}}\right)^2\right. \\ &\qquad \left. - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{x_i x_j}\left(\frac{1}{x_{i-1}} - \frac{1}{x_{i+1}}\right)\left(\frac{1}{x_{j-1}} - \frac{1}{x_{j+1}}\right)\right) \\ &= \frac{\boldsymbol{R}(x)}{6x_1^2x_2^2x_3^2}\sum_{i=1}^{3}(x_{i+1} - x_{i-1})^2 = \frac{1}{2(\boldsymbol{R}(x))^5}\sum_{i=1}^{3}\left(x_i - \frac{1}{3}\right)^2 \\ &= \frac{\boldsymbol{r}(x)}{2(\boldsymbol{R}(x))^5}, \qquad x \in \boldsymbol{\Delta}^3_+, \end{aligned}$$

where the last two equalities follow from (6.8).

7. Summary. In this paper, we place ourselves in the context of continuous semimartingales $\mu(\cdot)$ taking values in the d-dimensional simplex $\boldsymbol{\Delta}^d$. Each component of $\mu(\cdot)$ is interpreted as the relative capitalization of a company in an equity market, with respect to the whole market capitalization. We then study conditions on the volatility structure of $\mu(\cdot)$ that guarantee the existence of relative arbitrage opportunities with respect to the market. More precisely, we consider conditions

that bound a cumulative aggregation $\Gamma^{\boldsymbol{G}}(\cdot)$ of the volatilities of the individual components of $\mu(\cdot)$ from below. Here, the aggregation is done according to a so-called regular function $\boldsymbol{G} : \boldsymbol{\Delta}^d \to \mathbb{R}$, assumed to be sufficiently smooth. Then $\Gamma^{\boldsymbol{G}}(\cdot)$ is given by (3.2), namely

$$\Gamma^{\boldsymbol{G}}(\cdot) = -\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\int_0^{\cdot} D^2_{i,j}\boldsymbol{G}\big(\mu(t)\big)\,\mathrm{d}\langle\mu_i,\mu_j\rangle(t),$$

and the condition on the cumulative aggregated market volatility by (4.2):

$$\mathsf{P}\big(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^{\boldsymbol{G}}(t) - \eta t \text{ is nondecreasing}\big) = 1$$
$$\text{for some } \eta > 0.$$

Section 4 recalls the trading strategy $\boldsymbol{\varphi}^{\boldsymbol{G}}(\cdot)$, which is a strong relative arbitrage on the time horizon $[0, T]$ for all $T > \boldsymbol{G}(\mu(0))/\eta$, provided that (4.2) holds. It is important to note that this strategy $\boldsymbol{\varphi}^{\boldsymbol{G}}(\cdot)$ is "model-free": it does not depend on the specifications of a particular model for asset capitalizations or relative weights, and works for any continuous semimartingale model that satisfies (4.2).

Section 5 provides several sufficient conditions guaranteeing the existence of short-term relative arbitrage. First, Section 5.1 studies the question of strong short-term relative arbitrage. If a specific stock contributes to the overall market volatility, then Theorem 5.1 yields the existence of such a strong arbitrage opportunity. The corresponding trading strategy turns out to be independent of the model specification but will be dependent on the choice of time horizon. A similar construction is the underlying idea of Banner and Fernholz (2008). There, not a specific stock, but always the smallest one in terms of capitalization, contributes to the overall market volatility; see also Remark 5.3.

Section 5.2 yields sufficient conditions for the existence of short-term relative arbitrage, not necessarily strong. The first sufficient condition concerns the support of $\mu(\cdot)$ and assumes that it is, in a certain weak sense, time-homogeneous (Theorem 5.7). The second sufficient condition is on the strict nondegeneracy of the covariance process of $\mu(\cdot)$; see Theorem 5.10. Any of the two conditions yields, in conjunction with (4.2), the existence of a relative arbitrage opportunity for the time horizon $[0, T]$, for any $T > 0$. However, the corresponding trading strategies usually depend on the model specification (including the specification on drifts) and also on the time horizon T. Moreover, these sufficient conditions usually do not yield strong relative arbitrage. Nevertheless, if the market is complete, the strategies can be chosen to be strong relative arbitrages; see Remark 5.4.

Section 6 answers negatively the long-standing question, whether the condition in (4.2) yields indeed the existence of relative arbitrage opportunities with respect to the market on $[0, T]$, for any given real number $T > 0$. Specific counterexamples are constructed. In this section, we assume that $d = 3$, namely, that we are in a market of only three stocks. This is done for notational convenience, as a

lower-dimensional market can always be embedded in a higher-dimensional one. A smaller choice for d is not possible, as Proposition 5.13 yields that in the case $d = 2$ of two stocks the condition of (4.2) always yields the existence of relative arbitrage opportunities over any time horizon.

Section 6 differs from the earlier ones. In Section 5, we considered a fixed market with relative weight processes $\mu(\cdot)$ and formulated conditions on the market model that yield (possibly strong) relative arbitrage opportunities for short time horizons. These trading strategies might or might not depend on the specific characteristics of $\mu(\cdot)$. Now, in Section 6, we fix the regular function $\boldsymbol{G}$ and then construct market weights $\mu(\cdot)$ for which there exist no relative arbitrage opportunities over time horizon $[0, T]$ for some $T > 0$, but (4.2) holds. To be precise, the models we construct do not satisfy exactly (4.2) but only the local (in time) version (6.1). As pointed out in Remark 6.1, this is done for notational convenience only. Some additional technical assumptions are made on the regular function $\boldsymbol{G}$ in Section 6, most importantly that $\boldsymbol{G}$ is strictly concave and its second derivative is locally Lipschitz continuous.

The constructed market models $\mu(\cdot)$ not only prevent relative arbitrage and satisfy (6.1), but also yield that $\boldsymbol{G}(\mu(\cdot))$ is a deterministic function of time. That is, $\mu(\cdot)$ *flows along the level sets of the Lyapunov function* $\boldsymbol{G}$.

Moreover, Proposition 6.11 yields the existence of a market model, which satisfies the nondegeneracy conditions of Theorem 5.10 but *not strictly*, and does *not* allow for relative arbitrage. This shows that the conditions on $\mu(\cdot)$ in Theorem 5.10 are tight; see also Remark 6.12.

While this paper answers some old open questions, it suggests several new ones. The three most important, in our opinion, are the following:

1. Under (4.2) with $\eta = 1$, strong relative arbitrage opportunities exist on $[0, T]$ for all $T > \boldsymbol{G}(\mu(0))$. In Section 6, market models $\mu(\cdot)$ are constructed that satisfy (4.2) but do not admit relative arbitrage opportunities on $[0, T^\flat]$, for some real number $T^\flat = T^\flat(\mu(0)) \in (0, \boldsymbol{G}(\mu(0)))$. What can be said for time horizons $[0, T]$ with $T \in (T^\flat, \boldsymbol{G}(\mu(0))]$? In this connection, see also Remarks 6.6, 6.9 and 6.17.

2. The following question arises from the methodology used in Section 6 to construct the counterexamples. Assume a diffusion lives on a submanifold of $\mathbb{R}^d$, which is incompatible with a martingale structure (e.g., on a sphere). If we now want to turn the diffusion into a martingale, how will the submanifold evolve through $\mathbb{R}^d$ (e.g., it could turn into an expanding sphere)? In this connection, see also Remark 6.15.

3. Section 6 contains examples for diffusions, which could be turned into market models where short-term relative arbitrage is impossible but long-term relative arbitrage is possible. There are no additional frictions (e.g., trading costs, etc.) necessary to achieve this. Are there any interesting economic implications and, if so, what are they? For instance, can such models arise from equilibrium theory in an economy where agents have preferences with respect to different time horizons?

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E. R. FERNHOLZ
INTECH INVESTMENT MANAGEMENT
ONE PALMER SQUARE, SUITE 441
PRINCETON, NEW JERSEY 08542
USA
E-MAIL: bob@bobfernholz.com

I. KARATZAS
DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
NEW YORK, NEW YORK 10027
USA
AND
INTECH INVESTMENT MANAGEMENT
ONE PALMER SQUARE, SUITE 441
PRINCETON, NEW JERSEY 08542
USA
E-MAIL: ik1@columbia.edu
ikaratzas@intechjanus.com

J. RUF
DEPARTMENT OF MATHEMATICS
LONDON SCHOOL OF ECONOMICS
AND POLITICAL SCIENCE
COLUMBIA HOUSE, HOUGHTON ST
LONDON WC2A 2AE
UNITED KINGDOM
E-MAIL: j.ruf@lse.ac.uk