

## FROM TRANSIENCE TO RECURRENCE WITH POISSON TREE FROGS

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Consider the following interacting particle system on the  $d$ -ary tree, known as the *frog model*: Initially, one particle is awake at the root and i.i.d. Poisson many particles are sleeping at every other vertex. Particles that are awake perform simple random walks, awakening any sleeping particles they encounter. We prove that there is a phase transition between transience and recurrence as the initial density of particles increases, and we give the order of the transition up to a logarithmic factor.

**1. Introduction.** We study a system of branching random walks known as the frog model, and we discover a phase transition as the initial state becomes more saturated with particles. Similar phase transitions have been observed in related models, including activated random walk [9, 25], reinforced random walk [17], killed branching random walk [1] and the contact process [18].

The frog model starts with a single particle awake at the root of a graph and sleeping particles at the other vertices. The initial configuration of sleeping particles can be deterministic or random. Particles that are awake perform independent simple random walks in discrete time. When a vertex with sleeping particles is first visited, all of the particles at the site wake up and each begins its own walk. The name “frog model” was coined in 1996 by Rick Durrett; we continue the zoomorphism and refer to the particles as frogs. As with other interacting particle systems, the frog model is often motivated as a model for the spread of a rumor or infection (see [3], e.g.). It and its variants have also found interest as models of combustion [7, 22, 23], generally with particles moving in continuous time.

We call a realization of the frog model *recurrent* if the root is visited infinitely often by frogs and *transient* if not. Even if each individual frog is transient, the aggregate of visits to the root can still be infinite. For this reason, the transience or recurrence of the frog model gives a measurement of its growth, and the question of transience or recurrence for the frog model on a given graph is one of the most fundamental ones.

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The first ever published result on the frog model is that it is recurrent on  $\mathbb{Z}^d$  with one sleeping frog per site for all  $d$  [26]. In fact, the frog model on  $\mathbb{Z}^d$  is recurrent for any i.i.d. initial configuration of sleeping frogs [5]. It is natural to wonder if a sparser configuration changes the behavior. [20] exhibits a threshold at which a frog model with Bernoulli( $\alpha \|x\|^{-2}$ ) frogs at each  $x \in \mathbb{Z}^d$  switches from transience to recurrence. A similar phenomenon occurs when the walks have a bias in one direction: [14] finds that on  $\mathbb{Z}$ , the model is recurrent if and only if the number of sleeping frogs per site has infinite logarithmic moment. Recently, this result was partially extended to  $\mathbb{Z}^d$  in [10] and worked out in finer detail in [15].

Let  $\mathbb{T}_d$  denote the full infinite  $d$ -ary tree, in which the root has degree  $d$  and all other vertices degree  $d + 1$ . The question of transience or recurrence on  $\mathbb{T}_d$  is especially subtle. On one hand, the number of sleeping frogs grows exponentially with the distance from the root. On the other hand, each frog that wakes up has a drift away from the root; its probability of visiting the root shrinks exponentially as the starting vertex of the frog moves outward. The question of whether  $\mathbb{T}_d$  is transient for the one-per-site model is posed in [4] and again in [21] and [14]. Surprisingly, the answer depends on the degree of the tree. In [16], we prove that the one-per-site frog model is recurrent on the binary tree and transient on  $d$ -ary trees with  $d \geq 5$ .

We conjecture that the one-per-site frog model is recurrent for  $d = 3$  and transient for  $d = 4$ . While we would like to pin this down and complete the picture of transience and recurrence for the one-per-site frog model on trees, we believe that the most interesting aspect of this work is that the frog model on trees is teetering on the edge between recurrence and transience. The point of this paper is to demonstrate this more precisely. We consider the frog model on  $\mathbb{T}_d$  with i.i.d.  $\text{Poi}(\mu)$  sleeping frogs at each site. Our result is a phase transition between recurrence and transience as  $\mu$  varies.

**THEOREM 1.** *Consider the frog model on a  $d$ -ary tree with  $\text{Poi}(\mu)$  sleeping frogs per site. For all  $d \geq 2$ , there exists a critical value  $\mu_c(d) > 0$  such that the model is recurrent a.s. if  $\mu > \mu_c(d)$  and transient a.s. if  $\mu < \mu_c(d)$ . The critical value satisfies*

$$Cd < \mu_c(d) < C'd \log d$$

for some constants  $C$  and  $C'$ .

**PROOF.** By a straightforward coupling, the probability of recurrence is monotone in  $\mu$ . By [16], Theorem 4, the probability of recurrence is either 0 or 1. The theorem is then an immediate consequence of Propositions 6 and 15, where we prove recurrence and transience, respectively.  $\square$

Contrast our result with the frog model on  $\mathbb{Z}^d$ , which is recurrent for any i.i.d. configuration of sleeping frogs [5]. To show the existence of the recurrence phase,

we consider a restricted process that lets us take advantage of the recursive structure of  $\mathbb{T}_d$ . We then use a bootstrapping argument, showing that the number of returns to the root is stochastically larger and larger at each step. We establish the transience phase essentially by dominating the model with a branching random walk, using a similar argument as in [16]. As in that paper, the most difficult part is recurrence. Our result is an advance in that we are able to show recurrence on any  $d$ -ary tree with enough sleeping frogs. In [16], we prove recurrence only for  $d = 2$ , and the proof does not apply to a general choice of  $d$ ; even extending it to  $d = 3$  seems difficult. The argument here relies on having Poisson many sleeping frogs at each site, however, and thus neither result implies the other. A more detailed comparison between the recurrence proofs in the two papers is in Section 2.4.

*Further questions.* A nice general survey on the frog model can be found in [21]. Here, we pose four questions specifically related to the frog model on trees.

The question most directly related to our paper is to better estimate the critical value  $\mu_c(d)$ . We are interested in both the asymptotic behavior and precise values for small  $d$ .

OPEN QUESTION 2. What is the correct order of  $\mu_c(d)$  as  $d \rightarrow \infty$ ? Also, what is the value of  $\mu_c(d)$  for small  $d$ ?

We suspect that  $\mu_c(d) = \Theta(d)$ . As for the second question, the best bounds we can prove for  $d = 2$  are  $0.125 \leq \mu_c(2) \leq 1.13$  (see Section 2.4).

As a start at considering the frog model on less regular graphs, we would like to know if the analogue of our result holds on Galton–Watson trees.

OPEN QUESTION 3. Consider a frog model with  $\text{Poi}(\mu)$  frogs at each site of an infinite Galton–Watson tree. As  $\mu$  varies, does a phase transition occur between transience and recurrence?

We are also interested in the relationship between the frog model and the degree distribution of the tree.

OPEN QUESTION 4. Does the recurrence of the frog model on a Galton–Watson tree depend on the entire degree distribution or just the maximal degree? Concretely, consider a one-per-site frog model on a Galton–Watson tree where each vertex has probability  $p$  of having two children and probability  $1 - p$  of having five children. [16], Theorem 1 implies that this is recurrent when  $p = 1$  and transient when  $p = 0$ . Is it recurrent for any  $p < 1$ ?

This dependence on the maximal degree of the tree alone is seen in the contact process (see [18] and [19], Proposition 2.5).

Our next question comes from Itai Benjamini and concerns the frog model on finite trees. Define the *cover time* to be the expected time for every frog to wake up in a one-per-site frog model on the full  $d$ -ary tree with height  $n$ . We call this the cover time since it is equivalent to the time when every site is visited. A naive bound on the cover time is  $O(n^2 d^n)$ , the expected time for a single random walk to visit every site, as shown in [2]. We have an unpublished proof improving this to  $O(n^5 (d/\sqrt{2})^n)$ , but we suspect the correct value is polynomial.

OPEN QUESTION 5. Is the cover time for the one-per-site frog model on a  $d$ -ary tree of height  $n$  polynomial in  $n$ ?

Possibly the cover time on finite trees relates to the recurrence and transience properties on the corresponding infinite tree. For instance, it would be exciting to see that the cover time is polynomial in the height of the tree for  $d = 2$  but exponential for higher  $d$ . This would be reminiscent of the contact process, which behaves similarly on finite lattices and trees as on their infinite counterparts [8, 11–13].

**2. Recurrence.** We start with a sketch. Let  $\nu'$  be the law of the number of visits to the root in the frog model with  $\text{Poi}(\mu)$  frogs at each site. To get some regularity, we restrict the motion of awakened frogs to the nonbacktracking component of their ranges. Call this the *nonbacktracking frog model* (more details are in Section 2.1) and let  $\nu$  be the law of the number of visits to the root in this model. A coupling argument in Proposition 7 confirms the intuition that

$$(1) \quad \nu \preceq \nu'.$$

Here,  $\preceq$  denotes *stochastic dominance*, that is  $\nu([x, \infty)) \leq \nu'([x, \infty))$  for all  $x$ .

In Section 2.2, we define an operator  $\mathcal{A}$  under which the image of  $\nu$  has an interpretation in an even more restricted frog model. First, a bit of notation (see Figure 1) is necessary. Say the initial nonbacktracking frog moves down the tree from the root  $\emptyset$  to  $\emptyset'$  and then to  $v_1$ . Let  $v_2, \dots, v_d$  be the other children of  $\emptyset'$

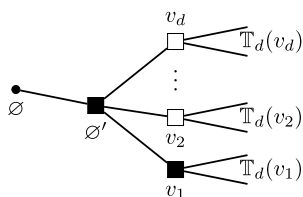


FIG. 1. The frog from  $\emptyset$  visits  $\emptyset'$  and  $v_1$ . Suppose at most one frog in the nonbacktracking frog model is allowed to enter each  $\mathbb{T}_d(v_i)$  and only frogs woken at  $\emptyset'$  and emerging from  $\mathbb{T}_d(v_1)$  can enter other subtrees. We see in Lemma 9 that the number of visits to  $\emptyset$  is stochastically fewer than  $\nu$  and is distributed as  $\mathcal{A}\nu$ .

and let  $\mathbb{T}_d(v_i)$  denote the subtree rooted at  $v_i$ . The measure  $\mathcal{A}\nu$  is the law of the number of visits to the root in the nonbacktracking frog model with two further restrictions:

- (i) At most one frog can enter  $\mathbb{T}_d(v_i)$  for each  $1 \leq i \leq d$ .
- (ii) Only frogs woken at  $\emptyset'$  and those emerging from  $\mathbb{T}_d(v_1)$  can enter the other  $\mathbb{T}_d(v_i)$ .

The advantage of (i) is that it makes the number of frogs emerging from the activated subtrees i.i.d. random variables. The advantage of (ii) is that it simplifies which subtrees become activated (see Lemma 8). Intuitively, these restrictions reduce the number of visits to the root. This is made rigorous in Lemma 9 where we prove that

$$(2) \quad \mathcal{A}\nu \preceq \nu.$$

We stress that this a special property of  $\nu$ . In fact, the essence of our argument is to show that when  $\mu$  is large enough, (2) can hold only if  $\nu = \delta_\infty$ .

Section 2.3 explores properties of  $\mathcal{A}$ . In Lemma 10, we show that  $\mathcal{A}$  is monotonic, meaning that for two probability measures  $\pi_1$  and  $\pi_2$ ,

$$(3) \quad \text{if } \pi_1 \preceq \pi_2, \text{ then } \mathcal{A}\pi_1 \preceq \mathcal{A}\pi_2.$$

Lemma 11 shows that  $\mathcal{A}$  acts nicely on the Poisson distribution. In fact, by writing the Poisson distribution in a nonstandard way (see Lemma 13), we can compare  $\mathcal{A}\text{Poi}(\lambda)$  with  $\text{Poi}(\lambda + \varepsilon)$ . We carry this out in Proposition 14, where we show that when  $\mu \geq 2(d + 1) \log d$ , there exists  $\varepsilon$  such that

$$(4) \quad \text{Poi}(\lambda + \varepsilon) \preceq \mathcal{A}\text{Poi}(\lambda)$$

for all  $\lambda \geq 0$ . This is where the value of  $\mu$  plays a role. Proving (4) reduces to comparing two binomial distributions with parameters depending on  $\mu$ .

Now we explain how (1), (2), (3) and (4) imply the recurrence part of Theorem 1.

**PROPOSITION 6.** *If  $\mu > 2(d + 1) \log d$ , then the frog model is recurrent a.s. on the  $d$ -ary tree with an initial configuration of  $\text{Poi}(\mu)$  sleeping frogs per vertex.*

**PROOF.** By (1), it suffices to prove that  $\nu$  is a point mass at infinity. From (2), we have

$$\text{Poi}(0) \preceq \mathcal{A}\nu \preceq \nu.$$

Statement (3) implies this relation is preserved under iterations of  $\mathcal{A}$ . Moreover, (4) lets us increase the Poisson term by  $\varepsilon$  with each iteration. In symbols, this says that for all  $n \geq 1$ ,

$$\text{Poi}(\varepsilon n) \preceq \mathcal{A}^n \nu \preceq \mathcal{A}^{n-1} \nu \preceq \dots \preceq \mathcal{A}\nu \preceq \nu.$$

Taking  $n \rightarrow \infty$  implies that  $\nu$  is a point mass at infinity, and so the frog model is recurrent almost surely.  $\square$

In the rest of this section, we will carry out this plan and prove statements (1)–(4). First, we give some notation. Recall that  $\preceq$  denotes stochastic domination. We also use the notation  $X \preceq Y$  to indicate that the law of  $X$  is stochastically dominated by the law of  $Y$ . An equivalent condition to stochastic dominance is that  $\pi_1 \preceq \pi_2$  if and only if there exists a coupling  $(X, Y)$  with  $X \sim \pi_1$ ,  $Y \sim \pi_2$ , and  $X \leq Y$  a.s. A thorough reference on stochastic domination is [24].

For a nonnegative random variable  $N$ , we use  $\text{Poi}(N)$  to denote a mixture of Poisson distributions; when we write  $X \sim \text{Poi}(N)$ , we mean that  $X$  is coupled with  $N$  such that the distribution of  $X$  conditional on  $N = n$  is  $\text{Poi}(n)$ . If  $N \sim \pi$ , we also use  $\text{Poi}(\pi)$  to denote the same Poisson mixture. We similarly use the notation  $\text{Bin}(N, p)$  and  $\text{Bin}(\pi, p)$ .

2.1. *The nonbacktracking frog model.* A random nonbacktracking walk on  $\mathbb{T}_d$  starting at a vertex  $x_0$  moves in its first step to a uniformly random neighbor of  $x_0$ . In all subsequent steps, it moves to a vertex chosen uniformly from all its neighbors except for the one it just arrived from.

Suppose that  $(S_n, n \geq 0)$  is a random nonbacktracking walk starting from  $x_0$ , stopped if it arrives at the root at step 1 or beyond. (If  $x_0$  is the root, then it is never stopped.) Define the *nonbacktracking frog model* just as the usual frog model, except that the motion of a frog waking at  $x_0$  is an independent copy of  $(S_n)$ , rather than a simple random walk. The advantage is that when a nonbacktracking frog moves away from the root, it will forever remain in the just-entered subtree. This gives the model more self-similarity. As shown in [16], Proposition 7,  $(S_n)$  can be coupled with a simple random walk on  $\mathbb{T}_d$  starting from  $x_0$  so that its path is a subset of the simple random walk’s path. This lets us relate the nonbacktracking and usual frog models, proving (1).

PROPOSITION 7. *Let  $\nu$  and  $\nu'$  be the laws of the number of returns to the root in the nonbacktracking and usual frog models on  $\mathbb{T}_d$ , respectively, both with  $\text{Poi}(\mu)$  sleeping frogs per vertex. Then  $\nu \preceq \nu'$ .*

PROOF. It suffices to show that we can couple the two models so that at least as many frogs visit the root in the usual model as in the nonbacktracking model. We construct the coupling as follows. For each vertex  $v \in \mathbb{T}_d$ , make the number of sleeping frogs on  $v$  identical in the two models. Make each frog’s path in the nonbacktracking model a subset of the corresponding frog’s path in the usual model as previously described. Thus, any frog woken in the nonbacktracking model is also woken in the usual model, and any visit to the root in the nonbacktracking model corresponds to a visit in the usual model.  $\square$

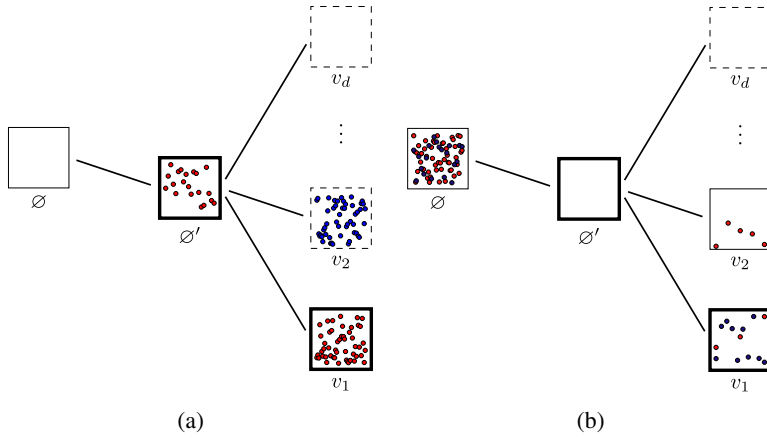


FIG. 2. An interacting particle system related to the frog model. Initially, the number of active particles at  $\emptyset'$  is distributed as  $\text{Poi}(\mu)$ , and the number of active particles at  $v_1$  is distributed according to some probability measure  $\pi$ . Active particles take random nonbacktracking steps until reaching a leaf. For each  $2 \leq i \leq d$ , if any of these particles reach  $v_i$ , then a new  $\pi$ -distributed batch of particles is released at  $v_i$ . These second-wave particles do not activate other vertices. (a) Initial state: particles at  $\emptyset'$  and  $v_1$  will move first and possibly release a second wave of particles from  $v_2, \dots, v_d$ . (b) Terminal state:  $\#\{\text{particles at } \emptyset\} \sim \mathcal{A}\pi$ .

For the remainder of this section, we only consider the nonbacktracking frog model. We record an observation: Suppose the initial frog in the nonbacktracking model steps from the root  $\emptyset$  to a child  $\emptyset'$ . Since frogs are stopped at the root, no other child of the root besides  $\emptyset'$  is ever visited, and all action occurs in the subtree rooted at  $\emptyset'$ .

**2.2. Formal definition of  $\mathcal{A}$ .** Fix a probability measure  $\pi$  on the nonnegative integers. We will define  $\mathcal{A}\pi$  to be the probability measure for the number of particles ending at  $\emptyset$  (see Figure 2) in the random system of nonbacktracking particles described below.

The setting for the particle system is a star graph, consisting of a central vertex connected to  $d + 1$  leaf vertices. In a slight abuse of notation, we reuse the vertex names from Figure 1, calling the central vertex  $\emptyset'$  and the leaves  $\emptyset$  and  $v_1, \dots, v_d$ . Let  $X \sim \text{Poi}(\mu)$  and  $X_1, \dots, X_d \sim \pi$ , all independent. Place  $X$  particles at  $\emptyset'$  and  $X_i$  particles at each  $v_i$ . Each particle if activated will perform an independent random nonbacktracking walk until it halts at a leaf.

Initially, only the particles at  $\emptyset'$  and at  $v_1$  are active. If one of these first-wave particles lands at  $v_i$  for  $i \geq 2$ , then the particles there are activated and begin independent nonbacktracking random walks until reaching a leaf. These second-wave particles do not activate other particles; only the first-wave particles have that power. The number of particles that finish at  $\emptyset$  in this system is a random variable, and we define  $\mathcal{A}\pi$  as its law. With these dynamics, we can summarize the system as follows:

- Particles at  $\emptyset'$  move to one of  $\{\emptyset, v_1, \dots, v_d\}$  each with probability  $1/(d + 1)$ .
- Particles at  $v_1$  move to one of  $\{\emptyset, v_2, \dots, v_d\}$  each with probability  $1/d$ .
- If a first-wave particle visits  $v_i$ , the particles at  $v_i$  move to  $\emptyset$  with probability  $1/d$ .

For  $2 \leq i \leq d$ , let  $E_i$  be the event that a first-wave particle ends at  $v_i$ . The following lemma follows from the definition of  $\mathcal{A}$ . Informally, it says that conditional on how many of the events  $E_2, \dots, E_d$  occur, the number of second-wave particles ending at  $\emptyset$  is a sum of independent thinned copies of  $\pi$ .

LEMMA 8. *Conditional on  $\sum_{i=2}^d \mathbf{1}_{E_i} = u$ , the number of second-wave particles ending at  $\emptyset$  is distributed as the sum of  $u$  independent  $\text{Bin}(\pi, 1/d)$ -distributed random variables.*

PROOF. If  $E_i$  occurs then by definition a  $\pi$ -distributed batch of particles is released at  $v_i$ . With probability  $1/d$  each released particle halts at  $\emptyset$ . As particles move independently, the total number is distributed as  $\text{Bin}(\pi, 1/d)$ . Since the second-wave particles cannot wake other sites, the total number of particles to arrive is distributed as claimed.  $\square$

Now, we show the connection between this operator and the frog model.

LEMMA 9. *Let  $\nu$  be the distribution of number of returns to the root in the non-backtracking frog model on the  $d$ -ary tree with sleeping frog distribution  $\text{Poi}(\mu)$ . Then  $\mathcal{A}\nu \leq \nu$ .*

PROOF. Let  $\mathbb{T}_d(x)$  denote the subtree of  $\mathbb{T}_d$  rooted at a given vertex  $x$ . Recall that no children of the root other than  $\emptyset'$ , the child visited by the initial frog, are ever visited. In light of this, it will be helpful to think of the nonbacktracking frog model as taking place on  $\emptyset \cup \mathbb{T}_d(\emptyset')$  rather than on all of  $\mathbb{T}_d$ .

We say that the frogs sleeping on some vertex  $v \in \mathbb{T}_d(v_1)$  wake within  $\mathbb{T}_d(v_1)$  if there exists a chain of vertices  $x_1, \dots, x_m = v$  all in  $\mathbb{T}_d(v_1)$  such that the initial frog starting from the root visits  $x_1$ , a frog starting at  $x_1$  visits  $x_2$ , and so on. More simply, a frog is woken within  $\mathbb{T}_d(v_1)$  if it would have been woken even if there were no frogs sleeping on any vertices outside of  $\mathbb{T}_d(v_1)$ .

We define some random variables counting frogs that might possibly visit the root. Let  $X \sim \text{Poi}(\mu)$  be the number of frogs sleeping on  $\emptyset'$ , which are woken by the initial frog. Let  $X_1$  be the number of frogs waking within  $\mathbb{T}_d(v_1)$  that visit  $\emptyset'$ . We claim that  $X_1$  is distributed as  $\nu$ . Indeed, when we consider frogs as waking only if they wake within  $\mathbb{T}_d(v_1)$  and relabel the vertices  $\{\emptyset'\} \cup \mathbb{T}_d(v_1)$  as  $\{\emptyset\} \cup \mathbb{T}_d(\emptyset')$ , we see a process identical in law to the original nonbacktracking frog model. Call the frogs counted by  $X$  and  $X_1$  the *first-wave frogs*.

For each  $2 \leq i \leq d$ , let  $E_i$  be the event that some of the frogs counted by  $X$  or  $X_1$  move to  $v_i$ . Conditional on  $E_i$ , arbitrarily choose one of these frogs that



visits  $v_i$  and call it  $f$ . We say that the frogs at  $v$  are woken within  $\mathbb{T}_d(v_i)$  if there exists a chain of vertices  $x_1, \dots, x_m = v$  in  $\mathbb{T}_d(v_i)$  such that  $f$  visits  $x_1$ , a frog starting at  $x_1$  visits  $x_2$ , and so on. Let  $X_i$  be the number of frogs waking within  $\mathbb{T}_d(v_i)$  that visit  $\emptyset'$ . By the same argument showing that  $X_1 \sim \nu$ , the distribution of  $X_i$  conditional on  $E_i$  is also  $\nu$ . Furthermore, for any  $\{i_1, \dots, i_k\} \subseteq \{2, \dots, d\}$ , the random variables  $X_{i_1}, \dots, X_{i_k}$  are conditionally independent given  $E_{i_1}, \dots, E_{i_k}$ , since each  $X_i$  is determined solely by the paths of the frogs sleeping in  $\mathbb{T}_d(v_i)$ . We call the frogs counted by  $X_2, \dots, X_d$  the *second-wave frogs*.

The first- and second-wave frogs all visit  $\emptyset'$ . We define  $V''$  as the number of these that move from there to  $\emptyset$ .

CLAIM.  $V'' \sim \mathcal{A}\nu$ .

PROOF. Our strategy is to show that the first-wave frogs behave identically as the first-wave particles, and then to show that the second-wave frogs conditional on the behavior of first-wave frogs behave the same as the second-wave particles conditional on the behavior of the first-wave particles.

For the first of these claims, consider the first-wave frogs, counted by  $X$  and  $X_1$ . Observe that  $X$  and  $X_1$  are independent with  $X \sim \text{Poi}(\mu)$  and  $X_1 \sim \nu$ , just as in the particle system defining  $\mathcal{A}\nu$ . The frogs counted by  $X$  move from  $\emptyset'$  independently to a random choice out of  $\emptyset, v_1, \dots, v_d$ , and the frogs counted by  $X_1$  move from  $\emptyset'$  independently to a random choice out of  $\emptyset, v_2, \dots, v_d$ , also matching the particle system. Thus, the locations of the first-wave frogs one step after leaving  $\emptyset'$  are distributed identically to the ending locations of the first-wave particles.

Now, condition on some arrangement of the first-wave frogs on  $\emptyset, v_1, \dots, v_d$  one step after leaving  $\emptyset$ . Suppose that  $u$  out of the vertices  $v_2, \dots, v_d$  are occupied by first-wave frogs in this arrangement. The number of second-wave frogs visiting  $\emptyset'$  conditional on this arrangement of first-wave frogs is a sum of  $u$  independent copies of  $\nu$ . Each second-wave frog that visits  $\emptyset'$  has an independent  $1/d$  chance of moving next to  $\emptyset$ . Thus, the number of second-wave frogs that visit  $\emptyset$  is the sum of  $u$  independent copies of  $\text{Bin}(\nu, 1/d)$ . This matches the conditional distribution of second-wave particles ending at  $\emptyset$  given in Lemma 8. Thus, the distribution of the number of first- and second-wave frogs visiting  $\emptyset$  is the same as the distribution of the number of first- and second-wave particles ending at  $\emptyset$ , which is by definition  $\mathcal{A}\nu$ .  $\square$

With this claim, the proof of the lemma is almost complete: Let  $V$  be the total number of visits to  $\emptyset$  in the nonbacktracking frog model. Since  $V'' \leq V$  with  $V'' \sim \mathcal{A}\nu$  and  $V \sim \nu$ , we have shown that  $\mathcal{A}\nu \preceq \nu$ .  $\square$

2.3. *Properties of  $\mathcal{A}$ .* We first show (3), monotonicity of  $\mathcal{A}$  with respect to stochastic dominance.

LEMMA 10. *If  $\pi_1 \preceq \pi_2$ , then  $\mathcal{A}\pi_1 \preceq \mathcal{A}\pi_2$ .*

PROOF. If  $\pi_1 \preceq \pi_2$ , then we can couple the two particle systems defining  $\mathcal{A}\pi_1$  and  $\mathcal{A}\pi_2$  so that the second particle system contains all the same particles as the first, moving identically, as well as additional ones. Thus, at least as many particles visit  $\emptyset$  in the second system as in the first, and  $\mathcal{A}\pi_1 \preceq \mathcal{A}\pi_2$ .  $\square$

Now, we describe the result of applying  $\mathcal{A}$  to a Poisson distribution, whose thinning property simplifies things.

LEMMA 11. *The distribution  $\mathcal{A}\text{Poi}(\lambda)$  is a mixture of Poisson distributions, given by*

$$(5) \quad \mathcal{A}\text{Poi}(\lambda) \sim \text{Poi}\left(\frac{(U + 1)\lambda}{d} + \frac{\mu}{d + 1}\right),$$

where

$$(6) \quad U \sim \text{Bin}\left(d - 1, 1 - \exp\left(-\frac{\lambda}{d} - \frac{\mu}{d + 1}\right)\right).$$

PROOF. In the particle process defining  $\mathcal{A}\text{Poi}(\lambda)$ , let  $Y_{u \rightarrow v}$  be the number of particles that start at  $u$  and finish at  $v$ , for  $u \in \{\emptyset', v_1, \dots, v_d\}$  and  $v \in \{\emptyset, v_1, \dots, v_d\}$ . Each of the  $\text{Poi}(\mu)$  particles starting at  $\emptyset'$  moves to a random neighbor. By Poisson thinning, the random variables  $Y_{\emptyset' \rightarrow v}$  for  $v \in \{\emptyset, v_1, \dots, v_d\}$  are independent and distributed as  $\text{Poi}(\mu/(d + 1))$ . Similarly,  $Y_{v_1 \rightarrow v}$  for  $v \in \{\emptyset, v_2, \dots, v_d\}$  are independent and distributed as  $\text{Poi}(\lambda/d)$ . These two collections of random variables are also independent of each other.

Thus, the number of first-wave particles that move to  $v_i$  for each  $2 \leq i \leq d$  are independent and distributed as  $\text{Poi}(\lambda/d + \mu/(d + 1))$ . Let  $U$  be the number of vertices out of  $\{v_2, \dots, v_d\}$  that are visited. As each vertex has an independent  $1 - \exp(-\lambda/d - \mu/(d + 1))$  chance of being visited, the distribution of  $U$  is as given in (6). And since  $U$  is determined by  $Y_{\emptyset' \rightarrow v_i}$  and  $Y_{v_1 \rightarrow v_i}$  for  $i = 2, \dots, d$ , it is independent of  $Y_{\emptyset' \rightarrow \emptyset}$  and  $Y_{v_1 \rightarrow \emptyset}$ .

By Lemma 8 and Poisson thinning, the number of second-wave particles ending at  $\emptyset$  is  $\text{Poi}(U\lambda/d)$ . The number of first-wave particles ending at  $\emptyset$  is  $Y_{\emptyset' \rightarrow \emptyset} + Y_{v_1 \rightarrow \emptyset}$ , independent of  $U$  and distributed as  $\text{Poi}(\lambda/d + \mu/(d + 1))$ . Summing these together yields (5).  $\square$

We are nearly in a position to establish that  $\mathcal{A}^n \text{Poi}(0)$  grows without limit as  $n \rightarrow \infty$ . First, we need two technical lemmas on the Poisson distribution.

LEMMA 12. *Let  $\bar{Z}_\lambda$  be distributed as  $\text{Poi}(\lambda)$  conditioned to be nonzero. If  $\lambda_1 \leq \lambda_2$ , then  $\bar{Z}_{\lambda_1} \preceq \bar{Z}_{\lambda_2}$ .*

PROOF. Consider the Radon–Nikodym derivative of the law of  $\bar{Z}_{\lambda_2}$  with respect to the law of  $\bar{Z}_{\lambda_1}$ ,

$$r(k) = \frac{\mathbf{P}[\bar{Z}_{\lambda_2} = k]}{\mathbf{P}[\bar{Z}_{\lambda_1} = k]} = \frac{1 - e^{-\lambda_1}}{1 - e^{-\lambda_2}} e^{\lambda_1 - \lambda_2} \left(\frac{\lambda_2}{\lambda_1}\right)^k.$$

The function  $r(k)$  is increasing, and it is straightforward to show that this implies that  $\bar{Z}_{\lambda_1} \preceq \bar{Z}_{\lambda_2}$  (or see [24], Theorem 1.C.1).  $\square$

LEMMA 13. Let  $\bar{Z}_{\lambda/n}^{(1)}, \bar{Z}_{\lambda/n}^{(2)}, \dots$  be independent and distributed as  $\text{Poi}(\lambda/n)$  conditioned to be nonzero. Let  $M$  be independent of these and be distributed as  $\text{Bin}(n, 1 - e^{-\lambda/n})$ , and let

$$Z = \sum_{i=1}^M \bar{Z}_{\lambda/n}^{(i)}.$$

Then  $Z$  is distributed as  $\text{Poi}(\lambda)$ .

PROOF. Decompose  $\text{Poi}(\lambda)$  as a sum of  $n$  independent copies of  $\text{Poi}(\lambda/n)$ . Let  $M$  be the number of these that are nonzero, and condition on  $M$  to get the desired representation.  $\square$

Finally, we prove (4).

PROPOSITION 14. If  $\mu > 2(d + 1) \log d$ , then there exists  $\varepsilon > 0$  such that

$$\text{Poi}(\lambda + \varepsilon) \leq \mathcal{A}\text{Poi}(\lambda)$$

for all  $\lambda \geq 0$ .

PROOF. Let  $X \sim \text{Poi}(\lambda + \varepsilon)$  for some  $\varepsilon > 0$  to be chosen later, and let  $Y \sim \mathcal{A}\text{Poi}(\lambda)$ . We start by decomposing  $X$  into a sum of Poissons conditioned to be nonzero. For any  $a$ , let  $\bar{Z}_a^{(1)}, \bar{Z}_a^{(2)}, \dots$  be distributed as  $\text{Poi}(a)$  conditioned to be nonzero, and let  $Z_a \sim \text{Poi}(a)$  (with no conditioning). Take all these random variables to be independent. By Lemma 13, we can write  $X$  as

$$(7) \quad X = Z_{(\lambda+\varepsilon)/d} + \sum_{i=1}^M \bar{Z}_{(\lambda+\varepsilon)/d}^{(i)},$$

where

$$M \sim \text{Bin}\left(d - 1, 1 - \exp\left(-\frac{\lambda + \varepsilon}{d}\right)\right).$$

We now turn to  $Y$ , which by Lemma 11 is distributed as

$$(8) \quad \text{Poi}\left(\frac{(U + 1)\lambda}{d} + m\right),$$

where  $m = \mu/(d + 1)$  and  $U \sim \text{Bin}(d - 1, 1 - \exp(-\lambda/d - m))$ . Let  $Y' \sim \text{Poi}((U + 1)(\lambda + m)/d)$ . For each  $u$ , the distribution of  $Y'$  conditional on  $U = u$  is stochastically dominated by the distribution of  $Y$  conditional on  $U = u$ , simply because  $\text{Poi}(a) \preceq \text{Poi}(b)$  when  $a \leq b$ . It follows that  $Y' \preceq Y$ . Thus, it suffices to show that  $X \preceq Y'$ . Decomposing  $Y'$  by Lemma 13 and using the same notation as before, we can write  $Y'$  as

$$(9) \quad Y' = Z_{(\lambda+m)/d} + \sum_{i=1}^N \bar{Z}_{(\lambda+m)/d}^{(i)}$$

with

$$N \sim \text{Bin}\left(U, 1 - \exp\left(-\frac{\lambda + m}{d}\right)\right).$$

These decompositions allow us to stochastically compare  $X$  and  $Y'$ . Assume that  $\varepsilon$  is chosen to be smaller than  $m$ . We claim that to show that  $X \preceq Y'$ , it suffices to show that  $M \preceq N$ . Indeed, we can then couple the random variables on the right-hand sides of (7) and (9) so that:

- (1)  $M \leq N$ ;
- (2)  $Z_{(\lambda+\varepsilon)/d} \leq Z_{(\lambda+m)/d}$ ;
- (3)  $\bar{Z}_{(\lambda+\varepsilon)/d}^{(i)} \leq \bar{Z}_{(\lambda+m)/d}^{(i)}$  for each  $i$ .

Property (2) is possible because  $\text{Poi}(a) \preceq \text{Poi}(b)$  if  $a \leq b$ , and (3) is possible by Lemma 12. Together, this yields a coupling of  $X$  and  $Y'$  with  $X \leq Y'$ .

Thus, it only remains to show that  $M \preceq N$ . Recalling that  $U$  is itself binomial, we have

$$\begin{aligned} N &\sim \text{Bin}\left(\text{Bin}\left(d - 1, 1 - \exp\left(-\frac{\lambda}{d} - m\right)\right), 1 - \exp\left(-\frac{\lambda + m}{d}\right)\right) \\ &= \text{Bin}\left(d - 1, \left(1 - \exp\left(-\frac{\lambda}{d} - m\right)\right)\left(1 - \exp\left(-\frac{\lambda + m}{d}\right)\right)\right). \end{aligned}$$

Since  $M$  and  $N$  are both binomial, proving  $M \preceq N$  reduces to comparing their parameters. The argument will be complete once we show for some  $\varepsilon > 0$  and all  $\lambda > 0$ ,

$$(10) \quad 1 - \exp\left(-\frac{\lambda + \varepsilon}{d}\right) \leq \left(1 - \exp\left(-\frac{\lambda}{d} - m\right)\right)\left(1 - \exp\left(-\frac{\lambda + m}{d}\right)\right).$$

Some basic calculus (see Lemma 16 in the Appendix) establishes that for all  $d \geq 2$ ,

$$e^{-2 \log d} + e^{-2 \log d/d} < 1.$$

Since  $m > 2 \log d$ , we can choose  $\varepsilon > 0$  such that

$$1 > \exp\left(-\frac{\varepsilon}{d}\right) \geq e^{-m} + e^{-m/d}.$$

Multiplying both sides of this inequality by  $e^{-\lambda/d}$  gives

$$\exp\left(-\frac{\lambda + \varepsilon}{d}\right) \geq \exp\left(-\frac{\lambda}{d} - m\right) + \exp\left(-\frac{\lambda + m}{d}\right).$$

Thus,

$$\begin{aligned} 1 - \exp\left(-\frac{\lambda + \varepsilon}{d}\right) &\leq 1 - \exp\left(-\frac{\lambda}{d} - m\right) - \exp\left(-\frac{\lambda + m}{d}\right) \\ &\leq \left(1 - \exp\left(-\frac{\lambda}{d} - m\right)\right)\left(1 - \exp\left(-\frac{\lambda + m}{d}\right)\right). \end{aligned}$$

Looking back at (10), we have shown that  $M \leq N$ .  $\square$

We have now proven (1)–(4), completing the proof of Proposition 6.

2.4. *Comparison to one-per-site results.* In [16], we proved that the frog model on a binary tree with one sleeping frog per site is recurrent. The proof has the same overarching idea as here: We use the self-similarity of the tree to obtain a recursive distributional relationship for the number of returns to the root. We then use this relationship in a bootstrapping argument, assuming that the number of visits to the root is stochastically larger than  $\text{Poi}(\lambda)$  and proving that it is in fact stochastically larger than  $\text{Poi}(\lambda + \varepsilon)$ .

The major difference between the two arguments is in the bootstrapping portion. The approach in this paper using traditional stochastic domination fails with the one-per-site frog model. The problem is that the distributions given by successively applying the analogue of the  $\mathcal{A}$  operator in the one-per-site model have finite support, and hence are never stochastically greater than any Poisson distribution. Our proof in [16] instead uses an exotic definition of stochastic dominance, where  $\pi_1$  is dominated by  $\pi_2$  if the probability generating function of  $\pi_1$  is greater than the probability generating function of  $\pi_2$ .

This generating function approach works better than the technique in this paper in some ways and worse in others. On one hand, it can handle both deterministic and random initial configurations. On the other hand, the generating function approach seems confined to small values of  $d$ . It relies on a purely analytic argument that is elementary but difficult. It seems impossible to apply this argument to an arbitrary choice of  $d$ . Even for  $d = 3$ , the generating functions to be analyzed become extremely complicated. The technical advance in this paper is the probabilistic argument we give in Proposition 14, which allows us to work on any  $d$ -ary tree.

**3. Transience.** The main idea of our proof of transience is to consider a *weight function* on the frog model. To analyze the weight function, we bound the frog model by a branching random walk. The weight function is the frog model analogue to a common martingale derived from branching random walk (see [6]).

PROPOSITION 15. *If  $\mathbf{E}\eta < \frac{(d-1)^2}{4d}$ , then the frog model with an independent copy of  $\eta$  frogs per site on  $\mathbb{T}_d$  is almost surely transient.*

PROOF. Let  $F_n$  be the set of frogs awake at time  $n$ . For  $f \in F_n$ , let  $|f|$  denote the level of  $f$  on the tree (that is, its distance from the root). We define a weight function

$$W_n = \sum_{f \in F_n} e^{-\theta|f|},$$

with  $\theta$  to be chosen shortly. Let

$$m = \frac{1}{d+1}e^\theta + \frac{d}{d+1}\mathbf{E}[\eta + 1]e^{-\theta}.$$

Before we explain the meaning of this, we minimize  $m$  by setting  $\theta = \log((\mathbf{E}\eta + 1)d)/2$ , making

$$m = \frac{2\sqrt{(\mathbf{E}\eta + 1)d}}{d + 1} < 1$$

under our assumption that  $\mathbf{E}\eta < \frac{(d-1)^2}{4d}$ .

The strategy of the proof now is to show that  $W_n \rightarrow 0$ , and hence that the root eventually stops being visited. The term  $m$  gives an upper bound for the expected contribution to  $W_{n+1}$  of a frog at time  $n$  in the following way: Suppose that at time  $n$ , some frog  $f$  is at level  $i$  of the tree for any  $i \geq 1$ . With probability  $1/(d + 1)$ , the next jump of  $f$  is toward the root, waking no frogs. With probability  $d/(d + 1)$ , the jump is away from the root, possibly waking up an  $\eta$ -distributed number of frogs. Thus, the expected contribution to  $W_{n+1}$  from  $f$  and any frogs it wakes at time  $n + 1$  is at most  $e^{-\theta i}m$ . If  $f$  is at the root at time  $n$ , then the expected contribution to  $W_{n+1}$  from  $f$  and the frogs it wakes is at most  $\mathbf{E}[\eta + 1]e^{-\theta}$ , which is bounded by  $m$  given our choice of  $\theta$ . Therefore,

$$\mathbf{E}[W_{n+1} | W_n] \leq \sum_{f \in F_n} e^{-\theta|f|}m = mW_n.$$

Thus,  $W_n/m^n$  is a positive supermartingale. By the martingale convergence theorem, it converges almost surely to a finite limit. Since  $m^n \rightarrow 0$ , we also have  $W_n \rightarrow 0$  a.s., which implies that eventually no frogs are present at the root.  $\square$

### APPENDIX

LEMMA 16.  $x^{-2} + x^{-2/x} < 1$  for all  $x \geq 2$ .

PROOF. Let  $f(x) = x^{-2} + x^{-2/x}$ . First, we show the inequality holds on the interval  $[2, 8]$ . Since  $x^{-2}$  is decreasing,

$$f(x) \leq \frac{1}{4} + x^{-2/x}.$$

It is easily checked that the maximum of  $x^{-2/x}$  on  $[2, 8]$  occurs at  $x = 8$  and is less than  $\frac{3}{4}$ .

Next, we consider  $x \geq 8$ . L'Hôpital's rule implies that  $\lim_{x \rightarrow \infty} f(x) = 1$ . Thus, it suffices to confirm that  $f(x)$  is increasing on  $[8, \infty)$ . We compute

$$f'(x) = 2x^{-(2/x)-2}(\log x - x^{(2/x)-1} - 1).$$

For  $x \geq 8$ , it holds that  $x^{(2/x)-1} < 1$ . Hence,

$$f'(x) \geq 2x^{-(2/x)-2}(\log x - 2),$$

which is positive on  $[8, \infty)$  since  $\log 8 > 2$ .  $\square$

## REFERENCES

- [1] AÏDÉKON, E., HU, Y. and ZINDY, O. (2013). The precise tail behavior of the total progeny of a killed branching random walk. *Ann. Probab.* **41** 3786–3878. [MR3161464](#)
- [2] ALDOUS, D. J. (1991). Random walk covering of some special trees. *J. Math. Anal. Appl.* **157** 271–283. [MR1109456](#)
- [3] ALVES, O. S. M., MACHADO, F. P. and POPOV, S. YU. (2002). The shape theorem for the frog model. *Ann. Appl. Probab.* **12** 533–546. [MR1910638](#)
- [4] ALVES, O. S. M., MACHADO, F. P. and POPOV, S. YU. (2002). Phase transition for the frog model. *Electron. J. Probab.* **7** no. 16, 21. [MR1943889](#)
- [5] ALVES, O. S. M., MACHADO, F. P., POPOV, S. YU. and RAVISHANKAR, K. (2001). The shape theorem for the frog model with random initial configuration. *Markov Process. Related Fields* **7** 525–539. [MR1893139](#)
- [6] BIGGINS, J. D. (1977). Martingale convergence in the branching random walk. *J. Appl. Probab.* **14** 25–37. [MR0433619](#)
- [7] COMETS, F., QUASTEL, J. and RAMÍREZ, A. F. (2009). Fluctuations of the front in a one dimensional model of  $X + Y \rightarrow 2X$ . *Trans. Amer. Math. Soc.* **361** 6165–6189. [MR2529928](#)
- [8] CRANSTON, M., MOUNTFORD, T., MOURRAT, J.-C. and VALESIN, D. (2014). The contact process on finite homogeneous trees revisited. *ALEA Lat. Am. J. Probab. Math. Stat.* **11** 385–408. [MR3249416](#)
- [9] DICKMAN, R., ROLLA, L. T. and SIDORAVICIUS, V. (2010). Activated random walkers: Facts, conjectures and challenges. *J. Stat. Phys.* **138** 126–142. [MR2594894](#)
- [10] DÖBLER, C. and PFEIFROTH, L. (2014). Recurrence for the frog model with drift on  $\mathbb{Z}^d$ . *Electron. Commun. Probab.* **19** no. 79, 13. [MR3283610](#)
- [11] DURRETT, R. and LIU, X. F. (1988). The contact process on a finite set. *Ann. Probab.* **16** 1158–1173. [MR0942760](#)
- [12] DURRETT, R. and SCHONMANN, R. H. (1988). The contact process on a finite set. II. *Ann. Probab.* **16** 1570–1583. [MR0958203](#)
- [13] DURRETT, R., SCHONMANN, R. H. and TANAKA, N. I. (1989). The contact process on a finite set. III. The critical case. *Ann. Probab.* **17** 1303–1321. [MR1048928](#)
- [14] GANTERT, N. and SCHMIDT, P. (2009). Recurrence for the frog model with drift on  $\mathbb{Z}$ . *Markov Process. Related Fields* **15** 51–58. [MR2509423](#)
- [15] GHOSH, A. P., NOREN, S. and ROITERSHTEIN, A. (2015). On the range of the transient frog model on  $\mathbb{Z}$ . Available at [arXiv:1502.02738](#).
- [16] HOFFMAN, C., JOHNSON, T. and JUNGE, M. (2015). Recurrence and transience for the frog model on trees. Available at [arXiv:1404.6238](#).
- [17] PEMANTLE, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16** 1229–1241. [MR0942765](#)

- [18] PEMANTLE, R. (1992). The contact process on trees. *Ann. Probab.* **20** 2089–2116. [MR1188054](#)
- [19] PEMANTLE, R. and STACEY, A. M. (2001). The branching random walk and contact process on Galton–Watson and nonhomogeneous trees. *Ann. Probab.* **29** 1563–1590. [MR1880232](#)
- [20] POPOV, S. YU. (2001). Frogs in random environment. *J. Stat. Phys.* **102** 191–201. [MR1819703](#)
- [21] POPOV, S. YU. (2003). Frogs and some other interacting random walks models. In *Discrete Random Walks (Paris, 2003)*. *Discrete Math. Theor. Comput. Sci. Proc.*, AC 277–288 (electronic). Assoc. Discrete Math. Theor. Comput. Sci., Nancy. [MR2042394](#)
- [22] RAMÍREZ, A. F. and SIDORAVICIUS, V. (2004). Asymptotic behavior of a stochastic combustion growth process. *J. Eur. Math. Soc. (JEMS)* **6** 293–334. [MR2060478](#)
- [23] ROLLA, L. T. and SIDORAVICIUS, V. (2012). Absorbing-state phase transition for driven-dissipative stochastic dynamics on  $\mathbb{Z}$ . *Invent. Math.* **188** 127–150. [MR2897694](#)
- [24] SHAKED, M. and SHANTHIKUMAR, J. G. (2007). *Stochastic Orders*. Springer, New York. [MR2265633](#)
- [25] SIDORAVICIUS, V. and TEIXEIRA, A. (2014). Absorbing-state transition for Stochastic Sand-piles and Activated Random Walks. Available at [arXiv:1412.7098](#).
- [26] TELCS, A. and WORMALD, N. C. (1999). Branching and tree indexed random walks on fractals. *J. Appl. Probab.* **36** 999–1011. [MR1742145](#)

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