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EXPONENTIAL MOMENTS OF AFFINE PROCESSES

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We investigate the maximal domain of the moment generating function of affine processes in the sense of Duffie, Filipović and Schachermayer [*Ann. Appl. Probab.* **13** (2003) 984–1053], and we show the validity of the affine transform formula that connects exponential moments with the solution of a generalized Riccati differential equation. Our result extends and unifies those preceding it (e.g., Glasserman and Kim [*Math. Finance* **20** (2010) 1–33], Filipović and Mayerhofer [*Radon Ser. Comput. Appl. Math.* **8** (2009) 1–40] and Kallsen and Muhle-Karbe [*Stochastic Process Appl.* **120** (2010) 163–181]) in that it allows processes with very general jump behavior, applies to any convex state space and provides both sufficient and necessary conditions for finiteness of exponential moments.

1. Introduction. This article investigates the maximal domain of the moment generating function of an affine process. An affine process is a time-homogeneous Markov processes X on a finite-dimensional state space $D \subset \mathbb{R}^d$ whose characteristic function has the following property: There exist a complex-valued function ϕ and a $\mathbb{C}^d$-valued function ψ such that

$$\Phi(t,u,x) := \mathbb{E}\big[e^{\langle u, X_t\rangle} \mid X_0 = x\big] = e^{\phi(t,u)+\langle \psi(t,u),x\rangle}, \tag{1.1}$$

for all $u \in i\mathbb{R}^d$, $t \geq 0$ and $x \in D$. This so-called *affine property* implies that the PDE

$$\frac{\partial}{\partial t}\Phi(t,u,x) = \mathcal{A}\Phi(t,u,x), \qquad \Phi(0,u,x) = \exp\big(\langle u,x\rangle\big),$$

where $\mathcal{A}$ denotes the infinitesimal generator of X, can be reduced to a system of nonlinear ODEs, commonly referred to as *generalized Riccati differential equations*, which are of the form

$$\frac{\partial}{\partial t}\phi(t,u) = F\big(\psi(t,u)\big), \qquad \phi(0,u) = 0, \tag{1.2a}$$

$$\frac{\partial}{\partial t}\psi(t,u) = R\big(\psi(t,u)\big), \qquad \psi(0,u) = u. \tag{1.2b}$$

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A natural and important question is whether formula (1.1) and the generalized Riccati system (1.2) can be extended to real exponential moments ($u \in \mathbb{R}^d$) or complex exponential moments ($u \in \mathbb{C}^d$). One might expect that if F and R can be suitably extended, for example, by analytic extension, then the exponential moment $\mathbb{E}^x[e^{\langle u, X_T\rangle}]$ is finite if and only if a solution to the extended Riccati system exists up to time T, and that in this case also (1.1) remains valid. A statement of this type is usually referred to as *affine transform formula*. Showing such a formula in full generality is far from trivial—difficulties include the fact that analytic extension of F and R may not be possible, that solutions of the extended Riccati equations might not be unique and that the differentiability of $t \mapsto \phi(t, u)$ and $t \mapsto \psi(t, u)$ is not obvious from (1.1). The latter problem of showing that differentiability of ϕ and ψ can be concluded from the definition of an affine processes is known as the *regularity problem* for affine processes; cf. Duffie, Filipović and Schachermayer (2003), Keller-Ressel, Schachermayer and Teichmann (2011), Cuchiero (2011).

Several articles have been concerned with showing the affine transform formula under different conditions on the process X or the state space D. In particular we mention the following contributions:

- Glasserman and Kim (2010) show the affine transform formula for real moments of affine diffusion processes on $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ under a mean-reversion condition;
- Filipović and Mayerhofer (2009) show the affine transform formula for real and complex moments of affine diffusion processes on $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$;
- Kallsen and Muhle-Karbe (2010) show that for affine semi-martingales on $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ existence of a solution to the extended Riccati system on $[0, T]$ implies the validity of the affine transform formula for real moments under a mild condition on the jump-measures;
- Spreij and Veerman (2010) show an affine transform formula for affine processes whose jump measures possess exponential moments of all orders and where the state space is a convex subset of $\mathbb{R}^d$;
- in the context of a stock price model with stochastic interest rates and possibility of default, Cheridito and Wugalter (2012) show an affine transform formula for affine processes with killing when the jump measures possess exponential moments of all orders.

In this article we generalize and unify most of these results. In particular we remove the condition that all exponential moments of the jump measures must exist, which is typically not fulfilled in applications; see the discussion in Section 3.5. Moreover we show that the existence of a minimal solution to the extended Riccati system is necessary and sufficient for the exponential transform formula to hold, while Kallsen and Muhle-Karbe (2010) covers only sufficiency. Finally our results apply to very general types of state spaces: The results on real exponential moments hold for affine processes on an arbitrary convex state space, and the results on complex exponential moments apply to affine processes on $D = \mathbb{R}_+^m \times \mathbb{R}^n$ and on $D = S_d^+$ (the positive semidefinite $d \times d$ matrices). These two state spaces [see Duffie, Filipović and Schachermayer (2003) and Cuchiero et al. (2011)] are of particular interest both from the theoretic viewpoint and from the applied one. The outline of this paper is as follows: In Section 2 we present general definitions, some useful notation and our main results:

- Theorem 2.14 proves the affine transform formula in terms of *minimal solutions* to the so-called *extended Riccati system*, which comes from considering (1.2) in the real domain. Here we only require the state space to be closed, convex and with nonempty interior. The proof of Theorem 2.14 is provided in Section 4.
- Theorem 2.26 extends the validity of the affine property (1.1) to complex moments $u = p + iz$, where $z \in \mathbb{R}^d$. This extension succeeds under the premise that the pth real moment is finite, or equivalently, that the extended Riccati equations are solvable until time T. The result holds for the state space $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ and—under some mild additional conditions—for the state space S_d^+. For the proof of Theorem 2.26, see Section 5.

In Section 3 several applications of our results to mathematical finance are outlined. Finally, Sections 4 and 5 contain the proofs of our main results for real moments and complex moments respectively.

2. Definitions and main results.

2.1. *Affine processes.* Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space, with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ a right-continuous filtration. We endow $\mathbb{R}^d$, $(d \geq 1)$ with an inner product $\langle \cdot, \cdot \rangle$ and let D be a nonempty convex subset of $\mathbb{R}^d$, which will act as the state space of the stochastic process X we are about to define. The state space D has a measurable structure given by its Borel σ-algebra $\mathcal{B}(D)$, and without loss of generality (see the explanation after Definition 2.2), we may assume that D contains 0 and that the linear span of D is the full space $\mathbb{R}^d$. Under this assumption it follows in particular that the interior D° of D is nonempty. Associated to D is the set

$$\mathcal{U} = \{u \in \mathbb{C}^d : x \mapsto e^{\langle x, u \rangle} \text{is bounded on } D\}. \tag{2.1}$$

Finally let $(\mathbb{P}^x)_{x \in D}$ be a family of probability measures on the filtered space $(\Omega, \mathcal{F}, \mathbb{F})$ and assume that $\mathcal{F}$ is complete with respect to $(\mathbb{P}^x)_{x \in D}$ in the sense of Blumenthal and Getoor (1968), Chapter I.5.

Let X be a càdlàg[2] $\mathbb{F}$-adapted time-homogeneous conservative Markov process with state space D. More precisely, writing

$$p_t(x, A) = \mathbb{P}^x(X_t \in A) \qquad (t \geq 0, x \in D, A \in \mathcal{B}(D)) \tag{2.2}$$

for the transition kernel of X, $p_t(x, A)$ satisfies the following:

[2]For convex state spaces, affine processes have càdlàg modifications, see Remark 2.5 below.

(a) $x \mapsto p_t(x, A)$ is $\mathcal{B}(D)$-measurable for all $t \geq 0$, $A \in \mathcal{B}(A)$,
(b) $p_t(x, D) = 1$ for all $t \geq 0, x \in D$,
(c) $p_0(x, \{x\}) = 1$ for all $x \in D$ and
(d) the Chapman–Kolmogorov equation

$$p_{t+s}(x, A) = \int_D p_t(y, A) p_s(x, dy)$$

holds for every $t, s \geq 0$ and $(x, A) \in D \times \mathcal{B}(D)$.

REMARK 2.1. Since X is càdlàg, the law of X under $\mathbb{P}^x$ is a probability measure on the Skorokhod space of càdlàg paths $\mathbb{D}(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$, for each $x \in D$. There will be no loss of generality by directly interpreting $\mathbb{P}^x$ as a measure on this path space.

DEFINITION 2.2 (Affine process). The process X is called *affine* with state space D, if its transition kernel $p_t(x, A)$ satisfies the following:

(i) it is stochastically continuous, that is, $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly for all $t \geq 0, x \in D$, and
(ii) there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\int_D e^{\langle u, z \rangle} p_t(x, d\xi) = \exp\bigl(\phi(t, u) + \langle x, \psi(t, u) \rangle\bigr) \tag{2.3}$$

for all $t \geq 0, x \in D$ and $u \in \mathcal{U}$.

REMARK 2.3. We explain why it is no loss of generality to assume that D contains 0 and linearly spans the whole space $\mathbb{R}^d$: For an arbitrary nonempty convex subset D of $\mathbb{R}^d$, let $\operatorname{aff}(D)$ be the smallest affine subspace of $\mathbb{R}^d$ that contains D, and let $(x_0, x_1, \ldots, x_k)$ be an affine basis of $\operatorname{aff}(D)$ such that $x_0 \in D$. Let $h : \operatorname{aff}(d) \to \mathbb{R}^k : x \mapsto A^\top (x - x_0)$ be the projection to canonical affine coordinates, that is, $h(x_0) = 0$ and $h(x_i) = e_i$ for each $i \in \{1, \ldots, k\}$. Set $\widetilde{D} = h(D) \subset \mathbb{R}^k$ and $\widetilde{X} = h(X)$. Then $\widetilde{D}$ is convex, contains 0 and linearly spans $\mathbb{R}^k$. It is easily verified that $\widetilde{X}$ is again an affine process with

$$\widetilde{\phi}(t, u) = \phi(t, Au) + \langle x_0, \psi(t, Au) - u \rangle, \tag{2.4}$$

$$\widetilde{\psi}(t, u) = A^+ \psi(t, Au), \tag{2.5}$$

where A^+ is the Pseudoinverse of A (or any other $k \times d$-matrix such that $A^+ A = \mathrm{id}_k$).

The next result shows that an affine process is a semimartingale with affine (differential) semimartingale characteristics.

THEOREM 2.4 [Cuchiero (2011)]. *Let X be an affine process with state space $D \subset \mathbb{R}^d$. Then for each $x \in D$, the process X is a $\mathbb{P}^x$-semimartingale with semimartingale characteristics*

$$A_t = \int_0^t a(X_{s-})\,ds, \tag{2.6a}$$

$$B_t = \int_0^t b(X_{s-})\,ds, \tag{2.6b}$$

$$\nu(\omega, dt, d\xi) = K\big(X_{t-}(\omega), d\xi\big)\,dt, \tag{2.6c}$$

where $a(x)$, $b(x)$ and $K(x, d\xi)$ are affine functions of the form

$$a(x) = a + x_1\alpha^1 + \cdots + x_d\alpha^d, \tag{2.7a}$$

$$b(x) = b + x_1\beta^1 + \cdots + x_d\beta^d, \tag{2.7b}$$

$$K(x, d\xi) = m(d\xi) + x_1\mu^1(d\xi) + \cdots + x_d\mu^d(d\xi), \tag{2.7c}$$

and for each $x \in D$ it holds that $a(x)$ is a positive semidefinite $d \times d$ matrix, $b(x)$ is a $\mathbb{R}^d$-vector and $K(x, d\xi)$ is a Radon measure on $\mathbb{R}^d$, satisfying

$$\int_{\mathbb{R}^d} \big(\|\xi\|^2 \wedge 1\big) K(x, d\xi) < \infty$$

and $K(x, \{0\}) = 0$.

PROOF. Follows from Cuchiero (2011), Theorems 1.4.8 and 1.5.4. □

REMARK 2.5. Note that several of the assumptions made at the beginning of the section could be slightly weakened: Following Cuchiero and Teichmann (2013) *any* affine process (satisfying a mild regularity property on ϕ, ψ which is automatically fulfilled for convex state spaces) has a càdlàg modification; moreover the $(\mathbb{P}^x)_{x \in D}$-completion of the filtration generated by an affine process is automatically right continuous. Note that it is unkown to this date, whether all affine processes are Feller. Hence the proof of the càdlàg modification in Cuchiero and Teichmann (2013) is not an immediate consequence of the Feller property, but more involved.

2.2. *Real moments of affine processes.*

DEFINITION 2.6. Given an affine process X and the associated functions $(a(x), b(x), K(x, d\xi))$ in (2.6), define for each $x \in D$ the function $\mathcal{R}_x : \mathbb{R}^d \to (-\infty, \infty]$ by

$$\begin{aligned}\mathcal{R}_x(y) = {} & \frac{1}{2}\langle y, a(x)y\rangle + \langle b(x), y\rangle \\ & + \int_{\mathbb{R}^d \setminus \{0\}} \big(e^{\langle \xi, y\rangle} - 1 - \langle h(\xi), y\rangle\big) K(x, d\xi),\end{aligned} \tag{2.8}$$

where $h(\xi) = \mathbf{1}_{\{|\xi|\le 1\}}\xi$.

For each fixed $x \in D$, the function $\mathcal{R}_x$ is a convex and lower semi-continuous function[3] that may take the value $+\infty$. As for any convex function, the effective domain $\mathcal{Y}_x$ is the set of arguments for which $\mathcal{R}_x$ takes finite values. Taking the intersection over all $x \in D$ leads to the following definition.

DEFINITION 2.7. Given an affine process X and the associated function $\mathcal{R}_x$ as in Definition 2.6, define

$$\mathcal{Y} = \bigcap_{x\in D}\left\{y \in \mathbb{R}^d : \int_{|\xi|\ge 1} e^{\langle y,\xi\rangle} K(x, d\xi) < \infty\right\}. \tag{2.9}$$

As an intersection of convex sets, also $\mathcal{Y}$ is convex. Moreover, $\mathcal{Y}$ contains 0 and hence is nonempty, because $\mathcal{R}_x(0) = 0$ for all $x \in D$.

Since the functions $a(x)$, $b(x)$ and $K(x, d\xi)$ are affine in x, we can decompose $\mathcal{R}_x$ into $\mathcal{R}_x(y) = F(y) + \langle R(y), x\rangle$. For arguments $y \in \mathcal{Y}$, the functions F and R are uniquely specified, since D contains 0 and d linearly independent points.

PROPOSITION 2.8. *Let X be an affine process with state space D. Then there exist functions $F : \mathcal{Y} \to \mathbb{R}$, $R : \mathcal{Y} \to \mathbb{R}^d$ such that*

$$\mathcal{R}_x(y) = F(y) + \langle R(y), x\rangle$$

for all $x \in D$, $y \in \mathcal{Y}$. Let $(e_1, \dots, e_d)$ be the canonical basis vectors in $\mathbb{R}^d$. Then we can write F and $R_i(y) := \langle R(y), e_i\rangle$ as

$$\begin{aligned} F(y) &= \frac{1}{2}\langle u, ay\rangle + \langle b, y\rangle \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \big(e^{\langle\xi,y\rangle} - 1 - \langle h(\xi), y\rangle\big) m(d\xi), \end{aligned} \tag{2.10a}$$

$$\begin{aligned} R_i(y) &= \frac{1}{2}\langle y, \alpha^i y\rangle + \langle \beta^i, y\rangle \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \big(e^{\langle\xi,y\rangle} - 1 - \langle h(\xi), y\rangle\big) \mu^i(d\xi), \end{aligned} \tag{2.10b}$$

with $h(\xi) = \mathbf{1}_{\{|\xi|\le 1\}}\xi$.

PROOF. The proof follows immediately from Definition 2.6 and Theorem 2.4. $\square$

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[3]Lower semi-continuity follows from Fatou's lemma applied to the integral with respect to $K(x, d\xi)$.

REMARK 2.9. Setting $x = 0$ in (2.8) yields that $F(y)$ is a convex and lower semi-continuous function of Lévy–Khintchine form. The same is not necessarily true for $R_1, \dots, R_d$, since the matrices α^i may not be positive semidefinite, or the measures μ^i may be signed measures.

We use the functions $F(y)$ and $R(y)$ to set up a system of ODEs associated to the affine process X. These equations play a key role in our main result.

DEFINITION 2.10 (Extended Riccati system). Let X be an affine process and F, R and $\mathcal{Y}$ be defined as in Definition 2.7 and Proposition 2.8. Let $T \ge 0$, $y \in \mathcal{Y}$ and let

$$p : t \mapsto p(t, y), q : t \mapsto q(t, y)$$

be C^1-functions mapping $[0, T]$ to $\mathbb{R}$ (resp., $\mathcal{Y}$) that satisfy

$$\frac{\partial}{\partial t} p(t, y) = F\big(q(t, y)\big), \qquad p(0, y) = 0, \tag{2.11a}$$

$$\frac{\partial}{\partial t} q(t, y) = R\big(q(t, y)\big), \qquad q(0, y) = y \tag{2.11b}$$

for all $t \in [0, T]$. Then we call (p, q) a solution (up to time T and with starting point y) of the extended Riccati system associated to X.

It is important to note that in general the function R is locally Lipschitz continuous only on the interior of $\mathcal{Y}$, but may fail to be Lipschitz continuous at the boundary of $\mathcal{Y}$. Hence solutions of (2.11) reaching or starting at the boundary of $\mathcal{Y}$ may not be unique. For this reason we add the following definition.

DEFINITION 2.11 (Minimal solution). Let X be an affine process, and let (p, q) a solution up of T starting at $y \in \mathcal{Y}$ to the associated extended Riccati system. We call (p, q) a *minimal* solution, if for any other solution $(\widetilde{p}, \widetilde{q})$ up to $\widetilde{T} \le T$ and starting at the same point $q(0, y) = \widetilde{q}(0, y) = y$ it holds that

$$p(t, y) + \big\langle q(t, y), x\big\rangle \le \widetilde{p}(t, y) + \big\langle \widetilde{q}(t, y), x\big\rangle \tag{2.12}$$

for all $t \in [0, \widetilde{T}]$ and $x \in D$.

REMARK 2.12. By setting $q_x(t, y) := p(t, y) + \langle q(t, y), x\rangle$, the extended Riccati system may be written in condensed form as

$$\frac{\partial}{\partial t} q_x(t, y) = \mathcal{R}_x\big(q(t, y)\big), \qquad q_x(0, y) = \langle y, x\rangle \qquad \forall x \in D. \tag{2.13}$$

In this notation the minimality property can we written as

$$q_x(t, y) \le \widetilde{q}_x(t, y) \qquad \forall x \in D, t \in [0, \widetilde{T}], y \in \mathcal{Y}.$$

REMARK 2.13. The following properties are easy to see: If for a given starting value $y \in \mathcal{Y}$ there is only one solution to the extended Riccati system, then it is automatically a minimal solution. Also, if for a given starting value a minimal solution (p, q) exists up to time T, it is automatically the unique minimal solution. Indeed, if there were another minimal solution $(\widetilde{p}, \widetilde{q})$, then

$$p(t, y) + \langle q(t, y), x\rangle = \widetilde{p}(t, y) + \langle \widetilde{q}(t, y), x\rangle$$

for all $t \in [0, T]$, $x \in D$. Since D contains d linearly independent points and 0, it follows that $p = \widetilde{p}$ and $q = \widetilde{q}$ in this case.

We can now formulate our main results on the behavior of exponential moments of affine processes.

THEOREM 2.14 (Real moments of affine processes). *Let X be an affine process on D, and let $T \geq 0$.*

(a) *Let $y \in \mathbb{R}^d$, and suppose that $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ for some $x \in D^\circ$. Then $y \in \mathcal{Y}$ and there exists a unique minimal solution (p, q) up to time T of the extended Riccati system* (2.11), *such that*

$$\mathbb{E}^x\big[e^{\langle y, X_t\rangle}\big] = \exp\big(p(t, y) + \langle q(t, y), x\rangle\big) \tag{2.14}$$

holds for all $x \in D$, $t \in [0, T]$.

(b) *Let $y \in \mathcal{Y}$, and suppose that the extended Riccati system* (2.11) *has solutions $(\widetilde{p}, \widetilde{q})$ that start at y and exist up to T. Then $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ and there exist unique minimal solutions (p, q) up to time T of the extended Riccati system such that* (2.14) *holds for all $x \in D$, $t \in [0, T]$.*

REMARK 2.15. We emphasize that in point (b) of the theorem $p = \widetilde{p}$ and $q = \widetilde{q}$ does not necessarily hold, that is, the candidate solutions $(\widetilde{p}, \widetilde{q})$ have to be replaced by the minimal solutions (p, q) in order for (2.14) to hold true.

The following corollary is a conditional version of Theorem 2.14 and thus extends the corresponding result [Filipović and Mayerhofer (2009), Theorem 3.3(iv)] for affine diffusions on canonical state-spaces:

COROLLARY 2.16. *Suppose that the conditions of either Theorem* 2.14(a) *or* (b) *are satisfied, and let (p, q) be the associated minimal solutions of the Riccati system* (2.11). *Then also $\mathbb{E}^x[e^{\langle q(T-t,y), X_t\rangle}] < \infty$ and*

$$\mathbb{E}^x\big[e^{\langle y, X_T\rangle} | \mathcal{F}_t\big] = \exp\big(p(T - t, y) + \langle q(T - t, y, X_t)\rangle\big)$$

holds for all $x \in D$, $t \in [0, T]$.

The next proposition provides a way to identify whether some solution $(\widetilde{p}, \widetilde{q})$ of the extended Riccati system is in fact the minimal solution.

PROPOSITION 2.17. *Let X be an affine process, and let $(\widetilde{p}, \widetilde{q})$ be a solution up to time $T \geq 0$ of the extended Riccati system associated to X. Each of the following conditions is sufficient for $(\widetilde{p}, \widetilde{q})$ to be the unique minimal solution*:

(a) *X is a diffusion process*;
(b) $\mathcal{Y} = \mathbb{R}^d$;
(c) *$\mathcal{Y}$ is open*;
(d) $\widetilde{q}(t, y) \in \mathcal{Y}^\circ$ *for all $t \in [0, T)$*.

PROOF. From Definition 2.7 of $\mathcal{Y}$ it follows that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d), that is, it is sufficient to show that (d) implies uniqueness of the solution $(\widetilde{p}, \widetilde{q})$. But R is locally Lipschitz on $\mathcal{Y}^\circ$, such that standard ODE results imply that $(\widetilde{p}, \widetilde{q})$ is the unique (and hence unique minimal) solution of the extended Riccati system (2.11) on $[0, T)$. Due to continuity, $\widetilde{q}$ is unique on the compact interval $[0, T]$ as well. □

REMARK 2.18. Condition (b) is equivalent to $\int_{|\xi|\geq 1} e^{\langle y,\xi\rangle} K(x, d\xi) < \infty$ for all $x \in D$, $y \in \mathbb{R}^d$, that is, to the jump measure having exponential moments of all orders. In this special case analogues of Theorem 2.14 have been shown in Spreij and Veerman (2010) and Cheridito and Wugalter (2012). This condition is restrictive, as it is typically not satisfied in applications; cf. Section 3.5.

We briefly discuss two important special cases, in which great simplifications of the results occur. These cases have been treated previously in the literature, but serve as a first "sanity check" of the main results of this article.

EXAMPLE 2.19 (Affine diffusion). Suppose that the affine process X is a diffusion. In this case $K(x, \cdot) = 0$ for all $x \in D$ and consequently $\mathcal{Y} = \mathbb{R}^d$ and the functions $F(y), R_1(y), \ldots, R_d(y)$ are quadratic polynomials (hence locally Lipschitz continuous everywhere). In this case any solution of the extended Riccati system is unique, and there is no need to introduce the concept of minimal solutions; see Proposition 2.17(a) above. Thus Theorem 2.14 holds true even with "minimal solution" replaced by "solution." For the case of affine diffusions on canonical state spaces, the analogue of Theorem 2.14 has been shown in Filipović and Mayerhofer [(2009), Theorem 3.3].

EXAMPLE 2.20 (Lévy process). Suppose that X is a Lévy process. Then X is an affine process with $R(y) = 0$ and with $F(y)$ equal to the Lévy exponent of X. Consequently $\mathcal{Y}$ is simply the effective domain of the Lévy exponent. The extended Riccati system has unique global solutions for each $y \in \mathcal{Y}$, which are given by $p(t, y) = tF(y)$ and $q(t, y) = y$ for $t \geq 0$. It follows from Theorem 2.14 that $\mathbb{E}^x[e^{\langle y,X_t\rangle}]$ is finite if and only if $y \in \mathcal{Y}$, and in case of finiteness we have $\mathbb{E}^x[e^{\langle y,X_t\rangle}] = \exp(tF(y) + \langle y, x\rangle)$. In particular, finiteness of exponential moments is a time-independent property; that is, for given $y \in \mathbb{R}^d$ the exponential moment $\mathbb{E}^x[e^{\langle y, X_t\rangle}]$ is either finite for all $t > 0$ or for no $t > 0$. Of course, all these results are well known in the case of Lévy processes and can be found, for example, in Sato [(1999), Theorem 25.17].

2.3. *Complex moments of affine processes.* In this subsection we give an analogue of Theorem 2.14 for complex exponential moments of X. The first step is to analytically extend the functions F and R. We introduce the following notation: For a set $A \subset \mathbb{R}^d$ write

$$S(A) := \{u \in \mathbb{C}^d : \operatorname{Re} u \in A\}$$

for the complex "strip" generated by A.

PROPOSITION 2.21. *Let X be an affine process, and suppose that $\mathcal{Y}^\circ \neq \varnothing$. Then, for every $x \in D$, the function $\mathcal{R}_x$ defined in (2.8) has an analytic extension to $S(\mathcal{Y}^\circ)$ which we also denote by $\mathcal{R}_x$. Moreover it holds that*

$$\mathcal{R}_x(u) = F(u) + \langle R(u), x\rangle, \qquad x \in D, u \in S(\mathcal{Y}^\circ),$$

where F, R are the analytic extensions of the functions defined in (2.10) to $S(\mathcal{Y}^\circ)$.

PROOF. Follows from standard results on Lévy–Khintchine-type functions; see, for example, Sato (1999), Theorem 25.17. □

DEFINITION 2.22 (Complex Riccati system). Let X be an affine process such that $\mathcal{Y}^\circ \neq \varnothing$, and let F, R be defined as in Proposition 2.21. Let $T \geq 0$, $y \in S(\mathcal{Y}^\circ)$, and let

$$\phi : t \mapsto \phi(t, y), \qquad \psi : t \mapsto \psi(t, y)$$

be C^1-functions mapping $[0, T]$ to $\mathbb{C}$ [resp., $S(\mathcal{Y}^\circ)$] that satisfy

$$\frac{\partial}{\partial t}\phi(t, y) = F(\psi(t, y)), \qquad \phi(0, y) = 0, \tag{2.15a}$$

$$\frac{\partial}{\partial t}\psi(t, y) = R(\psi(t, y)), \qquad \psi(0, y) = y \tag{2.15b}$$

for all $t \in [0, T]$. Then we call (ϕ, ψ) a solution (up to time T and with starting point u) of the complex Riccati system associated to X.

REMARK 2.23. Let us compare the complex Riccati system to the extended Riccati system (2.11a)–(2.11b). We observe that if $u \in S(\mathcal{Y}^\circ)$ is real valued, that is, has $\operatorname{Re} u = y$ and $\operatorname{Im} u = 0$, then any solution (ϕ, ψ) up to time T of the complex Riccati system is also a solution of the extended Riccati system; that is, setting $p(t, y) = \phi(t, u)$ and $q(t, y) = \psi(t, u)$ for all $t \in [0, T]$ defines a solution (p, q)

of the extended Riccati system. The reverse is not necessarily true. Furthermore we point out that for a given starting value u any solution (ϕ, ψ) of the complex Riccati system is automatically the unique solution. This is in contrast to the extended Riccati system, where solutions starting at the boundary may be nonunique. This difference is just a consequence of the fact that solutions of the complex Riccati system are restricted to stay in the open domain $S(\mathcal{Y}^\circ)$, on which F and R are locally Lipschitz.

ASSUMPTION 2.24. Let X be an affine process with state space D and assume that either:

(i) $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, or

(ii) $D = S_d^+$ and there exists some $x \in S_d^{++}$ such that $a(x)$ either vanishes, or it is nondegenerate.

REMARK 2.25. Note that in the notation of (2.7) $a(x)$ is given as a symmetric $\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}$ matrix. Of course we can also interpret it as quadratic form on S_d^+, which is more natural and, in particular, a coordinate free notion. A simple characterization of (ii) in terms of the admissible parameter set is given in Remark 5.11.

The analogue of Theorem 2.14 for complex moments reads as follows.

THEOREM 2.26 (Complex moments of affine processes). *Let X be an affine process that satisfies Assumption 2.24. Let $T \geq 0$, $u \in S(\mathcal{Y}^\circ)$ and suppose that the extended Riccati system (2.11a)–(2.11b) has a solution (p, q) with initial value* $\operatorname{Re} u$ *up to time T such that $q(t, \operatorname{Re} u) \in \mathcal{Y}^\circ$ for all $t \in [0, T]$. Then also the complex Riccati system (2.15) has a solution (ϕ, ψ) with initial value u up to time T, $\mathbb{E}^x[|e^{\langle u, X_t\rangle}|] < \infty$ and*

$$\mathbb{E}^x\big[e^{\langle u, X_t\rangle}\big] = \exp\big(\phi(t, u) + \langle \psi(t, u), x\rangle\big) \tag{2.16}$$

for all $x \in D$, $t \in [0, T]$.

3. Applications in mathematical finance. This section presents applications of our main results, Theorems 2.14 and 2.26, to mathematical finance in the spirit of Duffie, Filipović and Schachermayer (2003), Section 13. We consider the following generic setup: A traded asset S is modeled by the exponential of an affine factor process X with state space D, that is, $S = e^{\langle \theta, X\rangle}$ for some $\theta \in \mathbb{R}^d$. Moreover, bond prices are given through an affine short rate model of the form

$$r_t = L(X_t) = l + \langle \lambda, X_t\rangle,$$

where $l \in \mathbb{R}$ and $\lambda \in \mathbb{R}^d$. This setup includes, in particular, affine term structure models of interest rates [Cox, Ingersoll and Ross (1985), Dai and Singleton (2000), Duffie and Kan (1996), etc.], affine stochastic volatility models [Heston (1993), Bates (2000), Barndorff-Nielsen and Shephard (2001), etc.] and combinations with possible correlation of short rate and asset prices. Also credit risk can be included, when r_t is interpreted as a superposition of a risk-free short rate and an affine default intensity process; cf. Lando (1998). Moreover, we can cover a setup with multiple possibly dependent assets simply by setting $S^i = \exp\langle\theta_i, X\rangle$ for different $\theta_i \in \mathbb{R}^d$. For most applications the measures $(\mathbb{P}^x)_{x\in D}$ should be considered risk-neutral measures, although there are few cases where also the behavior under the physical measure is of relevance. Many problems of interest can be reduced to determining the $\mathcal{F}_t$-conditional expectations

$$Q_{T-t}g(x) = \mathbb{E}^x\big[e^{-\int_t^T L(X_s)\,ds} g(X_T) \mid \mathcal{F}_t\big], \tag{3.1}$$

for some measurable function $g : D \to \mathbb{R}$. In particular:

- $g \equiv 1$ corresponds to bond pricing;
- $g(x) = e^{\langle\theta,x\rangle}$ corresponds to checking for the martingale property of the discounted asset price;
- $g(x) = e^{\langle y\theta,x\rangle}$, $y \in \mathbb{R}$ corresponds to calculating expectations of the type $\mathbb{E}^x[S_t^y]$ which are relevant for evaluation of power utility and determining the time of "moment explosions."
- $g(x) = e^{\langle u,x\rangle}$, $u \in \mathbb{C}^d$ corresponds to Fourier methods for the pricing of European contingent claims.

For a more detailed account of the literature on affine processes in financial mathematics, we refer to Duffie, Filipović and Schachermayer (2003), Section 13; for an easy-to-read introduction to discounting and pricing techniques (using the Fourier–Laplace transform), we refer to Filipović and Mayerhofer (2009), Section 4. Let us also remark that already Duffie, Filipović and Schachermayer (2003), Section 11, gives sufficient conditions on an affine process such that the pricing operator Q_{T-t} is well defined, but the results only apply to the state space $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and conditions are less general than the ones we obtain.

To deal with the discounting term in (3.1) we use the extension-of-state-space approach outlined in Duffie, Filipović and Schachermayer [(2003), Section 11.2]. We define the extended state space $\widetilde{D} := D \times \mathbb{R}$. Let $(a, \alpha, b, \beta, m(d\xi), \mu(d\xi))$ be the parameters of X in the sense of Theorem 2.4. Following Duffie, Filipović and Schachermayer [(2003), Section 11.2], we have that $Z := (X, Y)$ where $Y_t := y + \int_0^t L(X_s)\,ds$ is an affine process on $\widetilde{D}$ with parameters $(a', \alpha', b', \beta', m'(d\xi), \mu'(d\xi))$ given by

$$a' = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad \alpha_i' = \begin{pmatrix} \alpha_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad b' = \begin{pmatrix} b \\ l \end{pmatrix}$$

and

$$\beta_i' = \begin{pmatrix} \beta_i \\ \lambda \end{pmatrix}, \qquad i = 1, \dots, d, \beta_{d+1}' = 0,$$

and finally

$$m'(d\xi) = m(d\xi) \times \delta_0(d\xi'), \qquad \mu_i'(d\xi) = \mu_i(d\xi) \times \delta_0(d\xi'),$$

where $\delta_0(d\xi')$ denotes the unit mass at 0. Let $F(u)$ and $R(u)$ be the functions associated with X through Proposition 2.8, and let $q \in \mathbb{C}$. Then we can introduce the new functions

$$F'(u,q) := F(u) + lq, \qquad R'(u,q) = R(u) + \lambda q,$$

which are related to the functions (F_Z, R_Z) of the extended process Z in the way that $R_Z = (R'(u,q), 0)$, while $F_Z(u,q) = F'(u,q)$. We consider now solutions $\phi(t,u,q)$ and $\psi(t,u,q)$ of the system

$$\partial_t \phi(t,u,q) = F'\big(\psi(t,u,q), q\big), \qquad \phi(0,u,q) = 0, \tag{3.2a}$$

$$\partial_t \psi(t,u,q) = R'\big(\psi(t,u,q), q\big), \qquad \psi(0,u,q) = u. \tag{3.2b}$$

Note that ψ still is d-dimensional. These solutions are related to the (not necessarily unique) solutions ϕ_Z, ψ_Z of the corresponding $(d+2)$-dimensional system associated with F_Z, R_Z as follows: $\phi_Z(t,(u,q)) = \phi(t,u,q)$ and $\psi_Z(t,(u,q)) = (\psi(t,u,q), q)$.

3.1. *Bond pricing in affine term structure models.* The following result is an immediate consequence of Theorem 2.14. As such it generalizes Duffie, Filipović and Schachermayer [(2003), Proposition 11.2], as well as Filipović and Mayerhofer [(2009), Theorem 4.1].

THEOREM 3.1. *Let $\tau > 0$. The following are equivalent*:

(1) $\mathbb{E}^x[e^{-\int_0^\tau L(s)\,ds}] < \infty$, *for some $x \in D^\circ$.*

(2) *For $q = -1$, there exists a solution $(\widetilde{\phi}, \widetilde{\psi})$ on $[0,\tau]$ to the generalized Riccati differential equations* (3.2a)–(3.2b) *with initial data $u = 0$.*

In any of the above cases, let us define $A(t) := -\phi(t,(0,-1))$, $B(t) := -\psi(t,(0,-1))$ from the unique minimal solution (ϕ,ψ) of equations (3.2a)–(3.2b).[4] *Then the price $P(t,T)$ of a zero-coupon bond is given, for all $0 \le t \le T \le \tau$, and all $x \in D$, by*

$$P(t,T) := \mathbb{E}^x\big[e^{-\int_t^T L(s)\,ds} \mid \mathcal{F}_t\big] = e^{-A(T-t) - \langle B(T-t), X_t\rangle}. \tag{3.3}$$

[4] It follows from Theorem 2.14 that if some solution exists on a nonempty interval $[0,T]$, so does the unique minimal solution.

3.2. *Martingale conditions.* Conditions for the exponentials of affine processes to be martingales have been obtained, for example, in Mayerhofer, Muhle-Karbe and Smirnov (2011). The following result extends known criteria and follows again from Theorem 2.14.

THEOREM 3.2. *Let $\widetilde{S} = e^{-\int_0^t L(X_t)} e^{\langle \theta, X_t \rangle}$ be the discounted asset price. Then the following holds*:

(1) *Suppose that $\theta \in \mathcal{Y}^\circ$, $F(\theta) = l$ and $R(\theta) = \lambda$. Then $(\widetilde{S}_t)_{t \geq 0}$ is a true martingale under any $\mathbb{P}^x$, $x \in D$.*

(2) *Let $x \in D^\circ$. The process $(\widetilde{S}_t)_{t \geq 0}$ is a true $\mathbb{P}^x$-martingale if and only if $\theta \in \mathcal{Y}$, $F(\theta) = l$, $R(\theta) = \lambda$ and $\phi(t, \theta, -1) = 0$ and $\psi(t, \theta, -1) = \theta$ are the unique minimal solutions of the Riccati equations* (3.2a)–(3.2b).

Using (3.1) it is clear that $\widetilde{S}$ is a $\mathbb{P}^x$-martingale if and only if $Q_t g(x) = g(x)$ for all $t \in \mathbb{R}_{\geq 0}$ and with $g(x) = e^{\langle \theta, x \rangle}$. Applying Theorem 2.14 to the extended process Z the above result follows immediately.

3.3. *Moment explosions.* Here we set $L = 0$ for simplicity. It is well understood that the existence of moments $\mathbb{E}[S_t^y]$ with $y \in \mathbb{R}$ is intimately connected to the shape of the implied volatility surface derived from the prices of options on the underlying S; cf. Lee (2004), Keller-Ressel (2011). Of particular interest is the time of moment explosion, that is, the quantity

$$T_+(y) = \sup\{t \geq 0 : \mathbb{E}[S_t^y] < \infty\}.$$

Applying again Theorem 2.14 we obtain the following:

PROPOSITION 3.3. *Let $S = \exp \langle \theta, X \rangle$ with $\theta \in \mathcal{Y}$, and let $y \in \mathbb{R}$.*

(1) *If $y\theta \in \mathcal{Y}^\circ$, then $T_+(y)$ is the maximal lifetime of the solution (p, q) of the extended Riccati system.*

(2) *If $y\theta \in \mathcal{Y}$, then $T_+(y)$ is the maximal lifetime of the unique minimal solution (p, q) of the extended Riccati system. If $y\theta \notin \mathcal{Y}$, then $T_+(y) = 0$.*

Related applications include the approximation of more complicated payoff functions by "power payoffs" [see Cheridito and Wugalter (2012)] and portfolio optimization involving power utility; see Muhle-Karbe (2009) and the references quoted therein.

3.4. *Option pricing.* In general, European option payoffs are nonlinear functions that do not fall under the setup of the previous subsction. Numerically expensive Monte Carlo simulations may be avoided by the method of Fourier pricing, if the characteristic function (or Fourier–Laplace transform) is given in closed form; cf. Carr and Madan (1999). This is the case for affine processes, and the key for applying Fourier pricing is our Theorem 2.26 on complex exponential moments. We provide here an extension of Theorem 10.5 from the book of Filipović [(2009), Chapter 10], which has been written in the context of affine diffusions, where certain simplifications occur (most importantly $\mathcal{Y}^\circ = \mathbb{R}^d$). For general affine processes with jumps we have to impose some stronger assumptions and obtain the following result. To allow for multi-asset options, we consider a generic payoff $g : D \to \mathbb{R}$ depending on all components of the underlying factor process X. In typical applications g will be of the more specific form $g(x) = h(e^{\langle \theta, x\rangle})$ with $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which can be accomodated in the theorem below by setting $q = 1$; see also Filipović [(2009), Theorem 10.6].

THEOREM 3.4. *Let X be an affine process satisfying Assumption 2.24. Assume there exists a $d \times q$ matrix K such that the payoff function g satisfies*

$$g(x) = \int_{\mathbb{R}^d} e^{\langle v + iK\lambda, x\rangle} \widetilde{g}(\lambda)\, d\lambda \tag{3.4}$$

for some integrable function $\widetilde{g} : \mathbb{R}^q \to \mathbb{C}$, $q \leq d$ and with $v \in \mathcal{Y}^\circ$. Suppose that (3.2a)–(3.2b) *has solutions on $[0, \tau]$ for initial data $u = 0$ and $u = v$, which stay in $\mathcal{Y}^\circ$ for all $t \leq \tau$. Then we have*

$$\mathbb{E}^x\big[e^{-\int_t^T L(s)\,ds} g(X_T)\big] = \int_{\mathbb{R}^q} e^{\phi(T-t, v+iK\lambda) + \langle \psi(T-t, v+iK\lambda), X(t)\rangle} \widetilde{g}(\lambda)\, d\lambda,$$

where (ϕ, ψ) are the unique solutions of (3.2a)–(3.2b) *with complex initial data $v + iK\lambda$.*

3.5. *Remarks on jump behavior and examples.* As discussed in the Introduction, a main contribution of this article is that the results apply to conservative affine processes with completely general jump measures. The condition that the jump measures possess exponential moments of all orders that is imposed in Spreij and Veerman (2010) and Cheridito and Wugalter (2012) is typically not fulfilled in financial modeling. Considering, for example, the jump measures of the models discussed in Cont and Tankov (2004), Chapter 4, the condition is satisfied only for the Merton model, but not for the Kou, variance gamma, normal inverse Gaussian, tempered stable and generalized hyperbolic models.

For some affine processes with jumps, the solution of the Riccati equations is known explicitly. In this case, even when not all exponential moments of the jump measures exist, ad-hoc arguments based on analyzing the singularities of the characteristic function can be used to find sufficient conditions for the validity of an affine transform formula; see Nicolato and Venardos (2003) for an example of this approach. While this ad-hoc approach does not give a satisfactory answer on the connection between exponential moments and solutions to the Riccati equations in general, it can be sufficient for applications. However, as the following examples illustrate, several models proposed in the literature on financial mathematics are based on affine processes, for which the Riccati equations do not allow for explicit solutions. In these cases previous results do not apply and also the ad-hoc approaches fail. Hence, Theorems 2.14 and 2.26 are essential for the applications outlined in the previous sections and cannot replaced by simpler arguments or existing results.

EXAMPLE 3.5. Wu (2011) models the S&P 500 index as

$$S_t = S_0 \exp(L_{\int_0^t v_u\,du}), \qquad t \in [0, T],$$

where L is a Lévy process of unit variance at unit time, and a time-change is induced by a general $\mathbb{R}_{\geq 0}$-valued affine process v_t-independent of L—with functional characteristics F, R; see Definition 5.3.[5] We assume a riskless rate of return r and denote the log-returns process by $Y_t := \log(S_t/S_0) = L_{\int_0^t v_u\,du}$. Writing g for the characteristic exponent of L, we have

$$\mathbb{E}[e^{iuY_t} \mid v_0 = v] = e^{iu\theta t}\mathbb{E}[e^{g(iu)\int_0^t v_u\,du}] = e^{iu\theta t + \phi(t, g(iu)) + v\psi(t, g(iu))},$$

where (ϕ, ψ) satisfy

$$\frac{\partial}{\partial t}\phi = F(\psi), \qquad \frac{\partial}{\partial t}\psi = R(\psi) + \zeta, \qquad \psi(0) = \phi(0) = 0 \tag{3.5}$$

with $\zeta = g(iu)$. Under the risk-neutral measure, $e^{-rt}S_t$ must be a martingale on $[0, T]$, whence

$$\mathbb{E}[e^{L_{\int_0^t v_u\,du}}] < \infty,$$

for each $t \in [0, T]$. Theorem 2.14 implies that the Riccati equations (3.5) with $\zeta = g(1)$ allow a minimal solution (p, q) on $[0, T]$. Furthermore, if $q(t)$ lies in $\mathcal{Y}^\circ$ for each $t \in [0, T]$, an application of Theorem 2.26 extends the validity of the affine property (1.1) to complex moments $u \in (1 + \varepsilon) + i\mathbb{R}$, where $\varepsilon > 0$. Hence the way is paved for pricing contingent claims on S by using, for example, the Fourier pricing technique.

EXAMPLE 3.6. Schneider, Sögner and Veza (2010) propose a model for pricing credit default swaps (CDS), where the hazard rate is a linear functional of an affine process (η, γ), given under the risk-neutral measure by

$$d\eta_t = \kappa_\eta(\gamma_t - \eta_t)\,dt + \sigma_\eta\sqrt{\eta_t}\,dW_{\eta,t} + dZ_t^1,$$

$$d\gamma_t = \kappa_\gamma(\zeta_\gamma - \gamma_t)\,dt + \sigma_\gamma\sqrt{\gamma_t}\,dW_{\gamma,t} + dZ_t^2.$$

The two components are correlated via the instantaneous drift and by simultaneous jumps of the compound Poisson process Z_t. The jump-size distributions of

[5] Wu (2011) also specifies a separate drift term, which we absorb into the drift of the Lévy process L.

the two components are assumed to be independent and exponentially distributed. Also here, a closed-form expression for the characteristic function of (η, γ) is not available. However, due to the exponential distribution of jump sizes, the domain $\mathcal{Y}^\circ$ takes a particular, simple form

$$\mathcal{Y}^\circ = (-\infty, \mu_\eta) \times (-\infty, \mu_\gamma),$$

where μ_η, μ_γ are the expected jump sizes of η, γ, respectively. In this case, one first produces a numerical solution of the extended Riccati system on $[0, T]$. By construction, this solution will lie in $\mathcal{Y}^\circ$. Combining this approximate solution with a global error bound we can find a $T' \leq T$ such that also the exact solution must exist and stay in $\mathcal{Y}^\circ$ on $[0, T']$. Theorem 2.14 then yields the existence of the associated real exponential moments. Having this solution, one can proceed to solve the ODE with complex initial data. Existence of these solutions and the validity of the corresponding affine transform formula is guaranteed by Theorem 2.26, and Theorem 3.4 can be used for Fourier pricing of contingent claims in this case of credit default swaps. An extension to state-dependent jump behavior is straightforward and can be similarly dealt by using Theorem 2.26.

We give a final example of an asset model for optimal portfolio choice with affine factors which exhibit a nontrivial correlation structure:

EXAMPLE 3.7. Leippold and Trojani (2010) propose an affine model (Y, X), where $Y_{i,t+h} - Y_{i,t} = \log(S_{i,t+h}/S_{i,t})$ $(i = 1, \dots, d)$ are log-returns for assets S_i $(i = 1, \dots, d)$, and X is a general $d \times d$ positive semidefinite affine jump-diffusion. They specify Y, X as a solution to the SDE

$$\begin{aligned} dY_t &= \left[r\mathbf{1} + X_t\eta - \tfrac{1}{2}\operatorname{diag}(X_t)\right]dt + \sqrt{X_t}\,dZ_t, \\ dX_t &= \left(\Omega\Omega^\top + MX_t + X_tM^\top\right)dt + \sqrt{X_t}\,dB_tQ + Q^\top dB_t^\top\sqrt{X_t} + dJ_t. \end{aligned}$$

Here $\operatorname{diag}(X_t) = (X_{t,11}, \dots, X_{t,dd})^\top$, $\mathbf{1} = (1, \dots, 1)^\top$, both Z and B are $d \times d$ standard Brownian motions, with a certain correlation structure defined by a correlation parameter $\rho \in \mathbb{R}^d$; see Leippold and Trojani (2010) for details. Moreover, J is a pure jump-process independent of (B, W), whose jump intensity is an affine function of X_t. The parameters are given by $\eta \in \mathbb{R}^d$ and M, Ω, Q are $d \times d$ matrices satisfying the constraint $\Omega\Omega^\top - (d-1)Q^\top Q \in S_d^+$, which guarantees a weak solution (Y, X) to the above SDE. In this model, closed-form solutions for the associated Riccati equations exist only in the absence of jumps in X (i.e., $J = 0$). Leippold and Trojani (2010) consider an investor with CRRA utility of terminal wealth w_T, trading in each of these asset S_i and with riskless investment oportunity at constant rate $r > 0$. It turns out that his/her value function is given by

$$V(t, w_t, X_t) = \frac{w_t^{1-\gamma}}{1-\gamma}\exp\bigl(\operatorname{tr}\bigl(A(T-t)X_t\bigr) + B(T-t)\bigr),$$

while the vector of optimal portfolio weights for the risky assets equals $\pi^*(t) = (\eta + 2A(T-t)Q^\top \rho)/\gamma$. The functions A, B satisfy a matrix-valued Riccati equation, and Leippold and Trojani (2010) make the salient assumption that (a) the value function is well defined at the optimal trading policy $\pi^*(t)$, which amounts to assuming the existence of exponential moments of the process (X, Y) for a certain initial value; (b) the associated extended Riccati equations are (uniquely) solvable and give the value of these exponential moments. Theorem 2.14 in this paper now gives sufficient and necessary conditions that allows us to check the validity of these assumptions.

4. Proofs for real moments of affine processes.

4.1. *Decomposability and dependency on the starting value.* Definition 2.2 of an affine process immediately implies a decomposability property of the laws $\mathbb{P}^x$ on the path space; see also Duffie, Filipović and Schachermayer [(2003), Theorem 2.15]. As in Duffie, Filipović and Schachermayer [(2003), Definition 2.14], we write $\mathbb{P} \star \mathbb{P}'$ for the image of $\mathbb{P} \times \mathbb{P}'$ under the measurable mapping $(\omega, \omega') \mapsto \omega + \omega' : (\Omega \times \Omega, \mathcal{F} \times \mathcal{F}) \to (\Omega, \mathcal{F})$.

PROPOSITION 4.1. *Let X be an affine process with state space D. Its probability laws $\mathbb{P}^x$ satisfy the following decomposability property: Suppose that x, ξ and $x + \xi$ are in D. Then*

$$\mathbb{P}^x \star \mathbb{P}^\xi = \mathbb{P}^0 \star \mathbb{P}^{x+\xi}. \tag{4.1}$$

PROOF. Write $\mathbf{u} = (u^1, \dots, u^N)$ for an ordered set of points $u^k \in \mathcal{U}$. Choosing some finite sequence $0 \le t_1 \le \cdots \le t_N$ in $\mathbb{R}_{\ge 0}$, define

$$f(x, \mathbf{u}) = \mathbb{E}^x\left[\exp\left(\sum_{k=1}^N \langle X_{t_k}, u^k\rangle\right)\right] \qquad \big(x \in D, (u^1, \dots, u^N) \in \mathcal{U}^N\big),$$

that is, $f(x, \mathbf{u})$ is the joint characteristic function of $(X_{t_1}, \dots, X_{t_n})$ under $\mathbb{P}^x$. Applying the affine property (2.3) recursively, we obtain

$$f(x, \mathbf{u}) = \exp\big(p(\mathbf{u}) + \langle x, q(\mathbf{u})\rangle\big), \tag{4.2}$$

where $p(\mathbf{u}) = p_1$ and $q(\mathbf{u}) = q_1$, with

$$p_{k-1} = \phi\big(t_k - t_{k-1}, q_k + u^k\big) + p_k, \qquad p_N = 0, \tag{4.3}$$

$$q_{k-1} = \psi\big(t_k - t_{k-1}, q_k + u^k\big), \qquad q_N = 0. \tag{4.4}$$

From (4.2) we derive that

$$f(x, \mathbf{u}) f(\xi, \mathbf{u}) = f(0, \mathbf{u}) f(x + \xi, \mathbf{u})$$

for all $\mathbf{u} = (u^1, \dots, u^N) \in \mathcal{U}^N$. Since the distribution of a stochastic process is determined by its finite-dimensional marginal distributions, this equality is equivalent to (4.1). $\square$

In the following, we set

$$g(t, y, x) = \mathbb{E}^x\big[e^{\langle y, X_t\rangle}\big] = \int_D e^{\langle y, \xi\rangle} p_t(x, d\xi),$$

for all $(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d$ and $x \in D$. Note that $g(t, y, x)$ is always strictly positive, but might take the value $+\infty$. By approximating $g(t, y, x)$ monotonically from below by bounded functions and using the Chapman–Kolmogorov equation, we derive that

$$g(t+s, y, x) = \int_D g(t, y, \xi) p_s(x, d\xi) \tag{4.5}$$

holds for all $t, s \in \mathbb{R}_{\geq 0}$, $y \in \mathbb{R}^d$ and $x \in D$, where $+\infty$ is allowed on both sides and in the integrand. The following lemma concerns the role of the starting value $X_0 = x$ of the affine process with regards to finiteness of exponential moments.

LEMMA 4.2. *Let X be an affine process on D, and let $(T, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d$. Then the following holds*:

(a) $\mathbb{E}^0[e^{\langle y, X_T\rangle}] = \infty$ *implies* $\mathbb{E}^x[e^{\langle y, X_T\rangle}] = \infty$ *for all* $x \in D^\circ$;
(b) $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ *for some* $x \in D^\circ$ *implies* $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ *for all* $x \in D$;
(c) $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ *for all* $x \in D$ *implies* $\mathbb{E}^x[e^{\langle y, X_t\rangle}] < \infty$ *for all* $t \in [0, T], x \in D$.

PROOF. As before we set $g(t, y, x) = \mathbb{E}^x[e^{\langle y, X_t\rangle}]$, which takes values in the extended positive half-line $(0, \infty]$. Using the decomposability property of X (cf. Proposition 4.1), we have

$$g(t, y, x) g(t, y, \xi) = g(t, y, 0) g(t, y, x+\xi)$$

for all $x, \xi \in D$ for which $x + \xi \in D$. Let x_* be an arbitrary point in D. Setting $x = \xi = x_*/2$ it follows that

$$g\left(t, y, \frac{x_*}{2}\right)^2 = g(t, y, 0) g(t, y, x_*). \tag{4.6}$$

We conclude that $g(t, y, 0) = \infty$ implies $g(t, y, \frac{x_*}{2}) = \infty$; hence (a) is verified for the point $x = x_*/2$. We introduce the affine process $X_{2,t} := X_t - x_*/2$ with state-space $D_2 := D - x_*/2$. Clearly $0 \in D_2$ and $g_2(t, y, z) := \mathbb{E}[\exp(\langle y, X_{2,t}\rangle) | X_{2,0} = z] = e^{-\langle y, x_*\rangle/2} g(t, y, z + x_*/2)$. Since $g_2(t, y, 0) = \infty$ and $x_*/2 \in D_2$, we may apply the same argument as above to g_2 instead of g and obtain that $g_2(t, y, x_*/4) = \infty$. But this means $g(t, y, 3x_*/4) = \infty$, and by iterating this procedure, we obtain that for each $k \geq 1$ $g(t, y, x_*(1 - 2^{-k})) = \infty$. Since x_* was an arbitrary point in the convex set D, (a) follows. We prove (b) by contraposition: Assume that $g(t, y, x) = \infty$ for some $x \in D$, and introduce the affine process $Z_t := X_t - x$ with state-space $D_x := D - x$. Clearly $0 \in D_x$ and $\mathbb{E}[e^{\langle y, Z_t\rangle} | Z_0 = 0] = e^{\langle y,x\rangle} g(t, y, x) = \infty$, by assumption. Applying (a) to the process Z yields that

$$g(t, y, \xi) = e^{-\langle y,\xi\rangle}\mathbb{E}\big[e^{\langle y,Z_t\rangle} | Z_0 = \xi - x\big] = \infty$$

for all $\xi \in D$, which completes the proof of (b).

To show (c) pick an arbitrary $x \in D^\circ$ and $\varepsilon > 0$. Since X has càdlàg paths, we can find $\delta > 0$ such that $\mathbb{P}^x(\|X_t - x\| < \varepsilon) \geq \frac{1}{2}$ for all $t \leq \delta$. With $p_t(x, d\xi)$ denoting the transition kernel of X, we can rewrite this as $p_t(x, B_\varepsilon(x)) \geq \frac{1}{2}$ for all $t \leq \delta$. We show assertion (c) for $t \in [T - \delta, T]$; the general case follows then by iteration. By (4.5),

$$g(T, y, x) = \int_D g(t, y, \xi) p_{T-t}(x, d\xi)$$

holds for all $(t, y) \in [0, T] \times \mathbb{R}^d$. By assumption, the left-hand side is finite, and we want to show that also $g(t, y, \xi)$ is finite for all $\xi \in D$ and $t \in [T - \delta, T]$. Assume for a contradiction that $g(t, y, \xi^*) = \infty$ for some $t \in [T - \delta, T]$ and $\xi^* \in D$. Then by Lemma 4.2(b) $g(t, y, \xi) = \infty$ for all $\xi \in D^\circ$. But $p_{T-t}(x, D^\circ) \geq p_{T-t}(x, B_\varepsilon(x)) \geq \frac{1}{2}$, and we conclude that $g(T, y, x) = \infty$, which is a contradiction. □

4.2. *From moments to Riccati equations*. In this section we prove Theorem 2.14(a), except for the minimality property of the Riccati solution.

LEMMA 4.3. *Let X be an affine process on D, and let $T \geq 0$. Suppose that for some $x \in D^\circ$ and $y \in \mathbb{R}^d$, it holds that $\mathbb{E}^x[e^{\langle y,X_T\rangle}] < \infty$. Then $y \in \mathcal{Y}$, and the following holds*:

(a) *There exist functions $t \mapsto p(t, y) \in \mathbb{R}$ and $t \mapsto q(t, y) \in \mathbb{R}^d$ such that (2.14) holds for all $x \in D$, $t \in [0, T]$.*

(b) $\mathbb{E}^x[e^{\langle q(T-t,y),X_t\rangle}] < \infty$ *for all $t \in [0, T]$ and*

$$\mathbb{E}^x\big[e^{\langle y,X_T\rangle} | \mathcal{F}_t\big] = \exp\big(p(T - t, y) + \langle q(T - t, y), X_s\rangle\big) \qquad \text{for all } x \in D. \tag{4.7}$$

(c) *The functions $p(t, y), q(t, y)$ satisfy the semi-flow equations*

$$p(T, y) = p(T - t, y) + p\big(t, q(T - t, y)\big), \qquad p(0, y) = 0, \tag{4.8a}$$

$$q(T, y) = q\big(t, q(T - t, y)\big), \qquad q(0, y) = y, \tag{4.8b}$$

for all $t \in [0, T]$.

PROOF. From Lemma 4.2(c) it follows that $\mathbb{E}^x[e^{\langle y,X_t\rangle}] < \infty$ for all $(t,x) \in [0,T] \times D$. Fix $t \in [0,T]$, and write $g(t,y,x) = \mathbb{E}^x[e^{\langle y,X_t\rangle}]$. Then by Proposition 4.1 $g(t,y,x)$ satisfies the functional equation (4.1). Since $g(t,y,0) > 0$ there exists $p(t,y) \in \mathbb{R}$ such that $g(t,y,0) = e^{p(t,y)}$. Set $h(t,y,x) = e^{-p(t,y)} g(t,y,x)$. Then $h(t,y,x)$ is finite for all $x \in D$ and satisfies Cauchy's functional equation

$$h(t,y,x)h(t,y,\xi) = h(t,y,x+\xi), \qquad x, \xi, x+\xi \in D.$$

We conclude that there exists $q(t,y) \in \mathbb{R}^d$ such that $h(t,y,x) = e^{\langle q(t,y),x\rangle}$ for all $x \in D$, and we have shown (2.14).

To show equation (4.7) note that by the Markov property of X,

$$\mathbb{E}^x\big[\mathbf{1}_{\{|X_t|\le n\}} e^{\langle y,X_t\rangle} | \mathcal{F}_s\big] = \int_{\{\xi\in D:\,|\xi|\le n\}} e^{\langle y,\xi\rangle} p_{t-s}(X_s, d\xi)$$

holds for all $n \in \mathbb{N}$, $x \in D$, $0 \le s \le t$. Using dominated convergence we may take the limit $n \to \infty$ and obtain equation (4.7) from (2.14). Taking (unconditional) expectations in (4.7) yields

$$\begin{aligned}\exp&\big(p(T,y) + \langle x, q(T,y)\rangle\big)\\ &= \exp\big(p(T-t,y) + p\big(t, q(T-t,x)\big) + \big\langle x, q\big(t, q(T-t,y)\big)\big\rangle\big),\end{aligned}$$

for all $t \ge 0$ and $x \in D$. Since D contains 0 and linearly spans $\mathbb{R}^d$, the semi-flow equations (4.8) follow. $\square$

Note that if $t \mapsto p(t,y)$ and $t \mapsto q(t,y)$ are differentiable with derivatives $F(y)$ and $R(y)$ at zero, then it follows by differentiating the semi-flow equations (4.8) that (p,q) is a solution of the extended Riccati system (2.11). The main difficulty thus is showing the differentiability of p and q. This is very similar to the *regularity problem* for affine processes, where the same question is asked regarding the functions $\phi(t,u)$ and $\psi(t,u)$ in Definition 2.2. Several solutions of the regularity problem have been given; see, for example, Keller-Ressel, Schachermayer and Teichmann (2011) and Keller-Ressel, Schachermayer and Teichmann (2013). Here we adapt the approach of Cuchiero (2011) to our setting.

We enlarge the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that it supports $d+1$ independent copies of the affine process X, which we denote by $X^0, \dots, X^d$. Without loss of generality it can be assumed that $X = X^0$. In what follows we will use the convention that upper indices correspond to the different instances of the process X, while lower indices correspond to the coordinate projections of a single process. For a vector $x^i \in D$ denote by $\mathbb{P}^{x^i}$ the probability $\mathbb{P}$ conditional on $\{X^i(0) = x^i\}$; that is, the process X^i starts at the point x^i with $\mathbb{P}^{x^i}$-probability 1. Similarly for an ordered set $\mathbf{x} = (x^0, \dots, x^d)$ of points in D, we denote by $\mathbb{P}^{\mathbf{x}}$ the probability $\mathbb{P}$ conditional on $\{(X^0(0) = x^0) \wedge \cdots \wedge (X^d(0) = x^d)\}$; that is, the processes $X^0, \dots, X^d$ start at the points $x^0, \dots, x^d$ respectively with $\mathbb{P}^{\mathbf{x}}$-probability 1.

LEMMA 4.4. *Let $X^0, \dots, X^d$ be $d+1$ independent copies of the affine process X. Furthermore let $\mathbf{x} = (x^0, \dots, x^d)$ be $d+1$ affinely independent points in D. Define the matrix-valued random function*

$$\Xi(t; \mathbf{x}, \omega) = \begin{pmatrix} 1 & X_1^0(t,\omega) & \cdots & X_d^0(t,\omega) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_1^d(t,\omega) & \cdots & X_d^d(t,\omega) \end{pmatrix}.$$

Then there exists $\delta > 0$ such that

$$\mathbb{P}^{\mathbf{x}}\big(\det \Xi(t; \mathbf{x}) \neq 0 \textit{ for all } 0 \le t \le \delta\big) > \tfrac{1}{2};$$

that is, $t \mapsto \Xi(t; \mathbf{x})$ stays regular on $[0, \delta]$ with $\mathbb{P}^{\mathbf{x}}$-probability at least $\frac{1}{2}$.

PROOF. Since $(x^0, \dots, x^d)$ are affinely independent, and $X^i(0) = x^i$ for all $i \in \{0, \dots, d\}$ with $\mathbb{P}^{\mathbf{x}}$-probability one, the matrix $\Xi(0; \mathbf{x})$ is regular $\mathbb{P}^{\mathbf{x}}$-almost surely. Define $a_t = \inf_{s \in [0,t]} |\det \Xi(s; \mathbf{x})|$. Since the processes X^i are right-continuous, also $\det \Xi(s; \mathbf{x})$ is, and hence even a_t. By dominated convergence also $b_t = \mathbb{P}^{\mathbf{x}}(a_t > 0)$ is right-continuous and has the starting value $b_0 = 1$. We conclude that there exists some $\delta > 0$ such that $b_\delta > \frac{1}{2}$, which completes the proof. □

The following proposition settles Theorem 2.14(a) apart from the minimality property of (p, q) as solutions of the extended Riccati system. The key ideas in the subsequent proof come from Cuchiero (2011), proofs of Lemma 1.5.3, Theorem 1.5.4.

PROPOSITION 4.5. *Let X be an affine process on D, and let $T \ge 0$. Let $y \in \mathbb{R}^d$, and suppose that $\mathbb{E}^x[e^{\langle y, X_T \rangle}] < \infty$ for some $x \in D^\circ$. Then $y \in \mathcal{Y}$ and there exist a solution (p, q) up to time T of the extended Riccati system (2.11), such that (2.14) holds for all $x \in D$, $t \in [0, T]$.*

PROOF. Recall that we are working on an extended probability space that supports $d+1$ independent copies $(X^0, \dots, X^d)$ of X. Let $\mathbf{x} = (x^0, \dots, x^d)$ be $d+1$ affinely independent points in D. By Theorem 2.4 each X^i is a $\mathbb{P}^{\mathbf{x}}$-semi-martingale with canonical semimartingale representation

$$X_t^i = x^i + \int_0^t b\big(X_{s-}^i\big)\, ds + N_t^i + \int_0^t \int_{\mathbb{R}^d} \big(\xi - h(\xi)\big) J^i(\omega; ds, d\xi), \tag{4.9}$$

where N_t^i is a local martingale and $J^i(\omega; dt, d\xi)$ is the Poisson random measure associated to the jumps of X^i with predictable compensator $K(X_{t-}^i, d\xi)\, dx$.

By Lemma 4.3 we know that

$$\mathbb{E}^{\mathbf{x}}\big[e^{\langle y, X_T^i \rangle} | \mathcal{F}_t\big] = \exp\big(p(T-t, y) + \langle q(T-t, y), X_t^i \rangle\big) \tag{4.10}$$

for each $i \in \{0, \dots, d\}$, $t \in [0, T]$. Let us denote $M_t^i = \mathbb{E}^{\mathbf{x}}[e^{\langle y, X_T^i \rangle} | \mathcal{F}_t]$. Clearly, each $t \mapsto M_t^i$ is a $\mathbb{P}^{\mathbf{x}}$-martingale for $t \le T$ and for each $i \in \{0, \dots, d\}$. Taking logarithms and arranging the equations in matrix form, we get

$$\begin{pmatrix} \log M_t^0(\omega) \\ \vdots \\ \log M_t^d(\omega) \end{pmatrix} = \begin{pmatrix} 1 & X_1^0(t,\omega) & \cdots & X_d^0(t,\omega) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_1^d(t,\omega) & \cdots & X_d^d(t,\omega) \end{pmatrix} \cdot \begin{pmatrix} p(T-t,y) \\ q_1(T-t,y) \\ \vdots \\ q_d(T-t,y) \end{pmatrix}, \tag{4.11}$$

and recognize on the right-hand side the matrix $\Xi(t; \mathbf{x}, \omega)$ from Lemma 4.4. The latter allows us to conclude that there exists a set $A \subset \Omega$ with $\mathbb{P}^{\mathbf{x}}(A) > \frac{1}{2}$ and some $\delta > 0$ such that $\Xi(t; \mathbf{x}, \omega)$ is invertible for all $t \in [0, \delta]$ and $\omega \in A$. Hence for $T' = T \wedge \delta$ we obtain

$$\begin{aligned} &\begin{pmatrix} 1 & X_1^0(t,\omega) & \cdots & X_d^0(t,\omega) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_1^d(t,\omega) & \cdots & X_d^d(t,\omega) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \log M_t^0(\omega) \\ \vdots \\ \log M_t^d(\omega) \end{pmatrix} \\ &\qquad = \begin{pmatrix} p(T'-t,y) \\ q_1(T'-t,y) \\ \vdots \\ q_d(T'-t,y) \end{pmatrix}, \end{aligned} \tag{4.12}$$

for all $t \in [0, T']$. All processes occurring on the left-hand side of equation (4.12) are semimartingales, hence also the right-hand side consists row-by-row of semimartingales for all $t \in [0, T']$. Since they are deterministic, the functions $t \mapsto p(t, y)$ and $t \mapsto q(t, y)$ are of finite variation on $[0, T']$. This implies in particular that they are almost everywhere differentiable and can be written as

$$p(T'-t,y) - p(T',y) = -\int_0^t dp(T'-s,y), \tag{4.13a}$$

$$q(T'-t,y) - q(T',y) = -\int_0^t dq(T'-s,y). \tag{4.13b}$$

Applying Itô's formula to the martingales $M_t^{i,y}$, we obtain

$$\begin{aligned} M_t^{i,y} = M_0^{i,y} &+ \int_0^t M_{s-}^{i,y} \bigl(-dp(T'-s,y) + \langle -dq(T'-s,y), X_{s-}^i \rangle\bigr) \\ &+ \int_0^t M_{s-}^{i,y} \Bigl\{ \langle q(T'-s,y), b(X_{s-}^i) \rangle \\ &\qquad + \frac{1}{2} \langle q(T'-s,y), a(X_{s-}^i) q(T'-s,y) \rangle \\ &\qquad + \int_D \bigl(e^{\langle q(T'-s,y), \xi \rangle} - 1 - \langle q(T'-s,y), h(\xi) \rangle\bigr) \end{aligned}$$

$$\times K\big(X^i_{s-}, d\xi\big)\bigg\}\, ds$$

$$+ \int_0^t M^{i,y}_{s-}\langle q(T'-s, y), dN^i_s\rangle$$

$$+ \int_0^t \int_D M^{i,y}_{s-}\big(e^{\langle q(T'-s,y),\xi\rangle} - 1 - \langle q(T'-s, y), h(\xi)\rangle\big)$$

$$\times \big(J(\omega, ds, d\xi) - K\big(X^i_{s-}, d\xi\big)\, ds\big),$$

for all $i \in \{0, \dots, d\}$. On the right-hand side, the last two terms are local martingales and the other terms are of finite variation. Hence the finite variation terms have to sum up to 0. Rewriting in terms of the functions $F(y)$ and $R(y)$ this means that

$$\begin{aligned} &-dp\big(T'-t, y\big) + \big\langle -dq\big(T'-t, y\big), X^i_{s-}\big\rangle \\ &\quad = F\big(q\big(T'-t, y\big)\big)\, dt + \big\langle X^i_{s-}, R\big(q\big(T'-t, y\big)\big)\big\rangle\, dt \end{aligned}$$

holds for almost all $t \in [0, T']$ $\mathbb{P}^{\mathbf{x}}$-a.s. Inserting into (4.13) and using the regularity of the matrix $\Xi(t, \mathbf{x}, \omega)$ on $[0, T']$, this yields

$$p\big(T'-t, y\big) - p\big(T', y\big) = -\int_0^t F\big(q\big(T'-s, y\big)\big)\, ds, \tag{4.14}$$

$$q\big(T'-t, y\big) - q\big(T', y\big) = -\int_0^t R\big(q\big(T'-s, y\big)\big)\, ds. \tag{4.15}$$

Applying the fundamental theorem of calculus, we have shown that (p, q) is a solution to the extended Riccati system (2.11) up to $T' = T \wedge \delta$, where δ was given by Lemma 4.4. To show the general case we conclude with an induction argument. Suppose that (p, q) are solutions of the extended Riccati system up to $T_k = T \wedge (k\delta)$. We show that they can be extended to solutions up to $T_{k+1} = T \wedge ((k+1)\delta)$. Set $\Delta_k = T_{k+1} - T_k$; clearly $\Delta_k \le \delta$. By Lemma 4.2, $\mathbb{E}^x[e^{\langle y, X_T\rangle}] < \infty$ implies that $\mathbb{E}^x[e^{\langle y, X_{T_{k+1}}\rangle}] < \infty$, and by Lemma 4.3 we have that $\mathbb{E}^x[e^{\langle q(y,T_k), X_{\Delta_k}\rangle}] < \infty$. Set $y' = q(y, T_k)$. Then, proceeding exactly as in the proof above, we obtain

$$\frac{\partial}{\partial t} p\big(t, y'\big) = F\big(q\big(t, y'\big)\big), \qquad p\big(0, y'\big) = 0, \tag{4.16a}$$

$$\frac{\partial}{\partial t} q\big(t, y'\big) = R\big(q\big(t, y'\big)\big), \qquad q\big(0, y'\big) = y' \tag{4.16b}$$

for all $t \in [0, \Delta_k]$. Using the flow property, this is equivalent to

$$\frac{\partial}{\partial t} p(t, y) = F\big(q(t, y)\big), \qquad p(0, y) = 0, \tag{4.17a}$$

$$\frac{\partial}{\partial t} q(t, y) = R\big(q(t, y)\big), \qquad q(0, y) = y \tag{4.17b}$$

for all $t \in [T_k, T_{k+1}]$. By the induction hypothesis (4.17) already holds for all $t \in [0, T_k]$, and we have shown that (p, q) is a solution of the extended Riccati system up to $T_{k+1} = T \wedge \delta(k+1)$. As this holds true for all $k \in \mathbb{N}$, the proof is complete. $\square$

4.3. *From Riccati equations to moments.* Using the result from above, the step from the extended Riccati system to the existence of moments is simple:

PROPOSITION 4.6. *Let X be an affine process taking values in D. Let $y \in \mathcal{Y}$, and suppose that the extended Riccati system (2.11) has a solution $(\widetilde{p}, \widetilde{q})$ that starts at y and exists up to $T \geq 0$. Then $\mathbb{E}^x[e^{\langle y, X_T \rangle}] < \infty$ and (2.14) holds for all $x \in D$, $t \in [0, T]$, where (p, q) is also a solution up to T to (2.11).*

PROOF. Using the solution $(\widetilde{p}, \widetilde{q})$ of the extended Riccati system (2.11), define for $t \in [0, T]$,

$$\widetilde{M}_t^y = \exp\bigl(\widetilde{p}(T-t, y) + \langle \widetilde{q}(T-t, y), X_t \rangle\bigr). \tag{4.18}$$

Applying Itô's formula to $\widetilde{M}_t^y$ and using the semimartingale representation (4.9), we see that

$$\begin{aligned}
\widetilde{M}_t^y = \widetilde{M}_0^y &+ \int_0^t \widetilde{M}_{s-}^y \bigl(-\widetilde{p}(T-s, y) + \langle -\widetilde{q}(T-s, y), X_{s-} \rangle\bigr)\, ds \\
&+ \int_0^t \widetilde{M}_{s-}^y \biggl(\langle \widetilde{q}(T-s, y), b(X_{s-}) \rangle \\
&\qquad + \frac{1}{2} \langle \widetilde{q}(T-s, y), a(X_{s-}) \widetilde{q}(T-s, y) \rangle \\
&\qquad + \int_D \bigl(e^{\langle \widetilde{q}(T-s, y), \xi \rangle} - 1 - \langle \widetilde{q}(T-s, y), h(\xi) \rangle\bigr) K(X_{s-}, d\xi) \biggr)\, ds \\
&+ \int_0^t \widetilde{M}_{s-}^y \langle \widetilde{q}(T-s, y), dN_s \rangle \\
&+ \int_0^t \int_D \widetilde{M}_{s-}^y \bigl(e^{\langle \widetilde{q}(T-s, y), \xi \rangle} - 1 - \langle \widetilde{q}(T-s, y), h(\xi) \rangle\bigr) \\
&\qquad \times \bigl(J(\omega, ds, d\xi) - K(X_{s-}, d\xi)\, ds\bigr).
\end{aligned}$$

The ds-terms can be simplified to

$$\begin{aligned}
&-\widetilde{p}(T-s, y) + \langle -\widetilde{q}(T-s, y), X_{s-} \rangle + F\bigl(\widetilde{q}(T-s, y)\bigr) + \langle R\bigl(\widetilde{q}(T-s, y)\bigr), X_{s-} \rangle \\
&\quad = 0,
\end{aligned}$$

and we conclude that $(\widetilde{M}_t^y)_{t \in [0,T]}$ is a local $\mathbb{P}^x$-martingale for all $x \in D$. It is also strictly positive, and hence it is a $\mathbb{P}^x$-supermartingale. Therefore

$$\mathbb{E}^x\bigl[e^{\langle y, X_T \rangle}\bigr] = \mathbb{E}^x\bigl[\widetilde{M}_T^y\bigr] \leq \widetilde{M}_0^y < \infty$$

for all $x \in D$. The second part of the assertion, and in particular the validity of equation (2.14) follows now by applying Proposition 4.5. $\square$

4.4. *Proof of Theorem* 2.14. Looking at Proposition 4.6 and Proposition 4.5 we see that Theorem 2.14 is almost proved. Only one issue in both parts of the theorem is not answered yet, namely the minimality of (p, q) in (2.14) as minimal (hence unique, see Remark 2.13) solution of the extended Riccati system. We start with the following lemma:

LEMMA 4.7. *Let (p, q) and $(\widetilde{p}, \widetilde{q})$ be given as in Proposition* 4.6. *Then for all $t \in [0, T]$ and $x \in D$,*

$$p(t, y) + \langle q(t, y), x\rangle \le \widetilde{p}(t, y) + \langle \widetilde{q}(t, y), x\rangle.$$

PROOF. Set $M_t^y = \exp(p(T - t, y) + \langle q(T - t, y), x\rangle)$, and define $\widetilde{M}_t^y$ as in (4.18). Then, for each $x \in D$ the process M^y is a $\mathbb{P}^x$-martingale [see (4.10) and below]; $\widetilde{M}^y$ is a $\mathbb{P}^x$-supermartingale, and they satisfy $M_T^y = \widetilde{M}_T^y$. Hence

$$M_t^y = \mathbb{E}^x[M_T^y|\mathcal{F}_t] = \mathbb{E}^x[\widetilde{M}_T^y|\mathcal{F}_t] \le \widetilde{M}_t^y$$

for all $t \in [0, T]$. Taking logarithms the claimed inequality follows. $\square$

PROOF OF THEOREM 2.14. Proof of (a): In view of Remark 2.13 we only need to show that the solution (p, q) of the Riccati system established in Proposition 4.5 is minimal. Let $(\widetilde{p}, \widetilde{q})$ be another solution on $[0, T']$ of the extended Riccati system, $T' \le T$. Then by Proposition 4.6 there exists (p^*, q^*) such that (2.14) holds for all $y \in D$ and $t \in [0, T']$, as is the case for (p, q). By taking logarithms of the respective right-hand sides of (2.14) and by applying Lemma 4.7, we see that

$$p + \langle q, y\rangle = p^* + \langle q^*, y\rangle \le \widetilde{p} + \langle \widetilde{q}, x\rangle,$$

on $[0, T']$ and for all $y \in D$. Hence by Definition 2.11 (p, q) is the minimal solution of the extended Riccati system, and we are done with part (a).

The proof of (b) follows immediately from Lemma 4.7, Definition 2.11 and Remark 2.13. $\square$

5. Proofs for complex moments of affine processes. In this section we show Theorem 2.26 on the existence of complex moments of affine processes, whose state space satisfies Assumption 2.24. The key to the proof is to relate the lifetime of solutions (ϕ, ψ) of the complex Riccati system (2.15a)–(2.15b) and the solutions (p, q) of the extended Riccati system (2.11a)–(2.11b). Unlike in preceding parts of the paper, we only solve for initial values in the interiors $y \in \mathcal{Y}^\circ$ [resp., $u \in S(\mathcal{Y}^\circ)$]. Also, in this section we need more precise knowledge about the restrictions on the parameters, which appear in the Riccati equations.

With S_d^+ we denote the $d \times d$ positive semidefinite matrices. Let $T_+(y)$ [resp., $T_+(u)$] denote the maximal lifetime of $t \mapsto (p(t, y), q(t, y))$ [resp., $t \mapsto$

$(\phi(t,u), \psi(t,u))]$.

PROPOSITION 5.1. *Suppose that Assumption* 2.24 *holds true, and let* $u \in S(\mathcal{Y}^\circ)$ *and* $y = \mathrm{Re}(u)$. *Then* $T_+(u) \geq T_+(y)$.

We split the proof into the two cases covered by Assumption 2.24, a state space D of the form $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and a state space of the form S_d^+. Note that $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and S_d^+ are convex cones. To apply certain results of Volkmann (1973) on multivariate ODE comparison, we introduce the following property:

DEFINITION 5.2. Let $K \subset \mathbb{R}^d$ be a proper closed, convex cone, and denote by $\preceq$ the induced partial order. Let $U \subset \mathbb{R}^d$. A function $f : U \to \mathbb{R}^d$ is called quasimonotone increasing (with respect to K), if for all $y, z \in U$ for which $y \preceq z$ and $\langle y, x\rangle = \langle z, x\rangle$ for some $x \in K$ it holds that $\langle f(y), x\rangle \leq \langle f(z), x\rangle$.

5.1. *State space* $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$. In this section we consider the "canonical state space" $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ from Duffie, Filipović and Schachermayer (2003). We use the index sets $I = \{1, 2, \dots, m\}$ and $J = \{m+1, \dots, d\}$ corresponding to the positive and to the real valued components of D respectively. Accordingly, R_I denotes the function $(R_1, \dots, R_m)$, and similarly R_J is constituted by the last n coordinates of R.

First, we recall the definition of the admissible parameter set for (conservative) affine processes on $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ from Duffie, Filipović and Schachermayer (2003):

DEFINITION 5.3. A set of $\mathbb{R}^d$-vectors $b, \beta^1, \dots, \beta^d$, positive semidefinite $d \times d$ matrices $a, \alpha^1, \dots, \alpha^d$, Lévy measures $m, \mu^1, \dots, \mu^d$ on $\mathbb{R}^d$, is called *admissible for* $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ if and only if

$$
\begin{aligned}
a_{kl} &= 0 && \text{for all } k \in I \text{ or } l \in I,\\
\alpha^j &= 0 && \text{for all } j \in J,\\
\alpha^i_{kl} &= 0 && \text{if } k \in I \setminus \{i\} \text{ or } l \in I \setminus \{i\};\\
b &\in D, &&\\
\beta^i_k - \int \xi_k \mu^i(d\xi) &\geq 0 && \text{for all } i \in I, k \in I \setminus \{i\},\\
\beta^i_k &= 0 && \text{for all } j \in J, k \in I;\\
\int_{|\xi| \leq 1} |\xi_I| m(d\xi) &< \infty, &&\\
\mu^j &= 0 && \text{for all } j \in J,\\
\int_{|\xi| \leq 1} |\xi_{I \setminus \{i\}}| \mu^i(d\xi) &< \infty. &&
\end{aligned}
$$

REMARK 5.4. The matrices $a, \alpha^1, \dots, \alpha^d$ are frequently referred to as *diffusion matrices*, the vectors $b, \beta^1, \dots, \beta^d$ as *drift vectors* and the Lévy measures $m, \mu^1, \dots, \mu^d$ as *jump measures*.

Let $R(y) = (R_1(y), \dots, R_d(y))$ be defined as in Proposition 2.8. The admissibility conditions imply that each $R_1(y), \dots, R_d(y)$ is a convex lower semicontinuous function of Lévy–Khintchine-type. Denoting $\mu^0(d\xi) := m(d\xi)$ we therefore have

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^d : \sum_{i=0}^{d} \int_{|\xi| \geq 1} e^{\langle y, \xi \rangle} \mu^i(d\xi) < \infty \right\}, \tag{5.1}$$

which is the intersection of the effective domains of $F, R_1, \dots, R_d$.

We start with the following crucial lemma:

LEMMA 5.5. *There exists a function g which is finite, nonnegative and convex on $\mathcal{Y}$ such that for all $u \in S(\mathcal{Y}^\circ)$ we have*

$$\operatorname{Re}(\langle \overline{u}_I, R_I(u) \rangle) \leq g(\operatorname{Re} u)(1 + |u_J|^2)(1 + |u_I|^2). \tag{5.2}$$

PROOF. It clearly sufficient to show

$$\operatorname{Re}(\overline{u}_i R_i(u)) \leq g_i(\operatorname{Re}(u))(1 + |u_J|^2)(1 + |u_I|^2)$$

individually for each $i \in I$ and with some nonnegative convex $g_i(\cdot)$ that is finite on $\mathcal{Y}$. In addition we may split $R_i(u)$ into the drift part, the diffusion part, a small-jump part and a large-jump part and show the inequality for each part separately. The drift and the large jump-part are the easiest to deal with. Using the Cauchy–Schwarz inequality we infer the existence of a positive constant C such that

$$\begin{aligned} \operatorname{Re}(\overline{u}_i(\langle \beta^i, u \rangle)) &\leq |u_i|(|\beta_I^i||u_I| + |\beta_J^i||u_I|) \\ &\leq C(1 + |u_J|^2)(1 + |u_I|^2) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \operatorname{Re}\left(\overline{u}_i \int_{|\xi|>1} e^{\langle \xi, u \rangle} \mu^i(d\xi)\right) &\leq |u_i| \int_{|\xi|>1} e^{\langle \xi, \operatorname{Re} u \rangle} \mu^i(d\xi) \\ &\leq \widetilde{g}_i(\operatorname{Re} u)(1 + |u_I|^2) \end{aligned} \tag{5.4}$$

for the large-jump part. Here $\widetilde{g}_i(z) := \int_{|\xi|>1} e^{\langle \xi, z \rangle} \mu^i(d\xi)$ clearly is a nonnegative convex function which is finite on $\mathcal{Y}$. To estimate the diffusion part we have to take into account the admissibility conditions, which tell us that α_{ij}^i is zero if $j \in I \setminus \{i\}$. Thus we obtain

$$\begin{aligned} \operatorname{Re}(\overline{u}_i \langle u, \alpha^i u \rangle) &= \alpha_{ii}^i |u_i|^2 \operatorname{Re} u_i + 2 \operatorname{Re}(|u_i|^2 \alpha_{iJ} u_J) + \operatorname{Re}(\overline{u}_i u_J^\top \alpha_{JJ}^i u_J) \\ &\leq C(1 + (\operatorname{Re} u_I)_+)(1 + |u_J|^2)(1 + |u_I|^2), \end{aligned} \tag{5.5}$$

as desired. The hardest term to estimate is the small-jump part. We follow the proof of Lemma 6.2 in Duffie, Filipović and Schachermayer (2003). As a shorthand notation we introduce $u_{I-} = u_{I\setminus\{i\}}$ and $u_{J+} = u_{J\cup\{i\}}$. First we do a Taylor expansion of the integrand $h(\xi) = e^{\langle \xi, u\rangle} - 1 - \langle \xi_{J+}, u_{J+}\rangle$ with $|\xi| \le 1$,

$$\begin{aligned} h(\xi) &= e^{\langle \xi, u\rangle} - e^{\langle \xi_{J+}, u_{J+}\rangle} + e^{\xi_i u_i}\big(e^{\langle \xi_J, u_J\rangle} - 1 - \langle \xi_J, u_J\rangle\big) \\ &\quad + \langle \xi_J, u_J\rangle\big(e^{\xi_i u_i} - 1\big) + e^{\xi_i u_i} - 1 - \xi_i u_i \\ &= e^{\langle \xi_{J+}, u_{J+}\rangle}\left(\int_0^1 e^{t\langle \xi_{I-}, u_{I-}\rangle}\, dt\right)\langle u_I, \xi_I\rangle \\ &\quad + e^{\xi_i u_i}\left(\int_0^1 (1-t)e^{t\langle \xi_J, u_J\rangle}\, dt\right)\sum_{j,k\in J}\xi_j\xi_k u_j u_k \\ &\quad + \left(\int_0^1 e^{t\xi_i u_i}\, dt\right)\xi_i u_i \sum_{j\in J}\xi_j u_j + \left(\int_0^1 (1-t)e^{t\xi_i u_i}\, dt\right)\xi_i^2 u_i^2. \end{aligned} \tag{5.6}$$

Next we calculate

$$\operatorname{Re}\big(\overline{u}_i h(\xi)\big) = K(u,\xi) + |u_i|^2\xi_i \int_0^1 (1-t)\operatorname{Re}\big(u_i\xi_i e^{tu_i\xi_i}\big)\, dt.$$

Since $|\xi| \le 1$, we get

$$\begin{aligned} \big|K(u,\xi)\big| &\le e^{(\operatorname{Re} u)_+}\big(|u_i|^2 + |u_I||u_J|^2 + |u_I||u_J|\big)\big(|\xi_{I-}| + |\xi_{J+}|^2\big) \\ &\le \big(1 + e^{(\operatorname{Re} u)_+}\big)\big(1 + |u_J|^2\big)\big(1 + |u_I|^2\big)\big(|\xi_{I-}| + |\xi_{J+}|^2\big) \end{aligned} \tag{5.7}$$

for the first term. For the second term we use Lemma 5.6 below and estimate

$$|u_i|^2\xi_i \int_0^1 (1-t)\operatorname{Re}\big(u_i\xi_i e^{tu_i\xi_i}\big)\, dt \le |u_i|^2\xi_i\big(e^{\xi_i(\operatorname{Re} u_i)_+} - 1\big). \tag{5.8}$$

Adding up (5.7) and (5.7), and integrating against the Lévy measure μ^i we obtain

$$\operatorname{Re}\left(\overline{u}_i \int_{|\xi|\le 1} h(\xi)\mu^i(d\xi)\right) \le \widehat{g}_i(\operatorname{Re} u)\big(1 + |u_J|^2\big)\big(1 + |u_I|^2\big) \tag{5.9}$$

with

$$\widehat{g}_i(y) = e^{y_+}\int_{|\xi|\le 1}\big(|\xi_{I-}| + |\xi_{J+}|^2\big)\mu^i(d\xi) + \int_{|\xi|\le 1}\xi_i\big(e^{\xi_i y_i} - 1\big)\mu^i(d\xi),$$

which is nonnegative, convex and finite for all $y \in \mathbb{R}^d$. Adding up (5.3)–(5.5) and (5.9) yields the desired estimate (5.2). □

LEMMA 5.6. *For any $z \in \mathbb{C}$,*

$$\int_0^1 (1-t)\operatorname{Re}\big(ze^{tz}\big)\, dt \le \big(e^{(\operatorname{Re} z)_+} - 1\big). \tag{5.10}$$

PROOF. For $\operatorname{Re} z \le 0$ the inequality was shown in Duffie, Filipović and Schachermayer (2003). Denote the left-hand side by $L(z)$. Writing $z = p + iq$ and evaluating the integral, we have that

$$\begin{aligned} L(z) &= \int_0^1 (1-t)e^{pt}\big(p\cos(qt) - q\sin(qt)\big) \\ &= \frac{1}{p^2+q^2}\big\{p\big(e^p\cos(q) - 1 - p\big) + q\big(e^p\sin(q) - q\big)\big\}. \end{aligned}$$

The expression is symmetric in q such that we may assume that $q \ge 0$. If in addition $p \le 0$, then using $\cos(q) \ge 1 - q^2/2$ and $\sin(q) \le 1$ we may estimate

$$L(z) \le \frac{1}{p^2+q^2}\big(p(e^p - 1 - p) - q^2 + q^2 e^p(1 - p/2)\big).$$

Since $e^p(1 - p/2) \le 1$ and $(e^p - 1 - p) \ge 0$, the right-hand side is smaller than 0 showing the lemma for $p \le 0$. If $p \ge 0$, we may use that $\cos(q) \le 1$, $\sin(q) \le q$ and $e^p - 1 \le pe^p$ to estimate

$$L(z) \le \frac{1}{p^2+q^2}\big\{p^2\big(e^p - 1\big) + q^2\big(e^p - 1\big)\big\} = \big(e^p - 1\big),$$

thus completing the proof. □

Recall Definition 5.2 of quasimonotonicity with respect to a convex cone K. Here, $K = \mathbb{R}^m_+$; in this particular setting, quasimonotonicity of a function $f : U \subset K \to \mathbb{R}^m$ can be expressed in coordinates and is equivalent to

$$y \preceq z, \text{ and } y_i = z_i \text{ for some } i \in \{1, \dots, m\} \Rightarrow f_i(y) = f_i(z).$$

LEMMA 5.7. *Let $y_J \in \mathbb{R}^n$. For each $t \ge 0$, $y_I \mapsto R_I(y_I, \psi_J(t, y_J))$ is quasimonotone increasing* (*with respect to the natural cone $\mathbb{R}^m_+$*) *on $\mathcal{Y}$.*

PROOF. See, for instance, Keller-Ressel (2009) or Mayerhofer, Muhle-Karbe and Smirnov (2011). □

We further need the following special property of $\mathcal{Y}^\circ$.

LEMMA 5.8. *If $y \in \mathcal{Y}^\circ$, $z \in \mathbb{R}^d$ and $z_I \preceq y_I$, $z_J = y_J$, then we also have $z \in \mathcal{Y}^\circ$.*

PROOF. We choose $\varepsilon > 0$ such that $B_\varepsilon(y) = \{w \in \mathbb{R}^d \mid |y - w| < \varepsilon\} \subset \mathcal{Y}$. By (5.1) we have for $i = 0, 1, \dots, d$,

$$\int_{|\xi|\ge 1} e^{\langle y+w,\xi\rangle}\mu^i(d\xi) < \infty, \qquad |w| < \varepsilon. \tag{5.11}$$

Note the Lévy measures μ^i are clearly positive and supported on D. Now for all $\xi \in D$ we have

$$\langle z + w, \xi \rangle = z_I^\top \xi_I + z_J^\top \xi_J + \langle w, \xi \rangle = z_I^\top \xi_I + y_J^\top \xi_J + \langle w, \xi \rangle \le \langle y + w, \xi \rangle$$

because $\xi_I \in \mathbb{R}_+^m$ and $z_i \le y_i$ for all $i \in I$, by assumption. Hence, by the monotonicity of the exponential we see that (5.11) holds with y replaced by z. Hence, once again by (5.1) we have $B_\varepsilon(z) \subset \mathcal{Y}$, that is, $z \in \mathcal{Y}^\circ$. □

We are now prepared to prove Proposition 5.1 under Assumption 2.24(i).

PROOF OF PROPOSITION 5.1 UNDER ASSUMPTION 2.24(i). By a straightforward check, for every $u \in \mathcal{Y}$,

$$\operatorname{Re}\big(R_i(u)\big) \le R_i\big(\operatorname{Re}(u)\big),$$

and by Lemma 5.7 we can apply the ODE comparison result of Volkmann (1973) to the first m coordinates of ψ, which let us conclude that $\operatorname{Re}(\psi_I(t,u)) \preceq \psi_I(t, \operatorname{Re}(u))$ for $t < T_+(u) \wedge T_+(\operatorname{Re}(u))$. In view of Lemma 5.8 we therefore have $T_+(u) \ge T_+(\operatorname{Re}(u))$, unless $|\psi(t,u)|$ explodes before $|\psi(t, \operatorname{Re}(u))|$ does. We show in the following that this cannot happen: By Lemma 5.5 we have

$$\begin{aligned}\frac{\partial}{\partial t}\big|\psi_I(t,u)\big|^2 &= 2\operatorname{Re}\big\langle\overline{\psi}_I(t,u), R_I\big(\psi(t,u)\big)\big\rangle \\ &\le g\big(\operatorname{Re}\psi(t,u)\big)\big(1 + \big|\psi_J(t,u)\big|^2\big)\big(1 + \big|\psi_I(t,u)\big|^2\big)\end{aligned}$$

with a function g which is finite on all of $\mathcal{Y}$. Since $\psi_J(t,u) \equiv \psi_J(t,u_J)$ exists globally as solution of a linear ordinary differential equation, we obtain by Gronwall's inequality applied to $(1 + |\psi_I(t,u)|^2)$ that

$$\big|\psi_I(t,u)\big| \le |u_I|^2 + \big(1 + |u_I|^2\big) \int_0^t h(s) e^{\int_0^s h(\xi)\, d\xi}\, ds,$$

where $h(t) := g(\operatorname{Re}\psi(t,u))(1 + |\psi_J(t,u)|^2)$. Hence we have shown $T_+(u) \ge T_+(\operatorname{Re}(u))$. □

5.2. *Matrix state spaces.* Let S_d be the space of symmetric real $d \times d$ matrices, endowed with the inner product $\langle x, y\rangle = \operatorname{tr}(xy)$, where tr denotes the trace operator. We further denote by $\mathbb{C}^{m \times n}$ the space of complex $m \times n$ matrices. We make the latter into a normed space by introducing a norm as $\|a\|^2 := \operatorname{tr}(a\bar{a}^\top)$. Here $^\top$ denotes matrix transposition, and $\bar{a}$ is the element-wise conjugate of the matrix a.

We start with the following observation, which is a generalization of Mayerhofer (2012), Lemma B.1.

LEMMA 5.9. *There exists a locally Lipschitz function $h : S_d \to \mathbb{R}_+$ such that for all $a \in \mathbb{C}^{m\times n}$ and for any $b \in S(S_n)$, we have*

$$\operatorname{Re}\operatorname{tr}\bigl(-b\bar{a}a^\top\bigr) \le h(\operatorname{Re} b)\cdot \|a\|^2. \tag{5.12}$$

PROOF. Recall that the projection $\pi : S_d \to S_d^+$ is a well defined, convex (hence locally Lipschitz continuous) map, which satisfies $\pi(z) \succeq z$ for all $z \in S_d$.

Let us write $a = a_1 + ia_2$ and $b = b_1 + ib_2$ with $a_1, a_2 \in \mathbb{R}^{m\times n}$ and $b_1, b_2 \in S_n$. Then we have

$$\begin{aligned}
\operatorname{Re}\operatorname{tr}\bigl(-b\bar{a}^\top a\bigr) &= \operatorname{Re}\operatorname{tr}\bigl(-(b_1+ib_2)\bigl(a_1^\top - ia_2^\top\bigr)(a_1+ia_2)\bigr)\\
&= \operatorname{tr}\bigl(-b_1\bigl(a_1^\top a_1\bigr)\bigr) + \operatorname{tr}\bigl(-b_1\bigl(a_2^\top a_2\bigr)\bigr) + 0\\
&\le \operatorname{tr}\bigl(\pi(-b_1)\bigl(a_1^\top a_1\bigr)\bigr) + \operatorname{tr}\bigl(\pi(-b_1)\bigl(a_2^\top a_2\bigr)\bigr)\\
&\le \bigl\|\pi(-b_1)\bigr\|\bigl(\|a_1\|^2 + \|a_2\|^2\bigr)\\
&= \bigl\|\pi(-b_1)\bigr\|\,\|a\|^2.
\end{aligned}$$

The last inequality holds in view of the Cauchy–Schwarz inequality. We now see that inequality (5.12) holds by setting $h(x) := \|\pi(-x)\|$. $\square$

Next we present the admissibility conditions for matrix-valued affine processes that have been established in Cuchiero et al. (2011). Note that in the case $d = 1$ it holds that $S_d^+ = \mathbb{R}_{\ge 0}$, that is, the one-dimensional case is already covered by the previous section. Therefore we may assume that $d \ge 2$, which leads to several simplifications of the parameter conditions. It has been shown in Mayerhofer (2012) that affine processes on S_d^+ ($d \ge 2$) do not exhibit jumps of infinite total variation. Compared with Cuchiero et al. (2011) this makes the use of a truncation function in the definition of R obsolete and also simplifies the very complicated (i.e., hard to check) necessary tradeoff between linear jump coefficient and drift; cf. Cuchiero et al. (2011), 2.11. In the following, $\preceq$ denotes the partial order on S_d induced by the cone S_d^+.

DEFINITION 5.10. An admissible parameter set $(\alpha, b, B, m(d\xi), \mu(d\xi))$ consists of:

- a linear diffusion coefficient $\alpha \in S_d^+$,
- a constant drift $b \in S_d^+$ satisfying

$$b \succeq (d-1)\alpha,$$

- a constant jump term: a Borel measure m on $S_d^+ \setminus \{0\}$ satisfying

$$\int_{S_d^+\setminus\{0\}} \bigl(\|\xi\| \wedge 1\bigr) m(d\xi) < \infty,$$

- a linear jump coefficient μ which is an S_d^+-valued, sigma-finite measure on $S_d^+ \setminus \{0\}$ satisfying

$$\int_{S_d^+\setminus\{0\}} (\|\xi\| \wedge 1)\mu(d\xi) < \infty$$

- and finally, a linear drift B, which is a linear map from S_d to S_d and "inward pointing" at the boundary of S_d^+. That is,

$$\operatorname{tr}(xB(u)) \geq 0 \qquad \text{for all } u, x \in S_d^+ \text{ with } \operatorname{tr}(ux) = 0.$$

REMARK 5.11. Using the notation $a(x)$ from (2.7) we have $a(x)(u) = 2\operatorname{tr}(xu\alpha u)$, and the following are equivalent:

(1) condition 2.24(ii);
(2) $\alpha = 0$ or α is invertible;
(3) either $a(x)$ vanishes for all $x \in S_d^+$, or it is nondegenerate for any $x \in S_d^+ \setminus \{0\}$.

The only nontrivial direction to prove is (1) $\Rightarrow$ (2). Assume, for a contradiction, that $\alpha \neq 0$, but α is degenerate. Then there exists $u \in S_d^+ \setminus \{0\}$ such that $u\alpha = \alpha u = 0$. But then $a(x)(u) = \operatorname{tr}(xu\alpha u) = 0$, for any x.

Note that Cuchiero et al. (2011) uses the Laplace transform to define the affine property, which introduces several changes of signs compared with our definition. To comply with the notation of Cuchiero et al. (2011), we introduce

$$\widehat{F}(y) = -F(-y), \qquad \widehat{R}(y) = -R(-y),$$

which can now be written as

$$\widehat{F}(y) = \operatorname{tr}(by) - \int_{S_d^+\setminus\{0\}} (e^{-\operatorname{tr}(y\xi)} - 1)m(d\xi),$$

$$\widehat{R}(y) = -2y\alpha y + B^\top(y) - \int_{S_d^+\setminus\{0\}} (e^{-\operatorname{tr}(y\xi)} - 1)\mu(d\xi).$$

Writing furthermore

$$\widehat{p}(t, y) = -p(t, -y), \qquad \widehat{q}(t, y) = -q(t, -y),$$

and similarly for ϕ and ψ, then, by Cuchiero et al. (2011),

$$\mathbb{E}^x\left[e^{-\operatorname{tr}(yX_t)}\right] = e^{-\widehat{p}(t,y)-\operatorname{tr}(\widehat{q}(t,y)x)}$$

for all $t \geq 0$, $y, x \in S_d^+$, and by Mayerhofer (2012) the exponents $(\widehat{p}, \widehat{q}) : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+ \times S_d^+$ solve the system of generalized Riccati equations

$$\frac{\partial}{\partial t}\widehat{p}(t, y) = \widehat{F}(\widehat{q}(t, y)), \tag{5.13a}$$

$$\frac{\partial}{\partial t}\widehat{q}(t, y) = \widehat{R}(\widehat{q}(t, y)), \tag{5.13b}$$

given initial data $\widehat{p}(0, y) = 0$, $\widehat{q}(0, y) = y$.

Since μ is an S_d^+-valued measure, $\operatorname{tr}(\mu)$ is a well-defined, nonnegative measure, naturally given by

$$\operatorname{tr}(\mu)(A) = \operatorname{tr}(\mu(A)).$$

Accordingly, the domain $\widehat{\mathcal{Y}} := -\mathcal{Y}$ is given by

$$\widehat{\mathcal{Y}} = \left\{ y \in S_d \;\middle|\; \int_{\|\xi\| \geq 1} e^{-\operatorname{tr}(y\xi)} \bigl(m(d\xi) + \operatorname{tr}(\mu)(d\xi)\bigr) < \infty \right\}. \tag{5.14}$$

The inclusion $\supseteq$ holds in view of the positive definiteness of the measure μ, while the inclusion $\subseteq$ follows from Mayerhofer (2012), Lemma 3.3. Similarly to the preceding section, we start with the following crucial estimate: I denotes the $d \times d$ unit matrix.

LEMMA 5.12. *Suppose that the diffusion coefficient satisfies $\alpha = I$ or $\alpha = 0$. Then there exists a locally Lipschitz continuous function g on $\widehat{\mathcal{Y}}^\circ$ such that for all $u \in S(\widehat{\mathcal{Y}}^\circ)$ we have*

$$\operatorname{Re}\bigl(\operatorname{tr}(\bar{u}\widehat{R}(u))\bigr) \leq g\bigl(\operatorname{Re}(u)\bigr)\bigl(1 + \|u\|^2\bigr). \tag{5.15}$$

PROOF. As in the proof of Lemma 5.5 we start with drift and big-jump parts. Clearly we have

$$\operatorname{Re}\operatorname{tr}\bigl(\bar{u}B^\top(u)\bigr) \leq G_1\bigl(1 + \|u\|\bigr)^2 \tag{5.16}$$

for some positive constant G_1. What concerns the big-jump parts, we have

$$\begin{aligned}
\operatorname{Re}\operatorname{tr}\left(\bar{u}\left(\int_{\|\xi\|>1} (e^{-\operatorname{tr}(u\xi)} - 1)\mu(d\xi)\right)\right) &\leq \|u\| \operatorname{tr}(\mu)\bigl(\{\xi : \|\xi\| > 1\}\bigr) \\
&\quad + \|u\| \int_{\|\xi\|>1} \bigl(e^{-\operatorname{tr}(\operatorname{Re} u\xi)}\bigr) \operatorname{tr}(\mu)(d\xi) && (5.17) \\
&\leq g_2\bigl(\operatorname{Re}(u)\bigr)\bigl(1 + \|u\|^2\bigr), && (5.18)
\end{aligned}$$

for some locally Lipschitz continuous function g_2. The integral (5.17) is finite, because $\operatorname{Re}(u) \in \widehat{\mathcal{Y}}$ by assumption. Here we have also used Mayerhofer (2012), Lemma 3.3. Note that we can set

$$g_2(y) := \operatorname{tr}(\mu)\bigl(\{\xi : \|\xi\| > 1\}\bigr) + \int_{\|\xi\|>1} \bigl(e^{-\operatorname{tr}(y\xi)}\bigr) \operatorname{tr}(\mu)(d\xi).$$

For $\alpha = 0$ we set $g_3 = 0$. If $\alpha = I$, we involve Lemma 5.9 and obtain

$$\operatorname{Re}\bigl(\bar{u}u^2\bigr) \leq g_3\bigl(\operatorname{Re}(u)\bigr)\|u\|^2, \tag{5.19}$$

where $g_3(\cdot) = h(\cdot) = \pi(-\cdot)$.

It remains to estimate the small-jump part. Using again Mayerhofer (2012), Lemma 3.3, we have

$$\begin{aligned}
&\operatorname{Re}\operatorname{tr}\Big(\bar{u}\int_{0<\|\xi\|\le 1}\big(e^{-\operatorname{tr}(u\xi)}-1\big)\mu(d\xi)\Big)\\
&\quad=\operatorname{Re}\operatorname{tr}\Big(\bar{u}\int_{0<\|\xi\|\le 1}\int_0^1\operatorname{tr}(u\xi)e^{-s\operatorname{tr}(u\xi)}\,ds\,\mu(d\xi)\Big)\\
&\quad\le\|u\|^2\int_{0<\|\xi\|\le 1}\int_0^1 e^{-s\operatorname{tr}(u\xi)}\|\xi\|\operatorname{tr}(\mu)(d\xi)\\
&\quad\le e^{\|\operatorname{Re}u\|}\|u\|^2\int_{0<\|\xi\|\le 1}\|\xi\|\operatorname{tr}(\mu)(d\xi)\\
&\quad\le g_4\big(\operatorname{Re}(u)\big)\big(1+\|u\|^2\big)
\end{aligned}$$

with

$$g_4(y):=e^{\|y\|}\int_{0<\|\xi\|\le 1}\|\xi\|\operatorname{tr}(\mu)(d\xi).$$

Summarizing the last estimate together with (5.16), (5.18) and (5.19) and setting

$$g(y):=G_1+g_2(y)+g_3(y)+g_4(y)$$

proves the assertion. $\square$

We provide two further lemmas:

LEMMA 5.13. *If $y\in\widehat{\mathcal{Y}}^\circ$, and $z\in S_d$ such that $z\succeq y$, then we also have $z\in\widehat{\mathcal{Y}}^\circ$.*

PROOF. Using (5.14) we infer the existence of some $\varepsilon>0$ such that for all $w\in B_\varepsilon(0)=\{w\in S_d\mid \|w\|<\varepsilon\}$, we have

$$\int_{\|\xi\|\ge 1}e^{-\operatorname{tr}((y+w)\xi)}\big(m(d\xi)+\operatorname{tr}(\mu)(d\xi)\big)<\infty.$$

The assumption of the lemma implies that $\langle z,\xi\rangle\ge\langle y,\xi\rangle$ for all $\xi\in S_d^+$. Furthermore, m and $\operatorname{tr}(\mu)$ are supported on S_d^+. Therefore we have

$$\begin{aligned}
&\int_{\|\xi\|\ge 1}e^{-\operatorname{tr}((z+w)\xi)}\big(m(d\xi)+\operatorname{tr}(\mu)(d\xi)\big)\\
&\quad\le\int_{\|\xi\|\ge 1}e^{-\operatorname{tr}((y+w)\xi)}\big(m(d\xi)+\operatorname{tr}(\mu)(d\xi)\big)<\infty
\end{aligned}$$

for all $w\in B_\varepsilon(0)$, which in view of (5.14) proves that $z\in\widehat{\mathcal{Y}}^\circ$. $\square$

LEMMA 5.14. *$\widehat{R}$ is quasimonotone increasing (with respect to S_d^+) on $\widehat{\mathcal{Y}}^\circ$.*

PROOF. The proof is analogous to the one of Cuchiero et al. (2011), Lemma 5.1, which states quasimonotonicity of $\widehat{R}$ on S_d^+. □

We are now prepared to prove Proposition 5.1 for $D = S_d^+$, $d \geq 2$:

PROOF. According to Cuchiero et al. (2011), Theorem 4.14, for any affine process X (with diffusion coefficient α) there exists a linear automorphism g of S_d^+ such that the affine process $Y = g(X)$ has diffusion coefficient $\widetilde{\alpha} = \operatorname{diag}(I_r, 0)$, where I_r is the $r \times r$ unit matrix, and $r = \operatorname{rank}(\alpha)$. According to our assumption $r = 0$ or $r = d$ (see Remark 5.11), and linear transformations do not affect the blow-up relation (between the real and complex-valued solutions) we are about to prove here. Hence we may without loss of generality assume that $\alpha = 0$ or $\alpha = I$.

For any $u \in S(\widehat{\mathcal{Y}}^\circ)$ we write $y = \operatorname{Re}(u)$. The quasimonotonicity of $\widehat{R}$ (Lemma 5.14) allows us to apply the multivariate comparison result by Volkmann (1973), and we conclude that for $t < T_+(u) \wedge T_+(y)$, we have $\operatorname{Re} \widehat{\psi}(t, u) \succeq \widehat{q}(t, y)$. In view of Lemma 5.13 we only need to show that $t \mapsto \|\widehat{\psi}(t, u)\|$ does not explode before $t \mapsto \|\widehat{q}(t, y)\|$. By Lemma 5.12, there exists a continuous function g such that

$$\operatorname{Re} \operatorname{tr}(\bar{u} \widehat{R}(u)) \leq g(\operatorname{Re}(u))(1 + \|u\|^2), \qquad u \in S(\widehat{\mathcal{Y}}^\circ).$$

Hence, we have for all $t < T_+(u) \wedge T_+(y)$,

$$\frac{\partial}{\partial t}(\|\widehat{\psi}(t, u)\|^2) = 2 \operatorname{Re} \operatorname{tr}(\overline{\widehat{\psi}(t, u)} \widehat{R}(\widehat{\psi}(t, u))) \leq g(\operatorname{Re}(\widehat{\psi}(t, u)))(1 + \|\widehat{\psi}(t, u)\|^2),$$

and by Gronwall's inequality, we obtain

$$\|\widehat{\psi}(t, u)\| \leq (1 + \|u\|^2) \int_0^t g(s) e^{\int_0^s g(\xi)\, d\xi}\, ds.$$

Hence we have shown that $T_+(u) \geq T_+(y)$. □

5.3. *Proof of Theorem* 2.26. The first part of Theorem 2.26 is proved in Proposition 5.1. For the proof of the second part, the validity of the complex transform formula (2.16), we utilize the concept of analytic continuation.

PROOF OF THEOREM 2.26. Consider the set

$$U := \{y \in \mathcal{Y}^\circ \mid T_+(y) > T\}.$$

By assumption U is nonempty, and from the standard existence and uniqueness theorem for ODEs it follows that U is open. Next, we show that U is convex. For $y_1, y_2 \in U$ it follows from Theorem 2.14(b) on real moments that $\mathbb{E}^x[e^{\langle y_1, X_T \rangle}] < \infty$ and $\mathbb{E}^x[e^{\langle y_2, X_T \rangle}] < \infty$ for all $x \in D$. Let $\lambda \in [0, 1]$ and set $y_\lambda = \lambda q_1 + (1 - \lambda) q_2$. By Hölder's inequality

$$\mathbb{E}^x[e^{\langle y_\lambda, X_T \rangle}] \leq \mathbb{E}^x[e^{\langle y_1, X_T \rangle}]^\lambda \cdot \mathbb{E}^x[e^{\langle y_2, X_T \rangle}]^{(1-\lambda)} < \infty$$

for all $x \in D$ and we conclude, using Theorem 2.14(a) that $y_\lambda \in U$ and hence that U is convex. Now set $U' := S(U) \subset \mathbb{C}^d$. From the properties of U we conclude that U' is nonempty, open and connected. By Proposition 5.1 we have $T_+(u') > T_+(\operatorname{Re} u')$ for all $u' \in U'$. Furthermore, since $u \mapsto R(u)$ and $u \mapsto F(u)$ are complex analytic on $\mathcal{Y}^\circ$, we have by Dieudonné (1969), Theorem 10.8.2, that the function

$$M(u) := e^{\phi(t,u)+\langle \psi(t,u),x\rangle}$$

is complex analytic on U' for all $t \leq T$. Furthermore, by Theorem 2.14 and by Remark 2.23, we have that

$$M(y) = e^{\phi(t,y)+\langle \psi(t,y),x\rangle} = \mathbb{E}^x\big[e^{\langle y,X_t\rangle}\big], \qquad y \in U, t \leq T.$$

We conclude that the function $\Phi(u): U' \to \mathbb{C}: u \mapsto \mathbb{E}^x[e^{\langle u,X_t\rangle}]$ is an analytic function, which coincides with $M(u)$ on the nonempty open subset $U \subset U'$. Hence by the principle of analytic continuation[6] [cf. Dieudonné (1969), (9.4.4)] $\mathbb{E}^x[e^{\langle u,X_t\rangle}] = M(u)$ on all of U', and the proof is complete. $\square$

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[6] Here we use that U' is open and connected.

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