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THE K-PROCESS ON A TREE AS A SCALING LIMIT OF THE GREM-LIKE TRAP MODEL

BY L. R. G. FONTES[1], R. J. GAVA[2] AND V. GAYRARD

University of São Paulo, Federal University of São Carlos and Aix Marseille Université

We introduce trap models on a finite volume k-level tree as a class of Markov jump processes with state space the leaves of that tree. They serve to describe the GREM-like trap model of Sasaki and Nemoto. Under suitable conditions on the parameters of the trap model, we establish its infinite volume limit, given by what we call a K-process in an infinite k-level tree. From this we deduce that the K-process also is the scaling limit of the GREM-like trap model on extreme time scales under a fine tuning assumption on the volumes.

1. Introduction. The long time behavior of slow dynamics in random environments and phenomena like *aging* is a research theme of recent interest. Trap models and related stochastic processes have been proposed as simple models where these issues can be studied and understood on a rigorous basis. Perhaps the simplest such models are Markov jump processes on given graphs with simple symmetric random walks as embedded chains. The mean jump times at the vertices are random i.i.d. parameters, with heavy tailed distribution, that may be seen as the depths of traps, playing the role of the random environment. The case of $\mathbb{Z}^d$ was extensively analyzed in the physics [14, 24] as well as mathematical literature [1, 6, 8, 16]. The case of the *complete graph* was introduced in [10] as a toy model for the aging behavior of the REM, and is well understood [9–11, 18, 19]. The actual REM dynamics (with Gibbs factors instead of the i.i.d. heavy tailed random variables) was studied in [2, 3], where it is shown that aging is the same as in the complete graph. Refined understanding of this dynamics on a wide range of time scales was obtained in [2, 3, 5, 7, 17, 20]. A natural next step to the analyzes of the REM is to consider correlated Hamiltonians, namely the p-spin SK models and the GREM. The p-spin dynamics was studied in [4, 12, 13] in a particular range of time scales and temperature parameters where aging is the same as in the REM. At present, however, there is no rigorous results about the GREM dynamics. The only results available are nonrigorous theoretical results [10, 25, 26] and concern

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trap-like models of this dynamics. In this work we consider one of these models, namely the GREM-like trap model introduced by Sasaki and Nemoto [25].

Let us first describe the model with a fixed deterministic environment, which we call the *trap model on a tree*, and come back to the GREM-like trap model after that. Let $M_1, \dots, M_k$ be positive integers, and consider a k-level rooted tree whose first generation has size M_1, and such that each vertex in generation $j-1$ has M_j offspring at generation j, $j = 2, \dots, k$. The state space of our model are the leaves of that tree. The parameters of the model are as follows. To each leaf vertex we will attach a positive parameter γ_k, dependent on the vertex. To interior vertices (not leaves), we attach probabilities p_j, $j = 1, \dots, k-1$, also dependent on the vertex, all of them positive, except for p_{root}, which vanishes. See Figure 1 below. The transition mechanism of the trap model is as follows. Once in a given leaf vertex x of the tree, the process waits an exponential time with mean $\gamma_k(x)$ and then jumps. The destination of the jump, another leaf vertex of the tree—let us call it y—is chosen as follows. An ancestor of x on the tree is first chosen by going up the path from x to the root, and independently flipping coins whose probabilities of heads are the p_j's encountered along the way, until tails come up for the first time. The corresponding stopping vertex is the chosen ancestor; let us call it z. We then choose y uniformly at random among the leaf vertices descending from z. The trap model on a tree is thus fully described.

The GREM-like trap model is the trap model on a (k-level) tree for which the γ_k's as well as the inverse of the p_j's, $j = 1, \dots, k-1$, are random variables, independent over the vertices, whose common distribution on a given level j is in the basin of attraction of a stable distribution with index α_j such that $0 < \alpha_j < 1$, $j = 1, \dots, k$. More detailed descriptions of both the trap model on a tree and the GREM-like trap model are done in Section 2.

We are interested in the long time behavior of the GREM-like trap model as the volume diverges. There is of course an issue of how time scales with the volumes,

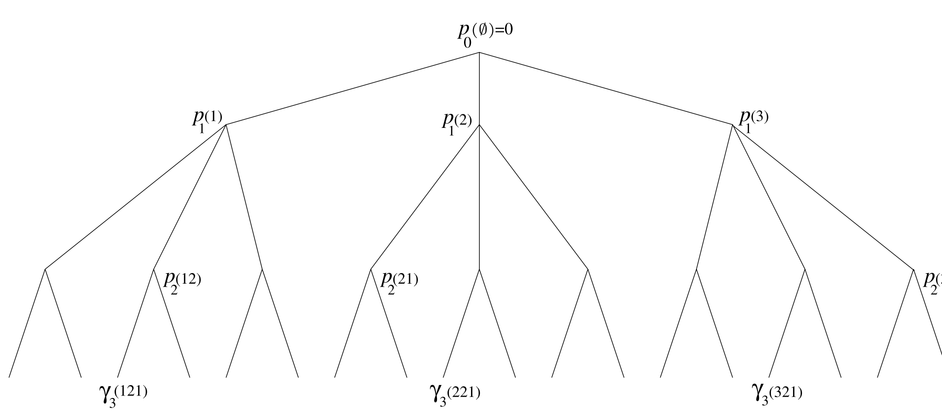

FIG. 1. *A representation of $\mathbb{T}_3^F$ with $M_1 = M_2 = 3$ and $M_3 = 2$, with coin tossing and waiting time parameters appearing beside or below a few vertices, respectively.*

and how the volumes M_j and the indices α_j, $j = 1, \dots, k$, relate to each other. In this paper we derive a scaling limit for the process at times of the order of the maxima of the γ_k's—we qualify this time scale as *extreme*. The α_j's will be taken in strictly increasing order. The volumes will be related to each other in what we call the *fine tuning* regime [see (5.5) on Section 5.5]. On the extreme time scale the process is close to equilibrium, so aging does not take place in this regime. In our setting, aging requires taking a second limit, after first sending the volume to infinity: the macroscopic time must then be sent to zero (as discussed, e.g., in [21]); this will be done in a follow-up paper. Alternatively, we may take a single limit, with a smaller time scale than the extreme one—this is done in [22]; see Remark 5.3 below.

A scaling limit for the GREM-like trap model is stated and proved in Section 5, under the conditions outlined above; see Theorem 5.2. In the same section, we state and prove a general infinite volume limit result for the trap model on a tree; see Theorem 5.4. The proof of Theorem 5.2 is obtained in Section 5.5 by verifying the conditions of Theorem 5.4.

In order to perform the infinite volume limit of the trap model on a tree, we consider an alternative description of that dynamics, since the original description does not straightforwardly suggest an infinite volume version. This is done in Section 3. This representation, a key element of the paper, immediately suggests an infinite volume limit version of the finite volume dynamics, introduced in Section 4.

2. The model. We describe the trap model on a tree in detail now. Let us start with the tree. Throughout, k will be a fixed integer in $\mathbb{N}_* := \{1, 2, \dots\}$. Consider k numbers $M_1, \dots, M_k \in \mathbb{N}_*$, sometimes below called volumes, and let $\mathcal{M}_j = \{1, \dots, M_j\}$, $\mathcal{M}|_j = \mathcal{M}_1 \times \cdots \times \mathcal{M}_j$, $j = 1, \dots, k$. Let us then consider the tree rooted at $\varnothing$

$$\mathbb{T}_k^F = \bigcup_{j=0}^{k} \mathcal{M}|_j, \tag{2.1}$$

where $\mathcal{M}|_0 = \{\varnothing\}$. We will use the notation $x|_j \equiv (x_1, \dots, x_j)$ for a generic element of $\mathcal{M}|_j$. We will also use the notation

$$\mathcal{M}|^j = \mathcal{M}_j \times \cdots \times \mathcal{M}_k, \tag{2.2}$$

$1 \le j \le k$, and let $x|^j \equiv (x_j, \dots, x_k)$ denote a generic element of $\mathcal{M}|^j$.

$\mathbb{T}_k^F$ is of course a finite tree, and this is emphasized in the notation by the use of the superscript "F." We understand the root to be at the 0th generation of $\mathbb{T}_k^F$, and $x|_j \in \mathcal{M}|_j$ to be in its jth generation, $1 \le j \le k$. We will sometimes use simply x for $x|_k$. Given $0 \le i < j \le k$, $x|_i \in \mathcal{M}|_i$ and $x'|_j \in \mathcal{M}|_j$, we will regard $x|_i$ as an *ancestor* of $x'|_j$ whenever $x|_i = x'|_i$.

Let $\bar{\mathbb{N}}_* = \mathbb{N}_* \cup \{\infty\}$. In the sections below, we will consider the infinite tree

$$\mathbb{T}_k = \bigcup_{j=0}^{k} \bar{\mathbb{N}}_*^j, \tag{2.3}$$

where $\bar{\mathbb{N}}_*^0 = \{\varnothing\}$, with generations and ancestors as in $\mathbb{T}_k^F$.

The dynamics we will consider is a continuous time Markov jump process on the set of leaves of $\mathbb{T}_k^F$, namely its kth generation $\mathcal{M}|_k$.

Let us describe the transition mechanism of the process. There will be a set of parameters for that. In order to distinguish this finite tree description from the later infinite tree one (to be presented in Section 4 below) on the one hand, and to emphasize the analogy between the two cases on the other hand, we continue resorting to the use the superscript "F" for the set of parameters of the finite volume process as well.

For $j = 1, \dots, k-1$, let

$$p_j^F : \mathcal{M}|_j \to (0, 1) \tag{2.4}$$

and

$$\gamma_k^F : \mathcal{M}|_k \to (0, \infty). \tag{2.5}$$

For $x \in \mathcal{M}|_k$, let $g_x \in \{0, 1, \dots, k-1\}$ be a random variable such that

$$P(g_x = i) = \bigl[1 - p_i^F(x|_i)\bigr] \prod_{j=i+1}^{k-1} p_j^F(x|_j), \tag{2.6}$$

where by convention $\Pi_{j=k}^{k-1} p_j^F(x|_j) = 1$ and $p_0^F \equiv 0$.

Let Z_k^F be a continuous time Markov chain on $\mathcal{M}|_k$ as follows. When Z_k^F is at $x \in \mathcal{M}|_k$, it waits an exponential time of mean $\gamma_k^F(x)$ and then jumps as follows. It first looks at a copy g_x' of g_x (at each time independent of the copies looked at previously). If $g_x' = j$, then, letting $a_x(j)$ denote the (only) ancestor of x on generation j of $\mathbb{T}_k^F$ [namely $a_x(j) = x|_j$], Z_k^F jumps uniformly at random to one of the *descendants* of $a_x(j)$ in $\mathcal{M}|_k$. In other words, given that $g_x' = j$, then the coordinates $x|_j (= a_x(j))$ of x are left unchanged, and the remainder coordinates are chosen uniformly at random on $\mathcal{M}|^{j+1}$. We may then say that the transition distribution of the jump chain of Z_k^F from x is the uniform distribution on the descendants of an ancestor of x whose generation is randomly chosen according to the distribution of g_x.

REMARK 2.1. We may understand the random variable g_x as follows. Let us attach coins to the sites of the tree that are not leaves, namely, the points of $\bigcup_{j=0}^{k-1} \mathcal{M}|_j = \mathbb{T}_k^F \setminus \mathcal{M}|_k$, in such a way that the probability of heads of the coin at site $x|_j \in \mathcal{M}|_j$ is $p_j^F(x|_j)$, $j = 1, \dots, k-1$; see Figure 1. The coin of the root has probability $p_0^F = 0$ of turning up heads. When it decides to jump from site $x \in \mathcal{M}|_k$, Z_k^F first flips successively the coins of $x|_{k-1}, \dots, x|_1, \varnothing$ (in that order) until it gets tails for the first time, and then it stops at the respective site. Notice that this procedure is almost surely well defined since $p_0^F = 0$. Given that $x|_j$ was the stopping site of the procedure, then $g_x = j$.

DEFINITION 2.2. We call Z_k^F a trap model on $\mathbb{T}_k^F$, or k-level trap model, with *waiting time* parameter γ_k^F and *activation* parameters $(p_j^F)_{j=1}^{k-1}$, and write

$$Z_k^F \sim TM(\mathbb{T}_k^F; \gamma_k^F; (p_j^F)_{j=1}^{k-1}).$$

Our main motivation in considering this model is in the particular case where the parameters are related to the following random variables. For $j = 1, \dots, k$, let $\tau_j := \{\tau_j(x|_j); x|_j \in \mathcal{M}|_j\}$ be an i.i.d. family of positive random variables in the domain of attraction of an α_j-stable law. Now consider the k-level trap model, with *waiting time* parameters $\gamma_k^F(x|_k) \equiv \tau_k(x|_k)$ and *activation* parameters $p_j^F(x|_j) \equiv 1/(1 + \tau_j(x|_j))$, $j = 1, \dots, k-1$. We call this model the *GREM-like trap model* on $\mathbb{T}_k^F$ with parameters τ_j, $j = 1, \dots, k$. We state a scaling limit result for this model in Section 5 below. In the next two sections we present supporting material for that result, as anticipated at the end of the Introduction.

3. A representation of the k-level trap model. In this section we will inductively construct a process X_k^F on $\mathcal{M}|_k$, under a particular choice of whose parameters it is a version of the k-level trap model of last section. As explained in the Introduction, this particular version will help us to formulate the infinite volume limit of the latter model.

The process of this section will involve a set of parameters $\gamma_j^F : \mathcal{M}|_j \to (0, \infty)$, $j = 1, \dots, k$, for given $M_1, \dots, M_k \in \mathbb{N}_*$.

In order to have our inductive construction go smoothly, we introduce an auxiliary process Y_j^F, for bookkeeping reasons only, as will be explained below. We will then have pairs (X_j^F, Y_j^F), $j = 1, \dots, k$. (The auxiliary process will not be needed in the infinite volume version of X_k^F to be introduced in Section 4.) We first define the process (X_1^F, Y_1^F). X_1^F is a continuous time Markov chain on $\mathcal{M}|_1 (= \mathcal{M}_1)$ that, when at $x_1 \in \mathcal{M}|_1$, waits an exponential time of mean $\gamma_1^F(x_1)$ and then jumps uniformly to a site in $\mathcal{M}|_1$. We will construct X_1^F in the following way.

Let $\mathcal{N}_1 = \{(N_r^{(x_1,1)})_{r \geq 0}, x_1 \in \mathbb{N}_*\}$ be i.i.d. Poisson processes of rate 1, and let $\sigma_i^{x_1,1}$ be the ith mark of $N^{(x_1,1)}$ (viewed as a point process), $i \geq 1$. We will call $\mathcal{S}_1^F = \{\sigma_i^{(x_1,1)}; x_1 \in \mathcal{M}_1, i \geq 1\}$ the set of *marks of the first level* of X_k^F. Let $\mathcal{T}_1 = \{T_s^{(1)}, s \in \mathbb{R}^+ := [0, \infty)\}$ be i.i.d. exponential random variables of rate 1. $\mathcal{N}_1$ and $\mathcal{T}_1$ are assumed independent.

For $s \in \mathcal{S}_1^F$, let $\xi_1^F(s) = x_1$ if $s = \sigma_j^{x_1,1}$ for some $x_1 \in \mathcal{M}_1$ and $j \geq 1$. Notice that ξ_1^F is well defined almost surely. Let us now define a measure μ_1^F on $\mathbb{R}^+$ as follows: $\mu_1^F(\{s\}) = \gamma_1^F(\xi_1^F(s))T_s^{(1)}$ if $s \in \mathcal{S}_1^F$ and $\mu_1^F(\mathbb{R}^+ \setminus \mathcal{S}_1^F) = 0$.

REMARK 3.1. We note that $\xi_1^F(s), s \in \mathcal{S}_1^F$, are i.i.d. uniform random variables in $\mathcal{M}_1$.

For $r \geq 0$, let

$$\Gamma_1^F(r) := \mu_1^F([0,r]). \tag{3.1}$$

For $t \geq 0$, let

$$\varphi_1^F(t) := (\Gamma_1^F)^{-1}(t) = \inf\{r \geq 0; \Gamma_1^F(r) > t\} \tag{3.2}$$

be the (right continuous) inverse of Γ_1^F.

Let us recall that $\bar{\mathbb{N}}_* = \{1, 2, \dots, \infty\}$. We define the process (X_1^F, Y_1^F) on $(\bar{\mathbb{N}}_*, \mathbb{R}^+)$ as follows. For $t \geq 0$,

$$(X_1^F, Y_1^F)(t) = (\xi_1^F(\varphi_1^F(t)), \varphi_1^F(t)). \tag{3.3}$$

Let us suppose (X_j^F, Y_j^F) is defined for $j = 1, \dots, l-1$, $l \leq k$.

DEFINITION 3.2. An interval $I \subset \mathbb{R}^+$ is a constancy interval of (X_j^F, Y_j^F) if (X_j^F, Y_j^F) is constant over I, that is,

$$(X_j^F, Y_j^F)(r) = (X_j^F, Y_j^F)(s) \qquad \text{for all } r, s \in I \tag{3.4}$$

and I is maximal with that property.

The maximality condition and right continuity of (X_j^F, Y_j^F) implies that $I = [a, b)$ for some $0 \leq a < b$. We are now ready to define (X_l^F, Y_l^F) for $2 \leq l \leq k$.

Let $\mathcal{I}_{l-1}^F$ be the collection of constancy intervals of (X_{l-1}^F, Y_{l-1}^F). Let also $\mathcal{N}_l = \{(N_r^{(x_l,l)})_{r \geq 0}, x_l \in \mathbb{N}_*\}$ be i.i.d. Poisson processes of rate 1. Let $\sigma_i^{x_l,l}$ the ith mark of $N^{(x_l,l)}$, $i \geq 1$. We will call $\mathcal{S}_l^F = \{\sigma_i^{(x_l,l)}; x_l \in \mathcal{M}_l, i \geq 1\}$ the set of Poisson marks of the lth level, and $\mathcal{R}_l^F = \{a; I = [a, b) \text{ and } I \in \mathcal{I}_{l-1}^F\}$ the set of extra marks of the lth level. Notice that $\mathcal{R}_l^F$ is the set of left endpoints of intervals of $\mathcal{I}_{l-1}$. We call $\mathcal{S}_l^F \cup \mathcal{R}_l^F$ the set of marks of the lth level; see Figure 2. Let $\mathcal{T}_l = \{T_s^{(l)}, s \in \mathbb{R}^+\}$ be i.i.d. exponential random variables of rate 1. $\mathcal{N}_l$ and $\mathcal{T}_l$ are assumed independent and are independent of $\mathcal{N}_j$ and $\mathcal{T}_j$ for $j < l$.

To each $s \in \mathcal{R}_l^F$, we associate a uniform random variable $U_l(s)$ on $\{1, \dots, M_l\}$. Assume that $\{U_l(s), s \in \mathcal{R}_l^F, l \geq 1\}$ are mutually independent and independent of the other random variables in the model. Let

$$\xi_l^F(s) = \begin{cases} x_l, & \text{if } s = \sigma_j^{(x_l,l)} \text{ for some } x_l \in \mathcal{M}_l \text{ and } j \geq 1, \\ U_l(s), & \text{if } s \in \mathcal{R}_l^F. \end{cases} \tag{3.5}$$

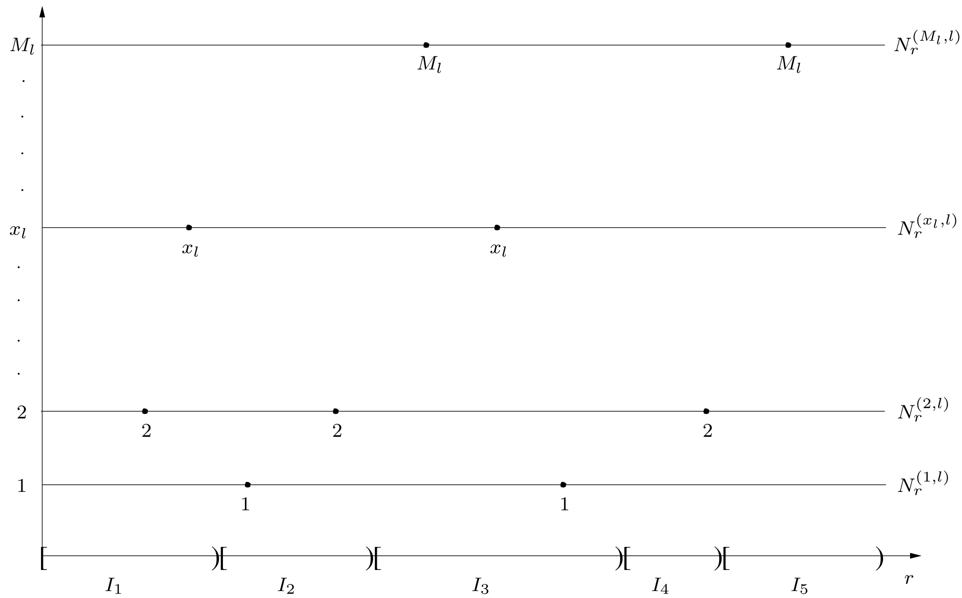

FIG. 2. *Representation of the timelines of the Poisson point processes entering the definition of* X_l^F, *one for each* $x \in \mathcal{M}_l$. *Another ingredient are the constancy intervals of* X_{l-1}^F, *here successively represented in the x-axis as* $I_1, \dots, I_5$.

We will call $\xi_l^F(s)$ the *label* of $s \in \mathcal{R}_l^F \cup \mathcal{S}_l^F$, $1 \le l \le k$ (where $\mathcal{R}_1^F = \varnothing$).

Notice that in the first case of (3.5) above $s \in \mathcal{S}_l^F$, and that ξ_l^F is well defined almost surely. Let us now define a measure μ_l^F on $\mathbb{R}^+$ as follows:

$$\mu_l^F(\{s\}) = \gamma_l^F(X_{l-1}^F(s), \xi_l^F(s))T_s^{(l)} \qquad \text{if } s \in \mathcal{S}_l^F \cup \mathcal{R}_l^F \tag{3.6}$$

and $\mu_l^F(\mathbb{R}^+ \setminus (\mathcal{S}_l^F \cup \mathcal{R}_l^F)) = 0$. Notice that $\mathcal{S}_l^F \cap \mathcal{R}_l^F = \varnothing$ almost surely.

REMARK 3.3. We note that $\xi_j^F(s), s \in \mathcal{S}_j^F \cup \mathcal{R}_j^F$, are i.i.d. uniform random variables in $\mathcal{M}_j$, $j = 1, \dots, l$.

For $r \ge 0$, let

$$\Gamma_l^F(r) := \mu_l^F([0, r]). \tag{3.7}$$

For $t \ge 0$, let

$$\varphi_l^F(t) := (\Gamma_l^F)^{-1}(t) = \inf\{r \ge 0 : \Gamma_l^F(r) > t\} \tag{3.8}$$

be the inverse of Γ_l^F.

We define the process (X_l^F, Y_l^F) on $(\mathcal{M}|_l, \mathbb{R}_+^l)$ as follows. For $t \ge 0$,

$$(X_l^F, Y_l^F)(t) = ((X_{l-1}^F(\varphi_l^F(t)), \xi_l^F(\varphi_l^F(t))); (Y_{l-1}^F(\varphi_l^F(t)), \varphi_l^F(t))); \tag{3.9}$$

see Figure 3.

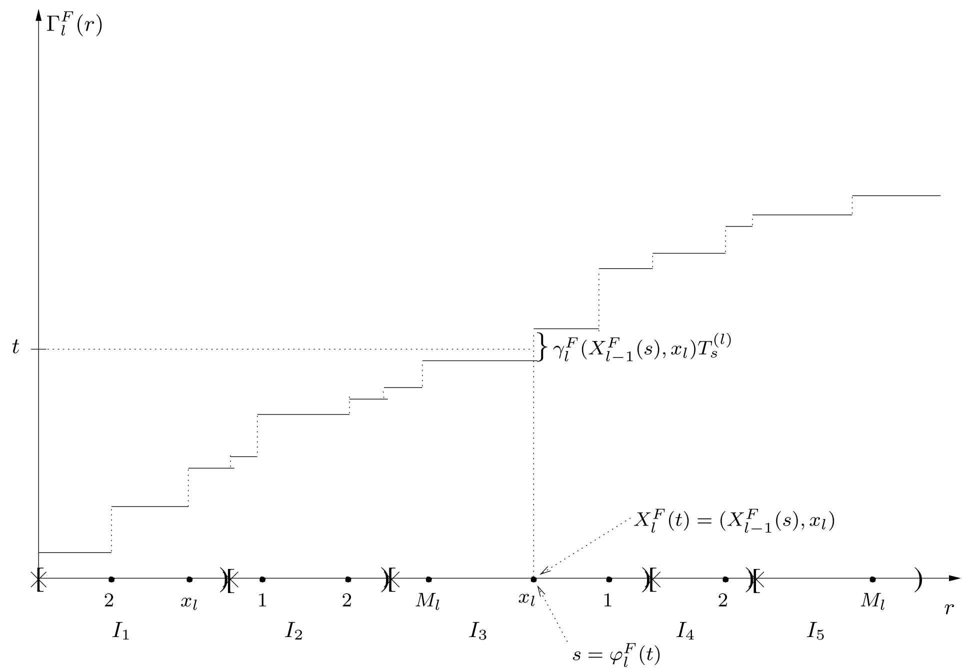

FIG. 3. *Illustration of the construction of X_k^F from the constancy intervals $I_1 = [a_1, b_1), \dots, I_5 = [a_5, b_5)$ of X_{k-1}^F on x-axis, Poisson marks and extra marks [here represented as crosses on left endpoints of the constancy intervals, with (not shown) labels $U_l(a_1), \dots, U_l(a_5)$, resp.]. The Poissonian points may be seen as projected down from timelines of Figure 2. The point "s" in the x-axis equals $\varphi_l^F(t)$.*

REMARK 3.4. We note that each interval I of $\mathcal{I}_j^F$, $j \geq 2$ can be identified with a jump of Γ_j^F, that is, $\mathcal{I}_j^F = \{[\Gamma_j^F(r-), \Gamma_j^F(r)) \neq \varnothing; r \geq 0\} = \{I_j^F(s) := [\Gamma_j^F(s-), \Gamma_j^F(s)); s \in \mathcal{S}_j^F \cup \mathcal{R}_j^F\}$, and the lengths of the intervals of $\mathcal{I}_j^F$, namely $\{|I_j^F(s)|, s \in \mathcal{S}_j^F \cup \mathcal{R}_j^F\}$, are independent exponential random variables with means $\{\gamma_j^F(X_{j-1}^F(s), \xi_j^F(s)), s \in \mathcal{S}_j^F \cup \mathcal{R}_j^F\}$, respectively.

At this point we may observe that our interest is in X_k^F; as anticipated above, Y_k^F is introduced for bookkeeping purposes, solely for the convenience of having the property mentioned in Remark 3.4. In Section 4 below we will introduce an infinite volume version of X_k^F for which the respective version of Y_k^F will not be needed explicitly, and thus not explicitly introduced.

REMARK 3.5. We note that the number of marks of $\mathcal{S}_j^F$ in each interval $I \in \mathcal{I}_{j-1}^F$ (when integrated with respect to the exponential interval length; see Remark 3.4 above) is a geometric random variable[1] with mean $M_j\gamma_{j-1}^F(X_{j-1}^F(s))$, where $s\in\mathcal{S}_{j-1}^F\cup\mathcal{R}_{j-1}^F$ is such that $I=I_{j-1}^F(s)$, and that each such interval has exactly one mark of $\mathcal{R}_j^F$ at its left end. So, the total number of marks (of $\mathcal{S}_j^F\cup\mathcal{R}_j^F$) within I is the above mentioned geometric variable plus one.

DEFINITION 3.6. We call X_k^F defined above the trap model on $\mathbb{T}_k^F$, or k-level trap model, with parameter set $\underline{\gamma_k^F}=\{\gamma_i^F;i=1,\dots,k\}$. Notation: $X_k^F\sim TM(\mathbb{T}_k^F;\underline{\gamma_k^F})$.

We now make the connection between the models of Sections 2 and 3, a key result of this paper, which in particular establishes that the latter is a representation of the former under the appropriate relationship of their respective set of parameters, thus justifying the common terminology.

LEMMA 3.7. *Let X_k^F be as above and Z_k^F be as in Section 2, that is,*

$$Z_k^F\sim TM\big(\mathcal{M}|_k;\gamma_k^F;(p_j^F)_{j=1}^{k-1}\big).$$

Suppose

$$p_j^F(x|_j):=\frac{1}{1+M_{j+1}\gamma_j^F(x|_j)} \tag{3.10}$$

for all $x|_j\in\mathcal{M}|_j$ and $j=1,\dots,k-1$. Then X_k^F and Z_k^F have the same distribution.

PROOF. We begin with the following remark concerning Z_k^F, which follows immediately from the construction of that process in Section 2.

REMARK 3.8. Let $Z_{k,i}^F$, $i=1,\dots,k$, be the ith coordinate of $Z_k^F=(Z_{k,1}^F,\dots,Z_{k,k}^F)$, and let $Z_k^F|_j=(Z_{k,1}^F,\dots,Z_{k,j}^F)$, $j=1,\dots,k$. As pointed out in Section 2 above, the jump chain of Z_k^F, let us call it $J_k^F=(J_{k,1}^F,\dots,J_{k,k}^F)$, with $J_k^F|_j=(J_{k,1}^F,\dots,J_{k,j}^F)$, $j=1,\dots,k$, can be described in terms of the sucessive flips of the coins of $J_k^F|_{k-1},\dots,J_k^F|_1$; see Remark 2.1. After n jumps of J_k^F, let us consider the event $A_{k-1}(n)=\{$flip of the coin of $J_k^F|_{k-1}(n)$ results in heads$\}$. In terms of the random variable $g_{J_k^F(n)}$, we have $A_{k-1}(n)=\{g_{J_k^F(n)}<k-1\}$. We now remark that, given $J_k^F(n)=x|_k$ and $A_{k-1}(n)$, the distribution of the jump from $J_k^F|_{k-1}(n)$ is the same as that from $J_{k-1}^F(n)$ given $J_{k-1}^F(n)=x|_{k-1}$. Since

[1] In this paper, we call a geometric random variable one whose probability function is given by $p^n(1-p)$, $n=0,1,\dots$, where p is a parameter in $(0,1)$, and whose mean is thus in terms of p given by $(1-p)/p$.

the distribution of a jump from $J^F_{k,k}(n)$ is always uniform in $\mathcal{M}_k$ independent from anything else, we have that, given $J^F_{k-1}(n) = x|_k$ and $A_{k-1}(n)$, the distribution of the jump from $J^F_k(n)$ is the same as the joint distribution of the jump from $J^F_{k-1}(n)$ given $J^F_{k-1}(n) = x|_{k-1}$, and an independent uniform random variable in $\mathcal{M}_k$.

For $k = 1$, we remark that the transition probabilities of X^F_1 are always uniform in $\mathcal{M}_1$, since the labels of the successive points of $\mathcal{S}^F_1$ have this property and are independent of each other. So, X^F_1 and Z^F_1 are processes with the same state space and transition probabilities, and the holding times are clearly matched. The result follows for $k = 1$.

We now proceed by induction on k. Let us suppose that the result holds for $k = K - 1$ for some $K \geq 2$. To show that $X^F_K \sim Z^F_K$, it is enough to identify the transition mechanisms of both processes. Let us thus fix a time $t \geq 0$ and a point $x|_K \in \mathcal{M}|_K$. Given that either process is at $x|_K$ at time t, then both jump times are exponentially distributed with mean $\gamma^F_K(x|_K)$. So far, we have an identification. Now let us identify the jump mechanisms of both processes.

Let $\mathfrak{N}_t$ denote the number of jumps of Z^F_K up to time t. As discussed in Remark 3.8 above, given $Z^F_K(t) = x|_K$ and $A_{K-1}(\mathfrak{N}_t)$ we have that $Z^F_K|_{K-1}(t)$ jumps as $Z^F_{K-1}(t)$ given $Z^F_{K-1}(t) = x|_{K-1}$, and $Z^F_{K,K}(t)$ jumps uniformly in $\mathcal{M}_K$, and the jumps of $Z^F_K|_{K-1}(t)$ and $Z^F_{K,K}(t)$ are independent. Let us identify a *coin tossing* mechanism in the jump of X^F_K. Let I be the interval of $\mathcal{I}^F_{K-1}$ containing $\varphi_K(t)$, and let $I' = (\varphi_K(t), \infty) \cap I$. The lack of memory of the exponential distribution of I (see Remark 3.4 above) implies that, given that $X^F_K(t) = x|_K$, then $|I'|$ is an exponential random variable with mean $\gamma^F_{K-1}(x|_{K-1})$. The mechanism in the X^F_K process that plays the role of (the first) coin tossing is whether or not I' contains at least one (Poisson) mark. Let us call $\bar{A}_{K-1}$ the event that I' contains no mark. This corresponds to the coin at $x|_{K-1}$ turning up heads. This means that $X^F_K|_{K-1}$ will take a jump, which we identify as a jump of X^F_{K-1}, independent of the (uniform in $\mathcal{M}_K$) accompanying jump of $X^F_{K,K}$.

At this point we should stress that a particular element of the construction of X^F_K plays a key role in this identification, namely, the inclusion of the marks of $\mathcal{R}^F_K$. This guarantees that each interval of constancy I of $\mathcal{I}^F_{k-1}$ gets at least one mark. Without these marks, the jump of $X^F_K|_{K-1}$ might not coincide with that of X^F_{K-1}—that would happen if (and only if) the first interval of $\mathcal{I}^F_{K-1}$ neighboring I to the right had got no mark.

By the induction hypothesis, $X^F_{K-1} \sim Z^F_{K-1} \sim Z^F_K|_{K-1}$; clearly $X^F_{K,K} \sim Z^F_{K,K}$. We close the argument in two steps. The first one is to show that

$$\begin{aligned} P\bigl(\bar{A}_{K-1} | X^F_K(t) = x|_K\bigr) &= \frac{1}{1 + M_K \gamma^F_{K-1}(x|_{K-1})} \\ &= P\bigl(A_{K-1}(\mathfrak{N}_t) | Z^F_K(t) = x|_K\bigr) \end{aligned} \tag{3.11}$$

and then use the hypothesis. But it follows from our discussion above and the construction of X_K^F that the left-hand side of (3.11) equals $P(N'_{T'}=0)$, where N' is a Poisson process of rate M_k and T' an independent exponential random variable with mean $\gamma_{K-1}^F(x|_{K-1})$, and a simple computation yields the result.

And the last step is to argue that, given $\bar{A}_{K-1}^c$, then $X_K^F|_{K-1}$ does not move, and we have only the uniform in $\mathcal{M}_K$ jump of $X_{K,K}^F$, which agrees with the corresponding move of Z_K^F given A_{K-1}^c. $\square$

4. *K*-process on a tree. K-processes on $\bar{\mathbb{N}}_*$ were introduced in [18] in the study of limits of trap models in the complete graph. They appear as scaling limits of the REM-like trap model in the complete graph. Below we introduce an extension of that model to a model on $\bar{\mathbb{N}}_*^k$, which we will view as the leaves of a tree with k generations, as done similarly in the previous sections. As anticipated in the Introduction and established in the next section, the process of this section turns up in limit results for the processes of the previous sections as volume diverges.

Let $\gamma_j:\mathbb{N}_*^j\to(0,\infty)$, $j=1,\dots,k$, be such that, making

$$\bar{\gamma}_j(x|_j):=\gamma_1(x|_1)\times\gamma_2(x|_2)\times\cdots\times\gamma_j(x|_j), \tag{4.1}$$

we have

$$\sum_{x|_j\in\mathbb{N}_*^j}\bar{\gamma}_j(x|_j)<\infty. \tag{4.2}$$

We will construct a process X_k on $\bar{\mathbb{N}}_*^k$ inductively, similarly as in Section 3. This will be a càdlàg process, similar to the ones we dealt with so far. First we define the process X_1. It is a continuous time Markov chain on $\bar{\mathbb{N}}_*$ described as follows.

Let $\mathcal{N}_1=\{(N_r^{(x_1,1)})_{r\ge0}, x_1\in\mathbb{N}_*\}$ be i.i.d. Poisson processes of rate 1. Let $\sigma_i^{x_1,1}$ be the ith mark of $N^{(x_1,1)}$, $i\ge1$. We will call $\mathcal{S}_1=\{\sigma_i^{(x_1,1)}; x_1\in\mathbb{N}_*, i\ge1\}$ the set of marks of the first level of X_k. Let $\mathcal{T}_1=\{T_s^{(1)}, s\in\mathbb{R}^+\}$ be i.i.d. exponential random variables of rate 1. $\mathcal{N}_1$ and $\mathcal{T}_1$ are assumed independent.

For $s\in\mathcal{S}_1$, let $\xi_1(s)=x_1$ if $s=\sigma_i^{x_1,1}$ for some $x_1\in\mathbb{N}_*$ and $i\ge1$. Notice that ξ_1 is well defined almost surely. Let us now define a measure μ_1 on $\mathbb{R}^+$ as follows:

$$\mu_1(\{s\})=\gamma_1(\xi_1(s))T_s^{(1)},\qquad \text{if } s\in\mathcal{S}_1 \quad\text{and}\quad \mu_1(\mathbb{R}^+\setminus\mathcal{S}_1)=0. \tag{4.3}$$

For $r\ge0$, let

$$\Gamma_1(r):=\mu_1([0,r]) \tag{4.4}$$

and, for $t\ge0$, let

$$\varphi_1(t):=\Gamma_1^{-1}(t)=\inf\{r\ge0:\Gamma_1(r)>t\} \tag{4.5}$$

be the inverse of Γ_1.

REMARK 4.1. Notice that μ_1 is almost surely a purely atomic measure whose set of atoms, $\mathcal{S}_1$, is a.s. countable and dense in $\mathbb{R}^+$. Moreover, from Lemma 4.6 below, it is a.s. σ-finite. These properties imply that $\Gamma_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a.s. strictly increasing and that its range $\Gamma_1(\mathbb{R}^+)$ is an uncountable set (since it is the image of an uncountable set, $\mathbb{R}^+$, by a 1 to 1 map) of Lebesgue measure zero. It follows from this and the independence and continuity of its constituents that any fixed deterministic r is a.s. a continuity point of Γ_1. It may also be checked that $\varphi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a.s. continuous.

In order to make the processes to be defined below càdlàg, we need the following general definition.

DEFINITION 4.2. Given a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $t \in \mathbb{R}$, we say that f is *upper locally constant* at t if there exists an $\epsilon > 0$ such that f is constant in $[t, t+\epsilon]$; let ULC_f denote the set $\{t \in \mathbb{R}^+ : f \text{ is upper locally constant at } t\}$.

We define X_1 on $\bar{\mathbb{N}}_*$ as follows. For $t \geq 0$

$$X_1(t) = \begin{cases} \xi_1(\varphi_1(t)), & \text{if } \varphi_1(t) \in \mathcal{S}_1 \text{ and } t \in \mathrm{ULC}_{\varphi_1}, \\ \infty, & \text{otherwise}. \end{cases} \tag{4.6}$$

Suppose X_j is defined for $j = 1, \dots, l-1$, $2 \leq l \leq k$. Let $\mathcal{N}_l = \{(N_r^{(x_l,l)})_{r\geq 0}, x_l \in \mathbb{N}_*\}$ be i.i.d. Poisson processes of rate 1. Let $\sigma_i^{x_l,l}$ the ith mark of $N^{(x_l,l)}$, $i \geq 1$. We will call $\mathcal{S}_l = \{\sigma_i^{(x_l,l)}; x_l \in \mathbb{N}_*, i \geq 1\}$ the set of Poisson marks of the lth level. Let $\mathcal{T}_l = \{T_s^{(l)}, s \in \mathbb{R}^+\}$ be i.i.d. exponential random variables of rate 1. $\mathcal{N}_l$ and $\mathcal{T}_l$ are assumed independent and are independent of $\mathcal{N}_j$ and $\mathcal{T}_j$ for $j < l$.

For $s \in \mathcal{S}_l$, let $\xi_l(s) = x_l$ if $s = \sigma_j^{(x_l,l)}$ for some $x_l \in \mathbb{N}_*$ and $j \geq 1$. Notice that ξ_l is well defined almost surely. Let us now define a measure μ_l on $\mathbb{R}^+$ as follows:

$$\mu_l(\{s\}) = \gamma_l(X_{l-1}(s), \xi_l(s)) T_s^{(l)}, \qquad \text{if } s \in \mathcal{S}_l \quad \text{and} \quad \mu_l(\mathbb{R}^+ \setminus \mathcal{S}_l) = 0.$$

For $r \geq 0$, let

$$\Gamma_l(r) := \mu_l([0, r]) \tag{4.7}$$

and for $t \geq 0$, let

$$\varphi_l(t) := \Gamma_l^{-1}(t) = \inf\{r \geq 0; \Gamma_l(r) > t\} \tag{4.8}$$

be the inverse of Γ_l. Γ_l will sometimes below be referred to as *the clock*. It may be (and has been, in the literature) also called *clock process* (in this case, at level l).

REMARK 4.3. Remark 4.1 holds with "1" replaced by "l." In particular, the range of Γ_l has a.s. Lebesgue measure zero, $l = 1, \dots, k$, and every fixed deterministic r is a.s. a continuity point of Γ_l.

We define the process X_l on $\bar{\mathbb{N}}_*^l$ as follows. For $t \geq 0$, let

$$X_l(t) = \begin{cases} \big(X_{l-1}(\varphi_l(t)), \xi_l(\varphi_l(t))\big), & \text{if } \varphi_l(t) \in \mathcal{S}_l \text{ and } t \in \mathrm{ULC}_{\varphi_l}, \\ \big(X_{l-1}(\varphi_l(t)), \infty\big), & \text{otherwise.} \end{cases}$$

DEFINITION 4.4. We call X_k defined just above the K-process on $\mathbb{T}_k$, or k-level K-process, with parameter set $\underline{\gamma_k} = \{\gamma_i; i = 1, \dots, k\}$. Notation: $X_k \sim K(\mathbb{T}_k, \underline{\gamma_k})$.

REMARK 4.5. Since we only have Poissonian marks in the above definition of X_k, we did not have the need of the second coordinate Y_k, as in finite volume, nor did we need to explicitly mention constancy intervals. The latter notion is nonetheless useful in this context (it will come up later, in one of our proofs of convergence), and is defined as follows. Given $1 \leq j \leq k$, an interval $I \subset \mathbb{R}^+$ is a constancy interval of X_j if it has positive length

$$X_j(r) = X_j(s) \qquad \text{for all } r, s \in I \text{ and } I \text{ is maximal.} \tag{4.9}$$

The maximality condition and right continuity of X_j implies that $I = [a, b)$ for some $0 \leq a < b$.

Pictures like those in Figures 2 and 3 might be drawn (or perhaps, more accurately, envisioned) for X_k, with minor changes: in the present case we would have an infinite sequence of time lines for the Poisson processes $(N_r^{(x_l,l)})_{r\geq 0}$, $x_l \in \mathbb{N}_*$, in Figure 2. In Figure 3, superscripts "F" should be dropped throughout; there would be no extra marks, and thus no crosses; the Poissonian marks would form a dense set of the x-axis; if one wanted to represent them, the constancy intervals would be such that there would be an infinite number of them in the neighborhood of any fixed one of them—or, more precisely, between any two distinct such intervals, there is a distinct such interval; the graph of Γ_l would be that of a strictly increasing function with a dense set of jumps (the Poissonian marks).

The next result makes the above construction a.s. well-defined for all times, and implies that X_k is never absorbed at any state. For its proof, let us introduce the notation

$$X_k = (X_{k,1}, \dots, X_{k,k}), \qquad k \geq 1,$$

making the coordinates of X_k explicit.

LEMMA 4.6. *We have that almost surely* $\Gamma_k(r) < \infty$ *for all* $r \in [0, \infty)$ *and* $\lim_{r\to\infty} \Gamma_k(r) = \infty$.

PROOF. As Γ_j is nondecreasing and unbounded for $j = 1, \dots, k$, it is sufficient to show that, for all $r \in (0, \infty)$, $\Gamma_k \circ \cdots \circ \Gamma_1(r) < \infty$ almost surely. Let

$$\Theta_k(r) := \Gamma_k \circ \cdots \circ \Gamma_1(r). \tag{4.10}$$

We will show by induction that

$$E\big(\Theta_k(r)\big) = r\sum_{x_1=1}^{\infty}\cdots\sum_{x_k=1}^{\infty}\gamma_1(x|_1)\cdots\gamma_k(x|_k) = r\sum_{x\in\mathbb{N}_*^k}\bar{\gamma}_k(x). \tag{4.11}$$

Since the right-hand side of (4.11) is finite by assumption [see (4.2) above], this closes the argument.

Equation (4.11) is immediate from the definition for $k=1$. Let us suppose that it holds up to $k-1$, for a fixed arbitrary $k\ge 2$. Let us consider the constancy intervals of $X_{k,1}$ (i.e., maximal intervals over which $X_{k,1}$ is constant): $\mathcal{I} = \{\text{constancy intervals of } X_{k,1}\subset[0,\Theta_k(r)]\}$. We can enumerate such intervals as $\mathcal{I} = \{I(s) := [\Gamma_1(s-),\Gamma_1(s)); s\in\mathcal{S}_1\cap[0,r]\}$. So,

$$\Theta_k(r) = \sum_{s\in\mathcal{S}_1\cap[0,r]}\big|I(s)\big| = \sum_{x_1=1}^{\infty}\sum_{i_1=1}^{N_r^{(x_1,1)}} L_{i_1}^{(x_1)}, \tag{4.12}$$

where $L_{i_1}^{(x_1)} := |I(\sigma_{i_1}^{(x_1,1)})|$, and we recall that $N^{(x_1,1)}$ is a Poisson process of rate 1. Now notice that, for every $x_1\in\mathbb{N}_*$, $N^{(x_1,1)}$ and $L^{(x_1)} := \{L_{i_1}^{(x_1)} : i_1\ge 1\}$ are independent, and $L^{(x_1)}$ is an i.i.d. family of random variables with $L_{i_1}^{(x_1)}\sim \Theta_{k-1}^{(x_1)}(\gamma_1(x_1)T_1^{(x_1)})$, where $\Theta_{k-1}^{(x_1)} = \Gamma_{k-1}\circ\cdots\circ\Gamma_1$ is the corresponding of Θ_k for a K process on $\mathbb{T}_k^{(x_1)}$, the $k-1$-level subtree of $\mathbb{T}_k$ rooted on x_1, and parameter set $\underline{\gamma_k^{(x_1)}} = \{\gamma_i(x_1,\cdot) : i = 2,\dots,k\}$.

Then

$$\begin{aligned}E\big(\Theta_k(r)\big) &= r\sum_{x_1=1}^{\infty}E\{\Theta_{k-1}^{(x|_1)}\big(\gamma_1(x|_1)T_1^{(x_1)}\big)\}\\ &= r\sum_{x_1=1}^{\infty}\gamma_1(x|_1)\sum_{x_2,\dots,x_k}\gamma_2(x|_2)\cdots\gamma_k(x|_k),\end{aligned} \tag{4.13}$$

where we have used that $T_{i_1}^{(x_1)}$, $i_1\ge 1$, are i.i.d. with mean 1 random variables, independent of all other random variables, and, in the second equality, the induction hypothesis. The coincidence of the right-hand sides of (4.11) and (4.13) closes the argument for the first assertion.

It follows readily from (4.12) and the independence of the summands on its right-hand side, and the fact that their distribution depend only on x_1, that a.s. $\Theta_k(r)\to\infty$ as $r\to\infty$, and the second assertion follows from this and the first assertion. $\square$

REMARK 4.7. We will on several occasions below, as we did right above, work with the compounded clock Θ_j rather than with the simple clock Γ_j or simple time. In finite volume, we will do the same with the finite volume version $\Theta_j^{(n)}$

[appearing below; see (5.26)]. This is (only) for convenience, since we can obtain simpler expressions to work with for quantities involving the compounded clocks, like (4.12) above, or (5.31)–(5.32) below, than ones for simple clocks or simple time. A typical argument (as the one above) will use the fact that a.s. $\Theta_j(r) < \infty$ for all r and $\Theta_j(r) \to \infty$ as $r \to \infty$ to go from a statement involving $\Theta_j(r)$ to one involving $\Gamma_j(r)$ or r. Notice that the definitions of X_k^F and X_k involve only simple clocks Γ_j^F and Γ_j, respectively; the composition behind Θ_j (and $\Theta_j^{(n)}$) helps with computations, however.

We next prove a property about the infinities of X_k. Even though this result is not used in what follows it, and is in some sense contained in the next result, Lemma 4.9, it sheds light on a characteristic of X_k which is worth pointing out.

LEMMA 4.8. *Let $k \geq 1$.*

(1) *The set of infinities of X_k has a.s. Lebesgue measure zero. More precisely, let $\mathfrak{I}_k = \bigcup_{i=1}^k \mathfrak{I}_{k,i}$, with $\mathfrak{I}_{k,i} := \{t \geq 0 : X_{k,i}(t) = \infty\}$; then, $\mathfrak{I}_k$ has a.s. Lebesgue measure zero.*

(2) *Almost surely, if $X_{k,i}(t) = \infty$ for some $t \geq 0$ and $i = 1, \dots, k$, then $X_{k,j}(t) = \infty$ for $i \leq j \leq k$.*

PROOF. From the construction of X_k it follows that $X_{k,k}(t) \in \mathbb{N}_*$, that is, is finite if and only $\varphi_k(t) \in \mathcal{S}_k$ and $t \in \mathrm{ULC}_{\varphi_k}$, which means that $t \in \bigcup_{s \in \mathcal{S}_k} [\Gamma_k(s)-, \Gamma_k(s)) = \mathbb{R}^+ \setminus \Gamma_k(\mathbb{R}^+)$. From Remark 4.3, it follows that $\mathfrak{I}_{k,k}$ has a.s. Lebesgue measure zero, and in particular both claims are established for $k = 1$.

Let us inductively suppose they hold for $k = K - 1$ for $K \geq 2$. By the reasoning of previous paragraph, we have that $\mathfrak{I}_{K,K}$ has a.s. Lebesgue measure zero. It is thus enough to consider $\mathfrak{I}_{K,i} \setminus \mathfrak{I}_{K,K}$ for $i < K$. Again by the reasoning of the previous paragraph and the construction of X_K, we have that the latter set is nonempty only if $\mathfrak{I}_{K-1,i} \cap \mathcal{S}_K \neq \varnothing$, but given the induction hypothesis, this a.s. does not happen for a.e. realization of $\mathcal{S}_K$, since Poisson processes of constant rate a.s. assign no point to sets of null Lebesgue measure. This means that a.s. $\mathfrak{I}_K = \mathfrak{I}_{K,K}$, and the induction step for the first claim follows. The latter equality implies in particular the second claim for $i = K - 1$ (which is the only remaining case if $K = 2$). Suppose now that $K \geq 3$ and $X_{K,i}(t) = \infty$ for some $i < K - 1$ and $t \geq 0$; since $X_{K,i}(t) = X_{K-1,i}(\varphi_K(t))$, we may apply the induction hypothesis to conclude that $X_{K,i}(t) = X_{K-1,i}(\varphi_K(t)) = X_{K-1,K-1}(\varphi_K(t)) = X_{K,K-1}(t) = \infty$, and from the conclusion of the previous sentence follows the induction step for the second claim. $\square$

The next result roughly states that once a coordinate of a K-process is large, then so are the subsequent ones. This is in line with the property stated in Lemma 4.8(2) above—in a way, it is a continuous extension of it. In the next section we establish a finite volume analogue; see Lemma 5.12 below.

LEMMA 4.9. *Let X_k be a k-level K-process. Given $T > 0$, not necessarily deterministic, and $m \geq 1$, there a.s. exists $\tilde{m} = \tilde{m}^{(k)}$ such that if $X_{k,i}(t) > \tilde{m}$ for some $1 \leq i < k$ and $t \in [0, T]$, then $X_{k,j}(t) > m$ for all $j = i+1, \dots, k$.*

PROOF. We will start by claiming that there a.s. exists $\hat{m}^{(k)}$ such that if $X_{k,i}(t) > \hat{m}^{(k)}$ for any $t \leq T$ and $i = 1, \dots, k-1$, then $X_{k,k}(t) > m$. This closes the argument when $k = 2$. For $k \geq 3$, we use an inductive argument.

For $m \in \mathbb{N}_*$ and $j = 1, \dots, k$, let

$$\tilde{\mathcal{S}}_j^{(m)} = \{\sigma_i^{(x,j)} : x = 1, \dots, m, i \geq 1\} \tag{4.14}$$

be the set of Poissonian marks of level j with labels at most m. Let us fix $T' > 0$ deterministic, and let $\mathcal{T}_{kj}(l)$ denote the set of times up to $\Theta_{k-1}(T')$ spent by $X_{k-1,j}$ above l, $j = 1, \dots, k-1$. By a similar reasoning as the one employed to prove (4.11), we may check that the expected Lebesgue measure of $\mathcal{T}_{kj}(l)$ equals

$$T' \sum_{x_1=1}^{\infty} \cdots \sum_{x_j=l+1}^{\infty} \cdots \sum_{x_{k-1}=1}^{\infty} \gamma_1(x|_1) \cdots \gamma_{k-1}(x|_{k-1}). \tag{4.15}$$

[We recall that the reason to work with the compounded clocks Θ_j's rather than the simple clocks Γ_j's or deterministic times is precisely to be able to derive a simple formula like the one in (4.15), which would be more complicated for simple clocks or deterministic times replacing $\Theta_{k-1}(T')$.] Since that Lebesgue measure is decreasing in l, and, as follows from our assumptions on $\underline{\gamma_k}$, the expression in (4.15) vanishes as $l \to \infty$, we have that the limit of that Lebesgue measure as $l \to \infty$ vanishes almost surely. We then have from elementary properties of Poisson processes that

$$\left\{\bigcup_{j=1}^{k-1} \mathcal{T}_{kj}(l)\right\} \cap \tilde{\mathcal{S}}_k^{(m)} = \varnothing \tag{4.16}$$

for all large enough l almost surely, so given $m \in \mathbb{N}_*$, we find $\bar{m}^{(k)} = \bar{m}^{(k)}(T')$ such that on $\{\Theta_k(T') > T\}$ if $X_{k,i}(t) > \bar{m}^{(k)}$ for some $t \leq T$ and $i = 1, \dots, k-1$, then, since on $\{\Theta_k(T') > T\}$ the trajectory of $X_{k,k}$ in $[0, T]$ depends only on the Poisson points of $\mathcal{N}_k$ on $[0, \Theta_{k-1}(T')]$, we have from (4.16) that $X_{k,k}(t) > m$. Since from second assertion of Lemma 4.6 $\bigcup_{T'>0}\{\Theta_k(T') > T\}$ has full measure, we may a.s. choose T' such $\{\Theta_k(T') > T\}$ occurs, and then choose $\hat{m}^{(k)} = \bar{m}^{(k)}(T')$, and the claim at the beginning of the proof follows.

As we have already argued, this in particular establishes the lemma for $k = 2$, by the choice $\tilde{m}^{(2)} = \hat{m}^{(2)}$. Let us assume that the lemma is established for $k - 1 \geq 2$. This means that given $m \geq 1$ there a.s. exists $\tilde{m}^{(k-1)}$ such that if $X_{k-1,i}(t) > \tilde{m}^{(k-1)}$ for some $1 \leq i < k-1$ and $t \in [0, \Theta_{k-1}(T')]$, then $X_{k-1,j}(t) > m$, where T' is as at the end of the previous paragraph [notice that we used $\Theta_{k-1}(T')$ as T here]. The claim of the lemma then follows by the choice $\tilde{m}^{(k)} = \tilde{m}^{(k-1)} \vee \hat{m}^{(k)}$, where $m^{(k)}$ is as in the claim at the beginning of the proof with $\Theta_{k-1}(T')$ replacing T. Indeed, if for some $t \in [0, T]$ we have $X_{k,i}(t) > \tilde{m}^{(k)}$, then, by the claim at the beginning, $X_{k,k}(t) > m$, and the claim of the lemma is established for $j = k$. If $i < j < k$, then since the trajectory of $(X_{k,i}, i = 1, \dots, k-1)$ in $[0, T]$ shadows that of X_{k-1} in $[0, T'']$ for some $T'' \le \Theta_{k-1}(T')$, meaning that there exists $t' \in [0, \Theta_{k-1}(T')]$ such that $(X_{k,i}(t), i = 1, \dots, k-1) = X_{k-1}(t')$, we have that $X_{k,j}(t) = X_{k-1,j}(t') > m$, by the induction hypothesis, and the argument is complete. □

5. Convergence.

5.1. *Scaling limit for the GREM-like trap model.* We start this section with our main result, the scaling limit for the GREM-like trap model in the fine tuning regime and extreme time scale. Let us go again, this time in more detail, over the definition of these terms; see the last paragraph of Section 2. [In this section, we replace the notation above with superscript "F," denoting finite volume, to a notation with superscript "(n)," to emphasize sequence dependence instead.] The parameters of the model of this subsection will be taken random, as described below.

For $j = 1, \dots, k$, let $\tau_j := \{\tau_j(x|_j); x|_j \in \mathcal{M}|_j\}$ be an i.i.d. family of random variables in the domain of attraction of an α_j-stable law. We suppose

$$0 < \alpha_1 < \cdots < \alpha_k < 1. \tag{5.1}$$

For $j = 1, \dots, k$, we will relabel τ_j obtaining $\tau_j^{(n)} = \{\tau_j^{(n)}(x|_j); x|_j \in \mathcal{M}|_j\}$, so that, for every $(x|_{j-1}) \in \mathcal{M}|_{j-1}$, $\{\tau_j^{(n)}(x|_j); x_j \in \mathcal{M}_j\}$ are the decreasing order statistics of $\{\tau_j(x|_j); x_j \in \mathcal{M}_j\}$.

For $j = 1, \dots, k$, $n \ge 1$ and $x|_{j-1} \in \mathbb{N}_*^{j-1}$, let

$$c_j^{(n)} = \big(G_j^{-1}(M_j^{-1})\big)^{-1}, \tag{5.2}$$

where G_j^{-1} is the (generalized) inverse of $G_j : [0, \infty) \to (0, 1]$ such that $G_j(t) = P(\tau_j(x|_j) > t)$, and make

$$\gamma_j^{(n)}(x|_j) = c_j^{(n)} \tau_j^{(n)}(x|_j), \qquad x|_j \in \mathcal{M}_j; \tag{5.3}$$

let also

$$\{\gamma_j(x|_j), x_j \in \mathbb{N}_*^j\} \tag{5.4}$$

denote independent j-parametrized Poisson point processes, with intensity measure given by $y^{-\alpha_j - 1}$, $y > 0$, in decreasing order.

The *fine tuning* regime mentioned above and at the Introduction corresponds to choosing $M_1^{(n)} = n$ and

$$M_{j+1}^{(n)} = \lfloor 1/c_j^{(n)} \rfloor, \qquad j = 1, \dots, k-1. \tag{5.5}$$

We will make this choice from now on.

Let

$$\tilde{\gamma}_j^{(n)}(x|_j) = \begin{cases} \gamma_j^{(n)}(x|_j), & \text{if } j = 1, \dots, k-1, \\ \tau_k^{(n)}(x|_k), & \text{if } j = k \end{cases} \tag{5.6}$$

and let $\tilde{X}_k^{(n)} \sim TM(\mathbb{T}_k^{(n)}; \underline{\tilde{\gamma}_k^{(n)}})$.

REMARK 5.1. One may readily check from Lemma 3.7 that, in terms of the coin tossing description, $\tilde{X}_k^{(n)} \sim TM(\mathbb{T}_k^{(n)}; \tau_k^{(n)}, (p_j^{(n)})_{j=1}^{k-1})$, where for $j = 1, \dots, k-1$ and $x|_j \in \mathcal{M}|_j$

$$p_j^{(n)}(x|_j) = \frac{1}{1 + \tau_j^{(n)}(x|_j)}. \tag{5.7}$$

With this description, and general finite $M_1, \dots, M_k$ [not necessarily satisfying (5.5)], we call $\tilde{X}_k^{(n)}$ the GREM-like trap model on $\mathbb{T}_k^{(n)}$ with parameters $\tau_j(x|_j)$, $j = 1, \dots, k$, $x|_j \in \mathcal{M}|_j$. [The relabeling performed in this subsection (cf. the definition given in the last paragraph of Section 2) is necessary for the existence of the limit.] In this guise, with a choice of $M_1 = \cdots = M_k$, the model was introduced and studied in [25, 26], with the derivation of infinite volume aging functions as the main motivation, with infinite volume limits taken first, and then an infinite time limit. See Remark 5.3 below.

Let us speed up $\tilde{X}_k^{(n)}$ by $c_k^{(n)}$, namely, let

$$X_k^{(n)} = \tilde{X}_k^{(n)}(t/c_k^{(n)}), \qquad t \geq 0. \tag{5.8}$$

This corresponds to the extreme time scale mentioned above and at the Introduction. One may readily check that $X_k^{(n)} \sim TM(\mathbb{T}_k^{(n)}; \underline{\gamma_k^{(n)}})$. Let $X_k \sim K(\mathbb{T}_k; \underline{\gamma_k})$.

THEOREM 5.2. *Let $X_k^{(n)}$ and X_k be as above. Then*

$$\big(X_k^{(n)}, \underline{\gamma_k^{(n)}}\big) \quad \Rightarrow \quad (X_k, \underline{\gamma_k}), \tag{5.9}$$

where $\Rightarrow$ means weak convergence in the product of Skorohod space with the space of finite measures in $\mathbb{N}_*^k$ equipped with the topology of weak convergence.

The Skorohod space in the above statement will be described in detail at the beginning of next subsection.

REMARK 5.3. As a note on the differences between the above result and those of [25, 26], let us point out that the choice of volume relations should not be very important in the context of [25, 26], since the volume limit is taken first, and then the time limit. One expects aging to take place in this regime, and that is what is behind the (explicit) results of [25, 26]. Our choice of volume/time relations is on the other hand essential in order to obtain the specific limit stated above. In particular, they represent not an aging time regime, but an ergodic time regime, that is, a time regime where the process is already close to equilibrium. (Aging is a phenomenon that instead takes place far from equilibrium.) In this sense, our results do not compare immediately to those in [25, 26], since they involve different time/volume regimes, where different behaviors take place. In [22], a smaller time regime is studied, where aging takes place, with results comparable to [25, 26]. Other choices of volume/time scaling may lead to different asymptotics (from the above one and conceivably also from [25, 26]).

5.2. *Infinite volume limit for the k-level trap model.* As anticipated in the Introduction, Theorem 5.2 will be proven in Section 5.5 below by verifying the conditions of an infinite volume limit result for k-level trap models. This is the object of this and the next two subsections. We may in this section, and in the following two subsections, think of the parameters of the model as deterministic. We will return to random parameters at the last subsection.

Let us consider a sequence of k-level trap models $X_k^{(n)}, n \geq 1$, on a sequence of finite trees $\mathbb{T}_k^{(n)}$, with volumes $M_1 = M_1^{(n)}, \dots, M_k = M_k^{(n)}$, and parameter sets $\underline{\gamma_k^{(n)}}$, respectively (see Definition 3.6), and prove a weak convergence result for that sequence under the Skorohod topology on $D(\bar{\mathbb{N}}_*^k, [0, \infty))$, the space of càdlàg functions from $[0, \infty)$ to $\bar{\mathbb{N}}_*^k$. As anticipated at the beginning of Section 5.1, we replace the superscript "F" used in the first sections by "(n)" everywhere to emphasize the dependence on n.

Before proceeding, let us briefly review the Skorohod topology. We start by equipping $\bar{\mathbb{N}}_*^k$ with the metric

$$d(x, y) = \max_{1 \leq j \leq k} \left| x_j^{-1} - y_j^{-1} \right|, \qquad x, y \in \bar{\mathbb{N}}_*^k, \tag{5.10}$$

where $\infty^{-1} = 0$, under which it is compact. The Skorohod metric on $D(\bar{\mathbb{N}}_*^k, [0, \infty))$ is as follows. For $f, g \in D(\bar{\mathbb{N}}_*^k, [0, \infty))$, let

$$\rho(f, g) = \inf_{\lambda \in \Lambda} \left[\phi(\lambda) \vee \int_0^\infty e^{-u} \rho(f, g, \lambda, u) \, du \right], \tag{5.11}$$

where

$$\rho(f, g, \lambda, u) = \sup_{t \geq 0} d\big(f(t \wedge u), g\big(\lambda(t) \wedge u\big)\big) \tag{5.12}$$

with Λ the class of *time distortions*: increasing Lipschitz functions from $[0, \infty)$ onto $[0, \infty)$, and $\phi : \Lambda \to [0, \infty)$ such that

$$\phi(\lambda) = \sup_{0 \leq s < t} \left| \log \frac{\lambda_t - \lambda_s}{t - s} \right|; \tag{5.13}$$

see Section 3.5 in [15].

In order to get our convergence result, we will impose the following conditions on the volumes and parameters. For $j = 1, \dots, k$, suppose that as $n \to \infty$

$$M_j^{(n)} \to \infty, \tag{5.14}$$

$$\gamma_j^{(n)}(x) \to \gamma_j(x) \qquad \text{for every } x \in \mathbb{N}_*^j \text{ and } \sum_{x \in \mathbb{N}_*^j} \bar{\gamma}_j^{(n)}(x) \to \sum_{x \in \mathbb{N}_*^j} \bar{\gamma}_j(x) \tag{5.15}$$

with $\gamma_j, \bar{\gamma}_j$ as in the beginning of Section 4 [see paragraph of (4.1), (4.2) above], $\gamma_j^{(n)} \equiv 0$ on $\mathbb{N}_*^j \setminus \mathcal{M}|_j$ and

$$\bar{\gamma}_j^{(n)}(x|_j) := \gamma_1^{(n)}(x|_1) \times \gamma_2^{(n)}(x|_2) \times \cdots \times \gamma_j^{(n)}(x|_j). \tag{5.16}$$

Our result will require additional conditions that look quite intricate. We state them now and discuss them, together with the above conditions, after we state the convergence result. We further suppose that for $j = 2, \dots, k$

$$\begin{aligned} \frac{1}{\prod_{p=1}^{j-1} M_{p+1}} \sum_{l=1}^{j-1} \sum_{x|_{j-1} \in \mathcal{M}|_{j-1}} \prod_{p=1}^{l-1} \bigl(M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) & \\ \times \prod_{p=l+1}^{j-1} \bigl(1 + M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) &\to 0 \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \frac{1}{\prod_{p=1}^{j-1} M_{p+1}} \sum_{l=1}^{j-1} \sum_{x|_j \in \mathcal{M}|_j} \gamma_j^{(n)}(x|_j) \prod_{p=1}^{l-1} \bigl(M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) & \\ \times \prod_{p=l+1}^{j-1} \bigl(1 + M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) &\to 0 \end{aligned} \tag{5.18}$$

as $n \to \infty$, where by convention

$$\prod_{p=1}^{0} \bigl(M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) = \prod_{p=j}^{j-1} \bigl(1 + M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) = 1.$$

Here, and many times below, we omit the superscript "(n)" from the notation for the volumes $M_1, \dots, M_k$.

We are ready to state our infinite volume limit result.

THEOREM 5.4. *For $n \geq 1$, let $X_k^{(n)}$ be the trap model on $\mathbb{T}_k^{(n)}$, with volumes $M_1^{(n)}, \dots, M_k^{(n)}$, and parameter sets $\underline{\gamma_k^{(n)}}$, respectively, satisfying conditions (5.14)–(5.18). Let X_k be the K-process on $\mathbb{T}_k$ with parameter set $\underline{\gamma_k}$. Then $X_k^{(n)} \to X_k$ weakly in Skorohod space as $n \to \infty$.*

We will see (from the proofs) that conditions (5.14)–(5.18) have the following significance. Obviously, (5.14) means that we are taking an infinite volume limit. Equation (5.15) implies that the contributions coming from the Poisson marks to the construction of $X_k^{(n)}$ converge (in a uniform way) to the respective contributions of (Poisson) marks of X_k. Finally, as will be seen in the arguments below, (5.17)–(5.18) imply the negligibility of the total contribution of the extra marks entering $X_k^{(n)}$. [Poisson and extra marks were introduced in the paragraph before (3.5) above.] It may be readily checked that in general neither are conditions (5.14)–(5.18) equivalent, nor do they follow from previous conditions; in the generality of the statement of Theorem 5.4, indeed, they need to be separately imposed.

REMARK 5.5. One way to gain insight into the meaning of (5.17)–(5.18) is as follows. In order to have a single condition, we start by writing the sum over $\mathcal{M}|_{j-1}$ in (5.17) as sum over $\mathcal{M}|_j$ with an extra term of $1/M_j$ multiplying each summand. We then sum the resulting expression to the one on the left of (5.18), getting

$$\begin{aligned}\frac{1}{\prod_{p=1}^{j-1} M_{p+1}} \sum_{l=1}^{j-1} \sum_{x|_j\in\mathcal{M}|_j} \prod_{p=1}^{l-1} \bigl(M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) \\ \times \prod_{p=l+1}^{j-1} \bigl(1+M_{p+1}\gamma_p^{(n)}(x|_p)\bigr) \\ \times \left(\frac{1}{M_j}+\gamma_j^{(n)}(x|_j)\right).\end{aligned} \tag{5.19}$$

Dividing now the double product inside the double sum in (5.19) by the product outside the double sum, and defining

$$\begin{aligned}\bar{\gamma}_{j,l}^{(n)}(x|_j) := {}& \prod_{p=1}^{l-1} \gamma_p^{(n)}(x|_p)\frac{1}{M_{l+1}} \\ & \times \prod_{p=l+1}^{j-1}\left(\frac{1}{M_{p+1}}+\gamma_p^{(n)}(x|_p)\right)\left(\frac{1}{M_j}+\gamma_j^{(n)}(x|_j)\right),\end{aligned} \tag{5.20}$$

we find that (5.17)–(5.18) are equivalent to the following condition. For $1\le l< j\le k$, as $n\to\infty$

$$\sum_{x|_j\in\mathcal{M}|_j} \bar{\gamma}_{j,l}^{(n)}(x|_j)\to 0. \tag{5.21}$$

Compare $\bar{\gamma}_{j,l}^{(n)}(x|_j)$ to $\bar{\gamma}_j^{(n)}(x|_j)$ and (5.21) to the second condition in (5.15).

The remainder of this section is organized as follows. We briefly start below, in this same subsection, with the proof of Theorem 5.4. The full proof will require a number of auxiliary results, which we collect in Section 5.3 below, before proceeding with the proof in Section 5.4 after that. And, as we already mentioned, Section 5.5 is devoted to the proof of Theorem 5.2.

PROOF OF THEOREM 5.4. We will argue by induction, using coupled versions of $X_k^{(n)}$ and X_k, and show convergence in probability for a subsequence. The coupling is going to be given by using common Poisson processes $\{N^{(x_i,i)}, x_i \in \mathbb{N}_*, i = 1, \dots, k\}$ and common exponential variables $\{T_s^{(i)}, s \in \mathbb{R}^+, i = 1, \dots, k\}$ in the construction of $X_k^{(n)}$ and X_k. The notation is detailed at the beginning of Section 5.4. It will be clear that the same can be done for every subsequence of (n), and that the limiting distribution for each subsubsequence does not depend on the subsequence. This then implies weak convergence of the original sequence.

Lemma 3.11 of [18] establishes the (convergence in probability; actually a.s. convergence) result for $k = 1$ and $\gamma_1^{(n)}(x)$ not depending on n as soon as $x \le M_1$. This result (convergence in probability) holds (with minor changes in argumentation, as sketched in the proof of Theorem 5.2 of [18]) in our case as well. It is also part of the argumentation of Lemma 3.11 and Theorem 5.2 of [18], and can also be readily checked independently, that for every $r \in [0, \infty)$,

$$\Gamma_1^{(n)}(r) \to \Gamma_1(r) \tag{5.22}$$

in probability as $n \to \infty$.

As part of our induction argument, we will then assume that for $j = 1, \dots, k-1$ and every $r \in [0, \infty)$,

$$X_j^{(n)} \to X_j, \tag{5.23}$$

$$\Gamma_j^{(n)}(r) \to \Gamma_j(r) \tag{5.24}$$

as $n \to \infty$ almost surely, possibly over a subsequence. $\square$

5.3. *Auxiliary results for the proof of Theorem* 5.4. We assume throughout that the hypotheses of Theorem 5.4 are in force.

Our first auxiliary result establishes that the contribution of extra marks and their descendants to $X_j^{(n)}$ is negligible as $n \to \infty$. That is the content of Lemma 5.6. To be precise, let $\mathcal{E}_2^{(n)} = \mathcal{R}_2^{(n)}$ and for $3 \le i \le k$

$$\mathcal{E}_i^{(n)} = \mathcal{R}_i^{(n)} \cup \{s \in \mathcal{S}_i^{(n)} : \varphi_{i-1}^{(n)}(s) \in \mathcal{E}_{i-1}^{(n)}\}. \tag{5.25}$$

$\mathcal{E}_i^{(n)}$ represents the extra marks of the ith level and the *descendants* of extra marks of previous levels in the ith level (i.e., Poisson marks belonging to constancy intervals originating from extra marks of the previous level or descendants of extra marks from levels before that).

LEMMA 5.6. *Assume that the induction hypotheses* (5.23)–(5.24) *hold. Then, for every* $r>0$ *and* $j=2,\dots,k$, *we have that*:

(a) $\min\{\xi_j^{(n)}(s):s\in\mathcal{E}_j^{(n)}\cap[0,r]\}\to\infty$ *in probability as* $n\to\infty$.

(b) $\mu_j^{(n)}(\mathcal{E}_j^{(n)}\cap[0,r])\to 0$ *in probability as* $n\to\infty$.

PROOF. For $r>0$, $j=1,\dots,k$, let

$$\Theta_j^{(n)}(r):=\Gamma_j^{(n)}\circ\cdots\circ\Gamma_1^{(n)}(r) \tag{5.26}$$

and define $K_j^{(n)}(r):=|\mathcal{E}_j^{(n)}\cap[0,\Theta_{j-1}^{(n)}(r)]|$, where (here) $|\cdot|$ stands for cardinality. An evaluation of $K_j^{(n)}(r)$ will play a crucial role in the proof. We begin with that.

It follows from induction hypothesis (5.24) that for $j=1,\dots,k-1$, $\Gamma_j^{(n)}(r)\to\infty$ as $r\to\infty$ in probability, uniformly in n. So it is enough to consider $\mathcal{E}_j^{(n)}\cap[0,\Theta_{j-1}^{(n)}(r)]$ instead of $\mathcal{E}_j^{(n)}\cap[0,r]$. In order to evaluate the cardinality of that set, as well as its contribution to $\mu_j^{(n)}([0,\Theta_{j-1}^{(n)}(r)])$, we start by describing the structure of $\mathcal{H}_1^{(n)}:=\mathcal{S}_1^{(n)}$, $\mathcal{H}_2^{(n)}:=\mathcal{S}_2^{(n)}\cup\mathcal{R}_2^{(n)},\dots,\mathcal{H}_k^{(n)}:=\mathcal{S}_k^{(n)}\cup\mathcal{R}_k^{(n)}$; at the same time, we will relabel the marks of those sets conveniently.

Each $s_1\in\mathcal{S}_1^{(n)}$ can be put in a one-to-one correspondence with its label $\xi_1^{(n)}(s_1)=x_1$ and *index* $i_1(s_1)=i_1\in\mathbb{N}_*$ via the relation $s_1=\sigma_{i_1}^{(x_1,1)}$. Using this correspondence, we see that to each mark s_1 of $\mathcal{S}_1^{(n)}$ there corresponds an interval $I_n^{(x_1,i_1)}$ of $\mathbb{R}^+$ of length $L_n^{(x_1,i_1)}=\gamma_1^{(n)}(x_1)T^{(x_1,i_1)}$. Such intervals form a partition of $\mathbb{R}^+$, and the random variables involved are independent when we vary s_1.

Now to each $s_1\in\mathcal{S}_1^{(n)}$, there corresponds marks of $\mathcal{S}_2^{(n)}$ belonging to the respective interval $I_n^{(x_1,i_1)}$, whose cardinality is a geometric random variable $G^{(x_1,i_1)}$ with mean $M_2\gamma_1^{(n)}(x_1)$, plus a mark of $\mathcal{R}_2^{(n)}$ at the left endpoint of $I_n^{(x_1,i_1)}$—recall Remark 3.5. Each such mark will be identified with $(x_1,i|_2)$, where (x_1,i_1) is the identifier of s_1, and $i_2\in\{1,\dots,G^{(x_1,i_1)}+1\}$, and we attach to it a random variable $U^{(x_1,i|_2)}$ with uniform distribution in $\mathcal{M}_2$, which corresponds to $\xi_2^{(n)}(s_2)$, for $(x_1,i|_2)\equiv s_2\in\mathcal{S}_2^{(n)}\cup\mathcal{R}_2^{(n)}$. We will identify the unique mark of $\mathcal{R}_2^{(n)}$ at the left endpoint of $I_n^{(x_1,i_1)}$ with $(x_1,i_1,1)$.

We now proceed inductively. For $3\le j\le k$, we assume we have identified each mark of $\mathcal{S}_{j-1}^{(n)}\cup\mathcal{R}_{j-1}^{(n)}$ as $(x_1,i|_{j-1})$, with $x_1\in\mathcal{M}_1$, $i_1\ge 1$, $i_l=1,\dots,1+G^{(x_1,i|_{l-1})}$, $l=2,\dots,j-1$, where, for $l\ge 3$, $G^{(x_1,i|_{l-1})}$ is geometric with mean $M_l\gamma_{l-1}^{(n)}(x_1,U^{(x_1,i|_2)},\dots,U^{(x_1,i|_{l-1})})$, with $U^{(x_1,i|_j)}\sim\mathrm{Uniform}(\mathcal{M}_j)$. The random variables of

$$\mathcal{U}_{j-1}:=\{U^{(x_1,i|_l)}:l=2,\dots,j-1;x_1,i_1,\dots,i_l\ge 1\}$$

are independent, and, given $\mathcal{U}_{j-1}$, so are those of

$$\{G^{(x_1,i|_l)}:l=1,\dots,j-1;x_1,i_1,\dots,i_l\ge 1\}.$$

Notice that $G^{(x_1,i|_j)}$ is independent of $U^{(x_1,i|_l)}$ as soon as $j < l$. Here $i_{j-1} = 1$ means $(x_1, i|_{j-1}) \in \mathcal{R}_{j-1}^{(n)}$; otherwise, $(x_1, i|_{j-1}) \in \mathcal{S}_{j-1}^{(n)}$. Then to each mark $(x_1, i|_{j-1})$ there corresponds an interval $I_n^{(x_1,i|_{j-1})}$ of $\mathbb{R}^+$ of length

$$L_n^{(x_1,i|_{j-1})} := \gamma_{j-1}^{(n)}\bigl(x_1, U^{(x_1,i|_2)}, \dots, U^{(x_1,i|_{j-1})}\bigr)T^{(x_1,i|_{j-1})}$$

with $\{T^{(x_1,i|_{j-1})}\}$ i.i.d. mean 1 exponential random variables independent of $\{G^{(x_1,i|_l)}, U^{(x_1,i|_l)}\}, l = 1, \dots, j-1$, such that $\{I_n^{(x_1,i|_{j-1})}\}$ is a partition of $\mathbb{R}^+$. The mark of $\mathcal{R}_j^{(n)}$ placed at the left end of $I_n^{(x_1,i|_{j-1})}$ is labeled $(x_1, i|_{j-1}, 1)$, and the marks of $\mathcal{S}_j^{(n)} \in I_n^{(x_1,i|_{j-1})}$, if any, are labeled $(x_1, i|_j), i_j = 2, \dots, 1 + G^{(x_1,i|_{j-1})}$, with $G^{(x_1,i|_{j-1})}$ a geometric random variable with mean $M_j\gamma_{j-1}^{(n)}(x_1, U^{(x_1,i|_2)}, \dots, U^{(x_1,i|_{j-1})})$. The random variables in $\{G^{(x_1,i|_{j-1})}\}$ are independent among themselves, and independent of the previous random variables. Finally, we assign to each $(x_1, i|_j)$ a random variable $U^{(x_1,i|_j)}$ uniformly distributed in $\mathcal{M}_j$, corresponding to $\xi_j^{(n)}(s_j)$, for $(x_1, i|_j) \equiv s_j \in \mathcal{S}_j^{(n)} \cup \mathcal{R}_j^{(n)}$, with $\{U^{(x_1,i|_j)}\}$ independent among themselves, and independent of previous random variables.

With this representation, we have labeled the marks of $\mathcal{H}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)]$, $j = 1, \dots, k$, as $(x_1, i|_j)$, $x_1 = 1, \dots, M_1$; $i_1 = 1, \dots, N_r^{(x_1)}$; $i_l = 1, \dots, 1 + G^{(x_1,i|_{l-1})}, 2 \le l \le j$, with $G^{(x_1,i|_l)}$ geometric with mean $M_2\gamma_1^{(n)}(x_1)$ when $l = 1$, and with mean $M_{l+1}\gamma_l^{(n)}(x_1, U^{(x_1,i_1)}, \dots, U^{(x_1,i|_l)})$, when $l = 2, \dots, j$, respectively. $U^{(x_1,i|_l)}$ is uniformly distributed on $\mathcal{M}_l$, $l = 2, \dots, j$. The random variables in the family

$$\mathcal{U}_j := \{U^{(x_1,i|_l)}; x_1, i_1, \dots, i_l \ge 1, l = 2, \dots, j\}$$

are independent, and given $\mathcal{U}_j$ so are those in

$$\{G^{(x_1,i|_l)}; x_1, i_1, \dots, i_l \ge 1, l = 1, \dots, j\}.$$

Notice that, as before, $G^{(x_1,i|_j)}$ is independent of $U^{(x_1,i|_l)}$ as soon as $j < l$.

The marks of $\mathcal{E}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)]$ are those $(x_1, i|_j)$ as above for which $i_l = 1$ for some $l = 2, \dots, j$. In order to write an expression for $K_j^{(n)}(r)$, we first view $\mathcal{E}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)]$ as the leaves of a forest (see Figure 4), the distinct trees of which have the marks labeled $(x_1, i|_l, 1), l = 1, \dots, j-1, i_1 = 1, \dots, N_r^{(x_1)}; i_l = 2, \dots, \tilde{G}^{(x_1,i|_{l-1})} := 1 + G^{(x_1,i|_{l-1})}, l = 2, \dots, j-1$, as roots; the tree rooted at $(x_1, i|_l, 1)$ consisting of, besides the root, marks whose labels form the set

$$\mathfrak{T}_j^{(x_1,i|_l)} := \bigl\{(x_1, i|_j^l) : i_m = 1, \dots, \tilde{G}^{(x_1,i|_{m-1}^l)}, m = l+2, \dots, j\bigr\}, \tag{5.27}$$

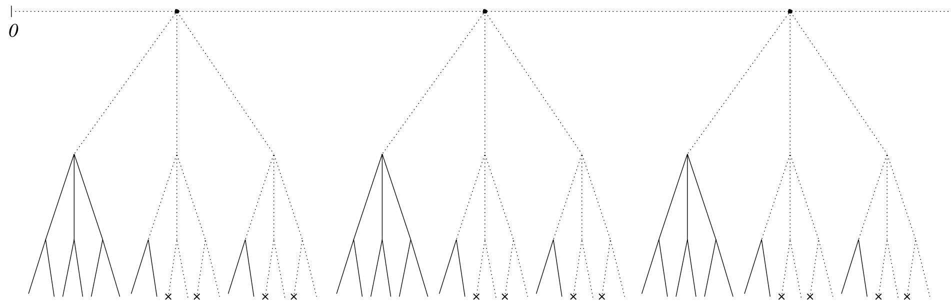

FIG. 4. *Schematic representation of portions of* $\bigcup_{i=1}^{4}\mathcal{H}_i^{(n)}$ *and the forest whose leaves are* $\mathcal{E}_4^{(n)}\cap[0,\Theta_3^{(n)}(r)]$. *Full points on horizontal dotted line represent* $\mathcal{H}_1^{(n)}\cap[0,r]$. *Successive generations of trees attached to each point of* $\mathcal{H}_1^{(n)}\cap[0,r]$ *represent* $\mathcal{H}_2^{(n)}\cap[0,\Theta_1^{(n)}(r)]$, $\mathcal{H}_3^{(n)}\cap[0,\Theta_2^{(n)}(r)]$ *and* $\mathcal{H}_4^{(n)}\cap[0,\Theta_3^{(n)}(r)]$, *respectively. Forests of extra marks and their descendants are shown in full lines and crosses.* (*Actual picture should look less regular, since the degrees of the vertices of the trees are independent random variables, which should be moreover large for large* n.)

where, for $1\le h\le j$,

$$i|_h^l=\begin{cases} i|_l, & \text{if } h\le l,\\ (i|_l,1), & \text{if } h=l+1,\\ (i|_l,1,i_{l+2},\dots,i_h), & \text{if } h>l+1.\end{cases}\tag{5.28}$$

Equation (5.27) is well defined whenever $l<j-1$; otherwise, each of the above mentioned trees consists of its root only.

REMARK 5.7. For each $l=1,\dots,j-1$, the roots of the above trees, namely the points labeled $(x_1,i|_l,1)$, with x_1 and $i|_l$ as described above, represent the extra marks of level $l+1$, as described in the paragraph before the one containing (3.5), now with a labeling suited to the computations to be performed below. The sites other than themselves on the trees of which they are the roots represent their descendants, corresponding to either Poissonian or extra marks originating of an extra mark at some level above.

Then the number of elements of $\mathcal{E}_j^{(n)}\cap[0,\Theta_{j-1}^{(n)}(r)]$ on the leaves of $\mathfrak{T}_j^{(x_1,i|_l)}$, $l=1,\dots,j-2$, $j\ge 3$, is given by

$$\sum_{i_{l+2}=1}^{\tilde G^{(x_1,i|_{l+1}^l)}}\cdots\sum_{i_j=1}^{\tilde G^{(x_1,i|_{j-1}^l)}}1\tag{5.29}$$

and their contribution to $\mu_j^{(n)}([0,\Theta_{j-1}^{(n)}(r)])$ amounts to

$$\sum_{i_{l+2}=1}^{\tilde G^{(x_1,i|_{l+1}^l)}}\cdots\sum_{i_j=1}^{\tilde G^{(x_1,i|_{j-1}^l)}}\gamma_j^{(n)}\big(x_1,U^{(x_1,i|_2^l)},\dots,U^{(x_1,i|_j^l)}\big)T^{(x_1,i|_j^l)},\tag{5.30}$$

where $\{T^{(x_1,i|_j)}\}$ are i.i.d. mean 1 exponential random variables, independent of all other random variables. So the size of $\mathcal{E}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)]$ is given by

$$K_j^{(n)}(r) = \sum_{l=1}^{j-1} \sum_{x_1=1}^{M_1} \sum_{i_1=1}^{N_r^{(x_1,1)}} \sum_{i_2=2}^{\tilde{G}^{(x_1,i_1)}} \cdots \sum_{i_l=2}^{\tilde{G}^{(x_1,i|_{l-1})}} \sum_{i_{l+2}=1}^{\tilde{G}^{(x_1,i|_{l+1}^l)}} \cdots \sum_{i_j=1}^{\tilde{G}^{(x_1,i|_{j-1}^l)}} 1 \tag{5.31}$$

and its contribution to $\mu_j^{(n)}([0, \Theta_{j-1}^{(n)}(r)])$ amounts to

$$\begin{aligned} &\mu_j^{(n)}\big(\mathcal{E}_j^{(n)} \cap \big[0, \Theta_{j-1}^{(n)}(r)\big]\big) \\ &\quad = \sum_{l=1}^{j-1} \sum_{x_1=1}^{M_1} \sum_{i_1=1}^{N_r^{(x_1,1)}} \sum_{i_2=2}^{\tilde{G}^{(x_1,i_1)}} \cdots \sum_{i_l=2}^{\tilde{G}^{(x_1,i|_{l-1})}} \sum_{i_{l+2}=1}^{\tilde{G}^{(x_1,i|_{l+1}^l)}} \cdots \\ &\qquad \sum_{i_j=1}^{\tilde{G}^{(x_1,i|_{j-1}^l)}} \gamma_j^{(n)}\big(x_1, U^{(x_1,i|_2^l)}, \ldots, U^{(x_1,i|_j^l)}\big) T^{(x_1,i|_j^l)}, \end{aligned} \tag{5.32}$$

where for $l = 1$ the sum $\sum_{i_2=2}^{\tilde{G}^{(x_1,i_1)}}$ should be absent in (5.31)–(5.32); for $l = j-1$, the expressions in (5.29)–(5.30) should be interpreted as 1 and

$$\gamma_j^{(n)}\big(x_1, U^{(x_1,i|_2)}, \ldots, U^{(x_1,i|_{j-1},1)}\big) T^{(x_1,i|_{j-1},1)},$$

respectively, and for $j = 2$ (5.31)–(5.32) should be, respectively, interpreted as

$$K_2^{(n)}(r) = \sum_{x_1=1}^{M_1} \sum_{i_1=1}^{N_r^{(x_1,1)}} 1,$$

$$\mu_2^{(n)}\big(\mathcal{E}_2^{(n)} \cap \big[0, \Theta_1^{(n)}(r)\big]\big) = \sum_{x_1=1}^{M_1} \sum_{i_1=1}^{N_r^{(x_1,1)}} \gamma_2^{(n)}\big(x_1, U^{(x_1,i_1,1)}\big) T^{(x_1,i_1,1)},$$

from which we readily get

$$\begin{aligned} E\big(K_2^{(n)}(r)\big) &= r M_1, \\ E\big(\mu_2^{(n)}\big(\mathcal{E}_2^{(n)} \cap \big[0, \Theta_1^{(n)}(r)\big]\big)\big) &= \frac{r}{M_2} \sum_{x|_2 \in \mathcal{M}|_2} \gamma_2^{(n)}(x|_2). \end{aligned} \tag{5.33}$$

For $j \geq 3$, by conditioning on $N_r^{(x_1,1)}, G^{(x_1,i_1)}, \ldots, G^{(x_1,i|_{j-2})}$, $U^{(x_1,i|_2)}, \ldots,$ $U^{(x_1,i|_{j-2})}$ (in the case of $j = 3$, $N_r^{(x_1,1)}$, $G^{(x_1,i_1)}$), and integrating on the remaining random variables, we get from (5.31)

$$\sum_{l=1}^{j-1}\sum_{x_1=1}^{M_1}\cdots\sum_{i_l=2}^{\tilde G^{(x_1,i|_{l-1})}}\sum_{i_{l+2}=1}^{\tilde G^{(x_1,i|_{l+1}^l)}}\cdots$$

$$\sum_{i_{j-1}=1}^{\tilde G^{(x_1,i|_{j-2}^l)}}\frac{1}{M_{j-1}}\sum_{x_{j-1}=1}^{M_{j-1}}\bigl(1+M_j\gamma_{j-1}^{(n)}\bigl(x_1,U^{(x_1,i|_2^l)},\dots,U^{(x_1,i|_{j-2}^l)},x_{j-1}\bigr)\bigr).$$

Proceeding inductively, we find

$$\begin{aligned}&E\bigl(K_j^{(n)}(r)\bigr)\\ &\quad=\frac{r}{\prod_{p=1}^{j-2}M_{p+1}}\sum_{l=1}^{j-1}\sum_{x|_{j-1}\in\mathcal{M}|_{j-1}}\prod_{p=1}^{l-1}M_{p+1}\gamma_p^{(n)}(x|_p)\\ &\qquad\times\prod_{p=l+1}^{j-1}\bigl(1+M_{p+1}\gamma_p^{(n)}(x|_p)\bigr).\end{aligned}\tag{5.34}$$

Similarly,

$$\begin{aligned}&E\bigl(\mu_j^{(n)}\bigl(\mathcal{E}_j^{(n)}\cap\bigl[0,\Theta_{j-1}^{(n)}(r)\bigr]\bigr)\bigr)\\ &\quad=\frac{r}{\prod_{p=1}^{j-1}M_{p+1}}\sum_{l=1}^{j-1}\sum_{x|_j\in\mathcal{M}|_j}\gamma_j^{(n)}(x|_j)\prod_{p=1}^{l-1}M_{p+1}\gamma_p^{(n)}(x|_p)\\ &\qquad\times\prod_{p=l+1}^{j-1}\bigl(1+M_{p+1}\gamma_p^{(n)}(x|_p)\bigr).\end{aligned}\tag{5.35}$$

We are now ready to argue our claims.

(a) Fix $j\in\{2,\dots,k\}$, $r>0$ and $L>0$. Then, using Jensen's inequality,

$$\begin{aligned}&P\bigl(\min\{\xi_j^{(n)}(s):s\in\mathcal{E}_j^{(n)}\cap\bigl[0,\Theta_{j-1}^{(n)}(r)\bigr]\}>L\bigr)\\ &\quad=\sum_{l=0}^{\infty}\biggl(1-\frac{L}{M_j}\biggr)^lP\bigl(K_j^{(n)}(r)=l\bigr)=E\biggl[\biggl(1-\frac{L}{M_j}\biggr)^{K_j^{(n)}(r)}\biggr]\\ &\quad\ge\biggl(1-\frac{L}{M_j}\biggr)^{E(K_j^{(n)}(r))}=\biggl\{\biggl(1-\frac{L}{M_j}\biggr)^{M_j/L}\biggr\}^{L[E(K_j^{(n)}(r))/M_j]}.\end{aligned}\tag{5.36}$$

Using (5.34), we find that the expression within square brackets on the right-hand side of (5.36) is the expression in (5.17), which goes to 0 as $n\to\infty$ by hypothesis. From (5.14), we have that the expression within curly brackets on the right-hand side of (5.36) is bounded away from zero as $n \to \infty$. It immediately follows that the probability on the left-hand side of (5.36) tends to 1 as $n \to \infty$, and part (a) of Lemma 5.6 is established.

(b) Given $\varepsilon > 0$, by Markov's inequality,

$$P\big(\mu_j^{(n)}(\mathcal{E}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)]) > \varepsilon\big) \le \varepsilon^{-1} E\big(\mu_j^{(n)}(\mathcal{E}_j^{(n)} \cap [0, \Theta_{j-1}^{(n)}(r)])\big) \tag{5.37}$$

and the result follows from (5.35) and (5.18). $\square$

REMARK 5.8. If $X_{k-1}^{(n)} \to X_{k-1}$ a.s. as $n \to \infty$ in Skorohod space, then, by Proposition 5.2 in Chapter 3 of [15] (page 118), we have that

$$\lim_{n\to\infty} X_{k-1}^{(n)}(s) = \lim_{n\to\infty} X_{k-1}^{(n)}(s-) = X_{k-1}(s) \tag{5.38}$$

for all $s \ge 0$ which is a continuity point of X_{k-1}.

LEMMA 5.9. *Assume that the induction hypotheses* (5.23)–(5.24) *hold, and let $r \in [0, \infty)$ be fixed. Then $\Gamma_k^{(n)}(r) \to \Gamma_k(r)$ in probability as $n \to \infty$.*

PROOF. The strategy is to separate the contribution of the extra marks and Poissonian marks with large labels from the remaining contributions. The convergence of the remaining main (as it turns out) contributions to the corresponding infinite volume contributions follows readily from the first part of (5.15), since there is only a fixed finite number of contributions involved. The negligibility of the total contribution of extra marks was established in Lemma 5.6 above, so we are left with establishing that of the total contribution of high label marks. Details follow.

Let $\Psi_k^{(n)}(r) = \mu_k^{(n)}((\mathcal{S}_k^{(n)} \setminus \mathcal{E}_k^{(n)}) \cap [0, r])$. Then

$$|\Gamma_k(r) - \Gamma_k^{(n)}(r)| \le |\Gamma_k(r) - \Psi_k^{(n)}(r)| + \mu_k^{(n)}(\mathcal{E}_k^{(n)} \cap [0, r]). \tag{5.39}$$

By Lemma 5.6(b), the second term on the right of (5.39) goes to 0 in probability as $n \to \infty$. We will argue that so does the first one. In preparation for this, let us take, for given $\varepsilon > 0$, $m_1 \in \mathbb{N}_*$ such that

$$\sum_{x_1 > m_1} \sum_{x|^2 \in \mathbb{N}_*^{k-1}} \bar{\gamma}_k(x|_k) \le \varepsilon/k \tag{5.40}$$

[recall the notation introduced around (2.2) above]. Proceeding inductively, with $m_1, \dots, m_{j-1}$, $2 \le j \le k-1$, fixed, choose m_j such that

$$\sum_{x_1=1}^{m_1} \cdots \sum_{x_{j-1}=1}^{m_{j-1}} \sum_{x_j > m_j} \sum_{x|^{j+1} \in \mathbb{N}_*^{j+1}}^{\infty} \bar{\gamma}_k(x|_k) \le \varepsilon/k \tag{5.41}$$

and with $m_1, \dots, m_{k-1}$ fixed, choose m_k such that

$$\sum_{x_1=1}^{m_1} \cdots \sum_{x_{k-1}=1}^{m_{k-1}} \sum_{x_k > m_k}^{\infty} \bar{\gamma}_k(x|_k) \le \varepsilon/k. \tag{5.42}$$

This procedure is well defined by (4.2).

Going back to the first term on the right of (5.39), we have that

$$\begin{aligned} &\big|\Gamma_k(r) - \Psi_k^{(n)}(r)\big| \\ &\quad \le \Bigg| \sum_{s \in \tilde{\mathcal{S}}_k^{(m_k)} \cap [0,r]} \big\{\gamma_k\big(X_{k-1}(s), \xi_k(s)\big) - \gamma_k^{(n)}\big(X_{k-1}^{(n)}(s), \xi_k(s)\big)\big\} T_s^{(k)} \Bigg| \qquad (5.43) \\ &\qquad + \sum_{s \in (\mathcal{S}_k \setminus \tilde{\mathcal{S}}_k^{(m_k)}) \cap [0,r]} \gamma_k\big(X_{k-1}(s), \xi_k(s)\big) T_s^{(k)} \qquad (5.44) \\ &\qquad + \sum_{s \in (\mathcal{S}_k^{(n)} \setminus \tilde{\mathcal{S}}_k^{(m_k)}) \cap [0,r]} \gamma_k^{(n)}\big(X_{k-1}^{(n)}(s), \xi_k(s)\big) T_s^{(k)} \qquad (5.45) \end{aligned}$$

[recall (4.14)]. We have used here the fact that, given the coupled construction of $X_k^{(n)}$ and X_k, we have that $\xi_k^{(n)}(s) = \xi_k(s)$ for Poisson points s.

The expression on the right-hand side of (5.43) converges to 0 in probability as n increases because it is a finite sum, and from the first part of (5.15), and since $X_{k-1}^{(n)}(s) = X_{k-1}(s)$ for all $s \in \tilde{\mathcal{S}}_k^{(m_k)} \cap [0,r]$ for all large enough n almost surely, as follows from Remark 5.8 above, and the fact that the points of $\tilde{\mathcal{S}}_k^{(m_k)}$ are almost surely continuity ponts of X_{k-1}.

Let B and C denote the expressions in (5.44) and (5.45), respectively.

To analyze B, we start by taking, for given $\eta > 0$, r_0' such that

$$P\big(\Theta_{k-1}(r_0') > r\big) \ge 1 - \eta. \tag{5.46}$$

This is allowed by the second assertion of Lemma 4.6. Now letting

$$\mathcal{A}_0 = \big(\mathcal{S}_k \setminus \tilde{\mathcal{S}}_k^{(m_k)}\big) \cap \big[0, \Theta_{k-1}(r_0')\big],$$

we define

$$\begin{aligned} \mathcal{A}_1 &= \big\{s \in \mathcal{A}_0 : X_{k-1,1}(s) > m_1\big\}, \\ \mathcal{A}_2 &= \big\{s \in \mathcal{A}_0 : X_{k-1,1}(s) \le m_1, X_{k-1,2}(s) > m_2\big\}, \\ &\ \ \vdots \\ \mathcal{A}_{k-1} &= \big\{s \in \mathcal{A}_0 : X_{k-1,1}(s) \le m_1, \dots, X_{k-1,k-2}(s) \le m_{k-2}, \\ &\qquad\qquad X_{k-1,k-1}(s) > m_{k-1}\big\}, \\ \mathcal{A}_k &= \big\{s \in \mathcal{A}_0 : X_{k-1,1}(s) \le m_1, \dots, X_{k-1,k-1}(s) \le m_{k-1}\big\}. \end{aligned} \tag{5.47}$$

Notice that by the definition of $\mathcal{A}_0$ and $\tilde{\mathcal{S}}_k^{(m_k)}$ (recall (4.14) above), we have that $\xi_k(s) > m_k$. Then, outside an event of probability at most η, we have that

$$B \le \sum_{i=1}^{k} \sum_{s\in\mathcal{A}_i} \gamma_k\big(X_{k-1}(s), \xi_k(s)\big) T_s^{(k)} \tag{5.48}$$

and following the same arguments used to establish (4.11), and using (5.40)–(5.42), we conclude that

$$\begin{aligned} &E\Bigg(\sum_{i=1}^{k}\sum_{s\in\mathcal{A}_i} \gamma_k\big(X_{k-1}(s), \xi_k(s)\big) T_s^{(k)}\Bigg) \\ &\quad \le r_0'\Bigg[\sum_{x_1>m_1}\sum_{x|^2\in\mathbb{N}_*^{k-1}} \bar{\gamma}_k(x|_k) + \sum_{x_1=1}^{m_1}\sum_{x_2>m_2}\sum_{x|^3\in\mathbb{N}_*^{k-2}} \bar{\gamma}_k(x|_k) \\ &\qquad\qquad + \cdots + \sum_{x_1=1}^{m_1}\cdots\sum_{x_{k-1}=1}^{m_{k-1}}\sum_{x_k>m_k} \bar{\gamma}_k(x|_k)\Bigg] \le r_0'\varepsilon, \end{aligned}$$

where the first inequality comes from ignoring the restriction $s \in \mathcal{A}_0$ in the first $k-1$ terms of the sum in i. This shows that $B \to 0$ in probability as $\epsilon \to 0$, since η is arbitrary.

The analysis of C is similar, with the dependence on n as a distinctive aspect. From induction hypothesis (5.24) and the second assertion of Lemma 4.6, given $\eta > 0$, there exists r_0 such that for all n sufficiently large,

$$P\big(\Theta_{k-1}^{(n)}(r_0) > r\big) \ge 1 - \eta; \tag{5.49}$$

recall (5.26). With such r_0 and the above choice of $m_1, \dots, m_k$, define

$$\begin{aligned} \mathcal{A}_0^{(n)} &= \big(\mathcal{S}_k^{(n)} \setminus \mathcal{S}_k^{(m_k)}\big) \cap \big[0, \Theta_{k-1}^{(n)}(r_0)\big], \\ \mathcal{A}_1^{(n)} &= \big\{s \in \mathcal{A}_0^{(n)} : X_{k-1,1}^{(n)}(s) > m_1\big\}, \\ \mathcal{A}_2^{(n)} &= \big\{s \in \mathcal{A}_0^{(n)} : X_{k-1,1}^{(n)}(s) \le m_1, X_{k-1,2}^{(n)}(s) > m_2\big\}, \\ &\ \ \vdots \\ \mathcal{A}_k^{(n)} &= \big\{s \in \mathcal{A}_0^{(n)} : X_{k-1,1}^{(n)}(s) \le m_1, \dots, X_{k-1,k-1}^{(n)}(s) \le m_{k-1}\big\}. \end{aligned} \tag{5.50}$$

Then

$$P\Bigg(C \le \sum_{i=1}^{k-1} \sum_{s\in\mathcal{A}_i^{(n)}} \gamma_k^{(n)}\big(X_{k-1}^{(n)}(s), \xi_k(s)\big) T_s^{(k)}\Bigg) \ge 1 - \eta \tag{5.51}$$

for all n large enough. By (5.15), we may take n suficiently large such that

$$
\begin{aligned}
&\left|\sum_{x_1>m_1}^{M_1}\sum_{x|^2\in\mathcal{M}|^2}\bar{\gamma}_k^{(n)}(x|_k)-\sum_{x_1>m_1}\sum_{x|^2\in\mathbb{N}_*^{k-1}}\bar{\gamma}_k(x|_k)\right|\le\varepsilon/k,\\
&\left|\sum_{x_1=1}^{m_1}\sum_{x_2=m_2+1}^{M_2}\sum_{x|^3\in\mathcal{M}|^3}\bar{\gamma}_k^{(n)}(x|_k)-\sum_{x_1=1}^{m_1}\sum_{x_2>m_2}\sum_{x|^3\in\mathbb{N}_*^{k-2}}\bar{\gamma}_k(x|_k)\right|\le\varepsilon/k, \qquad (5.52)\\
&\qquad\vdots\\
&\left|\sum_{x_1=1}^{m_1}\cdots\sum_{x_{k-1}=1}^{m_{k-1}}\sum_{x_k=m_k+1}^{M_k}\bar{\gamma}_k^{(n)}(x|_k)-\sum_{x_1=1}^{m_1}\cdots\sum_{x_{k-1}=1}^{m_{k-1}}\sum_{x_k>m_k}\bar{\gamma}_k(x|_k)\right|\le\varepsilon/k.
\end{aligned}
$$

Following the same arguments used to establish (4.11), and using (5.40)–(5.42) and (5.52), we get

$$
\begin{aligned}
&E\left(\sum_{i=1}^{k-1}\sum_{s\in\mathcal{A}_i^{(n)}}\gamma_k^{(n)}\big(X_{k-1}^{(n)}(s),\xi_k(s)\big)T_s^{(k)}\right)\\
&\quad\le r_0'\Bigg[\sum_{x_1=m_1+1}^{M_1}\sum_{x|^2\in\mathcal{M}|^2}\bar{\gamma}_k^{(n)}(x|_k)+\sum_{x_1=1}^{m_1}\sum_{x_2=m_2+1}^{M_2}\sum_{x|^3\in\mathcal{M}|^3}\bar{\gamma}_k^{(n)}(x|_k)\\
&\qquad\qquad+\cdots+\sum_{x_1=1}^{m_1}\cdots\sum_{x_{k-1}=1}^{m_{k-1}}\sum_{x_k=m_k+1}^{M_k}\bar{\gamma}_k^{(n)}(x|_k)\Bigg]\\
&\quad\le\big(r_0+r_0'\big)\varepsilon.
\end{aligned}
$$

This shows that $C\to 0$ in probability as we first take $n\to\infty$ and then $\epsilon\to 0$, since η is arbitrary, thus completing the proof. $\square$

COROLLARY 5.10. *The result of Lemma 5.9 still holds if we replace r on the left-hand side by r_n with $r_n\to r$ as $n\to\infty$, with (r_n) a deterministic sequence.*

PROOF. Let us write

$$
\big|\Gamma_k^{(n)}(r_n)-\Gamma_k(r)\big|\le\big|\Gamma_k^{(n)}(r_n)-\Gamma_k^{(n)}(r)\big|+\big|\Gamma_k^{(n)}(r)-\Gamma_k(r)\big|. \qquad (5.53)
$$

Using the hypothesis and the monotonicity of $\Gamma_k^{(n)}$, given $\delta>0$, we have that the first term on the right-hand side of (5.53) is bounded above by $\Gamma_k^{(n)}(r+\delta)-\Gamma_k^{(n)}(r-\delta)$ for all n large enough, which is in turn bounded above by

$$
\begin{aligned}
&\Gamma_k(r+\delta)-\Gamma_k(r-\delta)+\big|\Gamma_k^{(n)}(r+\delta)-\Gamma_k(r+\delta)\big|\\
&\qquad+\big|\Gamma_k^{(n)}(r-\delta)-\Gamma_k(r-\delta)\big|.
\end{aligned}
\qquad (5.54)
$$

Let $\eta > 0$ now be given. By Lemma 5.9, and using (5.53)–(5.54), we find that

$$\limsup_{n\to\infty} P\bigl(\bigl|\Gamma_k^{(n)}(r_n) - \Gamma_k(r)\bigr| > \eta\bigr) \le P\bigl(\Gamma_k(r+\delta) - \Gamma_k(r-\delta) > \eta/3\bigr) \tag{5.55}$$

and the result follows from r being almost surely a continuity point of Γ_k (see Remarks 4.1 and 4.3), since δ is arbitrary. $\square$

REMARK 5.11. The same argument, of course, works in the case when (r_n, r) are random and independent of $(\Gamma_k^{(n)}, \Gamma_k)$ and $r_n \to r$ almost surely as $n \to \infty$. This can be applied to establish that under the assumption of Lemma 5.9, we have that

$$\Theta_k^{(n)}(T) \to \Theta_k(T) \qquad \text{as } n \to \infty, \tag{5.56}$$

in probability for every $T \ge 0$.

The next result is a finite volume version of Lemma 4.9 in the above section.

LEMMA 5.12. *Given $k \ge 2$, $n \ge 1$, let $X_k^{(n)}$ be a trap model on $\mathbb{T}_k^{(n)}$. Suppose that the assumptions of Theorem 5.4 and the induction hypotheses (5.23)–(5.24) all hold. Then given $m \ge 1$ and $\epsilon > 0$, there exists $\tilde m = \tilde m^{(k)} \ge 1$ such that the event*

$$\begin{aligned} A_m^{(n)} = \{&\text{if } X_{k,i}^{(n)}(t) > \tilde m \text{ for some } i = 1,\dots,k-1 \text{ and } t \in [0,T], \\ &\qquad\qquad \text{then } X_{k,j}^{(n)}(t) > m \text{ for } j = i+1,\dots,k\} \end{aligned} \tag{5.57}$$

has probability bounded below by $1-\epsilon$ for all n large enough.

PROOF. We argue similarly as in the proof of Lemma 4.9, except that statements here hold *with high probability*, rather than almost surely.

By Remark 5.11 and the fact that $\lim_{r\to\infty}\Theta_j(r) = \infty$ for every $1 \le j \le k$, we may choose $T' > 0$ such that $\Theta_k^{(n)}(T') > T$ with probability at least $1-\epsilon/4$ uniformly in n. Now for $m \in \mathbb{N}_*$ and $j = 1,\dots,k$, let $\tilde{\mathcal{S}}_j^{(m)}$ be as in (4.14) above. For fixed $\ell \in \mathbb{N}_*$, let $\mathcal{T}_{kj}^{(n)}(\ell)$ denote the set of times up to $\Theta_{k-1}^{(n)}(T')$ spent by $X_{k-1,j}^{(n)}$ above ℓ, $j = 1,\dots,k-1$. Analogously as for the infinite volume case [see (4.15)], we may check that the expected Lebesgue measure of $\mathcal{T}_{kj}^{(n)}(\ell)$ equals

$$T' \sum_{x_1=1}^{M_1} \cdots \sum_{x_j=\ell+1}^{M_j} \cdots \sum_{x_{k-1}=1}^{M_{k-1}} \gamma_1^{(n)}(x|_1)\cdots\gamma_{k-1}^{(n)}(x|_{k-1}) \tag{5.58}$$

plus the contribution of the extra marks and their descendants. It follows from (5.15) that the $\limsup_{n\to\infty}$ of the expression in (5.58) vanishes as $\ell \to \infty$. By Lemma 5.6(b), the contribution of the extra marks and their descendants vanishes in probability as $n \to \infty$. It then follows from elementary properties of Poisson processes, that

$$\left\{\bigcup_{j=1}^{k-1} \mathcal{T}_{kj}^{(n)}(\ell)\right\} \cap \tilde{\mathcal{S}}_k^{(m)} = \varnothing \tag{5.59}$$

outside an event whose probability is bounded above by $\epsilon/4$ for all ℓ, n large enough. This statement is about Poissonian marks; but it also holds for extra marks by Lemma 5.6(a).

So, given $m \in \mathbb{N}_*$ and $\epsilon > 0$, we find $\hat{m}^{(k)}$ such that outside an event of probability smaller than $\epsilon/2$ for all n large enough, if $X_{k,i}^{(n)}(t) > \hat{m}^{(k)}$ for any $t \le T$, then $X_{k,k}^{(n)}(t) > m$. This in particular establishes the claim for $k = 2$ by the choice $\tilde{m}^{(2)} = \hat{m}^{(2)}$. Let us assume that the claim is established for $k-1$. Then substituting in that claim ϵ for $\epsilon/4$, and T for T'' such that $P(\Theta_{k-1}^{(n)}(T') \le T'') > 1 - \epsilon/4$ for all large enough n as T, and choosing $\tilde{m}^{(k)} = \tilde{m}^{(k-1)} \vee \hat{m}^{(k)}$, we find that it satisfies the claim for k. $\square$

5.4. *End of proof of Theorem* 5.4. For $j = 1, \dots, k$, $n \ge 1$, let $X_j^{(n)} \sim TM(\mathbb{T}_k^{(n)}, \underline{\gamma_j^{(n)}})$ and $X_j \sim K_j(\mathbb{T}_k, \underline{\gamma_j})$, and, for $k \ge 2$ fixed, suppose that $X_{k-1}^{(n)} \to X_{k-1}$ in probability as $n \to \infty$. We may then and will inductively suppose that

$$X_{k-1}^{(n')} \to X_{k-1} \qquad \text{a.s. as } n' \to \infty, \tag{5.60}$$

for a subsequence (n'). We will fix $\epsilon > 0$, $T > 0$ and $m \ge 1$ and choose T' and $\tilde{m}$ such that outside an event $\mathcal{E} = \mathcal{E}_{n'}$ of probability at most $\epsilon/2$ for all n' large enough, we have that the conclusions of Lemma 4.9 and 5.12 hold, and also that $\Theta_{k-1}(T') \wedge \Theta_{k-1}^{(n')}(T') > T$. We will also assume that $\tilde{m} \ge m$, and that the claims of Lemma 5.6 hold almost surely over (n').

On the way to showing the validity of (5.60) with k replacing $k-1$ (in probability), we now proceed to define appropriate time distortions $\lambda^{(n')}$; see the discussion on the Skorohod metric at the beginning of Section 5.2. Let us start by considering the constancy intervals of $X_{k-1,1}$ in $[0, \Theta_{k-1}(T'))$ with $X_{k-1,1} \le \tilde{m}$. These are defined to be the *rank-$\tilde{m}$ constancy intervals of the level* 1 *for* X_{k-1}. Proceeding inductively, given $2 \le \ell \le k-1$, for each rank-$\tilde{m}$ constancy interval I of level $\ell - 1$, we consider the constancy intervals of $X_{k-1,\ell}$ inside I such that $X_{k-1,\ell} \le \tilde{m}$. The collection of all such intervals obtained from all the rank-$\tilde{m}$ constancy intervals of level $\ell - 1$ for X_{k-1} form the set of rank-$\tilde{m}$ constancy intervals of level ℓ for X_{k-1}.

Let $a_1, \dots, a_{2L}$ denote the collection of all endpoints of all the rank-$\tilde{m}$ constancy intervals of level i for X_{k-1}, $i = 1, \dots, k-1$, in increasing order, and let $b_1, \dots, b_{2J}$ denote the collection of all endpoints of all the rank-$\tilde{m}$ constancy intervals of level $k-1$ for X_{k-1} in increasing order. See Figure 5.

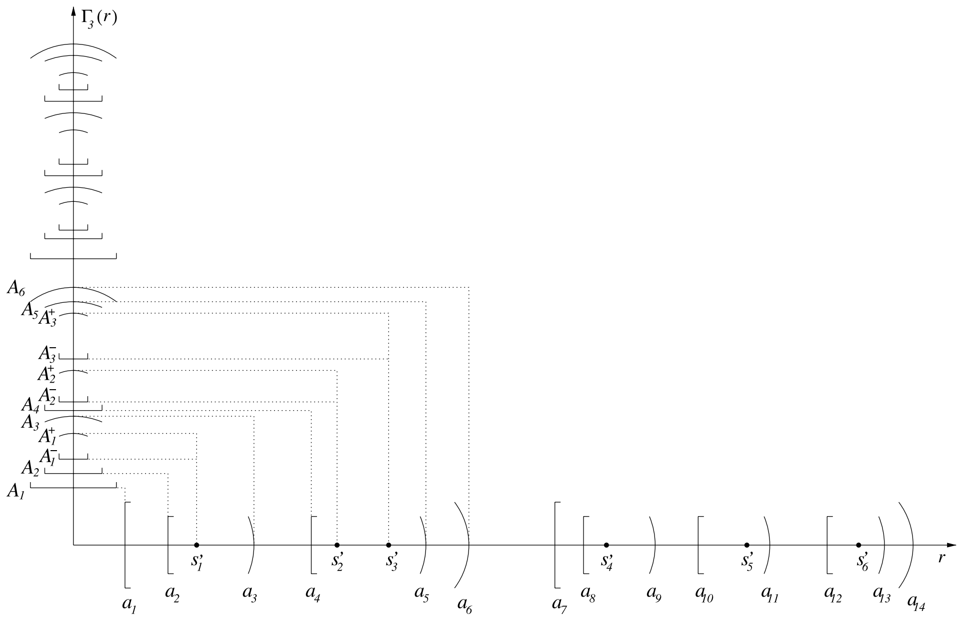

FIG. 5. *Depiction of objects appearing in the argument for the proof of Theorem 5.4 with* $k = 3$. *Rank-*$\tilde{m}$ *constancy intervals of the first level* ($[a_1, a_6)$ *and* $[a_7, a_{14})$) *and rank-*$\tilde{m}$ *constancy intervals of the second level* ($[a_2, a_3)$, $[a_4, a_5)$, $[a_8, a_9)$, $[a_{10}, a_{11})$ *and* $[a_{12}, a_{13})$) *for* X_2 *appear on the x-axis. Correspondingly on the y-axis, we have rank-*$\tilde{m}$ *constancy intervals of the first level* ($[A_1, A_6)$ *and another whose endpoints are not named in the picture*), *rank-*$\tilde{m}$ *constancy intervals of the second level* ($[A_2, A_3)$, $[A_4, A_5)$, *and others whose endpoints are not named in the picture*), *and rank-*$\tilde{m}$ *constancy intervals of the third level* ($[A_1^-, A_1^+)$, $[A_2^-, A_2^+)$, $[A_3^-, A_3^+)$, *and others whose endpoints are not named in the picture*) *for* X_3. *Some of the correspondences between the axes are indicated by dotted lines. We have also* $b_1 = a_2$, $b_2 = a_3$, $b_3 = a_4$, $b_4 = a_5$, $b_5 = a_8$, $b_6 = a_9$, $b_7 = a_{10}$, $b_8 = a_{11}$, $b_9 = a_{12}$ *and* $b_{10} = a_{13}$. *This picture is also good for* $X_3^{(n)}$, *with n large, and with all endpoint labels having superscripts "*(n)*."*

Let us also consider rank-$\tilde{m}$ constancy intervals of level i for $X_{k-1}^{(n')}$, with the paralell definition to the one above. By the assumption that Lemma 5.6 holds almost surely over (n'), and for n' large enough, there is one-to-one correspondence of the $a_1^{(n')}, \dots, a_{2L^{(n')}}^{(n')}$ and $a_1, \dots, a_{2L}$, with $L^{(n')} = L$ for all large n' and $a_i^{(n')}$ corresponding to a_i, and from (5.60),

$$a_i^{(n')} \to a_i \tag{5.61}$$

almost surely as $n' \to \infty$ for every $i = 1, \dots, 2L$.

Let now $A_i = \Gamma_k(a_i)$ and $A_i^{(n')} = \Gamma_k^{(n')}(a_i^{(n')})$, $i = 1, \dots, 2L$. See Figure 5. It follows from Lemma 5.9 (see Corollary 5.10 and Remark 5.11) that

$$A_i^{(n')} \to A_i \tag{5.62}$$

in probability as $n' \to \infty$, and we may assume a.s. convergence by taking a subsequence.

Let $\{s'_1, \dots, s'_Q\}$ be the enumeration in increasing order of $\bigcup_{i=1}^J \{\tilde{\mathcal{S}}_k^{(m)} \cap [b_{2i-1}, b_{2i})\}$, and let $A_i^- = \Gamma_k(s'_i-)$ and $A_i^+ = \Gamma_k(s'_i)$. See Figure 5. We note that the intervals $[A_i^-, A_i^+)$, $i = 1, \dots, Q$ are the rank-m constancy intervals of level k for X_k, whereas A_i, $i = 1, \dots, 2L$, are the endpoints of all the rank-$\tilde{m}$ constancy intervals of level i for X_k, $i = 1, \dots, k-1$.

We remark at this point that under our assumptions so far, we have that for all $i = 1, \dots, J$

$$\tilde{\mathcal{S}}_k^{(m)} \cap [b_{2i-1}, b_{2i}) = \tilde{\mathcal{S}}_k^{(m)} \cap [b_{2i-1}^{(n')}, b_{2i}^{(n')}) \tag{5.63}$$

almost surely for all n' large enough, where $b_i^{(n')} = a_j^{(n')}$ such that $a_j = b_i$.

Let $A_i^-(n') = \Gamma_k^{(n')}(s'_i-)$, $A_i^+(n') = \Gamma_k^{(n')}(s'_i)$. Then we have that for all large enough n', $[A_i^-(n'), A_i^+(n'))$, $i = 1, \dots, Q$ are the rank-m constancy intervals of level k for $X_k^{(n')}$, whereas $A_i^{(n')}$, $i = 1, \dots, 2L$, are the endpoints of all the rank-$\tilde{m}$ constancy intervals of level i for $X_k^{(n')}$, $i = 1, \dots, k-1$.

Let us now argue that

$$A_i^{\pm}(n') \to A_i^{\pm} \tag{5.64}$$

in probability as $n' \to \infty$ (and again we may assume a.s. convergence by taking a subsequence). It is enough to first note that almost surely $A_i^-(n') = \tilde{\Gamma}_k^{(n')}(s'_i)$, $A_i^- = \tilde{\Gamma}_k(s'_i)$, $A_i^+(n') = \tilde{\Gamma}_k^{(n')}(s'_i) + \gamma_k^{(n)}(X_{k-1}^{(n')}(s'_i), \xi_k(s'_i))$ and $A_i^+ = \tilde{\Gamma}_k(s'_i) + \gamma_k(X_{k-1}(s'_i), \xi_k(s'_i))$, where $\tilde{\Gamma}_k^{(n')}$ and $\tilde{\Gamma}_k$ are obtained from $\tilde{\mu}_k^{(n')}$ and $\tilde{\mu}_k$ as $\Gamma_k^{(n')}$ and $\Gamma_k^{(n')}$ are obtained from $\mu_k^{(n')}$ and μ_k, respectively, where $\tilde{\mu}_k^{(n')} = \mu_k^{(n')}$ and $\tilde{\mu}_k = \mu_k$ everywhere except at $\{s'_i\}$, where $\tilde{\mu}_k^{(n')}$ and $\tilde{\mu}_k$ both vanish. By the same arguments above we get $\tilde{\Gamma}_k^{(n')}(s'_i) \to \tilde{\Gamma}_k(s'_i)$ in probability, and the result follows upon noticing that $X_{k-1}^{(n')}(s'_i) = X_{k-1}(s'_i)$ for all large enough n' and using (5.15).

We are now ready to define our time distortion. Let $\lambda^{(n')} : [0, \infty) \to [0, \infty)$ be such that

$$\lambda^{(n')}(A_i) = A_i^{(n')}, \qquad \lambda^{(n')}(A_i^-) = A_i^-(n'), \qquad \lambda^{(n')}(A_i^+) = A_i^+(n') \tag{5.65}$$

and make it linear between successive points of $\mathcal{A} := \{A_i, i = 1, \dots, 2L; A_j^-, A_j^+, j = 1, \dots, Q\}$, and linear with inclination 1 from $\max \mathcal{A}$ on. Then $\lambda^{(n')}$ is almost surely well defined for all large enough n', and one readily checks that condition (5.65) implies that $\lambda^{(n')}$ maps rank-$\tilde{m}$ constancy intervals of level i for X_k to the corresponding rank-$\tilde{m}$ constancy intervals of level i for $X_k^{(n')}$, $i = 1, \dots, k$, given by the coupling. In particular, $X_{k,j}(\lambda^{(n')}(\cdot)) = X_{k,j}^{(n')}(\cdot)$, $j = 1, \dots i$, on those respective intervals. From the assumptions of the paragraph of (5.60), we then have that outside $\mathcal{E}$

$$\sup_{0\le u\le T} \rho\big(X_k^{(n')}, X_k, \lambda^{(n')}, u\big) \le 1/m \tag{5.66}$$

and by (5.62) and (5.64) and our construction and assumptions it follows that

$$\phi\big(\lambda^{(n')}\big) \to 0 \tag{5.67}$$

as $n' \to \infty$ almost surely, where ϕ is the time distortion function introduced in (5.13).

It follows from all of the above that for every fixed $\epsilon, T > 0$ and $m \in \mathbb{N}_*$ we may find a subsequence (n') such that

$$P\left(\rho\big(X_k^{(n')}, X_k\big) > \frac{1}{m} + e^{-T}\right) \le \epsilon \tag{5.68}$$

for all n' large enough, so we have that $X_k^{(n')} \to X_k$ in probability, and this readily implies the claim of Theorem 5.4.

5.5. *Proof of Theorem* 5.2. The strategy will be to work with a coupled version of $(\underline{\gamma_k^{(n)}}, \underline{\gamma_k})$, which we will call $(\underline{\hat\gamma_k^{(n)}}, \underline{\hat\gamma_k})$, such that almost surely for every $j = 1, \dots, k$ and $x|_j \in \mathbb{N}_*^j$

$$\hat\gamma_j^{(n)}(x|_j) \to \hat\gamma_j(x|_j) \qquad \text{as } n \to \infty \tag{5.69}$$

and then verify the remaining conditions of Theorem 5.4. (Recall that in the context of Theorem 5.2, the sets of parameters $\underline{\gamma_k^{(n)}}$ and $\underline{\gamma_k}$ are random.)

For the coupling, we use the construction of [19], Section 6, which we describe briefly, guiding the reader to that reference for more details.

Let $E_j(x|_j)$, $x|_j \in \mathbb{N}_*^j$, $j = 1, \dots, k$ be independent mean one exponential random variables, and, for $x|_{j-1} \in \mathbb{N}_*^{j-1}$ make

$$S_j(x|_j) = \sum_{i=1}^{x_j} E_j(x|_{j-1}, i), \tag{5.70}$$

where $x|_0$ is a void symbol. Let now

$$\hat\gamma_j^{(n)}(x|_j) = c_j^{(n)} G_j^{-1}\left(\frac{S_j(x|_j)}{S_j(x|_{j-1}, M_j + 1)}\right), \tag{5.71}$$

$$\hat\gamma_j(x|_j) = S_j(x|_j)^{-1/\alpha_j}. \tag{5.72}$$

From an elementary large deviation estimate, we may assume that

$$S_j(x|_{j-1}, M_j + 1) \le 2M_j \tag{5.73}$$

for all $x|_{j-1} \in \mathcal{M}|_{j-1}$ and n sufficiently large almost surely, recalling that the M_j's depend on n.

Then $\hat{\gamma}_j^{(n)}(x|_{j-1})$ and $\hat{\gamma}_j(x|_{j-1})$ are versions of $\gamma_j^{(n)}(x|_{j-1})$ and $\gamma_j(x|_{j-1})$, respectively [23]. Proposition 6.3 and Lemma 6.4 of [19] immediately imply (5.69), and also that almost surely for every $j = 1, \dots, k$ and $x|_{j-1} \in \mathbb{N}_*^{j-1}$

$$\sum_{x_j \in \mathcal{M}_j} \hat{\gamma}_j^{(n)}(x|_j) \to \sum_{x_j \in \mathbb{N}_*} \hat{\gamma}_j(x|_j) \qquad \text{as } n \to \infty. \tag{5.74}$$

The a.s. validity of the first part of (5.15) for all j, x, as well as that of the second part for $k = 1$, follow immediately.

In order to get the a.s. validity of the second part of (5.15) for general k, we argue as follows. We may suppose by induction that it holds for $k - 1$. We first write the sum in the second part of (5.15) more explicitly as follows:

$$\sum_{x_1} \hat{\gamma}_1^{(n)}(x_1) \cdots \sum_{x_j} \hat{\gamma}_j^{(n)}(x|_j) \cdots \sum_{x_k} \hat{\gamma}_k^{(n)}(x|_k), \tag{5.75}$$

and break each of the k sums (following the strategy of [19]; see proof of Proposition 6.3 thereof) in three parts, so that the jth sum is written as

$$\sum_{x_j}^{(1)} + \sum_{x_j}^{(2)} + \sum_{x_j}^{(3)}, \tag{5.76}$$

where given $\delta_j \in (0, 1)$, the first sum is over x_j such that $\hat{\gamma}_j(x|_j) > \delta_j$, the second sum is over $x|_j$ such that $M_j^{-1/\alpha_j} < \hat{\gamma}_k(x|_j) \le \delta_j$ and the third sum is over $x|_j$ such that $\hat{\gamma}_k(x|_j) \le M_j^{-1/\alpha_j}$.

It follows from (5.69) that

$$\sum_{x_1}^{(1)} \hat{\gamma}_1^{(n)}(x_1) \cdots \sum_{x_k}^{(1)} \hat{\gamma}_k^{(n)}(x|_k) \to \sum_{x_1}^{(1)} \hat{\gamma}_1(x_1) \cdots \sum_{x_k}^{(1)} \hat{\gamma}_k(x|_k) \tag{5.77}$$

as $n \to \infty$ almost surely, since these are sums over a fixed bounded set of terms. We will show that

$$\limsup_{\delta_1, \dots, \delta_k \to 0} \limsup_{n \to \infty} \sum_{x_1}^{(i_1)} \hat{\gamma}_1^{(n)}(x_1) \cdots \sum_{x_k}^{(i_k)} \hat{\gamma}_{i_k}^{(n)}(x|_k) = 0 \tag{5.78}$$

almost surely, for all $(i_1, \dots, i_k) \in \{1, 2, 3\}^k \setminus \{(1, \dots, 1)\}$. Since, again, $\sum_{x_j}^{(1)}$ are sums over a fixed bounded set of terms, and using the induction hypothesis, it is enough to consider sums

$$\sum_{x_1}^{(i_1)} \hat{\gamma}_1^{(n)}(x_1) \cdots \sum_{x_k}^{(i_k)} \hat{\gamma}_k^{(n)}(x|_k) \tag{5.79}$$

with $i_1 \in \{2, 3\}$.

Let us first consider the case where $i_1 = 2$ and $i_j \in \{1, 2\}$ for all $j = 2, \dots, k$. It follows from the arguments in the proof of Lemma 6.5 of [19] that given $\eta_j > 0$ there exists $C_j < \infty$ such that $\hat{\gamma}_j^{(n)}(x|_j) \le C_j[(\hat{\gamma}_j(x|_j))^{1-\eta_j} \vee (\hat{\gamma}_j(x|_j))^{1+\eta_j}]$, and we replace $\vee$ by $+$, thus obtaining an upper bound. We then have an upper bound for (5.79) in terms of 2^{k-1} sums of the form constant times

$$\sum_{x_1 : \hat{\gamma}_1(x_1) \le \delta_1} (\hat{\gamma}_1(x_1))^{1-\eta_1} \sum_{x_2} (\hat{\gamma}_2(x|_2))^{1\pm\eta_2} \cdots \sum_{x_k} (\hat{\gamma}_k(x|_k))^{1\pm\eta_k}. \tag{5.80}$$

Now, by choosing η_j small enough such that

$$\frac{\alpha_1}{1 \pm \eta_1} < \frac{\alpha_2}{1 \pm \eta_2} < \cdots < \frac{\alpha_k}{1 \pm \eta_k} < 1, \tag{5.81}$$

one readily checks, for example, by using Campbell's theorem, that for every x_1

$$\sum_{x_2} (\hat{\gamma}_2(x|_2))^{1\pm\eta_2} \cdots \sum_{x_k} (\hat{\gamma}_k(x|_k))^{1\pm\eta_k} \tag{5.82}$$

is an $\frac{\alpha_2}{1\pm\eta_2}$-stable random variable, and finally that the random variable in (5.80), which is decreasing in δ_1, converges in probability to 0 as $\delta_1 \to 0$. We conclude it converges almost surely to 0 as $\delta_1 \to 0$, and (5.78) follows for the case where $i_1 = 2$ and $i_j \in \{1, 2\}$ for all $j = 2, \dots, k$.

Let us now analyze the expression in (5.79) when $\mathcal{L} := \{j = 1, \dots, k : i_j = 3\} \ne \varnothing$. It is argued in [19] [see discussion leading to (6.20) in that reference] that for $j \in \mathcal{L}$, $\hat{\gamma}_j^{(n)}(x|_j)$ is almost surely bounded above by a deterministic constant times $c_j^{(n)}$ for all large enough n uniformly in $\mathcal{L}$. Let $k' = |\mathcal{L}|$ and let $i_1' < \cdots < i_{k'}'$ be an enumeration of $\mathcal{L}$, and let $i_1'' < \cdots < i_{k''}''$ be an enumeration of $\{1, \dots, k\} \setminus \mathcal{L}$, $k'' = k - k'$. Then, arguing as above, (5.79) may be bounded above by a sum of $2^{k''}$ terms of the form

$$\begin{aligned} &\prod_{j=1}^{k'} c_{i_j'}^{(n)} \sum_{x_{i_1'}=1}^{M_{i_1'}} \cdots \sum_{x_{i_{k'}'}=1}^{M_{i_{k'}'}} \Biggl\{ \sum_{x_{i_1''}=1}^{M_{i_1''}} (\hat{\gamma}_{i_1''}^{(n)}(x|_{i_1''}))^{1\pm\eta_{i_1''}} \cdots \\ &\qquad \sum_{x_{i_{k''}''}=1}^{M_{i_{k''}''}} (\hat{\gamma}_{i_{k''}''}^{(n)}(x|_{i_{k''}''}))^{1\pm\eta_{i_{k''}''}} \Biggr\}. \end{aligned} \tag{5.83}$$

Again, choosing η's small enough, we have that the random variables within braces are i.i.d. $\frac{\alpha_{i_1''}}{1\pm\eta_{i_1''}}$-stable ones, and since the outer sums are over $\prod_{j=1}^{k'} M_{i_j'}$ terms, and, as one may readily check, $\prod_{j=1}^{k'} c_{i_j'}^{(n)} (\prod_{j=1}^{k'} M_{i_j'})^{(1\pm\eta_{i_1''})/\alpha_{i_1''}}$ decays polynomially in n to 0 as $n \to \infty$, by a standard argument, we have that the expression in (5.83) decays almost surely to 0 as $n \to \infty$, and (5.74) follows for general k by first taking $n \to \infty$ and then $\delta_1, \dots, \delta_k \to 0$.

It remains to check (5.17) and (5.18) as strong limits for the $\hat{\gamma}$ representations of the respective γ's. This is done in much the same way as for checking (5.15) above, so we will be rather sketchy. First note that the expressions in (5.17) and (5.18) can, after dividing the M's on the denominator inside the sum, and expanding the resulting products

$$\prod_{p=l+1}^{j-1} \left(\frac{1}{M_{p+1}} + \hat{\gamma}_p^{(n)}(x|_p) \right), \tag{5.84}$$

be both written as a sum over a fixed number of terms of the form

$$\sum_{x_1} \check{\gamma}_1^{(n)}(x_1) \cdots \sum_{x_m} \check{\gamma}_\ell^{(n)}(x|_m), \tag{5.85}$$

where $1 \le m \le k$ and $\check{\gamma}_j^{(n)}(x|_j)$ is either $\hat{\gamma}_j^{(n)}(x|_j)$ or $1/M_{j+1} = c_j^{(n)}$ for all $j = 1, \dots, m$, with the latter case happening for at least one such j.

We can thus break each sum $\sum_{x_j}$ into three kinds as above [see (5.76)], with the superscript "(3)" applying also to the case where $\check{\gamma}_j^{(n)}(x|_j) = c_j^{(n)}$. The same arguments used above to estimate the latter cases of (5.79) [see the paragraph of (5.83)] apply, since there is always a sum of the third kind, and the result follows.

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REFERENCES

[1] BARLOW, M. T. and ČERNÝ, J. (2011). Convergence to fractional kinetics for random walks associated with unbounded conductances. *Probab. Theory Related Fields* **149** 639–673. MR2776627

[2] BEN AROUS, G., BOVIER, A. and GAYRARD, V. (2003). Glauber dynamics of the random energy model. I. Metastable motion on the extreme states. *Comm. Math. Phys.* **235** 379–425. MR1974509

[3] BEN AROUS, G., BOVIER, A. and GAYRARD, V. (2003). Glauber dynamics of the random energy model. II. Aging below the critical temperature. *Comm. Math. Phys.* **236** 1–54. MR1977880

[4] BEN AROUS, G., BOVIER, A. and ČERNÝ, J. (2008). Universality of the REM for dynamics of mean-field spin glasses. *Comm. Math. Phys.* **282** 663–695. MR2426140

[5] BEN AROUS, G. and GÜN, O. (2012). Universality and extremal aging for dynamics of spin glasses on subexponential time scales. *Comm. Pure Appl. Math.* **65** 77–127. MR2846638

[6] BEN AROUS, G. and ČERNÝ, J. (2007). Scaling limit for trap models on $\mathbb{Z}^d$. *Ann. Probab.* **35** 2356–2384. MR2353391

[7] BEN AROUS, G. and ČERNÝ, J. (2008). The arcsine law as a universal aging scheme for trap models. *Comm. Pure Appl. Math.* **61** 289–329. MR2376843

[8] BEN AROUS, G., ČERNÝ, J. and MOUNTFORD, T. (2006). Aging in two-dimensional Bouchaud's model. *Probab. Theory Related Fields* **134** 1–43. MR2221784

[9] BEZERRA, S. C., FONTES, L. R. G., GAVA, R. J., GAYRARD, V. and MATHIEU, P. (2012). Scaling limits and aging for asymmetric trap models on the complete graph and K processes. *ALEA Lat. Am. J. Probab. Math. Stat.* **9** 303–321. MR3069367

[10] BOUCHAUD, J. P. and DEAN, D. S. (1995). Aging on Parisi's tree. *J. Phys. I France* **5** 265–286.

[11] BOVIER, A. and FAGGIONATO, A. (2005). Spectral characterisation of ageing: The REM-like trap model in the complete graph. *Ann. Appl. Probab.* **15** 1997–2037.

[12] BOVIER, A. and GAYRARD, V. (2013). Convergence of clock processes in random environments and ageing in the p-spin SK model. *Ann. Probab.* **41** 817–847. MR3077527

[13] BOVIER, A., GAYRARD, V. and ŠVEJDA, A. (2013). Convergence to extremal processes in random environments and extremal ageing in SK models. *Probab. Theory Related Fields* **157** 251–283. MR3101847

[14] COMPTE, A. and BOUCHAUD, J. P. (1998). Localization in one-dimensional random walks. *J. Phys. A: Math. Gen.* **31** 6113–6121.

[15] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. MR0838085

[16] FONTES, L. R. G., ISOPI, M. and NEWMAN, C. M. (2002). Random walks with strongly inhomogeneous rates and singular diffusions: Convergence, localization and aging in one dimension. *Ann. Probab.* **30** 579–604. MR1905852

[17] FONTES, L. R. G. and LIMA, P. H. S. (2009). Convergence of symmetric trap models in the hypercube. In *XVth International Congress on Mathematical Physics, 2006, Rio de Janeiro. New Trends in Mathematical Physics* 285–297. Springer, Dordrecht.

[18] FONTES, L. R. G. and MATHIEU, P. (2008). K-processes, scaling limit and aging for the trap model in the complete graph. *Ann. Probab.* **36** 1322–1358. MR2435851

[19] GAYRARD, V. (2010). Aging in reversible dynamics of disordered systems. I. Emergence of the arcsine law in Bouchaud's asymmetric trap model on the complete graph. Available at arXiv:1008.3855v1 [math.PR] (longer version of [21]).

[20] GAYRARD, V. (2010). Aging in reversible dynamics of disordered systems. II. Emergence of the arcsine law in the random hopping time dynamics of the REM. Available at arXiv:1008.3849.

[21] GAYRARD, V. (2012). Convergence of clock process in random environments and aging in Bouchaud's asymmetric trap model on the complete graph. *Electron. J. Probab.* **17** 1–33. MR2959064

[22] GAYRARD, V. and GÜN, O. (2013). In preparation.

[23] LEPAGE, R., WOODROOFE, M. and ZINN, J. (1981). Convergence to a stable distribution via order statistics. *Ann. Probab.* **9** 624–632. MR0624688

[24] NIEUWENHUIZEN, T. M. and ERNST, M. H. (1985). Excess noise in a hopping model for a resistor with quenched disorder. *J. Stat. Phys.* **41** 773–801. MR0823623

[25] SASAKI, M. and NEMOTO, K. (2000). Analysis on aging in the generalized random energy model. *J. Phys. Soc. Jpn.* **69** 3045–3050.

[26] SASAKI, M. and NEMOTO, K. (2001). Numerical study of aging in the generalized random energy model. *J. Phys. Soc. Jpn.* **70** 1099–1104.

L. R. G. FONTES
IME-USP
RUA DO MATÃO 1010
CIDADE UNIVERSITÁRIA
05508-090, SÃO PAULO SP
BRAZIL
E-MAIL: lrenato@ime.usp.br

R. J. GAVA
CCET-UFSCAR
ROD. WASHINGTON LUIZ
KM 235, 13565-905, SÃO CARLOS SP
BRAZIL
E-MAIL: gavamat@yahoo.com.br

V. GAYRARD
CNRS, CMI, LAPT
AIX MARSEILLE UNIVERSITÉ
39 RUE F. JOLIOT CURIE 13453
MARSEILLE CEDEX 13
FRANCE
E-MAIL: veronique@gayrard.net