

A POSITIVE TEMPERATURE PHASE TRANSITION IN RANDOM HYPERGRAPH 2-COLORING¹

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Diluted mean-field models are graphical models in which the geometry of interactions is determined by a sparse random graph or hypergraph. Based on a nonrigorous but analytic approach called the “cavity method”, physicists have predicted that in many diluted mean-field models a phase transition occurs as the inverse temperature grows from 0 to ∞ [*Proc. National Academy of Sciences* **104** (2007) 10318–10323]. In this paper, we establish the existence and asymptotic location of this so-called condensation phase transition in the random hypergraph 2-coloring problem.

1. Introduction and results.

1.1. *Background and motivation.* Statistical mechanics models of “disordered system” such as glasses or spin-glasses are notoriously difficult to study analytically. Nonetheless, since the early 2000s physicists have developed an analytic but nonrigorous approach, the so-called *cavity method*, to put forward precise conjectures on an important class of models called *diluted mean-field models*. These are models where the geometry of interactions between individual “sites” is determined by a sparse random graph or hypergraph. Apart from models of inherent physical interest, the cavity method has since been applied to a wide variety of problems in combinatorics, computer science, information theory and compressive sensing [11, 15]. What these problems have in common is that there are “variables” and “constraints” whose mutual interaction is governed by a sparse random hypergraph. In effect, it has become an important research endeavour to provide a rigorous mathematical foundation for the cavity method. The present paper contributes to this effort.

Among the various predictions deriving from the cavity method, perhaps the most intriguing ones pertain to the existence and location of phase transitions. In particular, according to the cavity method in a variety of models there occurs a so-called *condensation phase transition*. This is a phenomenon that is ubiquitous in physics. Its role in the context of structural glasses goes back to the work of

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Kauzmann in the 1940s [10]. However, there are but a few rigorous results on the condensation phase transition in diluted mean-field models.

The aim of the present work is to establish the existence and asymptotic location of the condensation phase transition in a well-studied diluted mean-field model, the *random hypergraph 2-coloring problem*. To define this model, we recall that a k -uniform hypergraph H consists of a finite set V_H of vertices and a set E_H of edges, which are subsets of V_H of size k . For a k -uniform hypergraph H and a map $\sigma : V_H \rightarrow \{-1, 1\}$ we let $E_H(\sigma)$ be the number of edges $e \in E_H$ such that $|\sigma(e)| = 1$, that is, either all vertices of e are set to 1 or to -1 under σ . Thus, if we think of σ as a coloring of the vertices of H with two colors, then $E_H(\sigma)$ is the number of monochromatic edges. The Hamiltonian E_H gives rise to a Boltzmann distribution $\pi_{H,\beta}$ on the set of all maps $\sigma : V_H \rightarrow \{-1, 1\}$ in the usual way: we let

$$(1.1) \quad \pi_{H,\beta}[\sigma] = \frac{\exp(-\beta E_H(\sigma))}{Z_\beta(H)}$$

where $Z_\beta(H) = \sum_{\tau:V_H \rightarrow \{-1,1\}} \exp(-\beta E_H(\tau))$

is the partition function. We refer to β as the *inverse temperature*. Clearly, as $\beta \rightarrow \infty$ the Boltzmann distribution $\pi_{H,\beta}$ will place more and more weight on maps σ with fewer and fewer monochromatic edges. For a given hypergraph H , the key object of interest is the function $\beta \mapsto \frac{1}{n} \ln Z_\beta(H)$, the *free entropy*.

While the definition (1.1) makes sense for any hypergraph H , in the diluted mean-field model the hypergraph itself is random. More specifically, we consider the random hypergraph $H_k(n, p)$ on n vertices $V = \{1, \dots, n\}$, in which each of the $\binom{n}{k}$ possible hyperedges comprising of k distinct vertices is present with probability $p \in [0, 1]$ independently. Throughout the paper, we always let $\beta \in [0, \infty)$ and $p = d/\binom{n-1}{k-1}$, where $d > 0$ is a real number and $k \geq 3$ is an integer. The parameters d, k and β are going to remain fixed while we are going to let $n \rightarrow \infty$. The main objective is to determine

$$(1.2) \quad \Phi_{d,k}(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_\beta(H_k(n, p))],$$

the *free entropy density*. Of course, in (1.2) the expectation is over the choice of the random hypergraph $H_k(n, p)$.

An obvious question is whether the limit (1.2) exists for all d, k, β . That this is indeed the case follows from an application of the combinatorial interpolation method from [5]. Furthermore, a standard application of Azuma’s inequality shows that for any d, k, β the sequence $\{\frac{1}{n} \ln Z_\beta(H_k(n, p))\}_n$ converges to $\Phi_{d,k}(\beta)$ in probability.

1.2. *The main result.* In this paper, we establish the existence and approximate location of the condensation phase transition in random hypergraph 2-coloring.

More specifically, we are going to obtain a formula that determines the location of the condensation phase transition up to an error ε_k that tends to 0 for large k . This is the first (rigorous) result that determines the condensation phase transition within such accuracy in terms of the finite parameter β (the “positive temperature” case, in physics jargon).

We call $\beta_0 > 0$ *smooth* if there exists $\varepsilon > 0$ such that the function $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \mapsto \Phi_{d,k}(\beta)$ admits an expansion as an absolutely convergent power series around β_0 . Otherwise, we say that a *phase transition* occurs at β_0 . With these conventions, we have the following theorem.

THEOREM 1.1. *For any fixed number $C > 0$, there exists a sequence $\varepsilon_k > 0$ with $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ such that the following is true. Let*

$$\Sigma_{k,d}(\beta) = (\beta + 1) \exp(-\beta + k \ln 2) \ln 2 - 2 \left(\frac{d}{k} - 2^{k-1} \ln 2 + \ln 2 \right).$$

(i) *If $d/k < 2^{k-1} \ln 2 - \ln 2 - \varepsilon_k$, then any $\beta > 0$ is smooth and*

$$(1.3) \quad \Phi_{d,k}(\beta) = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))).$$

(ii) *If $2^{k-1} \ln 2 - \ln 2 + \varepsilon_k < d/k < 2^{k-1} \ln 2 + C$, then $\Sigma_{k,d}(\beta)$ has a unique zero $\beta_c(d, k) \geq k \ln 2$ and:*

- *any $\beta \in (0, \beta_c(d, k) + \varepsilon_k)$ is smooth and $\Phi_{d,k}(\beta)$ is given by (1.3),*
- *there occurs a phase transition at $\beta_c(d, k) + \varepsilon_k$,*
- *for $\beta > \beta_c(d, k) + \varepsilon_k$ we have*

$$\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))).$$

In summary, Theorem 1.1 shows that in the case that the “density” d/k of the random hypergraph is less than about $2^{k-1} \ln 2 - \ln 2$, there does not occur a phase transition for any finite β . By contrast, for slightly larger densities there is a phase transition. Its approximate location is given by $\beta_c(d, k)$. While in Theorem 1.1 this value is determined implicitly as the zero of $\Sigma_{k,d}(\beta)$, it is not difficult to obtain the expansion

$$(1.4) \quad \beta_c(d, k) = (k - 1) \ln 2 + \ln k + 2 \ln \ln 2 - \ln c + \delta_k,$$

where $c = d/k - 2^{k-1} \ln 2 + \ln 2$ and $\lim_{k \rightarrow \infty} \delta_k = 0$. Furthermore, the proof of Theorem 1.1 shows that there exists $c_1 > 0$ such that $\varepsilon_k \leq k^{c_1} 2^{-k}$. Thus, Theorem 1.1 determines the critical density from that on a phase transition starts to occur and the critical $\beta_c(d, k)$ up to an error term that decays exponentially with k .

1.3. *Discussion and related work.* In this section, we explain how Theorem 1.1 relates to the predictions based on the physicists’ “cavity method”. We also comment on further related work. As usual, we say that an event occurs *asymptotically almost surely* (a.a.s.) if its probability converges to 1 as $n \rightarrow \infty$.

1.3.1. *The “entropy crisis”.* Theorem 1.1 is perfectly in line with the picture sketched by the (nonrigorous) cavity method, and its proof is inspired by the physicists’ notion that the condensation phase transition results from an “entropy crisis” [12, 15]. More specifically, it is expected that already for densities much smaller than the one treated in Theorem 1.1, namely for d/k beyond about $2^{k-1} \ln k/k$ and for large enough β , the Boltzmann distribution can be approximated by a convex combination of probability measures corresponding to “clusters” of 2-colorings a.a.s. That is, there exist sets $\mathcal{C}_{\beta,1}, \dots, \mathcal{C}_{\beta,N} \subset \{-1, 1\}^n$ and small numbers $0 < \varepsilon < \delta$ such that:

- if $\sigma, \tau \in \mathcal{C}_{\beta,i}$ for some i , then $\langle \sigma, \tau \rangle > (1 - \varepsilon)n$,
- if $\sigma \in \mathcal{C}_{\beta,i}, \tau \in \mathcal{C}_{\beta,j}$ with $i \neq j$, then $\langle \sigma, \tau \rangle < (1 - \delta)n$.

Moreover, with $Z_{\beta,i} = \sum_{\tau \in \mathcal{C}_{\beta,i}} \exp(-\beta E_{H_k(n,p)}(\tau))$ the volume of $\mathcal{C}_{\beta,i}$, we have

$$\left\| \pi_{H_k(n,p),\beta}[\cdot] - \sum_{i=1}^N \frac{Z_{\beta,i}}{Z_{\beta}(H_k(n,p))} \cdot \pi_{H_k(n,p),\beta}[\cdot | \mathcal{C}_{\beta,i}] \right\|_{\text{TV}} < \exp(-\Omega(n)).$$

Given a hypergraph, the definition of the “clusters” $\mathcal{C}_{\beta,i}$ is somewhat canonical (under certain assumptions); we will formalise the construction in Section 3.

With the cluster decomposition in place, the physics story of how the condensation phase transition comes about goes as follows. If β is sufficiently small, we have $\max_{i \leq N} \ln Z_{\beta,i} \leq \ln Z_{\beta}(H_k(n,p)) - \Omega(n)$ a.a.s. That is, even the largest cluster only captures an exponentially small fraction of the overall mass $Z_{\beta}(H_k(n,p))$. Now, as we increase β (while d/k remains fixed), both $Z_{\beta}(H_k(n,p))$ and $\max_{i \leq N} Z_{\beta,i}$ decrease. But $Z_{\beta}(H_k(n,p))$ drops at a faster rate. In fact, for large enough densities d/k there might be a critical value β_* where the gap between $\max_{i \leq N} \ln Z_{\beta,i}$ and $\ln Z_{\beta}(H_k(n,p))$ vanishes. This β_* should mark a phase transition. This is because $\max_{i \leq N} \ln Z_{\beta,i}$ and $\ln Z_{\beta}(H_k(n,p))$ cannot both extend analytically to $\beta > \beta_*$, as otherwise we would arrive at the absurd conclusion that $\max_{i \leq N} Z_{\beta,i} > Z_{\beta}(H_k(n,p))$.

The proof of Theorem 1.1 is based on turning this “entropy crisis” scenario into a rigorous argument. To this end, we establish a rigorous version of the above “cluster decomposition” and, crucially, an estimate of the cluster volumes $Z_{\beta,i}$. The arguments that we develop for these problems partly build upon prior work from [1, 2, 6].

The key difference between [1, 2, 6] and the present work is the presence of the parameter β . More precisely, [1, 2, 6] dealt with *proper* hypergraph 2-colorings,

that is, maps $\sigma : V \rightarrow \{-1, 1\}$ such that $E_H(\sigma) = 0$. Thus, the Boltzmann distribution in those papers is just the uniform distribution over proper 2-colorings, and the partition function is the number of proper 2-colorings. In a sense, this corresponds to setting $\beta = \infty$ in the present setup. In particular, the only parameter present in [1, 2, 6] is the average degree d of the random hypergraph, whereas in the present paper we deal with a two-dimensional phase diagram governed by d and, additionally, β . Of course, from a “classical” statistical physics viewpoint it seems less natural to vary the parameter d that governs the geometry of the system and fix β than to fix d and vary β . Theorem 1.1 encompasses the latter case.

To prove Theorem 1.1, we extend some of the arguments from [1, 2, 6]. In particular, we provide a “finite- β ” version of the second moment arguments from [1, 6]. Independently of the present work, a similar extension was obtained by Achlioptas and Theodoropoulos [3]. In addition, we extend the argument for estimating the cluster size from [6] to the case of finite β . Moreover, the argument that we develop for inferring the condensation transition from the second moment method and the estimate of the cluster size draws upon ideas developed for the $\beta = \infty$ case in [1, 4, 6]. Especially with respect to the estimate of the cluster size, dealing with finite β requires substantial additional work and ideas.

1.3.2. *Prior work on condensation.* The first rigorous result on a genuine condensation phase transition in a diluted mean field model is due to Coja-Oghlan and Zdeborová [6], who dealt with the proper hypergraph 2-colorings (i.e., the $\beta = \infty$ case of the problem considered here). Thus, the only parameter in [6] is d . The main result of [6] is that there occurs a condensation phase transition at $d/k = 2^{k-1} \ln 2 - \ln 2 + \gamma_k$, where $\lim_{k \rightarrow \infty} \gamma_k = 0$. Up to the error term γ_k , the result confirms a prediction from [8]. Moreover, as Theorem 1.1 shows, the result from [6] matches the smallest density for which a condensation phase transition occurs for a finite β . In this sense, [6] determines the intersection of the “condensation line” in the two-dimensional phase diagram of Theorem 1.1 with the d -axis. Additionally, Bapst, Coja-Oghlan, Hetterich, Raßmann and Vilenchik [4] determined the condensation phase transition in the random graph coloring problem. This is the zero-temperature case of the Potts antiferromagnet on the Erdős–Rényi random graph. Thus, also in [4] the parameter β is absent.

The only prior (rigorous) paper that explicitly deals with the positive temperature case is the recent work of Contucci, Dommers, Giardina and Starr [7]. They study the k -spin Potts antiferromagnet on the Erdős–Rényi random graph with finite β and show that for certain values of the average degree a condensation phase transition exists. But to the extent that the results are comparable, [7] is less precise than Theorem 1.1. Indeed, a direct application of the approach from [7] to the present problem would determine $\beta_c(d, k)$ only up to an additive error of $\ln k$, rather than an error that diminishes with k . This is due to two technical differences between the present work and [7]. First, the second moment argument required

in the case of the k -spin Potts antiferromagnet is technically *far* more challenging than in the present case. In effect, an enhanced version of the second moment argument along the lines of [6] (with explicit conditioning on the cluster size) is not available in the Potts model. Second, [7] employs a conceptually less precise estimate of the cluster size than the one we derive. More precisely, [7] essentially neglects the entropic contribution to the cluster size, leading to under-estimate the typical cluster size significantly.

The condensation line at finite β in the Potts antiferromagnet on the Erdős–Rényi random graph was studied by Krzakala and Zdeborová [13] by means of nonrigorous techniques. They predict the location of the condensation line in terms of an intricate fixed-point problem. (While conjectured to yield the exact location of the phase transition for large enough average degrees d , no explicit expansion for large d such as the one of Theorem 1.1 was given.)

2. Preliminaries and notation. Because we take the limit $n \rightarrow \infty$ and due to the presence of the sequences $\varepsilon_k, \varepsilon'_k$, Theorem 1.1 is an asymptotic statement in both n and k . Therefore, throughout the paper we tacitly assume that both n, k are sufficiently large.

We use the standard O -notation when referring to the limit $n \rightarrow \infty$. Thus, $f(n) = O(g(n))$ means that there exist $C > 0, n_0 > 0$ such that for all $n > n_0$ we have $|f(n)| \leq C \cdot |g(n)|$. In addition, we use the standard symbols $o(\cdot), \Omega(\cdot), \Theta(\cdot)$. In particular, $o(1)$ stands for a term that tends to 0 as $n \rightarrow \infty$. We adopt the common notation that for the symbol $\Omega(\cdot)$ the sign matters, that is, $f(n) = \Omega(g(n))$ means that there exist $C > 0, n_0 > 0$ such that for all $n > n_0$ we have $f(n) \geq C \cdot g(n)$ whereas $f(n) = -\Omega(g(n))$ implies $-f(n) \geq C \cdot g(n)$ for all $n > n_0$.

Additionally, we use asymptotic notation with respect to k . To make this explicit, we insert k as an index. Thus, $f(k) = O_k(g(k))$ means that there exist $C > 0, k_0 > 0$ such that for all $k > k_0$ we have $|f(k)| \leq C \cdot |g(k)|$. Further, we write $f(k) = \tilde{O}_k(g(k))$ to indicate that there exist $C > 0, k_0 > 0$ such that for all $k > k_0$ we have $|f(k)| \leq k^C \cdot |g(k)|$. An analogous convention applies to $o_k(\cdot), \Omega_k(\cdot)$ and $\Theta_k(\cdot)$. Notice that here as well we have $\Omega_k(\cdot) \neq -\Omega_k(\cdot)$.

Throughout the paper, we set $p = d/\binom{n-1}{k-1}$. The *degree* of a vertex $v \in V$ in a hypergraph $H = (V, E)$ is the number of all edges $e \in E$ that contain v . We let $e(H)$ denote the total number of edges of the hypergraph H .

If L is an integer, then we write $[L]$ for the set $\{1, \dots, L\}$. Moreover, $\mathcal{H}(z) = -z \ln z - (1 - z) \ln(1 - z)$ denotes the entropy function. Further, we need the following instalment of the Chernoff bound.

LEMMA 2.1 ([9], page 29). *Assume that X_1, \dots, X_n are independent random variables such that X_i has a Bernoulli distribution with mean p_i . Let $\lambda = \mathbb{E}[X]$ and set $\phi(x) = (1 + x) \ln(1 + x) - x$. Then*

$$\mathbb{P}[X \geq \lambda + t] \leq \exp(-\lambda\phi(t/\lambda)), \quad \mathbb{P}[X \leq \lambda - t] \leq \exp(-\lambda\phi(-t/\lambda))$$

for any $t > 0$. In particular, $\mathbb{P}[X \geq t\lambda] \leq \exp(-t\lambda \ln(t/e))$ for any $t > 1$.

It is well known that $\ln Z_\beta$, the key quantity that we are interested in, enjoys the following ‘‘Lipschitz property’’.

FACT 2.2. Let H be a hypergraph and obtain another hypergraph H' from H by either adding or removing a single edge. Then $|\ln Z_\beta(H) - \ln Z_\beta(H')| \leq \beta$.

This Lipschitz property implies the following concentration bound for $\ln Z_\beta(H_k(n, p))$.

LEMMA 2.3. For any $\alpha > 0$ there is $\delta = \delta(\alpha) > 0$ such that

$$\mathbb{P}[|\ln Z_\beta(H_k(n, p)) - \mathbb{E}[\ln Z_\beta(H_k(n, p))]| > \alpha n] < \exp(-\delta n).$$

PROOF. This is immediate from Fact 2.2 and McDiarmid’s inequality [14], Theorem 3.8. \square

Throughout the paper, it will be convenient to work with two other random hypergraph models. More precisely, for integers $n, m > 0$ we let $H_k(n, m)$ denote the random hypergraph on the vertex set $[n]$ obtained by choosing exactly m edges without replacement uniformly at random from all possible edges, each comprising of k distinct vertices from $[n]$. This random hypergraph model will be used essentially in Section 5. The disadvantage of this model is the fact that the edges are not mutually independent. Therefore, to simplify calculations in Section 4 we let $H'_k(n, m)$ denote the random hypergraph on the vertex set $[n]$ obtained by choosing m edges uniformly and independently at random. In this model, we may choose the same edge more than once, however, the following statement shows that this is quite unlikely.

FACT 2.4. Assume that $m = m(n)$ is a sequence such that $m = O(n)$ and let \mathcal{A} be the event that $H'_k(n, m)$ has no multiple edges. Then $\mathbb{P}[\neg \mathcal{A}] = O(1/n^{k-2})$.

We relate the expected values of the partition functions of $H_k(n, m)$ and $H'_k(n, m)$ in Section 4.1.

3. Outline. Throughout this section let $0 \leq d/k \leq 2^{k-1} \ln 2 + O_k(1)$.

The proof of Theorem 1.1 is based on establishing the physicists’ notion of an ‘‘entropy crisis’’ rigorously. To this end, we are going to trace two key quantities. First, the free entropy density $\Phi_{d,k}(\beta)$, which mirrors the typical value of the partition function $Z_\beta(H_k(n, p))$. Second, the size of the ‘‘cluster’’ of a typical σ chosen from the Boltzmann distribution. More specifically, we are going to argue that it is sufficient to study the (appropriately defined) ‘‘cluster size’’ in a certain auxiliary probability space, the so-called ‘‘planted model’’. Ultimately, it will emerge that the

condensation phase transition marks the point where the cluster size in the planted model equals the typical value of $Z_\beta(H_k(n, p))$.

To implement this strategy, we begin by deriving upper and lower bounds on $\Phi_{d,k}(\beta)$ via the first and the second moment method. More precisely, in Section 4 we are going to prove the following.

PROPOSITION 3.1. *For any β , we have*

$$(3.1) \quad \Phi_{d,k}(\beta) \leq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))).$$

Moreover, if either $d/k \leq 2^{k-1} \ln 2 - 2$ and $\beta \geq 0$ or $d/k > 2^{k-1} \ln 2 - 2$ and $\beta \leq k \ln 2 - \ln k$, we have

$$(3.2) \quad \Phi_{d,k}(\beta) = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))).$$

Since the function $\beta \in [0, \infty) \mapsto \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$ is analytic, it follows that the least $\beta > 0$ for which (3.2) is violated marks a phase transition. Hence, in light of (3.1) we define

$$(3.3) \quad \beta_{\text{crit}}(d, k) = \inf \left\{ \beta > 0 : \Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) \right\}.$$

We have $\beta_{\text{crit}}(d, k) \in (0, \infty]$ and Proposition 3.1 readily implies the following lower bounds on $\beta_{\text{crit}}(d, k)$.

COROLLARY 3.2. *We have $\beta_{\text{crit}}(d, k) \geq k \ln 2 - \ln k$. If $d/k \leq 2^{k-1} \ln 2 - 2$, then $\beta_{\text{crit}}(d, k) = \infty$.*

The second main component of the proof of Theorem 1.1 is the analysis of the ‘‘cluster size’’ in the planted model. More precisely, for a hypergraph $H = (V_H, E_H)$ and a map $\sigma : V_H \rightarrow \{\pm 1\}$ we define the *cluster size* of σ in H as

$$\mathcal{C}_\beta(H, \sigma) = \sum_{\tau \in \{\pm 1\}^{V_H} : \langle \sigma, \tau \rangle \geq 2n/3} \exp(-\beta E_H(\tau)).$$

Thus, we sum up the contribution to the partition function of all those maps τ whose ‘‘overlap’’ $\langle \sigma, \tau \rangle = \sum_{v \in V_H} \sigma(v)\tau(v)$ with the given σ is big. Concerning the cluster size in $H_k(n, p)$, there is a concentration bound analogous to Lemma 2.3.

LEMMA 3.3. *For any $\sigma : [n] \rightarrow \{\pm 1\}$ and $\alpha > 0$, there is $\delta = \delta(\alpha, \sigma) > 0$ such that*

$$\mathbb{P}[\left| \ln \mathcal{C}_\beta(H_k(n, p), \sigma) - \mathbb{E}[\ln \mathcal{C}_\beta(H_k(n, p), \sigma)] \right| > \alpha n] < \exp(-\delta n).$$

PROOF. This follows from McDiarmid’s inequality [14], Theorem 3.8, and because we have $|\ln C_\beta(H, \sigma) - \ln C_\beta(H', \sigma)| \leq \beta$ for any σ if the hypergraph H' is obtained from the hypergraph H by either adding or removing a single edge. \square

Ideally, we would like to compare the cluster size of an assignment σ chosen from the Boltzmann distribution on $H_k(n, p)$ with the partition function $Z_\beta(H_k(n, p))$. Then according to the physicists’ “entropy crisis”, the condensation phase transition should mark the point β where $C_\beta(H_k(n, p), \sigma)$ is of the same order of magnitude as $Z_\beta(H_k(n, p))$. However, it seems difficult to calculate $C_\beta(H_k(n, p), \sigma)$ directly; the basic reason for this is that the Boltzmann distribution on a randomly generated hypergraph is a very difficult object to approach directly.

To sidestep this difficulty, we introduce another experiment whose outcome is much easier to study and that will emerge to be sufficient to pin down the condensation phase transition. This alternate experiment is the *planted model*. It is defined as follows. Let $\sigma : [n] \rightarrow \{-1, 1\}$ be a map chosen uniformly at random. Moreover, given d, k, β , set

$$p_1 = \frac{\exp(-\beta)}{1 - 2^{1-k}(1 - \exp(-\beta))} \cdot \frac{d}{\binom{n-1}{k-1}},$$

$$p_2 = \frac{1}{1 - 2^{1-k}(1 - \exp(-\beta))} \cdot \frac{d}{\binom{n-1}{k-1}}.$$

Now, obtain a random k -uniform hypergraph \mathbf{H} by inserting each edge that is monochromatic under σ with probability p_1 and each edge that is bichromatic under σ with probability p_2 independently. In symbols, for any hypergraph H with vertex set $[n]$ we have

$$\mathbb{P}[\mathbf{H} = H | \sigma] = p_1^{E_H(\sigma)} (1 - p_1)^{m_1} p_2^{e(H) - E_H(\sigma)} (1 - p_2)^{m_2},$$

where m_1 (resp., m_2) are the numbers of edges that are monochromatic (resp., bichromatic) under σ and are *not* in H .

The following proposition reduces the problem of determining $\beta_{\text{crit}}(d, k)$ to that of calculating $C_\beta(\mathbf{H}, \sigma)$. We will prove in Section 5.

PROPOSITION 3.4. *Assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta_0 \geq k \ln 2 - \ln k$. If for all $k \ln 2 - \ln k \leq \beta \leq \beta_0$ we have*

$$(3.4) \quad \lim_{\varepsilon \searrow 0} \liminf_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) \leq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) - \varepsilon \right] = 1,$$

then $\beta_0 \leq \beta_{\text{crit}}(d, k)$. Conversely, if

$$(3.5) \quad \lim_{\varepsilon \searrow 0} \liminf_{n \rightarrow \infty} \mathbb{P} \left[\frac{1}{n} \ln C_{\beta_0}(\mathbf{H}, \boldsymbol{\sigma}) \geq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta_0))) + \varepsilon \right] = 1,$$

then $\beta_0 \geq \beta_{\text{crit}}(d, k)$.

Finally, in Section 6 we are going to estimate the cluster size $C_\beta(\mathbf{H}, \boldsymbol{\sigma})$ to derive the following result.

PROPOSITION 3.5. *Assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \geq k \ln 2 - \ln k$. Then a.a.s. the cluster size in the planted model satisfies*

$$\frac{1}{n} \ln C_\beta(\mathbf{H}, \boldsymbol{\sigma}) = \frac{\ln 2}{2^k} - \frac{\beta \ln 2}{\exp(\beta)} + \tilde{O}_k(4^{-k}).$$

PROOF OF THEOREM 1.1. The result of the theorem in the case $d/k \leq 2^{k-1} \ln 2 - 2$ follows from Corollary 3.2. Let us thus assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$. Because we will use Proposition 3.4, we can also assume that $\beta \geq k \ln 2 - \ln k$. We write $c_k = d/k - 2^{k-1} \ln 2 + \ln 2$ and $b_k = \beta - k \ln 2$. With Proposition 3.5, we have a.a.s.

$$\begin{aligned} & \frac{1}{n} \ln C_\beta(\mathbf{H}, \boldsymbol{\sigma}) - \left(\ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) \right) \\ &= \left(\frac{\ln 2}{2^k} - (k \ln 2 + b_k) \ln 2 \frac{\exp(-b_k)}{2^k} \right) \\ & \quad - \left(\frac{\ln 2}{2^k} - \frac{c_k}{2^{k-1}} + \frac{\ln 2 \exp(-b_k)}{2^k} \right) + \tilde{O}_k(4^{-k}) \\ &= \frac{1}{2^k} [2c_k - (k \ln 2 + b_k + 1) \ln 2 \exp(-b_k)] + \tilde{O}_k(4^{-k}) \\ &= \frac{1}{2^k} [-\Sigma_{k,d}(\beta) + \tilde{O}_k(2^{-k})]. \end{aligned}$$

The equation $\Sigma_{k,d}(\beta) = 0$ has exactly one solution $\beta_c(d, k) \geq k \ln 2 - \ln k$ for $d/k > 2^{k-1} \ln 2 - \ln 2$, and no such solution for $d/k < 2^{k-1} \ln 2 - \ln 2$. Moreover, $\Sigma_{k,d}(\beta)$ is smooth for $d/k > 2^{k-1} \ln 2 - \ln 2 + 2^{-k}$, with derivatives of order $\Omega(k^{-4})$. Consequently, there is $\varepsilon_k = \tilde{O}_k(2^{-k})$ such that the following is true:

(i) If $d/k < 2^{k-1} \ln 2 - \ln 2 - \varepsilon_k$, then a.a.s. for all $\beta \geq k \ln 2 - \ln k$,

$$\frac{1}{n} \ln C_\beta(\mathbf{H}, \boldsymbol{\sigma}) \leq \left(\ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) \right) - \Omega(1).$$

(ii) If $d/k > 2^{k-1} \ln 2 - \ln 2 + \varepsilon_k$, then a.a.s. for all $\beta \geq k \ln 2 - \ln k$:

- if $\beta \leq \beta_c(d, k) - \varepsilon_k$ then

$$\frac{1}{n} \ln \mathcal{C}_\beta(\mathbf{H}, \sigma) \leq \left(\ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) \right) - \Omega(1),$$

- if $\beta \geq \beta_c(d, k) + \varepsilon_k$ then

$$\frac{1}{n} \ln \mathcal{C}_\beta(\mathbf{H}, \sigma) \geq \left(\ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) \right) + \Omega(1).$$

The proof of the theorem is completed by using Proposition 3.4. \square

4. The first and the second moment. *Throughout this section, we assume that $0 \leq d/k \leq 2^{k-1} \ln 2 + O_k(1)$. We let $m = \lceil dn/k \rceil$.*

In this section, we prove Proposition 3.1 and also lay the foundations for the proof of Proposition 3.4. Recall that $H_k(n, m)$ signifies the hypergraph on $[n]$ obtained by choosing m edges uniformly at random without replacement while for the hypergraph $H'_k(n, m)$ we choose m edges $\mathbf{e}_1, \dots, \mathbf{e}_m$ with replacement uniformly and independently at random, allowing for multiple edges.

4.1. *The first moment.* We begin with the following estimate of the first moment of Z_β in $H'_k(n, m)$.

LEMMA 4.1. *We have $\mathbb{E}[Z_\beta(H'_k(n, m))] = \Theta(2^n (1 - 2^{1-k}(1 - \exp(-\beta)))^m)$.*

The proof of Lemma 4.1 is straightforward, but we carry it out at leisure to introduce some notation that will be used throughout. For a map $\sigma : [n] \rightarrow \{-1, 1\}$, let

$$\text{Forb}(\sigma) = \binom{|\sigma^{-1}(-1)|}{k} + \binom{|\sigma^{-1}(1)|}{k}$$

be the number of “forbidden k -sets” of vertices that are colored the same under σ . The function $x \mapsto \binom{x}{k} + \binom{n-x}{k}$ is convex and takes its minimal value at $x = \frac{n}{2}$. Therefore,

$$(4.1) \quad \text{Forb}(\sigma) \geq 2 \binom{n/2}{k} = 2^{1-k} N (1 + O(1/n)) = 2^{1-k} N + O(N/n),$$

with $N = \binom{n}{k}$.

Let us call σ *balanced* if $||\sigma^{-1}(1)| - \frac{n}{2}| \leq \sqrt{n}$. Let $\text{Bal} = \text{Bal}_n$ be the set of all balanced maps $\sigma : [n] \rightarrow \{\pm 1\}$. Stirling’s formula yields $|\text{Bal}| = \Omega(2^n)$. If $\sigma \in \text{Bal}$, then

$$(4.2) \quad \text{Forb}(\sigma) \leq \binom{n/2 + \sqrt{n}}{k} + \binom{n/2 - \sqrt{n}}{k} = 2^{1-k} N + O(N/n).$$

For a hypergraph H , let

$$Z_{\beta, \text{bal}}(H) = \sum_{\sigma \in \text{Bal}} \exp(-\beta E_H(\sigma)).$$

PROOF OF LEMMA 4.1. By the independence of edges, we have

$$\begin{aligned} \mathbb{E}[\exp(-\beta E_{H'_k(n,m)}(\sigma))] &= \mathbb{E}\left[\prod_{i=1}^m \exp(-\beta \mathbf{1}_{e_i \in \text{Forb}(\sigma)})\right] \\ &= \prod_{i=1}^m \mathbb{E}[\exp(-\beta \mathbf{1}_{e_i \in \text{Forb}(\sigma)})] \\ &= (1 - N^{-1} |\text{Forb}(\sigma)| (1 - \exp(-\beta)))^m \\ &\leq (1 - 2^{1-k} (1 + O(1/n)) (1 - \exp(-\beta)))^m. \end{aligned}$$

Consequently,

$$(4.3) \quad \mathbb{E}[Z_{\beta}(H'_k(n,m))] = O(2^n (1 - 2^{1-k} (1 - \exp(-\beta)))^m).$$

If $\sigma \in \text{Bal}$, by (4.2) we have $\mathbb{E}[\exp(-\beta E_{H'_k(n,m)}(\sigma))] = \Omega((1 - 2^{1-k} (1 - \exp(-\beta)))^m)$. Therefore,

$$(4.4) \quad \begin{aligned} \mathbb{E}[Z_{\beta}(H'_k(n,m))] &\geq |\text{Bal}| \cdot \Omega((1 - 2^{1-k} (1 - \exp(-\beta)))^m) \\ &= \Omega(2^n (1 - 2^{1-k} (1 - \exp(-\beta)))^m). \end{aligned}$$

Thus, Lemma 4.1 follows from (4.3) and (4.4). \square

The following lemma relates the expectation of the partition functions of the models $H_k(n, m)$ and $H'_k(n, m)$.

LEMMA 4.2. We have $\mathbb{E}[Z_{\beta}(H_k(n, m))] = \Theta(\mathbb{E}[Z_{\beta}(H'_k(n, m))])$.

PROOF. Let \mathcal{A} be the event that $H'_k(n, m)$ has no multiple edges. Then, using Fact 2.4 we get

$$\mathbb{E}[Z_{\beta}(H'_k(n, m))] \geq \mathbb{E}[Z_{\beta}(H'_k(n, m)) | \mathcal{A}] \mathbb{P}[\mathcal{A}] \geq \mathbb{E}[Z_{\beta}(H_k(n, m))] (1 - o(1)),$$

implying that

$$(4.5) \quad \mathbb{E}[Z_{\beta}(H_k(n, m))] \leq O(1) \mathbb{E}[Z_{\beta}(H'_k(n, m))].$$

On the other hand, let $m_0 = \frac{2^{1-k} \exp(-\beta)}{1 - 2^{1-k} (1 - \exp(-\beta))} m$ and

$$f(x) = -x\beta - x \ln x - (1 - x) \ln(1 - x) + x \ln(2^{1-k}) + (1 - x) \ln(1 - 2^{1-k}).$$

We observe that f is strictly concave and attains its maximum at $x = \frac{m_0}{m}$ where it is equal to $\ln(1 - 2^{1-k}(1 - \exp(-\beta)))$. For $\sigma \in \text{Bal}$, we get with Stirling's formula

$$\begin{aligned}
 & \mathbb{E}[\exp(-\beta E_{H_k(n,m)}(\sigma))] \\
 &= \sum_{\mu} \mathbb{P}[E_{H_k(n,m)} = \mu] \exp(-\beta \mu) \\
 (4.6) \quad & \geq \sum_{\mu \in [m_0 - \sqrt{m}, m_0 + \sqrt{m}]} \exp(-\beta \mu) \frac{\binom{m}{\mu} (\text{Forb}(\sigma))^\mu (N - \text{Forb}(\sigma))^{m-\mu}}{N^m} \\
 &= \sum_{\mu \in [m_0 - \sqrt{m}, m_0 + \sqrt{m}]} \Theta_m\left(\frac{1}{\sqrt{m}}\right) \exp\left(mf\left(\frac{m_0}{m}\right)\right) \Theta(1) \\
 &= \Theta(1 - 2^{1-k}(1 - \exp(-\beta))^m).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (4.7) \quad \mathbb{E}[Z_\beta(H_k(n, m))] & \geq |\text{Bal}| \cdot \mathbb{E}[\exp(-\beta E_{H_k(n,m)}(\sigma))] \\
 & = \Omega(2^n (1 - 2^{1-k}(1 - \exp(-\beta))^m)).
 \end{aligned}$$

Combining (4.5), Lemma 4.1 and (4.7) proves the assertion. \square

As a further consequence of Lemma 4.1, we obtain the following.

COROLLARY 4.3. 1. We have $\Phi_{d,k}(\beta) \leq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$ for all d, β . 2. Assume that d, β are such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_\beta(H'_k(n, m))] < \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))).$$

Then $\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$.

PROOF. Let \mathcal{E} be the event that $|e(H_k(n, p)) - m| \leq \sqrt{n} \ln n$. Then we can couple the random hypergraphs $H_k(n, m)$ and $H_k(n, p)$ given \mathcal{E} as follows.

1. Choose a random hypergraph $H_0 = H_k(n, m)$.
2. Let $e = \text{Bin}(\binom{n}{k}, p)$ be a binomial random variable given that $|e - m| \leq \sqrt{n} \ln n$.
3. Obtain a random hypergraph H_1 from H_0 as follows:
 - If $e \geq m$, choose a set of $e - m$ random edges from all edges not present in H_0 and add them to H_0 .
 - If $e < m$, remove $m - e$ randomly chosen edges from H_0 .

The outcome H_1 has the same distribution as $H_k(n, p)$ given \mathcal{E} , and H_0, H_1 differ in at most $\sqrt{n} \ln n$ edges. Therefore, noting that $\frac{1}{n} |\ln Z_\beta| \leq \frac{d}{k} \beta + \ln 2$ with

certainly, we obtain with Fact 2.2:

$$\begin{aligned}
 \frac{1}{n} \mathbb{E} \ln Z_\beta(H_k(n, p)) &\leq \frac{1}{n} \mathbb{E} [\ln Z_\beta(H_1)] + \left(\frac{d}{k} \beta + \ln 2\right) \mathbb{P}[\neg \mathcal{E}] \\
 (4.8) \qquad \qquad \qquad &\leq \frac{1}{n} \mathbb{E} [\ln Z_\beta(H_0)] + \frac{\beta \ln n}{\sqrt{n}} + \left(\frac{d}{k} \beta + \ln 2\right) \mathbb{P}[\neg \mathcal{E}] \\
 &= \frac{1}{n} \mathbb{E} [\ln Z_\beta(H_k(n, m))] + \left(\frac{d}{k} \beta + \ln 2\right) \mathbb{P}[\neg \mathcal{E}] + o(1).
 \end{aligned}$$

Since $e(H_k(n, p))$ is a binomial random variable with mean $m + O(1)$, Lemma 2.1 implies that $\mathbb{P}[\neg \mathcal{E}] = o(1)$. Thus, by (4.8) and Jensen’s inequality,

$$\begin{aligned}
 \frac{1}{n} \mathbb{E} \ln Z_\beta(H_k(n, p)) &\leq \frac{1}{n} \mathbb{E} [\ln Z_\beta(H_k(n, m))] + o(1) \\
 &\leq \frac{1}{n} \ln \mathbb{E} [Z_\beta(H_k(n, m))] + o(1).
 \end{aligned}$$

The first assertion follows by Lemmas 4.1 and 4.2 and taking $n \rightarrow \infty$. Also the second assertion readily follows. \square

We conclude this section by observing that the contribution to Z_β of certain “exotic” σ is negligible. We begin with σ that are very imbalanced.

LEMMA 4.4. *For any $\varepsilon > 0$ there is $\delta > 0$ such that the following is true. Let \bar{B}_ε be the set of all $\sigma : [n] \rightarrow \{\pm 1\}$ such that $|\sigma^{-1}(1) - \frac{n}{2}| > \varepsilon n$. Moreover, let*

$$Z_{\beta, \bar{B}_\varepsilon}(H) = \sum_{\sigma \in \bar{B}_\varepsilon} \exp(-\beta E_H(\sigma)).$$

Then $\mathbb{E}[Z_{\beta, \bar{B}_\varepsilon}(H_k(n, m))] \leq \exp(-\delta n) \mathbb{E}[Z_\beta(H_k(n, m))]$.

PROOF. Stirling’s formula implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{1}{n} \ln |\bar{B}_\varepsilon| < \ln 2 - \delta$. Hence, (4.1) implies together with the independence of the edges that

$$\begin{aligned}
 \mathbb{E}[Z_{\beta, \bar{B}_\varepsilon}(H'_k(n, m))] &= \sum_{\sigma \in \bar{B}_\varepsilon} \mathbb{E}[\exp(-\beta E_{H'_k(n, m)}(\sigma))] \\
 &\leq |\bar{B}_\varepsilon| (1 - 2^{1-k} (1 - \exp(-\beta)))^m \\
 &\leq \exp(-\delta n) 2^n (1 - 2^{1-k} (1 - \exp(-\beta)))^m.
 \end{aligned}$$

The assertion follows from the remark that [as in equation (4.5)]

$$\mathbb{E}[Z_{\beta, \bar{B}_\varepsilon}(H_k(n, m))] = O(\mathbb{E}[Z_{\beta, \bar{B}_\varepsilon}(H'_k(n, m))]),$$

and from Lemma 4.2. \square

LEMMA 4.5. For any $\varepsilon > 0$, there is $\delta > 0$ such that the following is true. Let $m_0 = \frac{2^{1-k} \exp(-\beta)}{1-2^{1-k}(1-\exp(-\beta))}m$ and

$$Z_{\beta,\varepsilon}(H) = \sum_{\sigma:[n] \rightarrow \{\pm 1\}} \exp(-\beta E_H(\sigma)) \cdot \mathbf{1}_{|E_H(\sigma) - m_0| > \varepsilon m}.$$

Then $\mathbb{E}[Z_{\beta,\varepsilon}(H_k(n, m))] \leq \exp(-\delta n)\mathbb{E}[Z_\beta(H_k(n, m))]$.

PROOF. Let $M_0 = \{\mu \in [m] : |\mu - m_0| > \varepsilon m\}$. Moreover, for $\alpha > 0$ let B_α be the set of all $\sigma : [n] \rightarrow \{\pm 1\}$ such that $|\sigma^{-1}(1) - \frac{n}{2}| < \alpha n$. Then by Lemma 4.4 there exists $\delta > 0$ such that

$$(4.9) \quad \mathbb{E}[Z_{\beta,\varepsilon}(H_k(n, m))] \leq \exp(-\delta n)\mathbb{E}[Z_\beta(H_k(n, m))] + \sum_{\mu \in M_0} \sum_{\sigma \in B_\alpha} \exp(-\beta\mu)\mathbb{P}[E_{H_k(n,m)}(\sigma) = \mu].$$

As in the proof of Lemma 4.2, we define $f(x) = -x\beta - x \ln x - (1-x) \ln(1-x) + x \ln(2^{1-k}) + (1-x) \ln(1-2^{1-k})$ and find that for any $\gamma > 0$ we can choose $\alpha > 0$ small enough so that

$$\frac{1}{m} \ln(\exp(-\beta\mu)\mathbb{P}[E_{H_k(n,m)}(\sigma) = \mu]) \leq \gamma + f\left(\frac{\mu}{m}\right) \quad \text{for all } \sigma \in B_\alpha.$$

Because f is strictly concave and attains its maximum at $x = \frac{m_0}{m}$, there is $\delta' > 0$ such that

$$(4.10) \quad \sum_{\mu \in M_0} \sum_{\sigma \in B_\alpha} \exp(-\beta\mu)\mathbb{P}[E_{H_k(n,m)}(\sigma) = \mu] \leq \exp(-\delta' n)\mathbb{E}[Z_\beta(H_k(n, m))].$$

Finally, the assertion follows from (4.9) and (4.10). \square

4.2. *The second moment.* In Section 4.1, we derived an upper bound on $\Phi_{d,k}(\beta)$ by calculating the expectation of $Z_\beta(H'_k(n, m))$ (cf. Corollary 4.3). Here, we obtain for certain values of β and d a matching lower bound by estimating the second moment $\mathbb{E}[Z_{\beta,\text{bal}}(H'_k(n, m))^2]$. To this end, we define for $\alpha \in [-1, 1]$,

$$(4.11) \quad Z_\beta(\alpha) = \sum_{\sigma, \tau \in \text{Bal}:(\sigma, \tau) = \alpha n} \exp(-\beta(E_{H'_k(n,m)}(\sigma) + E_{H'_k(n,m)}(\tau))).$$

Thus, in (4.11) we sum over balanced pairs $\sigma, \tau : [n] \rightarrow \{\pm 1\}$ that agree on precisely $n((1 + \alpha)/2)$ vertices. Hence, we can express the second moment as

$$\begin{aligned} \mathbb{E}[Z_{\beta,\text{bal}}(H'_k(n, m))^2] &= \sum_{\sigma, \tau \in \text{Bal}} \mathbb{E}[\exp(-\beta(E_{H'_k(n,m)}(\sigma) + E_{H'_k(n,m)}(\tau)))] \\ &= \sum_{v=0}^n \mathbb{E}[Z_\beta(2v/n - 1)]. \end{aligned}$$

Consequently, we need to bound $Z_\beta(\alpha)$ for $-1 \leq \alpha \leq 1$. Recall that $\mathcal{H}(z) = -z \ln z - (1-z) \ln(1-z)$.

LEMMA 4.6. For $\alpha \in [-1, 1]$, we have

$$\frac{1}{n} \ln \mathbb{E}[Z_\beta(\alpha)] = \ln 2 + \Lambda_\beta(\alpha) - \frac{\ln n}{2n} + O(1/n),$$

where

$$\begin{aligned} \Lambda_\beta(\alpha) &= \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \frac{d}{k} \ln\left[1 - 2^{1-k}(1 - \exp(-\beta))\right] \\ &\quad \times \left[2 - (1 - \exp(-\beta)) \frac{(1+\alpha)^k + (1-\alpha)^k}{2^k}\right]. \end{aligned}$$

PROOF. Let e be a randomly chosen edge. Let $\sigma, \tau : [n] \rightarrow \{\pm 1\}$ be two balanced maps with overlap $\langle \sigma, \tau \rangle = \alpha n$. Let us write $\sigma \vDash e$ if $e \notin \text{Forb}(\sigma)$ (i.e., e is bichromatic under σ). By inclusion–exclusion,

$$\begin{aligned} \mathbb{P}[\sigma \vDash e], \mathbb{P}[\tau \vDash e] &= 1 - 2^{1-k} + O(1/n), \\ \mathbb{P}[\sigma, \tau \vDash e] &= 1 - 2^{2-k} + 2^{1-2k}((1+\alpha)^k + (1-\alpha)^k) + O(1/n). \end{aligned}$$

Hence, by the independence of edges,

$$\begin{aligned} \mathbb{E}[Z_\beta(\alpha)] &= \sum_{\sigma, \tau: \langle \sigma, \tau \rangle = \alpha n} \mathbb{E} \prod_{i=1}^m \exp[-\beta(\mathbf{1}_{\sigma \neq e_i} + \mathbf{1}_{\tau \neq e_i})] \\ &= \sum_{\sigma, \tau: \langle \sigma, \tau \rangle = \alpha n} (\mathbb{E}[\exp[-\beta(\mathbf{1}_{\sigma \neq e_1} + \mathbf{1}_{\tau \neq e_1})]])^m \\ &= 2^n \binom{n}{(1+\alpha)n/2} (\mathbb{P}[\sigma, \tau \vDash e_1] \\ &\quad + \exp(-\beta)(\mathbb{P}[\sigma \vDash e_1, \tau \not\vDash e_1] + \mathbb{P}[\sigma \not\vDash e_1, \tau \vDash e_1]) \\ &\quad + \exp(-2\beta) \cdot \mathbb{P}[\sigma, \tau \not\vDash e_1])^m \\ &= 2^n \binom{n}{(1+\alpha)n/2} (1 + O(1/n)) [1 - 2^{2-k}(1 - \exp(-\beta)) \\ &\quad + 2^{1-2k}(1 - \exp(-\beta))^2((1+\alpha)^k + (1-\alpha)^k)]^m. \end{aligned} \tag{4.12}$$

Furthermore, by Stirling’s formula,

$$\binom{n}{(1+\alpha)n/2} = O(n^{-1/2}) \exp\left(n\mathcal{H}\left(\frac{1+\alpha}{2}\right)\right). \tag{4.13}$$

The assertion follows by combining (4.12) and (4.13). \square

Hence, we need to study the function Λ_β . Since $\Lambda_\beta(\alpha) = \Lambda_\beta(-\alpha)$, $\alpha = 0$ is a stationary point. Moreover, with

$$s = s(\alpha, \beta) = 1 - 2^{1-k}(1 - \exp(-\beta)) \left[2 - (1 - \exp(-\beta)) \frac{(1+\alpha)^k + (1-\alpha)^k}{2^k}\right]$$

the first two derivatives of Λ_β work out to be

$$(4.14) \quad \Lambda'_\beta(\alpha) = \frac{\ln(1-\alpha) - \ln(1+\alpha)}{2} + \frac{2d}{4^k s} (\exp(-\beta) - 1)^2 ((1+\alpha)^{k-1} - (1-\alpha)^{k-1}),$$

$$(4.15) \quad \Lambda''_\beta(\alpha) = \frac{1}{\alpha^2 - 1} + \frac{2d(k-1)(\exp(-\beta) - 1)^2}{4^k s} ((1+\alpha)^{k-2} + (1-\alpha)^{k-2}) - \frac{dk(1 - \exp(-\beta))^4}{2^{4k-2} s^2} [(1+\alpha)^{k-1} - (1-\alpha)^{k-1}]^2.$$

In particular,

$$(4.16) \quad \Lambda''_\beta(0) = -1 + \tilde{O}_k(2^{-k}) < 0.$$

Hence, there is a local maximum at $\alpha = 0$. As a consequence, we have

$$\mathbb{E}[Z_\beta(H'_k(n, m))^2] = O(\mathbb{E}[Z_\beta(H'_k(n, m))]^2),$$

if Λ_β has a strict *global* maximum at $\alpha = 0$. More generally, we have the following.

LEMMA 4.7. *Assume that $\beta \geq 0$ and $J \subset [-1, 1]$ is a compact set such that $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \in J \setminus \{0\}$. Then*

$$\sum_{v=0}^n \mathbb{E}[Z_\beta(2v/n - 1)] \mathbf{1}_{2v/n-1 \in J} = O(\mathbb{E}[Z_\beta(H'_k(n, m))]^2).$$

PROOF. We start by observing that $\frac{\ln 2 + \Lambda_\beta(0)}{2} = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$. Hence, Lemma 4.1 yields

$$(4.17) \quad \exp[n(\ln 2 + \Lambda_\beta(0))] = O(\mathbb{E}[Z_\beta(H'_k(n, m))]^2).$$

Now, by (4.16), there exist $\eta, c > 0$ such that $\Lambda_\beta(\alpha) \leq \Lambda_\beta(0) - c\alpha^2$ for all $\alpha \in J_0 = J \cap (-\eta, \eta)$. Hence, by Lemma 4.6 and (4.17)

$$(4.18) \quad \begin{aligned} & \sum_{v=0}^n \mathbb{E}[Z_\beta(2v/n - 1)] \mathbf{1}_{2v/n-1 \in J_0} \\ &= O(n^{-1/2} 2^n) \sum_{v=0}^n \exp(n\Lambda_\beta(2v/n - 1)) \mathbf{1}_{2v/n-1 \in J_0} \\ &= O(2^n \exp(n\Lambda_\beta(0))) \sum_{v: |2v/n-1| < \eta} \frac{\exp(-nc(2v/n - 1)^2)}{\sqrt{n}} \\ &= O(2^n \exp(n\Lambda_\beta(0))) = O(\mathbb{E}[Z_\beta(H'_k(n, m))]^2). \end{aligned}$$

Further, let $J_1 = J \setminus (-\eta, \eta)$. Then J_1 is compact. Hence, there exists $\delta > 0$ such that $\Lambda_\beta(\alpha) < \Lambda_\beta(0) - \delta$ for all $\alpha \in J_1$. Therefore, Lemma 4.6 and (4.17) yield

$$\begin{aligned}
 \sum_{\nu=0}^n \mathbb{E}[Z_\beta(2\nu/n - 1)] \mathbf{1}_{2\nu/n-1 \in J_1} &= O(n2^n) \sup_{\alpha \in J_1} \exp(n\Lambda_\beta(\alpha)) \\
 (4.19) \qquad \qquad \qquad &= O(n2^n) \exp(n(\Lambda_\beta(0) - \delta)) \\
 &= O(\mathbb{E}[Z_\beta(H'_k(n, m))]^2).
 \end{aligned}$$

Finally, the assertion follows from (4.18) and (4.19). \square

Now we prove that for the set J from Lemma 4.7 we have at least $[-1 + 2^{-3k/4}, 1 - 2^{-3k/4}] \subset J$ for all $\beta \geq 0$.

LEMMA 4.8. *For $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \geq 0$ we have $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \neq 0$ with $|\alpha| \leq 1 - 2^{-3k/4}$.*

PROOF. We know that there is a local maximum at $\alpha = 0$. Moreover, we read off of (4.15) that $\Lambda''_\beta(\alpha) < 0$ if $|\alpha| < 1 - 6 \ln k/k$, and thus

$$\Lambda_\beta(0) > \Lambda_\beta(\alpha) \quad \text{for all } \alpha \in (-1 - 6 \ln k/k, 1 - 6 \ln k/k).$$

Further, we obtain from (4.14) for $|\alpha| \geq 1 - 6 \ln k/k$

$$\begin{aligned}
 \Lambda'_\beta(\alpha) &\leq \frac{\ln(1 - \alpha)}{2} + \frac{2d(1 - \exp(-\beta))^2(1 + \alpha)^{k-1}}{4^k(1 + O_k(2^{-k}))} \\
 &\leq \frac{\ln(1 - \alpha)}{2} + \frac{d(1 - \exp(-\beta))^2 \exp((1 + \alpha)(k - 1)/2)}{2^k(1 + O_k(2^{-k}))}.
 \end{aligned}$$

Hence, for k large enough $\Lambda'_\beta(\alpha) < 0$ if $|\alpha| < 1 - 2.01 \ln k/k$ and a similar estimate yields

$$(4.20) \qquad \qquad \Lambda'_\beta(\alpha) > 0 \quad \text{if } |\alpha| > 1 - 1.99 \ln k/k.$$

Thus, to proceed we need to evaluate Λ_β at $|\alpha| = 1 - \gamma \ln k/k$ for $\gamma \in [1.99, 2.01]$ and at $|\alpha| = 1 - 2^{-3k/4}$. We find

$$\Lambda_\beta(\alpha) = -\ln 2 + o_k(1)$$

for $|\alpha| = 1 - \gamma \ln k/k$ with $\gamma \in [1.99, 2.01]$ and $\Lambda_\beta(\alpha) = -\ln 2 + o_k(1)$ for $|\alpha| = 1 - 2^{-3k/4}$ proving the assertion. \square

LEMMA 4.9. *The function $\beta \mapsto \Lambda_\beta(\alpha) - \Lambda_\beta(0)$ is nondecreasing for $\alpha \neq 0$. In particular, if $d > 0$ and $\beta_0 \geq 0$ are such that $\Lambda_{\beta_0}(\alpha) < \Lambda_{\beta_0}(0)$ for all $\alpha \neq 0$, then $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \neq 0, 0 \leq \beta < \beta_0$.*

PROOF. The derivative of Λ_β with respect to β works out to be

$$\frac{\partial \Lambda_\beta}{\partial \beta} = \frac{d}{k} \cdot \frac{2^{2-2k}((1 + \alpha)^k + (1 - \alpha)^k) \exp(-\beta)(1 - \exp(-\beta)) - 2^{2-k} \exp(-\beta)}{1 - 2^{2-2k}(1 - \exp(-\beta)) + 2^{1-2k}(1 - \exp(-\beta))^2((1 + \alpha)^k + (1 - \alpha)^k)}.$$

Substituting $z = (1 + \alpha)^k + (1 - \alpha)^k$ and $b = 1 - \exp(-\beta)$ in the above, we obtain

$$g(z) = \frac{d}{k} \cdot \frac{2^{2-2k} b(1 - b)z - 2^{2-k}(1 - b)}{1 - 2^{2-2k} b + 2^{1-2k} b^2 z}.$$

Because a function $z \mapsto \frac{az-b}{cz+d}$ with $a, b, c, d \geq 0$ is nondecreasing, this completes the proof. \square

With these instruments in hand we identify regimes of d and β where $\Lambda_\beta(\alpha)$ takes its global maximum at $\alpha = 0$.

LEMMA 4.10. *Assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \leq k \ln 2 - \ln k$. Then $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ for all $\alpha \in [-1, 1] \setminus \{0\}$.*

PROOF. For $|\alpha| \leq 1 - 2^{-3k/4}$ this is the statement of Lemma 4.8. We write $\alpha = 1 - \delta$ with $\delta \in [0, 2^{-3k/4}]$. Let

$$f_\beta(\delta) = (1 - \exp(-\beta)) \left[2 - (1 - \exp(-\beta)) \frac{(2 - \delta)^k + \delta^k}{2^k} \right] \in [0, 2].$$

For $\beta = k \ln 2 - \ln k$, we have the expansion

$$\begin{aligned} f_\beta(\delta) &= \left(1 - \frac{k}{2^k} \right) \left[2 - \left(1 - \frac{k}{2^k} \right) \left(1 - k \frac{\delta}{2} + \tilde{O}_k(4^{-k}) \right) \right] \\ &= 1 + k \frac{\delta}{2} + \tilde{O}_k(4^{-k}). \end{aligned}$$

Therefore,

$$\begin{aligned} \Lambda_\beta(\alpha) &= -\frac{\delta}{2} \ln \left(\frac{\delta}{2} \right) - \left(1 - \frac{\delta}{2} \right) \ln \left(1 - \frac{\delta}{2} \right) \\ &\quad + \frac{d}{k} \ln \left[1 - 2^{1-k} \left[1 + k \frac{\delta}{2} + \tilde{O}_k(4^{-k}) \right] \right] \\ &= -\ln 2 - \frac{\delta}{2} \ln \delta + \frac{\delta}{2} - (k - 1) \frac{\delta}{2} \ln 2 + O_k(2^{-k}). \end{aligned}$$

The function $\delta \mapsto -\frac{\delta}{2} \ln \delta + \frac{\delta}{2} - (k - 1) \frac{\delta}{2} \ln 2$ is easily studied: it takes its maximum at $\delta_0 = 2^{1-k}$ for which it is equal to 2^{-k} . Hence, for $\alpha = 1 - \delta$ with $\delta \in [0, 2^{-3k/4}]$,

$$\Lambda_\beta(\alpha) \leq -\ln 2 + O_k(2^{-k}).$$

By symmetry, this also holds for $\alpha = -1 + \delta$ with $\delta \in [0, 2^{-3k/4}]$. By comparison,

$$\begin{aligned} \Lambda_\beta(0) &= \ln 2 + (2^{k-1} \ln 2 + O_k(1)) \ln \left(1 - 2^{2-k} + \frac{4k}{4^k} + O_k(4^{-k}) \right) \\ &= -\ln 2 + 2^{1-k} k \ln 2 + O_k(2^{-k}). \end{aligned}$$

Therefore, $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ for all $\alpha \neq 0$ if $\beta = k \ln 2 - \ln k$. Using Lemma 4.9, we can expand the result to all $\beta \leq k \ln 2 - \ln k$. \square

LEMMA 4.11. *Assume that $d/k \leq 2^{k-1} \ln 2 - 2$ and $\beta \geq 0$. Then $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ for all $\alpha \in [-1, 1] \setminus \{0\}$.*

PROOF. Let $r_k = O_k(1)$ such that $d/k = 2^{k-1} \ln 2 + r_k$. Define the function $\Lambda_\infty : [-1, 1] \rightarrow \mathbb{R}$ as

$$\alpha \mapsto \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \frac{d}{k} \ln(1 - 2^{2-k} + 2^{1-2k}((1+\alpha)^k + (1-\alpha)^k)).$$

Analogously to the proof of Lemma 4.10, we get $\Lambda_\infty(\alpha) \leq -\ln 2 - (\ln 2 + 2r_k - 1)2^{-k} + \tilde{O}_k(4^{-k})$ for all α and $\Lambda_\infty(0) = -\ln 2 - 2(\ln 2 + 2r_k)2^{-k} + \tilde{O}_k(4^{-k})$, which implies that for $r_k \leq -2$ we have $\Lambda_\infty(\alpha) < \Lambda_\infty(0)$ for all $\alpha \in [-1, 1] \setminus \{0\}$. Because the continuous functions Λ_β converge uniformly to Λ_∞ as $\beta \rightarrow \infty$, we conclude that there is $\beta_0 \geq 0$ such that for all $\beta > \beta_0$,

$$(4.21) \quad \Lambda_\beta(\alpha) < \Lambda_\beta(0) \quad \text{for all } \alpha \in [-1, 1] \setminus \{0\}.$$

Hence, Lemma 4.9 implies that (4.21) holds for all $\beta \geq 0$, as desired. \square

PROOF OF PROPOSITION 3.1. The first assertion follows directly from Corollary 4.3. Moreover, if d, β are such that for some n -independent number $C > 0$ we have

$$(4.22) \quad \mathbb{E}[Z_\beta(H'_k(n, m))^2] \leq C \cdot \mathbb{E}[Z_\beta(H'_k(n, m))]^2,$$

then the Paley–Zygmund inequality implies that

$$(4.23) \quad \begin{aligned} \mathbb{P}[Z_\beta(H'_k(n, m)) \geq \mathbb{E}[Z_\beta(H'_k(n, m))]/2] &\geq \frac{\mathbb{E}[Z_\beta(H'_k(n, m))]^2}{4\mathbb{E}[Z_\beta(H'_k(n, m))^2]} \\ &\geq \frac{1}{4C} > 0. \end{aligned}$$

Let \mathcal{A} be the event that $H'_k(n, m)$ has no multiple edges. Since \mathcal{A} occurs a.a.s. by Fact 2.4, (4.23) implies that

$$(4.24) \quad \mathbb{P}[Z_\beta(H'_k(n, m)) \geq \mathbb{E}[Z_\beta(H'_k(n, m))]/2 | \mathcal{A}] \geq \frac{1 - o(1)}{4C}.$$

Further, since the number $e(H_k(n, p))$ of edges in $H_k(n, p)$ has a binomial distribution with mean $m + O(1)$, Stirling’s formula implies that $\mathbb{P}[e(H_k(n, p)) = m] \geq$

$\Omega(n^{-1/2})$. Because given $e(H_k(n, p)) = m$, $H_k(n, p)$ is identically distributed as $H'_k(n, m)$ given \mathcal{A} , (4.24) implies that

$$(4.25) \quad \mathbb{P}[Z_\beta(H_k(n, p)) \geq \mathbb{E}[Z_\beta(H'_k(n, m))]/2] \geq \Omega(n^{-1/2}).$$

The concentration bound from Lemma 2.3 and (4.25) yields $\ln \mathbb{E}[Z_\beta(H'_k(n, m))] - \mathbb{E}[\ln Z_\beta(H_k(n, p))] - \ln 2 = o(n)$. Hence, if (4.22) is true, then

$$(4.26) \quad \frac{1}{n} \mathbb{E}[\ln Z_\beta(H_k(n, p))] \geq \frac{1}{n} \ln \mathbb{E}[Z_\beta(H'_k(n, m))] - o(1).$$

Finally, Lemma 4.7 and Lemma 4.11 imply that (4.22) holds for all $\beta \geq 0$ and $d/k \leq 2^{k-1} \ln 2 - 2$. Moreover, by Lemma 4.7 and Lemma 4.10 the bound (4.22) is true if $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \leq k \ln 2 - \ln k$. Thus, the assertion follows from (4.26). \square

5. The planted model. *The aim of this section is to prove Proposition 3.4. Throughout the section, we let $m = \lceil dn/k \rceil$. For $\varepsilon > 0$, we let B_ε be the set of all $\sigma : [n] \rightarrow \{\pm 1\}$ such that $|\sigma^{-1}(1) - \frac{n}{2}| < \varepsilon n$. Further, we let $\sigma : [n] \rightarrow \{\pm 1\}$ be a map chosen uniformly at random and \mathbf{H} be the random hypergraph obtained by inserting each edge that is monochromatic under σ with probability p_1 and each edge that is bichromatic with probability p_2 .*

5.1. *Quiet planting.* We begin with the second part of Proposition 3.4. The following statement relates the planted model to the random hypergraph $H_k(n, m)$. A similar statement has been obtained independently by Achlioptas and Theodoropoulos [3].

LEMMA 5.1. *Let $d > 0$ and $\beta \geq 0$. Assume that there is a sequence $(\mathcal{E}_n)_{n \geq 1}$ of events such that $\limsup_{n \rightarrow \infty} \mathbb{P}[\mathbf{H} \in \mathcal{E}_n]^{1/n} < 1$. Then $\mathbb{E}[Z_\beta(H_k(n, m)) \mathbf{1}_{\mathcal{E}_n}] \leq \exp(-\Omega(n)) \mathbb{E}[Z_\beta(H_k(n, m))]$.*

PROOF. Fix $\alpha > 0$ such that $\limsup_{n \rightarrow \infty} \mathbb{P}[\mathbf{H} \in \mathcal{E}_n]^{1/n} \leq \exp(-\alpha)$. To shorten the notation, we write $H_{n,m}$ for $H_k(n, m)$. For any $\varepsilon > 0$, we have the decomposition

$$(5.1) \quad \begin{aligned} & \mathbb{E}[Z_\beta(H_{n,m}) \mathbf{1}_{\mathcal{E}_n}] \\ &= \sum_{\sigma : [n] \rightarrow \{\pm 1\}} \mathbb{E}[\exp(-\beta E_{H_{n,m}}(\sigma)) \mathbf{1}_{\mathcal{E}_n}] \\ &\leq \sum_{\sigma \in B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_{n,m}}(\sigma)) \mathbf{1}_{\mathcal{E}_n}] + \sum_{\sigma \notin B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_{n,m}}(\sigma))]. \end{aligned}$$

To bound the first summand in (5.1), we let $m_0 = \frac{2^{1-k} \exp(-\beta)}{1-2^{1-k}(1-\exp(-\beta))}m$ and define the set $M_\varepsilon = \{\mu \in [m] : |\mu - m_0| < \varepsilon n\}$. Now, for any $\mu \in [m]$ we have

$$\begin{aligned} & \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\{E_{H_{n,m}}(\sigma) = \mu\} \cap \{H_{n,m} \in \mathcal{E}_n\}] \\ &= \sum_{\sigma \in B_\varepsilon} \mathbb{P}[H_{n,m} \in \mathcal{E}_n | E_{H_{n,m}}(\sigma) = \mu] \mathbb{P}[E_{H_{n,m}}(\sigma) = \mu]. \end{aligned}$$

Under the conditions $e(\mathbf{H}) = m$ and $E_{H_{n,m}}(\sigma) = E_{\mathbf{H}}(\sigma)$ for $\sigma : [n] \rightarrow \{\pm 1\}$, the two random hypergraphs $H_{n,m}$ and \mathbf{H} are identically distributed. Therefore,

$$\begin{aligned} & \mathbb{P}[H_{n,m} \in \mathcal{E}_n | E_{H_{n,m}}(\sigma) = \mu] \\ &= \mathbb{P}[\mathbf{H} \in \mathcal{E}_n | E_{\mathbf{H}}(\sigma) = \mu, e(\mathbf{H}) = m] \leq \frac{\mathbb{P}[\mathbf{H} \in \mathcal{E}_n]}{\mathbb{P}[E_{\mathbf{H}}(\sigma) = \mu, e(\mathbf{H}) = m]}. \end{aligned}$$

By standard concentration results, there is $\varepsilon > 0$ such that

$$\mathbb{P}[E_{\mathbf{H}}(\sigma) = \mu, e(\mathbf{H}) = m] \geq \exp\left(-\frac{\alpha}{2}n\right) \quad \text{for any } \sigma \in B_\varepsilon, \mu \in M_\varepsilon.$$

Hence, for any $\mu \in M_\varepsilon$:

$$\begin{aligned} & \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\{E_{H_{n,m}}(\sigma) = \mu\} \cap \{H_{n,m} \in \mathcal{E}_n\}] \\ & \leq \exp\left(\frac{\alpha}{2}n\right) \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\mathbf{H} \in \mathcal{E}_n] \mathbb{P}[E_{H_{n,m}}(\sigma) = \mu] \end{aligned}$$

and, therefore, letting $A = 2^n(1 - 2^{1-k}(1 - \exp(-\beta)))^m$, we get

$$\begin{aligned} & \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_{n,m}}(\sigma)) \mathbf{1}_{\mathcal{E}_n}] \\ &= \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp(-\beta \mu) \mathbb{P}[\{E_{H_{n,m}}(\sigma) = \mu\} \cap \{H_{n,m} \in \mathcal{E}_n\}] \\ (5.2) \quad & \leq \exp\left(-\frac{\alpha}{2}n\right) \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp(-\beta \mu) \mathbb{P}[E_{H_{n,m}}(\sigma) = \mu] \\ & \leq A \exp\left(-\frac{\alpha}{2}n\right). \end{aligned}$$

Furthermore, Lemma 4.5 shows that there is $\delta > 0$ such that

$$(5.3) \quad \sum_{\mu \notin M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp(-\beta \mu) \mathbb{P}[E_{H_{n,m}}(\sigma) = \mu] \leq A \exp(-\delta n).$$

To bound the second summand in (5.1), we get from Lemma 4.4 that there is $\delta' > 0$ such that

$$(5.4) \quad \sum_{\sigma \notin B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_{n,m}}(\sigma))] \leq A \exp(-\delta' n).$$

Combining the estimates (5.2), (5.3) and (5.4) in the decomposition (5.1) yields

$$\mathbb{E}[Z_\beta(H_{n,m})\mathbf{1}_{\mathcal{E}_n}] \leq A \exp(-\max(\alpha/2, \delta, \delta')n).$$

The assertion follows with Lemmas 4.1 and 4.2. \square

COROLLARY 5.2. *Let $d > 0$ and $\beta \geq 0$. Assume that there exists a sequence $(\mathcal{E}_n)_{n \geq 1}$ of events such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}[H_k(n, m) \in \mathcal{E}_n] = 1 \quad \text{while} \quad \limsup_{n \rightarrow \infty} \mathbb{P}[\mathbf{H} \in \mathcal{E}_n]^{1/n} < 1.$$

Then $\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$.

PROOF. Since $Z_\beta(H_k(n, m))^{1/n} \leq 2$ and $\mathbb{P}[H_k(n, m) \in \mathcal{E}_n] = 1 - o(1)$, Jensen's inequality yields

$$\begin{aligned} \mathbb{E}[Z_\beta(H_k(n, m))^{1/n}] &= \mathbb{E}[Z_\beta(H_k(n, m))^{1/n} \mathbf{1}_{\mathcal{E}_n}] + o(1) \\ &\leq \mathbb{E}[Z_\beta(H_k(n, m)) \mathbf{1}_{\mathcal{E}_n}]^{1/n} + o(1). \end{aligned}$$

Hence, under the assumptions of the corollary we obtain with Jensen's inequality and Lemma 5.1

$$\begin{aligned} \Phi_{d,k}(\beta) &\leq \limsup_{n \rightarrow \infty} \ln \mathbb{E}[Z_\beta(H_k(n, m))^{1/n}] \\ &\leq \exp(-\Omega(1)) \limsup_{n \rightarrow \infty} \mathbb{E}[Z_\beta(H_k(n, m))]^{1/n}. \end{aligned}$$

The result then follows from Lemmas 4.1 and 4.2. \square

5.2. An unlikely event. As a next step, we establish the following.

LEMMA 5.3. *Assume that (3.5) holds for some $\beta \geq k \ln 2 - \ln k$. Then there exists $z > 0$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(H_k(n, m)) \leq z\right] = 1, \quad \limsup_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq z\right]^{1/n} < 1.$$

The proof of Lemma 5.3, to which we dedicate the rest of this subsection, is an extension of the argument from [4], Section 6, to the case of finite β . We need the following concentration result.

LEMMA 5.4. For any fixed $d > 0$, $\beta \geq 0$, $\alpha > 0$ there are $\delta > 0$, $\delta' > 0$ such that the following is true. Suppose that $(\sigma_n)_{n \geq 1}$ is a sequence of maps $[n] \rightarrow \{\pm 1\}$. Then for all large enough n ,

$$\mathbb{P}[|\ln(Z_\beta(\mathbf{H})) - \mathbb{E}[\ln Z_\beta(\mathbf{H}) | \sigma = \sigma_n]| > \alpha n | \sigma = \sigma_n] \leq \exp(-\delta n)$$

and

$$\mathbb{P}[|\ln(C_\beta(\mathbf{H}, \sigma)) - \mathbb{E}[\ln C_\beta(\mathbf{H}, \sigma) | \sigma = \sigma_n]| > \alpha n | \sigma = \sigma_n] \leq \exp(-\delta' n).$$

PROOF. This is immediate from the Lipschitz property and McDiarmid’s inequality [14], Theorem 3.8. \square

We further need several statements about quantities in the planted model conditioned on σ being some fixed (balanced) coloring.

LEMMA 5.5. Assume that (3.5) is true for some $\beta \geq k \ln 2 - \ln k$. Then there exist a fixed number $\varepsilon > 0$ and a sequence σ_n of balanced maps $[n] \rightarrow \{\pm 1\}$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) > \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) + \varepsilon | \sigma = \sigma_n\right] = 1.$$

PROOF. By Stirling’s formula, there is an n -independent number $\delta > 0$ such that for sufficiently large n we have

$$(5.5) \quad \mathbb{P}[\sigma \in \text{Bal}] \geq \delta.$$

Let $A = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$. Using (3.5), we know that there is $\varepsilon > 0$ such that $\liminf_{n \rightarrow \infty} \mathbb{P}[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) > A + 3\varepsilon] \geq 0.9$. With the concentration bound from Lemma 3.3, we get

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) > A + 2\varepsilon\right] = 1.$$

Thus, with $p_n = \liminf_{n \rightarrow \infty} \max_{\sigma_n \in \text{Bal}} \mathbb{P}[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) > A + 2\varepsilon | \sigma = \sigma_n]$ and (5.5) we get

$$\begin{aligned} 1 &\leq \liminf_{n \rightarrow \infty} \left(\sum_{\sigma_n \in \text{Bal}} \mathbb{P}\left[\frac{1}{n} \ln C_\beta(\mathbf{H}, \sigma) > A + 2\varepsilon | \sigma = \sigma_n\right] \mathbb{P}[\sigma = \sigma_n] \right. \\ &\quad \left. + \sum_{\sigma_n \notin \text{Bal}} \mathbb{P}[\sigma = \sigma_n] \right) \\ (5.6) \quad &\leq \liminf_{n \rightarrow \infty} p_n \mathbb{P}[\sigma \in \text{Bal}] + \mathbb{P}[\sigma \notin \text{Bal}] \\ &\leq \liminf_{n \rightarrow \infty} p_n + 1 - \delta, \end{aligned}$$

implying that $\liminf_{n \rightarrow \infty} p_n \geq \delta$. Thus, the concentration bound from Lemma 5.4 yields

$$\lim_{n \rightarrow \infty} \max_{\sigma_n \in \text{Bal}} \mathbb{P} \left[\frac{1}{n} \ln \mathcal{C}_\beta(\mathbf{H}, \sigma) > A + \varepsilon \mid \sigma = \sigma_n \right] = 1$$

completing the proof. \square

LEMMA 5.6. *For any $\eta > 0$, there is $\delta > 0$ such that*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{P}[|\sigma^{-1}(1)| - n/2 > \eta n] \leq -\delta.$$

PROOF. This is immediate from the Chernoff bound. \square

For a set $S \subset V$ let $\text{Vol}(S|H)$ be the sum of the degrees of the vertices in S in the hypergraph H .

LEMMA 5.7. *For any $\gamma > 0$, there is $\alpha > 0$ such that for any set $S \subset [n]$ of size $|S| \leq \alpha n$ and any map $\sigma : [n] \rightarrow \{\pm 1\}$ we have $\limsup \frac{1}{n} \ln \mathbb{P}[\text{Vol}(S|\mathbf{H}) \geq \gamma n \mid \sigma = \sigma] \leq -\alpha$.*

PROOF. Let $(X_v)_{v \in [n]}$ be a family of independent random variables with distribution $\text{Bin}(\binom{n-1}{k-1}, 2p)$. Then for any σ and any $S \subset [n]$ the volume $\text{Vol}(S|\mathbf{H})$ is stochastically dominated by $X_S = 2k \sum_{v \in S} X_v$. Furthermore, $\mathbb{E}[X_S] = 4dk|S|$. Thus, for any $\gamma > 0$ we can choose an n -independent $\alpha > 0$ such that for any $S \subset [n]$ of size $|S| \leq \alpha n$ we have $\mathbb{E}[X_S] \leq \gamma n/2$. In fact, the Chernoff bound shows that by picking $\alpha > 0$ sufficiently small, we can ensure that $\mathbb{P}[\text{Vol}(S|\mathbf{H}) \geq \gamma n \mid \sigma = \sigma] \leq \mathbb{P}[X_S \geq \gamma n] \leq \exp(-\alpha n)$, as desired. \square

LEMMA 5.8. *Let $d > 0$ and $\beta \geq 0$. Assume that there exist numbers $z > 0$, $\varepsilon > 0$ and a sequence $(\sigma_n)_{n \geq 1}$ of balanced maps $[n] \rightarrow \{\pm 1\}$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_\beta(\mathbf{H}) \mid \sigma = \sigma_n] > z + \varepsilon.$$

Then $\limsup_{n \rightarrow \infty} \mathbb{P}[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq z]^{1/n} < 1$.

PROOF. Suppose that n is large enough so that $\frac{1}{n} \mathbb{E}[\ln Z_\beta(\mathbf{H}) \mid \sigma = \sigma_n] > z + \varepsilon/2$. Set $n_i = |\sigma_n^{-1}(i)|$ and let T be the set of all $\tau : [n] \rightarrow \{\pm 1\}$ such that $|\tau^{-1}(i)| = n_i$ for $i = \pm 1$. As Z_β is invariant under permutations of the vertices, we have

$$(5.7) \quad \frac{1}{n} \mathbb{E}[\ln Z_\beta(\mathbf{H}) \mid \sigma = \tau] = \frac{1}{n} \mathbb{E}[\ln Z_\beta(\mathbf{H}) \mid \sigma = \sigma_n] > z + \varepsilon/2$$

for any $\tau \in T$.

Let $\gamma = \varepsilon/(4\beta) > 0$. By Lemma 5.7, there exists $\alpha > 0$ such that for large enough n for any set $S \subset V$ of size $|S| \leq \alpha n$ and any $\sigma : [n] \rightarrow \{\pm 1\}$ we have

$$(5.8) \quad \mathbb{P}\left[\text{Vol}(S|\mathbf{H}) < \frac{\gamma n}{2} \mid \sigma = \sigma\right] \geq 1 - \exp(-\alpha n).$$

Fix such an $\alpha > 0$, and pick and fix a small $0 < \eta < \alpha/3$. By Lemma 5.6, there exists an (n -independent) number $\delta = \delta(\beta, \varepsilon, \eta) > 0$ such that

$$(5.9) \quad \mathbb{P}[\sigma \in B_\eta] \geq 1 - \exp(-\delta n).$$

Because σ_n is balanced, we have $|n_i - n/2| \leq \sqrt{n}$ for $i = \pm 1$. Therefore, if $\sigma \in B_\eta$, then it is possible to obtain from σ a map $\tau_\sigma \in T$ by changing the colors of at most $2\eta n$ vertices. Hence, if $\sigma \in B_\eta$ we let \mathbf{H}_{τ_σ} be the random hypergraph with planted coloring τ_σ . Further, let \mathbf{H}_σ be the hypergraph obtained by removing from \mathbf{H}_{τ_σ} each edge that is monochromatic under σ but not under τ_σ with probability $1 - \exp(-\beta)$ independently and inserting each edge that is monochromatic under τ_σ but not under σ with probability $(1 - \exp(-\beta))p_2$ independently. Then $\mathbf{H}_\sigma = \mathbf{H}$ in distribution.

Let S_σ be the set of vertices v with $\sigma(v) \neq \tau_\sigma(v)$. Our choice of η ensures that $|S_\sigma| < \alpha n$. Let Δ be the number of edges present in \mathbf{H}_{τ_σ} but not in \mathbf{H}_σ or vice versa. Then $\Delta \leq \text{Vol}(S_\sigma | \mathbf{H}_{\tau_\sigma}) + \text{Vol}(S_\sigma | \mathbf{H}_\sigma)$. Hence, with (5.8) there exists a constant $c > 0$ such that

$$(5.10) \quad \mathbb{P}[\Delta \leq \gamma n \mid \sigma \in B_\eta] \geq 1 - c \exp(-\alpha n).$$

Using (5.9), (5.10) and the fact that removing a single edge can reduce $\frac{1}{n} \ln Z_\beta$ by at most β/n , we obtain

$$(5.11) \quad \begin{aligned} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq z\right] &= \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_\sigma) \leq z\right] \\ &\leq \exp(-\delta n) + \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_\sigma) \leq z \mid \sigma \in B_\eta\right] \\ &\leq \exp(-\delta n) + c \exp(-\alpha n) \\ &\quad + \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_\sigma) \leq z \mid \sigma \in B_\eta, \Delta \leq \gamma n\right] \\ &\leq \exp(-\delta n) + c \exp(-\alpha n) \\ &\quad + \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_{\tau_\sigma}) - \gamma\beta \leq z \mid \sigma \in B_\eta, \Delta \leq \gamma n\right]. \end{aligned}$$

By the choice of γ , (5.9), (5.10) and (5.7), we have

$$\begin{aligned} &\mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_{\tau_\sigma}) - \gamma\beta \leq z \mid \sigma \in B_\eta, \Delta \leq \gamma n\right] \\ &\leq 2\mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}_{\tau_\sigma}) \leq z + \frac{\varepsilon}{4} \mid \sigma \in B_\eta\right] \end{aligned}$$

$$\begin{aligned} &\leq 3\mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq z + \frac{\varepsilon}{4} \mid \boldsymbol{\sigma} = \sigma_n\right] \\ &\leq 3\mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq \frac{1}{n} \mathbb{E}[\ln Z_\beta(\mathbf{H}) \mid \boldsymbol{\sigma} = \sigma_n] - \frac{\varepsilon}{4} \mid \boldsymbol{\sigma} = \sigma_n\right]. \end{aligned}$$

The assertion follows by combining this with (5.11) and Lemma 5.4. \square

PROOF OF LEMMA 5.3. Lemma 5.5 shows that there exist $\varepsilon > 0$ and balanced maps $\sigma_n : [n] \rightarrow \{\pm 1\}$ such that

$$(5.12) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln \mathcal{C}_\beta(\mathbf{H}, \boldsymbol{\sigma}) \geq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) + \varepsilon \mid \boldsymbol{\sigma} = \sigma_n\right] \\ = 1. \end{aligned}$$

Clearly, (5.12) implies that

$$(5.13) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \geq \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) + \varepsilon \mid \boldsymbol{\sigma} = \sigma_n\right] \\ = 1. \end{aligned}$$

Hence, with $z = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta))) + \varepsilon/2$, Lemma 5.8 and (5.13) yield

$$(5.14) \quad \limsup_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\mathbf{H}) \leq z\right]^{1/n} < 1.$$

By comparison, Lemma 4.1 and Lemma 4.2 imply

$$(5.15) \quad \lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(H_k(n, m)) \leq z\right] = 1.$$

Thus, the assertion follows from (5.14) and (5.15). \square

5.3. Tame colorings. To facilitate the proof of the first part of Proposition 3.4, we introduce a random variable that explicitly controls the ‘‘cluster size’’ $\mathcal{C}_\beta(H_k(n, m), \boldsymbol{\sigma})$. The idea of explicitly controlling the cluster size was introduced in [6] in the ‘‘zero temperature’’ case, and here we generalise it to the case of finite β . More precisely, we call $\sigma : [n] \rightarrow \{\pm 1\}$ *tame* in H if σ is balanced and if $\mathcal{C}_\beta(H, \sigma) \leq \mathbb{E}[Z_\beta(H)]$. Now, let

$$Z_{\beta, \text{tame}}(H_k(n, m)) = \sum_{\sigma : [n] \rightarrow \{-1, 1\}} \exp(-\beta E_{H_k(n, m)}(\sigma)) \cdot \mathbf{1}_{\sigma \text{ is tame}}.$$

LEMMA 5.9. *Assume that $0 \leq d/k \leq 2^{k-1} \ln 2 + O_k(1)$ is such that $\liminf_{n \rightarrow \infty} \frac{\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]}{\mathbb{E}[Z_\beta(H_k(n, m))]} > 0$. Then*

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]^2}{\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]^2} > 0.$$

PROOF. The proof is based on a second moment argument. Mimicking the notation of Section 4.2, we let

$$Z_{\beta, \text{tame}}(\alpha) = \sum_{\sigma, \tau: \langle \sigma, \tau \rangle = \alpha n} \exp(-\beta(E_{H_k(n, m)}(\sigma) + E_{H_k(n, m)}(\tau))) \cdot \mathbf{1}_{\sigma \text{ is tame}} \cdot \mathbf{1}_{\tau \text{ is tame}}.$$

Then it is clear that

$$\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))^2] = \sum_{\nu=0}^n \mathbb{E}[Z_{\beta, \text{tame}}(2\nu/n - 1)].$$

Furthermore, we have $Z_{\beta, \text{tame}}(\alpha) \leq Z_{\beta}(\alpha)$ for any α . We define $I = [-1 + 2^{-3k/4}, 1 - 2^{-3k/4}]$. Lemma 4.8 and Lemma 4.7 yield

$$(5.16) \quad \sum_{\alpha \in I} \mathbb{E}[Z_{\beta}(\alpha)] = O(\mathbb{E}[Z_{\beta}(H_k(n, m))]^2).$$

By the definition of “tame” we have

$$\begin{aligned} & \sum_{\alpha > 1 - 2^{-3k/4}} \mathbb{E}[Z_{\beta, \text{tame}}(\alpha)] \\ (5.17) \quad & \leq \mathbb{E} \left[\sum_{\sigma} \exp(-\beta E_{H_k(n, m)}(\sigma)) \cdot \mathbf{1}_{\sigma \text{ is tame}} \cdot C_{\beta}(H_k(n, m), \sigma) \right] \\ & \leq \mathbb{E} \left[\sum_{\sigma} \exp(-\beta E_{H_k(n, m)}(\sigma)) \cdot \mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))] \right] \\ & = O(\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]^2). \end{aligned}$$

Moreover, $\sum_{\alpha < -1 + 2^{-3k/4}} \mathbb{E}[Z_{\beta, \text{tame}}(\alpha)] = \sum_{\alpha > 1 - 2^{-3k/4}} \mathbb{E}[Z_{\beta, \text{tame}}(\alpha)]$ by symmetry. Hence, $\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))^2] = O(\mathbb{E}[Z_{\beta}(H_k(n, m))]^2)$ by equations (5.16) and (5.17).

Finally, the assertion follows from our assumption that $\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))] = \Omega(\mathbb{E}[Z_{\beta}(H_k(n, m))])$. \square

LEMMA 5.10. *Let $d > 0$ and $\beta \geq 0$ and assume that we have*

$$\limsup_{n \rightarrow \infty} \mathbb{P}[\sigma \text{ is not tame in } \mathbf{H}]^{1/n} < 1.$$

Then there is $c > 0$ such that $\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))] \geq \mathbb{E}[Z_{\beta}(H_k(n, m))]/c$.

PROOF. The proof is very similar to the proof of Lemma 5.1. We fix an $\alpha > 0$ such that $\limsup_{n \rightarrow \infty} \mathbb{P}[\sigma \text{ is not tame in } \mathbf{H}]^{1/n} \leq \exp(-\alpha) < 1$. For any $\varepsilon > 0$, we

have

$$\begin{aligned} & \mathbb{E}[Z_\beta(H_k(n, m)) - Z_{\beta, \text{tame}}(H_k(n, m))] \\ &= \sum_{\sigma: [n] \rightarrow \{\pm 1\}} \mathbb{E}[\exp(-\beta E_{H_k(n, m)}(\sigma)) \mathbf{1}_{\sigma \text{ is not tame in } H_k(n, m)}] \\ &\leq \sum_{\sigma \in B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_k(n, m)}(\sigma)) \mathbf{1}_{\sigma \text{ is not tame in } H_k(n, m)}] \\ &\quad + \sum_{\sigma \notin B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_k(n, m)}(\sigma))]. \end{aligned}$$

With m_0 and M_ε as in the proof of Lemma 5.1 and $\mathcal{A}(\sigma, \mu)$ the event $\{E_{\mathbf{H}}(\sigma) = \mu, e(\mathbf{H}) = m, |\sigma^{-1}(1)| = |\sigma^{-1}(1)|\}$, we fix an $\varepsilon > 0$ such that $\mathbb{P}[\mathcal{A}(\sigma, \mu)] > \exp(-\frac{\alpha}{2}n)$ for all $\sigma \in B_\varepsilon, \mu \in M_\varepsilon$. Then for any $\mu \in M_\varepsilon$:

$$\begin{aligned} & \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\{E_{H_k(n, m)}(\sigma) = \mu\} \cap \{\sigma \text{ is not tame in } H_k(n, m)\}] \\ &= \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\sigma \text{ is not tame in } H_k(n, m) | E_{H_k(n, m)}(\sigma) = \mu] \mathbb{P}[E_{H_k(n, m)}(\sigma) = \mu] \\ &= \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\sigma \text{ is not tame in } \mathbf{H} | \mathcal{A}(\sigma, \mu)] \mathbb{P}[E_{H_k(n, m)}(\sigma) = \mu] \\ &\leq \sum_{\sigma \in B_\varepsilon} \frac{\mathbb{P}[\sigma \text{ is not tame in } \mathbf{H}]}{\mathbb{P}(\mathcal{A}(\sigma, \mu))} \mathbb{P}[E_{H_k(n, m)}(\sigma) = \mu] \\ &\leq \exp\left(-\frac{\alpha}{2}n\right) \sum_{\sigma \in B_\varepsilon} \mathbb{P}[E_{H_k(n, m)}(\sigma) = \mu]. \end{aligned}$$

Letting $A = 2^n(1 - 2^{1-k}(1 - \exp(-\beta)))^m$, we get

$$\begin{aligned} & \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_k(n, m)}(\sigma)) \mathbf{1}_{\sigma \text{ is not tame in } H_k(n, m)}] \\ (5.18) \quad &= \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp(-\beta \mu) \mathbb{P}[\{E_{H_k(n, m)}(\sigma) = \mu\} \\ &\quad \cap \{\sigma \text{ is not tame in } H_k(n, m)\}] \leq A \exp\left(-\frac{\alpha}{2}n\right). \end{aligned}$$

Furthermore Lemma 4.5 shows that there is $\delta > 0$ such that

$$(5.19) \quad \sum_{\mu \notin M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp(-\beta \mu) \mathbb{P}[E_{H_k(n, m)}(\sigma) = \mu] \leq A \exp(-\delta n)$$

and we get from Lemma 4.4 that there is $\delta' > 0$ such that

$$(5.20) \quad \sum_{\sigma \notin B_\varepsilon} \mathbb{E}[\exp(-\beta E_{H_k(n, m)}(\sigma))] \leq A \exp(-\delta' n).$$

Combining the estimates (5.18), (5.19) and (5.20) and using Lemmas 4.1 and 4.2 yields

$$\begin{aligned} \mathbb{E}[Z_\beta(H_k(n, m)) - Z_{\beta, \text{tame}}(H_k(n, m))] &\leq A \exp(-\max(\alpha/2, \delta, \delta')n) \\ &\leq \exp(-\Omega(n))\mathbb{E}[Z_\beta(H_k(n, m))], \end{aligned}$$

which proves the assertion. \square

COROLLARY 5.11. *Assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and that $\beta_0 \geq k \ln 2 - \ln k$ is such that (3.4) holds for all $k \ln 2 - \ln k \leq \beta \leq \beta_0$. Then $\beta_{\text{crit}}(d, k) \geq \beta_0$.*

The proof of this corollary extends a “zero temperature” argument from [4], Section 5, to the case of $\beta \in [0, \infty)$.

PROOF OF COROLLARY 5.11. Assume for contradiction that β_0 is such that (3.4) holds for all $k \ln 2 - \ln k \leq \beta \leq \beta_0$ but $\beta_{\text{crit}}(d, k) < \beta_0$. By Corollary 3.2, we have $\beta_{\text{crit}}(d, k) \geq k \ln 2 - \ln k$. We pick and fix a number $\beta_{\text{crit}}(d, k) < \beta < \beta_0$. We let $A = \ln 2 + \frac{d}{k} \ln(1 - 2^{1-k}(1 - \exp(-\beta)))$. There exists $\varepsilon > 0$ such that

$$(5.21) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_\beta H_k(n, m)] < A - \varepsilon.$$

On the other hand, (3.4) and Lemma 3.3 ensure that we can apply Lemma 5.10 and find a number $c > 0$ such that

$$(5.22) \quad \mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))] \geq c \cdot \mathbb{E}[Z_\beta(H_k(n, m))].$$

Hence, $\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))^2] = O(\mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]^2)$ by Lemma 5.9. Using the Paley–Zygmund inequality, there is a number $C > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}[Z_{\beta, \text{tame}}(H_k(n, m)) \geq \mathbb{E}[Z_{\beta, \text{tame}}(H_k(n, m))]/2] \geq 1/C > 0.$$

With (5.22) and because $c/2 \cdot \mathbb{E}[Z_\beta(H_k(n, m))] > \exp(nA - n\varepsilon/3)$ we see that

$$\liminf_{n \rightarrow \infty} \mathbb{P}[Z_{\beta, \text{tame}}(H_k(n, m)) \geq \exp(nA - n\varepsilon/3)] > 0.$$

With Lemma 2.3, it follows that

$$\lim_{n \rightarrow \infty} \mathbb{P}[Z_{\beta, \text{tame}}(H_k(n, m)) \geq \exp(nA - 2n\varepsilon/3)] = 1.$$

With (5.21), we get the contradiction

$$A - \varepsilon > \liminf_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[\ln Z_{\beta, \text{tame}}(H_k(n, m))] \geq A - 2\varepsilon/3$$

which refutes our assumption that $\beta_{\text{crit}}(d, k) < \beta_0$. \square

PROOF OF PROPOSITION 3.4. The proposition is immediate from Corollary 5.2 combined with Lemma 5.3 and from Corollary 5.11. \square

6. The cluster size. *In this section, we prove Proposition 3.5. Throughout the section, we assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and that $\beta \geq k \ln 2 - \ln k$.*

In order to analyse the cluster size, we will show that there is a large set of vertices (the “core”) whose value cannot be changed without creating a large number of monochromatic edges. Hence, the contribution of these vertices to the cluster size can be controlled. Then we analyze the contribution of the remaining vertices.

The proof strategy broadly follows the argument for estimating the cluster size in the “zero temperature” case from [6]. However, the fact that we are dealing with a finite β causes significant complications. More precisely, one of the key features of the “zero temperature” case is the existence of “frozen variables”, that is, vertices that take the same color in all colorings in the cluster. Indeed, in the zero temperature case the problem of estimating the cluster size basically reduces to estimating the number of “frozen variables”. By contrast, in the case of finite β , frozen variables do not exist. In effect, we need to take a much closer look.

We let $\sigma : [n] \rightarrow \{\pm 1\}$ be a map chosen uniformly at random conditioned on the event that $\sigma \in \text{Bal}$ and \mathbf{H} be the random hypergraph obtained by inserting each edge that is monochromatic under σ with probability p_1 and each edge that is bichromatic with probability p_2 .

We say that a vertex v supports an edge $e \ni v$ under σ if $\sigma(e \setminus \{v\}) = \{-\sigma(v)\}$. In this case, we call e *critical*. Moreover, if $U \subset [n]$, then we say that an edge e of \mathbf{H} is U -endangered if $|\sigma(U \cap e)| = 1$ (i.e., the vertices in $U \cap e$ all have the same color).

For the first three subsections of this section, it will be convenient to introduce a slightly more general construction. Let $\omega \geq 0$ be fixed and let v_1, \dots, v_ω be vertices chosen uniformly at random without replacement from all vertices in \mathbf{H} . Let \mathbf{H}' be the hypergraph obtained from \mathbf{H} by removing v_1, \dots, v_ω and edges e involving one of these vertices. Without loss of generality, we can assume that $\{v_1, \dots, v_\omega\} = \{n - \omega + 1, \dots, n\}$. The edge set of \mathbf{H}' is thus $[n']$, with $n' = n - \omega$.

6.1. The core. Let $\text{core}(\mathbf{H}, \sigma)$ be the maximal set $V' \subset [n]$ of vertices such that the following two conditions hold.

CR1 Each vertex $v \in V'$ supports at least 100 edges that consist of vertices from V' only.

CR2 No vertex $v \in V'$ occurs in more than 10 edges that are V' -endangered under σ .

If V', V'' are sets that satisfy **CR1–CR2**, then so does $V' \cup V''$. Hence, the core is well-defined.

PROPOSITION 6.1. *A.a.s. $|\text{core}(\mathbf{H}, \sigma)| = n(1 - \tilde{O}_k(2^{-k}))$.*

To prove this proposition, we consider the following *whitening process* on the graph \mathbf{H}' whose result U is such that its complement $\bar{U} = [n'] \setminus U$ is a subset of $\text{core}(\mathbf{H}', \sigma)$.

WH1 Let W contain all vertices of \mathbf{H}' that either support fewer than 200 edges or that occur in more than 2 edges that are monochromatic under σ .

WH2 Let $U = W$ initially. While there is a vertex $v \in [n'] \setminus U$ such that:

- v occurs in more than 5 edges that are $[n'] \setminus U$ -endangered and contain a vertex from U , or
- v supports fewer than 150 edges containing vertices in $[n'] \setminus U$ only,

add v to U .

Proposition 6.1 will be a consequence of the following lemma, by taking $\omega = 0$ and noticing that $\text{core}(\mathbf{H}', \sigma)$ is a superset of the set \bar{U} .

LEMMA 6.2. *Let U be the outcome of the process **WH1–WH2** on \mathbf{H}' . Then $|U| = n' \tilde{O}_k(2^{-k})$ a.a.s.*

The rest of this subsection is dedicated to the proof of this lemma. We first bound the size of the set W generated by **WH1**.

LEMMA 6.3. *A.a.s. the set W contains $n' \tilde{O}_k(2^{-k})$ vertices.*

PROOF. Our assumptions on β and d ensure that the number of monochromatic edges that any fixed vertex v occurs in is binomially distributed with mean $\tilde{O}_k(2^{-k})$. Therefore, the probability that v occurs in more than 2 monochromatic edges is bounded by $\tilde{O}_k(2^{-2k})$. Furthermore, the number of edges that v supports is binomially distributed with mean $k \ln 2 + O_k(1)$. Hence, by the Chernoff bound the probability that v supports fewer than 200 edges is bounded by $\tilde{O}_k(2^{-k})$. Consequently,

$$(6.1) \quad \mathbb{E}[|W|] = n' \tilde{O}_k(2^{-k}).$$

Finally, either adding or removing a single edge from the hypergraph can alter the size of W by at most k . Therefore, (6.1) and Azuma’s inequality imply that $|W| = n' \tilde{O}_k(2^{-k})$ a.a.s., as desired. \square

In the next step, we state two results excluding some properties of small sets of vertices in \mathbf{H}' .

LEMMA 6.4. *A.a.s. the random hypergraph \mathbf{H}' enjoys the following property:*

$$(6.2) \quad \text{There is no set } T \neq \emptyset \text{ of vertices with } |T| \leq n'/k^8 \text{ such that at least } 0.9|T| \text{ vertices from } T \text{ occur in two or more } [n'] \setminus T \text{-endangered edges that contain another vertex from } T.$$

PROOF. For a set $T \subset [n']$ we define $\varepsilon = |T|/n'$ and we let $X_i(T)$ for $i \in \{2, \dots, k\}$ be the number of edges that are $[n'] \setminus T$ -endangered and contain exactly i vertices from T . Then $X_i(T)$ is stochastically dominated by a binomial random variable $\text{Bin}((1 + o(1))2^{i+1-k} \binom{\varepsilon n'}{i} \binom{n'}{k-i}, 2p)$. Indeed, there are $\binom{\varepsilon n'}{i}$ ways to choose i vertices from T and at most $\binom{(1-\varepsilon)n'}{k-i} \leq \binom{n'}{k-i}$ ways to choose $k - i$ vertices from $[n'] \setminus T$. Moreover, these $k - i$ vertices are required to have the same color and because we assumed that σ is balanced, this gives rise to the $(1 + o(1))2^{i+1-k}$ -factor. Let $X(T) = \sum_{i=2}^k X_i(T)$ be the total number of edges that are $[n'] \setminus T$ -endangered and contain at least two vertices from T . Then using the rough upper bound $\binom{n'}{k} 2p \leq n2^k \ln 2$ we obtain

$$(6.3) \quad \mathbb{E}[X(T)] = \sum_{i=2}^k \mathbb{E}[X_i(T)] \leq k\mathbb{E}[X_2(T)] \leq 3.6k^3 \varepsilon^2 n'.$$

Let $\mathcal{E}(T)$ be the event that $X(T) \geq 1.8|T|$. If the set T satisfies (6.2) then $\mathcal{E}(T)$ occurs. The Chernoff bound (Lemma 2.1) and the above upper bound (6.3) on $\mathbb{E}[X(T)]$ yield

$$\mathbb{P}[\mathcal{E}(T)] \leq \exp\left(-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right).$$

Hence, the probability of the event \mathcal{E} that there is a set T of size $|T| \leq n'/k^8$ such that $\mathcal{E}(T)$ occurs is bounded by

$$\begin{aligned} \mathbb{P}[\mathcal{E}] &\leq \sum_{T:|T| \leq n'/k^8} \mathbb{P}[\mathcal{E}(T)] \leq \sum_{1/n' \leq \varepsilon \leq 1/k^8} \binom{n'}{\varepsilon n'} \exp\left(-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right) \\ &\leq \sum_{1/n' \leq \varepsilon \leq 1/k^8} \left(\frac{2en'}{\varepsilon n'}\right)^{\varepsilon n'} \exp\left(-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right) \\ &\leq \sum_{1/n' \leq \varepsilon \leq 1/k^8} \exp(\varepsilon n'(5 + 5.6\ln(k) + 0.8\ln(\varepsilon))) = o(1), \end{aligned}$$

as claimed. \square

LEMMA 6.5. *A.a.s. the random hypergraph \mathbf{H}' enjoys the following property:*

(6.4) *There is no set $T \neq \emptyset$ of vertices of size $|T| \leq n'/k^6$ such that at least $0.09|T|$ vertices from T support at least 20 edges that contain another vertex from T .*

PROOF. For a set $T \subset [n']$ and a set $Q \subset [T]$, we let $\mathcal{E}(T, Q)$ be the event that each vertex $v \in Q$ supports at least 20 edges that contain another vertex from T . Let $\varepsilon = |T|/n'$. Then for each vertex v the number X_v of edges that v supports and that contain another vertex from T is stochastically dominated by a binomial

random variable $\text{Bin}((1 + o(1))2^{2-k}\varepsilon n' \binom{n'}{k-2}, p_2)$. Indeed, there are $\varepsilon n' - 1$ ways to choose another vertex $v' \neq v$ from T , and at most $\binom{n'}{k-2}$ ways to choose $k - 2$ further vertices to complete the edges. Moreover, these $k - 2$ vertices are required to have color $-\sigma(v)$, and because we assumed that σ is balanced this gives rise to the $(1 + o(1))2^{2-k}$ -factor. Furthermore, the random variables X_v are mutually independent, because the edges in question are distinct as they are supported by the distinguished vertex v . Therefore, using the rough upper bound $\binom{n'}{k} p_2 \leq n 2^k \ln 2$, we obtain

$$\begin{aligned}
 \mathbb{P}[\mathcal{E}(T, Q)] &\leq \prod_{v \in Q} \mathbb{P}[X_v \geq 20] \\
 (6.5) \quad &\leq \mathbb{P}\left[\text{Bin}\left((1 + o(1))2^{2-k}\varepsilon n' \binom{n'}{k-2}, p_2\right) \geq 20\right]^{|Q|} \\
 &\leq (k^2\varepsilon)^{20|Q|}.
 \end{aligned}$$

Now, let $\mathcal{E}(T)$ be the event that there is a set $Q \subset [T]$ of size $|Q| \geq 0.09|T|$ such that $\mathcal{E}(T, Q)$ occurs. Then (6.5) implies that

$$\mathbb{P}[\mathcal{E}(T)] \leq 2^{|T|} (k^2|T|/n')^{1.8|T|}.$$

Hence, the probability of the event \mathcal{E} that there is a set T of size $|T| \leq n'/k^6$ such that $\mathcal{E}(T)$ occurs is bounded by

$$\begin{aligned}
 \mathbb{P}[\mathcal{E}] &\leq \sum_{T:|T| \leq n'/k^6} \mathbb{P}[\mathcal{E}(T)] \leq \sum_{1 \leq t \leq n'/k^6} \binom{n'}{t} 2^t (k^2 t/n')^{1.8t} \\
 &\leq \sum_{1 \leq t \leq n'/k^6} \left(\frac{2\varepsilon n'}{t}\right)^t (k^2 t/n')^{1.8t} \leq \sum_{1 \leq t \leq n'/k^6} [2\varepsilon(t/n')^{0.8} k^{3.6}]^t = o(1),
 \end{aligned}$$

as claimed. \square

PROOF OF LEMMA 6.2. By Lemmas 6.4 and 6.5, we may assume that \mathbf{H}' enjoys the properties (6.2) and (6.4). We are going to argue that $|U| \leq k|W|$ a.a.s. Indeed, assume for contradiction that $|U| > k|W|$ and let U' be the set obtained by **WH2** when precisely $(k - 1)|W|$ vertices have been added to U ; thus, $|U'| = k|W|$. Then by construction each vertex $v \in U'$ has one of the following properties:

- (1) v belongs to W ,
- (2) or v occurs in two or more $[n'] \setminus U'$ -endangered edges,
- (3) or v supports at least 20 edges that contain another vertex from U' .

Let $U_0 \subset U'$ be the set of all $v \in U'$ that satisfy (1), let $U_1 \subset U' \setminus U_0$ be the set of all $v \in U' \setminus U_0$ that satisfy (2) and let $U_2 = U' \setminus (U_0 \cup U_1)$. There are two cases to consider.

Case 1. $|U_1| \geq 0.9|U'|$ then (6.2) implies that $|U'| > n'/k^8$.

Case 2. $|U_1| < 0.9|U'|$ then $|U_0| + |U_2| \geq 0.1|U'|$ and since $|U_0| = |W|$ and $|U'| = k|W|$ we have $|U_2| \geq 0.09|U'|$ for k large enough. Thus, (6.4) entails that $|U'| > n'/k^6$.

Hence, in either case we have $k|W| = |U'| > n'/k^8$, and thus $|W| > n'/k^9$. But by Lemma 6.3 we have $|W| = n' \tilde{O}_k(2^{-k})$ a.a.s. Thus, we conclude that $|U| \leq k|W| = n' \tilde{O}_k(2^{-k})$ a.a.s. \square

6.2. *The backbone.* We define the *backbone* $\text{back}(\mathbf{H}, \sigma)$ as the set of all vertices $v \in [n] \setminus \text{core}(\mathbf{H}, \sigma)$ such that the following two conditions hold.

BB1 v supports at least one edge e such that $e \setminus \{v\} \subset \text{core}(\mathbf{H}, \sigma)$.

BB2 v does not occur in a $\{v\} \cup \text{core}(\mathbf{H}, \sigma)$ -endangered edge.

Given \mathbf{H}' , we simply reconstruct \mathbf{H} (in distribution) by adding for each $i \in [\omega]$ each monochromatic edge involving v_i with probability p_1 , and each bichromatic edge involving v_i with probability p_2 . We let \mathcal{A} be the event that:

- no vertex $v \in [n']$ is incident with more than one edge containing a vertex from $\{v_1, \dots, v_\omega\}$, and
- there is no edge containing two vertices from $\{v_1, \dots, v_\omega\}$.

With the notation from the previous subsection we let \bar{U} be the complement of the set of vertices produced by the whitening process **WH1–WH2** applied to the hypergraph \mathbf{H}' . We note that $|\bar{U}| = n'(1 - \tilde{O}_k(2^{-k}))$ a.a.s. by Lemma 6.2. In addition, if \mathcal{A} occurs, then $\bar{U} \subset \text{core}(\mathbf{H}, \sigma)$. In this case, the following lemma states the probabilities for some events concerning the vertices $v_i, i \in [\omega]$.

LEMMA 6.6. *Assume that \mathcal{A} holds. Let $l \geq 0$ be fixed. Then the following statements are true for all $i \in [\omega]$:*

- (1) *The probability that v_i supports exactly l edges is $(1 + o(1)) \frac{\lambda^l}{l! \exp(\lambda)}$ where*

$$\lambda = \frac{d}{2^{k-1} - 1 + \exp(-\beta)} = k \ln 2 + \tilde{O}_k(2^{-k}).$$

- (2) *The probability that v_i occurs in exactly l monochromatic edges is $(1 + o(1)) \frac{(\lambda')^l}{l! \exp(\lambda')}$ where $\lambda' = \tilde{O}_k(2^{-k})$.*

- (3) *The probability that there exist exactly l edges blocking v_i and containing at least one vertex outside $\{v_i\} \cup \bar{U}$ is $(1 + o(1)) \frac{(\lambda'')^l}{l! \exp(\lambda'')}$ where $\lambda'' = \tilde{O}_k(2^{-k})$.*

- (4) *The probability that exactly l edges are $\{v_i\} \cup \bar{U}$ -endangered is $(1 + o(1)) \frac{(\lambda''')^l}{l! \exp(\lambda''')}$ where $\lambda''' = \tilde{O}_k(2^{-k})$.*

PROOF. For each $i \in [\omega]$ the number of edges that v_i supports is a binomial random variable $\text{Bin}(\binom{n-1}{k-1}(1 + o(1))2^{1-k}, p_2)$ and the number of monochromatic edges involving v_i is a binomial random variable $\text{Bin}(\binom{n-1}{k-1}(1 + o(1))2^{1-k}, p_1)$. Indeed, because we assumed that σ is balanced, there are $\binom{n-1}{k-1}(1 + o(1))2^{1-k}$ edges e involving v_i such that $\sigma(v) = -\sigma(v_i)$ [resp., $\sigma(v) = \sigma(v_i)$] for all $v \in e \setminus \{v_i\}$ and each of them is added independently at random with probability p_2 (resp., p_1). Hence, the Poisson approximation of the binomial distribution shows that the probability that v_i supports precisely l edges is $(1 + o(1))\frac{\lambda^l}{l! \exp(\lambda)}$ with

$$\lambda = \binom{n-1}{k-1} \frac{p_2}{2^{k-1}} = \frac{d}{2^{k-1} - 1 + \exp(-\beta)},$$

which proves assertion (1). Moreover, since $\beta = \Omega_k(k \ln 2)$ and $d = \tilde{O}_k(2^k)$, the probability that v_i occurs in precisely l monochromatic edges is $(1 + o(1))\frac{(\lambda')^l}{l! \exp(\lambda')}$ with

$$\lambda' = \binom{n-1}{k-1} \frac{p_1}{2^{k-1}} = \lambda \tilde{O}_k(2^{-k}) = \tilde{O}_k(2^{-k}).$$

This implies assertion (2).

The probability that in an edge blocking v_i at least one of the vertices is outside $\{v_i\} \cup \bar{U}$ is $\tilde{O}_k(2^{-k})$ by Lemma 6.2. Using (1), the number of edges blocking v_i and containing at least one vertex outside $\{v_i\} \cup \bar{U}$ is stochastically dominated by a $\text{Bin}(\binom{n-1}{k-1}\tilde{O}_k(4^{-k}), p_2)$ random variable. (3) then follows by the Poisson approximation.

If an edge e is $\{v_i\} \cup \bar{U}$ -endangered it is either monochromatic or such that $|(e \setminus \{v_i\}) \cap \bar{U}| \leq k - 2$. Given \mathbf{H}' , these two events are independent and the numbers of edges of each type are binomially distributed. The expected number of edges of the first type is $\tilde{O}_k(2^{-k})$ by (2). The expected number of edges of the second type is $\tilde{O}_k(2^{-k})$ by Lemma 6.3. Thus, (4) follows again from the Poisson approximation. □

6.3. *The rest.* Let $\text{rest}(\mathbf{H}, \sigma) = [n] \setminus (\text{core}(\mathbf{H}, \sigma) \cup \text{back}(\mathbf{H}, \sigma))$.

PROPOSITION 6.7. *A.a.s.* $|\text{rest}(\mathbf{H}, \sigma)| = n2^{-k}(1 + \tilde{O}_k(2^{-k}))$.

PROOF. $\text{rest}(\mathbf{H}, \sigma)$ contains at least all vertices that do not support an edge. Because the number of edges that a vertex supports is binomially distributed with mean $k \ln 2 + O_k(1)$, by the Chernoff bound we have $|\text{rest}(\mathbf{H}, \sigma)| \geq n2^{-k}(1 + \tilde{O}_k(2^{-k}))$ a.a.s. Now let $Y = \text{rest}(\mathbf{H}, \sigma)$ and let $\omega = \omega(n)$ be a slowly diverging function. Let $\varepsilon = \tilde{O}_k(2^{-k})$. We are going to show that

$$(6.6) \quad \mathbb{E}[Y(Y-1) \cdots (Y-\omega+1)] \leq \left(\frac{(1 + \varepsilon + o(1))n}{2^k} \right)^\omega.$$

This bound implies the assertion; indeed,

$$\begin{aligned} \mathbb{P}[Y > (1 + 2\varepsilon)n2^{-k}] &\leq \mathbb{P}[Y(Y - 1) \cdots (Y - \omega + 1) > ((1 + 2\varepsilon - o(1))n2^{-k})^\omega] \\ &\leq \frac{\mathbb{E}[Y(Y - 1) \cdots (Y - \omega + 1)]}{((1 + 2\varepsilon - o(1))n2^{-k})^\omega} \leq \left(\frac{1 + \varepsilon + o(1)}{1 + 2\varepsilon - o(1)}\right)^\omega = o(1). \end{aligned}$$

To prove (6.6), we observe that $Y(Y - 1) \cdots (Y - \omega + 1)$ is just the number of ordered ω -tuples of vertices belonging to neither the core nor the backbone—that is, belonging to Y . Hence, by symmetry and the linearity of expectation,

$$\mathbb{E}[Y(Y - 1) \cdots (Y - \omega + 1)] \leq n^\omega \mathbb{P}[v_1, \dots, v_\omega \in Y].$$

Thus, we are left to estimate $\mathbb{P}[v_1, \dots, v_\omega \in Y]$. If \mathcal{A} occurs, then $\bar{U} \subset \text{core}(\mathbf{H}, \sigma)$. If $\bar{U} \subset \text{core}(\mathbf{H}, \sigma)$ and $v_1, \dots, v_\omega \in Y$, then for any $i \in [\omega]$ one of the following must occur.

- (1) There is no edge blocking v_i that consists of vertices in $\{v_i\} \cup \bar{U}$ only.
- (2) v_i occurs in more than 10 edges that are $\{v_i\} \cup \bar{U}$ -endangered.
- (3) There are at least 200 edges blocking v_i but fewer than 100 of them consist of vertices in $\{v_i\} \cup \bar{U}$ only.
- (4) There are at most 200 edges blocking v_i and one edge e such that $v_i \in e$ and that is $\{v_i\} \cup \bar{U}$ -endangered.

Indeed, if a vertex v_i is in $\text{rest}(\mathbf{H}, \sigma)$ then it violates one of the conditions **CR1** and **CR2** and one of **BB1** and **BB2**. Therefore, we have to consider several cases. If v_i violates **BB1**, then (1) is true. If it violates **CR1** and **BB2**, then either (3) or (4) is true. If v_i violates **CR2** and one of **BB1** and **BB2**, then (2) is true.

Let \mathcal{B}_i be the event that one of the above is true for $i \in [\omega]$. By the principle of deferred decisions, we have $\mathbb{P}[\mathcal{A}] = 1 - O(\omega^2/n)$ and, therefore, we get

$$\mathbb{P}[v_1, \dots, v_\omega \in Y] \leq \mathbb{P}[v_1, \dots, v_\omega \in Y | \mathcal{A}] + o(1) \leq \mathbb{P}\left[\bigcap_{i=1}^\omega \mathcal{B}_i \mid \mathcal{A}\right] + o(1).$$

Given that there is no edge containing two vertices from v_1, \dots, v_ω , the events $\mathcal{B}_1, \dots, \mathcal{B}_\omega$ are mutually independent. Therefore, $\mathbb{P}[\bigcap_{i=1}^\omega \mathcal{B}_i | \mathcal{A}] = \mathbb{P}[\mathcal{B}_1 | \mathcal{A}]^\omega$. Given that \mathcal{A} occurs, by Lemma 6.6 the probability of event (1) is asymptotically equal to $2^{-k} + \tilde{O}_k(4^{-k})$ and the probabilities of events (2), (3) and (4) are asymptotically equal to $\tilde{O}_k(4^{-k})$. Hence, $\mathbb{P}[\mathcal{B}_1 | \mathcal{A}] = 2^{-k} + \tilde{O}_k(4^{-k})$ and $\mathbb{P}[v_1, \dots, v_\omega \in Y] \leq (2^{-k} + \tilde{O}_k(4^{-k}) + o(1))^\omega = ((1 + \varepsilon + o(1))2^{-k})^\omega$. \square

We define $\text{free}(\mathbf{H}, \sigma)$ as the set of all vertices $v \in \text{rest}(\mathbf{H}, \sigma)$ such that v occurs only in edges e such that $e \cap \text{core}(\mathbf{H}, \sigma)$ is bichromatic.

PROPOSITION 6.8. *A.a.s. $|\text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)| = n\tilde{O}_k(4^{-k})$. In particular, $|\text{free}(\mathbf{H}, \sigma)| = n(2^{-k} + \tilde{O}_k(4^{-k}))$.*

PROOF. We introduce $Y = |\text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)|$ and proceed just as in the proof of Proposition 6.7. To estimate $\mathbb{P}[v_1, \dots, v_\omega \in Y]$ we observe that if $\bar{U} \subset \text{core}(\mathbf{H}, \sigma)$ and $v_1, \dots, v_\omega \in Y$ then for any $i \in [\omega]$ one of the following must occur.

- (1) There is no edge blocking v_i that consists of vertices in $\{v_i\} \cup \bar{U}$ only and v_i occurs in at least one edge that is $\{v_i\} \cup \bar{U}$ -endangered.
- (2) v_i occurs in more than 10 edges that are $\{v_i\} \cup \bar{U}$ -endangered.
- (3) There are at least 200 edges blocking v_i but fewer than 100 of them consist of vertices in $\{v_i\} \cup \bar{U}$ only.
- (4) There are at most 200 edges blocking v_i and one edge e such that $v_i \in e$ and that is $\{v_i\} \cup \bar{U}$ -endangered.

Events (2), (3) and (4) are as in the proof of Proposition 6.7 and their probabilities are asymptotically equal to $\tilde{O}_k(4^{-k})$. By Lemma 6.6, the probability of (1) is $\tilde{O}_k(4^{-k})$ and the assertion follows. \square

In the last three subsections, we calculate the cluster size $\mathcal{C}_\beta(\mathbf{H}, \sigma)$ up to a small error term. We proceed by first eliminating the contribution of the vertices in the core and in a second step the contribution of the vertices in the backbone. Finally, we calculate the contribution of the vertices in $\text{rest}(\mathbf{H}, \sigma)$.

6.4. *Rigidity of the core.* In the following, we let $x = k^{-5}$. We first show that the cluster of σ under \mathbf{H} mostly consists of configurations at distance less than $2x$ from σ .

LEMMA 6.9. *A.a.s.*

$$\mathcal{C}_\beta(\mathbf{H}, \sigma) \sim \sum_{\tau \in \{-1, 1\}^n: (\sigma, \tau) \geq (1-x)n} \exp(-\beta E_{\mathbf{H}}(\tau)).$$

To prove this result, we recall the notation from Section 4. We need the following technical lemma.

LEMMA 6.10. *Let $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \geq k \ln 2 - \ln k$. Then $\sup_{\alpha \in [2/3, 1-k^{-5}]} \Lambda_\beta(\alpha) < \Lambda_\beta(1) - \Omega_k(k^{-5})$.*

PROOF. We observe that for $\alpha \in [1 - k^{-5}, 1 - k^{-7}]$,

$$(6.7) \quad \Lambda'_\beta(\alpha) = \frac{\ln(1 - \alpha)}{2} + \frac{d}{2^k} + \tilde{O}_k(2^{-k}) = k \ln 2 + O_k(\ln k) \geq 1.$$

An expansion of $\Lambda_\beta(\alpha)$ near $\alpha = 1$ gives $\Lambda_\beta(1 - k^{-7}) \leq \Lambda_\beta(1) + O_k(k^{-6})$ and together with (6.7) this implies

$$(6.8) \quad \Lambda_\beta(1 - k^{-5}) \leq \Lambda_\beta(1) - \Omega_k(k^{-5}).$$

Further, using that $\Lambda'_\beta(\alpha) > 0$ if $\alpha > 1 - 1.99 \ln k/k$ (as in the proof of Lemma 4.8) and (6.8) we obtain

$$(6.9) \quad \sup_{\alpha \in [1-1.99 \ln k/k, 1-k^{-5}]} \Lambda_\beta(\alpha) \leq \Lambda_\beta(1 - k^{-5}) \leq \Lambda_\beta(1) - \Omega_k(k^{-5}).$$

A study of $\Lambda_\beta(\alpha)$ also gives

$$(6.10) \quad \sup_{\gamma \in [1.99, 2.01]} \Lambda_\beta(1 - \gamma \ln k/k) \leq \Lambda_\beta(1) - \Omega_k(k^{-5})$$

and $\Lambda_\beta(\alpha) - \Lambda_\beta(1 - 2.01 \ln k/k) = \mathcal{H}(\frac{1+\alpha}{2}) + \tilde{O}_k((\frac{2}{2.01})^k) \leq 0$ for $\alpha \in [2/3, 1 - 2.01 \ln k/k]$, which leads to

$$(6.11) \quad \begin{aligned} & \sup_{\alpha \in [2/3, 1-2.01 \ln k/k]} \Lambda_\beta(\alpha) \\ & \leq \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \tilde{O}_k\left(\left(\frac{2}{2.01}\right)^k\right) + \Lambda_\beta(1 - 2.01 \ln k/k) \\ & \leq \Lambda_\beta(1) - \Omega_k(k^{-5}). \end{aligned}$$

Combining (6.9), (6.10) and (6.11) completes the proof of the assertion. \square

PROOF OF LEMMA 6.9. Let \mathcal{A} be the event that $|e(H_k(n, p)) - m| \leq m^{2/3}$. Given σ and $\alpha \in [-1, 1]$ and using Lemma 4.2 we have

$$\begin{aligned} & \mathbb{E} \left[\sum_{\tau \in \{-1, 1\}^n: \langle \sigma, \tau \rangle = \alpha n} \exp(-\beta E_{\mathbf{H}}(\tau)) \mid |e(\mathbf{H}) - m| \leq m^{2/3} \right] \\ & = \frac{\mathbb{E}[\sum_{\tau: \langle \sigma, \tau \rangle = \alpha n} \exp(-\beta E_{H_k(n, p)}(\sigma)) \exp(-\beta E_{H_k(n, p)}(\tau)) \mid \mathcal{A}]}{\mathbb{E}[\exp(-\beta E_{H_k(n, p)}(\sigma)) \mid \mathcal{A}]} \\ & \leq \frac{\mathbb{E}[Z_\beta(\alpha)]}{\mathbb{E}[Z_\beta(H'_k(n, m))]} \exp(O(m^{2/3})). \end{aligned}$$

In order to derive the last line, we used an observation similar to equation (4.5) and Lemma 4.2. We observe that by Lemma 4.5 we have a.a.s. $\mathcal{C}_\beta(\mathbf{H}, \sigma) \geq \exp(-\beta E_{\mathbf{H}}(\sigma)) \sim \exp(-n \tilde{O}_k(2^{-k}))$. Hence,

$$\begin{aligned} & \mathbb{E} \left[\sum_{\tau \in \{-1, 1\}^n: 2/3n \leq \langle \sigma, \tau \rangle < (1-x)n} \exp(-\beta E_{\mathbf{H}}(\tau)) \mid |e(\mathbf{H}) - m| \leq m^{2/3} \right] \\ & \leq \sum_{v=0}^n \frac{\mathbb{E}[Z_\beta(2v/n - 1)]}{\mathbb{E}[Z_\beta(H'_k(n, m))]} \mathbf{1}_{2v/n - 1 \in [2/3, (1-x)]} \exp(O(m^{2/3})) \\ & \leq \exp\left(n \left(\sup_{\alpha \in [2/3, 1-x]} \Lambda_\beta(\alpha) - \Lambda_\beta(1) + \tilde{O}_k(2^{-k}) \right)\right) \mathcal{C}_\beta(\mathbf{H}, \sigma) \\ & \leq \exp(-n \Omega_k(k^{-5})) \mathcal{C}_\beta(\mathbf{H}, \sigma) \end{aligned}$$

by Lemma 4.6 and by Lemma 6.10. It follows from Markov’s inequality that a.a.s.

$$\sum_{\tau \in \{-1,1\}^n: 2/3n \leq \langle \sigma, \tau \rangle < (1-x)n} \exp(-\beta E_{\mathbf{H}}(\tau)) = o(C_{\beta}(\mathbf{H}, \sigma)). \quad \square$$

We now approximate $C_{\beta}(\mathbf{H}, \sigma)$ based on the previous decomposition of the vertex set V . Given a k -uniform hypergraph $\mathbf{H}, \sigma : [n] \rightarrow \{\pm 1\}$, and three maps $\tau_{\text{core}} : \text{core}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$, $\tau_{\text{back}} : \text{back}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$ and $\tau_{\text{rest}} : \text{rest}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$, we define $E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ as $E_{\mathbf{H}}(\tau)$ for the unique τ whose restriction to $\text{core}(\mathbf{H}, \sigma)$ [resp., $\text{back}(\mathbf{H}, \sigma)$, $\text{rest}(\mathbf{H}, \sigma)$] is given by τ_{core} (resp., $\tau_{\text{back}}, \tau_{\text{rest}}$).

We introduce the “restricted” cluster size

$$C_{\beta}^{\text{back+rest}}(\mathbf{H}, \sigma) = \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp(-\beta E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})).$$

The summation is over $\tau_{\text{back}} : \text{back}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$ and $\tau_{\text{rest}} : \text{rest}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$. The aim of this section is to prove the following.

PROPOSITION 6.11. *A.a.s.*

$$\frac{1}{n} \ln C_{\beta}^{\text{back+rest}}(\mathbf{H}, \sigma) \leq \frac{1}{n} \ln C_{\beta}(\mathbf{H}, \sigma) \leq \frac{1}{n} \ln C_{\beta}^{\text{back+rest}}(\mathbf{H}, \sigma) + \exp(-88\beta).$$

In order to proceed, we first need a few additional results. We introduce the set $\mathcal{E}_{\mathbf{H}}(\tau, \sigma)$ of edges that:

- are supported by a vertex v such that $\tau_{\text{core}}(v) \neq \sigma_{\text{core}}(v)$,
- contain two or more vertices v' such that $\tau_{\text{core}}(v') \neq \sigma_{\text{core}}(v')$.

The following lemma is reminiscent of [6], Lemma 5.9.

LEMMA 6.12. *A.a.s. it holds that, for all $\tau : [n] \rightarrow \{\pm 1\}$ satisfying $\langle \sigma, \tau \rangle \geq (1-x)n$,*

$$|\mathcal{E}_{\mathbf{H}}(\tau, \sigma)| \leq 2|\{v : \sigma_{\text{core}}(v) \neq \tau_{\text{core}}(v)\}|.$$

PROOF. We claim that a.a.s. \mathbf{H} has the following property. Let $T \subset V$ be of size $|T| \leq n/(2e^3k^2\lambda^2)$. Then there are no more than $2|T|$ edges that are supported by a vertex in T and contain a second vertex from T . Indeed, by a first moment argument, with $|T| = tn$ the probability that there is a set T that violates the above property is bounded by

$$\begin{aligned} \binom{n}{tn} \binom{(1+o(1))\lambda n}{2tn} (kt^2)^{2tn} &\leq \left[(1+o(1)) \frac{e}{t} \left(\frac{\lambda e}{2t} \right)^2 (kt^2)^2 \right]^{tn} \\ &\leq ((1+o(1))t(e^3\lambda^2k^2))^{tn} = o(1). \end{aligned}$$

With $T = \{v : \sigma_{\text{core}}(v) \neq \tau_{\text{core}}(v)\}$ and $x = k^{-5}$, we have $|T| \leq 2xn < n/(2e^3k^2\lambda^2)$ which completes the proof. \square

LEMMA 6.13. *A.a.s. it holds that, for all $\tau : [n] \rightarrow \{\pm 1\}$ satisfying $\langle \sigma, \tau \rangle \geq (1 - x)n$,*

$$E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + 88 \text{dist}(\tau_{\text{core}}, \sigma_{\text{core}}).$$

PROOF. Denote for a vertex $v \in V$ and $\tau : [n] \rightarrow \{\pm 1\}$ by:

- $X(v)$ the number of critical (under σ) edges e supported by v such that $e \setminus \{v\} \subset \text{core}(\mathbf{H}, \sigma)$,
- $Y(v)$ the number of $\text{core}(\mathbf{H}, \sigma)$ -endangered edges containing v ,
- $M_{\tau}(v)$ the number of edges containing v that are monochromatic under $(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$.

We can lower bound $E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ in terms of $E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ as

$$(6.12) \quad E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + \sum_{v: \tau_{\text{core}}(v) \neq \sigma_{\text{core}}(v)} (X(v) - M_{\tau}(v)) - |\mathcal{E}_{\mathbf{H}}(\tau, \sigma)|.$$

Only edges that were $\text{core}(\mathbf{H}, \sigma)$ -endangered can be monochromatic under $(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$: $M_{\tau}(v) \leq Y(v)$. In particular,

$$(6.13) \quad \forall v \in \text{core}(\mathbf{H}, \sigma), \quad X(v) - M_{\tau}(v) \geq 90.$$

On the other hand, we can upper bound $|\mathcal{E}_{\mathbf{H}}(\tau, \sigma)|$ with Lemma 6.12. Replacing in (6.12) and using (6.13) gives

$$E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + 88 \text{dist}(\tau_{\text{core}}, \sigma_{\text{core}}),$$

a.a.s., completing the proof. \square

PROOF OF PROPOSITION 6.11. We first prove the lower bound on $\mathcal{C}_{\beta}(\mathbf{H}, \sigma)$. With Proposition 6.1, a.a.s. for all $(\tau_{\text{back}}, \tau_{\text{rest}})$ we have $\langle \sigma, (\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \rangle \geq (1 - x)n$. Hence, with Lemma 6.9. a.a.s.

$$\mathcal{C}_{\beta}(\mathbf{H}, \sigma) \geq \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp(-\beta E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})) = \mathcal{C}_{\beta}^{\text{back+rest}}(\mathbf{H}, \sigma).$$

To derive the upper bound, we write

$$(6.14) \quad \begin{aligned} \mathcal{C}_{\beta}(\mathbf{H}, \sigma) &\leq \sum_{\substack{\tau_{\text{core}}: \\ \langle \sigma_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp(-\beta E_{\mathbf{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})) \\ &\leq \sum_{\substack{\tau_{\text{core}}: \\ \langle \sigma_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \exp(-88\beta \text{dist}(\sigma_{\text{core}}, \tau_{\text{core}})) \mathcal{C}_{\beta}^{\text{back+rest}}(\mathbf{H}, \sigma), \end{aligned}$$

where the second inequality holds a.a.s. by Lemma 6.13. Finally,

$$\begin{aligned}
 (6.15) \quad & \sum_{\substack{\tau_{\text{core}}: \\ \langle \sigma_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \exp(-88\beta \text{dist}(\sigma_{\text{core}}, \tau_{\text{core}})) \\
 &= \sum_{i=0}^{xn/2} \binom{n}{i} \exp(-88\beta i) \leq \sum_{i=0}^n \binom{n}{i} \exp(-88\beta i) \\
 &= (1 + \exp(-88\beta))^n \leq \exp(n \exp(-88\beta)).
 \end{aligned}$$

Replacing with (6.16) in (6.14) completes the proof. \square

6.5. *Rigidity of the backbone.* We proceed one step further by eliminating the vertices in the backbone and comparing $\mathcal{C}_\beta^{\text{back+rest}}(\mathbf{H}, \sigma)$ to $\mathcal{C}_\beta^{\text{rest}}(\mathbf{H}, \sigma)$, where

$$\mathcal{C}_\beta^{\text{rest}}(\mathbf{H}, \sigma) = \sum_{\tau_{\text{rest}}} \exp(-\beta E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}})).$$

The sum is over $\tau_{\text{rest}} : \text{rest}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$. We prove the following result.

PROPOSITION 6.14. *A.a.s.*

$$\frac{1}{n} \ln \mathcal{C}_\beta^{\text{rest}}(\mathbf{H}, \sigma) \leq \frac{1}{n} \ln \mathcal{C}_\beta^{\text{back+rest}}(\mathbf{H}, \sigma) \leq \frac{1}{n} \ln \mathcal{C}_\beta^{\text{rest}}(\mathbf{H}, \sigma) + \tilde{O}_k(4^{-k}).$$

PROOF. The left inequality is obvious. To prove the right inequality, we observe that, by definition of the backbone, for any $\tau_{\text{back}} : \text{back}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$ and $\tau_{\text{rest}} : \text{rest}(\mathbf{H}, \sigma) \rightarrow \{\pm 1\}$, the following is true.

$$(6.16) \quad E_{\mathbf{H}}(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}}) + \text{dist}(\sigma_{\text{back}}, \tau_{\text{back}}).$$

Indeed for any vertex $v \in \text{back}(\mathbf{H}, \sigma)$ with $\sigma_{\text{back}}(v) \neq \tau_{\text{back}}(v)$ and any edge $e \ni v$:

- either v supports e and $e \setminus \{v\} \subset \text{core}(\mathbf{H}, \sigma)$, in which case the edge e is bichromatic under $(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}})$ and monochromatic under $(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$,
- or e is not $\{v\} \cup \text{core}(\mathbf{H}, \sigma)$ -endangered and is bichromatic both under $(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}})$ and under $(\sigma_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$.

Moreover, by the definition of $\text{back}(\mathbf{H}, \sigma)$ there is at least one edge of the first type for any $v \in \text{back}(\mathbf{H}, \sigma)$ with $\sigma_{\text{back}}(v) \neq \tau_{\text{back}}(v)$.

Using the definition of $\mathcal{C}_\beta^{\text{back+rest}}(\mathbf{H}, \sigma)$ and (6.16) yields

$$\begin{aligned}
 (6.17) \quad & \mathcal{C}_\beta^{\text{back+rest}}(\mathbf{H}, \sigma) \\
 & \leq \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp(-\beta \text{dist}(\sigma_{\text{back}}, \tau_{\text{back}})) \exp(-\beta E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}})) \\
 & \leq \sum_{\tau_{\text{back}}} \exp(-\beta \text{dist}(\sigma_{\text{back}}, \tau_{\text{back}})) \mathcal{C}_\beta^{\text{rest}}(\mathbf{H}, \sigma).
 \end{aligned}$$

The remaining sum can easily be upper-bounded:

$$\begin{aligned}
 \sum_{\tau_{\text{back}}} \exp(-\beta \text{dist}(\sigma_{\text{back}}, \tau_{\text{back}})) &= \sum_{i=0}^{|\text{back}(\mathbf{H}, \sigma)|} \binom{|\text{back}(\mathbf{H}, \sigma)|}{i} \exp(-\beta i) \\
 (6.18) \qquad \qquad \qquad &= (1 + \exp(-\beta))^{|\text{back}(\mathbf{H}, \sigma)|} \\
 &\leq \exp(\exp(-\beta) |\text{back}(\mathbf{H}, \sigma)|).
 \end{aligned}$$

The upper bound of Proposition 6.14 then follows from (6.17) and (6.18) combined with Proposition 6.1. \square

6.6. *The remaining vertices.* We finally deal with the vertices that belong neither to the core nor to the backbone. As anticipated in Proposition 6.8, most of them are free. This yields the following result.

PROPOSITION 6.15. *A.a.s.*

$$\frac{1}{n} \ln C_{\beta}^{\text{rest}}(\mathbf{H}, \sigma) = \frac{\ln 2}{2^k} - \beta \frac{E_{\mathbf{H}}(\sigma)}{n} + \tilde{O}_k(4^{-k}).$$

In order to prove the proposition, we need the following result. Let $M'_{\sigma}(v)$ be the number of monochromatic edges involving v in the configuration σ .

LEMMA 6.16. *A.a.s.*

$$\sum_{v \in \text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)} M'_{\sigma}(v) = n \tilde{O}_k(4^{-k}).$$

PROOF. We start with the following observation:

$$\sum_{v \in \text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)} M'_{\sigma}(v) \leq \sum_{v: M'_{\sigma}(v) > 2} M'_{\sigma}(v) + 2 |\text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)|.$$

The number of monochromatic edges involving a vertex v is a binomial random variable $\text{Bin}(\binom{n-1}{k-1}(1 + o(1))2^{k-1}, p_1)$. Hence $\sum_{v \in V: M'_{\sigma}(v) > 2} M'_{\sigma}(v) = n \tilde{O}_k(4^{-k})$. Applying Proposition 6.8 completes the proof. \square

PROOF OF PROPOSITION 6.15. By the definition of $\text{free}(\mathbf{H}, \sigma)$, the number of monochromatic edges $E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}})$ does not depend on the values $\tau_{\text{rest}}(v)$ for $v \in \text{free}(\mathbf{H}, \sigma)$. Consequently,

$$C_{\beta}^{\text{rest}}(\mathbf{H}, \sigma) \geq 2^{|\text{free}(\mathbf{H}, \sigma)|} \exp(-\beta E_{\mathbf{H}}(\sigma)).$$

Together with Proposition 6.8 this gives the lower bound on $\frac{1}{n} \ln C_{\beta}^{\text{rest}}(\mathbf{H}, \sigma)$. For the upper bound, we start with the general inequality

$$\frac{1}{n} \ln C_{\beta}^{\text{rest}}(\mathbf{H}, \sigma) \leq \frac{\ln 2}{n} |\text{rest}(\mathbf{H}, \sigma)| - \frac{\beta}{n} \inf_{\tau_{\text{rest}}} E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}}).$$

Because the number of monochromatic edges does not depend on the values of the vertices in $\text{free}(\mathbf{H}, \sigma)$ we have

$$\inf_{\tau_{\text{rest}}} E_{\mathbf{H}}(\sigma_{\text{core}}, \sigma_{\text{back}}, \tau_{\text{rest}}) \geq E_{\mathbf{H}}(\sigma) - \sum_{v \in \text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)} M'_{\sigma}(v).$$

Hence, we obtain

$$(6.19) \quad \begin{aligned} & \frac{1}{n} \ln C_{\beta}^{\text{rest}}(\mathbf{H}, \sigma) \\ & \leq \frac{\ln 2}{n} |\text{rest}(\mathbf{H}, \sigma)| - \beta \frac{E_{\mathbf{H}}(\sigma)}{n} + \frac{\beta}{n} \sum_{v \in \text{rest}(\mathbf{H}, \sigma) \setminus \text{free}(\mathbf{H}, \sigma)} M'_{\sigma}(v). \end{aligned}$$

The upper bound follows by combining (6.19) with Proposition 6.7 and Lemma 6.16. \square

6.7. *Proof of Proposition 3.5.* Combining Propositions 6.11, 6.14 and 6.15, we obtain that a.a.s.

$$(6.20) \quad \frac{1}{n} \ln C_{\beta}(\mathbf{H}, \sigma) = \frac{\ln 2}{2^k} - \beta \frac{E_{\mathbf{H}}(\sigma)}{n} + \tilde{O}_k(4^{-k}).$$

The number of monochromatic edges in the planted model is tightly concentrated by Chernoff bounds. Therefore, we get a.a.s.

$$E_{\mathbf{H}}(\sigma) = \binom{n}{k} 2^{1-k} p_1 (1 + o(1)) \sim \frac{\exp(-\beta)}{2^{k-1} - 1 + \exp(-\beta)} \frac{d}{k} n.$$

For $d/k = 2^{k-1} \ln 2 + O_k(1)$ and $\beta \geq k \ln 2 - \ln k$, we have $E_{\mathbf{H}}(\sigma) = \ln 2 \exp(-\beta) n + \tilde{O}_k(4^{-k}) n$. Inserting this in (6.20) yields a.a.s.

$$\frac{1}{n} \ln C_{\beta}(\mathbf{H}, \sigma) = \frac{\ln 2}{2^k} - \beta \ln 2 \exp(-\beta) + \tilde{O}_k(4^{-k}),$$

proving Proposition 3.5.

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