DISORDER AND WETTING TRANSITION: THE PINNED HARMONIC CRYSTAL IN DIMENSION THREE OR LARGER

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We consider the lattice Gaussian free field in $d + 1$ dimensions, $d = 3$ or larger, on a large box (linear size N) with boundary conditions zero. On this field, two potentials are acting: one, that models the presence of a wall, penalizes the field when it enters the lower half space and one, the *pinning potential*, rewards visits to the proximity of the wall. The wall can be soft, that is, the field has a finite penalty to enter the lower half-plane, or hard, when the penalty is infinite. In general, the pinning potential is disordered and it gives on average a reward $h \in \mathbb{R}$ (a negative reward is a penalty): the energetic contribution when the field at site x visits the pinning region is $\beta \omega_x + h$, $\{\omega_x\}_{x \in \mathbb{Z}^d}$ are i.i.d. centered and exponentially integrable random variables of unit variance and $\beta \geq 0$. In [*J. Math. Phys.* **41** (2000) 1211–1223], it is shown that, when $\beta = 0$ (i.e., in the nondisordered model), a delocalization-localization transition happens at $h = 0$, in particular the free energy of the system is zero for $h \leq 0$ and positive for $h > 0$. We show that, for $\beta \neq 0$, the transition happens at $h = h_c(\beta) := -\log \mathbb{E} \exp(\beta \omega_x)$, and we find the precise asymptotic behavior of the logarithm of the free energy density of the system when $h \searrow h_c(\beta)$. In particular, we show that the transition is of infinite order in the sense that the free energy is smaller than any power of $h - h_c(\beta)$ in the neighborhood of the critical point and that disorder does not modify at all the nature of the transition. We also provide results on the behavior of the paths of the random field in the limit $N \to \infty$.

1. Introduction.

In this paper we focus on the nature of the wetting transition for a d-dimensional harmonic crystal interacting with a substrate and on the effect of disorder on this transition.

The harmonic crystal, or lattice Gaussian free field (LGFF), is the basic model for surfaces with Hamiltonian given by the sum of the square of the gradients of the field. Its Gaussian nature makes it, in most of the cases, easier to analyze than other surface fields with gradient potential and conclusions drawn for LGFF are expected to remain valid for a larger class of field.

This is the case for the study of the wetting transition, which involves a competition between a repelling potential (possibly infinite) acting on the lower half-space

and an attracting one located on a band of finite width above this half-space. What one finds in the literature about this specific problem—the literature on LGFF is very vast since it naturally emerges in a variety of contexts (see [17, 19] and references therein)—can be resumed as follows:

- In the absence of attracting potential, a wall constraint in the lower half-plane induces a phenomenon of repulsion of entropic origin in dimension $d = 2$ and $d \geq 3$. The surface lies at a distance from the wall which is of order $\log N$ in dimension two and $\sqrt{\log N}$ in dimension three or larger when N is the size of the system [2, 8, 9, 15].
- An arbitrary small (positive) *pinning* potential in the intermediate band is sufficient to overcome this entropic repulsion when $d \geq 3$ [3] (that should be complemented by the correction in [7]) whereas when $d = 2$, the repulsion prevails even in the presence of a small positive potential [4]. So when $d \geq 3$, there is a transition when the potential switches from repulsive to attractive, while the transition happens at some positive value of the pinning potential.

In the present work, we analyze the phase transition for $d \geq 3$, with a twofold objective:

- We study the free energy behavior at the vicinity of the critical point and show that the transition is of infinite order.
- We investigate the effect of disorder on this phase transition and show that quenched and annealed critical points coincide. Moreover, we show that the critical behavior is not modified by the disorder.

We also prove that in the localized phase, the distribution of the field in the middle of large box converges when the boundary is sent to infinity to a translation covariant limiting distribution.

These results offer a sharp contrast with those obtained in the absence of half-space repulsion [12, 14] (see also [6] for a first contribution to the subject). In that case, the transition, which is of first order when $d \geq 3$ and of second order when $d = 2$ for the homogeneous case, becomes smoother in the disordered one (order two and infinity, respectively). These differences can be interpreted in the light of Harris criterion concerning disorder relevance [13]: for the wetting transition here, the homogeneous model has a smooth transition (the specific heat exponent is negative), and for this reason, disorder should be irrelevant, that is, it should not change the critical behavior, at least for perturbations of small amplitude. For the pinning transition studied in [12, 14], the specific heat exponent is positive, so the Harris criterion predicts disorder relevance. More details on the Harris criterion are in point (2) of the list after Theorem 2.2.

Note that the model is also defined in dimension one, that is, $1 + 1$: in that case, the harmonic crystal is simply a random walk with i.i.d. Gaussian increments. The $1 + 1$ dimensional behavior of the model is quite different and very similar to the random walk pinning model (the case where no wall is present) for which an

extended literature exists (see [16] for a treatment of the nondisordered case and [10, 11] and references therein for one-dimensional disordered pinning models).

2. Model and results.

2.1. *Wetting models*, *with and without disorder.* Given Λ a finite subset of $\mathbb{Z}^d$ (Λ is always going to be a hypercube and $d = 3, 4, \ldots$), we let $\partial \Lambda$ denote the internal boundary of Λ and $\overset{\circ}{\Lambda}$ the set of interior points of Λ, that is (with $\sim$ standing for nearest neighbor),

$$(2.1) \quad \partial \Lambda = \{x \in \Lambda : \text{there exists } y \notin \Lambda \text{ such that } x \sim y\} \quad \text{and} \quad \overset{\circ}{\Lambda} := \Lambda \setminus \partial \Lambda.$$

$\mathbf{P}_\Lambda^{\hat{\phi}}$ is the law of the LGFF on Λ (denoted by $\phi = \{\phi_x\}_{x \in \mathbb{Z}^d}$) with boundary conditions $\hat{\phi} \in \mathbb{R}^{\mathbb{Z}^d}$ on $\mathbb{Z}^d \setminus \overset{\circ}{\Lambda}$. Explicitly, $\phi_x = \hat{\phi}_x$ for $x \notin \overset{\circ}{\Lambda}$ and consider $\mathbf{P}_\Lambda^{\hat{\phi}}$ as a probability on $\mathbb{R}^{\overset{\circ}{\Lambda}}$ whose density is given by

$$(2.2) \qquad \mathbf{P}_\Lambda^{\hat{\phi}}(d\phi) \propto \exp\left(-\frac{1}{2} \sum_{\substack{(x,y) \in (\Lambda)^2 \setminus (\partial \Lambda)^2 \\ x \sim y}} \frac{(\phi_x - \phi_y)^2}{2}\right) \prod_{x \in \overset{\circ}{\Lambda}} d\phi_x,$$

where $\prod_{x \in \overset{\circ}{\Lambda}} d\phi_x$ denotes the Lebesgue measure on $\mathbb{R}^{\overset{\circ}{\Lambda}}$. For the particular case $\hat{\phi} \equiv u$, we write $\mathbf{P}_\Lambda^u$. In most of the cases,

$$(2.3) \qquad\qquad\qquad \Lambda = \Lambda_N := \{0, \ldots, N\}^d,$$

for some (usually large) $N \in \mathbb{N}$, so $\overset{\circ}{\Lambda}_N := \{1, \ldots, N-1\}^d$. We also introduce the notation $\widetilde{\Lambda}_N := \{1, \ldots, N\}^d$.

REMARK 2.1. Of course, $\mathbf{P}_\Lambda^{\hat{\phi}}$ is the finite volume LGFF. Much has been written about this field: we stress here that for $d \geq 3$ the $N \to \infty$ limit $\mathbf{P}^u$, with respect to the product topology, of $\mathbf{P}_{\Lambda_N}^u$ exists and it can be characterized as the Gaussian field with constant expectation u and covariance of ϕ_x and ϕ_y equal to the expected time spent in y by a simple symmetric random walk issued from x (for more on this very well-known issue we refer to [12], Section 2.9, and references therein). In particular, the variance of ϕ_x in the infinite volume limit does not depend on x and we denote it by σ_d^2. Moreover, the random walk representation holds also in finite volume—the walk is killed al the boundary—and this directly implies that the variance of ϕ_x grows as the region considered grows in the sense of set inclusion.

Given $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ a family of i.i.d. square integrable centered random variables (of law $\mathbb{P}$) with unit variance, we set, for all $\beta \in \mathbb{R}$,

$$(2.4) \qquad\qquad\qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_x}].$$

We call $I_\mathbb{P}$ the interval where $\lambda(\beta)$ is finite and assume that it contains a neighborhood of the origin. The two families of random variables, ω with law $\mathbb{P}$ and the LGFF ϕ with law $\mathbf{P}_\Lambda^{\hat{\phi}}$, are realized on a common probability space and they are independent.

For $x \in \mathbb{Z}^d$, set $\delta_x := \mathbf{1}_{[0,1]}(\phi(x))$ and $\rho_x := \mathbf{1}_{(-\infty,0)}(\phi(x))$. For $\beta \in I_\mathbb{P}$, $h \in \mathbb{R}$ and $K \in (0, \infty]$, we define a modified measure $\mathbf{P}_{N,h,K}^{\beta,\omega,\hat{\phi}}$ via

$$(2.5) \qquad \frac{d\mathbf{P}_{N,h,K}^{\beta,\omega,\hat{\phi}}}{d\mathbf{P}_N^{\hat{\phi}}}(\phi) = \frac{1}{Z_{N,h,K}^{\beta,\omega,\hat{\phi}}} \exp\left(\sum_{x \in \widetilde{\Lambda}_N} ((\beta\omega_x - \lambda(\beta) + h)\delta_x - K\rho_x) \right),$$

where

$$(2.6) \qquad Z_{N,h,K}^{\beta,\omega,\hat{\phi}} := \mathbf{E}_N^{\hat{\phi}}\left[\exp\left(\sum_{x \in \widetilde{\Lambda}_N} ((\beta\omega_x - \lambda(\beta) + h)\delta_x - K\rho_x) \right) \right].$$

In the homogeneous case, $\beta = 0$, we just drop from the superscripts β and ω.

2.2. *Main results.* We introduce the free energy (density) for every $K \in (-\infty, \infty]$, every $\beta \geq 0$ such that $\lambda(\beta) < \infty$ and every $h \in \mathbb{R}$ as

$$(2.7) \qquad \mathrm{F}_K(\beta, h) = \lim_{N \to \infty} \frac{1}{N^d} \mathbb{E} \log Z_{N,h,K}^{\beta,\omega,0}.$$

Theorem A.1 ensures that this limit exists, also as an almost sure limit if we drop the expectation with respect to the disorder. We note that, from the free energy viewpoint there is no point in paying attention to summing over $\widetilde{\Lambda}_N$ in the energy term $(\sum_{x \in \widetilde{\Lambda}_N} \ldots)$ defining the partition function: Λ_N or $\mathring{\Lambda}_N$ give the same free energy. Even more, the measure $\mathbf{P}_{N,h}^{\beta,\omega,\hat{\phi}}$ does not see these energy changes at the boundary. The choice of $\widetilde{\Lambda}_N$ enters the game in a nonnegligible way in relation to the super-additive property: at this stage, this important issue is just technical (see Appendix A).

Here is a simple but crucial observation: for every $K \in [0, \infty]$ (and every $\beta \in I_\mathbb{P}$ and every h),

$$(2.8) \qquad \mathrm{F}_K(\beta, h) \geq 0.$$

This follows simply from the fact that $-\log \mathbf{P}_N^0(\phi_x > 1$ for every $x \in \widetilde{\Lambda}_N) = o(N^d)$ [15]. The bound (2.8) combined with the fact that the convex function $\mathrm{F}_K(\beta, \cdot)$ is nondecreasing, tells us that there exists $h_{c,K}(\beta)$ (at this stage we drop the dependence on K for conciseness), a priori in $[-\infty, \infty]$, such that $\mathrm{F}_K(\beta, h) > 0$ if and only if $h > h_c(\beta)$. Elementary arguments directly yield that $h_c(\beta) \in [h_c(0), h_c(0) + \lambda(\beta)]$ and that $h_c(0) \in [0, \infty)$: $h_c(\beta) \geq h_c(0)$ is just a consequence of the *annealed bound* (Jensen's inequality)

$$(2.9) \quad \mathrm{F}(\beta, h) = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\log Z_{N,h,K}^{\beta,\omega,h}] \leq \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Z_{N,h,K}^{\beta,\omega,h}] = \mathrm{F}(0, h).$$

The bound $h_c(\beta) \leq h_c(0) + \lambda(\beta)$ follows by convexity of $\mathrm{F}(\cdot, h)$, using $\partial_\beta \mathrm{F}(0, h) = 0$ (see, e.g., [10], page 23). Finally, $h_c(0) \geq 0$ is just direct consequence of $Z_{N,h,K}^0 \leq 1$ if $h \geq 0$ and $h_c(0) \leq -\log C := -\log \mathbf{P}(\mathcal{N} \in [0, 2d])$, with $\mathcal{N}$ a standard Gaussian variable, follows from $\mathbf{P}_N^0(\phi_x \in [0, 1]$ for every $x \in \widetilde{\Lambda}_N) \geq C^{-N^d}$, which one derives by an easy nearest neighbors conditioning argument.

THEOREM 2.2. *For every $K \in (0, \infty]$, every $\beta \in I_\mathbb{P}$ we have*

$$(2.10) \qquad h_c(\beta) = 0.$$

Furthermore, if β belongs to the interior of $I_\mathbb{P}$ we have that for $h \searrow 0$

$$(2.11) \qquad \mathrm{F}_K(\beta, h) = \exp\left(\left(-\frac{\sigma_d^2}{2} + o(1)\right)\left(\log \frac{1}{h}\right)^2\right).$$

The proof of Theorem 2.2 is in Section 3 (Proposition 3.1: case $\beta = 0$ and upper bound estimate for $\beta > 0$) and in Section 4 (Proposition 4.1: lower bound estimate when $\beta > 0$).

It is worth pointing out that also in the case $K = 0$, treated in [12], we have that $h_c(\beta) = 0$, but (2.11) does not hold. In fact, in [12] it is shown that the critical behavior in that case is power law. More importantly, in [12] it is shown that in the $K = 0$ case disorder is relevant, that is, it changes the critical behavior (for any $\beta > 0$)—while (2.11) shows disorder irrelevance (more on this just below).

Moreover, Theorem 2.2 directly generalizes, by a scaling argument, to the case in which δ_x is defined by $b\mathbf{1}_{[0,a]}(\phi_x)$, with a and $b > 0$: (2.11) should simply be replaced by

$$(2.12) \qquad \mathrm{F}_K(\beta, h) = \exp\left(\left(-\frac{\sigma_d^2}{2a^2} + o(1)\right)\left(\log \frac{1}{h}\right)^2\right).$$

The novel content of Theorem 2.2 is twofold:

(1) It improves substantially what was known in the literature, and notably the results in [3], where the case $\beta = 0$ has been considered for a pinning potential of the form $b\mathbf{1}_{[0,a]}(\cdot)$, precisely the one addressed by (2.12). In our setup, the potential is rather $hb\mathbf{1}_{[0,a]}(\cdot)$, so we can set $b = 1$ and $h(> 0)$ plays the role of b. The result in [3] can be restated as $h_c(0) = 0$ with a lower bound on the free energy that has not been made explicit by the authors. However, one can extract from [3] (see Appendix B, Remark B.2) the lower bound: for every $c > 2$, there exists $h_0 > 0$ such that

$$(2.13) \qquad \mathrm{F}(0, h) \geq \exp(-h^{-c}),$$

for every $h \in (0, h_0]$. Hence (2.11) improves considerably this bound and provides a matching upper bound. In [3], also a singular limit of the model has been considered: we pick up this model at the end of Section 2.3.

(2) Our result also covers the disordered case and shows a strong form of disorder irrelevance, in agreement with the Harris criterion. A discussion of such a criterion, with review of a selection of the vast literature, can be found in [11], Section 5. But in a nutshell the criterion for disorder irrelevance is "$\nu d > 2$" (or *negative specific heat exponent* [11], Section 5), where ν is the exponent of the correlation length of the *pure* (i.e., $\beta = 0$) model approaching criticality. Much like for the renewal pinning model (see in particular [11], Section 2.4), the natural notion of correlation length $\ell(h)$ is given by the *minimal* linear size of the systems for which exponential growth sets in: so when $\ell(h)^d \mathrm{F}(0, h)$ is larger than a positive constant (say, one). So if $\mathrm{F}(0, h)$ vanishes as $h^{\widetilde{\nu}}$ for $h \searrow 0$, then $\ell(h)$ diverges with exponent $\nu = \widetilde{\nu}/d$. But Theorem 2.2 is saying that $\widetilde{\nu} = \infty$, hence $\infty = \nu d > 2$, so disorder is irrelevant. We point out that Theorem 2.2 finds an exact parallel in the results on loop exponent one renewal pinning models [1]: also in that case $\nu = \infty$ and disorder irrelevance is proven for every β, and not only for β below a threshold (relevance is expected to set in for β sufficiently large, when $\nu < \infty$; see [11], Section 5). This is why we speak of *strong* disorder irrelevance.

It is rather straightforward to extract from the convexity of the free energy that the system has a positive density of contacts if and only if $h > h_c(\beta)$ and, therefore, $h_c(\beta)$ is the critical point for a localization transition. One can also get a large deviation type estimate on the number of contacts in a large volume in the localized regime, like it is done in [3, 7] for the $\beta = 0$ case. And it is precisely in [3] that obtaining pointwise bounds on the field is cited as an open problem. An answer to this question can be found in [18], where quantitative pointwise bounds on the height of the field are obtained in the nondisordered setup and in the limit case of the model that goes under the name of δ-pinning model (see Appendix B for more on this model). Here, we present a result that is always in the direction of the pointwise control of the field, but in a different spirit. For this, we choose to work, only for the next statement (and its proof, see Section 5), with $\Lambda_N := \{-N, \ldots, N\}^d$ [and $\widetilde{\Lambda}_N := \{-N+1, \ldots, N\}^d$ in (2.5)]. The measures we consider are all viewed either as elements of the set of probability measures on $[0, \infty)^{\mathbb{Z}^d}$ or $[0, \infty]^{\mathbb{Z}^d}$, both equipped with the product topology and $[0, \infty]$ is equipped with the usual compactified topology. We use Θ_x for the translation operator, both for ϕ, that is, $(\Theta_x \phi)_y = \phi_{x+y}$, and for ω.

THEOREM 2.3. *Let us choose $\beta \in I_{\mathbb{P}}$:*

(1) *If $h > 0$, the sequence of averaged quenched probabilities $\{\mathbb{E} \mathbf{P}_{N,h,\infty}^{\omega,\beta,0}\}_{N=1,2,\ldots}$, probabilities on $[0, \infty)^{\mathbb{Z}^d}$, converges to a translation invariant limit. Moreover, for $\mathbb{P}$-almost every ω the sequence $\{\mathbf{P}_{N,h,\infty}^{\omega,\beta,0}\}_{N=1,2,\ldots}$, probabilities on $[0, \infty)^{\mathbb{Z}^d}$, converges to a translation covariant limit $\mathbf{P}_{\infty,h,\infty}^{\omega,\beta,0}$, that is $\mathbf{E}_{\infty,h,\infty}^{\omega,\beta,0}[f(\Theta_x \phi)] = \mathbf{E}_{\infty,h,\infty}^{\Theta_x \omega,\beta,0}[f(\phi)]$ for every bounded local f.*

(2) *If $h \leq 0$, the sequence of averaged quenched and quenched measures, both probabilities on $[0, \infty]^{\mathbb{Z}^d}$, converge to the probability concentrated on the singleton $\{\infty\}^{\mathbb{Z}^d}$.*

REMARK 2.4.   Let us observe here that the proof of Theorem 2.3 does not rely much on the assumption $d \geq 3$. When $d = 2$, a nontrivial covariant limit exists when $h > h_c(\beta)$ (as a consequence of [4] and of the annealed bound (2.9) $h_c(\beta) \geq h_c(0) > 0$ for all β) while for $h < h_c(\beta)$, the limit is concentrated on $\{\infty\}^{\mathbb{Z}^2}$ (the proof is identical). We cannot conclude in the case $h = h_c(\beta)$ since it is not known whether the localization transition is of first order or higher. In dimension one, results of the same type have been known since a while (see [11], Section 7.3)

The proof of Theorem 2.3 is in Section 5.

2.3. *Outline of the proof of Theorem 2.2.*   We will first treat, in Section 3, the homogeneous, or pure, model ($\beta = 0$). This both for presenting the easier case first and because $F_K(\beta, h) \leq F_K(0, h)$ so, in Section 4 dedicated to $\beta > 0$, we just need to provide a lower bound on $F_K(\beta, h)$.

The key point behind Theorem 2.2 is that a one site strategy turns out to be sufficient. Roughly, the idea is the following: We can imagine that, close to criticality, the field has very few pinned sites (note that this would not be the case if the transition were of first order, but we are at the stage of guessing). Therefore, also helped by the (entropic) repulsion effect of the (soft, $K < \infty$, or hard, $K = \infty$) wall and by the massless character of the field, the field is expected to be at a typical height $u \gg 1$, hence quite far from the levels that contribute to the energy. So the energy contributions are due to rare spikes downwards. The various sharp results on entropic repulsion for LGFF in $d \geq 3$, notably [2, 8, 9], support this intuition and also the fact that large excursions of the LGFF in $d \geq 3$ are essentially just isolated spikes (see [5] for a convergence result of these spikes to a Poisson process). For example, the expectation of $\max_{x \in \Lambda_N} \phi_x$, where ϕ the infinite volume centered LGFF in $d \geq 3$, is, to leading order as $N \to \infty$, the same as if ϕ were a collection of IID $\mathcal{N}(0, \sigma_d^2)$ random variables (recall that σ_d^2 is the variance of the one dimensional marginal of the infinite volume LGFF). So let us imagine that the field is repelled for h small to a height u very large and that we can look at the contribution of each variable as if they were independent. We are thus reduced to the computation of the contribution to the partition function of one site in this idealized setup:

$$
(2.14) \quad
\begin{aligned}
&\mathbf{P}(\phi_x > 1) + e^h \mathbf{P}(\phi_x \in [0, 1]) + e^{-K} \mathbf{P}(\phi_x < 0) \\
&\quad = 1 + (e^h - 1)\mathbf{P}(\phi_x \leq 1) + (e^{-K} - e^h)\mathbf{P}(\phi_x < 0).
\end{aligned}
$$

Now, we use $\phi_x \sim \mathcal{N}(u, \sigma_d^2)$, so $\phi_x = \sigma_d \mathcal{N} + u$ [recall that we use $\mathcal{N}$ for a $\mathcal{N}(0, 1)$ variable] and the standard asymptotic estimate

$$(2.15) \qquad \mathbf{P}(\mathcal{N} > t) \stackrel{t \to \infty}{\sim} \frac{1}{t\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right).$$

Since h is small and u is large, we can approximate (2.14) by

$$
\begin{aligned}
1 &+ \frac{h\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{(u-1)^2}{2\sigma_d^2}\right) - (1 - e^{-K})\frac{\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_d^2}\right) \\
&= 1 + \frac{\sigma_d}{u\sqrt{2\pi}} \exp\left(-\frac{u^2}{2\sigma_d^2}\right)\left[h \exp\left(\frac{u}{\sigma_d^2} - \frac{1}{2\sigma_d^2}\right) - (1 - e^{-K})\right].
\end{aligned}
$$

(2.16)

Up to now, we have not said anything about the value of u, but this computation says that a positive contribution to the free energy requires the term in the square brackets to be positive. And for this one needs to choose $u = \sigma_d^2 \log(1/h) + c$ for some positive constant c and choosing a much larger u would strongly penalize the gain, because of the prefactor $\exp(-u^2/(2\sigma_d^2))$. Note that the role played by K in this computation is marginal, as long as $K > 0$. This computation is suggesting that the one site contribution is for $h \searrow 0$,

$$(2.17) \qquad 1 + \exp\left(-\frac{(\sigma_d^2 + o(1))(\log\frac{1}{h})^2}{2}\right)$$

and, therefore,

$$
\begin{aligned}
(2.18) \qquad F_K(0, h) &= \log\left(1 + \exp\left(-\frac{(\sigma_d^2 + o(1))(\log\frac{1}{h})^2}{2}\right)\right) \\
&= \exp\left(-\frac{(\sigma_d^2 + o(1))(\log\frac{1}{h})^2}{2}\right).
\end{aligned}
$$

This is the main claim of Theorem 2.2 and in order to convert such an argument into a proof we will proceed separately for upper and lower bound. The upper bound is achieved by reducing the estimate to a model on a (very) spaced sublattice (via an application of the Hölder inequality): the Markov property of the LGFF at this point can be used to provide enough independence to obtain the bound we are after. For the lower bound, we exploit the fact (in Appendix A) that the free energy can be computed by choosing boundary conditions that are sampled from the infinite volume LGFF with an arbitrary average height u: in fact, with such boundary conditions the logarithm of the partition function forms a super-additive sequence, hence we can perform estimates for finite N to estimate from below the $N = \infty$ case; see (A.5). We then proceed by using Jensen's inequality in a way that a priori may seem very rough (we just compute the expectation of the energy term), but this turns out to be sufficient, thanks to the first step and the wise choice of the boundary mean u.

For what concerns the disordered case, the desired lower bound on $F_K(\beta, h)$ is achieved once again by exploiting super-additivity—we will work with a finite volume size that diverges as h tends to zero—and by choosing the boundary values at average height u. We choose $u = \sigma_d^2 \log(1/h) + c$. It was argued after (2.16) that this should be the value of u that maximizes the energy gain in the pure case and we choose to use the same value for the disordered case because we are aiming at showing disorder irrelevance. The volume size is chosen so that it is improbable to observe two or more contacts and then the estimate is performed on the partition function limited to the trajectories of the field that have at most one contact (and we send also K to infinity since, by monotonicity in K of the partition function, this is the worst case scenario). On this reduced free energy, we perform a second moment argument. A look at the proof shows that the variance term in this computation plays a very marginal role, reinforcing the idea that disorder is very irrelevant in this model. Even more, we are able to apply the second moment method without assuming that the second moment of $e^{\beta \omega_x}$ is finite. This is achieved with an accurate cut-off procedure. The core of the lower bound argument on $F_K(\beta, h)$ also for $\beta \neq 0$ is in any case the one site computation we have just sketched and the argument in Section 4 can be used verbatim (just set $\beta = 0$) to obtain another (somewhat more involved) proof of the lower bound presented in Section 3 for the nondisordered case.

3. The homogeneous case. The main result of this section, Proposition 3.1, implies Theorem 2.2 for $\beta = 0$ and provides the upper bound for the case $\beta > 0$.

PROPOSITION 3.1. *For every $K \in (0, \infty]$ in the limit $h \searrow 0$, we have*

$$(3.1) \qquad F_K(0, h) = \exp\left(\left(-\frac{\sigma_d^2}{2} + o(1)\right)\left(\log \frac{1}{h}\right)^2\right).$$

PROOF. We treat separately the upper and lower bound.
Upper bound. Since the partition function decreases as K increases, for the upper bound it suffices to prove the statement for a $K > 0$. It is also sufficient to consider $N = nL - 1$, with $n, L \in \mathbb{N}$, L even (both sufficiently large, say larger than 3 at this stage, but later on L is chosen fixed but arbitrarily large and n is sent to ∞) and the quantity

$$(3.2) \qquad Z_{n,L} := \mathbf{E}_N^0 \exp\left(\sum_{x \in \{L, L+1, \ldots, (n-1)L-1\}^d} (h\delta_x - K\rho_x)\right).$$

Note that the sum in the exponential does not range over the whole box $\widetilde{\Lambda}_N$, but there are only $O(n^{d-1}L^d)$ terms missing and for this reason for every fixed L we have

$$(3.3) \qquad \lim_{n \to \infty} \frac{1}{(Ln)^d} \log Z_{n,L} = \lim_{n \to \infty} \frac{1}{(Ln)^d} \log Z_{nL-1, h, K}^0 = F(0, h).$$

We now set for $v \in \{0, \dots, L-1\}^d =: B_L$

(3.4) $$\Lambda_{L,n}^v := (v + L\mathbb{Z}^d) \cap \{L, L+1, \dots, (n-1)L-1\}^d.$$

The family $\{\Lambda_{L,n}^v\}_{v \in B_L}$ is a partition of $\{L, L+1, \dots, (n-1)L-1\}^d$ and

$$Z_{n,L} = \mathbf{E}_N^0 \left[\exp\left(\sum_{v \in B_L} \sum_{x \in \Lambda_{L,n}^v} (h\delta_x - K\rho_x) \right) \right]$$

(3.5)

$$\leq \prod_{v \in B_L} \left(\mathbf{E}_N^0 \left[\exp\left(L^d \sum_{x \in \Lambda_{L,n}^v} (h\delta_x - K\rho_x) \right) \right] \right)^{L^{-d}},$$

by Hölder's inequality.

Step (3.5) allows focusing on models in which the pinning potential acts only on a L-sparse lattice. Then we condition on $\{\phi_y\}_{y \in \Gamma_{N,L}^v}$ where

(3.6)

$$\Gamma_{N,L}^v := \Big\{ y \in \Lambda_N : \text{there exists } x \in \Lambda_{N,L}^v \text{ such that}$$

$$\max_{i=1,\dots,d} |(y-x)_i| = (L/2) \Big\},$$

is just a grid that separates the sparse sites $x \in \Lambda_{L,n}^v$. By the Markov property of the LGFF, we readily see that $\{\phi_x\}_{x \in \Lambda_{L,n}^v}$ is a family of conditionally independent Gaussian variables. Their (conditional) mean is given by the harmonic extension of $\{\phi_y\}_{y \in \Gamma_{N,L}^v}$ to the full box and their variance is equal to the variance c_L^2 of free field with zero boundary conditions in the center of a box of side length L, so $\lim_{L \to \infty} c_L = \sigma_d$. Hence, with the notation $\mathcal{F}_A$ for the σ-algebra generated by $\phi_A := \{\phi_x\}_{x \in A}$, $A \subset \mathbb{Z}^d$, we obtain that almost surely

$$\mathbf{E}_N^0 \left[\exp\left(L^d \sum_{x \in \Lambda_{L,n}^v} (h\delta_x - K\rho_x) \right) \Big| \mathcal{F}_{\Gamma_{N,L}^v} \right]$$

(3.7)

$$\leq \left(\sup_{u \in \mathbb{R}} \mathbf{E} \exp(L^d h \mathbf{1}_{[0,1]}(c_L \mathcal{N} + u) - L^d K \mathbf{1}_{(-\infty,0)}(c_L \mathcal{N} + u)) \right)^{|\Lambda_{L,n}^v|},$$

which yields the same bound for the unconditional expectation. With the notation $P(a,b) = \mathbf{P}(\mathcal{N} \in (a,b))$, we have

$$\sup_{u \in \mathbb{R}} \mathbf{E} \exp(L^d h \mathbf{1}_{[0,1]}(c_L \mathcal{N} + u) - L^d K \mathbf{1}_{(-\infty,0)}(c_L \mathcal{N} + u))$$

(3.8) $$= 1 + \sup_{u \in \mathbb{R}} \left((e^{L^d h} - 1) P(-u, -u + 1/c_L) - (1 - e^{-L^d K}) P(-\infty, -u) \right)$$

$$\leq 1 + \sup_{u \in \mathbb{R}} \left(2 L^d h P(-u, -u + 1/c_L) - \frac{1}{2} P(-\infty, -u) \right),$$

where the last step we have used $h \leq L^{-d}$ and $(1 - \exp(-L^d K)) \geq 1/2$ (so we assume L larger than a suitable constant dependent on K). We note that the argument of the supremum in the last line in (3.8) is larger than zero if and only if

$$(3.9) \qquad \frac{P(-u, -u + 1/c_L)}{P(-\infty, -u)} > \frac{1}{4L^d h}.$$

We now observe that the function

$$(3.10) \qquad (-\infty, \infty) \ni u \mapsto \frac{P(-u, -u + a)}{P(-\infty, -u)} \stackrel{u \to \infty}{\sim} \exp\left(au - \frac{a^2}{2}\right),$$

is smooth and positive. Moreover, it goes to zero as $u \to -\infty$ and to ∞ as u goes to $+\infty$ (we cite as a fact that this function is increasing, but we do not use it in our proof). Therefore, for h small the supremum will be achieved for u large: in particular, (3.9) implies that we can restrict the supremum to

$$(3.11) \qquad u \geq \frac{1}{2c_L} + c_L \log\left(\frac{1}{5L^d h}\right) = u_0(h, L).$$

By using this information, we could get a sharp estimate on the supremum, but we will content ourselves with a much simpler estimate which is sufficient for our purposes. In fact, neglecting the negative term in (3.8), we obtain that given L and $a < c_L^2/2$ for h sufficiently small we have

$$(3.12) \qquad \begin{aligned} \sup_{u \geq u_0} &\ 2L^d h P(-u, -u + 1/c_L) \\ &= 2L^d h P(-u_0, -u_0 + 1/c_L) \\ &\leq \frac{3L^d h}{\sqrt{2\pi}} \exp\left(-\frac{u_0^2}{2}\right) \leq \exp(-a(\log(1/h))^2). \end{aligned}$$

Therefore, going back to (3.7), we have

$$(3.13) \quad \mathbf{E}_N^0\left[\exp\left(L^d \sum_{x \in \Lambda_{L,n}^v} (h\delta_x - K\rho_x)\right)\right] \leq \left(1 + \exp\left(-a\left(\log \frac{1}{h}\right)^2\right)\right)^{|\Lambda_{L,n}^v|},$$

and from (3.2),

$$(3.14) \qquad \begin{aligned} \frac{1}{(nL)^d} \log Z_{n,L} &\leq \frac{|\Lambda_{L,n}^v|}{(nL)^d} \log\left(1 + \exp\left(-a\left(\log \frac{1}{h}\right)^2\right)\right) \\ &\leq \frac{2}{L^d} \exp\left(-a\left(\log \frac{1}{h}\right)^2\right), \end{aligned}$$

again for h sufficiently small. By recalling that c_L can be chosen arbitrarily close to σ_d, we see that the proof of the upper bound is complete.

Lower bound. Also for the lower bound we work with $K \in (0, \infty)$, but this time we follow the K dependence of the bound. By Proposition A.2, precisely (A.5), and Jensen's inequality we obtain that for every u and every N

$$\text{(3.15)} \quad F_K(h) \geq \frac{1}{N^d} \hat{\mathbf{E}}^u \mathbf{E}_N^{\hat{\phi}} \left[\sum_{x \in \widetilde{\Lambda}_N} (h\delta_x - K\rho_x) \right] = \mathbf{E}^u[h\delta_x - K\rho_x]$$

$$= \mathbf{E}[h\mathbf{1}_{[0,1]}(\sigma_d \mathcal{N} + u) - K\mathbf{1}_{(-\infty,0)}(\sigma_d \mathcal{N} + u)],$$

where $\hat{\mathbf{E}}^u$ is the expectation that acts on the random field $\hat{\phi}$, which is just the infinite volume free field with mean u (hence, of law $\mathbf{P}^u$). Therefore, with the notation used for the upper bound we have that for every u

$$\text{(3.16)} \quad F_K(h) \geq hP(-u, -u + 1/\sigma_d) - KP(-\infty, -u).$$

We set $u = \sigma_d \log(1/h) + r$, with r to be determined. Therefore, for h sufficiently small [how small depends here on r since we require $u \geq C$ for some deterministic C to use the asymptotic statement (2.15)]

$$F_K(h)$$

$$\text{(3.17)} \quad \geq \frac{1}{u\sqrt{2\pi}} \left(\frac{h}{2} \exp\left(-\frac{1}{2}\left(u - \frac{1}{\sigma_d}\right)^2\right) - 2K \exp\left(-\frac{1}{2}u^2\right) \right)$$

$$= \frac{2h^{\sigma_d r} \exp(-r^2/2)}{(r + \sigma_d \log(1/h))\sqrt{2\pi}} \exp\left(-\frac{\sigma_d^2}{2}\left(\log\frac{1}{h}\right)^2\right) \left(\frac{e^{-1/(2\sigma_d^2) + r/\sigma_d}}{4} - K \right).$$

We now set $r = \frac{1}{2\sigma_d} + \sigma_d \log(4(K + 1))$ and we get to the explicit bound

$$\text{(3.18)} \quad F_K(h)$$

$$\geq \frac{2h^{\frac{1}{2} + \sigma_d^2 \log(4(K+1))} \exp(-\frac{1}{2}(\frac{1}{2\sigma_d} + \sigma_d \log(4(K + 1)))^2)}{(\frac{1}{2\sigma_d} + \sigma_d \log(4(K + 1)) + \sigma_d \log(1/h))\sqrt{2\pi}} e^{-\frac{\sigma_d^2}{2}(\log\frac{1}{h})^2}.$$

Therefore, for every $b > \sigma_d^2/2$ and every $K \in (0, \infty)$ there exists $h_0 > 0$ such that for every $h \in (0, h_0)$

$$\text{(3.19)} \quad F_K(h) \geq \exp\left(-b\left(\log\frac{1}{h}\right)^2\right).$$

This completes the proof of the lower bound, except for the case $K = \infty$.

For the lower bound in the case $K = \infty$, we observe that for K large r becomes large too and (3.18) holds for arbitrary $h_0 > 0$ as $K \to \infty$ [but in our formulas we want to have $\log(1/h) \geq 0$ so $h_0 = 1$]. Now remark that if we choose $K + 1 = \exp((\log(1/h))^{1/2})$ the ratio in right-hand side of (3.18) is bounded below by $\exp(-(\log(1/h))^a)$ for any $a > 3/2$ and h sufficiently small. So for every $b > \sigma_d^2/2$ there exists $h_1 > 0$ such that for every $h \in (0, h_1)$ (3.19)

holds. The conclusion is then an immediate consequence of Lemma A.3, because $F_\infty(0, h) \geq F_K(0, h) - \exp(-K)$. This completes the proof of the lower bound and, therefore, the proof of Proposition 3.1. $\square$

4. The disordered case.

PROPOSITION 4.1. *For every $\beta \in I_\mathbb{P}$, we have $h_c(\beta) = 0$. Moreover, if β is in the interior of $I_\mathbb{P}$, for every $K \in (0, \infty]$ and every $\varepsilon > 0$ there exists h_0 such that*

$$(4.1) \qquad F_K(\beta, h) \geq \exp\left(-(1 + \varepsilon)\frac{\sigma_d^2}{2}\left(\log\frac{1}{h}\right)^2\right),$$

for $h \in (0, h_0)$.

In this section, we set

$$(4.2) \qquad \xi_x := e^{\beta\omega_x - \lambda(\beta)},$$

and let ξ denote a variable which has the same distribution as all the ξ_x. Note that $\mathbb{E}\xi = 1$ and that the assumption that β is in the interior of $I_\mathbb{P}$ is equivalent to

$$(4.3) \qquad \text{"There exist } C > 0 \text{ and } \gamma > 1 \text{ such that}$$
$$\mathbb{P}(\xi \geq t) \leq Ct^{-\gamma} \text{ for every } t \geq 0.\text{"}$$

We also use the notation $\rho_x^+ := \mathbf{1}_{(-\infty, 1]}(\phi_x)$. For $\tilde{a} > 1$, we set

$$(4.4) \qquad u := \tilde{a}\sigma_d^2|\log h| \quad \text{and} \quad N = \exp(|\log h|^{3/2}).$$

A basic recurrent quantity in the proof is going to be

$$(4.5) \qquad P(u) := \mathbf{P}^u(\phi_0 \leq 1) \begin{cases} \overset{h\searrow 0}{\sim} \dfrac{\sigma_d}{u\sqrt{2\pi}}\exp\left(-\dfrac{(u-1)^2}{2\sigma_d^2}\right), \\ \geq \exp\left(-(1 + \varepsilon)\dfrac{\tilde{a}^2\sigma_d^2}{2}|\log h|^2\right), \end{cases}$$

where we have used (2.15) and the inequality, which holds for every $\varepsilon > 0$ and h sufficiently small, is directly obtained by inserting the value of u. Moreover, for every $c < 1$ we have $\mathbf{P}^u(\phi_0 \in [c, 1]) \sim P(u)$, for $u \to \infty$. Another relevant estimate is

$$(4.6) \qquad \frac{\mathbf{P}^u(\phi_0 \leq 1)}{\mathbf{P}^u(\phi_0 < 0)} \overset{h\searrow 0}{\sim} h^{-\tilde{a}}\exp(-1/(2\sigma_d^2)).$$

We introduce also the event

$$(4.7) \qquad G_u := \left\{\phi \in \mathbb{R}^{\mathbb{Z}^d} : \phi_x > \frac{u}{2} \text{ for } x \in \partial\Lambda_N\right\}.$$

Here and in the remainder of the proof, we avoid insisting on the fact that we should choose h such that $\exp(|\log h|^{3/2}) \in \mathbb{N}$: obtaining the estimate along this

subsequence yields the claim for $h \searrow 0$ by a direct estimate and using that $\mathrm{F}_K(\beta, \cdot)$ is nondecreasing. Alternatively, one can carry along the proof $N = \lfloor \exp(|\log h|^{3/2}) \rfloor$ and deal with the little nuisances that arise.

The following statement controls the contribution of the bad boundary configurations.

LEMMA 4.2. *For every d, there exist $C_d > 0$ and $h_0 > 0$ such that for every $K \geq 0$, $\beta \in I_{\mathbb{P}}$ and for $h \in [0, h_0)$ we have*

$$(4.8) \qquad \mathbb{E}\hat{\mathbf{E}}^u[(\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}})\mathbf{1}_{G_u^{\complement}}((\hat{\phi})] \geq -C_d(\lambda(\beta) \vee K)N^{d-1}P(u).$$

PROOF.   By Jensen's inequality,

$$\mathbb{E}\hat{\mathbf{E}}^u[(\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}})\mathbf{1}_{G_u^{\complement}}(\hat{\phi})]$$

$$(4.9) \qquad \geq (-\lambda(\beta) + h)\mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x; G_u^{\complement}\right] - K\mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \rho_x; G_u^{\complement}\right]$$

$$\geq -(\lambda(\beta) \vee K)\mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \rho_x^+; G_u^{\complement}\right],$$

with the notation $\mathbf{E}^u[\cdot; F] = \mathbf{E}^u[\cdot\mathbf{1}_F(\phi)]$. By the union bound and by making an elementary splitting for a $C > 0$, we have

$$
\begin{aligned}
\mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \rho_x^+; G_u^{\complement}\right] &\leq \sum_{\substack{x \in \widetilde{\Lambda}_N \\ y \in \partial\Lambda_N}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) \\
&\leq \sum_{\substack{x \in \widetilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| \leq C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2)
\end{aligned}
$$

$$(4.10) \qquad\qquad\qquad + \sum_{\substack{x \in \widetilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| > C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2)$$

$$\leq 2dCN^{d-1}P(u)$$

$$+ \sum_{\substack{x \in \widetilde{\Lambda}_N, y \in \partial\Lambda_N: \\ |x-y| > C}} \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2).$$

Now given $\eta > 0$ we have

$$(4.11) \qquad \mathbf{P}^u(\phi_x \leq 1, \phi_y \leq u/2) \leq \mathbf{P}^u((\phi_x + \eta\phi_y) \leq 1 + u\eta/2).$$

Now $\phi_x + \eta\phi_y$ is a Gaussian variable of mean $(1+\eta)u$. To compute its variance, we observe that the covariance between ϕ_x and ϕ_y is given by $G(x, y) = \sigma_d^2 p(x, y)$

with $p(x, y)$ the probability that a simple random walk issued from x hits y: since $p(x, y)$ vanishes when $|x - y|$ becomes large, we choose C so that $p(x, y) \leq 1/8$ when $|x - y| \geq C$. Hence we have for $\eta \leq 1/4$

$$(4.12) \qquad \mathrm{var}(\phi_x + \eta\phi_y) \leq \sigma_d^2(1 + \eta^2 + \eta/4) \leq \sigma_d^2(1 + \eta/2).$$

Using this information, we have for u sufficiently large

$$(4.13) \qquad \begin{aligned} \mathbf{P}^u\big((\phi_x + \eta\phi_y) \leq 1 + u\eta/2\big) &\leq \exp\left(-\frac{(u(1 + \eta/2) - 1)^2}{\sigma_d^2(2 + \eta)}\right) \\ &\leq P(u)\exp\big(-c(\eta)u^2\big) \leq N^{-d}P(u), \end{aligned}$$

where $c(\eta) = \eta/(8\sigma_d^2)$ and we have used the first line in (4.5) together with the relation between the parameters (4.4).

Hence, going back to (4.10), we see that

$$(4.14) \qquad \mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \rho_x^+; G_u^{\complement}\right] \leq \big(2d(C + 1)N^{d-1}\big)P(u).$$

By plugging this estimate into (4.9), we complete the proof. $\quad\square$

PROOF OF PROPOSITION 4.1.   We aim at producing a lower bound on $Z_{N,h,K}^{\beta,\omega,\hat{\phi}}$ for good boundary values $\hat{\phi}$ (Lemma 4.2 is going to take care of the bad ones). This will be achieved by a second moment approach: we give first the proof assuming that the second moment of ξ is finite, that is, $\lambda(2\beta) < \infty$, or $2\beta \in I_{\mathbb{P}}$. Then we will show how to relax this condition.

The second moment method is not applied directly to the partition function, but to a *reduced* version for which we allow at most one contact in $[0, 1]$ and none in $(-\infty, 0)$. Note in fact that

$$\begin{aligned} (4.15) \qquad Z_{N,h,K}^{\beta,\omega,\hat{\phi}} &\geq Z_{N,h,K}^{\beta,\omega,\hat{\phi}}\left(\left\{\phi : \sum_{x \in \widetilde{\Lambda}_N} \delta_x \in \{0, 1\}, \sum_{x \in \widetilde{\Lambda}_N} \rho_x = 0\right\}\right) \\ &= \mathbf{P}_N^{\hat{\phi}}(\phi_x > 1 \text{ for } x \in \widetilde{\Lambda}_N) \\ &\quad + \sum_{x \in \widetilde{\Lambda}_N} e^{\beta\omega_x - \lambda(\beta) + h}\mathbf{P}_N^{\hat{\phi}}\left(\delta_x = 1 \text{ and } \sum_{y \in \widetilde{\Lambda}_N \setminus \{x\}} \rho_y^+ = 0\right) \\ &=: Q_{N,h}^{\beta,\omega,\hat{\phi}}, \end{aligned}$$

and for conciseness we write $Q_N^{\omega,\hat{\phi}}$ for $Q_{N,h}^{\beta,\omega,\hat{\phi}}$: this is the reduced partition function. Note that the reduced partition function does not contain K and in fact (4.15) holds uniformly in $K \geq 0$.

The first observation on $Q_N^{\omega,\hat\phi}$ is that

$$Q_N^{\omega,\hat\phi} \geq \mathbf{P}_N^{\hat\phi}(\phi_x > 1 \text{ for } x \in \widetilde\Lambda_N)$$

(4.16)

$$= 1 - \mathbf{P}_N^{\hat\phi}\left(\bigcup_{x\in\widetilde\Lambda_N}\{\phi_x \leq 1\}\right) \geq 1 - \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}(\phi_x \leq 1),$$

and a direct estimate shows that, if $\hat\phi \in G_u$ (which ensures that the mean of ϕ_x under $\mathbf{P}_N^{\hat\phi}$ is bounded below by $|\log h|$ times a positive constant), we can find $c > 0$ such that

(4.17)
$$\mathbf{P}_N^{\hat\phi}(\phi_x \leq 1) \leq \exp(-c(\log h)^2),$$

for every $x \in \widetilde\Lambda_N$. Hence, for h sufficiently small we have $Q_N^{\omega,\hat\phi} \geq 1/2$ and, therefore,

(4.18)
$$\log Q_N^{\omega,\hat\phi} \geq (Q_N^{\omega,\hat\phi} - 1) - (Q_N^{\omega,\hat\phi} - 1)^2,$$

which leads to the bound (uniform in $K \geq 0$)

(4.19)
$$\mathbb{E}\hat{\mathbf{E}}^u\big[(\log Z_{N,h,K}^{\beta,\omega,\hat\phi})\mathbf{1}_{G_u}(\hat\phi)\big]$$
$$\geq \mathbb{E}\hat{\mathbf{E}}^u\big[(Q_N^{\omega,\hat\phi} - 1)\mathbf{1}_{G_u}(\hat\phi)\big] - \mathbb{E}\hat{\mathbf{E}}^u\big[(Q_N^{\omega,\hat\phi} - 1)^2\mathbf{1}_{G_u}(\hat\phi)\big],$$

on which we will concentrate our attention from here until the end of the proof.

First moment estimates (lower and upper bound). Let us observe that $\mathbb{E}(Q_N^{\omega,\hat\phi} - 1)$ is equal to

(4.20)
$$-\mathbf{P}_N^{\hat\phi}\left(\bigcup_{x\in\widetilde\Lambda_N}\{\phi_x \leq 1\}\right)$$
$$+ e^h \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}\big(\delta_x = 1 \text{ and } \phi_y > 1 \text{ for } y \in \widetilde\Lambda_N \setminus \{x\}\big),$$

and that this quantity, by the union bound, using also $e^h - 1 \geq h$ and the notation $F_x := \{\phi_y > 1 \text{ for } y \in \widetilde\Lambda_N \setminus \{x\}\}$, can be bounded below by

(4.21)
$$- \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}(\phi_x \leq 1) + \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}(\delta_x = 1) - \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}\big(\{\delta_x = 1\} \cap F_x^{\complement}\big)$$
$$+ h \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}(\delta_x = 1) - h \sum_{x\in\widetilde\Lambda_N}\mathbf{P}_N^{\hat\phi}\big(\{\delta_x = 1\} \cap F_x^{\complement}\big),$$

which we reorder into

$$\mathbb{E}(Q_N^{\omega,\hat{\phi}} - 1) \geq h \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) - (1+h) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x < 0)$$

(4.22)
$$- (1+h) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^{\complement}).$$

Now we observe that

$$(4.23) \quad \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x^{\complement}) \leq \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1) \max_{z \in [0,1]} \sum_{y \in \widetilde{\Lambda}_N \setminus \{x\}} \mathbf{P}_N^{\hat{\phi}}(\phi_y \leq 1 | \phi_x = z),$$

and we use the fact that the probability that a random walk issued from y hits $\partial \Lambda_N$ before visiting x is larger than the probability that a random walk in $\mathbb{Z}^d$ issued from y never hits x, and this latter probability q is positive. Hence the mean of ϕ_y, under $\mathbf{P}_N^{\hat{\phi}}(\cdot | \phi_x = z)$, is at least $qu/2$, because $\hat{\phi} \in G_u$ and, therefore, since the variance is bounded (by σ_d^2), there exists $c > 0$ such that $\mathbf{P}_N^{\hat{\phi}}(\phi_y \leq 1 | \phi_x = z) \leq \exp(-c(\log h)^2)$ and, therefore, the last term in (4.22) is negligible with respect to the first in the right-hand side of the same formula. Therefore, for $\hat{\phi} \in G_u$ and for h sufficiently small we have

$$(4.24) \qquad \mathbb{E}(Q_N^{\omega,\hat{\phi}} - 1) \geq \frac{4}{5} h \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1) - (1+h) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\phi_x < 0).$$

We will also need an upper bound on $\mathbb{E}(Q_N^{\omega,\hat{\phi}} - 1)$. For this, we restart from (4.20) and observe that

$$\mathbb{E}(Q_N^{\omega,\hat{\phi}} - 1) = -\mathbf{P}_N^{\hat{\phi}}\left(\bigcup_{x \in \widetilde{\Lambda}_N} \{\phi_x \leq 1\} \right) + e^h \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x)$$

(4.25)
$$\leq (e^h - 1) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\{\delta_x = 1\} \cap F_x)$$

$$\leq (e^h - 1) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1).$$

Second moment estimate. Recall that we assume here that $\mathbb{E}[\xi^2] < \infty$. First of all,

$$(4.26) \qquad \mathbb{E}[(Q_N^{\omega,\hat{\phi}} - 1)^2] = (\mathbb{E}(Q_N^{\omega,\hat{\phi}} - 1))^2 + \mathrm{var}_{\mathbb{P}}(Q_N^{\omega,\hat{\phi}}).$$

The variance term is easily computed and estimated:

$$\text{var}_{\mathbb{P}}(Q_N^{\omega,\hat{\phi}}) \leq e^{2h} \text{var}(\xi) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1)^2$$

(4.27)

$$\leq e^{2h} \text{var}(\xi) \max_{x' \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1).$$

For the square of the mean, the estimate is already in (4.25). Hence

$$\mathbb{E}[(Q_N^{\omega,\hat{\phi}} - 1)^2]$$

$$\leq (e^{2h} \text{var}(\xi) + (e^h - 1)^2 N^d) \max_{x' \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1)$$

(4.28)

$$\leq 2N^d \max_{x' \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_{x'} = 1) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1)$$

$$\leq \exp(-c(\log h)^2) \sum_{x \in \widetilde{\Lambda}_N} \mathbf{P}_N^{\hat{\phi}}(\delta_x = 1).$$

In both inequalities, we used that h is small [how small depends on var(ξ): we require $e^{2h} \text{var}(\xi) \leq N^d$ and $(e^h - 1)^2 \leq 1$] and that $N = \exp(|\log h|^{3/2})$, and in the last inequality we used $\hat{\phi} \in G_u$ in the same way as for (4.17). Note that the constant c does not depend on ξ. We have insisted on the role of ξ to prepare the generalization to the case in which ξ has unbounded second moment.

Lower bound on $\log Z$. We go back to (4.19): uniformly in $K \geq 0$

$$\mathbb{E}\hat{\mathbf{E}}^u[(\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}})\mathbf{1}_{G_u}(\hat{\phi})]$$

$$\geq \frac{4}{5}h \sum_{x \in \widetilde{\Lambda}_N} \hat{\mathbf{E}}^u[\mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1)\mathbf{1}_{G_u}(\hat{\phi})]$$

$$- (1+h)|\widetilde{\Lambda}_N|\mathbf{P}^u(\phi_0 < 0) - \exp(-c(\log h)^2)|\widetilde{\Lambda}_N|\mathbf{P}^u(\delta_0 = 1)$$

(4.29)

$$\geq \frac{4}{5}hN^d\mathbf{P}^u(\phi_0 \leq 1) - \frac{4}{5}h \sum_{x \in \widetilde{\Lambda}_N} \hat{\mathbf{E}}^u[\mathbf{P}_N^{\hat{\phi}}(\phi_x \leq 1)\mathbf{1}_{G_u^{\complement}}(\hat{\phi})]$$

$$- (1+h)N^d\mathbf{P}^u(\phi_0 < 0) - \exp(-c(\log h)^2)N^d\mathbf{P}^u(\delta_0 = 1)$$

$$\geq \frac{4}{5}hN^d P(u) - \frac{4}{5}h\mathbf{E}^u\left[\sum_{x \in \widetilde{\Lambda}_N} \rho_x^+; G_u^{\complement}\right] - h^b N^d P(u),$$

with $b \in (1, \widetilde{a})$ (recall that $\widetilde{a} > 1$), and h sufficiently small. In the last inequality, we have controlled from below the two terms in the line before the last one by $-h^b N^d P(u)$: this is because (4.6) tells us $\mathbf{P}^u(\phi_0 < 0) = O(h^{\widetilde{a}})\mathbf{P}^u(\phi_0 \leq 1)$, so the

first term in the line before the last one in (4.29) is much larger than the second and all this line is controlled as we claimed. The second term in the last line of (4.29) has been already treated in (4.14) and we readily see that it is negligible with respect to the first. Therefore, we get to

$$(4.30) \qquad \mathbb{E}\hat{\mathbf{E}}^u\big[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}}; G_u\big] \geq \frac{2}{3}hN^d P(u),$$

for h sufficiently small and by Lemma 4.2 for every $K \geq 0$

$$(4.31) \qquad \mathbb{E}\hat{\mathbf{E}}^u\big[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}}\big] \geq \Big(\frac{2}{3}h - C_d h^2(\lambda(\beta) \vee K)\Big)N^d P(u),$$

so the proof of Proposition 4.1, assuming $\mathbb{E}[\xi^2] < \infty$ is complete, for every $K > 0$. For the case $K = \infty$, we apply Lemma A.6—recall also (A.7)—with $K = h^{-1/2}$ and (4.31).

Relaxing the assumption $E[\xi^2] < \infty$. Let us assume now that $E[\xi^2] = \infty$, keeping of course the hypothesis $\beta \in I_{\mathbb{P}}$. We then replace ξ by $\xi_H := \min(\xi, H)$ in the partition function. Since $E[\xi^2] = \infty$, we have $E[\xi_H] < 1$. However, if we rescale u accordingly, all the computations of the above remain valid if one chooses

$$(4.32) \qquad h = -\log \mathbb{E}[\xi_H] + s,$$

with $s > 0$ and choose $N = \exp(|\log s|^{3/2})$. In this setup, s plays the role of h. We obtain in particular that there exists $s_0 := s_0(H, \varepsilon)$ such that for $s < s_0$

$$(4.33) \qquad \mathrm{F}_K\big(\beta, -\log \mathbb{E}[\xi_H] + s\big) \geq \exp\Big(-(1+\varepsilon)\frac{\sigma_d^2}{2}\Big(\log \frac{1}{s}\Big)^2\Big),$$

and from this we extract that $h_c(\beta) \leq -\log \mathbb{E}[\xi_H]$, which, sending $H \to \infty$, yields $h_c(\beta) = 0$.

To obtain a lower bound on the free energy, we assume that β is in the interior of $I_{\mathbb{P}}$ and we make explicit the estimate by carefully tracking the H dependence in the lower bound proof. Of course, the first moment estimates do not depend on the value of H, and browsing the part involving the second moment, we can check that the variance of ξ only intervenes in (4.27)–(4.28). It suffices that

$$(4.34) \qquad \exp(2h)\,\mathrm{var}(\xi_H) \leq N^d,$$

and since trivially $\mathrm{var}(\xi_H) \leq H^2$, $N \geq H$ suffices. Recalling (4.4), if $s \leq \exp(-(\log H)^{2/3})$ and if H is sufficiently large, how large depends on ε, (4.33) holds. On the other hand, we have from (4.3)

$$(4.35) \qquad -\log \mathbb{E}(\xi_H) = -\log\Big(1 - \int_H^\infty \mathbb{P}(\xi > t)\,dt\Big) \leq \frac{C}{\gamma - 1}H^{-(\gamma-1)}.$$

Hence for small h we can fix $s = h/2$ and $H = \exp(|\log h|^{3/2})$ and we deduce from (4.33) that

$$(4.36) \qquad \mathrm{F}_K(\beta, h) \geq \mathrm{F}_K\big(\beta, -\log \mathbb{E}[\xi_H] + h/2\big) \geq \exp\Big(-(1+\varepsilon)\frac{\sigma_d^2}{2}\Big(\log \frac{2}{h}\Big)^2\Big),$$

and the proof of Proposition 4.1 is now complete. $\quad\square$

REMARK 4.3. A look at the previous argument shows that it goes through even weakening a little (4.33), therefore, including cases in which $\lim_{t \to \infty} t^\gamma \mathbb{P}(\xi > t) = \infty$, for any $\gamma > 1$, but it is equal to zero for $\gamma = 1$. This means that in some cases it works also for β at the boundary of $I_\mathbb{P}$. However, if the tail decay is too weak [e.g., $\mathbb{P}(\xi \geq t) \geq (t(\log t))^{-1}(\log \log t)^{-2}$], (2.11) does not hold as it can be seen by applying and adapting the upper bound argument present in [12].

5. Infinite volume limit: Proof of Theorem 2.3. We recall that for Theorem 2.3, we have chosen $\Lambda_N = \{-N, \ldots, N\}^d$, and $\widetilde{\Lambda}_N$ accordingly. In this section, we always assume that $\beta \in I_\mathbb{P}$.

The first remark is that $\{\mathbf{P}_{N,h}^{\beta,\omega}\}_{N=1,2,\ldots}$ is increasing for the order induced by stochastic domination. Later on we will use also the more general statement

$$(5.1) \qquad\qquad \mathbf{P}_{\Lambda,h}^{\beta,\omega} \leq \mathbf{P}_{\Lambda',h}^{\beta,\omega} \qquad \text{if } \Lambda \supset \Lambda',$$

where $\leq$ stands here for stochastic domination. Hence we can couple the family of random variables $\phi^N = \{\phi^N\}_{x \in \mathbb{Z}^d}$ with law $\mathbf{P}_{N,h}^{\beta,\omega} := \mathbf{P}_{N,h,\infty}^{\beta,\omega}$ in a way that ϕ^N increases with N. Therefore, for every local continuous function $f : [0,\infty]^{\mathbb{Z}^d} \to \mathbb{R}$ we have that for every ω

$$(5.2) \qquad\qquad \lim_{N \to \infty} \mathbf{E}_{N,h}^{\beta,\omega}[f(\phi)] = \mathbf{E}_h^{\beta,\omega}[f(\phi)],$$

and, by the dominated convergence theorem, the same holds by taking the $\mathbb{E}$ expectation on both sides (we are using $\mathbf{E}_h^{\beta,\omega}$ for $\mathbf{E}_{\infty,h}^{\beta,\omega}$).

We are now going to argue that $\mathbf{P}_h^{\beta,\omega}$ satisfies the Markov property. Recall the notation $\mathcal{F}_A$ introduced right before (3.7). The aim is showing that for every finite subset Γ of $\mathbb{Z}^d$ and for every local bounded continuous $g : [0,\infty]^\Gamma \to \mathbb{R}$—in particular, the limit of $g(\phi_\Gamma)$, when $\phi_x \to \infty$ for every $x \in \Gamma$, exists and we call it $g(\infty)$—for all $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ we have that $\mathbf{P}_h^{\beta,\omega}(d\widetilde{\phi})$-a.s.:

$$(5.3) \quad \begin{aligned} &\mathbf{E}_h^{\beta,\omega}\big[g(\phi_\Gamma)|\mathcal{F}_{\Gamma^\complement}\big](\widetilde{\phi}) \\ &= \begin{cases} \dfrac{1}{Z_{\Gamma,h}^{\beta,\omega,\widetilde{\phi},+}}\mathbf{E}_\Gamma^{\widetilde{\phi},+}\Big[\exp\Big(\sum_{x \in \Gamma}(\beta\omega - \lambda(\beta) + h)\delta_x\Big)g(\phi_\Gamma)\Big] \\ \qquad \text{if } \max\phi_{\partial^+\Gamma} < \infty, \\ g(\infty) \\ \qquad \text{if } \max\phi_{\partial^+\Gamma} = \infty, \end{cases} \end{aligned}$$

where $\partial^+\Gamma := \{y \in \Gamma^\complement : \text{there exists } x \in \Gamma \text{ such that } y \sim x\}$ and $\mathbf{P}_\Gamma^{\widetilde{\phi},+}$ is the law of a free field ϕ with boundary condition $\widetilde{\phi}$ on $\mathbb{Z}^d \setminus \Gamma$ [recall (2.2)], conditioned to $\{\phi : \phi_x \geq 0$ for every $x \in \Gamma\}$. $Z_{\Gamma,h}^{\beta,\omega,\widetilde{\phi},+}$ is the obvious normalization constant associated to the Boltzmann term that appears on the right-hand side.

Call $G_{g,\Gamma}(\widetilde{\phi})$ the right-hand side of (5.3). Two important observations are:

(1) $G_{g,\Gamma}(\widetilde{\phi})$ depends only on $\phi_{\partial+\Gamma}$: this is the Markov property. We will then consider $G_{g,\Gamma}(\cdot)$ as a function from $[0,\infty]^{\partial+\Gamma}$ to $\mathbb{R}$.

(2) Of course, $\|G_{g,\Gamma}\|_\infty \le \|g\|_\infty$ and one directly verifies also the continuity of $G_{g,\Gamma}(\cdot)$.

To prove (5.3), it suffices to show that for every bounded local continuous $f : [0,\infty]^{\Gamma^\complement} \to \mathbb{R}$ we have that

$$(5.4) \qquad \mathbf{E}_h^{\beta,\omega}[f(\phi_{\Gamma^\complement})g(\phi_\Gamma)] = \mathbf{E}_h^{\beta,\omega}[f(\phi_{\Gamma^\complement})G_{g,\Gamma}(\phi)].$$

But, by continuity and boundedness of the integrands, in both sides of (5.4) we can replace $\mathbf{E}_h^{\beta,\omega}[\cdots]$ with $\lim_N \mathbf{E}_h^{N,\beta,\omega}[\cdots]$ and the finite volume statement is directly verified as soon as N is sufficiently large, since the locality of f implies that $f(\phi) = f(\phi')$ if $\phi_{\Lambda_N} = \phi'_{\Lambda_N}$ for N larger than a finite value that depends on f. So (5.3) holds and the infinite volume field we built satisfies the Markov property.

Next, we prove that the quenched measure $\mathbf{P}_h^{\beta,\omega}$ is translationally covariant and two results about the averaged quenched measure $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$. Translation covariance for the quenched limit probability and translation invariance of $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$ stem from the same argument that we give now. Using (5.1), one checks that that for $x \in \mathbb{Z}^d$, we have the following stochastic comparison for the translated measure for finite $N > |x|$:

$$(5.5) \qquad \mathbf{P}_{N-|x|,h}^{\beta,\omega} \le \mathbf{P}_{N,h}^{\beta,\omega}\Theta_x \le \mathbf{P}_{N+|x|,h}^{\beta,\omega},$$

where $\mathbf{E}_{N,h}^{\beta,\omega}\Theta_x[h(\phi)] = \mathbf{E}_{N,h}^{\beta,\omega}[h(\Theta_x\phi)]$ for every bounded local continuous function h. Translation covariance of the quenched measure follows by taking $N \to \infty$, translation invariance for the averaged quenched measure follows by taking the $\mathbb{E}$ expectation of the three terms in (5.5) and by sending $N \to \infty$.

For the second result on the averaged quenched measure, let ∂_h^- and ∂_h^+ denote, respectively, the left and right derivative with respect to h.

LEMMA 5.1. *For every h,*

$$(5.6) \qquad \mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0] \in \big[\partial_h^- \mathbf{F}_\infty(\beta,h), \partial_h^+ \mathbf{F}_\infty(\beta,h)\big].$$

LEMMA 5.2. *For every h, if $\{\phi_x\}_{x \in \mathbb{Z}^d}$ is distributed according to $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$, the random field $\{\delta_x\}_{x \in \mathbb{Z}^d}$, is (translation) ergodic.*

Let us see how these two lemmas, and the Markov property, allow to conclude the proof. First of all, ergodicity implies that

$$(5.7) \qquad \mathbb{E}\mathbf{P}_h^{\beta,\omega}(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1) \in \{0, 1\}.$$

Of course, $\mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0]$ is either zero or positive: Lemma 5.1 ensures that this dichotomy precisely corresponds to the localization transition, that is, to $h \leq 0$ and $h > 0$ by Theorem 2.2. It also corresponds to the dichotomy (5.7) by elementary arguments.

Consider first the case $h > 0$, that is, in the case in which the probability in (5.7) is equal to one and, therefore,

$$(5.8) \qquad \mathbf{P}_h^{\beta,\omega}\big(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1\big) = 1, \qquad \mathbb{P}(d\omega)\text{-a.s.}$$

We claim that, $\mathbb{P}(d\omega)$-a.s., $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) = 0$ for every y, hence that $\mathbf{P}_h^{\beta,\omega}$(there exists y such that $\phi_y = \infty) = 0$. In fact, reasoning by absurd, if there exists y such that $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) > 0$ then, by the Markov property (5.3), for every $x \neq y$ we have $\mathbf{P}_h^{\beta,\omega}(\phi_x = \infty, \phi_y = \infty) = \mathbf{P}_h^{\beta,\omega}(\phi_y = \infty) > 0$, so, iterating countably many times the argument, we see that, $\mathbb{P}(d\omega)$-a.s., $\mathbf{P}_h^{\beta,\omega}(\phi_y = \infty$ for every $y \in \mathbb{Z}^d) > 0$, which contradicts the statement (5.8). Therefore, the claim is proven and, therefore, we have also that $\mathbb{E}\mathbf{P}_h^{\beta,\omega}$(there exists y such that $\phi_y = \infty) = 0$.

On the other hand, if $h \leq 0$ we are in the case in which the probability in (5.7) is equal to zero. Hence

$$(5.9) \qquad \mathbf{P}_h^{\beta,\omega}\big(\text{there exists } x \in \mathbb{Z}^d : \delta_x = 1\big) = 0, \qquad \mathbb{P}(d\omega)\text{-a.s.}$$

In particular, for the same ω's for any x we have that $\mathbf{P}_h^{\beta,\omega}(\delta_x = 1) = 0$. By the Markov property (5.3), this implies that $\phi_y = \infty$ for at least a $y \sim x$. But we have just seen from the previous argument that $\mathbf{P}_h^{\beta,\omega}(\phi_x = \infty$ for every $x) = \mathbf{P}_h^{\beta,\omega}(\phi_y = \infty)$, which in this case is one.

The proof of Theorem 2.3 is complete.

PROOF OF LEMMA 5.1. Set $m = \mathbb{E}\mathbf{E}_h^{\beta,\omega}[\delta_0]$. We recall that $\mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega}[\delta_0]$ decreases as N grows. The limit is m by convergence in law [cf. (5.2)] because the discontinuity point of δ_0 is $\phi_0 = 1$ and $\mathbf{P}_{h,N}^{\beta,\omega}(\phi_0 = 1) = 0$ as one can see by conditioning on $\mathcal{F}_{\{0\}^\complement}$ and using (5.3): if the ϕ values on which we condition on the nearest neighbors of 0 are all finite, then the conditional measure has a density, otherwise the field at the origin takes the value ∞. Either way, this conditional probability is zero and the claim follows. By exploiting further the monotonicity under set inclusion of the measure, we directly see that for every $\varepsilon > 0$ we can find N_0 such that for $N > N_0$

$$(5.10) \qquad \mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega}[\delta_x] \in [m, m + \varepsilon],$$

for every $x \in \widetilde{\Lambda}_{N-N_0}$. But then

$$(5.11) \quad \partial_h \mathbb{E} \log Z_{N,h,\infty}^{\beta,\omega} = \mathbb{E}\mathbf{E}_{h,N}^{\beta,\omega}\left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x \right] \leq |\widetilde{\Lambda}_{N-N_0}|(m + \varepsilon) + |\widetilde{\Lambda}_N \setminus \widetilde{\Lambda}_{N-N_0}|,$$

and, therefore, the superior limit as $N \to \infty$ of the left-hand side, normalized by $|\widetilde{\Lambda}_N|$, is not larger than $m + \varepsilon$. Similarly,

$$(5.12) \qquad \partial_h \mathbb{E} \log Z_{N,h,\infty}^{\beta,\omega} = \mathbb{E} \mathbf{E}_{h,N}^{\beta,\omega} \left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x \right] \geq |\widetilde{\Lambda}_{N-N_0}| m,$$

and the inferior limit of this left-hand side, normalized by $|\widetilde{\Lambda}_N|$, is not smaller than m. $\square$

PROOF OF LEMMA 5.2. Let A be a translation invariant event in the σ-algebra generated by $\{\delta_x\}_{x \in \mathbb{Z}^d}$. We can approximate the event by A_M which just depends on $\{\delta_x\}_{x \in \widetilde{\Lambda}_M}$ in a way that

$$(5.13) \qquad \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A \Delta A_M) \leq \varepsilon.$$

Furthermore, we can choose $N > M$ so large that

$$(5.14) \qquad \mathbb{E} \mathbf{E}_{N,h}^{\beta,\omega} \left[\sum_{x \in \widetilde{\Lambda}_M} \delta_x \right] - \mathbb{E} \mathbf{E}_h^{\beta,\omega} \left[\sum_{x \in \widetilde{\Lambda}_M} \delta_x \right] \leq \varepsilon.$$

This is a consequence of the convergence of the sequence of measures [cf. (5.2)], because $\mathbb{E} \mathbf{P}_h^{\beta,\omega}(\bigcup_{x \in \widetilde{\Lambda}_M} \{\phi_x = 1\}) = 0$ as can be seen by a conditioning argument like in the very beginning of the proof of Lemma 5.1.

Now let v_N be a vector with all entries 0 except one that is equal to $3N$.

$$(5.15) \qquad \Gamma(N) := \Lambda_N \cup \Theta_{v_N} \Lambda_N.$$

Note that Γ_N is composed of two disjoint boxes, and thus that under $\mathbf{P}_{\Gamma(N),h}^{\beta,\omega}$, $\{\phi_x\}_{x \in \Lambda_N}$ and $\{\phi_x\}_{x \in \Theta_{v_N} \Lambda_N}$ are independent, so

$$(5.16) \qquad \begin{aligned}
\mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega}[A_M \cap \Theta_{v_N} A_M] \\
= \mathbb{E}[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}(A_M) \mathbf{P}_{\theta_{v_N} \Lambda_N,h}^{\beta,\omega}(\Theta_{v_N} A_M)] \\
= \mathbb{E}[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}[A_M]] \mathbb{E}[\mathbf{P}_{\theta_{v_N} \Lambda_N,h}^{\beta,\omega}(\Theta_{v_N} A_M)] \\
= (\mathbb{E}[\mathbf{P}_{\Lambda_N,h}^{\beta,\omega}[A_M]])^2,
\end{aligned}$$

where in the first equality we used the Markov property and for the second we used independence of the environment in the two boxes.

We assume $N > 2M$ so that $\widetilde{\Lambda}_M \cup \Theta_{v_N} \widetilde{\Lambda}_M \subset \Gamma(N)$ and the distance of both Λ_M and $\Theta_{v_N} \widetilde{\Lambda}_M$ to the boundary of $\Gamma(N)$ is more than $N/2$. We have

$$(5.17) \qquad \mathbb{E} \mathbf{E}_{\Gamma(N),h}^{\beta,\omega} \left[\sum_{x \in \widetilde{\Lambda}_M \cup \Theta_{v_N} \widetilde{\Lambda}_M} \delta_x \right] - \mathbb{E} \mathbf{E}_h^{\beta,\omega} \left[\sum_{x \in \widetilde{\Lambda}_M \cup \Theta_{v_N} \widetilde{\Lambda}_M} \delta_x \right] \leq 2\varepsilon,$$

as can be directly extracted from (5.14) because the first addendum in the left-hand side can be written as the sum of two terms on which we can apply (5.14) after using translation invariance. Now by stochastic domination, we know that there exists a monotone coupling between the two probabilities. For such a coupling $\{\delta_x^1\}_{x \in \widetilde{\Lambda}_M \cup \Theta_{v_N} \widetilde{\Lambda}_M}$ and $\{\delta_x^2\}_{x \in \widetilde{\Lambda}_M \cup \Theta_{v_N} \widetilde{\Lambda}_M}$ coincide with probability at least $1 - 2\varepsilon$ [we use Markov's inequality together with (5.17)]. As a consequence, we have

$$(5.18) \qquad \left| \mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega}(A_M \cap \Theta_{v_N} A_M) - \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M \cap \Theta_{v_N} A_M) \right| \leq 2\varepsilon.$$

Note that in the same manner as for (5.14)—the boundary of A_M is a subset of $\bigcup_{x \in \widetilde{\Lambda}_M} \{\phi_x = 1\}$—we have also that, for M sufficiently large,

$$(5.19) \qquad \left| \mathbb{E} \mathbf{P}_{N,h}^{\beta,\omega}(A_M) - \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M) \right| \leq \varepsilon.$$

By putting everything together [using the triangle inequality and (5.16)], we obtain

$$
\begin{aligned}
(5.20) \quad & \left| \mathbb{E} \mathbf{P}^{\beta,\omega}(A) - \mathbb{E} \mathbf{P}^{\beta,\omega}(A)^2 \right| \\
&= \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A \cap \Theta_{v_N} A) - \mathbf{P}^{\beta,\omega}(A)^2 \right| \\
&\leq \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A \cap \Theta_{v_N} A) - \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M \cap \Theta_{v_N} A_M) \right| \\
&\quad + \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M \cap \Theta_{v_N} A_M) - \mathbb{E} \mathbf{P}_{\Gamma(N),h}^{\beta,\omega}(A_M \cap \Theta_{v_N} A_M) \right| \\
&\quad + \left| \mathbb{E} \mathbf{P}_{N,h}^{\beta,\omega}(A_M)^2 - \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M)^2 \right| + \left| \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A_M)^2 - \mathbb{E} \mathbf{P}_h^{\beta,\omega}(A)^2 \right| \\
&\leq 8\varepsilon.
\end{aligned}
$$

The last inequality comes from the fact that all four terms are smaller than 2ε, the first from (5.13) and translation invariance, the second from (5.18), the third from (5.19) and the last one from (5.13). Since $\varepsilon > 0$ is arbitrary, we obtain that $\mathbb{E} \mathbf{P}^{\beta,\omega}(A) \in \{0, 1\}$. $\quad \square$

APPENDIX A: FREE ENERGY: EXISTENCE AND OTHER ESTIMATES

THEOREM A.1. *For every $K \in (-\infty, \infty]$, every $\beta \in I_{\mathbb{P}}$ and every $h \in \mathbb{R}$ we have that the limit*

$$(A.1) \qquad \lim_{N \to \infty} \frac{1}{N^d} \log Z_{N,h,K}^{\beta,\omega,0}$$

exists $\mathbb{P}(d\omega)$-a.s. and in $L^1(\mathbb{P})$ and the limit is not random.

Therefore, (2.7) provides a definition of $\mathrm{F}_K(\beta, h)$.

PROOF. As long as K is finite, the arguments in [6] go through and they yield the result. For $K = \infty$, we observe that, since the partition function decreases as

K increases for every K [so, in particular, $F_\infty(\beta, h) = \lim_{K\to\infty} F_K(\beta, h) \geq 0$],

$$\text{(A.2)} \qquad \limsup_{N\to\infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \leq F_K(\beta, h),$$

$\mathbb{P}(d\omega)$-a.s. On the other hand, Lemma A.3 ensures that

$$\text{(A.3)} \qquad \liminf_{N\to\infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \geq F_K(\beta, h) - r(K),$$

with $\lim_{K\to\infty} r(K) = 0$ and this gives (A.1) in the $\mathbb{P}(d\omega)$-a.s. sense. The $L^1(\mathbb{P})$ limit can then be obtained by an application of the dominated convergence theorem. $\square$

As an important technical tool, we have the following analog of [12], Proposition 4.2: the proof is a direct generalization because the potential terms are bounded.

PROPOSITION A.2. *For any value of $u \in \mathbb{R}$, $K \in \mathbb{R}$, $h \in \mathbb{R}$ and $\beta \in I_\mathbb{P}$ we have*

$$\text{(A.4)} \qquad \lim_{N\to\infty} \frac{1}{N^d} \mathbb{E}\hat{\mathbf{E}}^u[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}}] = F_K(\beta, h).$$

Moreover, for any u and N one has

$$\text{(A.5)} \qquad \frac{1}{N^d} \mathbb{E}\hat{\mathbf{E}}^u[\log Z_{N,h,K}^{\beta,\omega,\hat{\phi}}] \leq F_K(\beta, h).$$

LEMMA A.3. *For every K, h and $\beta \in I_\mathbb{P}$, we have that the bound*

$$
\text{(A.6)} \quad
\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N^d} \log Z_{N,h,\infty}^{\beta,\omega,0} \\
&\qquad \geq F_K(\beta, h) - \mathbb{E}\big[\log\big(1 + \exp\big(-K + (\beta\omega_1 - \lambda(\beta) + h)_-\big)\big)\big],
\end{aligned}
$$

holds $\mathbb{P}(d\omega)$-a.s. Moreover, (A.6) still holds if $\log Z_{N,h,\infty}^{\beta,\omega,0}$ in the left-hand side is replaced by $\mathbb{E}\log Z_{N,h,\infty}^{\beta,\omega,0}$.

Of course, $\mathbb{E}[\log(1 + \exp(-K + (\beta\omega_1 - \lambda(\beta) + h)_-))] = o(1)$ as $K \to \infty$ by the dominated convergence theorem, but the estimate is quantitative. In fact, for every $\beta \geq 0$ and $h \geq 0$, we can find $c = c_\beta > 0$ such that for every K sufficiently large we have

$$\text{(A.7)} \qquad \mathbb{E}\big[\log\big(1 + \exp\big(-K + (\beta\omega_1 - \lambda(\beta) + h)_-\big)\big)\big] \leq \exp(-c_\beta K).$$

In fact, it is immediate to see that $c_0 = 1$. For $\beta > 0$, it is sufficient to argue for $h = 0$ and with $Y = \beta\omega_1 - \lambda(\beta)$ we have

$$
\begin{aligned}
\mathbb{E}[\log(1 &+ \exp(-K + (Y)_-))] \\
&\leq \log(1 + \exp(-K/2)) \\
&\quad + \mathbb{E}[\log(1 + \exp(-K + (Y)_-)); Y < -K/2] \\
&\leq \log(1 + \exp(-K/2)) + \mathbb{E}[(\log 2 + (Y)_-); Y < -K/2],
\end{aligned}
$$
(A.8)

and observe that, since by assumption there exists $a > 0$ such that $\lambda(-a\beta) < \infty$

$$
\begin{aligned}
\mathbb{P}(Y < -K/2) &= \mathbb{P}\left(-a\beta\omega_1 > \frac{a}{2}K - a\lambda(\beta)\right) \\
&\leq \exp\left(\lambda(-a\beta) + a\lambda(\beta) - \frac{a}{2}K\right).
\end{aligned}
$$
(A.9)

The conclusion, that is, (A.7) is now obtained by applying the Cauchy–Schwarz inequality to the very last term in (A.8).

PROOF. We start with observing that the left-hand side in (A.6) is $\mathbb{P}(d\omega)$-a.s. equal to

$$
\textsc{f}_K(\beta, h) + \liminf_{N \to \infty} \frac{1}{N^d} \log \mathbf{P}_{N,h,K}^{\beta,\omega,0}(\phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N),
$$
(A.10)

and we have to bound from below the inferior limit in the last expression. For this, we observe that if we set $E_A^- := \{\phi \in \mathbb{R}^{\mathring{\Lambda}_N} : \phi_x < 0 \text{ for } x \in A \text{ and } \phi_x \geq 0 \text{ for every } x \in \mathring{\Lambda}_N \setminus A\}$, we have [with the concise notation $Y_x := \beta\omega_x - \lambda(\beta) + h$]

$$
\mathbf{P}_{N,h,K}^{\beta,\omega,0}(E_A^-) = \exp(-K|A|) \int_{E_A^-} \frac{\exp(\sum_{x \in \widetilde{\Lambda}_N} Y_x \delta_x)}{Z_{N,h,K}^{\beta,\omega,0}} \mathbf{P}_N^0(d\phi),
$$
(A.11)

and by performing the change of variables $\widetilde{\phi}_x = -\phi_x$ if $x \in A$ and $\widetilde{\phi}_x = \phi_x$ otherwise, we see that

$$
\begin{aligned}
\int_{E_A^-} \exp&\left(\sum_{x \in \widetilde{\Lambda}_N} Y_x \delta_x\right) \mathbf{P}_N^0(d\phi) \\
&\leq \exp\left(\sum_{x \in A} Y_x\right) \int_{E_\varnothing^-} \exp\left(\sum_{x \in \widetilde{\Lambda}_N} Y_x \delta_x\right) \mathbf{P}_N^0(d\phi),
\end{aligned}
$$
(A.12)

because such a transformation has (absolute value) of the Jacobian determinant equal to one, $\sum_{x,y}(\widetilde{\phi}_x - \widetilde{\phi}_y)^2 \leq \sum_{x,y}(\phi_x - \phi_y)^2$, where the sums are over the nearest neighbor (x, y) in $\Lambda_N^2 \setminus (\partial\Lambda_N)^2$, and $\sum_{x \in \widetilde{\Lambda}_N} Y_x \delta_x$ under such transformation can decrease of at most $\sum_{x \in A}(Y_x)_-$. Since of course $E_\varnothing^- = \{\phi : \phi_x \geq 0 \text{ for }$

every $x \in \overset{\circ}{\Lambda}_N\}$ and since $\sum_{A \subset \overset{\circ}{\Lambda}_N} \mathbf{P}_{N,h,K}^{\beta,\omega,0}(E_A^-) = 1$ we see that

$$(A.13) \quad 1 \leq \left(\sum_{A \subset \overset{\circ}{\Lambda}_N} \prod_{x \in A} \exp(-K + (Y_x)_-) \right) \mathbf{P}_{N,h,K}^{\beta,\omega,0}(\phi_x \geq 0 \text{ for every } x \in \overset{\circ}{\Lambda}_N),$$

and since the sum is equal to $\prod_{x \in \overset{\circ}{\Lambda}_N}(1 + \exp(-K + (Y_x)_-))$, the claim in Lemma A.3 follows by applying the law of large numbers to the family of L^1 i.i.d. random variables $\{\log(1 + \exp(-K + (Y_x)_-))\}_{x \in \mathbb{Z}^d}$. $\quad \square$

APPENDIX B: ABOUT THE δ-PINNING MODEL

In [3], the nondisordered model that goes under the name of δ-pinning is also considered. It corresponds to the $a \searrow 0$ case of the $b\mathbf{1}_{[0,a]}(\cdot)$ potential of (2.12)— we set $h = 1$ because $b > 0$ takes the role of h—under the constraint

$$(B.1) \qquad\qquad a(\exp(b) - 1) = e^J,$$

where J is a fixed real number. Note that for $a \searrow 0$ we have $b = -\log a + J + O(a)$. In particular, $b \to \infty$ in this limit and it is straightforward to see that the measure $\exp(b\mathbf{1}_{[0,a]}(\phi))\,d\phi$ tends to $d\phi + e^J \delta_0^{\mathrm{D}}(d\phi)$, with δ_0^{D} the Dirac delta measure in zero. The limit model has a number of nice features, but the model is not critical at any value of J, as it is proven in [3]: delocalization arises only for $J \to -\infty$. The correction in [7] does not impact the δ-pinning part of [3].

Before introducing in detail the δ-pinning, let us remark that for $a > 0$ fixed a straightforward application of Theorem 2.2 [see (2.12)] shows that the model with potential $b\mathbf{1}_{[0,a]}(\cdot)$ and $K \in (0, \infty]$ is critical at $b = 0$. Therefore, if we parametrize the model with J via (B.1), we see that the critical value $J_c(a)$ is, for a small, close to $-\log a$. and this is clearly in the direction of the result in [3] that $J_c = -\infty$ in the δ-pinning limit. However, our proofs cannot be adapted in a straightforward way (see Remark B.3), but let us define more explicitly the model and state a result.

To introduce the δ-pinning model, let us introduce $p_{\Lambda_N}(\phi)$, which is the density of the measure $\mathbf{P}_{\Lambda_N}^0(d\phi)$; cf. (2.2). We consider for $J \in \mathbb{R}$ the partition function

$$(B.2) \qquad Z_{N,J}^+ = \int_{[0,\infty)^{\overset{\circ}{\Lambda}_N}} p_{\Lambda_N}(\phi) \prod_{x \in \overset{\circ}{\Lambda}_N} (d\phi_x + e^J \delta_0^{\mathrm{D}}(d\phi_x)).$$

We set

$$(B.3) \qquad\qquad \mathrm{F}^+(J) = \lim_{N \to \infty} \frac{1}{N^d} \log Z_{N,J}^+.$$

We refer to [3] and references therein for the existence of this limit and for other properties, notably the fact that $\log Z_{N,-\infty}^+ = o(N^d)$, which implies that $\mathrm{F}^+(J) \geq 0$ for every J. In [3], it is proven also that $\mathrm{F}^+(J) > 0$ for every J.

PROPOSITION B.1.   *For every $\varepsilon > 0$,*

(B.4) $$\exp\left(-e^{-(2+\varepsilon)J}\right) \leq F^+(J) \leq \exp\left(-\frac{1}{4d}e^{-2J}\right),$$

for $J < 0$ and $|J|$ large (how large depends on d and, for the first inequality, also on ε).

Of course, Proposition B.1 implies the more informal

(B.5) $$F^+(J) = \exp\left(-e^{-(2+o(1))J}\right).$$

PROOF.   The lower bound is a quantitative version of the result in [3]. More precisely, it uses [3], Proposition 3, and the constant $c_1 > 0$ in there, in the special case of $t = 0$ (not affected by what is observed in [7]). This result implies that the free energy is bounded below by

(B.6) $$(J + c_0 + c_1 \log \log \Delta)/\Delta^d,$$

for Δ sufficiently large. The constant c_0 depends only on the dimension and, by looking at [3], first formula on page 1220, we see that $c_1 < 1/2$, but it can be chosen arbitrarily close to $1/2$, and this implies the lower bound in (B.4) because we can choose $\Delta = \exp(\exp(-(2+\varepsilon)J))$ and (B.6) becomes, to leading order for $J \to -\infty$, equal to $J(1 - (2+\varepsilon)c_1)\exp(-d\exp(-(2+\varepsilon)J))$ and by choosing $c_1 > 1/(2+\varepsilon)$ we conclude.

For the upper bound, consider the measure $\mathbf{P}^0_{\Lambda_N}(d\phi)$ and note that the conditional density of ϕ_y, with $y \in \overset{\circ}{\Lambda}_N$, given its $2d$ neighbors is the density of the univariate Gaussian variable $\frac{1}{\sqrt{2d}}\mathcal{N} + u$, with u the average of the $2d$ neighbors. We can then perform the integration on this variable and we see that if we define

(B.7) $$C^+(J) := \sup_{u \geq 0}\left(\mathbf{P}\left(\frac{1}{\sqrt{2d}}\mathcal{N} + u \geq 0\right) + \sqrt{2d}e^J g(-\sqrt{2d}u)\right),$$

where $g(\cdot)$ is the density of the standard Gaussian variable $\mathcal{N}$, we have that

(B.8) $$Z^+_{N,J} \leq C^+(J) \int_{[0,\infty)^{\overset{\circ}{\Lambda}_N \setminus \{y\}}} p_{\Lambda_N}(\phi) \prod_{x \in \overset{\circ}{\Lambda}_N \setminus \{y\}} \left(d\phi_x + e^J \delta^{\mathrm{D}}_0(d\phi_x)\right).$$

By iteration, we therefore obtain that $Z^+_{N,J} \leq C^+(J)^{(N-1)^d}$ from which we conclude that

(B.9) $$F^+(J) \leq \log C^+(J) \leq \sup_{v \geq 0}(\sqrt{2d}e^J g(v) - \mathbf{P}(\mathcal{N} > v)),$$

where the second inequality follows from (B.7). We are interested in the behavior for $J \to -\infty$ and we can therefore restrict the supremum to $v \geq v_0$ with $v_0 > 0$ arbitrary, because the expression we are maximizing is negative for $v \in [0, v_0]$ and

$-J$ sufficiently large. We can then use (2.15) and we see that there exists $J_0 < 0$ such that for $J < J_0$

(B.10)
$$\log C^+(J) \le \sup_{v \ge 0} g(v)\left(\sqrt{2d}e^J - \frac{1}{2v}\right) \le g\left(\frac{1}{2\sqrt{2d}}e^{-J}\right)$$
$$\le \exp\left(-\frac{1}{4d}e^{-2J}\right),$$

where the second inequality is obtained using the fact that $(\sqrt{2d}e^J - \frac{1}{2v}) \le 1$ (for $J \le J_0$), and is negative if $v \le \frac{1}{2\sqrt{2d}}e^{-J}$. This completes the proof of (B.4). □

REMARK B.2. The argument in [3] yields the free energy estimate (B.6) also for the case $a > 0$, that is, before the limit (say, $a \le 1$). By using (B.1), one easily rewrites (B.6) in terms of a and b and from this, for $a > 0$ fixed and $b = h \searrow 0$, one obtains (2.13).

REMARK B.3. If we repeat the heuristic arguments of Section 2.3, using $g_{\sigma_d}(\cdot)$ for the density of $\sigma_d \mathcal{N} \sim \mathcal{N}(0, \sigma_d^2)$, we get to the one site computation for the free energy analog to (2.16):

(B.11) $$g_{\sigma_d}(u)e^J - \mathbf{P}(\sigma_d \mathcal{N} > -u) = 1 + g_{\sigma_d}(u)\left(e^J - \frac{\sigma_d^2(1 + o(1))}{u}\right),$$

for u large. This is optimized by $u = (1 + o(1))\sigma_d^2 e^{-J}$, which yields for $J \to -\infty$

(B.12) $$\mathrm{F}^+(J) = \exp\left(-\frac{1}{2}(\sigma_d^2 + o(1))\exp(-2J)\right),$$

which is consistent with Proposition B.1, but much sharper. Both the key arguments that we use in this paper—diluting step to exploit the weak correlation of large excursions for the upper bound and restriction to finite sizes via superadditivity for the lower bound—do not appear to withstand the $a \searrow 0$ limit and the validity of (B.12) is an open issue.

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