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EXTREMAL LAWS FOR THE REAL GINIBRE ENSEMBLE

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The real Ginibre ensemble refers to the family of $n \times n$ matrices in which each entry is an independent Gaussian random variable of mean zero and variance one. Our main result is that the appropriately scaled spectral radius converges in law to a Gumbel distribution as $n \to \infty$. This fact has been known to hold in the complex and quaternion analogues of the ensemble for some time, with simpler proofs. Along the way we establish a new form for the limit law of the largest real eigenvalue.

1. Introduction. Ginibre (1965) introduced the basic non-Hermitian ensembles of random matrix theory. These are $n \times n$ matrices M comprised of independent (and standardized) real, complex or quaternion Gaussian entries and are clear analogues of the Gaussian orthogonal, unitary and symplectic ensembles (G{O/U/S}E).

The results of Ginibre (1965) include explicit formulas for the joint density of eigenvalues of M, in both the complex and quaternion cases. The real-entried case posed serious technical hurdles, due largely to the fact that the real line itself receives positive mass in this setting, and the determination of the joint eigenvalue density remained open until Edelman (1997), Lehmann and Sommers (1991). These papers found conditional densities for the real Ginibre ensemble eigenvalues, that is, formulas for the joint law given a predetermined number of real eigenvalues. Even with these in hand, the expressions proved sufficiently complicated that the finite-dimensional correlation functions—the basic tool(s) required to obtain limit theorems for local eigenvalue statistics—were only determined in the last few years. Borodin and Sinclair (2007) rigorously established that the eigenvalues of the real Ginibre ensemble form a Pfaffian point process: there are 2×2 skew matrix kernels for the real/real, complex/complex and real/complex correlations from which the general k-point correlation is built as a $2k \times 2k$ Pfaffian. These formulas along with the connected skew orthogonal polynomials discovered by Forrester and Nagao (2007) allowed Borodin–Sinclair to derive the scaling limits for the kernels at both the (real and complex) bulk and the (real and complex)

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edge [Borodin and Sinclair (2009)]. [We note that Forrester and Nagao (2007) also presents the real/real and complex/complex correlations, as well as some asymptotics for the one and two-point functions. Concurrently, Sommers (2007) reported the scaling limits of the kernels in the bulk.]

Here we are after a scaling limit for the spectral radius in the real Ginibre ensemble. Among other motivations, this may be viewed as one possible refinement of the circular law. The latter refers to the fact that the normalized counting measure of the scaled eigenvalues converges (weakly almost surely) to the uniform measure on the unit disk. This result has a rather long history, starting with the work of Girko (1984) which was made rigorous by Bai (1997), and culminating in the universality (in terms of entry distributions) theorems of Götze and Tikhomirov (2010) and then Tao and Vu (2010). On a local scale, considerable progress has been made on the universality of the $n \uparrow \infty$ bulk correlations (even all the way up to the edge) for both real and complex entries; see Bourgade, Yau and Yin (2012a, 2012b), Tao and Vu (2012).

Our main result is the following:

THEOREM 1.1. *Denote by R_n the spectral radius of the real Ginibre ensemble, and set $\gamma_n = \log(n/(2\pi(\log n)^2))$. Then, as $n \to \infty$,*

$$\sqrt{4\gamma_n}\left(R_n - \sqrt{n} - \sqrt{\frac{\gamma_n}{4}}\right) \Rightarrow G,$$

where G is the Gumbel law with distribution function $F_G(t) = e^{-(1/2)e^{-t}}$.

One can certainly adjust the scaling so that the limiting distribution function takes the more standard form of $e^{-e^{-t}}$. It is written this way for comparison: at the same scaling the limiting spectral radius in the complex Ginibre ensemble is also Gumbel, with distribution function $e^{-e^{-t}}$. A similar result holds in the quaternion case. The universality of the limiting Gumbel law for spectral radius in any setting (real, complex, or quaternion) has not been addressed.

The analog of Theorem 1.1 for complex Ginibre is a triviality. The eigenvalues of the complex Ginibre ensemble form the canonical radially symmetric determinantal process on the complex plane, and it is the case that the moduli of the points of such a process are independent. Stated in this generality this fact can be found in Hough et al. (2009), Chapter 7, but had been observed earlier by Kostlan (1992) specifically for complex Ginibre. While a Gumbel law is the only possible scaling limit for the extremal point, the precise scalings were worked out in Rider (2003) where the author was unaware of Kostlan's result, but rediscovered and used a consequence thereof. In the quaternion case there is nothing like this determinantal trick. Still the expectations of certain eigenvalue class functions factor nicely, which turns out to be enough; see again Rider (2003).

For real Ginibre, there appear to be no shortcuts toward pinning down the fluctuations of the spectral radius. Instead we return to the standard random matrix theory machinery of tracing through the limiting Fredholm determinant/Pfaffian formulas for the related gap probabilities (which are available once the correlation functions are known). The proof of Theorem 1.1 follows from determining the real and complex gaps separately:

THEOREM 1.2. *Let $z_1, \dots, z_n$ be the eigenvalues of M, then*

$$\mathbb{P}\Big(\max_{k\,:\,z_k\in\mathbb{C}/\mathbb{R}} |z_k| \le \sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{t}{\sqrt{4\gamma_n}}\Big) \to e^{-(1/2)e^{-t}}$$

for any $t \in \mathbb{R}$ as $n \to \infty$, where again $\gamma_n = \log(n/(2\pi(\log n)^2))$.

THEOREM 1.3. *Introduce the integral operator T with kernel*

$$T(x, y) = \frac{1}{\pi}\int_0^\infty e^{-(x+u)^2} e^{-(y+u)^2}\, du. \tag{1.1}$$

Let χ be the indicator of (t, ∞). Then, as $n \to \infty$,

$$\mathbb{P}\Big(\max_{k\,:\,z_k\in\mathbb{R}} z_k \le \sqrt{n} + t\Big) \to \sqrt{\det(I - T\chi)\Gamma_t}, \tag{1.2}$$

where Γ_t is built as follows. Set $g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $G(x) = \int_{-\infty}^x g(y)\,dy$ and denote by $R(\cdot,\cdot)$ the kernel of the resolvent operator $(I - T\chi)^{-1}$. Then

$$\Gamma_t = (1 - a_t)\Big(1 - \frac{1}{2}\int_{-\infty}^t R(x, t)\,dx\Big) + \frac{1}{2}(1 - b_t)\int_{-\infty}^t (I - T\chi)^{-1} g(x)\,dx$$

for $a_t = \int_t^\infty G(x)(I - T\chi)^{-1} g(x)\,dx$ and $b_t = (I - T\chi)^{-1} G(t)$.

Theorem 1.1 then just recasts the result for complex points; the largest real eigenvalue simply lives on a smaller scale. (Obviously the largest negative real eigenvalue exhibits the same limit law.) To make this explicit, we check here that the probability of the largest point in absolute value being real tends to zero. With that event denoted by A and $c_n = 1/4\sqrt{\gamma_n}$,

$$\begin{aligned}\mathbb{P}(A) &\le \mathbb{P}\Big(A, \max_k |\lambda_k| \ge n^{1/2} + c_n\Big) + \mathbb{P}\Big(\max_k |\lambda_k| \le n^{1/2} + c_n\Big)\\ &\le \mathbb{P}\Big(\max_{k\,:\,\lambda_k\in\mathbb{R}} \lambda_k \ge n^{1/2} + M\Big)\\ &\quad + \mathbb{P}\Big(\max_{k\,:\,\lambda_k\in\mathbb{C}/\mathbb{R}} |\lambda_k| \le n^{1/2} + \sqrt{\gamma_n/4} - M/\sqrt{4\gamma_n}\Big)\end{aligned}$$

for any large M as $n \uparrow \infty$. And by choice of M, the lim(sup) of the right-hand side can be made arbitrarily small by the outcomes of Theorems 1.2 and 1.3. Note by definition the right-hand side of (1.2) is a distribution function, and so tends to zero as t, here M, tends to infinity. This can also be seen directly from the simple fact that $T\chi$ goes to zero in trace norm in the same parameter limit.

From a technical standpoint the above means that we never have to consider the mixed real/complex correlations. The same calculation behind Theorem 1.2 produces the full Poisson point process surrounding the Gumbel limit. Since the complex eigenvalues occur in conjugate pairs and (again) the real eigenvalues are in sub-scaling the relevant statement is as follows.

COROLLARY 1.4. *Rescale the eigenvalues* $\{z_k\}$ *lying in the* (*strict*) *upper half plane as in* $z'_k = (r'_k, \theta'_k)$ *with* $r'_k = \sqrt{4\gamma_n}(|z_k| - \sqrt{n} - \sqrt{\frac{\gamma_n}{4}})$, $\theta'_k = \arg(z_k)$. *The resulting point process converges*, *in the sense of finite-dimensional distributions*, *to the Poisson random measure with intensity* $\frac{1}{2\pi}e^{-r}$ *on* $(-\infty, \infty) \times (0, \pi)$.

A convergence result along the lines of Theorem 1.3 for the largest real point was presented in Forrester and Nagao (2007). There the right-hand side of (1.2) was left in terms of the Fredholm Pfaffian of a 2×2 matrix operator. Here, besides rigorously establishing the appropriate norm convergence, the factorization in terms of scalar operators coincides with the initial form of $\beta = 1$ Tracy–Widom (TW_1) distribution function as originally found for GOE. Indeed, the resulting structure of the above limit law is precisely the same as that form of TW_1: T replaces the Airy kernel, and the Gaussian density $g(\cdot)$ plays the role of the Airy function $\mathrm{Ai}(\cdot)$. Unsurprisingly, our derivation of (1.2) follows Tracy and Widom (1996) quite closely.

One point of interest is that T, like the Airy or related Bessel operator, is product Hilbert–Schmidt, and so trace class, on any positive half-line. (Additional properties of T are needed to show that the factors which make up Γ_t are sensible.) Unlike the Airy or Bessel cases, however, T does not posses a Christoffel–Darboux form and so is not integrable in the sense of Deift (1999), or at least not in this simple way. This presents at least one roadblock in obtaining a "closed" expression for (1.2), say something in terms of a single special function like the Painlevé formulation of the Tracy–Widom laws. Even a characterizing PDE or system of ODEs has eluded us. The full large deviations of (1.2) are also open. The right tail, as $t \to \infty$, is easily seen to have a Gaussian shape; this again was pointed out in Forrester and Nagao (2007). The determination of the left tail lies deeper and will be pursued in a later paper. This seems worthwhile given the separate interest in the limiting largest real point due to its applications in the stability analysis of certain biological systems [May (1972)].

The next section recalls what is needed of the real Ginibre correlation functions and gap probabilities. The limit laws for the largest complex and real points are derived separately in Sections 3 and 4. Section 5 reports on some numerical simulations and discusses additional open questions.

2. Determinants. The results of Borodin and Sinclair (2009) lead to (Fredholm) determinantal formulas for the relevant gap probabilities in the $n \times n$ real Ginibre ensemble M. We state things in the particular form that we need.

PROPOSITION 2.1. *The probability $\mathbb{P}_{\mathbb{C},n}(t)$ of there being no complex eigenvalues of M of modulus greater than $t > 0$ is given by*

$$\mathbb{P}_{\mathbb{C},n}(t)^2 = \det(I - K_n \chi) \tag{2.1}$$

in which K_n is a 2×2 operator defined on $L^2(\mathbb{C}_+) \oplus L^2(\mathbb{C}_+)$, $\mathbb{C}_+ = \{z : \mathrm{Im}(z) > 0\}$, cut down by the indicator function $\chi = \chi_{\{|z|>t\}}$. In the standard notation,

$$K_n = \begin{bmatrix} S_n & DS_n \\ -IS_n & S_n^{\mathsf{T}} \end{bmatrix} \tag{2.2}$$

with the various operators defined most easily through their kernels

$$S_n(z, w) = \frac{i e^{-(1/2)(z-\bar{w})^2}}{\sqrt{2\pi}} (\bar{w} - z)\phi(z)\phi(w) e^{-z\bar{w}} \mathfrak{e}_{n-2}(z\bar{w}), \tag{2.3}$$

$DS_n(z, w) = -i S_n(z, \bar{w})$ and $IS_n(z, w) = i S_n(\bar{z}, w)$. The shorthand

$$\phi(z) = \sqrt{\mathrm{erfc}\big(\sqrt{2}|\mathrm{Im}(z)|\big)}, \qquad \mathfrak{e}_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!},$$

is used, where as usual $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$.

Note that while Borodin and Sinclair (2009) considers only even values of n ($n = 2M$ there), the subsequent results of Sinclair (2009) shows that the formulas remain unchanged for n odd. This can be loosely understood by considering that any "extra" particle must be real.

For gaps on the real line we have the following.

PROPOSITION 2.2. *The probability of there being no real eigenvalues of M larger than $t > -\infty$ is given by*

$$\mathbb{P}_{\mathbb{R},n}(t)^2 = \det(I - K_n \chi),$$

where again K_n is a 2×2 operator now defined on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and $\chi = \chi_{[t,\infty)}$. The overall form of K_n is similar to the complex case

$$K_n = \begin{bmatrix} S_n & DS_n \\ -IS_n + \epsilon & S_n^{\mathsf{T}} \end{bmatrix}. \tag{2.4}$$

Here ϵ is the operator

$$\epsilon f(x) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{sgn}(y - x) f(y)\, dy \tag{2.5}$$

and the basic kernel is

$$S_n(x,y) = \frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}} e^{-xy} \mathfrak{e}_{n-2}(xy) + \frac{x^{n-1}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-2}e^{-(1/2)u^2}\,du, \tag{2.6}$$

in terms of which $DS_n = \delta S_n^{\mathsf{T}}$ *where* δ *acts by differentiation on the first variable, and*:

(1) *when n is even*

$$IS_n(x,y) = \epsilon S_n(x,y);$$

(2) *when n is odd,*

$$\begin{aligned} IS_n(x,y) &= \epsilon S_n(x,y) + \frac{1}{2^{n/2}\Gamma(n/2)}\int_0^y u^{n-1}e^{-(1/2)u^2}\,du \\ &= \epsilon S_{n-1}(x,y) \\ &\quad + \frac{1}{\sqrt{2\pi}(n-2)!}\int_0^x\int_0^y (w-u)(wu)^{n-2}e^{-(1/2)u^2-(1/2)w^2}\,du\,dw. \end{aligned}$$

In both cases ϵ *also acts on the first variable.*

While the first form of the IS_n kernel for n odd may be more attractive, the second is better for asymptotics. Both structures are also valid for, for instance, for odd n GOE. In particular, in the first form the correction term may be expressed as $\frac{\pi_{n-1}}{s_{n-1}}$ for π_{n-1} the relevant skew-orthogonal polynomial and s_{n-1} its normalizer (see below), and the same holds for GOE with appropriate substitutions for π_{n-1} and s_{n-1}.

Note that since the ϵ operator is not trace class, what is meant by the Fredholm determinant of (2.4) is at first not clear. However, this is a standard technicality in the world of $\beta = 1$ ensembles, and how to mollify things or smooth out the ϵ is well understood; see, for instance, Tracy and Widom (1998), Section VIII.

PROOF OF PROPOSITION 2.2. The n even case is a restatement of Borodin and Sinclair (2009), Theorem 8. The n odd case was stated without proof in Forrester and Mays (2009) (based on their derivation of the correlation functions of $\beta = 1$ ensembles of odd order); see also Sommers and Wieczorek (2008). We include the odd n case here since the details of the derivation have not before appeared in the literature. We appeal to Sinclair (2009), Section 7 (providing yet another derivation of the correlation kernel for $\beta = 1$ ensembles of odd order), using the particulars of Ginibre's real ensemble.

Recall the weighted skew-orthogonal polynomials for Ginibre's real ensemble [Forrester and Nagao (2008)] given by

$$\pi_{2j}(x) := e^{-x^2/2}x^{2j}; \qquad \pi_{2j+1}(x) := e^{-x^2/2}(x^{2j+1} - 2jx^{2j-1})$$

with the normalization

$$r_j := \langle \pi_{2j}, \pi_{2j+1} \rangle = 2\sqrt{2\pi}(2j)!.$$

We will not explicitly use the skew-inner product, but the interested reader is referred to Sinclair (2007). One further set of normalizations (which does not arise for even n) is

$$s_{2j} := \int_{-\infty}^{\infty} \pi_{2j}(x)\,dx = 2^{j+1/2}\Gamma(j+1/2).$$

(In general, similar normalizations are necessary for the odd degree skew-orthogonal polynomials, but in this case these vanish.) In particular,

$$\frac{s_{2j}}{r_j} = \frac{2^{j+1/2}\Gamma(j+1/2)}{2\sqrt{2\pi}(2j)!} = \frac{1}{2^{j+1}j!}, \tag{2.7}$$

where the last identity uses the Gamma function duplication formula. We will also need

$$\epsilon\pi_{2j+1}(x) = e^{-x^2/2}x^{2j} \quad \text{and} \quad \epsilon\pi_{n-1}(x) = -2^{n/2-1}\operatorname{sgn}(x)\gamma\left(\frac{n}{2}, \frac{x^2}{2}\right).$$

It suffices to establish (2.6) for n odd, since the DS_n and (the modified) IS_n terms follow immediately from Sinclair (2009). For S_n, Sinclair [(2009), page 31] implies

$$S_n(x,y) = S_{n-1}(x,y) - 2\frac{\pi_{n-1}(x)}{s_{n-1}}\sum_{j=0}^{J-1}\frac{s_{2j}}{r_j}\epsilon\pi_{2j+1}(y) + 2\frac{\epsilon\pi_{n-1}(y)}{s_{n-1}}\sum_{j=0}^{J-1}\frac{s_{2j}}{r_j}\pi_{2j+1}(x) + \frac{\pi_{n-1}(x)}{s_{n-1}}.$$

Here, we use

$$2\sum_{j=0}^{J-1}\frac{s_{2j}}{r_j}\epsilon\pi_{2j+1}(y) = e^{-y^2/2}\sum_{j=0}^{J-1}\frac{1}{2^j j!}y^{2j} = e^{-y^2/2}\mathfrak{e}_{J-1}(y^2/2) \tag{2.8}$$

and

$$\frac{\epsilon\pi_{n-1}(y)}{s_{n-1}} = -\frac{\operatorname{sgn}(y)}{2}\frac{\gamma(n/2, y^2/2)}{\Gamma(n/2)} \tag{2.9}$$

to write

$$S_n(x,y) = S_{n-1}(x,y) + \frac{e^{-x^2/2}x^{n-1}}{2^{n/2}\Gamma(n/2)}\big(1 - e^{-y^2/2}\mathfrak{e}_{J-1}(y^2/2)\big) - \frac{2^{(n/2)-1}}{\sqrt{2\pi}(n-2)!}e^{-x^2/2}x^{n-2}\operatorname{sgn}(y)\gamma\left(\frac{n}{2},\frac{y^2}{2}\right).$$

Next, note that

$$1 - e^{x^2/2}\mathfrak{e}_{J-1}(x^2/2) = \frac{1}{2^{(n-3)/2}\Gamma((n-1)/2)}\int_0^x u^{n-2}e^{-(1/2)u^2}\,du.$$

More simply, $\gamma(\frac{n}{2},\frac{x^2}{2}) = 2^{-(n/2)+1}\int_0^{|x|}u^{n-1}e^{-(1/2)u^2}$, so that

$$S_n(x,y) = S_{n-1}(x,y) + \frac{x^{n-1}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-2}e^{-(1/2)u^2}\,du - \frac{x^{n-2}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-1}e^{-(1/2)u^2}\,du.$$

Since $n-1$ is even we can substitute for S_{n-1} to get

$$\begin{aligned}S_n(x,y) = {}& \frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}}e^{-xy}\mathfrak{e}_{n-3}(xy)\\ &+ \frac{x^{n-2}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-3)!}\int_0^y u^{n-3}e^{-(1/2)u^2}\,du\\ &+ \frac{x^{n-1}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-2}e^{-(1/2)u^2}\,du\\ &- \frac{x^{n-2}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-1}e^{-(1/2)u^2}\,du.\end{aligned} \tag{2.10}$$

Integration by parts on the second term yields

$$\frac{x^{n-2}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-3)!}\int_0^y u^{n-3}e^{-(1/2)u^2}\,du = \frac{e^{-(1/2)x^2}e^{-(1/2)y^2}(xy)^{n-2}}{\sqrt{2\pi}(n-2)!} + \frac{x^{n-2}e^{-(1/2)x^2}}{\sqrt{2\pi}(n-2)!}\int_0^y u^{n-1}e^{-(1/2)u^2}\,du.$$

The first of these terms takes

$$\frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}}e^{-xy}\mathfrak{e}_{n-3}(xy) \quad \text{to} \quad \frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}}e^{-xy}\mathfrak{e}_{n-2}(xy)$$

and the second cancels the last term in (2.10) to produce the advertised formula. □

2.1. *From correlations to gap probabilities.* Once again, what Borodin and Sinclair (2009) and Sinclair (2009) establish are Pfaffian formulas for the k-point (any combination of real and complex) correlation functions. For completeness we briefly review how to go from the correlations to the above determinantal gap formulas.

The situation is similar to the classical $\beta = 1$ (GOE) situation, but is complicated by the presence of both real and complex eigenvalues. Letting L represent the number of real eigenvalues and M the number of complex conjugate eigenvalues, there is a different joint eigenvalue density for each pair (L, M) with $L + 2M = n$. Representing this density as $\Omega_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \to [0, \infty)$ [the exact formula for which can be found in Lehmann and Sommers (1991), Edelman (1997)], the normalization constant for the ensemble is given by

$$Z_n = \sum_{\substack{(L,M)\\ L+2M=n}} \frac{1}{L!M!2^M} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \Omega_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})\, d\mu_{\mathbb{R}}^L(\boldsymbol{\alpha})\, d\mu_{\mathbb{C}}^M(\boldsymbol{\beta}),$$

where $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{R}}^L$ are Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^L$, and $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{C}}^M$ are defined analogously.

We define the ℓ, m correlation function $R_{\ell,m} : \mathbb{R}^\ell \times \mathbb{C}^m \to [0, \infty)$ by

$$\begin{aligned} R_{\ell,m}&(\mathbf{x}, \mathbf{z}) \\ &= \frac{1}{Z_n} \sum_{\substack{(L,M)\\ L\ge\ell, M\ge m}} \frac{1}{(L-\ell)!(M-m)!2^{M-m}} \\ &\qquad\qquad \times \int_{\mathbb{R}^{L-\ell}} \int_{\mathbb{C}^{M-m}} \Omega_{L,M}(\mathbf{x} \vee \boldsymbol{\alpha}, \mathbf{z} \vee \boldsymbol{\beta})\, d\mu_{\mathbb{R}}^{L-\ell}(\boldsymbol{\alpha})\, d\mu_{\mathbb{C}}^{M-m}(\boldsymbol{\beta}), \end{aligned}$$

where, for instance, $\mathbf{x} \vee \boldsymbol{\alpha} = (x_1, \ldots, x_\ell, \alpha_1, \ldots, \alpha_{L-\ell})$. That is, the ℓ, m correlation function is a weighted sum of all marginal densities formed by integrating out $L - \ell$ real variables and $M - m$ complex variables from all $\Omega_{L,M}$ for which this makes sense.

The main result of Borodin and Sinclair (2009) demonstrates the existence of three matrix kernels $K_n^{\mathbb{R},\mathbb{R}}$, $K_n^{\mathbb{C},\mathbb{C}}$ and $K_n^{\mathbb{R},\mathbb{C}}$ (and its transpose $K_n^{\mathbb{C},\mathbb{R}}$) such that

$$R_{\ell,m}(\mathbf{x}, \mathbf{z}) = \operatorname{Pf}\begin{bmatrix} [K_n^{\mathbb{R},\mathbb{R}}(x_j, x_k)]_{j,k=1}^{\ell\times\ell} & [K_n^{\mathbb{R},\mathbb{C}}(x_j, z_t)]_{j,t=1}^{\ell\times m} \\ [K_n^{\mathbb{C},\mathbb{R}}(z_s, x_k)]_{s,j=1}^{m\times\ell} & [K_n^{\mathbb{C},\mathbb{C}}(z_s, z_t)]_{s,t=1}^{m\times m} \end{bmatrix}.$$

Now consider a $C \subseteq \mathbb{C}$ which is invariant under complex conjugation, written as the disjoint union $C = A \cup B$ where $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{C} \setminus \mathbb{R}$. Conditioning on the number of real and complex conjugate pairs of eigenvalues, we have that

$$\begin{aligned} \mathbb{P}_{C,n} &:= P(\text{no eigenvalues in } C) \\ &= \sum_{\substack{(L,M)\\ L+2M=n}} P\big(\text{exactly } L \text{ real eigenvalues and all eigenvalues in } C^c\big) \end{aligned}$$

$$= \frac{1}{Z_n} \sum_{\substack{(L,M)\\ L+2M=n}} \frac{1}{L!M!2^M} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \left\{ \prod_{j=1}^{L} (1-\chi_A(\alpha_j)) \prod_{k=1}^{M} (1-\chi_B(\beta_k)) \right\} \times \Omega_{L,M}(\boldsymbol{\alpha},\boldsymbol{\beta})\, d\mu_{\mathbb{R}}^L(\boldsymbol{\alpha})\, d\mu_{\mathbb{C}}^M(\boldsymbol{\beta})$$

with χ_A and χ_B the characteristic functions of A and B. Expanding the products in the integrand and simplifying leads to

$$\mathbb{P}_{C,n} = \sum_{\substack{(\ell,m)\\ \ell+2m\le n}} \frac{(-1)^{\ell+m}}{\ell!m!2^m} \int_{A^\ell} \int_{B^m} R_{\ell,m}(\mathbf{x},\mathbf{z})\, d\mu_{\mathbb{R}}^\ell(\mathbf{x})\, d\mu_{\mathbb{C}}^m(\mathbf{z}).$$

In particular, when $B=\varnothing$, that is, when we are interested in the probability that there are no eigenvalues in some subset A of $\mathbb{R}$, but we place no restrictions on the complex eigenvalues,

$$\begin{aligned}\mathbb{P}_{A,n} &= \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell!} \int_{A^\ell} R_{\ell,0}(\mathbf{x},-)\, d\mu_{\mathbb{R}}^\ell(\mathbf{x})\\ &= \sum_{\ell=1}^{n} \frac{(-1)^\ell}{\ell!} \int_{A^\ell} \mathrm{Pf}\big[K_n^{\mathbb{R},\mathbb{R}}(x_j,x_k)\big]_{j,k=1}^{\ell\times\ell}\, d\mu_{\mathbb{R}}^\ell(\mathbf{x}).\end{aligned}$$

Similarly, if $A=\varnothing$,

$$\begin{aligned}\mathbb{P}_{B,n} &= \sum_{m=1}^{n/2} \frac{(-1)^m}{m!} \int_{B^m} \mathrm{Pf}\left[\frac{1}{2} K_n^{\mathbb{C},\mathbb{C}}(z_s,z_t)\right]_{s,t=1}^{m\times m} d\mu_{\mathbb{C}}^m(\mathbf{z})\\ &= \sum_{m=1}^{n/2} \frac{(-1)^m}{m!} \int_{(B^+)^m} \mathrm{Pf}\big[K_n^{\mathbb{C},\mathbb{C}}(z_s,z_t)\big]_{s,t=1}^{m\times m}\, d\mu_{\mathbb{C}}^m(\mathbf{z}),\end{aligned}$$

where B^+ is the component of B which lies in the upper half plane. Each of the last two displayed equations defines the Fredholm Pfaffian of the indicated matrix kernel K_n, or, in symbols $\mathrm{Pf}(J-K)$ where $J = \left[\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right] \otimes I$, and I is the $n\times n$ identity matrix. One may then invoke the relationship between the Fredholm Pfaffian and the Fredholm determinant [Rains (2000), Borodin and Kanzieper (2007)],

$$\det(I+JK) = \mathrm{Pf}(J-K)^2.$$

3. Complex points. Returning to (2.2) and (2.3) the correct scaling can be implemented as in $z, w \mapsto Z_n, W_n$,

$$\begin{aligned} Z_n(r,\theta) &= \left(\sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{r}{\sqrt{4\gamma_n}}\right) e^{i\theta},\\ W_n(s,\eta) &= \left(\sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{s}{\sqrt{4\gamma_n}}\right) e^{i\eta}, \end{aligned} \tag{3.1}$$

where $\gamma_n = \log(n/(2\pi(\log n)^2))$, and it is implicit that $r, s > -\gamma_n$. Next, we replace $K_n(z, w)$ by

$$\widetilde{K}_n(r, \theta; s, \eta) = \sqrt{\frac{|Z_n||W_n|}{4\gamma_n}} K_n(Z_n, W_n), \tag{3.2}$$

the variable change occurring entry-wise in $\widetilde{K}_n$, which acts on

$$\mathcal{L}_t = L^2(\mathcal{T}) \oplus L^2(\mathcal{T}), \qquad \mathcal{T} = [t, \infty) \times (0, \pi) \ni (r, \theta), (s, \eta) \qquad t > -\infty.$$

The restriction to $\mathcal{T}$ is from now on assumed in the definition of the transformed $\widetilde{K}_n$.

The typical procedure (which is followed in the real case) is to identify a limit kernel/operator K on $\mathcal{T}$ for which $\widetilde{K}_n \to K$ in trace norm, concluding the convergence of $\det(I - K_n)$. Here instead, though $\widetilde{K}_n$ is trace class (it is finite rank), it is more convenient to cast things in the Hilbert–Schmidt norm, in which $\widetilde{K}_n$ vanishes in the limit. This prompts the introduction of the regularized determinant

$$\det_2(I + A) = \det\big((I + A)e^{-A}\big).$$

In particular, if A is Hilbert–Schmidt with eigenvalues $\{\lambda_k(A)\}_{k\geq 0}$, there is the evaluation $\det_2(I + A) = \prod_{k=0}^{\infty}(1 + \lambda_k)e^{-\lambda_k}$ [Gohberg, Goldberg and Krupnik (2000), Section IV.7] allowing us to write

$$\det(I - \widetilde{K}_n) = \det_2(I - \widetilde{K}_n)e^{-\operatorname{tr}\widetilde{K}_n}.$$

With a matrix kernel, the trace is just the sum of the traces of the diagonal entries. The proof of Theorem 1.2 is then completed via the basic estimate, with $\|\cdot\|$ the Hilbert–Schmidt norm,

$$\big|\det_2(I + A) - \det_2(I + B)\big| \leq \|A - B\| \exp\big(\tfrac{1}{2}\big(\|A\| + \|B\| + 1\big)^2\big); \tag{3.3}$$

again see Gohberg, Goldberg and Krupnik (2000), Section IV.7, along with the next lemma.

LEMMA 3.1. *We have that* $\|\widetilde{S}_n\|_2 \to 0$ *as* $n \to \infty$ *while*

$$\operatorname{tr}(\widetilde{S}_n) \to \tfrac{1}{2}e^{-t}.$$

In addition, $\|\widetilde{DS}_n\|_2 = \|\widetilde{IS}_n\|_2 \to 0$. *All norms are with respect to* $L^2(\mathcal{T} \times \mathcal{T})$.

In particular, estimate (3.3) gives the desired result upon choosing $A = \widetilde{K}_n$ and $B = 0$. For a different way to understand Theorem 1.2, the proof of Lemma 3.1 will show that on bounded sets, the kernel $\widetilde{S}_n$ is well approximated (after an unimportant conjugation) by

$$S_\kappa(r, \theta; s, \eta) = \frac{\kappa}{2\pi} \frac{e^{-(1/2)(r+s)}}{(1 + \kappa)e^{i(\theta-\eta)} - 1}, \tag{3.4}$$

in which $\kappa = \kappa(n)$ tends to zero as $n \to \infty$. [So, formally, the limit operator has kernel $\frac{1}{2\pi}\chi_{\{\theta\}}(\eta)e^{-(1/2)(r+s)}$. Again though, we do not attempt to carry out a proof in this manner.] Here the decoupling of the moduli and phases of the points is made explicit, as is the asymptotic independence of neighboring phases (on the scale κ). The kernels for $\widetilde{DS}_n$ and $\widetilde{IS}_n$ will be shown to exhibit a shaper decay. One can also check that $\det(I - S_\kappa) \sim e^{-(1/2)e^{-t}}$ from the series definition of the Fredholm determinant.

As for the proof of Lemma 3.1, we will use the next three estimates on the polynomial $\mathfrak{e}_n(z)$.

LEMMA 3.2 [Wimp-Boyer and Goh (2007)]. *Uniformly in $t \geq 0$,*

$$e^{-nt}\mathfrak{e}_n(nt) = \mathbb{1}_{0 \leq t < 1} + \frac{1}{\sqrt{2}}\frac{\mu(t)t}{t-1}\operatorname{erfc}(\sqrt{n}\mu(t))\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right),$$

where $\mu(t) = \sqrt{t - \log t - 1}$ is taken positive for all t.

LEMMA 3.3 [Bleher and Mallison (2006)]. *For small enough $\delta > 0$, let now $\mu(z) = \sqrt{z - \log z - 1}$ be uniquely defined as analytic in $|z-1| < \delta$ with $\mu(1 + x) > 0$ for $0 < x < \delta$. Then, for any $M > 1$ it holds that with $n \to \infty$,*

$$e^{-nz}\mathfrak{e}_n(nz) = \frac{1}{2\sqrt{2}\mu'(z)}\operatorname{erfc}(\sqrt{n}\mu(z))\left(1 + O\left(\frac{1}{n(z-1)}\right)\right)$$

for z satisfying $\frac{M}{\sqrt{n}} \leq |z-1| \leq \delta$ and $|\arg(z-1)| \leq \frac{2\pi}{3}$.

LEMMA 3.4 [Kriecherbauer et al. (2008)]. *For any $0 < \alpha < 1/2$, set $U = \{|z-1| \leq n^{-\alpha}\}$, and denote by D the unit disk. Then*

$$\mathfrak{e}_{n-1}(nz) = e^n z^n \frac{1}{\sqrt{2\pi n}(1-z)}\left(1 + O\left(\frac{1}{n|1-z|^2}\right)\right) \qquad \text{for } z \in D^c - U$$

and all $n > 1$.

Both Bleher and Mallison (2006) and Kriecherbauer et al. (2008) contain more detailed and complete asymptotics along the lines stated in Lemmas 3.3 and 3.4; we record only what is used here. Also, as is easy to check, the appraisals of all three lemmas apply without change to $\mathfrak{e}_{n-2}$ (rather than say $\mathfrak{e}_n$, $\mathfrak{e}_{n-1}$) for n large enough. Last we should point out that Lemma 3.2 may be arrived at by combining Lemmas 3.3 and 3.4 (the asymptotics must be consistent after all). It is convenient though to have a single result to quote for the full range of the real argument.

PROOF OF PROPOSITION 3.1. This is split into five steps. Throughout, C is a large positive constant that may change from one line to the next.

Step 1 is the trace calculation, integrating

$$\widetilde{S}_n(r,\theta;r,\theta) = \frac{|Z_n|}{2\sqrt{\gamma_n}} S_n(Z_n,Z_n)$$
$$= \frac{1}{\sqrt{2\pi}} \frac{|Z_n|}{\sqrt{\gamma_n}} \operatorname{Im}(Z_n) e^{2\operatorname{Im}(Z_n)^2} \operatorname{erfc}\big(\sqrt{2}\operatorname{Im}(Z_n)\big) e^{-|Z_n|^2} \mathfrak{e}_{n-2}\big(|Z_n|^2\big)$$

with again $Z_n = (\sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{r}{\sqrt{4\gamma_n}})e^{i\theta}$ over $(r,\theta)\in\mathcal{T}$, recall (3.1), (3.2).

Since, for real $y>0$ we have that $\operatorname{erfc}(y) \le \frac{1}{\sqrt{\pi}y}e^{-y^2}$ while $\operatorname{erfc}(y) = \frac{1}{\sqrt{\pi}y}e^{-y^2}(1+O(1/y^2))$ for $y\to\infty$,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \operatorname{Im}(Z_n) e^{2\operatorname{Im}(Z_n)^2} \operatorname{erfc}\big(\sqrt{2}\operatorname{Im}(Z_n)\big) = \frac{1}{2\pi}, \tag{3.5}$$

pointwise on $\mathcal{T}$, with the left-hand side being bounded by the right for all large n.

Next, with

$$t_n = \frac{1}{n}|Z_n|^2 = 1 + \frac{\sqrt{\gamma_n} + r/\sqrt{\gamma_n}}{\sqrt{n}} + \frac{(\sqrt{\gamma_n}+r/\sqrt{\gamma_n})^2}{4n}, \tag{3.6}$$

Lemma 3.2 implies that

$$e^{-|Z_n|^2}\mathfrak{e}_{n-2}\big(|Z_n|^2\big) = \frac{1}{\sqrt{2}} \frac{\mu(t_n)t_n}{t_n-1} \operatorname{erfc}\big(\sqrt{n}\mu(t_n)\big)\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right) \tag{3.7}$$

uniformly on $\mathcal{T}$. And since $\mu(1+\varepsilon) = \frac{\varepsilon}{\sqrt{2}}(1+O(\varepsilon))$ for $0\le\varepsilon\ll 1$, (3.6) and (3.7) produce

$$\frac{|Z_n|}{\sqrt{\gamma_n}} e^{-|Z_n|^2}\mathfrak{e}_{n-2}\big(|Z_n|^2\big) = e^{-r}\big(1+o(1)\big) \qquad \text{uniformly for } r = o(\sqrt{\gamma_n}). \tag{3.8}$$

Here we have used that

$$\frac{\sqrt{n}}{\sqrt{2\pi}\,\gamma_n} e^{-\gamma_n/2} \to 1, \tag{3.9}$$

which in effect dictates the choice of γ_n. For the tail we have the following bounds: with $r\ge 0$ and n large enough so that $\gamma_n\ge 1$ while $\gamma_n^{3/2}/n^{1/2}\le 1/2$,

$$\frac{|Z_n|}{\sqrt{\gamma_n}} e^{-|Z_n|^2}\mathfrak{e}_{n-2}\big(|Z_n|^2\big) \le \frac{\sqrt{n}}{\gamma_n} t_n^2 \exp\big(-n\mu^2(t_n)\big)$$
$$\le C\frac{\sqrt{n}}{\gamma_n} r^2 \exp\left(-\frac{n}{2}\left(\sqrt{\frac{\gamma_n}{n}} - \frac{\gamma_n}{n}\right)\left(\frac{\sqrt{\gamma_n}+r/\sqrt{\gamma_n}}{\sqrt{n}}\right)\right)$$
$$\le Cr^2 e^{-r/4}.$$

The second line uses the inequality

$$\varepsilon - \log(1+\varepsilon) \ge \tfrac{1}{2}(\delta - \delta^2)\varepsilon \qquad \text{for all } \varepsilon \ge \delta \text{ and } 0 \le \delta < 1 \tag{3.10}$$

with the choices $\varepsilon = t_n - 1$ and $\delta = \sqrt{\gamma_n/n}$. Hence, dominated convergence coupled with (3.5) and (3.8) yields

$$\operatorname{tr}(\widetilde{S}_n) = \int_{\mathcal{T}} \widetilde{S}_n(r,\theta;r,\theta)\,dr\,d\theta = \frac{1}{2}e^{-t}\big(1+o(1)\big)$$

as required.

Step 2 considers $\widetilde{S}_n$ away (though just barely) from the diagonal. All further nontrivial behavior occurs when the argument of $\mathfrak{e}_{n-2}(Z_n\overline{W}_n)$ is in a small neighborhood of $\frac{1}{n}Z_n\overline{W}_n = 1$ for which we can invoke Lemma 3.3. For given $\theta \in (0,\pi)$ consider the set

$$\mathcal{N}_{\theta,n} = \left\{(\eta, r, s) : \eta \in (0,\pi), |\theta - \eta| \le \frac{1}{n^{1/4}\sqrt{\gamma_n}}, t \le r, s \le n^{1/4}\right\} \cap \mathcal{T},$$

in connection to which it will be useful to make the definition

$$\varepsilon = \frac{1}{n^{1/4}\sqrt{\gamma_n}}. \tag{3.11}$$

Similar to before, define $z_n = \frac{1}{\sqrt{n}}Z_n$, $w_n = \frac{1}{\sqrt{n}}W_n$, and record

$$\begin{aligned} z_n\bar{w}_n = \bigg(1 + \sqrt{\frac{\gamma_n}{n}} + \frac{r+s}{2\sqrt{\gamma_n n}} \\ + \frac{(\sqrt{\gamma_n} + r/\sqrt{\gamma_n})(\sqrt{\gamma_n} + s/\sqrt{\gamma_n})}{4n}\bigg)e^{-i(\theta-\eta)}. \end{aligned} \tag{3.12}$$

Note $|z_n| - 1, |w_n| - 1 = O(\varepsilon)$ on $\mathcal{N}_{\theta,n}$, and that $z_n\bar{w}_n$ satisfy the assumptions of Lemma 3.3 there,

$$\frac{1}{C}\sqrt{\frac{\gamma_n}{n}} \le |1 - z_n\bar{w}_n| \le C\varepsilon, \qquad \big|\arg(1 - z_n\bar{w}_n)\big| \le \pi/2.$$

As we still have the estimate $\operatorname{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}z}(1 + O(1/z^2))$ for $\arg(z) < 3\pi/4$, it holds that

$$e^{-Z_n\overline{W}_n}\mathfrak{e}_{n-2}(Z_n\overline{W}_n) = \frac{1}{\sqrt{2\pi n}}\frac{z_n\bar{w}_n}{z_n\bar{w}_n - 1}e^{-n\mu^2(z_n\bar{w}_n)}\left(1 + O\left(\frac{1}{\sqrt{n\gamma_n}}\right)\right), \tag{3.13}$$

since $\mu'(z) = \frac{z-1}{2z\mu(z)}$. The rational term in (3.13) may be bounded roughly as

$$\left|\frac{z_n\bar{w}_n}{z_n\bar{w}_n - 1}\right| \le C\sqrt{\frac{n}{\gamma_n}} \qquad \text{on } \mathcal{N}_{\theta,n}. \tag{3.14}$$

And combining the exponent in (3.13) with $e^{-(1/2)(Z_n-\overline{W}_n)^2}\phi(Z_n)\phi(W_n)$, S_n has the overall exponential factor of n times

$$\begin{aligned}(3.15)\qquad & -\tfrac{1}{2}(z_n-\bar{w}_n)^2-\mathrm{Im}(z_n)^2-\mathrm{Im}(w_n)^2-\mu(z_n\bar{w}_n)^2\\ &\quad =1-\tfrac{1}{2}\big(|z_n|^2+|w_n|^2\big)+\log\big(|z_n||w_n|\big)\\ &\qquad +i\big(\big(\mathrm{Re}(z_n)\,\mathrm{Im}(z_n)+\arg(z_n)\big)-\big(\mathrm{Re}(w_n)\,\mathrm{Im}(w_n)+\arg(w_n)\big)\big),\end{aligned}$$

where we are assuming $\mathrm{Im}(z_n),\mathrm{Im}(w_n)>0$. The relevant part of (3.15) satisfies

$$\begin{aligned}(3.16)\qquad & 1-\frac{1}{2}\big(|z_n|^2+|w_n|^2\big)+\log\big(|z_n||w_n|\big)\\ &\quad \le -\frac{1}{2}\big[\big(|z_n|-1\big)^2+\big(|w_n|^2-1\big)^2\big]\\ &\qquad -\left(\frac{1}{2}\big(|z_n|+|w_n|\big)-1\right)\left(\frac{1}{2}\sqrt{\frac{\gamma_n}{n}}-\frac{\gamma_n}{4n}\right)\\ &\quad \le -\frac{\gamma_n}{2n}-\frac{r+s}{8n}-\frac{r^2+s^2}{8\gamma_n n}+\frac{\gamma_n^{3/2}}{n^{3/2}}\end{aligned}$$

for all $r,s\ge 0$ and $\gamma_n/n\le 1$. Here again (3.10) is used (twice) with $\varepsilon=|z_n|-1$, $|s_n|-1$ and $\delta=\frac{1}{2}\sqrt{\frac{\gamma_n}{n}}$. (For r, $s<0$, or in particular just bounded, a Taylor expansion produces a better bound, with $\frac{r+s}{8n}$ replaced by $\frac{r+s}{2}$.)

Finally, there remains the prefactor,

$$\begin{aligned}(3.17)\qquad & \frac{1}{4\pi}\sqrt{\frac{n}{\gamma_n}}\times\sqrt{\frac{|z_n||w_n|}{\mathrm{Im}(z_n)\,\mathrm{Im}(w_n)}}\,i(\bar{w}_n-z_n)\\ &\quad =\frac{1}{2\pi}\sqrt{\frac{n}{\gamma_n}}\times\frac{\sin((\theta+\eta)/2)+\delta_n}{\sqrt{\sin(\theta)\sin(\eta)}}e^{-(i/2)(\theta-\eta)},\end{aligned}$$

where δ_n is an additive error term that satisfies $\delta_n=O(n^{-1/4}\gamma_n^{-1/2})=O(\varepsilon)$ for $r,s\le n^{1/4}$.

Combining the above [and recalling (3.9)], we have the upper bound

$$(3.18)\qquad \big|\widetilde{S}_n(r,\theta;s,\eta)\big|\le C\frac{\sin((\theta+\eta)/2)+\varepsilon}{\sqrt{\sin(\theta)\sin(\eta)}}e^{-(r+s)/8}\qquad \text{on } \mathcal{N}_{\theta,n}.$$

And since

$$(3.19)\qquad \int_{\varepsilon}^{\pi/2}\int_{\theta-\varepsilon}^{\theta+\varepsilon}\frac{(\sin((\theta+\eta)/2)+\varepsilon)^2}{\sin(\theta)\sin(\eta)}\,d\eta\,d\theta\le C\varepsilon$$

(this for any small $\varepsilon>0$) with a similar bound in a neighborhood of $\theta=\pi$, it follows that

$$\int_{\varepsilon}^{\pi-\varepsilon}d\theta\int_{\mathcal{N}_{n,\theta}}|\widetilde{S}_n|^2\le C\varepsilon,$$

where now recall that $\varepsilon=\varepsilon_n\downarrow 0$.

For the integral over $0 < \theta < \varepsilon$, we go back to the start and bound the $\operatorname{erfc}(a)$ appearing in each copy of $\phi(a)$ in a simpler way: for $a > 0$, $\operatorname{erfc}(a) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(t+a)^2}\,dt \le e^{-a^2}$. This removes the integrability issues due to the inverse sines in (3.18) at the expense of an additional factor of $\sqrt{n}$. Importantly though this keeps the over all exponent from (3.15) unchanged. In particular, the estimate

$$|\widetilde{S}_n(r,\theta;s,\eta)| \le C\sqrt{n}\left(\sin\left(\frac{\theta+\eta}{2}\right)+\varepsilon\right)e^{-(r+s)/8},$$

is also available on any $\mathcal{N}_{\theta,n}$. On the region of current interest, $\sin(\frac{\theta+\eta}{2}) = O(\varepsilon)$, and so

$$\int_0^\varepsilon \int_{\mathcal{N}_{\theta,n}} |\widetilde{S}_n|^2 \le Cn\varepsilon^4,$$

explaining in part the choice that ε should decay a bit faster than $n^{1/4}$.

REMARK. If we further restrict $r, s = o(\sqrt{\gamma_n})$ the bounds (3.14) and (3.16) can be supplanted by

$$\frac{z_n\bar{w}_n}{z_n\bar{w}_n - 1} = \frac{e^{i(\theta-\eta)}}{(1+\sqrt{\gamma_n/n})e^{i(\theta-\eta)} - 1}\big(1+o(1)\big)$$

and

$$1 - \frac{1}{2}\big(|z_n|^2 + |w_n|^2\big) + \log\big(|z_n||w_n|\big) = -\frac{\gamma_n}{2n} - \frac{r+s}{2n} + o\left(\frac{1}{n}\right),$$

respectively. Also considering a fixed $\theta \in (0,\pi)$ with $|\theta - \eta| = o(1)$, one has $\frac{\sin((\theta+\eta)/2)}{\sqrt{\sin(\theta)\sin(\eta)}} = 1 + o(1)$.

Therefore, setting $f_n(r,\theta) = n(\operatorname{Re}(z_n)\operatorname{Im}(z_n)) + (n + 1/2)\arg(z_n)$ [recall (3.15)], there is the estimate

$$e^{if_n(r,\theta)}\widetilde{S}_n(r,\theta;s,\eta)e^{-if_n(s,\eta)} = \frac{1}{2\pi}\sqrt{\frac{\gamma_n}{n}}\frac{e^{-(r+s)/2}}{(1+\sqrt{\gamma_n/n})e^{i(\theta-\eta)} - 1}\big(1+o(1)\big),$$

in the "bulk" of $\mathcal{T}$, as advertised in (3.4), with $\kappa = \sqrt{\frac{\gamma_n}{n}}$.

Step 3 considers again $r, s \le n^{1/4}$, but keeps ε and η away from the diagonal via $|\theta - \eta| > \varepsilon$, the latter defined in (3.11). With now $|z_n\bar{w}_n - 1| \ge \frac{1}{C}\varepsilon$ [see (3.12)] we have that

$$\begin{aligned}
&\big|e^{-n(\operatorname{Im}(z_n)^2+\operatorname{Im}(w_n)^2)}e^{-(n/2)(z_n-\bar{w}_n)^2}e^{nz_n\bar{w}_n}\mathfrak{e}_{n-2}(nz_n\bar{w}_n)\big| \\
&\quad \le C\frac{1}{\varepsilon}\exp\left(n\left(1 - \frac{1}{2}|z_n|^2 - \frac{1}{2}|w_n|^2 + \log|z_n||w_n|\right)\right) \\
&\quad \le C\frac{\gamma_n}{n\varepsilon}e^{-(r+s)/8}.
\end{aligned} \tag{3.20}$$

Lemma 3.4 is responsible for the first inequality, producing the same exponent as in step 2. Line 2 then reuses estimate (3.16) from that step in conjunction with (3.9).

Next, while the considerations behind (3.17) still hold, here the $\sin(\frac{\theta+\eta}{2})+\varepsilon$ is of little use. Instead we are lead to the bound

$$|\widetilde{S}_n(r,\theta;s,\eta)| \le C\sqrt{\frac{\gamma_n}{n}}\frac{1}{\varepsilon}\frac{e^{-(r+s)/8}}{\sqrt{\sin(\theta)\sin(\eta)}} \tag{3.21}$$
$$\text{on } |\theta-\eta|>\varepsilon \text{ and } r,s\le n^{1/4}.$$

First keeping θ and η away from the origin, let

$$\mathcal{O}_n = \{\varepsilon<\theta,\eta<\pi/2, |\theta-\eta|>\varepsilon r, s\le n^{1/4}\}$$

for which we have that

$$\int_{\mathcal{O}_n} |\widetilde{S}_n|^2 \le C\frac{\gamma_n}{n\varepsilon^2}\left(\int_\varepsilon^{\pi/2}\frac{d\theta}{\sin(\theta)}\right)^2 \le C\frac{\gamma_n^4}{n^{1/4}},$$

having substituted (3.11). The same bound holds for the integral over the set analogous to $\mathcal{O}_n$ but with $\theta,\eta<\pi-\varepsilon$.

To finish, as in step 2 we control the integral over the region where say $0\le\theta<\varepsilon$ by altering our initial bound on the ϕ function(s). Here though this is done in just the variable near the singular point. To illustrate, the bound

$$|\widetilde{S}_n(r,\theta;s,\eta)| \le C\sqrt{\frac{\gamma_n}{n}}\frac{n^{1/4}}{\varepsilon}\frac{e^{-(r+s)/8}}{\sqrt{\sin(\eta)}} \equiv C\gamma_n\frac{e^{-(r+s)/8}}{\sqrt{\sin(\eta)}},$$

is again valid throughout the region described in (3.21), but useful only when θ is small where it produces

$$\int_{\mathcal{O}_n\cap\{\theta<\varepsilon\}} |\widetilde{S}_n|^2 \le C\gamma_n^2\int_0^\varepsilon |\log(\theta+\varepsilon)|\,d\theta \le C\frac{\gamma_n^2}{n^{1/4}}.$$

Once more, like considerations apply to θ near π (and also of course to the situation where θ and η change roles).

Step 4 dispenses of the case that either r or s is greater than $n^{1/4}$. Once again $|z_n\bar{w}_n-1|\ge ||z_n\bar{w}_n|-1|\ge C\varepsilon$ and

$$|e^{-n(\mathrm{Im}(z_n)^2+\mathrm{Im}(w_n)^2)}e^{-(1/2)(z_n-\bar{w}_n)^2}e^{nz_n\bar{w}_n}\mathfrak{e}_{n-2}(nz_n\bar{w}_n)| \le C\frac{\gamma_n^{1/2}}{n^{3/4}}e^{-(r+s)/8},$$

exactly as in (3.20), now just employing the definition of ε. The relevant bound on the kernel becomes

$$|\widetilde{S}_n(r,\theta;s,\eta)| \le Cn^{1/4}\gamma_n(1+|r|+|s|)e^{-(r+s)/8}.$$

Here we have again used the simplified bound $\phi(a)\le e^{-a^2/2}$, as well as the even rougher estimate $|z_n-\bar{w}_n|\le(C+|r|+|s|)$ in the prefactor. In any case it is enough. The square integral of the above restricted to $\{r \vee s > n^{1/4}\}$ is dominated by $Ce^{-n^{1/4}/C}$.

Step 5 is to note that everything above applies to $\widetilde{DS}_n$ (or $\widetilde{IS}_n$) with one notable change. The appearance of $(w_n - z_n)$ [or $(\bar{w}_n - \bar{z}_n)$] in (3.17) rather than $(\bar{w}_n - z_n)$, leads to the replacement of the factor $[\sin(\frac{\theta+\eta}{2}) + \varepsilon]$ with $[\sin(\frac{\theta-\eta}{2}) + \varepsilon]$. The latter is $O(\varepsilon)$ on any $\mathcal{N}_{n,\theta}$, producing an additional decay along the diagonal. This completes the proof. $\square$

We close this section with the following:

PROOF OF COROLLARY 1.4. One needs to show that, for any nonnegative $f(r, \theta)$ supported on $\{r > t\}$,

$$\lim_{n\to\infty} \mathbb{E}\biggl[\prod_{z_k \in \mathbb{C}_+} e^{-f(r_k', \theta_k')}\biggr] = \exp\biggl[-\int_t^\infty \int_0^\pi (1 - e^{-f(r,\theta)}) \frac{1}{2\pi} e^{-r}\, dr\, d\theta\biggr], \tag{3.22}$$

recall the scaling $z_k \mapsto z_k' = (r_k', \theta_k')$ from the statement. But, by the above, the square of the expectation on the left is $\det(I - K_n(1 - e^{-f}))$. Since $|1 - e^{-f}|$ is bounded by χ, all the estimates in the previous proof apply with the result being the exponential of the trace of $-K_n(1 - e^{-f})$. This is exactly the right-hand side (3.22). $\square$

4. Real points.

We run through the calculation over even values of n, returning to the modifications required for n odd at the end.

4.1. *n even.* First the determinant of (2.4) at finite n is reduced to that of a scalar operator. Throughout this section any χ appearing on its own denotes $\chi = \chi_{\{t<x<\infty\}}$.

LEMMA 4.1. *With K_n defined in* (2.4) *we have that*

$$\det(I - K_n\chi) = \det(I - T_n\chi)\det(I - W_n). \tag{4.1}$$

Here T_n is the symmetric part of S_n [*recall* (2.6),

$$T_n(x, y) = \frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}} e^{-xy} \mathfrak{e}_{n-2}(xy)] \tag{4.2}$$

and W_n is a finite rank operator defined in (4.4).

This step in particular mimics Tracy and Widom's treatment of GOE [Tracy and Widom (1998)] quite closely. Next, introducing the scaling as in

$$\widetilde{T}_n(x, y) = T_n(\sqrt{n} + x, \sqrt{n} + y), \tag{4.3}$$

the convergence of the first factor in (4.1), and more, is dealt with by the following [the point being that $\det(I - \widetilde{T}_n\chi) = \det(I - \chi\widetilde{T}_n\chi)$].

LEMMA 4.2. *For all $t > -\infty$, the L^2 operator $\chi \widetilde{T}_n \chi$ converges in trace norm to $T\chi$ with the kernel for T defined in* (1.1). *Further, $\chi \widetilde{T}_n \chi \to \chi T_n \chi$ and $(I - \chi \widetilde{T}_n \chi)^{-1} \to (I - \chi T \chi)^{-1}$ in L^1, L^2 and L^∞ operator norms.*

The last step deals with the W_n operator appearing in the second factor of (4.1). The proof of Lemma 4.1 will show that W_n is of the form

$$W_n = \alpha_1 \otimes \beta_1 + \alpha_2 \otimes \beta_2 \tag{4.4}$$

in which

$$\begin{aligned} &\alpha_1 = (I - T_n\chi)^{-1}\phi_n, \qquad \beta_1 = \chi\psi_n, \\ &\alpha_2 = \tfrac{1}{2}\big((\psi_n, I-\chi)(I - T_n\chi)^{-1}\phi_n + (I - T_n\chi)^{-1}T_n(I-\chi)\big), \\ &\beta_2 = \delta_t - \delta_\infty, \end{aligned}$$

$\phi_n(x) = \kappa_n \int_0^x u^{n-2} e^{-(1/2)u^2}\,du$ and $\psi_n(x) = \kappa_n' x^{n-1} e^{-(1/2)x^2}$ (with certain constants κ_n, κ_n'). The determinant of $I - W_n$ is then comprised explicitly of the L^2-inner products $(\alpha_i, \beta_j)_{1\le i,j\le 2}$, and what we need is the following:

LEMMA 4.3. *After scaling as in* (4.3), *the inner products $(\alpha_i, \beta_j)_{1\le i,j\le 2}$ converge to their formal limits.*

The object identified as Γ_t in the statement of Theorem 1.3 is just the expansion of "$\det(I - W_\infty)$."

Before the proofs of Lemmas 4.1, 4.2 and 4.3, we verify, as indicated in the Introduction, that the kernel T does not have a "Christoffel–Darboux" structure. Gérard Letac showed us this short argument. The question is whether there exist functions F and G (which can assumed to be C^2) for which

$$\int_0^\infty e^{-(x+u)^2} e^{-(y+u)^2}\,du = \frac{F(x)G(y) - F(y)G(x)}{x-y}.$$

The answer is no. Since the left-hand side is of the form $e^{-x^2}e^{-y^2}H(x+y)$, it is enough to prove that

$$H(x+y) = \frac{F(x)G(y) - F(y)G(x)}{x-y}$$

is impossible except for F and G proportional, and so $H = 0$. Making the change of variables $t = x - y$ and $s = x + y$ we find that

$$\frac{\partial}{\partial t}\frac{1}{t}\big(F(s+t)G(s-t) - F(s-t)G(s+t)\big) = 0.$$

This implies that

$$t \mapsto F(s+t)G(s-t) - tF(s+t)G'(s-t) + tF'(s+t)G(s-t)$$

is an even function. Differentiating this function with respect to t and setting $t = 0$ yields $F'(s)G(s) = F(s)G'(s)$ which was the claim.

PROOF OF LEMMA 4.1. We start with the matrix kernel

$$K_n = \begin{bmatrix} S_n & \delta S_n^{\mathsf{T}} \\ -\epsilon S_n + \epsilon & S_n^{\mathsf{T}} \end{bmatrix}. \tag{4.5}$$

This is exactly analogous to the kernel for GOE given by Tracy and Widom (1998), and we follow the strategy laid out in Section II of that paper. First, since $\delta\epsilon = -I$,

$$\chi K_n \chi = \begin{bmatrix} \chi\delta & 0 \\ 0 & \chi \end{bmatrix} \begin{bmatrix} -\epsilon S_n \chi & S_n^{\mathsf{T}}\chi \\ (-\epsilon S_n + \epsilon)\chi & S_n^{\mathsf{T}}\chi \end{bmatrix}.$$

Using the famous $\det(\mathbf{I} - \mathbf{AB}) = \det(\mathbf{I} - \mathbf{BA})$ trick, we find the Fredholm determinant of $\chi K_n \chi$ equals that of

$$\begin{bmatrix} -\epsilon S_n \chi & S_n^{\mathsf{T}}\chi \\ (-\epsilon S_n + \epsilon)\chi & S_n^{\mathsf{T}}\chi \end{bmatrix} \begin{bmatrix} \chi\delta & 0 \\ 0 & \chi \end{bmatrix} = \begin{bmatrix} -\epsilon S_n \chi\delta & S_n^{\mathsf{T}}\chi \\ -\epsilon S_n \chi\delta + \epsilon\chi\delta & S_n^{\mathsf{T}}\chi \end{bmatrix}.$$

The determinant is further unaffected if we subtract the first row from the second and then add the second column to the first, resulting in

$$\begin{bmatrix} S_n^{\mathsf{T}}\chi - \epsilon S_n \chi\delta & S_n^{\mathsf{T}}\chi \\ \epsilon\chi\delta & 0 \end{bmatrix}.$$

The best way to understand this is to note this pair of moves is affected by $K_n \mapsto P K_n P^{-1}$ with $P = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus

$$\begin{aligned} \mathbb{P}_{n,\mathbb{R}}(t)^2 &= \det \begin{bmatrix} I - S_n^{\mathsf{T}}\chi + \epsilon S_n \chi\delta & -S_n^{\mathsf{T}}\chi \\ -\epsilon\chi\delta & I \end{bmatrix} \\ &= \det \begin{bmatrix} I - S_n^{\mathsf{T}}\chi + \epsilon S_n \chi\delta - S_n^{\mathsf{T}}\chi\epsilon\chi\delta & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

which follows by "row reducing" the matrix. And since one may check that $\epsilon S_n = S_n^{\mathsf{T}}\epsilon$, we find that for even n,

$$\mathbb{P}_{n,\mathbb{R}}(t)^2 = \det\bigl(I - S_n^{\mathsf{T}}\chi + S_n^{\mathsf{T}}(1-\chi)\epsilon\chi\delta\bigr). \tag{4.6}$$

The above manipulations have been carried out completely formally, with no attention as to in which space(s) the operators/determinants reside. The needed technical details may be taken (yet again) verbatim from Tracy and Widom (1998); see Section VIII.

Now for even n recall the kernel

$$\begin{aligned} S_n^{\mathsf{T}}(x,y) &= \frac{1}{\sqrt{2\pi}} e^{-(1/2)(x-y)^2} e^{-xy} \mathfrak{e}_{n-2}(xy) \\ &\quad + \frac{1}{\sqrt{2\pi}} \frac{y^{n-1}e^{-y^2/2}}{(n-2)!} \int_0^x u^{n-2} e^{-(1/2)u^2}\,du \\ &= T_n(x,y) + U_n(x,y), \end{aligned}$$

where we have previously defined T_n as the symmetric part of S_n, and consider now U_n the remainder. For later it will be useful to express U_n (as an operator) as

$$U_n = \phi_n \otimes \psi_n = \left(\kappa_n \int_0^x u^{n-2} e^{-(1/2)u^2}\, du\right) \otimes \left(\kappa_n' y^{n-1} e^{-(1/2)y^2}\right), \tag{4.7}$$

where $\kappa_n = \sqrt{\frac{n^{1/2}}{\sqrt{2\pi}(n-2)!}}$ and $\kappa_n' = \sqrt{\frac{n^{-1/2}}{\sqrt{2\pi}(n-2)!}}$. Keep in mind of course that $\phi_n = \epsilon\phi_n'$, $\phi_n'(x) = \kappa_n x^{n-2} e^{-(1/2)x^2}$.

Next, introduce the resolvent

$$(I - T_n\chi)^{-1} = I + R_n \quad \text{or} \quad R_n = T_n\chi(I - \chi T_n \chi)^{-1} \tag{4.8}$$

with kernel $R_n(\cdot,\cdot)$. The determinant (4.6) factors as in

$$\begin{aligned}\mathbb{P}_{n,\mathbb{R}}(t)^2 = \det(I - T_n\chi)\det\big(I - &\big((I - T_n\chi)^{-1}\phi_n\big) \otimes (\chi\psi_n) \\ &+ (I - T_n\chi)^{-1} S_n^{\mathsf{T}}(1-\chi)\epsilon\chi\delta\big),\end{aligned} \tag{4.9}$$

having used $A(B \otimes C)D = (AB) \otimes (D^{\mathsf{T}}C)$. The second term (in which $\epsilon\chi\delta$ appears) is simplified by considering the commutator

$$[\chi, \delta] = -(\delta_t \otimes \delta_t - \delta_\infty \otimes \delta_\infty),$$

where δ_a is the dirac delta. Since again $\epsilon\delta = -I$,

$$(1-\chi)\epsilon[\chi,\delta] = (1-\chi)\epsilon\chi\delta = -(1-\chi)(\epsilon_t \otimes \delta_t - \epsilon_\infty \otimes \delta_\infty)$$

with now

$$\epsilon_t(x) = \tfrac{1}{2}\operatorname{sgn}(t-x) \qquad \big(\text{so } \epsilon_\infty(x) \equiv \tfrac{1}{2}\big).$$

Thus

$$(I + R_n)S_n^{\mathsf{T}}(I-\chi)\epsilon\chi\delta = -\tfrac{1}{2}(I+R_n)S_n^{\mathsf{T}}(1-\chi) \otimes (\delta_t - \delta_\infty).$$

Expanding out the S_n and noting $(A \otimes B)(C \otimes D) = (B, C)(A \otimes D)$, the second factor in (4.9) is the determinant of the identity minus the finite rank operator

$$\begin{aligned}W_n &= \big((I - T_n\chi)^{-1}\phi_n\big) \otimes (\chi\psi_n) \\ &\quad + \tfrac{1}{2}\big((\psi_n, 1-\chi)(I - T_n\chi)^{-1}\phi_n + (I - T_n\chi)^{-1}T_n(1-\chi)\big) \otimes (\delta_t - \delta_\infty),\end{aligned}$$

to which we apply the well-known fact

$$\det(I - W_n) = \det\big(\delta_{i,j} - (\alpha_i, \beta_j)\big)_{1\le i,j\le 2}$$

with, as announced above,

$$\begin{aligned}&\alpha_1 = (I - T_n\chi)^{-1}\phi_n, \qquad \beta_1 = \chi\psi_n, \\ &\alpha_2 = \tfrac{1}{2}\big((\psi_n, 1-\chi)(I - T_n\chi)^{-1}\phi_n + (I - T_n\chi)^{-1}T_n(1-\chi)\big), \\ &\beta_2 = \delta_t - \delta_\infty.\end{aligned} \tag{4.10}$$

Here (α_i, β_j) are regular L^2-inner products. (We have reused δ many times—here it is the standard Kronecker delta.) In particular, all components comprising the original $\det(I - K_n\chi)$ are well defined.

We conclude with a few simplifications. First,

$$(\alpha_1, \beta_1) = \big(\chi\phi_n, (I - \chi T_n\chi)^{-1}\psi_n\big),$$
$$(\alpha_1, \beta_2) = \big((I - T_n\chi)^{-1}\phi_n\big)(t) - \phi_n(\infty). \tag{4.11}$$

In the second line $(I - T_n\chi)^{-1}\phi_n = \phi_n + T_n\chi(I - \chi T_n\chi)^{-1}\phi_n$ is used. Next, since $(T_n + T_n R_n^{\mathsf{T}})\chi = R_n$, it follows that

$$\big((I + R_n)T_n(1-\chi), \chi\psi_n\big) = (1 - \chi, R_n\psi_n),$$
$$\big((I + R_n)T_n(1-\chi), \delta_t\big) = \big(I - \chi, R_n(\cdot, t)\big)$$

and, with $c_n = (\psi_n, 1 - \chi)$,

$$(\alpha_2, \beta_1) = \frac{1}{2}\bigg(c_n(\alpha_1, \beta_1) - c_n + \int_{-\infty}^{t} (I + R_n)\psi_n(x)\,dx\bigg),$$
$$(\alpha_2, \beta_2) = \frac{1}{2}\bigg(c_n(\alpha_1, \beta_2) + \int_{-\infty}^{t} R_n(x, t)\,dx\bigg). \tag{4.12}$$

Combining (4.11) and (4.12), $\det(I - W_n)$ equals

$$\big(1 - (\alpha_1, \beta_1)\big)\bigg(1 - \frac{1}{2}\int_{-\infty}^{t} R_n(x,t)\,dx\bigg) - \frac{1}{2}(\alpha_1, \beta_2)\int_{-\infty}^{t}(I + R_n)\psi_n(x)\,dx;$$

note the c_n factor has dropped out. This last expression has precisely the same structure as equation (41) in Tracy and Widom (1998). □

PROOF OF LEMMA 4.2. We employ the trace convergence criteria of Theorem 2.20 of Simon (2005), showing that

$$\int_t^\infty \widetilde{T}_n(x,x)\,dx \to \int_t^\infty T(x,x)\,dx \tag{4.13}$$

and

$$\int_t^\infty\int_t^\infty f(x)\widetilde{T}_n(x,y)g(y)\,dx\,dy \to \int_t^\infty\int_t^\infty f(x)T(x,y)g(y)\,dx\,dy \tag{4.14}$$

for all $f, g \in L^2([t,\infty))$. Both follow from the pointwise convergence of $\widetilde{T}_n$ to T, and an easy domination.

Though $\widetilde{T}_n \to T$ pointwise already demonstrated in Borodin and Sinclair (2009), it is useful here to establish some local uniformity. Start with the expression

$$\widetilde{T}_n(x,y) = \frac{e^{-(1/2)(x-y)^2}}{\sqrt{2\pi}} e^{-n\eta_n}\mathfrak{e}_{n-2}(n\eta_n), \qquad \eta_n = \eta_n(x,y) = 1 + \frac{x+y}{\sqrt{n}} + \frac{xy}{n}.$$

Repeating the estimate from Lemma 3.2 (as used in step 1 of Lemma 3.1) yields

$$\widetilde{T}_n(x,y) = \frac{e^{-(1/2)(x-y)^2}}{2\sqrt{\pi}} \frac{\mu(\eta_n)\eta_n}{\eta_n - 1} \operatorname{erfc}(\sqrt{n}\mu(\eta_n))\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right),$$

uniformly, granted that $\eta_n \geq 0$ which holds say for x and y bounded and n large enough. Since $\sqrt{n}\mu(\eta_n) \to \frac{x+y}{\sqrt{2}}$ for uniformly for x and y on compacts, it follows $\widetilde{T}_n(x,y) \to \frac{e^{-(1/2)(x-y)^2}}{2\sqrt{2\pi}} \operatorname{erfc}(\frac{x+y}{\sqrt{2}})$ in the same fashion. A change of variables shows this object is equivalent to T. By the smoothness of the functions involved, we also have a constant C so that $\widetilde{T}_n(x,y) \leq CT(x,y)$ for all x and y bounded.

For the rest of domination, on diagonal $\eta_n = (1 + x/\sqrt{n})^2$ is always nonnegative, and we can continue as above. In particular if $x \geq 1$,

$$\frac{\mu(\eta_n)\eta_n}{\eta_n - 1} \operatorname{erfc}(\sqrt{n}\mu(\eta_n)) \leq \frac{\eta_n}{\sqrt{n}(\eta_n - 1)} e^{-n\mu^2(\eta_n)} \leq x e^{-x^2},$$

since $\mu^2((1+a)^2) \geq a^2$. This is enough to conclude that (4.13) holds. Off diagonal, let $x + y \geq 1$, go back to the definition of $\widetilde{T}_n(x,y)$, and note quite simply that $|e^{-n\eta_n}\mathfrak{e}_{n-2}(n\eta_n)| \leq e^{-n\eta_n + n|\eta_n|}$. Hence, when $xy > 0$ as well we have that $\widetilde{T}_n(x,y) \leq e^{-(1/2)(x-y)^2}$ which controls that range of the integral in (4.14): $\int e^{-(1/2)(x-y)^2} f(y)\,dy \in L^2$ for $f \in L^2$. On the other hand, if, for instance, $x > 0$ and $y < 0$ (requiring $t < 0$), the same observation gives $\widetilde{T}_n(x,y) \leq e^{-(1/2)x^2} e^{4|t|x}$ which suffices for the remaining variable range in (4.14).

The trace norm convergence certainly implies the L^2 operator norm convergence of $\chi\widetilde{T}_n\chi$. More directly though, any symmetric kernel operator of the form $\chi M\chi$ has $L^p \mapsto L^p$ norm, for $p = 1, 2, \infty$ bounded as in

$$\|\chi M\chi\| \leq \sup_{y>t} \int_t^\infty |M(x,y)|\,dx;$$

the $L^2 \mapsto L^2$ bound, less familiar than the $L^2 \otimes L^2$ kernel norm bound, is due to Holmgren [Lax (2002), Section 16.1]. The estimates above will then imply that $\chi\widetilde{T}_n\chi \to \chi T\chi$ in the L^1 and L^∞ operator norms too. As for the resolvents, it is enough to check that

$$\sup_{y>t} \int_t^\infty T(x,y)\,dx \leq \sup_{y>t} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(u+y)^2}\,du = \frac{1}{\sqrt{\pi}} \int_t^\infty e^{-u^2}\,du < 1$$

for $t > -\infty$. □

PROOF OF LEMMA 4.3. The scaling is to shift t by $\sqrt{n}$ in all appearances of χ (and so R_n). This shift then filters into ϕ_n, ψ_n and so on by changing variables in each (α_i, β_j). Again all scaled functions/operators are decorated with tildes.

With now $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $G(x) = \int_{-\infty}^{x} g(s)\,ds$ one readily finds: pointwise as $n \to \infty$,

$$\tilde{\psi}_n(x) \to g(x), \qquad \tilde{\phi}_n(x) \to G(x). \tag{4.15}$$

The first convergence also takes place in $L^1 \cap L^2$, while the latter can be considered to hold in $\mathbb{R} + L^2(s, \infty)$ for any $s > -\infty$ upon writing $\phi_n(x) = \phi_n(\infty) - \int_x^\infty \phi_n'$.

Allied considerations will also show that

$$\widetilde{T}_n(x,t) \to T(x,t), \qquad \int_{-\infty}^{\infty} \widetilde{T}_n(x,y)\,dy \to \int_{-\infty}^{\infty} T(x,y)\,dy, \tag{4.16}$$

pointwise and in $(L^1 \cap L^2)(t, \infty)$.

Starting with (α_1, β_1) we find that

$$\begin{aligned}(\tilde{\alpha}_1, \tilde{\beta}_1) &= \big(\tilde{\phi}_n \chi, (I - \chi \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n\big) \\ &= \big(\tilde{\phi}_n(\infty)\chi, \tilde{\psi}_n\big) - \big((\tilde{\phi}_n(\infty) - \tilde{\phi}_n)\chi, (I - \chi \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n\big).\end{aligned}$$

The convergence of $\tilde{\phi}_n(\infty) \to 1$ can be viewed as holding uniformly, and, by the second item in (4.15), $\tilde{\phi}_n(\infty) - \tilde{\phi}_n(\cdot)$ converges in $L^2(t, \infty)$. Further, $(I - \chi \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n(\cdot) \to (I - \chi T \chi)^{-1} g(\cdot)$ in $L^1 \cap L^2$ by the first item in (4.15) and Lemma 4.2.

The next three items are treated similarly (to each other). Take $\int_{-\infty}^{t} R_n(x,t)\,dx$, and rewrite the scaled version as

$$\begin{aligned}&\int_{-\infty}^{t} \widetilde{R}_n(x,t)\,dx \\ &\qquad = \int_{-\infty}^{t} \widetilde{T}_n \chi (I - \chi \widetilde{T}_n \chi)^{-1} \widetilde{T}_n(x,t)\,dx + \int_{-\infty}^{t} \widetilde{T}_n(x,t)\,dx.\end{aligned} \tag{4.17}$$

The first term is the $L^2(t, \infty)$ inner product of the functions

$$\int_{-\infty}^{t} \widetilde{T}_n(x, \cdot)\,dx \quad \text{and} \quad (I - \chi \widetilde{T}_n \chi)^{-1} \widetilde{T}_n(\cdot, t), \tag{4.18}$$

each of which converges in $L^2(t, \infty)$ by (4.16). The second term converges to $\int_{-\infty}^{t} T(x,t)\,dx$ also by (4.16). [As in Tracy and Widom (1998), it is most convenient to see this by writing $\int_{-\infty}^{t} R = (\int_{-\infty}^{\infty} - \int_t^\infty) R_n$ before applying the identity inherent in (4.17).]

The term $\int_{-\infty}^{t} (I - T_n \chi)^{-1} \psi\,dx$ is easier. Now we have that

$$\int_{-\infty}^{t} (I + \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n\,dx = \int_{-\infty}^{t} \widetilde{T}_n \chi (I - \chi \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n(x)\,dx + \int_{-\infty}^{t} \tilde{\psi}_n(x)\,dx.$$

The only real change is the replacement of $(I - \chi \widetilde{T}_n \chi)^{-1} \widetilde{T}_n(\cdot, t)$, appearing in (4.17), with $(I - \chi \widetilde{T}_n \chi)^{-1} \tilde{\psi}_n(\cdot)$. We already have noted that this tends to its formal limit in L^2. Finally, $\int_{-\infty}^{t} \tilde{\psi}_n(x)\,dx$ is the same as $\tilde{\phi}_n(t)$ up to trivial factors and also converges to $G(t)$.

Returning to (α_1, β_2), we only need deal with

$$\widetilde{T}_n\chi(I-\chi\widetilde{T}_n\chi)^{-1}\tilde{\phi}_n(t) = \int_t^\infty \widetilde{T}_n(t,x)(I-\chi\widetilde{T}_n\chi)^{-1}\tilde{\phi}_n(x)\,dx.$$

Again, one can decompose $\tilde{\phi}_n(x) = \tilde{\phi}_n(\infty) - (\tilde{\phi}_n(\infty) - \tilde{\phi}_n(x))$ and alternatively use the $L^1(t,\infty)$ or $L^2(t,\infty)$ convergence of $\widetilde{T}_n(t,\cdot)$ coupled with the $L^2 + L^\infty$ convergence of $(I-\chi\widetilde{T}_n\chi)^{-1}\tilde{\phi}_n(x)$. $\square$

4.2. *n odd.* By Proposition 2.2, when n is odd

$$\mathbb{P}_{\mathbb{R},n}(t)^2 = \det(I - K_n\chi),$$

where

$$K_n = \begin{bmatrix} S_n & \delta S_n^\mathsf{T} \\ -\epsilon S_{n-1} + \epsilon + (\phi_n \otimes \varphi_n - \varphi_n \otimes \phi_n) & S_n^\mathsf{T} \end{bmatrix},$$

where ϕ_n is as in (4.7) and $\varphi = \epsilon\psi_n$ with also ψ as in (4.7).

Performing the same maneuvers as for the even n kernel, we find that $\mathbb{P}_{\mathbb{R},n}(t)^2$ (n odd) equals the determinant of

$$\begin{aligned} I &- S_n^\mathsf{T}\chi - \tfrac{1}{2}S_n^\mathsf{T}(1-\chi)\otimes(\delta_t - \delta_\infty) \\ &+ S_n^\mathsf{T}\chi(S_n^\mathsf{T} - S_{n-1}^\mathsf{T})\epsilon\chi\delta + S_n^\mathsf{T}\chi(\phi_n\otimes\varphi_n - \varphi_n\otimes\phi_n)\chi\delta. \end{aligned}$$

Again we can factor out the $(I - T_n\chi)$, and are left with two extra components as compared with (the second factor on the right-hand side of) equation (4.9).

To see that the first extra component, featuring $S_n^\mathsf{T} - S_{n-1}^\mathsf{T}$, gives no contribution to the determinant in the limit note the following. From before $\chi(\widetilde{T}_n - \widetilde{T}_{n-1})\chi \to 0$, in trace as well as the L^1 and L^∞ operator norms, so this piece may be taken out as a perturbation to the (previously factored) $(I - T_n\chi)$. Also, we have that $\tilde{\phi}_n - \tilde{\phi}_{n-1} \to 0$, and $\tilde{\psi}_n - \tilde{\psi}_{n-1} \to 0$ [pointwise and in $\mathbb{R} + L^2(s,\infty)$ or L^2, resp.]. The latter two differences will appear as factors in some appropriate "α's" and "β's" in the limiting finite rank determinant, so it is enough that all L^2-inner products in which they figure are zero.

Similarly, writing

$$\phi_n\otimes\varphi_n - \varphi_n\otimes\phi_n = (\phi_n - \varphi_n)\otimes\varphi_n - \varphi_n\otimes(\phi_n - \varphi_n),$$

the vanishing of $\tilde{\phi}_n - \tilde{\varphi}_n$ pointwise and in $\mathbb{R} + L^2(s,\infty)$ is sufficient to conclude that any limiting inner product in which these terms enter in will also be zero. This completes the verification.

5. Numerics and open questions. Not having a closed form for the limiting distribution of the largest real point prompted us to carry out some straightforward simulations of the matrix ensembles, resulting in a few notable observations surrounding this object, as well as the finite n behavior of both the largest real and (in absolute value) complex points. (From now on we write just "largest complex point or eigenvalue"—that we mean in absolute value should be understood.)

Figure 1 compares the histograms for the largest real and complex points at $n = 36, 64$ and 100. Noticeable right away is the heavy left tail in the real point distribution. One might have expected that the tail going into the bulk of the spectrum would be lighter than that held down simply by the Gaussian weight. On the other hand, recall that there are only $O(\sqrt{n})$ real eigenvalues for $n \uparrow \infty$ [Edelman, Kostlan and Shub (1994)], and so rather weak level repulsion along the real line. As the right tail of this law can be seen to have Gaussian decay, a reasonable conjecture is that the limiting left tail is exponential to leading order.

A closer look at Figure 1 also shows that the empirical distribution of the largest complex point appears far more symmetric than its limiting Gumbel shape would suggest. Figure 2 focuses in on the $n = 100$ case and highlights that at least for moderate n the real point distribution is heavier tailed to the right as well. This gets right into some basic questions on the speed of convergence for these laws.

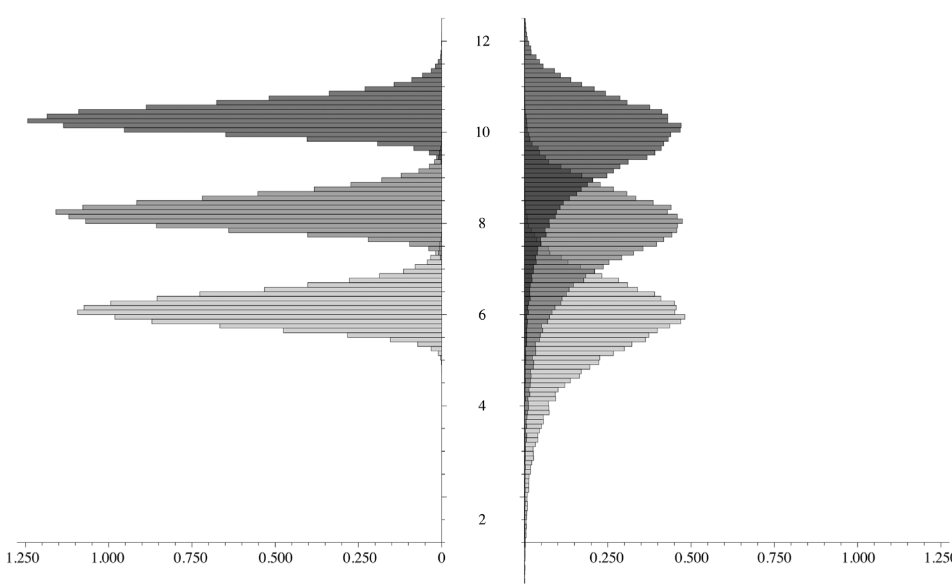

FIG. 1. *Normalized histograms for the modulus of the largest complex eigenvalue* (*left*) *and the largest real eigenvalue of* 40,000 *random* $n \times n$ *matrices for, from lightest to darkest, each of* $n = 36, 64$ *and* 100.

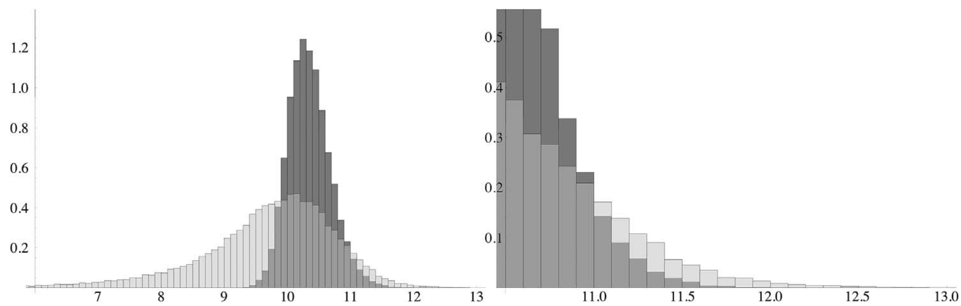

FIG. 2. *Histograms of the largest complex* (*dark*) *and real* (*light*) *eigenvalues of* 40,000 *random* 100×100 *with an enlarged picture of the right hand tails*.

Experts of RMT anticipate fast convergence of finite n statistics to their limiting distribution. Here though the Gumbel shape of the spectral radius will not "kick-in" until n is considerably large. What dictates the phenomena is that the real point is centered about $\sqrt{n}$ while the complex point is centered about $\sqrt{n} + \sqrt{\gamma_n}$ with $\gamma_n = \log(\frac{n}{2\pi(\log n)^2})$. Even as written, this is a purely asymptotic statement, γ_n is not even positive until $n \approx 165$, and in any case one sees that n has to be much larger still until the competing real/complex distributions separate.

A first question might then be what is the chance, at finite n, that the spectral radius comes from a real point? Figure 3 graphs this empirical probability, indicating this is a slowly decaying function, still just under 0.4 for $n = 100$. Quantifying this analytically appears challenging. To start, one would need sharp control of the mixed (real/complex) gap probability, given by the Fredholm determinant/Pfaffian of a 4×4 matrix kernel operator.

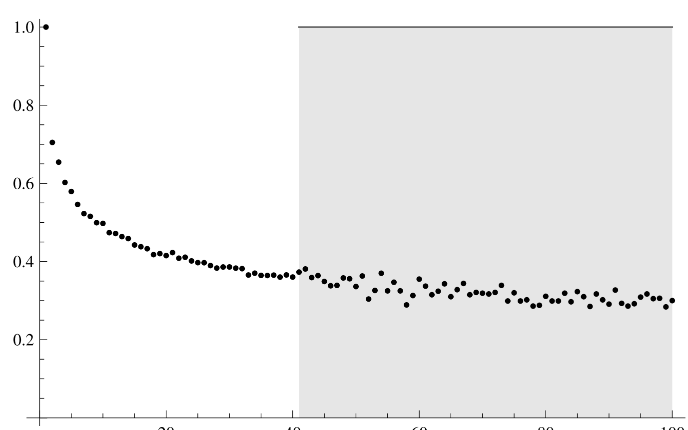

FIG. 3. *The proportion of samples for which the largest eigenvalue was real as a function of* n. *The unshaded region was sampled* 10,000 *times for each* n; *the shaded region* 1000 *times for each* n.

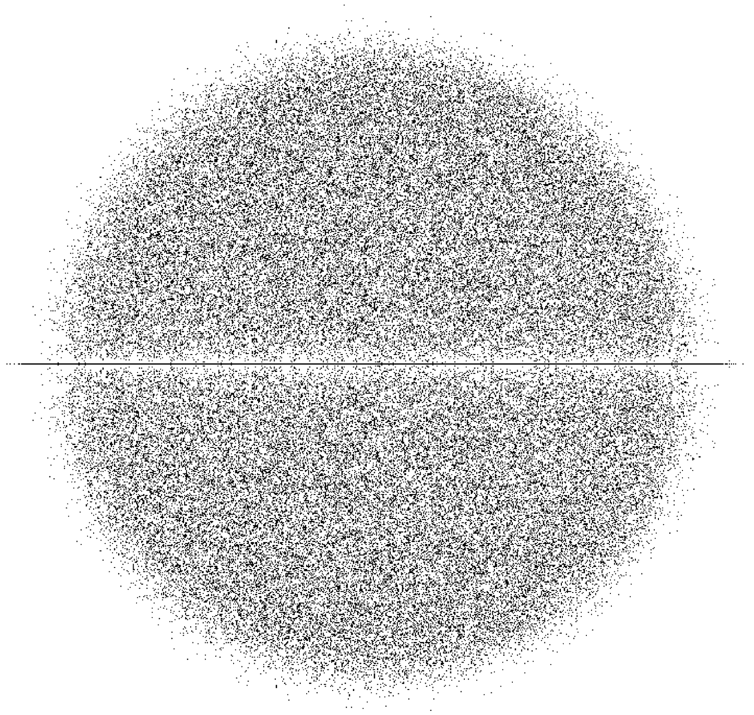

FIG. 4. *A plot of the eigenvalues of* 1000 *random* 100×100 *matrices.*

Together with Figure 2, the phenomena appears to be as follows. At moderate n (e.g., $n = 100$), the largest eigenvalue is most likely complex. However, in the situation where the largest eigenvalue *is* real, it is more likely to be larger than if it were complex. With hindsight one can see this in the most famous picture attached to the real Ginibre ensemble which we repeat an instance of here in Figure 4. The striking feature is the so-called "Saturn effect," based on which alone a person might be forgiven for having conjectured that the largest eigenvalue would be real, with probability one, as $n \uparrow \infty$. Rather, the Saturn effect is a phenomenon which appears from plotting the eigenvalues of many matrices simultaneously. Eventually, the complex points overwhelm the $O(\sqrt{n})$ on the real line.

In summary, one cannot expect the Gumbel law to be a good approximation for the spectral radius at small n. And this is due to more than just the mixture of the separate largest real and complex point laws. Figures 1 and 2 already show that the largest complex point distribution itself is not well approximated by its limiting Gumbel. Figure 5 makes this more transparent. The issue with $\sqrt{\gamma_n}$ not being sensible for smaller n is circumvented by appealing to equation (3.9), $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2\pi}\gamma_n} e^{-\gamma_n/2} = 1$, which more or less defines γ_n. For numerical comparisons then we take γ_n to be the solution γ of $\gamma^2 e^{\gamma} = \frac{n}{2\pi}$.

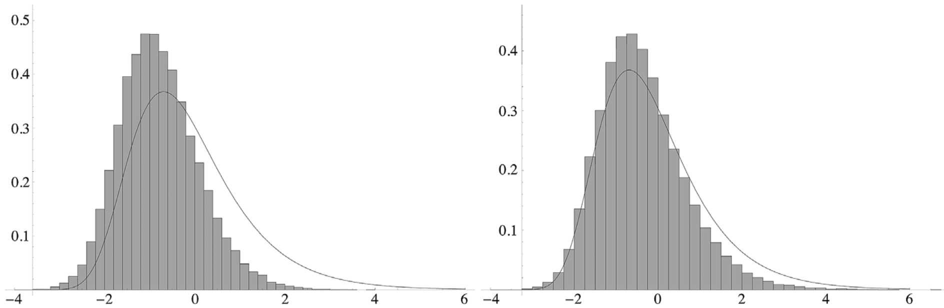

FIG. 5. *A scaled and shifted histogram of the largest complex eigenvalue* (*left*) *and the spectral radius* (*right*) *of* 40,000 100×100 *matrices as compared to the Gumbel density* $\frac{1}{2}e^{-t-(e^{-t}/2)}$.

More confounding in Figure 5 is that a scaled spectral radius appears better approximated by the Gumbel distribution than does the largest complex eigenvalue. This though must be purely superficial. It again comes back to the fact that when the largest eigenvalue is real it tends to be larger than were it complex, thickening the right tail of this histogram to look more Gumbel. As emphasized many times, however, this phenomenon vanishes in the large n limit.

In light of all this, a fair question that remains is how to engineer a decent fluctuation approximation for the spectral radius at finite n.

Simpler questions such as determining just how slow the speed of convergence of the largest complex point is to its Gumbel limit would also be interesting. Working through the proof of Section 3 only produces an $O((\log n)^{-1})$ speed estimate. There is no reason to expect this is close to optimal. On the other hand, Figure 6 compares the spectral radius in the $n = 100$ complex Ginibre ensemble (in which there are no real points with probability one) to its corresponding Gumbel limit. The fit is far more satisfying. Studying the proof from Rider (2003) of this limit theorem gives an $O(\frac{(\log n)^2}{\sqrt{n}})$ speed.

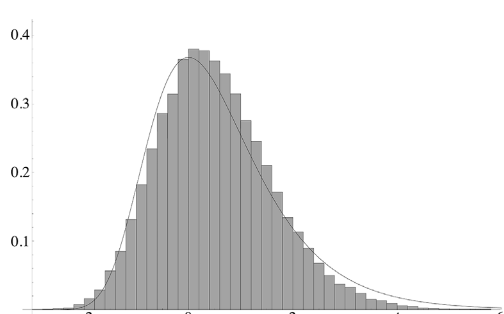

FIG. 6. *A histogram of the spectral radius of* 40,000 *complex random* 100×100 *matrices as compared to the Gumbel density* $e^{-t-e^{-t}}$.

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REFERENCES

BAI, Z. D. (1997). Circular law. *Ann. Probab.* **25** 494–529. MR1428519

BLEHER, P. and MALLISON, R. JR. (2006). Zeros of sections of exponential sums. *Int. Math. Res. Not. IMRN* Art. ID 38937, 49. MR2264712

BORODIN, A. and KANZIEPER, E. (2007). A note on the Pfaffian integration theorem. *J. Phys. A* **40** F849–F855. MR2396206

BORODIN, A. and SINCLAIR, C. D. (2007). Correlation functions of ensembles of asymmetric real matrices. Preprint. Available at arXiv:0706.2670.

BORODIN, A. and SINCLAIR, C. D. (2009). The Ginibre ensemble of real random matrices and its scaling limits. *Comm. Math. Phys.* **291** 177–224. MR2530159

BOURGADE, P., YAU, H. T. and YIN, J. (2012a). Local circular law for random matrices. Preprint. Available at arXiv:1206.1449.

BOURGADE, P., YAU, H. T. and YIN, J. (2012b). The local circular law II: The edge case. Preprint. Available at arXiv:1206.3187.

BOYER, R. and GOH, W. M. Y. (2007). On the zero attractor of the Euler polynomials. *Adv. in Appl. Math.* **38** 97–132. MR2288197

DEIFT, P. (1999). Integral operators. In *Differential Operators and Spectral Theory* (V. Buslaev, M. Solomyak and D. Yafaev, eds.). *American Mathematical Society Translations*, *Series* 2 **189** viii+285. Amer. Math. Soc., Providence, RI. MR1730498

EDELMAN, A. (1997). The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law. *J. Multivariate Anal.* **60** 203–232. MR1437734

EDELMAN, A., KOSTLAN, E. and SHUB, M. (1994). How many eigenvalues of a random matrix are real? *J. Amer. Math. Soc.* **7** 247–267. MR1231689

FORRESTER, P. J. and MAYS, A. (2009). A method to calculate correlation functions for $\beta = 1$ random matrices of odd size. *J. Stat. Phys.* **134** 443–462. MR2485724

FORRESTER, P. J. and NAGAO, T. (2007). Eigenvalue statistics of the real Ginibre ensemble. *Phys. Rev. Lett.* **99** 050603.

FORRESTER, P. J. and NAGAO, T. (2008). Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble. *J. Phys. A* **41** 375003, 19. MR2430570

GINIBRE, J. (1965). Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.* **6** 440–449. MR0173726

GIRKO, V. L. (1984). The circular law. *Teor. Veroyatn. Primen.* **29** 669–679. MR0773436

GOHBERG, I., GOLDBERG, S. and KRUPNIK, N. (2000). *Traces and Determinants of Linear Operators*. *Operator Theory*: *Advances and Applications* **116**. Birkhäuser, Basel. MR1744872

GÖTZE, F. and TIKHOMIROV, A. (2010). The circular law for random matrices. *Ann. Probab.* **38** 1444–1491. MR2663633

HOUGH, J. B., KRISHNAPUR, M., PERES, Y. and VIRÁG, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. *University Lecture Series* **51**. Amer. Math. Soc., Providence, RI. MR2552864

KOSTLAN, E. (1992). On the spectra of Gaussian matrices. *Linear Algebra Appl.* **162/164** 385–388. MR1148410

KRIECHERBAUER, T., KUIJLAARS, A., MCLAUGHLIN, K. and MILLER, P. (2008). Locating the zeros of partial sums of e^z with Riemann–Hilbert methods. In *Proceedings of the Conference "Integrable Systems*, *Random Matrix Theory*, *and Applications" Held at New York University*, *New York*, *May* 22–26, 2006 (J. Baik, T. Kriecherbauer, L.-C. Li, K. D. T. R. McLaughlin and C. Tomei, eds.). *Contemporary Mathematics* **458** xiv+430. Amer. Math. Soc., Providence, RI. MR2424215

LAX, P. D. (2002). *Functional Analysis*. Wiley, New York. MR1892228

LEHMANN, N. and SOMMERS, H.-J. (1991). Eigenvalue statistics of random real matrices. *Phys. Rev. Lett*. **67** 941–944. MR1121461

MAY, R. M. (1972). Will a large complex system be stable? *Nature* **238** 413–414.

RAINS, E. M. (2000). Correlation functions for symmetrized increasing subsequences. Preprint. Available at arXiv:math/0006097.

RIDER, B. (2003). A limit theorem at the edge of a non-Hermitian random matrix ensemble. *J. Phys. A* **36** 3401–3409. MR1986426

SIMON, B. (2005). *Trace Ideals and Their Applications*, 2nd ed. *Mathematical Surveys and Monographs* **120**. Amer. Math. Soc., Providence, RI. MR2154153

SINCLAIR, C. D. (2007). Averages over Ginibre's ensemble of random real matrices. *Int. Math. Res. Not. IMRN* **2007** 1–15.

SINCLAIR, C. D. (2009). Correlation functions for $\beta = 1$ ensembles of matrices of odd size. *J. Stat. Phys*. **136** 17–33. MR2525224

SOMMERS, H.-J. (2007). Symplectic structure of the real Ginibre ensemble. *J. Phys. A* **40** F671–F676. MR2371225

SOMMERS, H.-J. and WIECZOREK, W. (2008). General eigenvalue correlations for the real Ginibre ensemble. *J. Phys. A* **41** 405003, 24. MR2439268

TAO, T. and VU, V. (2010). Random matrices: Universality of ESDs and the circular law. *Ann. Probab.* **38** 2023–2065. MR2722794

TAO, T. and VU, V. (2012). Random matrices: Universality of local spectral statistics of non-Hermitian matrices. Preprint. Available at arXiv:1206.1893.

TRACY, C. A. and WIDOM, H. (1996). On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys*. **177** 727–754. MR1385083

TRACY, C. A. and WIDOM, H. (1998). Correlation functions, cluster functions, and spacing distributions for random matrices. *J. Stat. Phys*. **92** 809–835. MR1657844

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