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ON GERBER–SHIU FUNCTIONS AND OPTIMAL DIVIDEND DISTRIBUTION FOR A LÉVY RISK PROCESS IN THE PRESENCE OF A PENALTY FUNCTION

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This paper concerns an optimal dividend distribution problem for an insurance company whose risk process evolves as a spectrally negative Lévy process (in the absence of dividend payments). The management of the company is assumed to control timing and size of dividend payments. The objective is to maximize the sum of the expected cumulative discounted dividend payments received until the moment of ruin and a penalty payment at the moment of ruin, which is an increasing function of the size of the shortfall at ruin; in addition, there may be a fixed cost for taking out dividends. A complete solution is presented to the corresponding stochastic control problem. It is established that the value-function is the unique *stochastic solution* and the pointwise smallest *stochastic supersolution* of the associated HJB equation. Furthermore, a necessary and sufficient condition is identified for optimality of a single dividend-band strategy, in terms of a particular Gerber–Shiu function. A number of concrete examples are analyzed.

1. Optimal control of Lévy risk models. The spectrally negative Lévy risk model.

Recall the classical Cramér–Lundberg model

$$X_t - X_0 = \eta t - S_t, \qquad S_t = \sum_{k=1}^{N_t} C_k - \lambda m t, \tag{1.1}$$

which is used in collective risk theory (e.g., Gerber [20]) to describe the surplus $X = \{X_t, t \in \mathbb{R}_+\}$ of an insurance company. Here, $X_0 \geq 0$ is the initial level of reserves, C_k are i.i.d. positive random variables representing the claims made, $N = \{N_t, t \in \mathbb{R}_+\}$ is an independent Poisson process with intensity λ modeling the times at which the claims occur, and pt, with $p := \eta + \lambda m$, represents the premium income up to time t, with profit rate $\eta > 0$ and mean $m < \infty$ of C_1.

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In later years, model (1.1) was generalized to the "perturbed model"

$$X_t - X_0 := \sigma B_t + \eta t - S_t, \tag{1.2}$$

where B_t denotes an independent standard Brownian motion, which models small scale fluctuations of the risk process.

Since the jumps of X are all negative, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta(X_t - X_0)}]$ exists for all $\theta \geq 0$ and $t \in \mathbb{R}_+$, and is log-linear in t, defining thus a function $\psi(\theta)$ satisfying $\mathbb{E}[\mathrm{e}^{\theta(X_t - X_0)}] = \mathrm{e}^{t\psi(\theta)}$ with

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + \eta\theta + \int_{\mathbb{R}_+ \setminus \{0\}} (\mathrm{e}^{-\theta x} - 1 + \theta x)\nu(\mathrm{d}x), \tag{1.3}$$

where $\nu(\mathrm{d}x) = \lambda F_C(\mathrm{d}x)$, $x \in \mathbb{R}_+$, with F_C the distribution function of C_1, is the "Lévy measure" of the compound Poisson process S_t, and $\eta = \psi'(0)$ is the mean of $X_1 - X_0$.

The cumulant exponent $\psi(\theta)$ is well defined, at least on the positive half-line, where it is strictly convex with the property that $\lim_{\theta \to \infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. The right-inverse function of ψ is denoted by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

An important generalization is to replace the process S in (1.2) by a general subordinator [a nondecreasing Lévy process, with Lévy measure $\nu(\mathrm{d}x)$, $x \in \mathbb{R}_+$, which may have infinite mass]. Under this model, the "small fluctuations" can arise either continuously, due to the Brownian motion, or due to the infinite jump-activity.

Taking S to be a pure jump-martingale with i.i.d. increments and negative jumps with Lévy measure $\nu(\mathrm{d}x)$, one arrives thus to a general integrable spectrally negative Lévy process $X = \{X_t, t \in \mathbb{R}_+\}$, that is, a stochastic process that has stationary independent increments, no positive jumps and càdlàg paths, such that X_t integrable for any $t \in \mathbb{R}_+$, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ is the natural filtration generated by X satisfying the usual conditions of right-continuity and completeness; see Bertoin [12], Kyprianou [25], Sato [35]. The assumption that $X_t - X_0$ has finite mean for any fixed $t > 0$ is equivalent to the requirement that the Lévy measure ν satisfies the integrability condition

$$\nu_{1,\infty} := \int_{[1,\infty)} x\nu(\mathrm{d}x) < \infty. \tag{1.4}$$

To avoid degeneracies, the case that X has monotone paths is excluded. The (possibly random) initial value X_0 is assumed to be nonnegative. Conditioning the probability measure $\mathbb{P}$ on the value of X_0 gives rise to the family of probability measures $\{\mathbb{P}_x, x \in \mathbb{R}_+\}$ that satisfy $\mathbb{P}_x[X_0 = x] = 1$.

An alternative characterization of spectrally negative Lévy processes is via the "q-harmonic homogeneous scale function" $W^{(q)}$, a nondecreasing function defined on the real line that is 0 on $(-\infty, 0)$, continuous on $\mathbb{R}_+$, with Laplace transform $\mathcal{L}W^{(q)}$ given by

$$\mathcal{L}W^{(q)}(\theta) = \big(\psi(\theta) - q\big)^{-1}, \qquad \theta > \Phi(q). \tag{1.5}$$

Despite the diversity of possible path behaviors displayed by spectrally negative Lévy processes, a wide variety of results may be elegantly expressed in a unifying manner via the homogeneous scale function $W^{(q)}$, bypassing thus "probabilistic complexity" via unified analytic methods. This paper further illustrates this aspect, by unveiling the way the scale function intervenes in a quite complex control problem.

De Finetti's dividend problem. Under the assumption that the increments of the surplus process have positive mean, the Lévy risk model has the unrealistic property that it converges to infinity with probability one.

In answer to this objection, De Finetti [15] introduced the risk process with dividends

$$U_t^\pi = X_t - D_t^\pi, \qquad t \geq 0, \tag{1.6}$$

where π is an "admissible" dividend control policy and D_t^π denotes the cumulative amount of dividends that has been transferred to a beneficiary up to time t, and where $U_{0-}^\pi = X_0 \geq 0$ is the initial capital.

Writing $\tau^\pi = \inf\{t \in \mathbb{R}_+ : U_t^\pi < 0\}$ for the time at which ruin occurs, the objective is to maximize the expected cumulative dividend payments until the time of ruin

$$v_*(x) := \sup_{\pi \in \Pi} \mathbb{E}_x\left[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\, \mathrm{d}D_t^\pi\right],$$

with $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot | X_0 = x]$ and where Π denotes the set of all admissible strategies, and $q > 0$ is the discount rate.

Note that ruin may be either exogeneous or endogeneous (i.e., caused by a claim or by a dividend payment). A dividend strategy is admissible if ruin is always exogeneous, or more precisely, an admissible dividend strategy $D^\pi = \{D_t^\pi, t \in \mathbb{R}_+\}$ is a right-continuous $\mathbf{F}$-adapted stochastic process that will satisfy that, at any time preceding the epoch of ruin, a dividend payment is smaller than the size of the available reserves, that is, for any $t \leq \tau^\pi$,

$$\begin{cases} \Delta D_t^\pi := D_t^\pi - D_{t-}^\pi \leq \big(X_t - D_{t-}^\pi\big) \vee 0, \text{ and} \\ D_t^{\pi(c)} - D_u^{\pi(c)} \leq p(t-u), \qquad \forall u \in [0,t) \text{ if } \nu_{0,1} < \infty, \end{cases} \tag{1.7}$$

where $D^{\pi(c)}$ denotes the continuous part of D^π, $\nu_{0,1} := \int_{(0,1)} x\nu(\mathrm{d}x)$ and $p := \eta + \nu_{0,1} + \nu_{1,\infty}$. In the second line in (1.7) it is stated that if the jump-part of X is of bounded variation, it is not admissible to pay dividends at a rate larger than the premium rate p at any time t that there are no reserves (i.e., $U_t^\pi = 0$), as this would lead to immediate ruin.

Single barrier policies. Recall first the simplest case when there are no transaction costs. One possible dividends distribution policy is the "barrier policy" π_b of transferring all surpluses above a given level b, which results in the value

$$v_b(x) := v_{\pi_b}(x) = \mathbb{E}_x\left[\int_{[0,\tau_b)} \mathrm{e}^{-qt}\,\mathrm{d}D_t^b\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(b)}, \qquad x \in [0, b]$$

and $v_b(x) = x - b + v_b(b)$ for $x > b$, where $\tau_b = \inf\{t \geq 0 : X_t < D_t^b\}$, and $D^b = D^{\pi_b}$ is a local time-type strategy, given explicitly in terms of X by $D_{0-}^b = 0$ and

$$D_t^b = \sup_{s \leq t}(X_s - b)^+, \qquad t \in \mathbb{R}_+, \tag{1.8}$$

with $x^+ = \max\{x, 0\}$. As this equation shows, a nonzero optimal barrier must be an inflection point of the scale function if the latter is smooth.

Multiple bands policies. However, single barrier strategies might not be optimal; cf. Gerber [18, 19]. The optimal strategy may be a "multi-bands strategy," involving several "continuation bands" $[a_i, b_i)$, $i = 0, 1, \ldots,$ with upper reflecting boundaries b_i, separated by "lump-sum dividend taking bands" $[b_i, a_{i+1})$, $i = 0, 1, \ldots,$ of jumping to the next reflecting barrier below b_i, by paying all the excess as a lump-sum payment; see also Hallin [22], who formulated a system of time dependent integro-differential equations associated to multi-bands policies. Azcue and Muler [7] established the optimality of multi-bands strategies under the Cramér–Lundberg model in the presence of proportional and excess-of-loss reinsurance, adopting a viscosity approach. A direct approach was developed in Schmidli [37] where a recursive algorithm was provided to find, in terms of solutions to certain integro-differential equations, the value function of the optimal dividend problem under the Cramér–Lundberg model in the absence of a penalty. Recently, Albrecher and Thonhauser [1] proved the optimality of bands strategies, in the case that the reserves attract a fixed interest rate.

Gerber showed also that for exponential claims (and with no constraints on the dividends rate), the optimal policy involved only one barrier (and one continuation band); however, constructing examples where more than one band was necessary remained an open problem for a long time.

Optimality conditions for single barrier strategies. The interest in bands strategies was reawakened by Azcue and Muler [7], who produced the first example (with Gamma claims) in which a single constant barrier is not optimal. Let

$$b^* = \sup\{b > 0 : W^{(q)\prime}(b) \leq W^{(q)\prime}(x) \text{ for all } x\} \tag{1.9}$$

denote the last global minimum of the derivative of the q-scale function.

Avram et al. [6] showed that

$$(\Gamma v_{b^*} - q v_{b^*})(x) \leq 0 \qquad \text{for all } x > b^*, \tag{1.10}$$

where Γ denotes the infinitesimal generator of X, is a sufficient optimality condition for the single barrier strategy under a general spectrally negative Lévy model.

In fact, conditions (1.9)–(1.10) is both necessary and sufficient, as follows by examining the variational inequality characterizing the problem; see Loeffen [27], Lemmas 1, 2.

A simpler sufficient condition for the optimality of single band policies was obtained by Loeffen [27, 28] (with and without transaction costs), who showed that it is enough to check that the last local minimum of the q-scale function is also a global minimum. Even more direct optimality conditions in terms of the Lévy measure ν were provided by Kyprianou et al. [26], and Loeffen and Renaud [29], who showed, respectively, that log-convexity of the density and of the survival functions suffice (the second condition is more general). Note that the second result allowed also for an affine penalty function with slope less than unity, and that both results imply complete monotonicity of the Lévy density, and constitute therefore powerful generalizations of Gerber's unicity result [18, 19].

It turns out that b^* in (1.9) is always the right endpoint of the first continuation band. As already demonstrated in the rather terse example in Azcue and Muler ([7], page 274), left and right endpoints of subsequent bands can in principle be determined recursively (the former by ensuring the "smoothness" of the value function, and the latter similarly with b^*, by selecting last global maxima of updated value functions, adjusted by using the values of previous bands as stopping penalties). A characterization of points of nondifferentiability was provided in Schmidli [37]. However, an explicit smoothness condition (7.9) in terms of scale functions seems not to have been reported previously.

Quite paradoxically, it is possible that beyond the lump-sum dividend taking band following the first continuation band, waiting for higher barriers $b_i, i \geq 2$, may become again optimal. The level a_2 where the second continuation band starts may be determined by examining the family of functions $G_2^{(a)}(b)$ defined in (7.9), which are computed from a second Gerber–Shiu function, which uses the first value functions as stopping penalties, and so on, leading ultimately to all the optimal band levels; see Section 11.

Fixed transaction costs. It is interesting to consider also the effect of adding fixed transaction cost $K > 0$ that are not transferred to the beneficiaries when dividends are being paid. The objective of the beneficiaries becomes then to maximize $v_{\pi,K}(x)$, that is, $v_*(x) = \sup_{\pi\in\Pi} v_{\pi,K}(x)$ with

$$v_{\pi,K}(x) = \mathbb{E}_x\left[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\,\mathrm{d}D_t^\pi - K\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\,\mathrm{d}N_t^\pi\right],$$

where $N^\pi = \{N_t^\pi, t \in \mathbb{R}_+\}$ is the stochastic process that counts the number of jumps of D^π in the interval $[0, t]$,

$$N_t^\pi = \#\{s \in [0,t] : \Delta D_s^\pi > 0\}, \qquad t \in \mathbb{R}_+. \tag{1.11}$$

The introduction of a fixed transaction cost $K > 0$ has the usual effect of changing the optimal reflection boundaries b into strips $[b_-, b_+]$, so that when $U_t = b_+$, a lump-sum dividend $b_+ - b_-$ is paid, and the reserves process is diminished to the lower "entrance" point b_-. To emphasize this disappearance of reflection barriers, the term *band* will be used throughout when $K > 0$, and also when more than one barrier is present.

The typical optimal dividend strategy consists of "lump-sum payments" (see, e.g., Alvarez and Virtanen [2] and Thonhauser and Albrecher [42]), with π of the form $\pi = \{(J_k, T_k), k \in \mathbb{N}\}$, where $0 \leq T_1 \leq T_2 \leq \cdots$ is an increasing sequence of $\mathbf{F}$-stopping times representing the times at which dividend payments are made, and $J_i \geq K$ is a sequence of positive $\mathcal{F}_{T_i}$-measurable random variables representing the sizes of the dividend payments. Then

$$D_t^\pi = \sum_{k=1}^{N_t^\pi} J_k,$$

where $N_t^\pi = \#\{k : T_k \leq t\}$ is the number of times that dividends have been paid by time t.

For single bands policies for example, the dividend distribution consists of the fixed amount $J_i = b_+ - b_-$.

Balancing dividends and ruin penalties. Several alternative objectives have been proposed recently, involving final penalties $w(x)$ at ruin (see Dickson and Waters [16], Gerber et al. [21] and Zajic [43]), or continuous payoffs until ruin; see Albrecher and Thonhauser [41], Cai et al. [14]. For example, the case where the insurance company is bailed out by the beneficiaries every time that there is a shortfall in the reserves was investigated in Avram et al. [6] and in Kulenko & Schmidli [23]. This paper continues the investigation of the impact of a general penalty and fixed transaction costs on the optimal dividends policy. The considered objective is to maximize the expected cumulative discounted dividend payments until the moment of ruin less the penalty, which is an increasing function of the shortfall at the moment of ruin, by controlling the timing and size of dividend payments. This problem is phrased as an optimal control problem, which will be solved by constructing explicitly a solution of the associated Hamilton–Jacobi–Bellman (HJB) equation, in terms of scale functions of the Lévy process X.

Stochastic solutions. Given results concerning the smoothness of scale functions (see, e.g., Kyprianou et al. [24]), it is not to be expected that the candidate value-function is a classical solution of the HJB equation. In fact, it will turn out that the candidate value function is continuous but not C^1 on $\mathbb{R}_+ \setminus \{0\}$ if X has bounded variation, and is C^1 but not C^2 on $\mathbb{R}_+ \setminus \{0\}$, if X has unbounded variation. To verify optimality of the candidate optimal value-function under weak regularity conditions, a probabilistic approach is adopted in this paper. It is established that the value-function is the unique *stochastic solution* of the HJB equation corresponding to the optimal control problem under consideration. The notion of stochastic solution may informally be considered as a probabilistic counterpart of the analytical notion of viscosity solution: While viscosity sub- and supersolutions are defined in terms of pointwise approximations by smooth solutions to the variational inequalities associated to the HJB equation, stochastic super- and subsolutions are phrased in terms of super- and submartingale properties of related stochastic processes. The version of the notion of stochastic solution deployed here is an adaptation of Stroock and Varadhan's [40] classical notion, which was originally introduced in the setting of linear parabolic PDEs, to the current setting; see Definition 4.1. A stochastic version of Perron's method using the stochastic solution concept was recently developed in Bayraktar and Sîrbu [10] for the case of linear parabolic PDEs.

The viscosity solution method is a classical approach that has been used extensively in the study of existence and uniqueness of solutions to HJB equations; cf. Bardi and Capuzzo-Dolcetta [9] and Fleming and Soner [17] for general treatments. The HJB equation (3.6) corresponding to the stochastic control problem considered in the current paper is a nonlinear integro-differential equation with constant coefficients and with a gradient constraint, which is of first or second order depending on whether or not a Gaussian component is present in the dynamics of X. Due to the negative jumps of X and the boundary condition on the negative half-axis (the specified penalty at the epoch of ruin), one is led to the notion of constraint viscosity solutions which, in the context of different optimization problems, has been developed for first order equations by Sayah [36] and Soner [39], and for second order equations in Alvarez and Tourin [3], Benth et al. [11] and Pham [33]. In, for example, Azcue and Muler [7, 8] and Albrecher and Thonhauser [1], dividend optimization problems are studied under the Cramér–Lundberg model using the viscosity solution method.

By deploying probabilistic tools from among others martingale theory, analogues are derived of key results from viscosity solution theory. In particular, existence and uniqueness of a stochastic solution to the HJB equation is shown (Theorem 12.1), where the uniqueness is established deploying a comparison principle (Proposition 12.6). A (local) verification theorem (Theorem 4.4) is derived as tool for verifying optimality of a constructed candidate value-function, as direct consequence of a dual representation of the value function as pointwise minimum of stochastic supersolutions (Proposition 4.3).

Gerber–Shiu functions. A key point in the presented approach is the decomposition of the candidate value function preceding and within a continuation band $[a, b]$

$$v_{a,b}(x) = \begin{cases} f(x), & x < a, \\ F(x) + W^{(q)}(x)G(a, b), & x \in [a, b], \end{cases} \tag{1.12}$$

into a *nonhomogeneous* solution $F(x)$, which will be called a *Gerber–Shiu function*, and the product of the homogeneous scale function $W^{(q)}(x)$ and a "barrier-influence" function $G(a, b)$ defined in (6.2), which needs to be maximized at b and be smooth at a.

Note that the function G in the decomposition (1.12) is only determined up to a constant, but becomes fixed once F has been selected; see (7.7).

To ensure smoothness at a, it seems then natural to use a "smooth Gerber–Shiu function" $F_f(x)$ associated to a given penalty $f(x), x \in (-\infty, a)$. Informally, $F_f(x)$ is the "smooth nonhomogeneous solution" of the Dirichlet problem on $\{x \geq a\}$ with boundary condition $f(x), x \in (-\infty, a)$. More precisely, it is defined in Definitions 5.1 and 5.2 in Section 5 by subtracting a multiple of the homogeneous scale function $W^{(q)}(x)$ out of the solutions of either the two-sided, or the reflected exit problem, such that the remaining part is *continuous* on $\mathbb{R}$ if f is continuous, and *continuously differentiable* on $\mathbb{R}$ if f is continuously differentiable on $\mathbb{R}_-$ and X has unbounded variation. This results in the explicit formula (5.4).

For exponential penalties $w(x) = e^{xv}$, the Gerber–Shiu function takes a simple form (5.17), which may be used also as a generating function for the expected payoffs associated to polynomial penalties $x^k, k = 0, 1, \dots$.

Decomposition (1.12) with $F_f(x)$ chosen to fit the imposed penalty $f(x) = w(x)$ already determines the value function on the first continuation band (and the value function in the lump-sum dividend taking bands surrounding it); see Proposition 7.2 and Theorem 7.6. It also yields a necessary and sufficient criterion for optimality of two-dividend barrier policies with one barrier at zero, which is analogous to (1.10); see Theorem 10.3.

Contents. The remainder of the paper is organized as follows. Sections 2 and 3 are devoted to the formulation of the dividend-penalty and the corresponding HJB equation. In Section 4 the definition of stochastic solution is given in this context, and a verification result is established. Section 5 is concerned with Gerber–Shiu functions, and Sections 6 and 7 are devoted to single and two-band strategies. Section 8 is devoted to a key auxiliary result (Lemma 8.1). Conditions for optimalty of single and two-band strategies and a construction of the candidate value-function in terms of scale functions are given in Sections 9, 10 and 11. The optimal value function is shown to be the unique stochastic solution of the HJB equation in Section 12. Some examples are analyzed in Section 13. Some of the proofs are deferred to the Appendix.

2. The dividend-penalty control problem. Assume that the beneficiaries control the timing and size of dividend payments made by the company, and are liable to pay at the moment τ^π of ruin the penalty $-w(U^\pi_{\tau^\pi})$, which may be used to cover (part of) the claim that led to insolvency, where w is a penalty.

DEFINITION 2.1. (i) For any $a \in \mathbb{R}$, denote by $\mathcal{R}_a$ the set of càdlàg[3] functions $w : (-\infty, a] \to \mathbb{R}$ that are left-continuous at a, admit a finite first left-derivative

[3] càdlàg = right-continuous with left-limits.

$w'_-(a)$ at a and satisfy the integrability condition

$$\sup_{y>1} \int_{[y,\infty)} \sup_{u\in[y-1,y]} \big|w(a+u-z)\big|\nu(\mathrm{d}z) < \infty. \tag{2.1}$$

(ii) A *penalty* $w:\mathbb{R}_- \to \mathbb{R}_-$, with $\mathbb{R}_- = (-\infty, 0]$, is a function from the set $\mathcal{R}_0$ that is increasing. The collection of penalties is denoted by $\mathcal{P}$.

The beneficiaries seek to maximize the sum of the expected discounted cumulative dividend payments and an expected penalty payment by paying out dividends according to an admissible policy. The present value of the penalty payment discounted at rate $q > 0$, considered as function of the level of initial reserves, is called the *Gerber–Shiu penalty function* associated to the penalty w, and is given by

$$\mathcal{W}_w^\pi(x) := \mathbb{E}_x\big[\mathrm{e}^{-q\tau^\pi} w\big(U^\pi_{\tau^\pi}\big)\big], \qquad x \in \mathbb{R}_+.$$

For any penalty $w \in \mathcal{P}$, it holds that, for any level of initial capital $x \in \mathbb{R}_+$, $\mathcal{W}_w^\pi(x)$ is bounded uniformly over $\pi \in \Pi$; see Lemma 3.3.

The objective of the beneficiaries of the insurance company is described by the following stochastic control problem:

$$v_*(x) = \sup_{\pi\in\Pi} v_\pi(x), \qquad v_\pi(x) := \mathcal{W}_w^\pi(x) + \mathbb{E}_x\bigg[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\mu_K(\mathrm{d}t)\bigg], \tag{2.2}$$

for $x \in \mathbb{R}_+$, where Π denotes the set of admissible dividend policies π and μ_K is the (signed) random measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ defined by

$$\mu_K^\pi\big([0,t]\big) = D_t^\pi - KN_t^\pi, \tag{2.3}$$

with N_t^π and D_t^π equal to the counting process defined in (1.11) and the cumulative amount of dividends that has been paid out by time t, respectively. It is assumed throughout that w is a penalty ($w \in \mathcal{P}$) and that there is positive net income, $\eta := \mathbb{E}[X_1] > 0$. A solution to the stochastic control problem in (2.2) consists of a pair (u, π^*) of a function $u:\mathbb{R}_+ \to \mathbb{R}$ and a policy $\pi^* \in \Pi$ satisfying $v_*(x) = u(x) = v_{\pi^*}(x)$ for all $x \in \mathbb{R}_+$.

3. Dynamic programming and HJB equation. The analysis of the stochastic optimal control problem (2.2) starts from the observation that the value function v_* satisfies a dynamic programming equation.

PROPOSITION 3.1. (i) *Extending v_* to the negative half-axis by $v_*(x) = w(x)$ for $x < 0$, we have for any $\tau \in \mathcal{T}$, the set of* $\mathbf{F}$*-stopping times*, $v_*(x) = \sup_{\pi\in\Pi} v_{\pi,\tau}(x)$ *where*

$$v_{\pi,\tau}(x) := \mathbb{E}_x\bigg[\mathrm{e}^{-q(\tau\wedge\tau^\pi)} v_*\big(U^\pi_{\tau\wedge\tau^\pi}\big) + \int_{[0,\tau\wedge\tau^\pi]} \mathrm{e}^{-qs}\mu_K^\pi(\mathrm{d}s)\bigg]. \tag{3.1}$$

(ii) *For any fixed* $\pi \in \Pi$, *the process* $V^\pi = \{V_t^\pi, t \in \mathbb{R}_+\}$ *given by*

$$V_t^\pi := e^{-q(t\wedge\tau^\pi)} v_*(U^\pi_{t\wedge\tau^\pi}) + \int_{[0,t\wedge\tau^\pi]} e^{-qs} \mu_K^\pi(ds) \tag{3.2}$$

is an $\mathbf{F}$-*supermartingale.*

REMARK 3.2. Note that the integration domains $[0, \tau \wedge \tau^\pi]$ and $[0, t \wedge \tau^\pi]$ in (3.1) and (3.2) are consistent with the domain $[0, \tau^\pi)$ in (2.2) as $\mu_K(\{\tau^\pi\})$ is equal to 0 for any policy $\pi \in \Pi$.

The proof of Proposition 3.1(i) follows by straightforward adaptation of classical arguments (see, e.g., [7], pages 276–277), while that of Proposition 3.1(ii) is deferred to Appendix A.

The next step is to identify the HJB equation in the current setting. As the beneficiaries may decide to pay out part of the reserves immediately as lump-sum dividend the value function v_* satisfies in addition to the dynamic programming equation the following gradient condition (see Lemma 3.3):

$$v_*(x) - v_*(y) \geq (x - y - K) \qquad \text{for al } x, y > 0 \text{ with } x > y, \tag{3.3}$$

or equivalently,

$$\begin{aligned} &\mathrm{d}_{v_*}(x) \geq 1 \qquad \text{for all } x > 0, \text{ with for any function } g:\mathbb{R} \to \mathbb{R}, \\ &\mathrm{d}_g(x) = \inf_{y\in(0,x)} \frac{g(x) - g(x-y) + K}{y}, \qquad x > 0. \end{aligned} \tag{3.4}$$

Note that in the case $K = 0$ and when $v_*|_{\mathbb{R}_+\setminus\{0\}}$ is in $C^1(\mathbb{R}_+ \setminus \{0\})$ the gradient constraint in (3.3) is equivalent to the condition

$$v_*'(x) \geq 1 \qquad \text{for all } x > 0.$$

Rather than to pay out dividends immediately, the beneficiaries may decide to postpone such payments to a future epoch. Provided the value function v_* were sufficiently regular, it would hold at level x of the reserves that $\mathbb{E}_x[e^{-q(t\wedge T_0^-)} v_*(X_{t\wedge T_0^-})] = v_*(x) + t(\Gamma v_*(x) - q v_*(x)) + o(t)$ for $t \searrow 0$, where $T_0^- = \inf\{t \geq 0 : X_t < 0\}$, and Γ denotes the infinitesimal generator of the Feller semi-group of X which acts on $f \in C_c^2(\mathbb{R}_+)$ as (cf. Sato [35], Theorem 31.5)

$$\Gamma f(x) = \frac{\sigma^2}{2} f''(x) + \eta f'(x) + \int_{\mathbb{R}_+\setminus\{0\}} [f(x-y) - f(x) + y f'(x)] \nu(dy), \tag{3.5}$$

for $x \in \mathbb{R}_+$, where f' denotes the derivative of f and $\eta = \psi'(0)$. Heuristically, this suggests that v_* satisfies $\Gamma v_*(x) - q v_*(x) \leq 0$ at any $x > 0$, and that it is not optimal to postpone a dividend payment at level x in case $\Gamma v_*(x) - q v_*(x) < 0$.

As far as the boundary condition at $x = 0$ is concerned, it follows from (2.2) that $v_*(0) = w(0)$ if and only if ruin is immediate with zero initial capital (i.e., $\tau^\pi = 0$ $\mathbb{P}_0$ a.s.), which is precisely the case if X has paths of unbounded variation. Thus the boundary condition at $x = 0$ is imposed precisely if the Gaussian coefficient σ^2 is strictly positive or the Lévy measure ν does not finitely integrate x around 0 ($\nu_{0,1} = \infty$). In particular, in the case of the Cramér–Lundberg model or when X has paths of finite variation, $v_*(0)$ is in general different from $w(0)$.

By the above discussion, one is led to the following form of the HJB equation associated to the optimal control problem (2.2), expressed in a unified manner for general cost $K \geq 0$:

$$\max\{\Gamma g(x) - qg(x), 1 - \mathrm{d}_g(x)\} = 0, \qquad x > 0, \tag{3.6}$$

subject to the boundary condition

$$\begin{cases} g(x) = w(x), & \text{for all } x < 0, \text{ and} \\ g(0) = w(0), & \text{in the case } \{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\}, \end{cases} \tag{3.7}$$

where the function d_g is defined in (3.4).

3.1. *Properties of the value function.* For later reference a number of properties of the value function are collected below.

LEMMA 3.3. (i) *The function $x \mapsto v_*(x)$ is continuous on $\mathbb{R}_+$, and v_* satisfies equation* (3.3).

(ii) *For any $q > 0$, $x \in \mathbb{R}_+$ and $w \in \mathcal{P}$, there exists a $C \in \mathbb{R}_+ \setminus \{0\}$ such that the following bound holds true*:

$$\mathbb{E}_x\left[\sup_{t\in\mathbb{R}_+,\pi\in\Pi}\left\{\mathrm{e}^{-qt}U_t^\pi \mathbf{1}_{\{t<\tau^\pi\}} + \int_0^t \mathrm{e}^{-qs}\,\mathrm{d}D_s^\pi + \int_0^t \mathrm{e}^{-qs}(\overline{X}_s - \underline{X}_s)\,\mathrm{d}s\right\}\right] \\ + \sup_{y\in\mathbb{R}_+}\sup_{\pi\in\Pi}\mathbb{E}_y\left[\mathrm{e}^{-q\tau}\left|w\left(U_\tau^\pi\right)\right|\right] < C,$$

with $\overline{X}_t = \sup_{s\leq t} X_s$ and $\underline{X}_t = \inf_{s\leq t} X_s$ denoting the supremum and infimum of X_s over the $s \in [0,t]$.

(iii) *v_* is dominated by an affine function*: *for any $x \in \mathbb{R}_+$, $v_*(0) - K \leq v_*(x) - x \leq \frac{1}{\Phi(q)}$, and the process $V^\pi = \{V_t^\pi, t \in \mathbb{R}_+\}$ defined in* (3.2) *is a uniformly integrable* (*UI*) **F**-*supermartingale.*

The proof of part (i) is deferred to Appendix B.

PROOF OF LEMMA 3.3(II). The following bounds hold true:

$$\sup_{t\in\mathbb{R}_+} \mathrm{e}^{-qt}U_t^\pi \mathbf{1}_{\{t<\tau^\pi\}} \leq \sup_{t\in\mathbb{R}_+} \mathrm{e}^{-qt}X_t \leq \sup_{t\in\mathbb{R}_+}\int_t^\infty q\mathrm{e}^{-qs}\overline{X}_s\,\mathrm{d}s. \tag{3.8}$$

Since the running supremum $\overline{X}_{\mathbf{e}_q}$ at an independent exponential random time $\mathbf{e}_q$ with mean q^{-1} under $\mathbb{P}_0$ follows an exponential distribution with parameter $\Phi(q)$ (e.g., Bertoin [12], Corollary VII.2), the expectation under $\mathbb{P}_x$ of the expression on the right-hand side of (3.8) is bounded by $x + 1/\Phi(q)$.

The compensation formula applied to the Poisson point process $(\Delta X_t, t \in \mathbb{R}_+)$, the monotonicity of w and the fact that $w(0)$ is nonpositive yield that the following inequalities holds true, for any $x \in \mathbb{R}_+$:

$$\begin{aligned}\mathbb{E}_x\big[\mathrm{e}^{-q\tau^\pi} w\big(U^\pi_{\tau^\pi}\big)\big] &\geq w(-1) + \mathbb{E}_x\big[\mathrm{e}^{-q\tau^\pi} w\big(U^\pi_{\tau^\pi}\big)\mathbf{1}_{\{U^\pi_{\tau^\pi}<-1\}}\big] \\ &= w(-1) + \int_0^\infty \int_0^\infty w(y-z)\mathbf{1}_{\{y-z<-1\}}\nu(\mathrm{d}z)\tilde{R}^q_x(\mathrm{d}y),\end{aligned}$$

where $\tilde{R}^q_x(\mathrm{d}y)$ denote the q-potential measure of U^π under $\mathbb{P}_x$, $\tilde{R}^q_x(\mathrm{d}y) = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(U^\pi_t \in \mathrm{d}y, t < \tau^\pi)$. The right-hand side of (3.9) is bounded below, as w satisfies the integrability condition (2.1) (as $w \in \mathcal{P}$). $\square$

PROOF OF LEMMA 3.3(III). In the case $K = 0$ integration by parts, the non-negativity of w and condition (1.7) of "no exogenous ruin" imply that

$$\begin{aligned}v_\pi(x) &\leq \mathbb{E}_x\bigg[\int_{[0,\tau^\pi)} \mathrm{e}^{-qt}\,\mathrm{d}D^\pi_t\bigg] = \mathbb{E}_x\bigg[\int_0^{\tau^\pi} q\mathrm{e}^{-qs}D^\pi_s\,\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}D^\pi_{\tau^\pi}\bigg] \\ &\leq \mathbb{E}_x\bigg[\int_0^{\tau^\pi} q\mathrm{e}^{-qs}X_s\,\mathrm{d}s + \mathrm{e}^{-q\tau^\pi}X_{\tau^\pi-}\bigg] \leq \mathbb{E}_x\bigg[\int_0^\infty q\mathrm{e}^{-qs}\overline{X}_s\,\mathrm{d}s\bigg],\end{aligned}$$

which is equal to $x + \frac{1}{\Phi(q)}$ since, as noted before, $\overline{X}_{\mathbf{e}_q} \sim \mathrm{Exp}(\Phi(q))$ under $\mathbb{P}_0$. In the case $K > 0$, then the above bound remains valid since the value $v_*(x)$ decreases if the transaction cost K increases. The lower bound for the value-function follows from part (i) (with $x = 0$). The uniform integrability of V^π is a consequence of the fact that V^π is dominated by an integrable random variable, in view of the bounds in parts (ii). $\square$

3.2. *Generator and boundary condition.* From the HJB equation (3.6) one would expect that, on any interval I on which the restriction $v_*|_I$ has unit derivative, the function $\Gamma v_* - qv_*$ is nonpositive. Below this function is expressed explicitly in terms of the characteristic triplet of X. More generally, in the next result the form is specified of the generator applied to the functions $\widetilde{\ell}^w_{a,b} : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$, given by

$$\widetilde{\ell}^w_{a,b}(z) = \begin{cases} \ell_{a,b}(z), & z \geq a, \\ w(z), & z < a, \end{cases}$$
$$\text{with } \ell_{a,b} : [a,\infty) \to \mathbb{R}\text{: } \ell_{a,b}(x) = b(x-a) + w(a),$$

where $w : (-\infty, a] \to \mathbb{R}$ is a Borel-function satisfying the integrability condition

$$\forall x > a : \int_{(x-a,\infty)} \big|w(x-z)\big|\nu(\mathrm{d}z) < \infty. \tag{3.9}$$

For any such function w and any $a \in \mathbb{R}$, the operator ${}_a\Gamma^w_\infty : C^2([a,\infty)) \to D((a,\infty))$ is defined as follows: for $x > a$,

$$\begin{aligned} {}_a\Gamma^w_\infty f(x) = {} & \frac{\sigma^2}{2} f''(x) + \big(\eta + \overline{\nu}_1(x-a)\big) f'(x) - \big(q + \overline{\nu}(x-a)\big) f(x) \\ & + \int_{(0,x-a]} [f(x-y) - f(x) + f'(x)y] \nu(\mathrm{d}y) \\ & + \int_{(x-a,\infty)} w(x-y)\nu(\mathrm{d}y), \end{aligned} \tag{3.10}$$

where $\overline{\nu}(x) = \nu((x,\infty))$ and $\overline{\nu}_1(x) = \int_{(x,\infty)} y\nu(\mathrm{d}y)$. It follows by comparison with form (3.5) of the infinitesimal generator Γ that for any $f \in C^2_c(\mathbb{R})$ with $f|_{(-\infty,a]} = w$ it holds $(\Gamma f - qf)(x) = {}_a\Gamma^w_\infty g(x)$ for $x > a$ with $g = f|_{[a,\infty)}$. The form of the generator applied to $\ell_{a,b}$ is given in the following result:

LEMMA 3.4. *Let $a, b \in \mathbb{R}$ and let w be any Borel function satisfying integrability condition* (3.9). (i) *For any $x > a$, $({}_a\Gamma^w_\infty \ell_{a,b})(x)$ is given by*

$$\begin{aligned} & \eta\ell'_{a,b}(x) - q\ell_{a,b}(x) + \int_{\mathbb{R}_+\setminus\{0\}} [\widetilde{\ell}^w_{a,b}(x-z) - \ell_{a,b}(x) + z\ell'_{a,b}(x)]\nu(\mathrm{d}z) \\ & \quad = b\eta - q\big(b(x-a) + w(a)\big) \\ & \qquad + \int_{(x-a,\infty)} \{w(x-z) - w(a) + b(z+a-x)\}\nu(\mathrm{d}z). \end{aligned} \tag{3.11}$$

(ii) *Suppose $({}_a\Gamma^w_\infty \ell_{a,b})(x) \le 0$ for all $x > a$ and* $\sup_{x>a} \int_{(x-a,\infty)} |w(x-z) - w(a) + b(z+a-x)|\nu(\mathrm{d}z) < \infty$. *Then $\{\mathrm{e}^{-q(t\wedge T_a^-)}\widetilde{\ell}^w_{a,b}(X_{t\wedge T_a^-}), t \in \mathbb{R}_+\}$ is an* $\mathbf{F}$-*supermartingale.*

PROOF. (i) The assertion directly follows from the form (3.10) of the operator ${}_y\Gamma^w_\infty$.

(ii) An application of Itô's lemma [which is justified since $\ell_{a,b}$ is $C^2([a,\infty))$] shows that the following process is an $\mathbf{F}$-local martingale:

$$\mathrm{e}^{-q(t\wedge T_a{}^-)}\widetilde{\ell}^w_{a,b}(X_{t\wedge T_a^-}) - \int_0^{t\wedge T_a^-} \mathrm{e}^{-qs}\, {}_a\Gamma^w_\infty \ell_{a,b}(X_s)\,\mathrm{d}s. \tag{3.12}$$

Hence the assumptions (together with the fact $\int_0^{T_a^-} \mathbf{1}_{\{X_s=a\}}\,\mathrm{d}s = 0$ $\mathbb{P}$-a.s.) imply the asserted supermartingale property. □

4. Stochastic solutions of the HJB equation. While, as was mentioned in the Introduction, it is in general not to be expected that the HJB equation in (3.6) admits a classical solution, it will be shown in Section 12.1 that the optimal value-function v_* is the unique *stochastic solution* to the HJB equation. A real-valued function g with domain $\mathbb{R}$ and sublinear growth, satisfying the boundary condition (3.7) and the gradient constraint $\mathrm{d}_g(x) \geq 1$ for all $x > 0$, will be called a stochastic solution of the HJB equation given in (3.6) if the stochastic processes

$$\begin{aligned} \overline{M}^{g,T_I} &:= \{\mathrm{e}^{-q(t\wedge T_I)} g(X_{t\wedge T_I}), t \in \mathbb{R}_+\}, \\ T_I &:= \inf\{t \geq 0 : X_t \notin I\}, \end{aligned} \tag{4.1}$$

with $\inf \varnothing = \infty$, are $\mathbf{F}$-martingales for any closed interval I contained in $\mathcal{C}_g$, the "no dividend region" corresponding to the function g,

$$\mathcal{C}_g := \{x \in \mathbb{R}_+ \setminus \{0\} : \mathrm{d}_g(x) > 1\}, \tag{4.2}$$

and are $\mathbf{F}$-supermartingales for any closed interval I contained in $\mathbb{R}_+ \setminus \{0\}$.

More specifically, the notions of (local) stochastic (super-, sub-) solutions are defined as follows:

DEFINITION 4.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a càdlàg function satisfying the boundary condition (3.7) and the linear growth condition

$$\sup_{x \in \mathbb{R}_+} \frac{|g(x)|}{x+1} < \infty. \tag{4.3}$$

(i) g is a *local stochastic supersolution on the closed interval* $I \subset \mathbb{R}_+$ of the HJB equation (3.6) if

$$\overline{M}^{g,T_I} \text{ is a UI } \mathbf{F}\text{-supermartingale and } \mathrm{d}_g(x) \geq 1 \text{ for any } x \in I \setminus \{0\}.$$

(ii) g is called a *local stochastic subsolution on the closed interval* $I \subset \mathcal{C}_g$ of the HJB equation (3.6) if

$$\overline{M}^{g,T_I} \text{ is a UI } \mathbf{F}\text{-submartingale.}$$

(iii) g is a *stochastic supersolution* [*stochastic subsolution*] of the HJB equation if g is a local stochastic supersolution on $\mathbb{R}_+$ [local stochastic subsolution on I for all closed intervals $I \subset \mathcal{C}_g$], respectively.

(iv) g is a *stochastic solution* of the HJB equation if g is both a stochastic supersolution and a stochastic subsolution of the HJB equation.

REMARK 4.2. (i) The optimal value-function v_* is a stochastic supersolution. This follows as a direct consequence of Lemma 3.3(i), (iii) (taking π equal to the "waiting strategy" $\pi_\varnothing$ of not paying any dividends) and Doob's Optional Stopping theorem.

(ii) The terms "stochastic supersolution" and "stochastic subsolution" are justified by the fact that stochastic supersolutions dominate stochastic subsolutions (under some regularity condition); see Proposition 12.6.

(iii) When g is a local stochastic supersolution on a finite partition of intervals of $\mathbb{R}_+$, a global super-martingale property holds true on $\mathbb{R}_+$, provided that g is differentiable at the boundaries of the intervals when X has unbounded variation; see Corollary 8.2.

The following global representation of the optimal value function v_* in terms of the collection of stochastic supersolutions provides a key step in the solution of the optimal control problem in (2.2):

PROPOSITION 4.3. (i) *The value function v_* is the smallest stochastic supersolution of the HJB equation* (3.6)

$$v_*(x) = \min_{g \in \mathcal{G}^+} g(x), \tag{4.4}$$

for all $x \in \mathbb{R}_+$, where $\mathcal{G}^+$ denotes the family of stochastic supersolutions of the HJB equation (3.6).

(ii) *For any $a, b \in \mathbb{R}_+$ with $a < b$, representation* (4.4) *remains valid for all $x \in (-\infty, b]$ if the set $\mathcal{G}^+$ is replaced by the set $\mathcal{G}^+_{a,b}$ of local stochastic supersolutions g on $[a, b]$ satisfying the condition*

$$\begin{cases} g(x) = v^*(x), & \text{for all } x \in [0, a) \cup \{b\}, \text{ and in addition}, \\ g(a) = v^*(a), & \text{if } X \text{ has unbounded variation}. \end{cases} \tag{4.5}$$

Proposition 4.3, the proof of which is given in Section 4.1, yields the following *(local) verification theorem*, which is one of the main results of the paper:

THEOREM 4.4. (i) *If there exist $a, b \in \mathbb{R}_+$ with $b > a \geq 0$, $\pi \in \Pi$ and $g \in \mathcal{G}^+$ satisfying $g(x) = v_{\pi,\tau_a^\pi}(x)$ for all $x \in [a, b]$, with $\tau_a^\pi = \inf\{t \geq 0 : U_t^\pi < a\}$, then it holds $v_*(x) = v_{\pi,\tau_a^\pi}(x)$ for all $x \in [a, b]$.*

(ii) *In particular, if there exist $\pi \in \Pi$ and $g \in \mathcal{G}^+$ satisfying $g(x) = v_\pi(x)$ for all $x \in \mathbb{R}_+$, then $g = v_*$ and π is an optimal strategy.*

PROOF. In view of the dynamic programming equation (3.1), it follows that v_* dominates v_{π,τ_a^π}, while the dual representation (4.4) in Proposition 4.3 implies $v_*(x) \leq g(x)$ for all $x \in \mathbb{R}_+$, so that when g is equal to v_{π,τ_a^π} on the interval $[a, b]$, it follows that $v^*(x) = g(x) = v_{\pi,\tau_a^\pi}(x)$ for all $x \in [a, b]$, which establishes part (i). Part (ii) follows by a similar line of reasoning. $\square$

This verification result will be used in the piecewise construction of the value-function v_*, in Sections 6–11. It can also be used to deduce that the value function is affine for large levels of the reserves if ν is finite.

PROPOSITION 4.5. *Let the measure* ν *have finite mass. For some* $y \in \mathbb{R}_+$, *the function* v_* *restricted to* $[y, \infty)$ *takes the form*

$$v_*(x) = x - y + v_*(y) \qquad \text{for any } x - y \in \mathbb{R}_+, \tag{4.6}$$

and it is optimal to immediately pay out a lump-sum dividend for all sufficiently large levels of the reserves.

PROOF. The local verification theorem [Theorem 4.4(i)] in conjunction with Lemma 3.4 imply that condition in (4.6) holds if the supremum $m_* := \sup_{x>y} \int_{(x-y,\infty)} |v_*(x-z) - v_*(y) + z + y - x| \nu(\mathrm{d}z)$ is finite and

$$\text{for all } y \in \mathbb{R}_+ \text{ sufficiently large} \qquad \{\forall x > y : ({}_y\Gamma_\infty^{v_*} \ell_{y,1})(x) \le 0\}. \tag{4.7}$$

This is verified next. The expression for ${}_y\Gamma_\infty^{v_*} \ell_{y,1}$ in (3.11) for $x > y$ can be bounded above by

$$\eta - q(x - y + v_*(y)) + \int_{(x-y,x)} |v_*(x-z) - v_*(y) + z + y - x| \nu(\mathrm{d}z)$$
$$+ \int_{(x,\infty)} |w(x-z) - v_*(y) + z + y - x| \nu(\mathrm{d}z).$$

Hence, in view of (3.3), the linear bounds in Lemma 3.3(iii) and the monotonicity of w, the first and second integrals are bounded above by a constant times $\lambda(1+m)$ and by $\int_{(0,\infty)} |w(-z)| \nu(\mathrm{d}z) + \lambda(|y - v^*(y)|) + \lambda m$ with $\lambda = \nu(0, \infty)$ and $\lambda m = \int_{(0,\infty)} x \nu(\mathrm{d}x)$. Since the integral with w as integrand is finite [as $w \in \mathcal{P}$ satisfies (2.1)] it follows that m_* is finite. Moreover, as $v_*(y) \to \infty$ and $v^*(y) - y$ is bounded as $y \to \infty$ [Lemma 3.3(iii)], it is clear that (4.7) is satisfied, and the proof is complete. □

4.1. *Proof of the dual representation.* The proof of Proposition 4.3 is based on a representation of v_* as the point-wise minimum of a class of "controlled" supersolutions of the HJB equation.

DEFINITION 4.6. For any closed interval I, a Borel-measurable function $H : \mathbb{R} \to \mathbb{R}$ is called a *controlled supersolution for the stochastic control problem* (2.2) *on the closed interval* I if it holds for any $\pi \in \Pi$ that

$$\widetilde{M}_t^{H,\pi} := \mathrm{e}^{-q(\tau_I^\pi \wedge t)} H(U_{\tau_I^\pi \wedge t}^\pi) + \int_{[0,\tau_I^\pi \wedge t]} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s) \tag{4.8}$$

is a UI $\mathbf{F}$-supermartingale, with $\tau_I^\pi = \inf\{t \ge 0 : U_t^\pi \notin I\}$, subject to boundary condition

$$\begin{cases} H(x) \ge v_*(x), & \text{for } x < y := \min I \text{ and } x = z := \sup I \text{ if } z < \infty, \text{ and,} \\ H(y) \ge v_*(y), & \text{if } X \text{ has unbounded variation.} \end{cases}$$

The family of such functions will be denoted by $\mathcal{H}_I$.

PROPOSITION 4.7. *For any closed interval I the value-function v_* restricted to I admits the following representation*:

$$v_*(x) = \min_{H \in \mathcal{H}_I} H(x) \qquad \textit{for all } x \in \mathbb{R}_+.$$

PROOF. The proof rests on standard arguments. Fix $x \in \mathbb{R}_+$, a closed interval I in $\mathbb{R}_+$, and let H be any element of $\mathcal{H}_I$, and $\pi \in \Pi$ any admissible policy. The supermartingale property and uniform integrability (Definition 4.6) yield

$$\begin{aligned} H(x) &\geq \lim_{t\to\infty} \mathbb{E}_x\left[\mathrm{e}^{-q(\tau_I^\pi \wedge t)} H\left(U^\pi_{\tau_I^\pi \wedge t}\right) + \int_{[0,\tau_I^\pi \wedge t]} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s)\right] \\ &\geq \mathbb{E}_x\left[\mathrm{e}^{-q\tau_I^\pi} v_*\left(U^\pi_{\tau_I^\pi}\right) + \int_{[0,\tau_I^\pi]} \mathrm{e}^{-qs} \mu_K^\pi(\mathrm{d}s)\right], \end{aligned}$$

where the convention $\exp\{-\infty\} = 0$ is used. Taking the supremum over $\pi \in \Pi$ and using the dynamic programming equation (Proposition 3.1) show that $H(x) \geq v_*(x)$. Since $H \in \mathcal{H}_I$ was arbitrary, it holds thus

$$\inf_{H \in \mathcal{H}_I} H(x) \geq v_*(x).$$

The inequality in the display is in fact an equality since v_* is a member of $\mathcal{H}_I$, by virtue of Doob's optional stopping theorem and the fact that V^π is a UI supermartingale [Lemma 3.3(iii)]. $\square$

The proof of the representations of the value function v_* in Proposition 4.3 rests on the fact that for any admissible policy $\pi \in \Pi$ and stochastic supersolution there exists a corresponding "controlled" supermartingale.

LEMMA 4.8 (Shifting lemma). *Let $I \subset \mathbb{R}_+$ be any closed interval. If g is a local stochastic supersolution on I, then g is a controlled supersolution on I.*

Given the shifting lemma, the proof of the dual representations in Proposition 4.3 can be completed as follows:

PROOF OF PROPOSITION 4.3. (i) The representation follows from Proposition 4.7 in view of the following two observations: (a) $\mathcal{G}^+$ is contained in $\mathcal{H}_{[0,\infty)}$ [Remark 4.2(i)] and (b) v_* is an element of the set $\mathcal{G}^+$ [by Lemma 3.3(iii)].

(ii) The proof is analogous to that of part (i), using the facts $\mathcal{G}^+_{a,b} \subset \mathcal{H}_{[a,b]}$ [Lemma 4.8(ii)] and $v_* \in \mathcal{G}^+_{a,b}$ [by Remark 4.2(i) and Doob's Optional Stopping theorem]. $\square$

PROOF OF LEMMA 4.8. Fix arbitrary $\pi \in \Pi$ and $s, t \in \mathbb{R}_+$ with $s < t$. Note that $\widetilde{M}^{g,\pi}$ is $\mathbf{F}$-adapted (as g is a Borel-measurable), while $\widetilde{M}^{g,\pi}$ is UI by the linear growth condition and Lemma 3.3. Furthermore, the following (in)equalities hold true:

$$\mathbb{E}[\widetilde{M}_t^{g,\pi}|\mathcal{F}_{s\wedge\tau^\pi}] \overset{\text{(a)}}{=} \lim_{n\to\infty} \mathbb{E}[\widetilde{M}_t^{g,\pi_n}|\mathcal{F}_{s\wedge\tau^\pi}] \overset{\text{(b)}}{\leq} \lim_{n\to\infty} \widetilde{M}_{s\wedge\tau^\pi}^{g,\pi_n} \overset{\text{(c)}}{=} \widetilde{M}_{s\wedge\tau^\pi}^{g,\pi} \overset{\text{(d)}}{=} \widetilde{M}_s^{g,\pi},$$

where the sequence $(\pi_n)_{n\in\mathbb{N}}$ of strategies is defined by $\pi_n = \{D_t^{\pi_n}, t \in \mathbb{R}_+\}$ with $D_0^{\pi_n} = D_0^\pi$ and

$$D_u^{\pi_n} = \begin{cases} \sup\{D_v^\pi : v < u, v \in \mathbb{T}_n\}, & 0 < u < \tau^\pi, \\ D_{\tau^\pi-}^{\pi_n}, & u \geq \tau^\pi, \end{cases}$$

with $\mathbb{T}_n := (\{t_k := s + (t-s)\frac{k}{2^n}, k \in \mathbb{Z}\} \cup \{0\}) \cap \mathbb{R}_+$. Since s and t are arbitrary, it thus follows that $\widetilde{M}^{g,\pi}$ is a $\mathbf{F}$-supermartingale.

The remainder of the proof is devoted to the verification of the (in)equalities (a)–(d) in above display. (a) Note that the sequence $(D^{\pi_n})_n$ is monotone ($D^{\pi_n} \leq D^{\pi_{n+1}}$ for $n \in \mathbb{N}$) and tends to D^π as n tends to infinity, and D^{π_n} is equal to $D_{\tau^\pi-}^{\pi_n}$ on the interval $[\tau^\pi, \infty)$, for each $n \in \mathbb{N}$. Thus the monotone convergence theorem (MCT) in combination with an integration-by-parts implies $\int_{[0,\tau^\pi\wedge t]} \mathrm{e}^{-qs}\,\mathrm{d}D_s^{\pi_n} \nearrow \int_{[0,\tau^\pi\wedge t]} \mathrm{e}^{-qs}\,\mathrm{d}D_s^\pi$. Also, in the case $K > 0$, it holds $\int_{[0,\tau^\pi\wedge t]} \mathrm{e}^{-qs}\,\mathrm{d}N_s^{\pi_n} \nearrow \int_{[0,\tau^\pi\wedge t]} \mathrm{e}^{-qs}\,\mathrm{d}N_s^\pi$. Hence, by right-continuity of the function g, it holds

$$\widetilde{M}_{t\wedge\tau^\pi}^{g,\pi_n} \longrightarrow \widetilde{M}_{t\wedge\tau^\pi}^{g,\pi} \qquad \text{as } n \to \infty, \mathbb{P}\text{-a.s.} \tag{4.9}$$

As the collection $(\widetilde{M}_{t\wedge\tau^\pi}^{g,\pi_n})_n$ is UI, Lebesgue's dominated convergence theorem implies that the equality (a) holds true. Equality (c) is a consequence of the pointwise convergence in (4.9) (which also holds with t replaced by s), while (d) follows since it holds $\widetilde{M}_s^{g,\pi} = \widetilde{M}_{s\wedge\tau^\pi}^{g,\pi}$ (by definition of the process $\widetilde{M}^{g,\pi}$).

Finally, inequality (b) is verified, in what constitutes the key step of the proof. Denote $T_i := \tau^\pi \wedge t_i$ and $M = \widetilde{M}^{g,\pi_n}$, $D = D^{\pi_n}$, and observe that the folowing decomposition holds true:

$$M_t - M_s = \sum_{i=1}^{2^n} Y_i + \sum_{i=1}^{2^n} Z_i$$

$$\text{with } Y_i = \mathrm{e}^{-qT_i} g(X_{T_i} - D_{T_{i-1}}) - \mathrm{e}^{-qT_{i-1}} g(X_{T_{i-1}} - D_{T_{i-1}}),$$

with $Z_i = \mathrm{e}^{-qT_i}(g(X_{T_i} - D_{T_i}) - g(X_{T_i} - D_{T_{i-1}}) + \Delta D_{T_i} - K)\mathbf{1}_{\{\Delta D_i > 0\}}$ and $\Delta D_i = D_{T_i} - D_{T_{i-1}}$. The strong Markov property of X and the definition of U imply that $\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}]$ is equal to

$$\begin{aligned} &\mathrm{e}^{-qT_{i-1}}\mathbb{E}[\mathrm{e}^{-q(T_i-T_{i-1})} g(U_{T_{i-1}} + X_{T_i} - X_{T_{i-1}}) - g(U_{T_{i-1}})|\mathcal{F}_{T_{i-1}}] \\ &\quad = \mathrm{e}^{-qT_{i-1}}\mathbb{E}_{U_{T_{i-1}}}[\mathrm{e}^{-q\tau_i} g(X_{\tau_i}) - g(X_0)], \end{aligned} \tag{4.10}$$

with $\tau_i = T_i \circ \theta_{T_{i-1}}$, where θ denotes the translation-operator. The right-hand side of (4.10) is nonpositive as a consequence of the supermartingale property (4.1) (with $I = \mathbb{R}_+$) and Doob's optional stopping theorem. Furthermore, in view of the bound $\mathrm{d}_g(x) \geq 1$ for any $x \in \mathbb{R}_+ \setminus \{0\}$ it follows that all the Z_i are nonpositive in the case $X_{T_i} - D_{T_i} \geq 0$, while, in the case $X_{T_i} - D_{T_i} < 0$, it holds that Z_i is zero, since $T_i = \tau^\pi$, so that, by construction, $\Delta D_i = D^{\pi_n}_{\tau^\pi} - D^{\pi_n}(\tau_n^+) = 0$ with $\tau_n^+ = \sup\{v < \tau^\pi : v \in \mathbb{T}_n\}$. Hence, the tower-property of conditional expectation yields

$$\mathbb{E}[M_t - M_s | \mathcal{F}_s] \leq \sum_{i=1}^{2^n} \mathbf{1}_{\{T_{i-1} > s\}} \mathbb{E}\big[\mathbb{E}[Y_i | \mathcal{F}_{T_{i-1}}] | \mathcal{F}_s\big] \leq 0.$$

This establishes inequality (b), and the proof is complete. □

5. Gerber–Shiu functions. A key-ingredient for the solution of the optimal control problem (2.2) is a family of martingales given in terms of *Gerber–Shiu functions*, a nonstandard terminology; see Definitions 5.1 and 5.2. While the (homogeneous) q-scale function $W^{(q)}$ is defined to be equal to 0 on the set $(-\infty, 0)$, Gerber–Shiu functions are "inhomogeneous q-scale functions" corresponding to nonzero boundary conditions w on the negative half-line.

The definition of Gerber–Shiu functions is phrased in terms of w and $W^{(q)}$ of which next a number of well-known properties are recalled that will be deployed in the sequel; refer to the review article Kyprianou et al. [24], Chapters 2, 3, for proofs and references. The function $W^{(q)}$ [see (1.5) for its definition] is a "q-harmonic function" for the process X stopped at first entrance into $(-\infty, 0)$. Specifically, for any $a \in \mathbb{R}$, the stopped process

$$\begin{aligned}&\big(\mathrm{e}^{-q(t\wedge T_a^-)} W^{(q)}(X_{t\wedge T_a^-} - a), t \in \mathbb{R}_+\big)\\ &\qquad\text{is an } \mathbf{F}\text{-martingale, with } T_a^- := T_{[a,\infty)} = \inf\{t \in \mathbb{R}_+ : X_t < a\}.\end{aligned} \tag{5.1}$$

Furthermore, the function $W^{(q)}$ is well-known to be continuous and nondecreasing on $[0, \infty)$, and right- and left-differentiable on $(0, \infty)$, with the right-derivative and left-derivative at $x > 0$ denoted by $W^{(q)\prime}(x)$ and $W_-^{(q)\prime}(x)$, respectively, which are right- and left-continuous and satisfy

$$W^{(q)\prime}(x) \leq W_-^{(q)\prime}(x), \qquad x > 0, \tag{5.2}$$

by continuity and log-concavity of $W^{(q)}|_{\mathbb{R}_+}$. In particular, if $\nu_{0,1}$ [which was defined in (1.7)] is infinite, the function $W^{(q)}|_{(0,\infty)}$ is C^1, while $W^{(q)}|_{(0,\infty)}$ is C^2 with $W^{(q)\prime}(0+) = \frac{2}{\sigma^2}$ if the Gaussian coefficient σ^2 is strictly positive.

A function will be referred to as a Gerber–Shiu function if it satisfies the following conditions:

DEFINITION 5.1. Given $a \in \mathbb{R}$ and a *pay-off* $w : (-\infty, a] \to \mathbb{R}$ with $w \in \mathcal{R}_a$, the function $F : \mathbb{R} \to \mathbb{R}$ is called a *Gerber–Shiu function for payoff* w if $F(x - a) = w(x)$ for $x < a$, and

$$\big(\mathrm{e}^{-q(t\wedge T_a^-)} F(X_{t\wedge T_a^-} - a), t \in \mathbb{R}_+\big) \qquad \text{is an } \mathbf{F}\text{-martingale.} \tag{5.3}$$

Of course, such a function F is not unique (as the addition of multiples of $W^{(q)}$ to a Gerber–Shiu function yields another Gerber–Shiu function). It is shown below that there exists a special choice F_w of Gerber–Shiu function that is *continuous* on $\mathbb{R}$ for continuous payoffs w and *continuously differentiable* on $\mathbb{R}$ if X has unbounded variation and w is continuously differentiable (recall that $W^{(q)}$ is continuous nor continuously differentiable on $\mathbb{R}$ in general). The function F_w is defined as follows:

DEFINITION 5.2. Let $q \geq 0$ and $w \in \mathcal{R}_0$. The function $F_w : \mathbb{R} \to \mathbb{R}$ is given by $F_w(x) = w(x)$ for $x < 0$, and by

$$F_w(x) = w(0) + w'_-(0)x - \int_0^x W^{(q)}(x-y) J_w(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}_+, \text{ with} \tag{5.4}$$

$$J_w(x) = \big({}_0\Gamma^w_\infty \ell_{0,w'_-(0)}\big)(x), \tag{5.5}$$

where ${}_0\Gamma^w_\infty \ell_{0,w'_-(0)}$ is given in (3.11) [with $a = 0$ and $b = w'_-(0)$].

The following result confirms that the function $F_{{}_aw}$ is a Gerber–Shiu function that "inherits" the continuity/differentiability from the function w, where, for any function f and $a \in \mathbb{R}$, ${}_af$ denotes the composition of f with the translation-operator θ_a,

$$ {}_af := f \circ \theta_a := f(\cdot + a). \tag{5.6}$$

THEOREM 5.3. *Let $a \in \mathbb{R}$ and $w \in \mathcal{R}_a$. Then ${}_aw \in \mathcal{R}_0$ and the function $F_{{}_aw}$ is a Gerber–Shiu function for payoff w satisfying*

$$\begin{cases} F_{{}_aw}(0) = w(a), & \\ F'_{{}_aw}(0+) = w'_-(a), & \text{in the case } \sigma^2 > 0 \text{ or } \nu_{0,1} = \infty. \end{cases} \tag{5.7}$$

Furthermore, $F_{{}_aw}|_{\mathbb{R}_+}$ is right-differentiable, with right-derivative at $x \in \mathbb{R}_+$ denoted by $F'(x)$. If ${}_aw$ is continuous, then $F_{{}_aw}$ is continuous, and, in the case $w \in C^1(\mathbb{R}_-)$ and $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$, it holds $F_{{}_aw} \in C^1(\mathbb{R})$.

An example of a Gerber–Shiu function is the *Gerber–Shiu penalty function* $\mathcal{V}_w$ corresponding to penalty w

$$\mathcal{V}_w(x) = \mathbb{E}_x\big[\mathrm{e}^{-qT_0^-} w(X_{T_0^-})\big],$$

which admits the following explicit expression in terms of the functions $W^{(q)}$ and F_w (see Biffis and Kyprianou [13] for an equivalent representation of $\mathcal{V}_w$ in terms of $W^{(q)}$):

PROPOSITION 5.4 (Gerber–Shiu penalty function). *Let $w \in \mathcal{R}_0$. For any $x \in \mathbb{R}$ it holds*

$$\mathcal{V}_w(x) = F_w(x) - W^{(q)}(x)\kappa_w \qquad \text{with} \tag{5.8}$$

$$\kappa_w := \left[\frac{\sigma^2}{2}w'(0-) + \frac{q}{\Phi(q)}w(0) - \mathcal{L}w_\nu(\Phi(q))\right], \tag{5.9}$$

where $\mathcal{L}w_\nu$ denotes the Laplace transform of the function $w_\nu(x) = \int_{(x,\infty)}[w(x - z) - w(0)]\nu(\mathrm{d}z)$, $x > 0$.

For later reference two further exit identities are recorded that are also expressed in terms of $W^{(q)}$ and F_w. First, the two-sided exit identity of X on the interval $[a, b]$ which involves the distribution of the pair $(T_{a,b}, X_{T_{a,b}})$ where $T_{a,b} := T_{[a,b]} = T_a^- \wedge T_b^+$, with $T_b^+ := T_{(-\infty,b]} = \inf\{t \in \mathbb{R}_+ : X_t > b\}$, denotes the first exit time from the interval $[a, b]$. Second, a *absorption/reflection exit identity* on the interval $[a, b]$ which concerns the law of the pair $(\tau_a(b), Y^b_{\tau_a(b)})$ and the expected local time up to $\tau_a(b)$ at the level b of Y^b where $\tau_a(b) = \inf\{t \in \mathbb{R}_+ : Y_t^b < a\}$ denotes the first-passage time into the set $(-\infty, a)$ of the process $Y^b = \{Y_t^b, t \in \mathbb{R}_+\}$ given by

$$Y_t^b = X_t - \overline{X}_t^b \qquad \text{with } \overline{X}_t^b = \sup_{s \le t}(X_t - b) \vee 0. \tag{5.10}$$

The identities are given as follows:

PROPOSITION 5.5. *Given $a \in \mathbb{R}$ and a* pay-off $w : (-\infty, a] \to \mathbb{R}$ *with $w \in \mathcal{R}_a$, the following hold for all $b, \delta, \beta \in \mathbb{R}$ with $a < b < \infty$ and $x \in (a, b)$:*

$$\begin{aligned} &\mathbb{E}_x\big[\mathrm{e}^{-qT_{a,b}} w(X_{T_a^-})\mathbf{1}_{\{T_a^- < T_b^+\}}\big] + \delta\mathbb{E}_x\big[\mathrm{e}^{-qT_b^+}\mathbf{1}_{\{T_a^- > T_b^+\}}\big] \\ &\qquad = F_{{}_a w}(x - a) + W^{(q)}(x - a)\frac{\delta - F_{{}_a w}(b - a)}{W^{(q)}(b - a)}, \end{aligned} \tag{5.11}$$

$$\begin{aligned} &\mathbb{E}_x\big[\mathrm{e}^{-q\tau_a(b)} w(Y^b_{\tau_a(b)})\big] + \beta\mathbb{E}_x\left[\int_{[0,\tau_a(b)]} \mathrm{e}^{-qs}\,\mathrm{d}\overline{X}_s^b\right] \\ &\qquad = F_{{}_a w}(x - a) + W^{(q)}(x - a)\frac{\beta - F'_{{}_a w}(b - a)}{W^{(q)\prime}(b - a)}. \end{aligned} \tag{5.12}$$

The proofs of Theorem 5.3 and Proposition 5.4 rests on the following auxiliary results (shown in Section 5.1):

LEMMA 5.6. *Let $w \in \mathcal{R}_0$. The function $F_w|_{\mathbb{R}_+}$ real-valued and continuous and admits the following alternative representation: for $x \ge 0$,*

$$\begin{aligned} F_w(x) = \frac{\sigma^2 w'_-(0)}{2}W^{(q)}(x) + w(0)Z^{(q)}(x) - \int_0^x W^{(q)}(x - y)w_\nu(y)\,\mathrm{d}y& \\ \text{with } Z^{(q)}(x) = 1 + \textstyle\int_0^x W^{(q)}(y)\,\mathrm{d}y.& \end{aligned} \tag{5.13}$$

In particular, it holds $F_w(0) = w(0)$ *and* $\int_0^x |w_\nu(y)|\,\mathrm{d}y < \infty$ *for any* $x \geq 0$, *and in the case that* X *has bounded variation* $w_\nu(0+) < \infty$.

LEMMA 5.7. *Let* $w \in \mathcal{R}_0$. (i) $F_w(x)/W^{(q)}(x) \to \kappa_w$ *as* $x \to \infty$.

(ii) $F_w(x)$ *is left- and right-differentiable at any* $x > 0$ *with right-derivative at* $x > 0$ *given by*

$$\begin{aligned} F_w'(x) &= w_-'(0) - \int_{[0,x)} J_w(x-y)W^{(q)}(\mathrm{d}y) \\ &= F_{w,-}'(x) - W^{(q)}(0)\big(J_w(x+) - J_w(x-)\big), \end{aligned} \tag{5.14}$$

where $F_{w,-}'(x)$ *denotes the left-derivative of* F_w *at* x. *In particular,* $F_w'(0) = w_-'(0)$ *if* X *has unbounded variation*, *and* $F_w'(0) = w_-'(0) - W^{(q)}(0)J_w(0+)$ *if* X *has bounded variation*.

(iii) *The function* $x \mapsto F_w'(x)$ *is right-continuous on* $\mathbb{R}_+ \setminus \{0\}$, *and is* C^1 *on* $\mathbb{R}_+ \setminus \{0\}$ *in the case* $w \in C^1(\mathbb{R}_-)$.

Given these two results the proofs of Proposition 5.4 and Theorem 5.3 can be completed as follows:

PROOF OF PROPOSITION 5.4. Writing $\mathcal{V}_w(x) = w(0)\mathcal{V}_{\mathbf{1}}(x) + \mathbb{E}_x[\mathrm{e}^{-qT_0^-}(w(X_{T_0^-}) - w(0))]$, where $\mathbf{1}$ denotes the function on $\mathbb{R}_-$ that is constant equal to one, and applying the compensation formula (e.g., Bertoin [12], Chapter O) to the Poisson point process $(\Delta X_t, t \in \mathbb{R}_+)$ yields the following expressions for any $x \in \mathbb{R}_+$:

$$\begin{aligned} \mathcal{V}_w(x) - w(0)\mathcal{V}_{\mathbf{1}}(x) &= \int_{[0,\infty)} \int_{(y,\infty)} \big(w(y-z) - w(0)\big)\nu(\mathrm{d}z)U^q(x,\mathrm{d}y) \\ &= W^{(q)}(x)\mathcal{L}w_\nu\big(\Phi(q)\big) - \int_0^x W^{(q)}(x-y)w_\nu(y)\,\mathrm{d}y, \end{aligned} \tag{5.15}$$

$$U^q(x,\mathrm{d}y) = \big[W^{(q)}(x)\mathrm{e}^{-\Phi(q)y} - W^{(q)}(x-y)\big]\,\mathrm{d}y, \qquad y > 0,$$

where $U^q(x,\mathrm{d}y)$ denotes the q-potential measure of X under $\mathbb{P}_x$ killed upon entering $(-\infty,0)$. It follows from Lemmas 5.6 and 5.7 that the integrals in (5.15) are finite. Deploying the form of the Laplace transform of T_0^-, $\mathcal{V}_{\mathbf{1}}(x) = Z^{(q)}(x) - q\Phi(q)^{-1}W^{(q)}(x)$, and the definition of F_w leads to (5.8) [since the term $\frac{\sigma^2}{2}w'(0-)W^{(q)}(x)$ cancels]. $\square$

PROOF OF PROPOSITION 5.5. Denote the left-hand side of (5.12) by $\mathcal{U}_{w,\beta}^{a,b}(x)$, and let $e_{0,a}$ be the function with domain $(-\infty,a]$ that is constant equal to 1. Another application of the compensation formula yields the following representation of $\mathcal{U}_w^{a,b}(x)$ for $x \in [a,b]$:

$$\mathcal{U}_{w,\beta}^{a,b}(x) - w(0)\mathcal{U}_{e_{0,a},0}^{a,b}(x) - \beta\mathcal{U}_{0,1}^{a,b}(x)$$

$$= \int_{[a,b]}\int_{(y,\infty)} \big(w(y-z) - w(0)\big)\nu(\mathrm{d}z) R_{a,b}^q(x,\mathrm{d}y) \qquad \text{with}$$

$$R_{a,b}^q(x,\mathrm{d}y) = \frac{W^{(q)}(x-a)}{W^{(q)\prime}(b-a)} W^{(q)}(b-\mathrm{d}y) - W^{(q)}(x-y)\,\mathrm{d}y,$$

$$\mathcal{U}_{e_{0,a},0}^{a,b}(x) = \mathbb{E}_x\big[\mathrm{e}^{-q\tau_a(b)}\big] = Z^{(q)}(x-a) - q\frac{W^{(q)}(x-a)}{W^{(q)\prime}(b-a)} W^{(q)}(b-a),$$

$$\mathcal{U}_{0,1}^{a,b}(x) = \mathbb{E}_x\left[\int_{[0,\tau_a(b)]} \mathrm{e}^{-qs}\,\mathrm{d}\overline{X}_s^b\right] = \frac{W^{(q)}(x-a)}{W^{(q)\prime}(b-a)},$$

where $R_{a,b}^q(x,\mathrm{d}y)$, $y \in [a,b]$, is the q-resolvent measure of Y^b killed upon entering $(-\infty,a)$ (from Pistorius [34], Theorem 1) and the final two identities in the previous display are from Avram et al. ([4], Theorem 1, [6], Proposition 1). Combining these expressions with representation (5.13) of F_w and taking note of the fact that the term $\frac{\sigma^2}{2}{}_a w'(0-)W^{(q)}(x)$ again cancels yields that (5.12) holds true. Equation (5.11) follows by a similar line of reasoning. $\square$

PROOF OF THEOREM 5.3. That $F_{{}_a w}$ is a Gerber–Shiu function follows from (5.8) (with F_w replaced by $F_{{}_a w}$), the strong Markov property of X and the martingale property (5.1) of $W^{(q)}$. The martingale property (5.3) was shown in Proposition 5.4. The asserted continuity follows from the relation (5.7) combined with the continuity of ${}_a w$ and $F_{{}_a w}|_{\mathbb{R}_+}$ (Theorem 5.3). The assertion that $F_{{}_a w}$ is $C^1(\mathbb{R})$ is a consequence of the following two observations: (i) $F_{{}_a w}|_{\mathbb{R}_+\setminus\{0\}}$ is $C^1(\mathbb{R}_+ \setminus \{0\})$ [by Lemma 5.7(ii)]; (ii) ${}_a w$ is $C^1(\mathbb{R}_-)$ (by assumption) and $w'_-(a) = {}_a w'_-(0) = F'_{{}_a w}(0)$ [by Lemma 5.7(ii)]. $\square$

5.1. *Proofs of Lemmas 5.6 and 5.7.*

PROOF OF LEMMA 5.6. First it is verified that the function on the right-hand side of (5.13) is continuous on $\mathbb{R}_+$. This follows from the continuity on $\mathbb{R}_+$ of $W^{(q)}(x)$, $Z^{(q)}(x)$ and of the final term in (5.4), as functions of x. The continuity of the integral is a consequence of Lebesgue's dominated convergence theorem and the finiteness of $\int_0^x |w_\nu(y)|\,\mathrm{d}y$ for any $x \geq 0$, which in turn holds as w is càdlàg and left-differentiable at 0 ($w \in \mathcal{R}_0$) and ν satisfies the integrability condition $\int_0^1 z^2\nu(\mathrm{d}z) < \infty$. Furthermore, in the case that X has paths of bounded variation, it holds that $\int_0^1 z\nu(\mathrm{d}z)$ is finite, and a similar line of reasoning yields that $w_\nu(0+)$ is finite.

As it follows by a similar argument that also F_w is continuous on $\mathbb{R}_+$ it suffices to show that the Laplace transforms of the right-hand side of (5.13) and of (5.4) coincide in order to verify the representation (5.13). Note that the Laplace transform $\mathcal{L}|\tilde{w}_\nu|(\theta)$ of $|\tilde{w}_\nu|$ is finite for any $\theta > 0$ in view of the integrability condition (2.1) and since $\int_0^1 |w_\nu(y)|\,\mathrm{d}y$ is finite. Taking the Laplace transform of (5.4), using the forms (1.3) and (1.5) of the Laplace exponent $\psi(\theta)$ and the Laplace transform $\mathcal{L}W^{(q)}$ and rearranging terms yields

$$\begin{aligned}
\mathcal{L}F_w(\theta) &= \mathcal{L}W^{(q)}(\theta)\left[\frac{\sigma^2}{2}w'_-(0) + \frac{\psi(\theta)}{\theta}w(0) - \mathcal{L}w_\nu(\theta)\right], \qquad \theta > \Phi(q),\\
&= \theta^{-1}\cdot w(0) + \theta^{-2}\cdot w'_-(0) - \big(\psi(\theta) - q\big)^{-1}\mathcal{L}J_w(\theta),\\
\mathcal{L}J_w(\theta) &= \theta^{-1}\cdot\big[\psi'(0)w'_-(0) - qw(0)\big] + \mathcal{L}\tilde{w}_\nu(\theta) - \theta^{-2}\big[qw'_-(0)\big],\\
\mathcal{L}\tilde{w}_\nu(\theta) &= \mathcal{L}w_\nu(\theta) + w'_-(0)\cdot\theta^{-2}\int_{(0,\infty)}\big[\mathrm{e}^{-\theta x} - 1 + \theta x\big]\nu(\mathrm{d}x).
\end{aligned} \tag{5.16}$$

Termwise inverting (5.16) yields the expression (5.13).

By letting $x \to 0$ in (5.13), in combination with the facts $\sigma^2 W^{(q)}(0+) = 0$ and $Z^{(q)}(0+) = 1$ and the fact that the integral tends to zero (again by Lebesgue's dominated convergence theorem), it follows that $F_w(0) = w(0)$. □

PROOF OF LEMMA 5.7. (i) The limit (5.9) follows from (5.4) or (5.13) using $W^{(q)}(x) \sim \mathrm{e}^{\Phi(q)x}/\psi'(\Phi(q))$ as $x \to \infty$.

(ii) Observe first that J_w is càdlàg on $\mathbb{R}_+ \setminus \{0\}$, by noting that $w_\nu(x)$ is càdlàg at any $x > 0$ [as a consequence of the facts that w is càdlàg, left-differentiable at zero, and satisfies the integrability condition (2.1)].

The continuity of $W^{(q)}$ on $\mathbb{R}_+$, (2.1) and the finiteness of $\int_0^1 |w_\nu(y)|\,\mathrm{d}y$ (Lemma 5.6) imply that the integral $\int_0^x |W^{(q)}(x-y)J_w(y)|\,\mathrm{d}y$ is finite for any $x > 0$. A change of the order of integration in (5.13), justified by Fubini's theorem, implies for $x > 0$ the integral $\int_0^x J_w(x-y)W^{(q)}(y)\,\mathrm{d}y$ is equal to

$$W^{(q)}(0)\int_0^x J_w(u)\,\mathrm{d}u + \int_0^x\int_0^{x-z} J_w(u)\,\mathrm{d}u\,W^{(q)\prime}(z)\,\mathrm{d}z.$$

As a consequence, it follows that the right- and left-derivatives $F'_w(x)$ and $F'_{w,-}(x)$ are equal to $w'_-(0) - \int_0^x J_w((x-z)\pm)W^{(q)\prime}(z)\,\mathrm{d}z - W^{(q)}(0)J_w(x\pm)$, respectively, at any $x > 0$. Thus the difference $F'_w(x) - F'_{w,-}(x)$ is as stated in (5.14). An application of Lebesgue's dominated convergence theorem implies that the integral in the previous line converges to zero when x tends to 0. The right-continuity of J_w and the fact that $W^{(q)}(0)$ is 0 precisely if X has unbounded variation, yields the stated form of $F'_w(0)$.

(iii) The right-continuity follows from the right-continuity of J_w on $\mathbb{R}_+ \setminus \{0\}$ and Lebesgue's dominated convergence theorem. In the case $w \in C^1(\mathbb{R}_-)$, a similar argument as at the start of part (ii) implies that J_w is continuous on $\mathbb{R}_+$. It follows thus from (5.14) that $F'_w(x)$ is continuous at any $x > 0$. □

5.2. *Exponential and polynomial boundary conditions*. For later reference it is noted that in the case that the payoff w is exponential, $w(x) = \mathrm{e}^{xv}$ for some $v \in \mathbb{R}$, or is a monomial, $w(x) = x^k$, the solutions of the two-sided and mixed absorbing/reflected exit problems simplify and can be expressed in terms of the functions $Z^{(q,v)}$ and Z_k that are specified as follows:

DEFINITION 5.8. (i) For $q, v \in \mathbb{R}_+$, the function $Z^{(q,v)} : \mathbb{R} \to \mathbb{R}$ is defined by $Z^{(q,v)}(x) = \mathrm{e}^{vx}$ for $x < 0$, and by

$$Z^{(q,v)}(x) = \mathrm{e}^{vx} + \big(q - \psi(v)\big) \int_0^x \mathrm{e}^{v(x-y)} W^{(q)}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}_+. \tag{5.17}$$

(ii) With n_0 the largest integer such that $\int_{(-\infty,-1)} |x|^n \nu(\mathrm{d}x) < \infty$, the related family of functions $Z_k : \mathbb{R} \to \mathbb{R}$, $k = 0, \ldots, n$, is defined by

$$Z_k(x) = \frac{\partial^k}{\partial v^k}\bigg|_{v=0+} Z^{(q,v)}(x). \tag{5.18}$$

As suggested above, $Z^{(q,v)}$ and Z_k are in fact Gerber–Shiu functions of the exponential and monomial pay-offs $e_v, p_k : \mathbb{R}_- \to \mathbb{R}$, which for any $v \in \mathbb{R}$ and $k = 1, \ldots, n_0$ are given by $e_v(x) := \mathrm{e}^{vx}$ and $p_k(x) := x^k$.

COROLLARY 5.9. *For any $q > 0$, $v \in \mathbb{R}$ and $k = 1, \ldots, n_0$, $Z^{(q,v)}$ and Z_k are Gerber–Shiu functions with payoffs $e_{v,a} := {}_a e_v$ and $p_{k,a} = {}_a p_k$, the translations of e_v and p_k, respectively.*

PROOF. The assertion concerning $Z^{(q,v)}$ directly follows from Theorem 5.3 since the function $Z^{(q,v)}$ is equal to the Gerber–Shiu function F_w corresponding to $w = e_v$. The two functions coincide since both are continuous on $\mathbb{R}_+$ and it holds

$$\mathcal{L}F_{e_v}(\theta) = \mathcal{L}Z^{(q,v)}(\theta) = \big(\psi(\theta) - q\big)^{-1} \frac{\psi(\theta) - \psi(v)}{\theta - v}. \tag{5.19}$$

The proof of the assertion concerning Z_k is similar and omitted. $\square$

REMARK 5.10. (i) For $v \geq 0$, the function $x \mapsto Z^{(q,v)}(x)$ is strictly increasing on $\mathbb{R}_+$. In particular, for $x > 0$ and $v > \Phi(q)$, $Z^{(q,v)\prime}(x)$ is equal to

$$Z^{(q,v)\prime}(x) = \big(\psi(v) - q\big) \int_x^\infty \mathrm{e}^{v(x-y)} W^{(q)\prime}(y)\,\mathrm{d}y, \tag{5.20}$$

which can be derived from (1.5) and (5.17) by integration by parts.

(ii) The map $v \mapsto v^{-1} Z^{(q,v)\prime}(x)$ is completely monotone[4] on $(\Phi(q), \infty)$, for any $x > 0$. That this is the case follows from the observation that $v \mapsto$

[4] A function $f : (a, \infty) \to \mathbb{R}_+ \setminus \{0\}$, $a \in \mathbb{R}$, is *completely monotone* if $(-1)^{k-1} f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and $x > a$, where $f^{(k)}$ denotes the kth derivative with respect to x.

$v^{-1}Z^{(q,v)}(x)$ is the Laplace transform of some measure on $\mathbb{R}_+$ which is shown next. From the definition of $Z^{(q,v)}$ it follows that the derivative $Z^{(q,v)\prime}(x)$ at $x > 0$ satisfies

$$Z^{(q,v)\prime}(x) = vZ^{(q,v)}(x) + \big(q - \psi(v)\big)W^{(q)}(x).$$

Inserting the forms of the Laplace transforms of $W^{(q)}|_{\mathbb{R}_+}$ and $Z^{(q,v)}|_{\mathbb{R}_+}$ [given in (1.5) and (5.19)], it follows

$$\begin{aligned}\mathcal{L}Z^{(q,v)\prime}(\theta) = {} & \frac{q}{\psi(\theta) - q} \\ & + \frac{\theta v}{\psi(\theta) - q}\bigg[\frac{\sigma^2}{2} + \int_0^\infty \frac{\mathrm{e}^{-\theta y} - \mathrm{e}^{-vy}}{v - \theta}\overline{\nu}(y)\,\mathrm{d}y\bigg].\end{aligned} \tag{5.21}$$

Inversion of the Laplace transform in (5.21) and the observation

$$\int_0^\infty \frac{\mathrm{e}^{-\theta y} - \mathrm{e}^{-vy}}{v - \theta}\overline{\nu}(y)\,\mathrm{d}y = \int_0^\infty \int_0^\infty \mathrm{e}^{-\theta s - vt}\overline{\nu}(s + t)\,\mathrm{d}t\,\mathrm{d}s,$$

yield the following expression for $v^{-1}Z^{(q,v)\prime}(x)$ at any $x > 0$:

$$\frac{q}{v}W^{(q)}(x) + \frac{\sigma^2}{2}W^{(q)\prime}(x) + \int_0^\infty \int_{[0,x]} \mathrm{e}^{-vt}\overline{\nu}(x - y + t)W^{(q)}(\mathrm{d}y)\,\mathrm{d}t.$$

By inspection it follows that, for any $x > 0$, the function $v \mapsto v^{-1}Z^{(q,v)\prime}(x)$ is the Laplace transform of a measure on $[0, \infty)$, which implies the stated complete monotonicity.

6. Single dividend-band strategies. The analysis of various strategies starts with the case of single dividend-band strategies. In the absence of transaction costs such a barrier strategy at level $b = (b_-, b_+)$, denoted by π_b, specifies to pay out the minimal amount of dividends to keep the reserves $U^b := U^{\pi_b}$ below the level $b_+ = b_-$, while, in the case $K > 0$, π_b prescribes to pay out a lump-sum $b_+ - b_- > 0$ each time that the reserves U^b reach the level b_+. More formally, in the cases $K = 0$ and $K > 0$ the forms of the strategy $\pi_b = \{D_t^b, t \in \mathbb{R}_+\}$ are given by (1.8) [with $b = b_+ = b_-$] and by

$$D_t^b = \big(U_0^b - b_-\big) + (b_+ - b_-)N_t^b, \qquad N_t^b = \#\big\{s \in (0, t] : U_{s-}^b = b_+\big\}, \qquad t \in \mathbb{R}_+,$$

respectively. As a consequence, it follows that the value $v_b(x) := v_{\pi_b}(x)$ associated to the single dividend band strategy π_b at a nonzero level b when X_0 is equal to x is given by

$$v_b(x) = \mathbb{E}_x\bigg[\int_0^{\tau_b} \mathrm{e}^{-qt}\mu_K^b(\mathrm{d}t) + \mathrm{e}^{-q\tau_b}w\big(U_{\tau_b}^b\big)\bigg],$$

with $\mu_K^b := \mu_K^{\pi_b}$, $U^b := U^{\pi_b}$ and $\tau^b = \tau^{\pi_b} = \inf\{t \in \mathbb{R}_+ : U_t^b < 0\}$. The function v_b can be expressed in terms of the homogeneous and inhomogeneous scale functions $W^{(q)}$ and F_w as follows:

PROPOSITION 6.1. *For $b_+ > b_- \geq 0$ and $x \in [0, b_+]$ and with $F = F_w$ it holds*

$$v_b(x) = \begin{cases} w(x), & x < 0, \\ W^{(q)}(x)G(b_-, b_+) + F(x), & x \in [0, b_+], \\ x - b_+ + v_b(b_+), & x > b_+, \end{cases} \tag{6.1}$$

$$G(b_-, b_+) := \begin{cases} \dfrac{b_+ - b_- - K - (F(b_+) - F(b_-))}{W^{(q)}(b_+) - W^{(q)}(b_-)}, & K > 0, b_+ > b_-, \\ \dfrac{1 - F'(b_+)}{W^{(q)\prime}(b_+)}, & K = 0, b_+ = b_-. \end{cases} \tag{6.2}$$

REMARK 6.2. Note that in the case $K > 0$ and $X_0 = x > b_+$ the strategy π_b prescribes an immediate lump-sum dividend payment of size $x - b_-$, which is in agreement with the value $v_b(x)$ for $x > b_+$,

$$v_b(b_+) = v_b(b_-) + b_+ - b_- - K \Rightarrow v_b(x) = x - b_- - K + v_b(b_-), \qquad x > b_+.$$

PROOF OF PROPOSITION 6.1. Consider the case $K > 0$. Since no dividend payment takes place before X reaches the level b_+ it follows that $\{X_t, t \leq T_{0,b_+}\}$ and $\{U_t^{b_+}, t \leq \tau^{\pi_b}\}$ have the same law. The strong Markov property of X and the absence of positive jumps then yield that for $x \in [0, b_+]$ $v_b(x)$ is equal to

$$\begin{aligned} &\mathbb{E}_x\big[\mathrm{e}^{-qT_{b_+}^+}(v_b(b_-) + \Delta b - K)\mathbf{1}_{\{T_{b_+}^+ < T_0^-\}}\big] + \mathbb{E}_x\big[\mathrm{e}^{-qT_0^-} w(U_{T_0^-})\mathbf{1}_{\{T_{b_+}^+ > T_0^-\}}\big] \\ &\quad = \frac{W^{(q)}(x)}{W^{(q)}(b_+)}\big[v_b(b_-) + \Delta b - K\big] + \bigg[F(x) - F(b_+)\frac{W^{(q)}(x)}{W^{(q)}(b_+)}\bigg], \end{aligned}$$

with $F = F_w$, where the second line follows from Proposition 5.5 (applied with $w \equiv 0$ and with $\delta = 0$). Evaluating the expression in the display at $x = b_-$, solving the resulting linear equation for $v(b_-)$ and inserting the result yields the stated form. The case $K = 0$ follows by a similar line of reasoning, using (5.12) in Proposition 5.5. $\square$

Next the candidate optimal levels are described. The form of G suggests to define the level $b^* = (b_-^*, b_+^*)$ as a maximizer of $G(x, y)$ over all $x, y \geq 0$ in the case $K > 0$, and similarly, to define b_+^* as a maximizer of $G(x, x)$ over all $x \geq 0$ in the case $K = 0$.

REMARK 6.3. Observe that in the case $K > 0$ and G is C^1, the partial right derivatives of $G(x, y)$ are given by

$$\begin{aligned} &\frac{\partial G}{\partial x}(x, y) = \frac{W^{(q)\prime}(x)}{W^{(q)}[x, y]}\big[G(x, y) - G^\#(x)\big], \\ &\frac{\partial G}{\partial y}(x, y) = -\frac{W^{(q)\prime}(y)}{W^{(q)}[x, y]}\big[G(x, y) - G^\#(y)\big], \qquad G^\#(x) := \frac{1 - F'(x)}{W^{(q)\prime}(x)}, \end{aligned} \tag{6.3}$$

and with $W^{(q)}[x,y] := W^{(q)}(y) - W^{(q)}(x)$. Therefore, in this case, an interior maximum (x^*, y^*) will satisfy $G(x^*, y^*) = G^{\#}(x^*) = G^{\#}(y^*)$, and a candidate optimum may be found by fixing $d = y - x$, and optimizing the left endpoint $x(d)$ for fixed d [graphically, this would amount to determining the highest value of the function $G^{\#}$ where the "width" $y(d) - x(d)$ of the function $G^{\#}$ is d].

In the case $K > 0$, fix therefore $d > 0$, and let

$$b^*(d) = \sup\{b \geq 0 : G(b, b+d) \geq G(x, x+d)\ \forall x \geq 0\} \tag{6.4}$$

denote the last global maximum of $G(x, x+d)$.

Define next d^* to be the last global maximum of $G(b^*(y), b^*(y) + y)$

$$d^* = \sup\{d \geq 0 : G(b^*(d), b^*(d) + d) \geq G(b^*(y), b^*(y) + y)\ \forall y \geq 0\},$$

where $\inf \varnothing = +\infty$.

The candidate optimal levels are then defined as follows:

$$b^* = (b^*_-, b^*_+) \qquad \text{with } b^*_- = b^*(d^*), b^*_+ = b^*(d^*) + d^*. \tag{6.5}$$

In the absence of transaction cost ($K = 0$), set

$$b^*_+ = b^*_- = \sup\{b \geq 0 : G^{\#}(b) \geq G^{\#}(x)\ \forall x \geq 0\}. \tag{6.6}$$

THEOREM 6.4. *It holds $b^*_+ < \infty$ and*

$$v_*(x) = W^{(q)}(x) G^{\#}(b^*_+) + F(x), \qquad x \in [0, b^*_+], \tag{6.7}$$

where $F = F_w$. In particular, it is optimal to adopt the strategy π_{b^*} while the reserves are not larger than b^*_+.

The proof rests on the following auxiliary result that concerns explicit expressions linking the operator ${}_a\Gamma^w_\infty$ with the function G and the scale functions F_w and $W^{(q)}$. This relation is also deployed in the formulation of necessary and sufficient optimality conditions for optimality of band policies in Sections 9–11.

LEMMA 6.5. *Let $c > 0$, and for any $b_+ \geq b_- \geq 0$ (with $b_+ \neq b_-$ in the case $K > 0$) define*

$$J_{v_b} : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}, \qquad J_{v_b}(y) = ({}_{b_+}\Gamma^{v_b}_\infty v_b)(y), \qquad y > 0.$$

(i) *The following identity holds true*:

$$\begin{aligned} &W^{(q)\prime}(b^*_+ + c)[G(b^*_-, b^*_+ + c) - G(b^*_-, b^*_+)] \\ &\quad = \int_{[0,c)} {}_{b^*_+}J_{v_{b^*}}(c-y) W^{(q)}(\mathrm{d}y) \\ &\quad = v'_{b^*,-}(b^*_+) - F'_{b^*_+ v_b}(c). \end{aligned} \tag{6.8}$$

In particular, it holds

$$\int_{[0,c)} {}_{b_+^*}J_{v_{b^*}}(c-y)W^{(q)}(\mathrm{d}y) < 0 \qquad \forall c > 0, \tag{6.9}$$

and the functions $y \mapsto G(b^-, y)$ *and* $y \mapsto G^\#(y)$ *are decreasing for all* y *sufficiently large.*

(ii) *Denoting* $G_{b_-}(x) := G(b_-, x)$, *the Laplace transform of the function* $g : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ *given by* $g(x) = {}_{b_+}J_{v_b}(x)$ *is equal to*

$$\mathcal{L}g(\theta) = +\frac{\mathrm{e}^{\theta b_+}}{\theta}\int_{(b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G_{b_-}(\mathrm{d}z), \qquad \theta > \Phi(q). \tag{6.10}$$

In particular, g *is nonpositive precisely if* $\theta \mapsto -\mathcal{L}g(\theta + \Phi(q))$ *is completely monotone.*

REMARK 6.6. The integral in (6.10) is to be interpreted as a Lebesgue–Stieltjes integral. This follows as a consequence of the form of G_{b_-} and the fact that the functions $W^{(q)}$ and $1/W^{(q)\prime}$ are of bounded variation (which follows in turn as $W^{(q)}$ is increasing and $W^{(q)\prime}$ is logconcave).

The proof of Lemma 6.5 is given in Appendix C.

PROOF OF THEOREM 6.4. *b_+^* is finite, and the supremum is attained:* Note that, for any $x > 0$, it holds $G^\#(x) \geq G^\#(x-)$, by virtue of the form (6.3) of $G^\#(x)$, and the inequalities $W^{(q)\prime}(x) \geq W_-^{(q)\prime}(x)$ [from (5.2)] and $F'(x) \geq F'_-(x)$ [from (5.14)], where $W_-^{(q)\prime}(x)$, $F'_-(x)$ denote the left-derivatives at x. In view of the facts that the map $x \mapsto G^\#(x)$ defined in (6.2) is right-continuous and monotone decreasing for all x sufficiently large (Lemma 6.5), it then follows that there exists an $x^* \in \mathbb{R}_+$ such that $\sup_{x \geq 0} G^\#(x) = G^\#(x^*)$. In the case that K is strictly positive, G attains its maximum at some $(x^*, y^*) \in Q := \{(z_1, z_2) \in \mathbb{R}^2 : 0 \leq z_1 < z_2\}$, since (a) $G(x, y)$ is continuous at any (x, y) in Q, (b) monotone decreasing for y sufficiently large and fixed x [Proposition 6.5(iii)], (c) tends to minus infinity if $y \searrow x$ and (d) tends to the constant κ_w in (5.9) if $|x| + |y|$ tends to infinity such that $x < y$.

Verification of optimality: Assume for the moment that the function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by the right-hand side of (6.7) is a supersolution in the sense of Definition 4.1. Under this assumption h dominates the value-function v_* (by Proposition 4.3). In fact, since $h(x)$ is equal to the value $v_{b^*}(x)$ of the strategy π_{b^*} for any level x of initial reserve smaller or equal to b_+^*, the local verification theorem, Theorem 4.4(i), implies that $h(x)$ is equal to the optimal value $v_*(x)$ for all $x \in [0, b_+^*]$.

Next it is shown that h is a supersolution by verifying the following two facts: (a) $\mathrm{e}^{-q(t\wedge T_0^-)}h(X_{t\wedge T_0^-})$ is a martingale, and (b) h satisfies the inequality

$$h(x) - h(y) \geq x - y - K \qquad \text{for any } 0 \leq y < x.$$

Fact (a) follows from the martingale properties of F_w and $W^{(q)}$ (see Proposition 5.4), while (b) follows on account of the definitions of b^* and $G^\#$. Indeed, if $K = 0$ and $x > 0$, $h'(x) = W^{(q)\prime}(x)G^\#(b^*) - F_w'(x)$ is bounded below by

$$W^{(q)\prime}(x)G_*(x) - F_w'(x) = 1, \tag{6.11}$$

while, if $K > 0$ and $x > y > 0$, $h(x) - h(y) = (W^{(q)}(x) - W^{(q)}(y))G(b_-^*, b_+^*) - F_w(x) + F_w(y)$ is bounded below by

$$\begin{aligned} h(x) - h(y) &\geq \big(W^{(q)}(x) - W^{(q)}(y)\big)G(y,x) - F_w(x) + F_w(y) \\ &= x - y - K. \end{aligned} \tag{6.12}$$

Displays (6.11) and (6.12) imply $h(x) - h(y) \geq x - y - K$ for any $K \geq 0$ and $x, y \geq 0$ with $x \geq y$. This completes the proof. □

7. Two-band strategies and a mixed optimal stopping/control problem. The policy π_{b^*} considered in the previous section may be optimal for any level of the reserves, and not just for small levels as shown in Theorem 6.4—necessary and sufficient conditions for this to be the case are given in Section 9. In this section the complementary case is considered that it is optimal to have a *second* dividend band. The problem of finding the optimal levels of the second dividend band differs from the single-band optimization problem in the following two respects:

(i) at any time t prior to the time of ruin it is possible to make a lump-sum payment to bring the reserves down to the level b_-^* defined in (6.5), yielding a pay-off of $U_t - b_-^* + v_{b^*}(b_-^*) - K$, and

(ii) it will not be optimal to place a dividend band at levels close to b_+^*.

The observation in (i) in combination with the dynamic programming principle (Proposition 3.1) and Theorem 6.4 yield the representation

$$v_*(x) = \sup_{\pi\in\Pi,\tau\in\mathcal{T}} \mathbb{E}_x\left[\int_{[0,\tau\wedge\tau)} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q(\tau_{b^*}^\pi\wedge\tau)}v_{b^*}\big(U^\pi_{\tau^\pi_{b^*}\wedge\tau}\big)\right], \tag{7.1}$$

where $\tau_{b^*}^\pi = \inf\{t \geq 0 : U_t^\pi < b_+^*\}$. This section is devoted to a stochastic control problem that is closely related to (7.1), $V_*^f(x) = \sup_{\pi\in\Pi,\tau\in\mathcal{T}} V_{\tau,\pi}^f(x)$, where

$$V_{\tau,\pi}^f(x) = \mathbb{E}_x\left[\int_{[0,\tau^\pi\wedge\tau)} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q(\tau^\pi\wedge\tau)}f\big(U^\pi_{\tau^\pi\wedge\tau}\big)\right], \tag{7.2}$$

where, as before $\tau^\pi = \inf\{t \geq 0 : U_t^\pi < 0\}$, and $f:\mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following conditions:

$$f|_{\mathbb{R}_+} \text{ is given by } f(x) = x + c \text{ for } x \in \mathbb{R}_+, \text{ for some } c \in \mathbb{R}, \tag{7.3}$$

$$f'_-(0) \geq 1, \tag{7.4}$$

$$J_{\bar{w}}(u) := {}_0\Gamma_\infty^{\bar{w}} f(u) > 0 \text{ for some } u > 0, \text{ with } \bar{w} = f|_{\mathbb{R}_-}, \tag{7.5}$$

$$\text{for all } c \in \mathbb{R}_+ \setminus \{0\}, \int_{[0,c)} J_{\bar{w}}(c-y) W^{(q)}(\mathrm{d}y) < 0. \tag{7.6}$$

It will be shown that, under (7.5), it is not optimal to stop immediately ($V_*^f \not\equiv f$), while, under (7.6), the dividend barrier strategy with level at 0 is not optimal ($V_*^f \not\equiv V_{\tau^\pi,\pi_0}^f$). In particular, in the setting of the stochastic control problem in (7.1) conditions in (7.3)–(7.6) are satisfied:

LEMMA 7.1. *If it holds $v_{\pi_{b^*}}(x) < v_*(x)$ for some $x > b_+^*$, then the function $f:\mathbb{R} \to \mathbb{R}$ defined by $f(x) = v_{b^*}(b_+^* + x)$ satisfies the stated conditions in* (7.3)–(7.6).

PROOF. First, note that the conditions in (7.3)–(7.4) hold since $v_{b^*}|_{[b_+^*,\infty)}$ is affine with unit slope and $v'_{b^*,-}(b^*)$ is larger or equal to one (with equality when $W^{(q)}$ and F_w are differentiable at b^*). Also, condition (7.6) holds by (6.9) in Lemma 6.5. Furthermore, it is shown in Theorem 9.1 in Section 9 that if condition (7.5) was not satisfied, then $v_{b^*} = v_*$, which would be in contradiction with the assumed existence of an x larger than b_+^* satisfying $v_{b^*}(x) < v_*(x)$. $\square$

Next a candidate optimal policy is specified for the mixed optimal stopping/optimal control problem in (7.2). Strategies for this optimization problem consist of pairs (τ, π) of an $\mathbf{F}$-stopping time τ and a policy π from the set Π. The discussion at the beginning of the section [especially item (ii)] in conjunction with Lemma 7.1 suggests to consider candidate optimal strategies of the form $(\tau_a^{\pi_b}, \pi^b)$, $a < b_+$: such policies specify to pay out dividends according to a single dividend-band strategy π_b at levels (b_-, b_+) until the first moment $\tau_a^{\pi_b} = \inf\{t \geq 0 : U_t^{\pi_b} < a\}$ that U^{π_b} falls below the level $a > 0$ at which moment one should stop. Another strategy that is worth considering in the case $K > 0$ is to refrain from paying dividends until the first moment that the reserves process exits a finite interval $[a, b_+]$ and to stop then; such strategies are denoted by $(\pi^\varnothing, T_{a,b_+})$ for $a < b_+$. The value functions associated to the strategies $(\tau_a^{\pi_b}, \pi^b)$ and $(\pi^\varnothing, T_{a,b_+})$ are given by

$$V_{a,b_-,b_+}^f(x) = \mathbb{E}_x\left[\int_{[0,\tau_a^{\pi_b})} \mathrm{e}^{-qt} \mu_K^b(\mathrm{d}t) + \mathrm{e}^{-q\tau_a^{\pi_b}} f\left(U_{\tau_a^{\pi_b}}^b\right)\right],$$

and $V_{a,b_+}^{f,\varnothing}(x) = \mathbb{E}_x[\mathrm{e}^{-qT_{a,b_+}} f(X_{T_{a,b_+}})]$, with $\mu_K^b = \mu_K^{\pi_b}$. In the following result, which can be derived by a line of reasoning that is similar to the one used in the proof of Proposition 6.1, the functions V^f_{a,b_-,b_+} and $V^{f,\varnothing}_{a,b_+}$ are explicitly expressed in terms of scale functions and the families of functions $(y,z)\mapsto G^{(a)}_f(y,z)$, $G^{(a)}_{f,\varnothing}(y,z)$, $a\geq 0$, that are defined as follows:

$$G^{(a)}_f(b_-,b_+)=\begin{cases}\dfrac{b_+-b_--K-F^{(a)}[b_--a,b_+-a]}{W^{(q)}[b_--a,b_+-a]}, & K>0,\\ G^{(a)}_{f,\#}(b_+):=\dfrac{1-F^{(a)\prime}(b_+-a)}{W^{(q)\prime}(b_+-a)}, & K=0,\end{cases}\tag{7.7}$$

$$G^{(a)}_{f,\varnothing}(b_+)=\frac{f(b_+)-F^{(a)}(b_+-a)}{W^{(q)}(b_+-a)},\tag{7.8}$$

where $F^{(a)}=F_{{}_af}$ is the Gerber–Shiu function for payoff ${}_af=f(a+\cdot)$, $F^{(a)}[x,y]=F^{(a)}(y)-F^{(a)}(x)$ and, as before, $W^{(q)}[x,y]=W^{(q)}(y)-W^{(q)}(x)$.

PROPOSITION 7.2. *For any $b_-,b_+,a\in\mathbb{R}_+$ satisfying $b_+\geq b_-\geq a$ the following representations hold true*:

$$V^f_{a,b_-,b_+}(x)=\begin{cases}F^{(a)}(x-a)=f(x), & x\in[0,a),\\ W^{(q)}(x-a)G^{(a)}_f(b_-,b_+)+F^{(a)}(x-a), & x\in[a,b_+],\\ x-b_++V^f_{a,b_-,b_+}(b_+), & x\in(b_+,\infty);\end{cases}$$

$$V^{f,\varnothing}_{a,b_+}(x)=\begin{cases}F^{(a)}(x-a)=f(x), & x\notin[a,b_+],\\ W^{(q)}(x-a)G^{(a)}_{f,\varnothing}(b_+)+F^{(a)}(x-a), & x\in[a,b_+].\end{cases}$$

Next the candidate optimal levels are described. Focusing first on the case that dividends are paid and fixing the level a for the moment, and similarly as in the case of the single dividend-band strategies, let $\beta^*_f(a)=(\beta^*_{f,-}(a),\beta^*_{f,+}(a))$ denote the (largest) maximizer of the function $G^{(a)}_f$. In the case $K>0$ we set

$$\beta^*_{f,-}(a)=\beta^*_f\big(a,\delta^*_f(a)\big),\qquad \beta^*_{f,+}(a)=\beta^*_f\big(a,\delta^*_f(a)\big)+\delta^*_f(a),$$

$$\beta^*_f(a,d)=\sup\big\{b\geq a: G^{(a)}_f(b,b+d)\geq G^{(a)}_f(x,x+d)\ \forall x\geq 0\big\},$$

$$\delta^*_f(a)=\sup\big\{d\geq 0: G^{(a),*}_f(d)\leq G^{(a),*}_f(y)\ \forall y\geq 0\big\},$$

with $G^{(a),*}_f(d):=G^{(a)}_f(\beta^*_f(a,d),\beta^*_f(a,d)+d)$, while, in the case $K=0$, we define

$$\beta^*_{f,+}(a)=\beta^*_{f,-}(a)=\beta^*_{f,\#}(a):=\sup\big\{b\geq a: G^{(a)}_{f,\#}(b)\geq G^{(a)}_{f,\#}(x)\ \forall x\geq 0\big\}.$$

The candidate optimal specification α^*_f of the stopping level a and the candidate optimal level β^*_f are given by

$$\alpha^*_f=\inf\big\{a\geq 0: G^{(a,*)}_f\big(\delta^*_f(a)\big)>0\big\}\qquad \text{in the case } K>0,\tag{7.9}$$

(7.10) $\alpha_f^* = \inf\{a \geq 0 : G_{f,\#}^{(a)}(\beta_{f,\#}^*(a)) > 0\}$ in the case $K = 0$,

(7.11) $\beta_f^* = (\beta_{f,-}^*, \beta_{f,+}^*), \qquad \beta_{f,-}^* = \beta_{f,-}^*(\alpha_f^*), \qquad \beta_{f,+}^* = \beta_{f,+}^*(\alpha_f^*).$

Next consider the strategy to continue without paying dividends and stop upon exiting a finite interval. It will turn out that in the case $K = 0$ such a strategy is never optimal; see Remark 7.5.

In the case $K > 0$ define

(7.12) $\beta_{f,\varnothing}^*(a) = \sup\{b \geq a : G_{f,\varnothing}^{(a)}(b) \geq G_{f,\varnothing}^{(a)}(x)\ \forall x \geq 0\},$

(7.13) $\alpha_{f,\varnothing}^* = \inf\{a \geq 0 : G_{f,\varnothing}^{(a)}(\beta_{f,\varnothing}^*(a)) > 0\}, \qquad \beta_{f,\varnothing}^* = \beta_{f,\varnothing}^*(\alpha_{f,\varnothing}^*).$

The levels $\beta_{f,+}^*$, $\beta_{f,\varnothing}^*$ and $\alpha_{f,\varnothing}^*$ given above are finite and strictly positive.

LEMMA 7.3. *Suppose that f satisfies the conditions in* (7.3)–(7.6) *and denote* $\bar{w} = f|_{\mathbb{R}_-}$.

(i) $K = 0$: $0 < \alpha_f^* \leq \beta_{f,+}^* < \infty$ *and* $G_{f,\#}^{(\alpha_f^*)}(\beta_f^*) = 0$, *and* ${}_0\Gamma_\infty^{\bar{w}} f(u) \leq 0$ *for all* $u \in (0, \alpha_f^*)$.

Furthermore, if X has unbounded variation, it holds $\alpha_f^* < \beta_{f,+}^*$.

(ii) $K > 0$: $0 < \alpha_{f,\varnothing}^* \leq \beta_{f,\varnothing}^* < \infty$ *and* $G_{f,\varnothing}^{(\alpha_{f,\varnothing}^*)}(\beta_{f,\varnothing}^*) = 0$, *and it holds* ${}_0\Gamma_\infty^{\bar{w}} f(u) \leq 0$ *for all* $u \in (0, \alpha_{f,\varnothing}^*)$.

Furthermore, if it holds in addition $\alpha_f^* < \infty$, *then* $0 < \alpha_f^* < \beta_{f,+}^* < \infty$ *and* $G_f^{(\alpha_f^*)}(\beta_f^*) = 0$.

REMARK 7.4 (Smooth and continuous fit). The choice of α_f^* coincides with what would be obtained by applying the *principles of continuous* and *smooth fit* from the theory of optimal stopping (see Peskir and Shiryaev [32], Chapter IV.9), which suggest that in the mixed optimal stopping/stochastic control problem (7.2) it can be expected that V^f be *continuous/continuously differentiable* at a level α_f^* if α_f^* is irregular/regular for $(-\infty, \alpha_f^*)$ for X, respectively, where π_* denotes the optimal strategy. Since it is well-known that α_f^* is regular for $(-\infty, \alpha_f^*)$ for X if and only if X has unbounded variation, this heuristic yields

$$\begin{cases} \alpha_f^* \text{ satisfies } V_{\alpha^*,\beta^*}^{f\prime}(\alpha_f^*+) = f'(\alpha_f^*-), & \text{if } X \text{ has unbounded variation,} \\ \alpha_f^* \text{ satisfies } V_{\alpha^*,\beta^*}^{f}(\alpha_f^*) = f(\alpha_f^*), & \text{if } X \text{ has bounded variation.} \end{cases}$$

The first equation in the display is equivalent to the expression in (7.9) in view of the form of $V_{a,b}^f$ and the facts (i) $F'_{{}_a f}(0) = f'_-(a)$ for any $a > 0$ and (ii) $W_+^{(q)\prime}(0) \in (0, \infty]$. The second equation in the display can also be equivalently expressed as (7.9), in view of (i′) the form of V_{a,b_-,b_+}^f in Proposition 7.2 and (ii′) the fact that $W^{(q)}(0)$ is strictly positive precisely if X has bounded variation. A similar remark holds true for the level $\alpha_{f,\varnothing}^*$.

REMARK 7.5. (i) In the case $K = 0$ it is straightforward to verify that any strategy of the form $(\pi_\varnothing, T_{a,b_+})$, for $a, b \in \mathbb{R}_+$ with $0 < a < b_+$, is not optimal [indeed, the minimal slope of the value function u of such a strategy is smaller than one, since u satisfies $u(b_+) - u(0) = b_+$, given that $u(b_+) = f(b_+)$, $u(0) = f(0)$ and f is affine with unit slope].

(ii) In the case $K > 0$ and $\alpha^*_{f,\varnothing} < \alpha^*_f$, the definition of α^*_f, Proposition 7.2 and Lemma 7.3(ii) imply

$$V(x) := V^{f,\varnothing}_{\alpha^*_{f,\varnothing},\beta^*_{f,\varnothing}}(x) \geq V^{f}_{\alpha^*_{f,\varnothing},\beta^*_f(\alpha^*_{f,\varnothing})}(x), \qquad x \in [0, \beta^*_{f,\varnothing}].$$

Note that the nonpositivity of $G_f^{(\alpha^*_{f,\varnothing})}(\beta^*(\alpha^*_{f,\varnothing}))$ implies $\mathrm{d}_V(x) \geq 1$ for all $x > 0$.

(iii) In the case $K > 0$ and $\alpha^*_{f,\varnothing} \geq \alpha^*_f$ a similar argument using the definition of $\alpha^*_{f,\varnothing}$ in conjunction with Proposition 7.2 and Lemma 7.3(ii) implies

$$V^f_{\alpha^*_f,\beta^*_f}(x) \geq V^{f,\varnothing}_{\alpha^*_f,\beta^*_{f,\varnothing}(\alpha^*_f)}(x), \qquad x \in [0, \beta^*_f].$$

PROOF OF LEMMA 7.3. (i) Consider the function $\overline{G}:\mathbb{R}_+ \to \mathbb{R}$ defined by $\overline{G}(a) = \sup_{b\geq 0} G^{(a)}_{f,\#}(b)$. The fact that α^*_f is positive and finite is a consequence of the intermediate value theorem and the following three assertions concerning $\overline{G}$:

(a) $\overline{G}(0) < 0$;
(b) there exists an $a_0 \in \mathbb{R}_+ \setminus \{0\}$ such that $\overline{G}(a_0) > 0$;
(c) the function $a \mapsto \overline{G}(a)$ is continuous at $a \in [0, a_0]$.

Next these three assertions are verified. Assertion (a) follows from the definition of $G^{(0)}$ in (7.7), the form of $F^{(a)\prime}$ [in (5.14)] and conditions (7.4) and (7.6).

To verify assertion (b) it suffices to find a_0 and b with $a_0 < b$ satisfying $G^{(a_0)}_{f,\#}(b) > 0$, or equivalently $F^{(a_0)\prime}(b - a_0) < 1$ (in view of the form of $G^{(a_0)}_{f,\#}$). By the form of $F^{(a_0)\prime}$ and the fact $f'(a_0) \geq 1$ it suffices to show $\int_{[0,b-a_0)} J_{\tilde{w}}(b - a_0 - y)W^{(q)}(\mathrm{d}y) > 0$ with $\tilde{w} = {}_{a_0}f$ for some $a_0 < b$, which is equivalent to the condition $\int_{[0,b-a_0)} J_{\overline{w}}(b - y)W^{(q)}(\mathrm{d}y) > 0$ for some $a_0 < b$, as it holds $J_{\overline{w}}(b - y) = J_{\tilde{w}}(b - a_0 - y)$.

To see that the latter condition is satisfied, note that right-continuity of the map $J_{\overline{w}}$ and (7.5) imply that there exists an interval $I = [u_-, u_+]$, with $0 < u_- < u_+$, such that $J_{\overline{w}}(y) > 0$ for all $y \in I$; taking $a_0 := u_-$ and $b := u_+$ it thus follows that the integral $\int_{[0,b-a_0)} J_{\overline{w}}(b - y)W^{(q)}(\mathrm{d}y)$ is strictly positive, and the proof of assertion (b) is complete.

To verify that assertion (c) holds fix $a \geq 0$, and note $V^f_{a,\beta^*(a)}(x) = W^{(q)}(x) \times \overline{G}(a) + F^{(a)}(x - a)$ for $x \in [a, \beta^*_+(a)]$. By reasoning analogous to the proof of Theorem 6.4 the following identity can be shown to hold:

$$V^f_{a,\beta^*(a)}(x) = \sup_{(\pi,\tau)\in\Pi(\beta^*_+)} \mathbb{E}_x\left[\int_{[0,\tau^\pi_a\wedge\tau]} \mathrm{e}^{-qt}\,\mathrm{d}D^\pi_t + \mathrm{e}^{-q(\tau^\pi_a\wedge\tau)} f\big(U^\pi_{\tau^\pi_a\wedge\tau}\big)\right],$$

where $\Pi(\beta_+^*)$ is the set of the strategies (π, τ) that is such that the stochastic process $\{U_{t\wedge\tau}^\pi, t \in \mathbb{R}_+\}$ stays below the level β_+^*. Let $a_1, a_2 \in \mathbb{R}_+$ be such that $a_2 < a_1 < \min\{\beta^*(a_1), \beta^*(a_2)\}$ and fix $x_0 \in (a_1, \min\{\beta^*(a_1), \beta^*(a_2)\})$. To show the continuity of $\overline{G}(a)$ we show next that $V_{a_1,\beta^*(a_1)}^f(x_0) - V_{a_2,\beta^*(a_2)}^f(x_0)$ tends to 0 when $a_2 - a_1 \to 0$.

By an application of the triangle inequality it follows that the difference $|V_{a_1,\beta^*(a_1)}^f(x_0) - V_{a_2,\beta^*(a_2)}^f(x_0)|$ is bounded above by

$$\sup_{\pi\in\Pi} \mathbb{E}_{x_0}\left[\int_{[\tau_{a_1}^\pi, \tau_{a_2}^\pi]} \mathrm{e}^{-qt}\, \mathrm{d}D_t^\pi + \left|\mathrm{e}^{-q\tau_{a_2}^\pi} f(U_{\tau_{a_2}^\pi}^\pi) - \mathrm{e}^{-q\tau_{a_1}^\pi} f(U_{\tau_{a_1}^\pi}^\pi)\right|\right]. \tag{7.14}$$

Since $\mathbb{P}_{x_0}(U_{\tau_{a_1}^\pi} \in [a_2, a_1)) = \mathbb{P}_{x_0}(\tau_{a_1}^\pi < \tau_{a_2}^\pi)$ converges to zero if $a_1 - a_2 \searrow 0$, it follows that also the random variable under the expectation tends to zero $\mathbb{P}_{x_0}$-a.s. if $a_1 - a_2 \searrow 0$. Since this random variable is dominated by an integrable random variable, uniformly for all $(\pi, \tau) \in \Pi(\beta_+^*)$, Lebesgue's dominated convergence theorem implies that the right-hand side of (7.14) tends to zero when $a_1 - a_2 \searrow 0$. To see that the random variable is dominated, recall that f is affine, and note that $\mathrm{e}^{-q\tau_{a_1}^\pi} D_{\tau_{a_1}^\pi}^\pi \vee \mathrm{e}^{-q\tau_{a_2}^\pi} D_{\tau_{a_2}^\pi}^\pi \vee \int_{[\tau_{a_1}^\pi, \tau_{a_2}^\pi]} \mathrm{e}^{-qt}\, \mathrm{d}D_t^\pi$ is bounded above by

$$\int_{[0,\infty)} \mathrm{e}^{-qt}\, \mathrm{d}D_t^\pi \le \int_0^\infty q\mathrm{e}^{-qt} D_t^\pi\, \mathrm{d}t \le \int_0^\infty q\mathrm{e}^{-qt}\overline{X}_t\, \mathrm{d}t$$

with $\overline{X}_t = \overline{X}_t^0 = \sup_{s\in[0,t]} X_s \vee 0$, which is equal to $\mathbb{E}_{x_0}[\overline{X}_{\mathbf{e}_q}] = x_0 + \Phi(q)^{-1}$, where $\mathbf{e}_q$ is an independent exponential random time, and

$$\mathbb{E}_{x_0}\left[\left|\mathrm{e}^{-q\tau_a^\pi} X_{\tau_a^\pi}\right|\right] \le \mathbb{E}_{x_0}\left[\mathrm{e}^{-q\tau_a^\pi}(\overline{X}_{\tau_a^\pi} - \underline{X}_{\tau_a^\pi})\right] \le 2x_0 + \mathbb{E}_{x_0}[\overline{X}_{\mathbf{e}_q} - \underline{X}_{\mathbf{e}_q}] < \infty,$$

with $\underline{X}_t = \inf_{0\le s\le t} X_s \wedge 0$, where the finiteness follows from the bound $\mathbb{E}_{x_0}[\underline{X}_{\mathbf{e}_q}] \ge \mathbb{E}_0[\underline{X}_{\mathbf{e}_q}] = \mathbb{E}_0[X_{\mathbf{e}_q}] - \mathbb{E}_0[\overline{X}_{\mathbf{e}_q}]$ (which follows from the Wiener–Hopf factorization) and the fact $\mathbb{E}_0[X_{\mathbf{e}_q}] = \psi'(0)/q$.

The finiteness of $\beta_{f,+}^*(\alpha_f^*)$ follows by a line of reasoning that is analogous to the one that was used in the proof of Theorem 6.4, while the relation $\beta_{f,+}^*(\alpha_f^*) \ge \alpha_f^*$ follows by definition of $\beta_{f,+}^*(\alpha_f^*)$. Finally, in the case $K = 0$ and $\{\sigma^2 > 0$ or $\nu_{0,1} = \infty\}$ the equality $\alpha^* = \beta_+^*(\alpha^*)$ would imply that $V_{\alpha^*,\beta^*}^f \equiv f$; however, since there exists a u such that ${}_0\Gamma_\infty^f f(u) > 0$ by (7.5), an argument as above shows that, for some α, β, $V_{\alpha,\beta}^f(x) > f(x)$ for $x \in (\alpha, \beta)$, which yields a contradiction. A similar argument shows ${}_0\Gamma_\infty^{\bar{w}} f(u) \le 0$ for all $u \in (0, \alpha_f^*)$.

The proof of part (ii) is analogous to that of part (i), and is omitted. $\square$

The solution of the stochastic control problem in (7.2) for small levels of the reserves is given as follows:

THEOREM 7.6. *Suppose that f satisfies conditions* (7.3)–(7.6).

(i) *When either* $K = 0$ *or* $\{K > 0$ *and* $\alpha^*_{f,\varnothing} \geq \alpha^*_f\}$, *it holds* $V^f_*(x) = V^f_{\alpha^*_f,\beta^*_f}(x)$ *for any* $x \in [0, \beta^*_{f,+}]$. *While the reserves are smaller than* $\beta^*_{f,+}$ *it is optimal to adopt the policy* $(\tau^{\pi_{\beta^*}}_{\alpha^*}, \pi_{\beta^*})$.

(ii) *In the case* $\{K > 0$ *and* $\alpha^*_{f,\varnothing} < \alpha^*_f\}$ *it holds* $V^f_*(x) = V^{f,\varnothing}_{\alpha^*_{f,\varnothing},\beta^*_{f,\varnothing}}(x)$ *for any* $x \in [0, \beta^*_{f,\varnothing}]$. *While the reserves are smaller than* $\beta^*_{f,\varnothing}$ *it is optimal to adopt the policy* $(T_{\alpha^*_{f,\varnothing},\beta^*_{f,\varnothing}}, \pi^{\varnothing})$.

In particular, it holds

$$V^f_*(x) = \begin{cases} f(x), & x \in [0, a^*), \\ F^{(a^*)}(x - a^*), & x \in [a^*, b^*], \end{cases} \tag{7.15}$$

where $F^{(a^*)} = F_{a^*f}$ *and* $(a^*, b^*) = (\alpha^*_f, \beta^*_{f,+})$ *in the cases* $K = 0$ *or* $\{K > 0$ *and* $\alpha^*_{f,\varnothing} \geq \alpha^*_f\}$, *and* $(a^*, b^*) = (\alpha^*_{f,\varnothing}, \beta^*_{f,\varnothing})$ *in the case* $\{K > 0$ *and* $\alpha^*_{f,\varnothing} < \alpha^*_f\}$.

The proof of Theorem 7.6 rests an auxiliary result concerning the combination of locally defined martingales into a globally defined one, which is developed in the next section.

8. Pasting lemma. The verification that a given stochastic solution satisfies a global martingale property relies on "martingale pasting," which is the property (shown below) that, for a given function g, the combination of two supermartingales of type (4.1) on two adjacent closed intervals I_1 and I_2 gives rise to a supermartingale defined on the union $I_1 \cup I_2$, provided that, in the case that X has unbounded variation, g is differentiable at the intersection $I_1 \cap I_2$ of I_1 and I_2.

LEMMA 8.1. *Let* $(I_i)_{i=1}^n$ *be a finite collection of closed intervals with disjoint interiors satisfying* $\bigcup_{i=1}^n I_i = \mathbb{R}_+$, *and let* $g : \mathbb{R} \to \mathbb{R}$ *be a càdlàg function satisfying boundary condition* (3.7) *and growth condition* (4.3). *Assume in addition that* g *is differentiable at any* $x > 0$ *with* $x \in \bigcup_{i=1}^n \partial I_i$[5] *if* X *has unbounded variation. If*

$$S^{T_{I_i}} = \{e^{-q(t\wedge T_{I_i})} g(X_{t\wedge T_{I_i}}), t \in \mathbb{R}_+\} \qquad \text{are } \mathbf{F}\text{-supermartingales}, \tag{8.1}$$

for $i = 1, \ldots, n$, *then*

$$S = \{e^{-q(t\wedge T_{\mathbb{R}_+})} g(X_{t\wedge T_{\mathbb{R}_+}}), t \in \mathbb{R}_+\} \qquad \text{is a UI } \mathbf{F}\text{-supermartingale}. \tag{8.2}$$

The pasting lemma implies in particular that a global super-martingale property holds for sufficiently regular stochastic supersolutions:

[5]For any set A, $\partial A = \overline{A} \setminus A^o$ is the boundary of A, where $\overline{A}$, A^o denote the closure and interior of A.

COROLLARY 8.2. *Assume that g is a local stochastic supersolution on I_i, $i = 1, \dots, n$, for some finite collection of closed intervals $(I_i)_{i=1}^n$ with $\bigcup_{i=1}^n I_i = \mathbb{R}_+$ and $I_i^o \cap I_j^o = \varnothing$ for $i \neq j$. If X has unbounded variation, suppose in addition that g is differentiable at any $x > 0$ with $x \in \bigcup_{i=1}^n \partial I_i$. Then* (8.2) *holds true.*

PROOF OF LEMMA 8.1. In view of the observations that S is $\mathbf{F}$-adapted and UI [by Lemma 3.3(ii), as g satisfies the linear growth condition], it suffices to show that $\mathbb{E}[S_t|\mathcal{F}_s] \leq S_s$ for any $s, t \in \mathbb{R}_+$ with $s < t$. For the ease of presentation, only the verification in the case of a collection of closed intervals the form $\{[0, a], [a, \infty)\}$ for some $a > 0$ is considered, as the general case follows by a similar line of reasoning.

Fix thus $s, t \in \mathbb{R}_+$ arbitrary with $s < t$ and suppose first that X has bounded variation. Then a is irregular for $(-\infty, a)$ for X, so that the following collection of stopping times $(T_i)_{i \in \mathbb{N} \cup \{0\}}$ forms a discrete set:

$$T_0 := 0, \qquad T_{2i} := T_{[0,a]} \circ \theta_{T_{2i-1}}, \qquad T_{2i-1} = T_{[a,\infty)} \circ \theta_{T_{2i-2}}, \qquad i \in \mathbb{N}, \tag{8.3}$$

where θ denotes the translation operator. The strong Markov property of X and the tower property of conditional expectation imply that, on the event $\{s \leq T_{i-1}, T_{i-1} < \infty\}$, $i \in \mathbb{N}$, $\mathbb{E}[S_{t\wedge T_i} - S_{t\wedge T_{i-1}}|\mathcal{F}_s]$ is equal to

$$\begin{aligned} &\mathbb{E}\big[\mathbb{E}[S_{t\wedge T_i} - S_{t\wedge T_{i-1}}|\mathcal{F}_{T_{i-1}}]|\mathcal{F}_s\big] \\ &\quad = \mathbb{E}\big[\mathbf{1}_{\{t > T_{i-1}\}} \mathrm{e}^{-qT_{i-1}} \\ &\qquad \times \big\{\mathbb{E}_{X_{t\wedge T_{i-1}}}\big[\mathrm{e}^{-qR_v} g(X_{R_v})|\mathcal{F}_s\big]|_{v=T_{i-1}\wedge t} - g(X_{t\wedge T_{i-1}})\big\}\big], \end{aligned} \tag{8.4}$$

with $R_v = (T_i \wedge t) \circ \theta_v$, where the expectation on the right-hand side is nonpositive in view of Doob's optional stopping theorem [which holds in view of the uniform integrability of S and the assumed supermartingale property (8.1)]. Since $T_n \to \infty$ $\mathbb{P}$-a.s. as $n \to \infty$ (recalling $\inf \varnothing = \infty$ and $X_t \to \infty$ as $t \to \infty$) and S is UI, it follows $\mathbb{E}[S_t - S_s|\mathcal{F}_s] = \lim_{n\to\infty} \mathbb{E}[S_t^{T_n} - S_s^{T_n}|\mathcal{F}_s]$ is equal to the limit as $n \to \infty$ of

$$\sum_{j=1}^n \mathbf{1}_{\{T_{j-1} < s \leq T_j\}} \left\{ \mathbb{E}\big[(S_{T_j\wedge t} - S_{T_j\wedge s})|\mathcal{F}_s\big] + \sum_{i=j+1}^n \mathbb{E}\big[(S_{t\wedge T_i} - S_{t\wedge T_{i-1}})|\mathcal{F}_s\big] \right\},$$

which is nonpositive.

Suppose next that X has unbounded variation. For any given $\varepsilon > 0$, denote by $(T_i')_{i\in\mathbb{N}\cup\{0\}}$ the sequence of subsequent entrance times into the sets $[a - \varepsilon, a + \varepsilon]$ and $\mathbb{R} \setminus [a - 2\varepsilon, a + 2\varepsilon]$,

$$\begin{aligned} &T_0' := 0, \qquad T_{2i-1}' := T_{\mathbb{R}\setminus[a-\varepsilon,a+\varepsilon]} \circ \theta_{T_{2i-2}'}, \\ &T_{2i}' := T_{[a-2\varepsilon,a+2\varepsilon]} \circ \theta_{T_{2i-1}'}, \qquad i \in \mathbb{N}, \end{aligned}$$

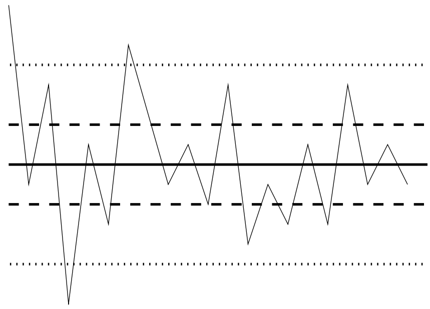

FIG. 1. *The martingale increments commence when X enters the inner band* (*dashed*) *and stop when X leaves the outer band* (*dotted*).

(see Figure 1). For any $t \in \mathbb{R}_+$, decompose S_t as $S_t - S_0 = S_t^{(1,\varepsilon)} + S_t^{(2,\varepsilon)}$ with

$$S_t^{(1,\varepsilon)} = \sum_{i\geq 1}[S_{t\wedge T'_{2i}} - S_{t\wedge T'_{2i-1}}], \qquad S_t^{(2,\varepsilon)} = \sum_{i\geq 1}[S_{t\wedge T'_{2i-1}} - S_{t\wedge T'_{2i-2}}].$$

The conditional expectation $\mathbb{E}[S_t^{(1,\varepsilon)} - S_s^{(1,\varepsilon)}|\mathcal{F}_s]$, which concerns increments of S during the periods that X spends in the band $[a-2\varepsilon, a+2\varepsilon]$, vanishes as $\varepsilon \searrow 0$, as shown in the following result:

LEMMA 8.3. *We have* $\lim_{n\to\infty} \mathbb{E}[S_t^{(1,\varepsilon_n)} - S_s^{(1,\varepsilon_n)}|\mathcal{F}_s] \leq 0$ *a.s. for some sequence* $(\varepsilon_n)_n$ *with* $\varepsilon_n \searrow 0$.

The proof of Lemma 8.3 is given below. Since $S^{(2,\varepsilon)}$ is a UI super-martingale for any $\varepsilon > 0$ (which follows by the line of the reasoning given in the first part of the proof), we thus have that $\mathbb{E}[S_t|\mathcal{F}_s]$ is equal to

$$\lim_{n\to\infty} \mathbb{E}\big[S_t^{(1,\varepsilon_n)}|\mathcal{F}_s\big] + \lim_{n\to\infty} \mathbb{E}\big[S_t^{(2,\varepsilon_n)}|\mathcal{F}_s\big] \leq \lim_{n\to\infty}\big(S_s^{(1,\varepsilon_n)} + S_s^{(2,\varepsilon)}\big),$$

which is equal to S_s. As s and t were arbitrary, the proof is complete. $\square$

Lemma 8.3 can be established deploying the properties of Gerber–Shiu functions:

PROOF OF LEMMA 8.3. Let $\varepsilon > 0$ be given and, for any $t \geq 0$ write $S_t^{(1,\varepsilon)} = \Sigma_t^{(1,\varepsilon)} + \Sigma_t^{(2,\varepsilon)} + \Sigma_t^{(3,\varepsilon)}$ with $\Sigma_t^{(1,\varepsilon)} = \sum_{i\geq 1} g(X_{t\wedge T'_{2i}})[\mathrm{e}^{-q(t\wedge T'_{2i})} - \mathrm{e}^{-q(t\wedge T'_{2i-1})}]$,

$$\Sigma_t^{(2,\varepsilon)} = \sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T'_{2i-1})}\big[\mathbb{E}\big[g(X_{t\wedge T'_{2i}})|\mathcal{F}_{t\wedge T'_{2i-1}}\big] - g(X_{t\wedge T'_{2i-1}})\big]$$

and $\Sigma_t^{(3,\varepsilon)} = \sum_{i\geq 1} \mathrm{e}^{-q(t\wedge T'_{2i-1})}[g(X_{t\wedge T'_{2i}}) - \mathbb{E}[g(X_{t\wedge T'_{2i}})|\mathcal{F}_{t\wedge T'_{2i-1}}]]$. We next estimate these three sums.

In view of growth condition (4.3), it follows that there exist positive real numbers a and b satisfying $\{\forall x \in \mathbb{R}_+, |g(x)| \le ax + b\}$, so that the following estimate holds:

$$\left|\Sigma_t^{(1,\varepsilon)}\right| \le (a\overline{X}_{t\wedge\tau_\pi} + b) \int_0^{t\wedge\tau_\pi} \mathrm{e}^{-qs} \mathbf{1}_{\{X_s \in (a-2\varepsilon, a+2\varepsilon)\}} \,\mathrm{d}s, \qquad t \ge 0.$$

The absolute continuity of the potential measure of X and the integrability of $\overline{X}_t$ for any $t \ge 0$ implies that, as $\varepsilon \searrow 0$, the left-hand side tends to zero $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ (by Lebesgue's dominated convergence theorem).

The next step is the observation that the following estimate holds (as a consequence of the differentiability of g at a):

LEMMA 8.4. *Let $\eta > 0$ and $q \ge 0$. There exists a $\widetilde{C} > 0$ such that for all $\varepsilon > 0$ sufficiently small, $L(x) = \mathbb{E}_x[\mathrm{e}^{-qT_{a-2\varepsilon,a+2\varepsilon}} g(X_{T_{a-2\varepsilon,a+2\varepsilon}})] - g(x)$ satisfies*

$$\sup_{x\in[a-2\varepsilon,a+2\varepsilon]} L(x) \le \varepsilon \cdot C(\varepsilon), \qquad C(\varepsilon) := \widetilde{C}\left[\eta + W^{(q)}(4\varepsilon)\right]. \tag{8.5}$$

The proof of Lemma 8.4 is given below.

The triangle inequality and the strong Markov property imply that $|\Sigma_t^{(2,\varepsilon)}|$ is bounded by the sum $\sum_{i\ge1} \mathrm{e}^{-q(t\wedge T'_{2i-1})}|(\widetilde{L}_1 + \widetilde{L}_2)(t - t\wedge T'_{2i-1}, X_{t\wedge T'_{2i-1}})|$ where $\widetilde{L}_1(t,x) = \mathbb{E}_x[(g(X_t) - g(x))\mathbf{1}_{\{T>t\}}]$ and $\widetilde{L}_2(t,x) = \mathbb{E}_x[(g(X_T) - g(x))\mathbf{1}_{\{T\le t\}}]$ with $T = T_{a-2\varepsilon,a+2\varepsilon}$ may be decomposed as $\widetilde{L}_2(t,x) = A_1 - A_2$ with $A_1 = L(x)$, and

$$A_2 = \mathbb{E}_x\left[\left(g(X_T) - g(x)\right)\mathbf{1}_{\{t<T\}}\right] = \mathbb{E}_x\left[L(X_t)\mathbf{1}_{\{t<T\}}\right].$$

To estimate $|\Sigma_t^{(2,\varepsilon)}|$ we split it into two sums. It is straightforward to check that the sum involving the terms $\widetilde{L}_1$ is bounded by $\mathbb{E}_x[|g(X_t) - g(X_\rho)|\mathbf{1}_{\{t<\rho'\}}]$ where $\rho = \sup\{u \le t : X_u \in (a-\varepsilon, a+\varepsilon)\}$ and $\rho' = \inf\{t > \rho : X_t \notin [a-2\varepsilon, a+2\varepsilon]\}$, which in turn is bounded by $C'\varepsilon$ for some constant C' (as g is differentiable in a).

Furthermore, it follows from Lemma 8.4 that $\widetilde{L}_2(t,x)$ is bounded by $2\varepsilon C(\varepsilon)$. Observe next that the number of terms in the sum $\Sigma^{(2,\varepsilon)}$ is bounded by $1 + D_t^-(\varepsilon) + U_t^+(\varepsilon)$, where $D_t^-(\varepsilon)$ and $U_t^+(\varepsilon)$ denote the numbers of down-crossings of the band $(a-2\varepsilon, a-\varepsilon)$ and upcrossings of $(a+\varepsilon, a+2\varepsilon)$ by X before time t. Thus the expectation of $|\Sigma_t^{(2,\varepsilon)}|$ can be bounded as follows:

$$\mathbb{E}_x\left[\left|\Sigma_t^{(2,\varepsilon)}\right|\right] \le 2\varepsilon\mathbb{E}_x\left[1 + D_t^-(\varepsilon) + U^+(\varepsilon)\right]C(\varepsilon) + C'\varepsilon. \tag{8.6}$$

Since X is a Lévy process with positive drift, X is a submartingale, so that the upcrossing lemma implies that the expected number of upcrossings of the band $(c,d) = (a+\varepsilon, a+2\varepsilon)$ by time t does not grow faster than ε^{-1},

$$\varepsilon \cdot \mathbb{E}_x\left[U_t^+(\varepsilon)\right] \le \mathbb{E}_x\left[(X_t - d)^+\right] - \mathbb{E}_x\left[(X_0 - c)^+\right].$$

Thus, it follows that $\varepsilon \cdot \mathbb{E}_x[U_t^+(\varepsilon)]$ remains bounded as $\varepsilon \to 0$. As the number of downcrossings $D_t^-(\varepsilon)$ of the band $(a-2\varepsilon, a-\varepsilon)$ is bounded by $2+U_t^+(\varepsilon)$; also $\varepsilon \cdot \mathbb{E}_x[D_t^-(\varepsilon)]$ remains bounded as $\varepsilon \to 0$. Since $C(\varepsilon)$ tends to η as $\varepsilon \to 0$ [as $W^{(q)}(0)=0$ when X has unbounded variation], it thus follows from (8.6) that $\mathbb{E}_x[|\Sigma_t^{(2,\varepsilon)}|]$ tends to 2η as ε tends to zero. As η is arbitrary, we conclude $\lim_{\varepsilon\searrow 0} \mathbb{E}_x[|\Sigma_t^{(2,\varepsilon)}|]=0$.

Next we turn to the sum $\Sigma^{(3,\varepsilon)}$. We have the decomposition $\mathbb{E}[\Sigma_t^{3,\varepsilon} - \Sigma_s^{3,\varepsilon}|\mathcal{F}_s] = \sum_{j\geq 1} \mathbf{1}_{\{T_{2j-2}\leq s<T_{2j}\}} B_j$ with $B_j = \mathrm{e}^{-q(t\wedge T_{2j-1}}(E[g(X_{T_{t\wedge T_{2j}}})|\mathcal{F}_s] - E[g(X_{T_{t\wedge T_{2j}}})|\mathcal{F}_{t\wedge T_{2j-1}}]$. Reasoning as above we find that the sum convergences to 0 in $L^1(\mathbb{P})$ when $\varepsilon \to 0$. Finally, an application of the Borel–Cantelli lemma (recalling $S^{(1,\varepsilon)} = \sum_{i=1}^3 \Sigma^{(i,\varepsilon)}$) yields the existence of a sequence (ε_n), $\varepsilon_n \to 0$, such that $\mathbb{E}[S_t^{(1,\varepsilon_n)} - S_s^{(1,\varepsilon_n)}|\mathcal{F}_s] \to 0$ a.s. as $n\to\infty$. $\square$

PROOF OF LEMMA 8.4. By rearranging terms observe that $L(x)$ can be written as $L(x) = g(a)R_0(x) + g'(a)R_1(x) + R(x) - \tilde{w}(x)$ with $\tilde{w}(x) := g(x) - g(a) - g'(a)(x-a)$, $R(x) := \mathbb{E}_x[\mathrm{e}^{-qT_{a-2\varepsilon,a+2\varepsilon}}\tilde{w}(X_{T_{a-2\varepsilon,a+2\varepsilon}})]$, $R_0(x) := \mathbb{E}_x[\mathrm{e}^{-qT_{a-2\varepsilon,a+2\varepsilon}}] - 1$ and

$$R_1(x) := \mathbb{E}_x\big[\mathrm{e}^{-qT_{a-2\varepsilon,a+2\varepsilon}}(X_{T_{a-2\varepsilon,a+2\varepsilon}} - a)\big] - (x-a).$$

Next the terms $R_0(x)$, $R_1(x)$ and $R(x)$ are estimated. Given $\eta > 0$, let $\delta > 0$ satisfy $|\tilde{w}(y)/(y-a)| < \eta$, whenever $|y-a| < \delta$ (such a δ exists as g is assumed to be differentiable at a). Then, for any ε sufficiently small and any $x \in [a-2\varepsilon, a+2\varepsilon]$, the bounds $|\tilde{w}(x)| \leq 2\eta\varepsilon$ and $|R(x)| \leq |R_2(x)| + \eta|R_3(x)|$ hold, with

$$\begin{aligned} &R_i(x) = \mathbb{E}_x\big[\mathrm{e}^{-qT_{a-2\varepsilon,a+2\varepsilon}} w_i(X_{T_{a-2\varepsilon,a+2\varepsilon}})\big], \qquad i=2,3, \\ &w_2(x) = \tilde{w}(x)\mathbf{1}_{(-\infty,a-\delta]}(x), \qquad w_3(x) = (x-a)\mathbf{1}_{(a-\delta,0]}(x), \qquad x \leq a. \end{aligned} \tag{8.7}$$

From expression (5.11), with the replacements $a \to a-2\varepsilon$, $b \to a+2\varepsilon$ and $w \to \tilde{w}_i \in \mathcal{R}_0$ for $i=0,\dots,3$ given by $\tilde{w}_i = {}_{a-2\varepsilon}w_i$ with $w_i : (-\infty, a-2\varepsilon] \to \mathbb{R}$ specified in (8.7) and by $w_0(x) := 1$ and $w_1(x) := x-a+2\varepsilon$, and the fact that $W^{(q)}$ is increasing, it is straightforward to verify that, for any $x \in [a-2\varepsilon, a+2\varepsilon]$,

$$\big|R_i(x)\big| \leq 2 \max_{z\in[0,4\varepsilon]} \big|F_{\tilde{w}_i}(z) - \tilde{w}_i(0) - \tilde{w}'_{i,-}(0)z\big|, \qquad i=0,1,2. \tag{8.8}$$

Since the functions $J_{\tilde{w}_i}$, $i=0,1,2$, given in (5.5) with $w \to \tilde{w}_i$, are bounded, by J_∞ say, and $W^{(q)}$ is increasing, it follows from the form (5.4) of F_w that $|F_{\tilde{w}_i}(z) - \tilde{w}_i(0) - \tilde{w}'_{i,-}(0)z|$, $i=0,1,2$, $z\in[0,4\varepsilon]$, is bounded by

$$J_\infty \int_0^z W^{(q)}(z-y)\,\mathrm{d}y \leq J_\infty \cdot 4\varepsilon \cdot W^{(q)}(4\varepsilon). \tag{8.9}$$

Combining (8.8) and (8.9) yields that the functions $R_i(x)$, $i=0,1,2$, are each bounded by $J_\infty \cdot 8\varepsilon W^{(q)}(4\varepsilon)$ for any $x \in [a-2\varepsilon, a+2\varepsilon]$. Similarly, it follows from the facts $F_{\tilde{w}_3}(0) = \tilde{w}_3(0) = 0$ and $F'_{\tilde{w}_3}(0+) = \tilde{w}'_{3,-}(0) = 1$ (Theorem 5.3) that, for all ε sufficiently small, $|R_3(x)| \le C_1\varepsilon$, for all x in the interval $[a-2\varepsilon, a+2\varepsilon]$ for some constant $C_1 > 0$. Combining the estimates for $\tilde{w}(x)$ and $R_0(x), \dots, R_3(x)$ with the form of $L(x)$ completes the proof. $\square$

9. Optimality conditions for single dividend-band strategies. A necessary and sufficient condition for the optimality of the single band policy $\pi_{\underline{b}^*}$ at levels $\underline{b}^* := b_1^* = (b_-^*, b_+^*)$ defined in (6.5)–(6.6) can be expressed in terms of the function $G^* : (b_-^*, \infty) \to \mathbb{R}$ given by

$$G^*(y) = G(b_-^*, y) = \begin{cases} \dfrac{y - b_-^* - K - (F(y) - F(b_-^*))}{W^{(q)}(y) - W^{(q)}(b_-^*)}, & \text{if } K > 0, \\ G^{\#}(x) = \dfrac{1 - F'(x)}{W^{(q)\prime}(x)}, & \text{if } K = 0. \end{cases} \tag{9.1}$$

This condition can be expressed in terms of the function $Z^{(q,v)}$ that was defined in Definition 5.8.

THEOREM 9.1. (i) *The single-band policy $\pi_{\underline{b}^*}$ at level $\underline{b}^* = b_1^*$ is optimal for the stochastic control problem* (2.2) *if and only if*

$${}_{b_+^*}(\Gamma_\infty^{\overline{w}} v_{b^*} - q v_{b^*})(x) \le 0 \qquad \text{for all } x > b_+^* \text{ and with } \overline{w} = v_{b^*}, \tag{9.2}$$

where the operator ${}_{b_+^*}\Gamma_\infty^{\overline{w}}$ is defined in (3.10), *or equivalently, if and only if $\Xi^* : (\Phi(q), \infty) \to \mathbb{R}$ is completely monotone, where*

$$\Xi^*(\theta) = -\frac{e^{\theta b_+^*}}{\theta} \int_{(b_+^*, \infty)} e^{-\theta z} Z^{(q,\theta)\prime}(z) G^*(dz), \qquad \theta > \Phi(q). \tag{9.3}$$

(ii) *In particular, if G^* is nonincreasing on (b_+^*, ∞), then the strategy π_{b^*} is optimal.*

Theorem 9.1(ii) yields a useful simple sufficient optimality condition:

COROLLARY 9.2. (i) *The unimodality of the function G^* implies the optimality of single dividend-band policies.*

(ii) *In particular, in the case $K = 0$ and if $G^{\#}$ is monotone decreasing, then the "lump-sum" strategy π_0 is optimal.*

REMARK 9.3. In the absence of transaction costs, the function Ξ^* in (9.3) can be equivalently expressed as

$$\Xi^*(\theta) = G^{\#}(b_+^*) L_0(\theta) + \frac{(\psi(\theta) - q)}{\theta^2} \mathbb{E}[F'(b_+^* + \mathbf{e}_\theta) - F'(b_+^*)],$$

$$L_0(\theta) := \frac{\psi(\theta) - q}{\theta^2} \mathbb{E}[W^{(q)\prime}(b_+^* + \mathbf{e}_\theta) - W^{(q)\prime}(b_+^*)],$$

where $\mathbf{e}_\theta$ denotes an independent exponential random variable with mean θ^{-1}. In particular, if the penalty is zero and there are no transaction cost ($w = K = 0$), the necessary and sufficient optimality condition simplifies to the complete monotonicity of $L_0(\theta)$ on the interval $(\Phi(q), \infty)$. This observation appears new even in this particular case.

REMARK 9.4 (Lump-sum strategy). In the absence of transaction cost ($K = 0$), the "lump-sum" strategy π_0 is to "pay out all the reserves to the beneficiaries and subsequently pay all the premiums as dividends, until the moment of ruin." Note that π_0 is a single dividend-band strategy at level 0. In the case that X is given by the Cramér–Lundberg model, the first jump (claim) arrives after an independent exponential time $\mathbf{e}_\lambda$ with finite mean λ^{-1}, so that the value v_0 is equal to

$$\begin{aligned} v_0(x) &= \mathbb{E}_x\left[x + p\int_0^{\mathbf{e}_\lambda} \mathrm{e}^{-qt}\,\mathrm{d}t + \mathrm{e}^{-q\mathbf{e}_\lambda} w(\Delta X_{\mathbf{e}_\lambda})\right] \\ &= \mathbb{E}_x\left[x + \frac{p}{q}\left(1 - \mathrm{e}^{-q\mathbf{e}_\lambda}\right) + \mathrm{e}^{-q\mathbf{e}_\lambda}\big(w(\Delta X_{\mathbf{e}_\lambda}) - w(0)\big) + w(0)\mathrm{e}^{-q\mathbf{e}_\lambda}\right], \end{aligned}$$

which is equal to $x + \frac{p + w_\nu(0) + \lambda w(0)}{\lambda + q}$, where $\Delta X_{\mathbf{e}_\lambda} = X(\mathbf{e}_\lambda) - X(\mathbf{e}_\lambda -)$, and $w_\nu : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ is defined in Proposition 5.4. If X_0 is zero and X has infinite activity or nonzero Gaussian component, ruin occurs immediately if strategy π_0 is followed ($\tau^{\pi_0} = 0$, $\mathbb{P}_0$-a.s.) and $v_0(x) = x + w(0)$.

Hence, the value of the lump-sum strategy is equal to $v_0(x) = (x + \gamma_w) \times \mathbf{1}_{[0,\infty)}(x) + w(x)\mathbf{1}_{(-\infty,0)}(x)$ with $\gamma_w = v_0(0)$ given by

$$\begin{cases} \dfrac{1}{q + \overline{\nu}}[p + w_\nu(0) + \overline{\nu} w(0)], & \text{if } \overline{\nu} := \nu(\mathbb{R}_+) < \infty \text{ and } \sigma = 0, \\ w(0), & \text{if } \overline{\nu} = \infty \text{ or } \sigma > 0. \end{cases}$$

If $G^{\#}$ is monotone decreasing, it attains its maximum over $\mathbb{R}_+$ at zero, and the function Ξ is completely monotone, so that π_0 is optimal [Theorem 9.1(ii)].

REMARK 9.5. In the following result (proved in Appendix D) explicit sufficient conditions are given in terms of the penalty w and the Lévy density ν for optimality of a single barrier strategy at a positive level:

COROLLARY 9.6. *In the case* $\{K = 0$ *and* $b_1^* > 0\}$, *if* ν *admits a convex density* ν' *and the penalty* w *is* severe [*i.e.*, $w(0) \le \gamma_w$ *and* $w(x + y) - w(y) \le x$ *for all* $x, y \in \mathbb{R}_-$], *then the strategy* $\pi_{b_1^*}$ *is optimal.*

Note that a penalty w is severe if (i) the penalty at 0 is at least the value of the lump-sum strategy at 0 and (ii) the slope of the penalty is at least one.

PROOF OF THEOREM 9.1, PART (I). The equivalence of the conditions (9.2) and (9.3) directly follows due to Lemma 6.5(iii).

Proof of sufficiency of (9.2): It suffices to show that v_{b^*} is a stochastic supersolution, as then the local verification theorem (Theorem 4.4) implies that v_{b^*} is equal to the value-function v_*. The supersolution property of v_{b^*} follows by combining the pasting lemma (Lemma 8.1) with the following facts:

(a) $\exp\{-q(t \wedge T^-_{b^*_+})\}v_{b^*}(X(t \wedge T^-_{b^*_+}))$ is an $\mathbf{F}$-supermartingale [by (9.2) and Lemma 3.4(ii)],

(b) $\exp\{-q(t \wedge T_{0,b^*_+})\}v_{b^*}(X(t \wedge T_{0,b^*_+}))$ is an $\mathbf{F}$-martingale [by the form of v_{b^*} in (6.7) and the martingale properties of $W^{(q)}$ and F_w in Proposition 3.1] and

(c) if X has unbounded variation, v_{b^*} is differentiable at b^*_+ [in view of the form of v_{b^*} in (6.7)].

Proof of necessity of (9.2): Suppose that the condition in (9.2) is not satisfied. Since $x \mapsto ({}_{b^*_+}\Gamma^{\overline{w}}_{\infty} v_{b^*} - q v_{b^*})(x)$ is right-continuous at any x with $x > b^*_+$, it follows that there exists an open interval (α, β) contained in (b^*_+, ∞) with $({}_{b^*_+}\Gamma^{\overline{w}}_{\infty} v_{b^*} - q v_{b^*})(x) > 0$ for $x \in (a, b)$. Define a strategy $\tilde{\pi}$ as follows: whenever U_t does not take a value in the interval (α, β), operate according to π_{b^*}, and while the reserve process U_t takes a value in the interval (α, β), do not pay any dividends. Then $S_t := \mathrm{e}^{-q(t \wedge T_{\alpha,\beta})}(v_{\tilde{\pi}}(X_{t\wedge T_{\alpha,\beta}}) - v_{b^*}(X_{t\wedge T_{\alpha,\beta}}))$ is an $\mathbf{F}$-supermartingale, and the following holds true [cf. (3.12)] for any $x \in (\alpha, \beta)$:

$$v_{\tilde{\pi}}(x) - v_{b^*}(x) \geq \mathbb{E}_x[S_t - S_0] = \mathbb{E}_x\left[\int_0^{t\wedge T_{\alpha,\beta}} \mathrm{e}^{-qs}\big({}_{b^*_+}\Gamma^{\overline{w}}_{\infty} v_{b^*} - q v_{b^*}\big)(X_s)\,\mathrm{d}s\right] > 0.$$

Hence it follows that π_{b_*} is not an optimal policy, and the proof is complete. □

PROOF OF THEOREM 9.1, PART (II). The statement follows by combining part (i) with the next result. □

LEMMA 9.7. *If $x \mapsto G^*(x)$ is nonincreasing on (b^*_+, ∞), then $\Xi(\theta)$ is completely monotone on $(\Phi(q), \infty)$.*

PROOF. If the function G^* is nonincreasing, then the function Ξ is completely monotone in view of the form of Ξ given in (9.3), the complete monotonicity of $\theta^{-1}\mathrm{e}^{\theta(b-x)}Z^{(q,\theta)\prime}(x)$ [cf. Remark 5.10(ii)] and the following facts:

(i) A function $f : (c, \infty) \to \mathbb{R}_+$, $c > 0$, is completely monotone if and only if f is the Laplace transform of a measure supported on $[0, \infty)$.

(ii) If $f(\theta)$ is the Laplace transform of the measure μ supported on $[0, \infty)$, then for any $c > 0$, $\mathrm{e}^{-\theta c} f(\theta)$ is the Laplace transform of the translated measure $y \mapsto \mathbf{1}_{\{y \geq c\}}\mu(\mathrm{d}(y - c))$.

(iii) The Laplace transform of the measure $n(\mathrm{d}y) = \int_{[b,\infty)} \mu_x(\mathrm{d}y)m(\mathrm{d}x)$ supported on $[0, \infty)$ is given by $\mathcal{L}n(\theta) = \int_{[b,\infty)} \mathcal{L}\mu_x(\theta)m(\mathrm{d}x)$ where $(\mu_x, x > b)$, $b \in \mathbb{R}$, is a collection measures supported on $[0, \infty)$. □

10. Optimality conditions for solutions to the mixed optimal stopping/ control problem. The Hamilton–Jacobi–Bellman equation associated to the stochastic control problem in (7.2) differs from (3.6) by the inclusion of the additional requirement that the value-function should be larger than the function f [reflecting the fact that (7.2) is a mixed optimal stopping/control problem]; hence, the HJB equation corresponding to (7.2) is given by

$$\max\{\mathcal{L}g(x) - qg(x), f(x) - g(x), 1 - \mathrm{d}_g(x)\} = 0, \qquad x > 0, \tag{10.1}$$

$$\begin{cases} g(x) = f(x), & \text{for all } x < 0, \\ g(0) = f(0), & \text{in the case } \{\sigma^2 > 0 \text{ or } \nu_{0,1} = \infty\}, \end{cases} \tag{10.2}$$

where $\mathrm{d}_g(x)$ is defined in (3.4). Stochastic supersolutions g of the HJB equation in (10.1) and (10.2) are defined as in Definition 4.1, with the additional requirement $g \geq f$. By a line of reasoning similar to that used in the proof of Theorem 4.4, it follows that a local verification result for the stochastic control problem (7.2) holds true:

COROLLARY 10.1. *Let g be a stochastic supersolution of the HJB equation in* (10.1) *and* (10.2). *If there exist c, a, b_-, b_+ satisfying $0 \leq c \leq a \leq b_- \leq b_+$ and $g(x) = V^f_{a,b_-,b_+}(x)$ $\{g(x) = V^{f,\varnothing}_{a,b_+}(x)\}$ for any $x \in [c, b_+]$, then it holds $V^f_*(x) = V^f_{a,b_-,b_+}(x)$ for all $x \in [c, b_+]$ $\{V^f_*(x) = V^{f,\varnothing}_{a,b_+}(x)$ for all $x \in [c, b_+]\}$, respectively.*

Given this verification result the proof of Theorem 7.6 can be completed. A key step in the proof is the following property of the function f:

LEMMA 10.2. *Suppose that f satisfies the conditions in* (7.3)–(7.6), *and denote $\bar{w} = f|_{\mathbb{R}_-}$. It holds ${}_0\Gamma^{\bar{w}}_\infty f(u) \leq 0$ for all $u \in (0, \alpha(K))$ with $\alpha(0) := \alpha^*_f$ and $\alpha(K) := \alpha^*_{f,\varnothing}$ for $K > 0$.*

PROOF OF THEOREM 7.6. (i) Since $V^f_{\alpha^*_f,\beta^*_f}$ is the value-function of the strategy $(\tau^{\pi_{\beta^*}}_{\alpha^*}, \pi_{\beta^*})$, Corollary 10.1 implies that, to prove the assertion, it suffices to show that $V^f_{\alpha^*_f,\beta^*_f}$ is a supersolution of the HJB equation in (10.1) and (10.2). Next the various conditions are verified.

Analogously to the proof of Theorem 6.4, it follows from the definition of β^*_f and the form of the function $V = V^f_{\alpha^*_f,\beta^*_f}$ given in Proposition 7.2 that the following inequality holds:

$$V(x) - V(y) \geq x - y - K \tag{10.3}$$

for all $x, y \geq 0$ satisfying $x \geq y \geq \alpha^*_f$. In view of the fact $V'(x) = f'(x) = 1$ for $x \in (0, \alpha^*_f)$, it follows that the inequality in (10.3) is in fact valid for all x and y satisfying $x \geq y \geq 0$.

To see that the V dominates the function f,

$$V(x) \geq f(x), \qquad x \geq 0, \tag{10.4}$$

note first that it holds $V(0) = f(0)$ (a direct consequence of the form of V in Proposition 7.2 and $\alpha_f^* > 0$ by Lemma 7.3). In the case $K = 0$, (10.4) is hence a special case of (10.3) (with $y = 0$). In the case $\{K > 0$ and $\alpha_{f,\varnothing}^* \geq \alpha_f^*\}$, the definitions of $\alpha_{f,\varnothing}^*$, $\beta_{f,\varnothing}^*$ and $G_{f,\varnothing}^{(a)}$, the positivity of $W^{(q)}(x)$ imply

$$\begin{aligned} &G_{f,\varnothing}^{(a)}(b) \leq 0 \qquad \text{for all } a \in [0, \alpha_{f,\varnothing}^*] \text{ and } b \in [0, \beta_{f,\varnothing}^*] \\ &\quad\iff\quad F^{(a)}(x-a) \geq f(x) \qquad \text{for all } x \in [0, \beta_{f,\varnothing}^*(a)] \text{ and } a \in [0, \alpha_{f,\varnothing}^*], \end{aligned}$$

which yields the inequality in (10.4), in view of the facts $V(x) = F^{(a)}(x-a)$ for all $x \leq b := \beta_{f,+}^*$ [by Proposition 7.2 and Lemma 7.3(i) and the fact $\beta_{f,+}^* \leq \beta_{f,\varnothing}^*$ which holds by Lemma 7.3(ii)], and $V|_{[b,\infty)}$ is affine (Proposition 7.2).

In view of the observations

$$\mathrm{e}^{-q(t\wedge T_{0,\alpha_f^*})} f(X_{t\wedge T_{0,\alpha_f^*}}) \qquad \text{is an } \mathbf{F}\text{-supermartingale, and} \tag{10.5}$$

$$\mathrm{e}^{-q(t\wedge T_{\alpha_f^*}^-)} F^{(\alpha_f^*)}(X_{t\wedge T_{\alpha_f^*}^-} - \alpha_f^*) \qquad \text{is an } \mathbf{F}\text{-martingale,} \tag{10.6}$$

and the differentiability of $F^{(\alpha_f^*)}(x)$ at $x = 0$ if X has unbounded variation $[F^{(\alpha_f^*)\prime}(0) = f_-'(\alpha_f^*)$, by Lemma 5.7], it follows from the pasting lemma (Lemma 8.1)

$$\mathrm{e}^{-q(t\wedge T_0^-)} F^{(\alpha_f^*)}(X_{t\wedge T_0^-} - \alpha_f^*) \qquad \text{is an } \mathbf{F}\text{-supermartingale.} \tag{10.7}$$

Here, the supermartingale property in (10.5) follows from Lemma 7.3(i), by a line of reasoning that is similar to the one used in the proof of Lemma 3.4, while the martingale property in (10.6) follows from Proposition 5.4.

The supermartingale property in (10.7) and the inequalities in (10.3) and (10.4) imply that $F^{(\alpha_f^*)}(x - \alpha_f^*)$ is a stochastic supersolution for the stochastic control problem in (7.2), which completes the proof of (i).

(ii) The line of reasoning is analogous to the one in part (i) (see Remark 7.5) and is therefore omitted. □

10.1. *Optimality conditions for two-band policies*. When a single band strategy is not globally optimal for the stochastic control problem in (2.2), it is not optimal to pay out a lump-sum dividend at all levels above b_+^* but is instead optimal to postpone paying dividends when the reserves process is in a certain subset of (b_*^+, ∞). This section is concerned with the necessary and sufficient conditions for optimality of a policy with only one additional band. Consider the candidate optimal two-band strategy $\pi_{\underline{a}^*,\underline{b}^*}$ at the levels $\underline{a}^* = (0, a_2^*)$ and $\underline{b}^* = (b_1^*, b_2^*)$ where the levels $b_1^* = (b_-^*, b_+^*)$ associated to the first band have been defined in (6.5)–(6.6), and where the levels associated to the second band are given by

$$\{a_2^*, b_2^*\} = b_{1,+}^* + \begin{cases} \{\alpha_{w^*}^*, (\beta_{w^*,-}^*, \beta_{w^*,+}^*)\}, & \text{if } K = 0 \\ & \text{or } \{K > 0 \text{ and } \alpha_{w^*,\varnothing}^* \geq \alpha_{w^*}^*\}; \\ \{\alpha_{v_{b_1^*},\varnothing}^*, (b_-^*, \beta_{w^*,\varnothing}^*)\}, & \text{if } \{K > 0 \text{ and } \alpha_{w^*,\varnothing}^* < \alpha_{w^*}^*\}, \end{cases}$$

where $w^* := {}_{b_{1,+}^*}v_{b_1^*}$ and the levels $\alpha_{w^*}^*, \alpha_{w^*,\varnothing}^*, \beta_{w^*,-}^*, \beta_{w^*,+}^*$ and $\beta_{w^*,\varnothing}^*$ are defined in (7.9)–(7.12).

Necessary and sufficient conditions for the two-band policy $\pi_{\underline{a}^*,\underline{b}^*}$ to be (globally) optimal are expressed in terms of the functions Ξ^* defined in (9.3) and the function

$$\Xi^{**} = \begin{cases} \Xi_{a_2^*,b_2^*}(w^*), & \text{if } K = 0 \text{ or } \{K > 0 \text{ and } \alpha_{w^*,\varnothing}^* \geq \alpha_{w^*}^*\}, \\ \Xi_{a_2^*,b_2^*}^{\varnothing}(w^*), & \text{if } \{K > 0 \text{ and } \alpha_{w^*,\varnothing}^* < \alpha_{w^*}^*\}. \end{cases}$$

Here for any a, b_- and b_+ with $a \leq b_- \leq b_+$ and $f \in \mathcal{R}_0$ the functions $\Xi_{a,b_-,b_+}(f)$ and $\Xi_{a,b_+}^{\varnothing}(f)$ are given by

$$\Xi_{a,b_-,b_+}(f) : \theta \mapsto -\frac{e^{\theta b_+}}{\theta} \int_{(b_+,\infty)} e^{-\theta z} Z^{(q,\theta)\prime}(z) G_{f,b_-}^{(a)}(dz),$$

$$\Xi_{a,b}^{\varnothing}(f) : \theta \mapsto -\frac{e^{\theta b}}{\theta} \int_{(b,\infty)} e^{-\theta z} Z^{(q,\theta)\prime}(z) G_{f,\varnothing}^{(a)}(dz),$$

where, for any $z \geq b_-$, $G_{f,b_-}^{(a)}(z) := G_f^{(a)}(b_-, z)$, and the functions $G_{f,\varnothing}^{(a)}$ and $G_f^{(a)}$ have been defined in (7.8) and (7.7).

Before stating the optimality condition for this two-band policy, we first state a condition for (global) optimality of the policies $(\tau_{\alpha_f^*}^{\pi_{\beta_f^*}}, \pi_{\beta_f^*})$ and $(T_{\alpha_{f,\varnothing}^*,\beta_{f,\varnothing}^*}, \pi^{\varnothing})$ in the auxiliary stochastic control problem in (7.2).

THEOREM 10.3. *Suppose that f satisfies the conditions in* (7.3)–(7.6).

(i) *Suppose that it holds either $K = 0$ or $\{K > 0$ and $\alpha_{f,\varnothing}^* \geq \alpha_f^*\}$. Then the strategy $(\tau_{\alpha_f^*}^{\pi_{\beta_f^*}}, \pi_{\beta_f^*})$ is optimal for the stochastic optimal control problem in* (7.2) *if and only if the function $\Xi_{\alpha_f^*,\beta_{f,-}^*,\beta_{f,+}^*}(f)$ is completely monotone.*

(ii) *Suppose that it holds $\{K > 0$ and $\alpha_{f,\varnothing}^* < \alpha_f^*\}$. Then the strategy $(T_{\alpha_{f,\varnothing}^*,\beta_{f,\varnothing}^*}, \pi^{\varnothing})$ is optimal for the stochastic optimal control problem in* (7.2) *if and only if the function $\Xi_{\alpha_{f,\varnothing}^*,\beta_{f,\varnothing}^*}(f)$ is completely monotone.*

The proof of Theorem 10.3 is omitted as it is analogous to the proof of Theorem 9.1(i).

REMARK 10.4. As in the proof of Lemma 6.5, it can be shown that the complete monotonicity of the function $\Xi_{\alpha^*_f,\beta^*_{f,-},\beta^*_{f,+}}(f)$ is equivalent to the condition

$$ {}_0\Gamma^w_\infty V^f_*(x) - qV^f_*(x) \le 0 \qquad \text{for all } x > \beta^*_{f,+}. \tag{10.8}$$

Similarly, it follows that the complete monotonicity of $\Xi_{\alpha^*_{f,\varnothing},\beta^*_{f,\varnothing}}(f)$ is equivalent to (10.8) with $\beta^*_{f,+}$ replaced by $\beta^*_{f,\varnothing}$.

The relationship between the stochastic control problems in (2.2) and (7.2) (cf. the discussion at the beginning of Section 7) immediately yields necessary and sufficient optimality conditions for the two-band strategy $\pi_{\underline{a}^*,\underline{b}^*}$:

COROLLARY 10.5. (i) *The two-band strategy* $\pi_{\underline{a}^*,\underline{b}^*}$ *at finite levels* $\underline{a} = (0, a^*_2)$ *and* $\underline{b} = (b^*_1, b^*_2)$ *is optimal for* (2.2) *if and only if* Ξ^* *is* not *completely monotone and* Ξ^{**} *is completely monotone*.

(ii) *If* Ξ^* *is not completely monotone then the levels* a^*_2 *and* $b^*_{2,+}$ *are finite*, *and it is optimal to adopt the two-band strategy* $\pi_{\underline{a}^*,\underline{b}^*}$ *while the reserves are below* $b^*_{2,+}$, *and it holds* (*with* $F^{(a^*_{2,+})}_* = F_{a^*_{2,+}} v_*$)

$$ v_*(x) = \begin{cases} W^{(q)}(x)\dfrac{1 - F'_w(b^*_{1,+})}{W^{(q)\prime}(b^*_{1,+})} + F_w(x), & x \in [0, b^*_{1,+}], \\ x - b^*_{1,+} + v_*(b^*_{1,+}), & x \in (b^*_{1,+}, a^*_{2,+}), \\ F^{(a^*_{2,+})}_*(x - a^*_{2,+}), & x \in [a^*_{2,+}, b^*_{2,+}]. \end{cases} \tag{10.9}$$

11. Multi dividend-band policies: The recursion for the dividend-band levels.

A flexible class of dividend strategies are the so-called multi dividend-band strategies, which generalize the single and two-band strategies, and are specified as follows:

DEFINITION 11.1. The *multi dividend-band strategy* $\pi_{\underline{a},\underline{b}}$, associated to sequences $\underline{a} = (a_n)_n$, $\underline{b}^- = (b^-_n)_n$, $\underline{b}^+ = (b^+_n)_n$ with $a_n, b^-_n, b^+_n \in [0,\infty]$ satisfying the intertwining conditions

$$ a_1 = 0 \le b^+_1 < a_2 \le b^+_2 < \cdots < a_n \le b^+_n < \cdots, \qquad b^-_n \le b^+_n, $$

is described as follows:

(i) when $U^{\underline{a},\underline{b}} := U^{\pi_{\underline{a},\underline{b}}} = y \in (b^+_n, a_{n+1})$, make a lump-sum payment $y - b^-_n$;

(ii) when $U^{\underline{a},\underline{b}} = b^+_n$, make a lump-sum payment $b^+_n - b^-_n$, if $K > 0$, and pay the minimal amount to keep $U^{\underline{a},\underline{b}}$ below $b^-_n = b^+_n$ if $K = 0$;

(iii) while $U^{\underline{a},\underline{b}} \in [a_n, b^+_n)$, do not pay any dividends.

The strategy $\pi^{\underline{a},\underline{b}}$ is called an N-dividend-bands strategy if $b^+_N < \infty = a_{N+1}$.

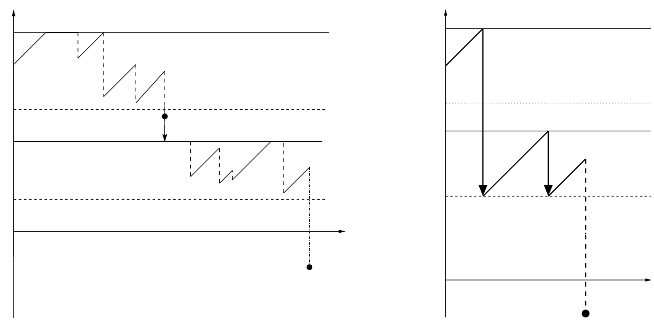

FIG. 2. *Illustrated in the figure on the left is a path of the risk process U^π in the absence of transaction cost ($K = 0$) for a three-band strategy with the lowest level b_1^+ equal to zero. The figure on the right pictures a path of the risk process U^π in the case $K > 0$, and π is a two-band strategy with $b_2^- = b_1^-$. The vertical dashed stretches represent the claims, while lump-sum dividend payments are indicated by arrows. At the moment τ of ruin a penalty payment $w(U_\tau)$ is required that is a function of the shortfall U_τ.*

A multi dividend-band strategy $\pi_{\underline{a},\underline{b}}$ consists of paying out "the minimal amount to keep $U_t^{\underline{a},\underline{b}}$ below the boundary $b(t)$," where

$$b(t) := b^+_{\rho(t)} \qquad \text{with } \rho(t) = \min\{i \in \mathbb{N} : U_t^{\underline{a},\underline{b}} < a_i\}.$$

In this case, while the boundary $b(t)$ is constant, $U_t^{\underline{a},\underline{b}}$ is equal to the process X reflected at the level $b(t)$ and the corresponding cumulative dividend payments $D_t^{\underline{a},\underline{b}}$ are equal to a local time of $U_t^{\underline{a},\underline{b}}$ at $b(t)$. In the case of a positive fixed transaction cost K the "reflection boundaries" b_n^+ widen to strips $[b_n^-, b_n^+]$, and the "local time" type payments are replaced by lump-sum payments $b_n^+ - b_n^-$ where b_n^- may lie below a_{n-1}; see Figure 2.

11.1. *Construction of the candidate solution of the stochastic control problem.* The dynamic programming equation satisfied by the optimal value function is recursive in nature, due to the presence of only negative jumps in both the uncontrolled reserves process X and the controlled reserves process U^π for any admissible policy π. In conjunction with the form of the optimal strategy of the mixed optimal stopping/stochastic control problem (7.1), this suggests that the candidate optimal policy for the stochastic control problem takes in general the form of a multi-dividend-band strategy $\pi_{\underline{a}^*,\underline{b}^*}$ at certain levels $\underline{a}^*$, $\underline{b}^*$. By repeatedly solving mixed-optimal stopping/stochastic control problems of the form (7.2) with suitably updated reward functions f, these levels $\underline{a}^*$, $\underline{b}^*$ can be identified, as summarized in the following recursive procedure:

	Recursion to construct the candidate optimal band levels
[0.]	**Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \{0\}$, $\underline{b}^* \leftarrow \{b^*\}$, $f \leftarrow_{b_+^*} v_b^*$ and $\Xi \leftarrow \Xi^*(f)$, where $\Xi^*(f)$ is given by (9.3).
[1.]	**If** Ξ is completely monotone, **set** $\underline{a}^* \leftarrow \underline{a}^* \cup \{\infty\}$. **Return** $\{\underline{a}, \underline{b}\}$.
[2.]	**Else if** $K = 0$ or **if** $\{K > 0$ and $\alpha^*_{f,\varnothing} \geq \alpha^*_f\}$ **define** $(a^*_{i+1}, b^*_{i+1}) \leftarrow (b^*_{i,+} + \alpha^*_f, b^*_{i,+} + \beta^*_f)$, where the levels α^*_f and β^*_f are defined in (7.9) and (7.11). **Else if** $\{K > 0$ and $\alpha^*_{f,\varnothing} < \alpha^*_f\}$ **define** $(a^*_{i+1}, b^*_{i+1}) \leftarrow (b^*_{i,+} + \alpha^*_{f,\varnothing}, \{b^{**}_{i,-}, b^*_{i,+} + \beta^*_{f,\varnothing}\})$ with $b^{**}_{i,-} = \inf\{b^*_{i,-} : V_{\underline{a}^*,\underline{b}^*}(b^*_{i,+} + \beta^*_{f,\varnothing}) - V_{\underline{a}^*,\underline{b}^*}(b^*_{i,-}) = \beta^*_{f,\varnothing} + b^*_{i,+} - b^*_{i,-} - K\}$, where the levels $\alpha^*_{f,\varnothing}$ and $\beta^*_{f,\varnothing}$ are defined in (7.12).
[3.]	**Set** $\underline{a}^* \leftarrow \underline{a} \cup \{a^*_{i+1}\}$, $\underline{b}^* \leftarrow \underline{b} \cup \{b^*_{i+1}\}$, $f \leftarrow_{b^*_{i+1,+}} V_{\underline{a}^*,\underline{b}^*}$, $\Xi \leftarrow \Xi_{\underline{a}^*,\underline{b}^*}(f)$, $i \leftarrow i+1$.
[4.]	**Go to** step 1.

REMARK 11.2. There may exist a limit point $\gamma_* = \lim_{i\to\infty} b^*_{i,+} = \lim_{i\to\infty} a^*_i$ of the band levels. In this case the procedure will converge to the value-function $V_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}$ corresponding to the levels $\underline{\tilde{a}}^* = (a^*_i)$, $\underline{\tilde{b}}^* = (b^*_i)$, and needs to be re-started as follows:

[0.′] **Set** $i \leftarrow 1$, $\underline{a}^* \leftarrow \underline{\tilde{a}}^*$, $\underline{b}^* \leftarrow \underline{\tilde{b}}^*$, $f \leftarrow_{\gamma^*} V_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}$, $\Xi \leftarrow \Xi_{\underline{\tilde{a}}^*,\underline{\tilde{b}}^*}(f)$.

In the following result (proved at the end of the section) it is confirmed that the constructed candidate policy $\pi_{\underline{a}^*,\underline{b}^*}$ is indeed optimal:

THEOREM 11.3. *The multi-dividend-band strategy $\pi_{\underline{a}^*,\underline{b}^*}$ is an optimal strategy for the control problem in* (2.2) *and the optimal value function is given by $v^* = v_{\pi_{\underline{a}^*,\underline{b}^*}} = V_{\underline{a}^*,\underline{b}^*}$, with*

$$V_{\underline{a}^*,\underline{b}^*}(x) := \begin{cases} W^{(q)}(x)C^*_i + F_w(x), & x \in [a^*_i, b^*_{i,+}], i \geq 1, \\ x - b^*_{i,+} + V_{\underline{a}_*,\underline{b}_*}(b^*_{i,+}), & x \in (b^*_{i,+}, a^*_{i+1}), i \geq 1, \end{cases} \tag{11.1}$$

for some constants C^*_i, where the functions $f_i : \mathbb{R}_- \to \mathbb{R}$ are given by $f_i(x) = V_{\underline{a}^*,\underline{b}^*}(a^*_{i-1} + x)$, $i > 1$, with $f_1 = w$.

REMARK 11.4. In Shreve et al. ([38], page 74), an explicit example is given of an optimal control problem in a diffusion setting in which a multi-dividend-band strategy is optimal with countably many bands. Azcue and Muler [8] provide an example of an optimal strategy with infinitely many bands below a finite level, for the classical De Finetti dividend problem with bounded dividend rates in the setting of a compound Poisson process. It is an open problem to construct an explicit example in which a multi-dividend-band strategy with countably many bands is optimal in the dividend-penalty problem.

11.2. *Proof of Theorem* 11.3. Denote by $\underline{v}_* = (v_{i,j})_{(i,j)}$, $\underline{a}^* = (a^*_{i,j})_{(i,j)}$ and $\underline{b}^* = (b^*_{i,j})_{(i,j)}$ the sequence of value-functions and band levels generated by the algorithm in Section 11.1, where the index (i, j) refers to the ith iteration of the algorithm in the jth run of the algorithm (i.e., it has been restarted $j-1$ times; cf. Remark 11.2). In particular, it follows that $v_{i,j}$ is given by

$$v_{i,j}(x) = \begin{cases} V_{\underline{a}^*,\underline{b}^*}(x), & x \in [0, b^*_{i,j,+}], \\ x - b^*_{i,j,+} + v_{i,j}(b^*_{i,j,+}), & x > b^*_{i,j,+}. \end{cases} \tag{11.2}$$

In the following result (which implies Theorem 11.3) it is established that $\pi_{\underline{a}^*,\underline{b}^*}$ is an optimal strategy for (2.2):

PROPOSITION 11.5. (i) *For a given pair (i, j) of iteration and run, $v_{i,j}$ is equal to the value-function $v_{\underline{a}^*_{i,j},\underline{b}^*_{i,j}}$ of the multi-dividend-band strategy $\pi_{\underline{a}^*_{i,j},\underline{b}^*_{i,j}}$ at levels $\underline{a}^*_{i,j} = (0, a^*_{1,1}, \dots, a^*_{i-1,j}, \infty)$ and $\underline{b}^*_{i,j} = (b^*_{1,1}, \dots, b^*_{i,j})$.*

(ii) *For each pair (ℓ, k) that is smaller than (j, i) in the lexico-graphical order, $v_{(k,\ell)}(x) = v_*(x)$ for all $x \le b^*_{k,\ell,+}$.*

(iii) *The optimal value function v_* is equal to the value function $V_{\underline{a}^*,\underline{b}^*}$ of the strategy $\pi_{\underline{a}^*,\underline{b}^*}$.*

PROOF. (i) The strong Markov property of the process $U = U^{\pi_{\underline{a}^*_{i,j},\underline{b}^*_{i,j}}}$ applied at the stopping time $\tau = \tau^{\pi}_{a^*_{i-1,j}}$ implies the relation

$$v_{k,\ell}(x) = \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-qt} \mu^{\pi}_K(\mathrm{d}t) + v_{k-1,\ell}(U_\tau)\right], \tag{11.3}$$

for $k \le j$, $\ell \le i$, with $\pi = \pi_{\underline{a}^*_{i,j},\underline{b}^*_{i,j}}$. As $v_{k,\ell}(x)$ is increasing in k, it follows that $v_{\infty,\ell}(x) := \lim_{k\to\infty} v_{k,\ell}(x)$ exists, for any $\ell \le j-1$. By applying again the strong Markov property it follows that $v_{1,\ell+1}$ satisfies, for any $l \le j-1$, $\pi = \pi_{\underline{a}^*_{i,j},\underline{b}^*_{i,j}}$,

$$v_{1,\ell+1}(x) = \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-qt} \mu^{\pi}_K(\mathrm{d}t) + v_{\infty,\ell}(U_\tau)\right]. \tag{11.4}$$

The form of $v_{i,j}$ then follows by induction, starting from the expression for a single dividend band strategy and using the form of the value-function of the auxiliary stochastic control problem in (7.2) [subsequently applied with pay-off functions $f(x) = v_{\pi_{\underline{a}^*_{k,\ell},\underline{b}^*_{k,\ell}}}(b^*_{k,\ell,+} + x)$, and performing induction in k for fixed ℓ and using the relation (11.4)].

(ii) By induction it follows that, for any k, $v_*(x) = v_{(k,1)}(x)$ for all $x \le b^*_{k,1,+}$. Indeed, note that Corollary 10.5 implies $v_{(2,1)}(x) = v_*(x)$ for all $x \le b^*_{2,1,+}$. Furthermore, that the induction step holds is verified as follows: Assuming that $v_{(k-1,1)}(x) = v_*(x)$ for all $x \le b^*_{k-1,1,+}$ for some pair k, Theorem 7.6 with $f =$

$b^*_{k-1,1,+}$ v_* in conjunction with the relation in (11.3) implies that $v_{(k,1)}(x) = v_*(x)$ for $x \le b^*_{k,1,+}$.

The assertion in (ii) thus follows by induction in $\ell > 1$, following a line of reasoning that is analogous to the one applied in the previous paragraph but with the function w replaced by $v_{\infty,\ell-1}$.

(iii) Since $v_{i,j}(x) = V_{\underline{a}^*,\underline{b}^*}(x)$ for all $x \le a^*_{i-1,j}$ [from (11.2)], it follows by virtue of part (ii) that $v_*(x) = V_{\underline{a}^*,\underline{b}^*}(x)$ for all $x \le a^*_{i-1,j}$. Since the sequence $(a_{i,j})_{i,j}$ is strictly increasing and ultimately tends to infinity (cf. step 2 of the algorithm and Lemma 7.3), it follows that $v_*(x)$ is equal to $V_{\underline{a}^*,\underline{b}^*}(x)$, for any fixed $x \in \mathbb{R}_+$. $\square$

12. Existence and uniqueness of stochastic solutions. In this section the optimal value function v_*, which was identified in the previous section, is shown to be a stochastic solution of the HJB equation (3.6). From the form (11.1) and properties of $W^{(q)}$ and of Gerber–Shiu functions, it follows that $v_*(x)$ is left- and right-differentiable at any $x > 0$. Furthermore, it was shown in Lemma 3.3 that $v_*(x)$ is continuous at any $x \in \mathbb{R}_+$. In particular, the function $g = v_*$ is continuous and left-differentiable at the "right-boundary" $\partial^+\mathcal{C}_g := \{b_1, b_2, \ldots\}$ of the set $\mathcal{C}_g$ (which was defined in (4.2) and where the interior $\mathcal{C}^o_g$ of $\mathcal{C}_g$ is denoted by $\mathcal{C}^o_g = \bigcup_n (a_n, b_n)$ for some $a_n, b_n \in [0,\infty]$ with $a_n < b_n$) and thus satisfies the following property:

(12.1) If $K = 0$, $g(x)$ is continuous and left-differentiable at any $x \in \partial^+\mathcal{C}_g$.

The HJB equation (3.6) admits a unique stochastic solution satisfying the regularity condition (12.1):

THEOREM 12.1. *The value function v_* is the unique stochastic solution of the HJB equation* (3.6) *satisfying* (12.1).

PROOF (EXISTENCE). As v_* is a stochastic supersolution [by Remark 4.2(i)] and v_* satisfies (12.1) (as discussed in above paragraph), it suffices to show that v_* is also a stochastic subsolution.

Note that, in view of the form (11.1), the interior $\mathcal{C}^o_{v_*}$ of the set $\mathcal{C}_{v_*}$ is identified as $\mathcal{C}^o_{v_*} = \bigcup_n (a^*_n, b^*_{n,+})$. Therefore, in view of (11.1) and the martingale properties of $W^{(q)}$ and of the Gerber–Shiu functions (Proposition 3.1), Doob's optional stopping theorem implies that v_* is a local stochastic subsolution of the HJB equation (3.6) on any closed interval $I \subset \mathcal{C}_{v_*}$, which shows that v_* is a stochastic subsolution. $\square$

12.1. *Proof of uniqueness.* Given a stochastic supersolution g of the HJB equation, an admissible candidate optimal strategy $\pi(g)$ can be described as follows:

DEFINITION 12.2. To a stochastic solution g of HJB equation (3.6) are associated:

(i) the policy $\pi(g) = \{D_t^{\pi(g)}, t \in \mathbb{R}_+\} \in \Pi$, given in terms of the sets $\mathcal{C}_g$ and $\mathcal{D}_g := \mathbb{R}_+ \setminus \mathcal{C}_g$,
(ii) the controlled process $U = U^{\pi(g)}$ and
(iii) the level $y^*(v) := \sup\{u \in [0, v] : g(v) - g(v-u) + K = u\}$ (with $\sup \varnothing = 0$), that are specified as follows:
  (a) In the case $K = 0$, let $D = D^{\pi(g)}$ be the increasing right-continuous $\mathbf{F}$-adapted process that satisfies

$$\begin{cases} U_t = X_t - D_t \in \overline{\mathcal{C}}_g, & \text{for any } t \in [0, \tau^{\pi(g)}), \\ \displaystyle\int_{[0,\tau^{\pi(g)})} \mathbf{1}_{\{s : X_s - D_{s-} \notin \overline{\mathcal{D}}_g\}}(t)\, \mathrm{d}D_t = 0, & \end{cases}$$

  where $\mathbf{1}_A$ denotes the indicator function of the set A and $\overline{\mathcal{C}}_g$ and $\overline{\mathcal{D}}_g$ denote the closures of $\mathcal{C}_g$ and $\mathcal{D}_g$;
  (b) in the case $K > 0$, pay out $\Delta D_t = y^*(X_t - D_{t-})$ at time t if $X_t - D_{t-} \in \mathcal{D}_g$ and $y^*(X_t - D_{t-}) > 0$;
  (c) otherwise, pay no dividends.

REMARK 12.3. The Skorokhod embedding lemma implies that the strategy $\pi(g) = \{D_t^{\pi(g)}, t \in \mathbb{R}_+\}$ described in Definition 12.2(iii)(a) is equal to

$$D_t^{\pi(g)} = \sup_{s \in [0, t\wedge\tau^{\pi(g)}]} \big(X_s - b(s)\big) \vee 0, \qquad b(s) = b_{\iota(s)}$$

with $\iota(s) = \inf\{n \in \mathbb{N} : X_s - D_{s-}^{\pi(g)} \le a_n\}$, given the representation $\overline{\mathcal{D}}_g = \bigcup_{n\ge1}[b_n, a_n]$. In particular, it follows that the policy defined in Definition 12.2 is a multi-dividend band strategy.

LEMMA 12.4. *Let g be a stochastic solution of the HJB in (3.6) satisfying (12.1). Then the process $\widetilde{M}^{g,\pi_*,\tau_{\mathbb{R}_+}^{\pi_*}}$ with $\pi_* = \pi(g)$, defined in Lemma 4.8 and Definition 12.2, is a UI $\mathbf{F}$-submartingale.*

The proof of Lemma 12.4 is based on the following auxiliary result:

LEMMA 12.5. *Let $a > 0$ be given and suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is such that $g|_{\mathbb{R}_-} \in \mathcal{P}$, $g|_{\mathbb{R}_+}$ is càdlàg, and g is continuous and left-differentiable at $a > 0$. If $M = \{M_t, t \in \mathbb{R}_+\}$ with $M_t = \mathrm{e}^{-q(t\wedge T_{0,a})} g(X_{t\wedge T_{0,a}})$ is an $\mathbf{F}$-martingale, then $Z = \{Z_t, t \in \mathbb{R}_+\}$ with*

$$Z_t = \mathrm{e}^{-q(t\wedge\tau_0)} g\big(Y^a_{t\wedge\tau_0}\big) - g\big(Y^a_0\big) - g'_-(a) \int_{[0,t\wedge\tau_0]} \mathrm{e}^{-qs}\, \mathrm{d}\overline{X}^a_s$$

is an $\mathbf{F}$-martingale, where $g'_-(a)$ denotes the left-derivative of g at a.

The proof of this result rests on an application of Itô's lemma and a density argument. Details are omitted since these follow straightforwardly from [31], Proposition 1.

PROOF OF LEMMA 12.4. The proof is a modification of the proof of Lemma 4.8. As, by Lemma 4.8, $\widetilde{M}^{g,\pi(g)}$ is a UI supermartingale, it suffices to verify that $\widetilde{M}^{g,\pi(g)}$ is in fact a martingale. Note that the set of distinct epochs $\tilde{\mathbb{T}}$ at which lump-sum dividend payments occur is countable,

$$\tilde{\mathbb{T}} = \{\tilde{T}_i : \Delta D_{\tilde{T}_i} > 0\} \qquad \text{with } \tilde{T}_i = \inf\{t > \tilde{T}_{i-1} : X_t - D_{t-}^{\pi(g)} \in \mathcal{D}_g\},$$

for $i \in \mathbb{N}$ with $\tilde{T}_0 = 0$ and $\inf \varnothing = \infty$. The form of the strategy $\pi(g)$ implies that the sequence $(U_{\tilde{T}_i})_i$ is decreasing with $U_{\tilde{T}_i} - U_{\tilde{T}_{i-1}} > 0$ on the set $\{\tilde{T}_i < \infty\}$. In particular, it follows that, also in this case, $\tilde{\mathbb{T}}$ is countable.

Writing $D = D^{\pi(g)}$ and $M = \widetilde{M}^{g,\pi(g)}$, fixing arbitrary $t, s \in \mathbb{R}_+$ with $s < t$ and denoting $T_i = \tilde{T}_i \wedge t$, we have $M_t = \sum_{i\ge 1} Y_i + \sum_{i \ge 0} Z_i$ with Y_i given by

$$\mathrm{e}^{-qT_i} g(X_{T_i} - D_{T_i-}) - \mathrm{e}^{-qT_{i-1}} g(X_{T_{i-1}} - D_{T_{i-1}}) - \int_{(T_{i-1},T_i)} \mathrm{e}^{-qs}\,\mathrm{d}D_s, \tag{12.2}$$

and $Z_i = \mathrm{e}^{-qT_i}(g(X_{T_i} - D_{T_i}) - g(X_{T_i} - D_{T_i-}) + \Delta D_i - K)\mathbf{1}_{\{\Delta D_i > 0\}}$ with $\Delta D_i = D_{T_i} - D_{T_{i-1}}$. By definition of the strategy $\pi(g)$ it is straightforward to verify that $Z_i = 0$ for all i.

In the case $K > 0$ the integral term in (12.2) vanishes, and we have $D_{T_{i-1}} = D_{T_i-}$ for $i \ge 0$. By reasoning as in Lemma 4.8 it follows that the equality in (4.10) holds. By combining (4.10) with the fact that g is a stochastic solution, Doob's optional stopping theorem and the definition of T_i, we have

$$\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}] = \mathrm{e}^{-qT_{i-1}} \mathbb{E}_{U_{T_{i-1}}}[\mathrm{e}^{-q\tau_i} g(X_{\tau_i}) - g(X_0)] = 0,$$

with $\tau_i = T_i \circ \theta_{T_{i-1}}$. The tower property hence yields $\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0$. Since s, t were arbitrary, it thus follows that M is a martingale.

If $K = 0$, the definition of $\pi(g)$ implies that the process $\{U_{T_{i-1}+t}, t < T_i - T_{i-1}\}$ conditional on $\mathcal{F}_{T_{i-1}}$ has the same law as the process $\{Y_t^b, t < \tau_b(a)\}$ with $X_0 = b = U_{T_{i-1}}$ and $\tau_b(a) = \inf\{t \ge 0 : Y_t^b < a\}$, conditional on $U_{T_{i-1}}$, where Y^b is independent of $U_{T_{i-1}}$. The strong Markov property of Y^a implies that $\mathbb{E}[Y_i|\mathcal{F}_{T_{i-1}}]$ is equal to

$$\mathrm{e}^{-qT_{i-1}} \mathbb{E}_{U_{T_{i-1}}}\left[\mathrm{e}^{-q\tau_b(a)} g\big(Y^b_{\tau_b(a)}\big) - g(Y_0) - \int_{(0,\tau_b(a))} \mathrm{e}^{-qs}\,\mathrm{d}\overline{X}_s^b\right].$$

This expectation is positive in view of Lemma 12.5 and the fact that $g'_-(a) \ge 1$ [as $\mathrm{d}_g(a) \ge 1$ and g is left-differentiable at a]. Again, an application of the tower property yields $\mathbb{E}[M_t - M_s|\mathcal{F}_s] \ge 0$, and it follows that, in this case, M is a submartingale. $\square$

The stated uniqueness follows as a consequence of the following comparison principle:

PROPOSITION 12.6. *Let h be any stochastic subsolution satisfying* (12.1), *and let g be any stochastic supersolution of the HJB equation* (3.6). *Then $g \geq h$.*

PROOF OF THEOREM 12.1 (UNIQUENESS). Let h be any stochastic solution of the HJB equation. Since, by the dual representation in Proposition 4.3, v_* is the minimal stochastic supersolution of the HJB and h is a stochastic supersolution, it follows $v_* \leq h$. Furthermore, the stochastic comparison principle in Proposition 12.6 implies $v_* \geq h$ (as h and v_* are stochastic sub- and supersolutions of the HJB). Thus it holds $v_* = h$, and uniqueness is established. □

PROOF OF PROPOSITION 12.6. Let g and h be a stochastic supersolution and stochastic subsolution, and denote by $\pi(h)$ the policy corresponding to h given in Definition 12.2. Since the processes $\widetilde{M}^{v_*,\pi(h)}$ and $\widetilde{M}^{h,\pi(h)}$ [defined in (4.8)], are a supermartingale and a submartingale (by Lemmas 4.8 and 12.4), Doob's optional stopping theorem implies for $x \in \mathbb{R}_+$

$$v_*(x) - h(x) \geq \lim_{t\to\infty} \mathbb{E}_x\bigl[\widetilde{M}^{v_*,\pi(h)}_{t\wedge\tau^{\pi(h)}} - \widetilde{M}^{h,\pi(h)}_{t\wedge\tau^{\pi(h)}}\bigr]. \tag{12.3}$$

The right-hand side of (12.3) is equal to 0, since $\widetilde{M}^{v_*,\pi(h)}$ and $\widetilde{M}^{h,\pi(h)}$ are UI, and satisfy the boundary condition

$$\widetilde{M}^{v_*,\pi(h)}_{\tau^{\pi(h)}} = \widetilde{M}^{h,\pi(h)}_{\tau^{\pi(h)}} = \mathrm{e}^{-q\tau^{\pi(h)}} w\bigl(U^{\pi(h)}_{\tau^{\pi(h)}}\bigr),$$

and $\mathbb{P}_x(\tau^{\pi(h)} < \infty) = 1$ for all $x \in \mathbb{R}_+$. This completes the proof. □

13. Examples.

13.1. *General computations for processes with rational Laplace exponent.* The determination of the optimal policy starts with the identification of the last global maximum of the barrier influence function G. For example, in the presence of an exponential penalty $w(x) = c\mathrm{e}^{vx}$ or a linear penalty $w(x) = cx + c_0$, we must compute the extrema of the functions

$$G^{(v)}(x) := \frac{1 - cZ^{(q,v)\prime}(x)}{W^{(q)\prime}(x)}, \qquad G_1(x) := \frac{1 - cZ_1'(x) - c_0 q W^{(q)}(x)}{W^{(q)\prime}(x)}, \tag{13.1}$$

respectively.

Therefore, the first step will be computing the homogeneous and generating scale functions $W^{(q)}(x)$, $Z^{(q,v)}(x)$, for processes with rational Laplace exponent. Assume the typical case

$$W^{(q)}(x) = \sum A_i \mathrm{e}^{\zeta_i(q)x},$$

with $A_i \in \mathbb{R}$ and the roots $\zeta_i(q)$ of the Cramér–Lundberg equation $\psi(\zeta) = q$ being distinct.

This implies $Z^{(q,v)}(x) = \mathrm{e}^{vx}(1 + (q - \psi(v)) \int_0^x \mathrm{e}^{-vy} W^{(q)}(y)\,\mathrm{d}y)$ is equal to

$$\mathrm{e}^{vx} + \big(q - \psi(v)\big) \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x} - \mathrm{e}^{vx}}{\zeta_i(q) - v} = \big(\psi(v) - q\big) \sum_i \frac{A_i}{v - \zeta_i(q)} \mathrm{e}^{\zeta_i(q)x},$$

using that $\sum \frac{A_i}{v - \zeta_i(q)} = \frac{1}{\psi(v) - q}$. In particular, $Z^{(q)}(x) = q \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)}$ and

$$Z_1(x) = \overline{Z}^{(q)}(x) - \psi'(0)\overline{W}^{(q)}(x) = q \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i^2(q)} - \psi'(0) \sum_i A_i \frac{\mathrm{e}^{\zeta_i(q)x}}{\zeta_i(q)},$$

$$Z^{(q,v)}(x) = Z^{(q)}(x) + \sum_i A_i \mathrm{e}^{\zeta_i(q)x} \frac{v}{v - \zeta_i(q)} \left(\frac{\psi(v)}{v} - \frac{q}{\zeta_i(q)} \right).$$

The simplest examples may be completely analyzed by studying the sign of the functions that are given by $D^{\#}(x) = -G^{\#\prime}(x) W^{(q)\prime}(x)^2$, and $D^*(x) = -G^{*\prime}(x) W^{(q)\prime}(x)^2$, which determine the critical point b^* (in particular whether it is 0), and the eventual unimodality after b^*, which implies optimality of the single barrier policy. To alleviate notation, the #, $*$ will be omitted in this section, since the function considered can always be inferred from the absence/presence of transaction costs.

For exponential and affine penalties, the corresponding functions are given by $D^{(v)}(x) = -G^{(v)\prime}(x) W^{(q)\prime}(x)^2$ and $D_1(x) = -G_1'(x) W^{(q)\prime}(x)^2$. By straightforward calculations we find

$$\begin{aligned} D^{(v)}(x) &= W^{(q)\prime\prime}(x)\big(1 - cZ^{(q,v)\prime}(x)\big) + cZ^{(q,v)\prime\prime}(x) W^{(q)\prime}(x) \\ &= \sum_j A_j \zeta_j(q)^2 \mathrm{e}^{\zeta_j(q)x} + c\big(\psi(v) - q\big) \sum_j \sum_{k>j} d_{j,k}^{(v)} A_j A_k \mathrm{e}^{(\zeta_j(q) + \zeta_k(q))x}, \\ D_1(x) &= \sum_j A_j \zeta_j(q)^2 \mathrm{e}^{\zeta_j(q)x} - cq \sum_j \sum_{k>j} d_{1;j,k} A_j A_k \mathrm{e}^{(\zeta_j(q) + \zeta_k(q))x} \\ &\quad + \big(c\psi'(0) - c_0 q\big) \sum_j \sum_{k>j} \big(\zeta_j(q) - \zeta_k(q)\big)^2 A_j A_k \mathrm{e}^{(\zeta_j(q) + \zeta_k(q))x}, \end{aligned}$$

with $d_{j,k}^{(v)} \frac{\zeta_j(q)\zeta_k(q)(\zeta_j(q) - \zeta_k(q))^2}{(v - \zeta_j(q))(v - \zeta_k(q))}$ and $d_{1;j,k} = \frac{(\zeta_j(q) + \zeta_k(q))}{\zeta_j(q)\zeta_k(q)} (\zeta_j(q) - \zeta_k(q))^2$.

[Note that the coefficients of c and $c\psi'(0) - c_0 q$ are the intervening Wronskians, and that the function $D^{(v)}(x) - W^{(q)\prime\prime}(x)$ is a generating function for the corresponding functions obtained with polynomial penalties.]

13.2. *Cramér–Lundberg model with exponential jumps.* Consider next the Cramér–Lundberg model (1.1) with exponential jump sizes with mean $1/\mu$, jump rate λ, and Laplace exponent $\psi(s) = ps - \lambda s/(\mu + s)$. The homogeneous scale function is

$$W^{(q)}(x) = A_+ e^{\zeta^+(q)x} - A_- e^{\zeta^-(q)x},$$

where $A_\pm = p^{-1}\frac{\mu+\zeta^\pm(q)}{\zeta^+(q)-\zeta^-(q)}$, and $\zeta^+(q) = \Phi(q)$, $\zeta^-(q)$ are the largest and smallest roots of the polynomial $(\psi(s) - q)(s + \mu) = ps^2 + s(p\mu - \lambda - q) - q\mu$:

$$\zeta^\pm(q) = \frac{q + \lambda - \mu p \pm \sqrt{(q + \lambda - \mu p)^2 + 4pq\mu}}{2p}.$$

Hence, it follows

$$\begin{aligned}
Z^{(q)}(x) &= q\left(\frac{A_+}{\zeta^+(q)} e^{\zeta^+(q)x} - \frac{A_-}{\zeta^-(q)} e^{\zeta^-(q)x}\right) \\
&= \frac{(q - \zeta^-(q))e^{\zeta^+(q)x} + (\zeta^+(q) - q)e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)}, \\
Z^{(q,v)}(x) &= Z^{(q)}(x) + \lambda \frac{v}{v+\mu} \frac{e^{\zeta^+(q)x} - e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)}, \\
D^{(v)}(x) &= \alpha_+ e^{\zeta^+(q)x} - \alpha_- e^{\zeta^-(q)x} + c\alpha_v e^{(\zeta^+(q)+\zeta^-(q))x},
\end{aligned}$$

with $\alpha_+ = A_+(\zeta_+(q))^2 > 0$, $\alpha_- = A_-(\zeta_-(q))^2 > 0$, $C = (\mu + \zeta_+(q))(\mu + \zeta_-(q)) = \frac{\lambda\mu}{p} > 0$, and

$$\alpha_v = \frac{p}{v+\mu} \frac{C}{p^2} \frac{q\mu}{p} = \frac{\lambda q \mu^2}{p^3} \frac{1}{v+\mu} > 0.$$

Then, differentiating $v \mapsto Z^{(q,v)}(x)$, $v \mapsto \alpha_v$ or by (13.2) and using that $(\zeta^+(q) + \zeta^-(q))/(\zeta^+(q)\zeta^-(q)) = \psi'(0)/q - 1/\mu$ yields

$$\begin{aligned}
Z_1(x) &= \lambda\mu^{-1} \frac{e^{\zeta^+(q)x} - e^{\zeta^-(q)x}}{\zeta^+(q) - \zeta^-(q)} = C_+ e^{\zeta^+(q)x} + C_- e^{\zeta^-(q)x}, \\
D_1(x) &= \alpha_+ e^{\zeta^+(q)x} - \alpha_- e^{\zeta^-(q)x} + \alpha_1 e^{(\zeta^+(q)+\zeta^-(q))x},
\end{aligned}$$

where $C_\pm = \pm\lambda\mu^{-1}(\zeta^+(q) - \zeta^-(q))^{-1}$ and

$$\begin{aligned}
\alpha_1 &= A_+ A_- (\zeta^+ - \zeta^-)^2 \left(cq\frac{\zeta^+ + \zeta^-}{\zeta^+\zeta^-} - c\psi'(0) + c_0 q\right) \\
&= \frac{C}{p^2}\left(c_0 q - c\frac{q}{\mu}\right) = \frac{\lambda q}{p^3}(c_0\mu - c).
\end{aligned}$$

Recall next that in the absence of penalty and costs [$w(x) = K = 0$], the function $W^{(q)\prime}(x) = G(x)^{-1}$ is unimodal (see Avram et al. [6]) with global minimum at b^* given by

$$\frac{1}{\zeta^+(q)-\zeta^-(q)}\begin{cases}\log\dfrac{\zeta^-(q)^2(\mu+\zeta^-(q))}{\zeta^+(q)^2(\mu+\zeta^+(q))}, \\ \qquad \text{if } W^{(q)\prime\prime}(0)<0 \Leftrightarrow (q+\lambda)^2<p\lambda\mu, \\ 0, \qquad \text{if } W^{(q)\prime\prime}(0)\geq 0 \Leftrightarrow (q+\lambda)^2\leq p\lambda\mu.\end{cases}$$

[Since $W^{(q)\prime\prime}(0) \sim \zeta^+(q)^2(\mu+\zeta^+(q)) - \zeta^-(q)^2(\mu+\zeta^-(q))/(\zeta^+(q)-\zeta^-(q)) = (q+\lambda)^2 - p\lambda\mu$, the optimal strategy is always the barrier strategy at level b^*.]

It is verified next that the functions $G^{(v)}$ and G_1 continue to be unimodal when w is exponential or affine and $K = 0$, as a consequence of Lemma 13.1 below, and hence single barrier policies continue to be optimal, in view of Lemma 9.2 (in the case of affine penalties this has already been established in [5, 29]).

LEMMA 13.1. *Let $\alpha_i, \lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ satisfy $\alpha_1 > 0 > \alpha_3$, and $\lambda_1 > \lambda_2 > \lambda_3$. Then the function $f(x) := \sum_{i=1}^3 \alpha_i e^{\lambda_i x}$ has a unique root c^* of $f(c^*) = 0$, and it holds $f'(c^*) > 0$, and*

$$f(x) > 0 \qquad \textit{for all } x > c^*.$$

Furthermore, if $h : \mathbb{R}_+ \to \mathbb{R}$ is such that $h'(x) = k(x)f(x)$ for $x > 0$, where $k : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$, then h is unimodal.

PROOF. The function $g(x) := e^{-\lambda_3 x} f(x)$ tends to $+\infty$ and to $\alpha_3 < 0$ as $x \to \pm\infty$. If it holds $\alpha_2 \geq 0$, g is strictly convex and strictly increasing. In the case $\alpha_2 < 0$, g attains a minimum at the unique root of g'. In both cases the equation $g(c) = 0$ admits a unique root c, and it holds $g'(c) > 0$. Hence it holds that c is a unique root of $f(c) = 0$, with $f'(c) > 0$ and with $f(x) > 0$ for $x > c$. In particular, h has a unique stationary point where it attains a maximum, so that it is unimodal. $\square$

The optimal level b^* is characterized as follows:

(i) For $K = 0$ and in the case of an exponential penalty, $b^*_{v,+} = 0$ if and only if

$$G^{(v)\prime}(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq -c\lambda q \frac{\mu^2}{v+\mu},$$

as follows from the expression for $D^{(v)}(x)$. Similarly, in the case of linear penalty, it holds $b^*_{1,+} = 0$ if and only if

$$G_1'(0) \leq 0 \Leftrightarrow (q+\lambda)^2 - \lambda\mu p \geq \lambda q(c - c_0\mu),$$

in view of the expression for $D_1(x)$. If b^*_+ is positive, it is a stationary point, and hence solves the equation

$$G^{(v)\prime}(b) = 0 \Leftrightarrow 0 = D^{(v)}(b) = \alpha_+ e^{\zeta^+(q)b} - \alpha_- e^{\zeta^-(q)b} + c\alpha_v e^{(\zeta^+(q)+\zeta^-(q))b},$$

if the penalty w is exponential and

$$G_1'(b) = 0 \Leftrightarrow 0 = D_1(b) = \alpha_+ e^{\zeta^+(q)b} - \alpha_- e^{\zeta^-(q)b} + \alpha_1 e^{(\zeta^+(q)+\zeta^-(q))b},$$

if w is an affine penalty.

(ii) Suppose next $K > 0$. Then b_+^* is strictly positive as a consequence of the positive transaction cost K, and the optimal levels (b_-^*, b_+^*) are given by $(b_-^*, b_-^* + d^*)$ where (b, d) maximizes over $(b, d) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{0\}$ the function

$$\widetilde{G}^{(v)} : (b, d) \mapsto \frac{d - K - B_+ e^{\zeta^+(q)b}(e^{\zeta^+(q)d} - 1) + B_- e^{\zeta^-(q)b}(e^{\zeta^-(q)d} - 1)}{A_+ e^{\zeta^+(q)b}(e^{\zeta^+(q)d} - 1) - A_- e^{\zeta^-(q)b}(e^{\zeta^-(q)d} - 1)}$$

if w is an exponential penalty, and the function

$$\widetilde{G}_1 : (b, d) \mapsto \frac{d - K - C_+ e^{\zeta^+(q)b}(e^{\zeta^+(q)d} - 1) + C_- e^{\zeta^-(q)b}(e^{\zeta^-(q)d} - 1)}{A_+ e^{\zeta^+(q)b}(e^{\zeta^+(q)d} - 1) - A_- e^{\zeta^-(q)b}(e^{\zeta^-(q)d} - 1)}$$

if w is an affine penalty.

The following result sums up the form of the optimal dividend policy:

LEMMA 13.2. *Consider a Cramér–Lundberg process* (1.1) *with exponential jump sizes with mean* $1/\mu$, *and fixed cost* $K \geq 0$. *The optimal dividend policy is given by a* single dividend-band strategy π_{b^*} *for the following Gerber–Shiu penalties* w:

(a) *Exponential penalties*: $w(x) = ce^{xv}$, *with* $v, c < 0$ *such that the integrability condition* (2.1) *is satisfied*.
   (i) *In the case* $\{K = 0$ *and* $(q + \lambda)^2 - \lambda\mu p \geq -c\lambda q \frac{\mu^2}{v+\mu}\}$, *then* $b^* = 0$.
   (ii) *In the case* $\{K = 0$ *and* $(q + \lambda)^2 - \lambda\mu p < -c\lambda q \frac{\mu^2}{v+\mu}\}$, *then* b^* *is the unique solution* $b \in \mathbb{R}_+ \setminus \{0\}$ *of the equation* $D^{(v)}(b) = 0$.
   (iii) *In the case* $K > 0$, *we have* $b_+^* = b_-^* + d^*$ *where* b_-^* *and* d^* *maximize over* $b \geq 0$, $d > 0$, *the function* $\widetilde{G}^{(v)}$.

(b) *Affine penalties*: $w(x) = cx + c_0$, *with* $c \geq 0$ *and* $c_0 \leq 0$ *such that* (2.1) *is satisfied*.
   (i) *In the case* $\{K = 0$ *and* $(q + \lambda)^2 - \lambda\mu p \geq \lambda q(c - c_0\mu)\}$, *then we have* $b^* = 0$.
   (ii) *In the case* $\{K = 0$ *and* $(q + \lambda)^2 - \lambda\mu p < \lambda q(c - c_0\mu)\}$, *then* b^* *is the unique solution* $b \in \mathbb{R}_+ \setminus \{0\}$ *of the equation* $D_1(b) = 0$.
   (iii) *In the case* $K > 0$, *we have* $b_+^* = b_-^* + d^*$ *where* $b_{1,-}^* \geq 0$ *and* $d^* > 0$ *maximize over* (b, d), *the function* $\widetilde{G}_1$.

13.3. *Cramér–Lundberg model with Erlang jumps*. Suppose next that X is given by the Cramér–Lundberg model (1.1) with the Erlang (n, μ) jump sizes.

The corresponding Laplace exponent is $\psi(s) = ps + \frac{\lambda\mu^n}{(\mu+s)^n} - \lambda$, and by Laplace inversion it follows that its q-scale function is given by

$$W^{(q)}(x) = \sum_{j=0}^{n} A_j e^{\zeta_j(q)x}, \qquad A_j = \frac{(\zeta_j(q)+\mu)^n}{p\prod_{k\neq j}(\zeta_j(q)-\zeta_k(q))}, \qquad x \geq 0,$$

where $\zeta_0(q) > 0 > \zeta_1(q) > -\mu > \zeta_2(q) > \cdots$ are the $n+1$ roots of the Cramér–Lundberg equation $\psi(\zeta) = q$.

Let $K = 0$ and $w(x) = ce^{vx}$ an exponential penalty ($c < 0$), and denote by b the point where $G^{(v)}$ attains its maximum. In general a single dividend-band strategy may not be optimal. A necessary and sufficient criterion for optimality of π_b is the complete monotonicity of the function $\Xi_v : (\Phi(q), \infty) \to \mathbb{R}_+$ given by

$$\Xi_v(s) = \frac{\psi(s)-q}{s} \cdot e^{sb} \int_b^\infty e^{-sz}\big(W^{(q)\prime}(z)G^*(b) - [1-F'(z)]\big)\,dc,$$

$$I(s) = s^{-1}\left[ps + \frac{\lambda\mu^n}{(\mu+s)^n} - \lambda - q\right],$$

$$I_v(s) = I_0(s) - c\sum_{j>i} k_{i,j}^{(v,q)}(s) A_j A_i e^{(\zeta_i(q)+\zeta_j(q))b},$$

$$I_0(s) = \int_0^\infty e^{-sx}\big[W^{(q)\prime}(b+x) - W^{(q)\prime}(b)\big]\,dx = \sum_{j=0}^{n} A_j k_{1,i,j}^{(q)}(s) e^{\zeta_j(q)b},$$

where $k_{i,j}^{(v,q)}(s) = \frac{(\zeta_j(q)-\zeta_i(q)^2(v-\zeta_i(q)-\zeta_j(q))}{(s-\zeta_j(q))(s-\zeta_i(q))(v-\zeta_j(q))(v-\zeta_i(q))}$ and $k_{1,i,j}^{(q)}(s) = \frac{\zeta_j(q)^2}{s(s-\zeta_j(q))}$. If in addition there is no penalty ($w = 0$), the expressions simplify. If b denotes the value where the minimum of $W^{(q)\prime}$ is attained, π_b is optimal precisely if $\Xi_0 : (\Phi(q), \infty) \to \mathbb{R}_+$ is completely monotone, where $\Xi_0(s) = I(s) \cdot I_0(s)$.

The Azcue–Muler example. Consider next the example in Azcue and Muller [7], with pure Erlang claims of order $n = 2$, with $\mu = 1$, $\lambda = 10$, $p = \frac{107}{5}$, $q = \frac{1}{10}$, $\theta = \frac{7}{100}$ and Laplace exponent $\psi(s) - q = ps + \lambda(\frac{\mu}{\mu+s})^2 - \lambda - q = \frac{p}{(\mu+s)^2}(s + \zeta_1)(s+\zeta_2)(s-\zeta_0)$, with $\zeta_0 \approx 0.0396$, $\zeta_1 \approx 0.0794$, $\zeta_2 \approx 1.4882$. In addition we consider a linear penalty $w(x) = cx$, $c \in \mathbb{R}_+$. We will analyze below four particular cases $c \in \{0, 0.2, 0.6, 1.0\}$. In cases $c \in \{0.6, 1.0\}$ the optimal strategy is a single dividend band strategy at level b_1, while in the cases $c \in \{0, 0.2\}$ it is optimal to adopt a two-band strategy with $b_1 = 0$ (in the case $c = 0$ we thus recover the form of the optimal strategy found in [7]). The parameters of the optimal strategies are summarized in Table 1 (with v_2 denoting the difference of the value function and the identity $x \mapsto x$ at the end of the nonempty continuation band).

In the cases $c \in \{0.6, 1\}$ a plot of the function G_1 defined in (13.1) reveals that G_1 is monotone decreasing on the right of the level at which attains its unique global maximum which implies the optimal strategy is a single-dividend band strategy at this level (Theorem 9.1). In the cases $c \in \{0, 0.2\}$ a plot of G_1 shows

TABLE 1

The values of the optimal band levels under a linear penalty $w(x) = cx$

	b_1	v_2	a_2	b_2
$c = 0$	0	2.44	1.83	10.45
$c = 0.2$	0	1.72	1.90	10.47
$c = 0.6$	10.96	1.71	∞	∞
$c = 1.0$	11.37	1.30	∞	∞

that this function attains its global maximum at 0 but also attains a second local maximum at some strictly positive level, so that the optimal value function is given by

$$v(x) = \begin{cases} x + v_1, & b_1 = 0 \le x < a_2, \\ F_1(x - a_1), & x \in [a_2, b_2], \\ x + v_2, & x > b_2. \end{cases}$$

Here $v_2 = -b_2 + F_1(b_2 - a_2)$ and $v_1 = \frac{p-20c}{q+\lambda} = \frac{214-200c}{101}$ is the value of the strategy (at zero) of paying all premiums as dividends until the moment the first claim arrives, which is also the moment of ruin, and $F_1(x)$ is given by

$$F_1(x) = p(a_2 + v_1)W^{(q)}(x) - \int_0^x W^{(q)}(x - y)\big[f_{\nu,a_2}(y)\big]\,\mathrm{d}y,$$

$$f_{\nu,a}(y) = \int_0^a (a - z + v_0)k(y + z)\,\mathrm{d}z + c\int_a^\infty (a - z)k(y + z)\,\mathrm{d}z,$$

where $k(y) = \lambda\mu^2 y\mathrm{e}^{-\mu y}$ denotes the Lévy density at y.

The function v is the value function of a two-band strategy at levels (b_0, a_1, b_1) with $b_0 = 0$. The unknowns a_1, b_1 are determined by the optimality equations $F_1'((b_1 - a_1)-) = 1$ and $F_1''((b_1 - a_1)-) = 0$ which yield the following system of two nonlinear equations for a_1 and b_1:

$$\begin{aligned} 1 &= p(a_1 + v_0)W^{(q)\prime}(b_1 - a_1) - p^{-1} f_{\nu,a_1}(b_1) \\ &\quad - \int_0^{b_1 - a_1} W^{(q)\prime}(b_1 - a_1 - y) f_{\nu,a_1}(y)\,\mathrm{d}y, \\ 0 &= p(a_1 + v_0)q W^{(q)\prime\prime}(b_1 - a_1) - p^{-1} f'_{\nu,a_1}(b_1) \\ &\quad - W^{(q)\prime}(0) f_{a_1,\nu}(b_1) - \int_0^{b_1 - a_1} W^{(q)\prime\prime}(b_1 - a_1 - y) f_{\nu,a_1}(y)\,\mathrm{d}y, \end{aligned}$$

with $W^{(q)\prime}(0) = \frac{101}{10} \cdot \frac{25}{107^2}$. The two-band strategies at the levels $(a_1, b_1) = (1.83, 10.45)$ $[c = 0]$ and $(a_1, b_1) = (1.90, 10.47)$ $[c = 0.2]$ are indeed optimal since it holds $({}_{b_1}\Gamma^w_\infty v - qv)(y) \le 0$ for all $y > b_1$ and $({}_0\Gamma^w_\infty v - qv)(y) \le 0$ for all $y \in (0, a_1)$.

APPENDIX A: PROOF OF DYNAMIC PROGRAMMING EQUATION

PROOF OF LEMMA 3.1(II). Fix arbitrary $\pi \in \Pi$, $x \in \mathbb{R}_+$ and $s, t \in \mathbb{R}_+$ with $s < t$. The process V_t^π is $\mathcal{F}_t$-measurable, and is UI on account of Lemma 3.3. Fix arbitrary $\pi \in \Pi$, $x \in \mathbb{R}_+$. Define by $W^\pi = \{W_s^\pi, s \in \mathbb{R}_+\}$ the value-process $W_s^\pi = \text{ess.sup}_{\tilde{\pi} \in \Pi_s} J_s^{\tilde{\pi}}$ with

$$J_s^{\tilde{\pi}} = \mathbb{E}\left[\int_{[0,\tau^{\tilde{\pi}})} \mathrm{e}^{-qu} \mu_K^{\tilde{\pi}}(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde{\pi}}} w\big(U_{\tau^{\tilde{\pi}}}^{\tilde{\pi}}\big) \Big| \mathcal{F}_s\right], \tag{A.1}$$

where $\Pi_s = \{\tilde{\pi} = (\pi, \overline{\pi}) = \{D_u^{\pi,\overline{\pi}}, u \in \mathbb{R}_+\} : \overline{\pi} \in \Pi\}$, and $D^{\pi,\overline{\pi}}$ is given in terms of the process $D^{\overline{\pi}}(x)$ of cumulative dividends of the strategy $\overline{\pi}$ corresponding to initial capital $X_0 = x$ by

$$D_u^{\pi,\overline{\pi}} = \begin{cases} D_u^\pi, & u \in [0, s); \\ D_s^\pi + D_{u-s}^{\overline{\pi}}(U_s^\pi), & u \geq s. \end{cases}$$

It follows that V^π is a supermartingale as direct consequence of the following $\mathbb{P}$-a.s. relations:

$$\text{(a)} \quad V_s^\pi = W_s^\pi, \qquad \text{(b)} \quad W_s^\pi \geq \mathbb{E}[W_t^\pi | \mathcal{F}_s],$$

where W^π is the process defined in (A.1).

Proof of (b): The identity follows by classical arguments. Since the family of random variables $\{J_t^{\tilde{\pi}}, \tilde{\pi} \in \Pi_t\}$ is directed upwards, it follows from Neveu [30] that there exists a sequence $\pi_n \in \Pi_t$ such that $J_t^{\tilde{\pi}_n} \uparrow W_t^\pi$. Since $\Pi_t \subset \Pi_s$ it follows that W_s^π dominates $J_s^{\pi_n} = \mathbb{E}[J_t^{\pi_n} | \mathcal{F}_s]$, so that monotone convergence implies that we have

$$W_s^\pi \geq \lim_n \mathbb{E}[J_t^{\pi_n} | \mathcal{F}_s] = \mathbb{E}[W_t^\pi | \mathcal{F}_s].$$

Proof of (a): The form of $D^{\tilde{\pi}}$ implies that, conditional on U_s^π, $\{D_u^{\tilde{\pi}} - D_s^{\tilde{\pi}}, u \geq s\}$ is independent of $\mathcal{F}_s$. On account of the Markov property of X it also follows that conditional on U_s^π, $\{U_u^{\tilde{\pi}} - U_s^{\tilde{\pi}}, u \geq s\}$ is independent of $\mathcal{F}_s$. As a consequence, we have the following identity on the set $\{s < \tau^\pi\}$:

$$\begin{aligned} &\mathbb{E}\left[\int_{[0,\tau^{\tilde{\pi}})} \mathrm{e}^{-qu} \mu_K^{\tilde{\pi}}(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\tilde{\pi}}} w\big(U_{\tau^{\tilde{\pi}}}^{\tilde{\pi}}\big) \Big| \mathcal{F}_s\right] \\ &\quad = \mathrm{e}^{-qs} \mathbb{E}_{U_s^\pi}\left[\int_{[0,\tau^{\overline{\pi}})} \mathrm{e}^{-qu} \mu_K^{\overline{\pi}}(\mathrm{d}u) + \mathrm{e}^{-q\tau^{\overline{\pi}}} w\big(U_{\tau^{\overline{\pi}}}^{\overline{\pi}}\big)\right] + \int_{[0,s]} \mathrm{e}^{-qu} \mu_K^\pi(\mathrm{d}u) \\ &\quad = \mathrm{e}^{-qs} v_{\overline{\pi}}(U_s^\pi) + \int_{[0,s]} \mathrm{e}^{-qu} \mu_K^\pi(\mathrm{d}u). \end{aligned}$$

In particular, $\mathbb{P}_x$-a.s. the following representation holds true:

$$J_s^{\tilde{\pi}} = \mathrm{e}^{-q(s \wedge \tau^\pi)} v_{\overline{\pi}}(U_{s \wedge \tau^\pi}^\pi) + \int_{[0, s \wedge \tau^\pi]} \mathrm{e}^{-qu} \mu_K^\pi(\mathrm{d}u),$$

which yields the following $\mathbb{P}_x$-a.s. representation for W_s^π:

$$W_s^\pi = \int_{[0,s\wedge\tau^\pi]} \mathrm{e}^{-qu}\mu_K^\pi(\mathrm{d}u) + \mathrm{e}^{-q(s\wedge\tau^\pi)} \operatorname*{ess.sup}_{\tilde{\pi}=(\pi,\overline{\pi})\in\Pi_s} v_{\overline{\pi}}(U^\pi_{s\wedge\tau^\pi}). \tag{A.2}$$

In view of the definitions of Π_s and v_*, the essential supremum in (A.2) is $\mathbb{P}$-a.s. equal to $v_*(U^\pi_{s\wedge\tau^\pi})$, which implies that, $\mathbb{P}$-a.s., $W_s^\pi = V_s^\pi$. $\square$

APPENDIX B: PROOF OF PROPERTIES OF VALUE FUNCTION

PROOF OF LEMMA 3.3(I). Let $x > y$. Denote by $\pi_\varepsilon(y)$ an ε-optimal strategy for the case $U_0 = y$. Then a possible strategy is to immediately pay out $x - y$ and subsequently to adopt the strategy $\pi_\varepsilon(y)$, so that the following holds:

$$v_*(x) \geq x - y - K + v_{\pi_\varepsilon}(y) \geq v_*(y) - \varepsilon + x - y - K.$$

Since this inequality holds for any $\varepsilon > 0$, the stated lower bound follows.

To prove the stated continuity we first establish an upper bound for the difference $v_*(x) - v_*(y)$ with $x > y$. Let $\tilde{\pi}_\varepsilon(x)$ denote an ε-optimal strategy for the case $U_0 = x$ for a given $\varepsilon > 0$. Then a possible strategy is to refrain from paying any dividends until the first time that the reserves hit the level x, and to subsequently follow the policy $\tilde{\pi}_\varepsilon$. Hence $v_*(y)$, $x \geq y$, is bounded below by

$$\begin{aligned}&\frac{W^{(q)}(y)}{W^{(q)}(x)}\big(v_{\tilde{\pi}_\varepsilon}(x) - F_w(x)\big) + F_w(y)\\ &\qquad \geq \frac{W^{(q)}(y)}{W^{(q)}(x)}\big(v^*(x) - \varepsilon - F_w(x)\big) + F_w(y).\end{aligned}$$

Rearranging and letting ε tend to zero yields the upper-bound

$$v_*(x) - v_*(y) \leq \left(1 - \frac{W^{(q)}(y)}{W^{(q)}(x)}\right)\big[v_*(x) - F_w(x)\big] + F_w(x) - F_w(y). \tag{B.1}$$

In the case $K = 0$, continuity of $W^{(q)}|_{\mathbb{R}_+\setminus\{0\}}$, the lower bound from part (i) and (B.1) yield that v_* is continuous on $\mathbb{R}_+$. In the case $K > 0$ continuity of v_* on $\mathbb{R}_+$ follows by combining the upper bound in (B.1) with a different lower bound that is derived next.

For fixed $\varepsilon > 0$ and given initial reserves $U_0 = y$ for some $y > x$, a possible strategy is to adopt $\tilde{\pi}_\varepsilon(x)$ until the first moment that the reserves U fall below $\delta := y - x$, and to follow then a waiting strategy $\pi_\varnothing$ (of not paying any dividends).

Taking $\pi = \tilde{\pi}_\varepsilon(x)$ it follows by the monotonicity of w that $v_*(y) - v_*(x)$ for $y \geq x$ is bounded below by

$$\begin{aligned}
&\mathbb{E}_y\biggl[\int_0^{\tau_\delta^\pi} \mathrm{e}^{-qt}\mu_K^\pi(\mathrm{d}t) + \mathrm{e}^{-q\tau_\delta^\pi} w\bigl(U_{\tau_\delta^\pi}^\pi\bigr)\mathbf{1}_{\{\tau_\delta^\pi = \tau_0^\pi\}} + \mathrm{e}^{-q\tau_\delta^\pi} v_{\pi_\varnothing}\bigl(U_{\tau_\delta^\pi}^\pi\bigr)\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\biggr] \\
&\quad - v_*(x) \\
&\qquad = \mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_\delta^\pi}\bigl(w\bigl(U_{\tau_\delta^\pi}^\delta\bigr) - w\bigl(U_{\tau_\delta^\pi}^\delta - \delta\bigr)\bigr)\mathbf{1}_{\{\tau_\delta^\pi = \tau_0^\pi\}}\bigr] + f_\varepsilon(x, y) \\
&\qquad\quad + v_\pi(x) - v_*(x) \geq -\varepsilon + f_\varepsilon(x, y),
\end{aligned}$$

where $\tau_\delta^\pi = \inf\{t \geq 0 : U_t^\pi < \delta\}$ and

$$f_\varepsilon(x, y) = \mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_\delta^\pi}\bigl(\mathcal{V}_w\bigl(U_{\tau_\delta^\pi}^\pi\bigr) - w\bigl(U_{\tau_\delta^\pi}^\pi - \delta\bigr)\bigr)\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\bigr].$$

Assume for the moment that $f_\varepsilon(x, y)$ tends to zero when $\delta = y - x$ tends to 0. Given this assumption and the bound in (B.1) it follows (since ε was arbitrary)

$$\liminf_{|x-y|\to 0}\bigl[v_*(y) - v_*(x)\bigr] \geq 0. \tag{B.2}$$

Similarly, it can be shown $\limsup_{|x-y|\to 0}[v_*(y) - v_*(x)] \leq 0$. Combining the two limits yields that $v_*(x)$ is continuous at each $x \in \mathbb{R}_+$.

Finally, the claim that $f_\varepsilon(x, y)$ tends to zero is verified. First, note the estimate

$$f_\varepsilon(x, y) \leq \Bigl(\sup_{x\in[0,\delta]} \mathcal{V}_w(x) - w(-\delta)\Bigr)\mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_\delta^\pi}\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\bigr]. \tag{B.3}$$

If X has unbounded variation, then the left-continuity of w at zero and the fact $\mathcal{V}_w(0+) = w(0)$ combined with the inequality in equation (B.3) imply $f_\varepsilon(x, y) \to 0$ when $\delta = y - x \to 0$. If X has bounded variation, $v_{\pi_w}(0)$ is (in general) not equal to $w(0)$, and it is next shown that the second factor in equation (B.3) tends to zero if $\delta \to 0$. Note that the policy $\tilde{\pi}_\varepsilon(x)$, being element of Π, consists of at most countably many dividends payments almost surely. Denoting the times of the dividend payments by $\tau_1, \tau_2, \ldots,$ and the values of $U^{\tilde{\pi}_\varepsilon(x)}$ at those times by $U_1, U_2, \ldots,$ the strong Markov property of X implies

$$\begin{aligned}
\mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_\delta^\pi}\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi\}}\bigr] &= \sum_i \mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_\delta^\pi}\mathbf{1}_{\{\tau_\delta^\pi < \tau_0^\pi, \tau_\delta^\pi \in [\tau_i, \tau_{i+1})\}}\bigr] \\
&\leq \sum_i \mathbb{E}_y\bigl[\mathrm{e}^{-q\tau_i}\mathbf{1}_{\{\tau_i < \tau_0^\pi\}}\mathbb{E}_{U_i}\bigl[\mathrm{e}^{-qT_\delta^-}\mathbf{1}_{\{T_\delta^- < T_0^-\}}\bigr]\bigr].
\end{aligned}$$

As X has bounded variation, we have $\mathbb{P}_x(X(T_\delta^-) < \delta) = 1$ for all $x \in [\delta, \infty)$ so that it follows that, for any $x \in [\delta, \infty)$, the probability $\mathbb{P}_x(T_\delta^- < T_0^-) = \mathbb{P}_x(0 < X(T_\delta^-) < \delta)$ tends to zero as δ tends to zero. Lebesgue's dominated convergence theorem implies that the right-hand side of the previous display converges to zero when δ tends to 0. This completes the proof of the claim in (B.2) $\square$

APPENDIX C: PROOF OF ANALYTICAL OPTIMALITY CRITERION

PROOF OF LEMMA 6.5. (i) First consider the case $K = 0$. The proof is based on the following identity that holds for any $c > 0$ and any $x \le b_+^* + c$:

$$\begin{aligned}\mathbb{E}_x&\left[\mathrm{e}^{-q(t\wedge\tau)}v_b(U_{t\wedge\tau}) + \int_{[0,t\wedge\tau]} \mathrm{e}^{-qs}\,\mathrm{d}D_s\right] - v_b(x)\\ &= \mathbb{E}_x\left[\int_0^{t\wedge\tau} \mathrm{e}^{-qs}\big({}_{b_+}\Gamma_\infty^{\overline{w}} v_b\big)(U_{s-})\mathbf{1}_{\{U_{s-}>b_+\}}\,\mathrm{d}s\right],\end{aligned}\tag{C.1}$$

with $b = b^*$, $b_+ = b_+^*$ and $\tau = \tau^{\pi(b_-^*,b_+^*+c)}$, $\overline{w} = v_{b^*}$, $\mu_K = \mu_K^{\pi(b_-^*,b_+^*+c)}$, $D = D^{\pi(b_-^*,b_+^*+c)}$, $U = U^{\pi(b_-^*,b_+^*+c)}$. The proof of (C.1) is similar to the proof of Lemma 3.4(ii) and is omitted.

Letting $t \to \infty$ in (C.1) Lebesgue's dominated convergence theorem implies for $x \in [0, b_+^* + c]$

$$\begin{aligned} v_{b^*+c}(x) - v_{b^*}(x) &= \mathbb{E}_x\left[\int_0^{\tau_{b^*+c}} \mathrm{e}^{-qs}\big[{}_{b_+^*}\Gamma_\infty^{\overline{w}} v_{b^*}\big]\big(U_{s-}^{b^*+c}\big)\mathbf{1}_{\{U_{s-}^{b^*+c}>b_+^*\}}\,\mathrm{d}s\right]\\ &= \int_{(b_+^*,b_+^*+c]} \big[{}_{b_+^*}\Gamma_\infty^{\overline{w}} v_b\big](y) R^q_{0,b_+^*+c}(x,\mathrm{d}y) \qquad \text{with}\\ R^q_{0,b_+^*+c}(x,\mathrm{d}y) &= \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x\big(Y_t^{b_+^*+c} \in \mathrm{d}y, t < \tau_0\big)\,\mathrm{d}t.\end{aligned}$$

Inserting the explicit expressions from (6.1) and Pistorius [34], Theorem 1 (see also proof of Proposition 5.5) for v_b^*, v_{b^*+c} and $R^q_{0,b_+^*+c}(x,\mathrm{d}y)$ yields for $x \in x \in [0, b_+^*]$

$$\begin{aligned} W^{(q)}(x)&\big[G\big(b_+^*+c\big) - G\big(b_+^*\big)\big]\\ &= W^{(q)}(x)\int \big[{}_{b_+^*}\Gamma_\infty^{\overline{w}} v_{b^*}\big](y)\frac{W^{(q)}(b_+^*+c-\mathrm{d}y)}{W^{(q)\prime}(b_+^*+c)},\end{aligned}$$

where the integral is over the interval $(b_+^*, b_+^*+c]$ with $G = G_{b_-^*}$ and using that $W^{(q)}(x)$ is equal to 0 for $x < 0$. Changing coordinates in the integral and using that $W^{(q)}(x)$ is strictly positive at any $x > 0$ yields the first equality in (6.8). The second equality in (6.8) follows by the representation in (5.14). The second statement is a direct consequence of (6.8) and the fact $\{G(b_-^*, b_+^*+c) < G(b_-^*, b_+^*)\ \forall c > 0\}$ (from the definition of d^* as *last* supremum). The proof of the case $K > 0$ is similar and omitted.

The ultimate monotonicity of $G(b^-, y)$ and $G^\#(y)$ follows from the fact that ${}_{b_+}\Gamma_\infty^w v_b(x)$ tends to minus infinity when $x \to \infty$ (by Lemma 3.4).

(ii) Taking the Laplace transform in c in (6.8) and using the form of the Laplace transform of $W^{(q)}$ yields that, for $\theta > \Phi(q)$ and with $G = G_{b_-}$,

$$\begin{aligned}\mathcal{L}g(\theta)\cdot\frac{\theta}{\psi(\theta)-q} &= \int_{[0,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)\big[G(b_+ + c) - G(b_+)\big]\,\mathrm{d}c \\ &= \int_{[0,\infty)}\int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(b_+ + c)\,\mathrm{d}c\, G(b_+ + \mathrm{d}z) \\ &= \mathrm{e}^{\theta b_+}\int_{[b_+,\infty)}\int_{[z,\infty)} \mathrm{e}^{-\theta c} W^{(q)\prime}(c)\,\mathrm{d}c\, G(\mathrm{d}z) \\ &= \frac{\mathrm{e}^{\theta b_+}}{\psi(\theta)-q}\int_{[b_+,\infty)} \mathrm{e}^{-\theta z} Z^{(q,\theta)\prime}(z) G(\mathrm{d}z),\end{aligned}$$

by a change of the order of integration, which is justified by Fubini's theorem, and the form (5.20) of $Z^{(q,\theta)\prime}(z)$. The second assertion follows since a function $f:(c,\infty)\to\mathbb{R}$ with $c>0$ is completely monotone if and only if it is the Laplace transform of a nonnegative measure supported on $\mathbb{R}_+$. $\square$

APPENDIX D: ON OPTIMALITY OF SINGLE BAND STRATEGIES

PROOF OF COROLLARY 9.6. In view of verification Theorem 4.4, it suffices to verify that it holds $J(x)\le 0$ for any $x>0$ with $J(x) := ({}_{b_+^*}\Gamma_\infty^{\tilde{w}} v_{b^*})(b_+^* + x)$. This assertion follows once the following three facts are verified:

(i) J is concave on $\mathbb{R}_+\setminus\{0\}$,
(ii) $J(0+)=0$ and
(iii) $J'(0+)\le 0$.

To show (i) note that under the stated assumptions, for $y\in(0,b)$, $[v(b-y) - v(b)+y]\le 0 \Leftrightarrow v(b)-v(b-y)\ge y$ (as $K=0$), and for $y\ge b$ it holds $w(b-y) - v(0) - b + y \le 0$ and $v(0)-v(b)+b\le 0$ which yields that $w(b-y)-v(b)\le y$ for $y\ge b$. As v' is convex, and a mixture of convex functions with positive weights is again convex, it follows that J is concave on $\mathbb{R}_+\setminus\{0\}$.

Given (ii), statement (iii) follows since if $J'(0+)$ were positive, $(J(x) - J(0+))/x = J(x)/x$ would be positive for all x sufficiently small which would be in contradiction with (6.8).

To see that (ii) holds, note that, from (6.8), $\int_{[0,c]} J(c-y)W^{(q)}(\mathrm{d}y)\le 0$ for all $c>0$ sufficiently small. Thus since J is continuous on $\mathbb{R}_+\setminus\{0\}$ (as it is concave) it follows $J(0+)\le 0$. To complete the proof it is next shown that also $J(0+)\ge 0$.

First consider the case that σ^2 is strictly positive: The observations that, for any $b>0$, $\mathrm{e}^{-q(t\wedge T_{0,b})}v_b(X_{t\wedge T_{0,b}})$ is a martingale with $v_b\in C^2$ together with Itô's lemma yield that $({}_0\Gamma_\infty^{w} v_b)(x) = 0$ for all $x\in(0,b_+)$ which in turn implies that

$J(0+) = {}_0\Gamma_\infty^w v_b(b_+) = 0$ on account of the continuity of $x \mapsto ({}_0\Gamma_\infty^w v_{b^*})(x)$ at $x = 0$.

Consider next the case $\sigma^2 = 0$, which follows by approximation. By adding a small Brownian component with variance $\sigma^2 > 0$ to X and subsequently letting $\sigma^2 \to 0$, it can be shown that in this case $J(0+) \geq 0$: If $\sigma \searrow 0$, the continuity theorem implies that the scale functions $W^{(q)(\sigma)}$ and $F_w^{(\sigma)}$ of the perturbed process $X^{(\sigma)} := X + \sigma B$ (where B is a Brownian motion independent of X) and the corresponding derivatives $W^{(q)(\sigma)\prime}$ and $F_w^{(\sigma)\prime}$ converge pointwise to the corresponding (derivatives of) scale functions of X at any point of continuity. Denote by $J^{(\sigma)}$ the function J with the function v replaced by the function $v^{(\sigma)}$ corresponding to the perturbed process $X^{(\sigma)}$. An application of Fatou's lemma, which is justified on account of the bounds in Lemma 3.3, then yields that

$$0 = \lim_{\sigma \searrow 0} J^{(\sigma)}(x) \leq J(x) \qquad \text{for any } x > 0.$$

The proof is complete. $\square$

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