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ACHLIOPTAS PROCESS PHASE TRANSITIONS ARE CONTINUOUS

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It is widely believed that certain simple modifications of the random graph process lead to discontinuous phase transitions. In particular, starting with the empty graph on n vertices, suppose that at each step two pairs of vertices are chosen uniformly at random, but only one pair is joined, namely, one minimizing the product of the sizes of the components to be joined. Making explicit an earlier belief of Achlioptas and others, in 2009, Achlioptas, D'Souza and Spencer [*Science* **323** (2009) 1453–1455] conjectured that there exists a $\delta > 0$ (in fact, $\delta \geq 1/2$) such that with high probability the order of the largest component "jumps" from $o(n)$ to at least δn in $o(n)$ steps of the process, a phenomenon known as "explosive percolation."

We give a simple proof that this is not the case. Our result applies to all "Achlioptas processes," and more generally to any process where a fixed number of independent random vertices are chosen at each step, and (at least) one edge between these vertices is added to the current graph, according to any (online) rule.

We also prove the existence and continuity of the limit of the rescaled size of the giant component in a class of such processes, settling a number of conjectures. Intriguing questions remain, however, especially for the product rule described above.

1. Introduction and results. At a Fields Institute workshop in 2000, Dimitris Achlioptas suggested a class of variants of the classical random graph process, defining a random sequence $(G(m))_{m\geq 0}$ of graphs on a fixed vertex set of size n, usually explained in terms of the actions of a hypothetical purposeful agent: start at step 0 with the empty graph. At step m, two potential edges e_1 and e_2 are chosen independently and uniformly at random from all $\binom{n}{2}$ possible edges [or from those edges not present in $G(m-1)$]. The agent must select one of these edges, setting $G(m) = G(m-1) \cup \{e\}$ for $e = e_1$ or e_2. Any possible strategy, or "rule," for the agent gives rise to a random graph process. Such processes are known as "Achlioptas processes."

If the agent always chooses the first edge, then (ignoring the minor effect of repeated edges) this is, of course, the classical random graph process, studied implicitly by Erdős and Rényi and formalized by Bollobás. In this case, as is well known, there is a phase transition around $m = n/2$. More precisely, writing $L_1(G)$ for the number of vertices in the (a, if there is a tie) largest component of a graph G,

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Erdős and Rényi [8] showed that there is a function $\rho = \rho^{\mathrm{ER}}: [0, \infty) \to [0, 1)$ such that for any fixed $t \geq 0$, whenever $m = m(n)$ satisfies $m/n \to t$ as $n \to \infty$, then $L_1(G(m))/n \xrightarrow{\mathrm{p}} \rho(t)$, where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability. Moreover, $\rho(t) = 0$ for $t \leq 1/2$, $\rho(t) > 0$ for $t > 1/2$ and $\rho(t)$ (the solution to a simple equation) is continuous at $t = 1/2$ with right-derivative 4 at this point.

Achlioptas originally asked whether the agent could shift the critical point of this phase transition by following an appropriate edge-selection rule. One natural rule to try is the "product rule": of the given potential edges, pick the one minimizing the product of the sizes of the components of its endvertices. This rule was suggested by Bollobás as the most likely to delay the critical point.

Bohman and Frieze [3] quickly showed, using a much simpler rule, that the transition could indeed be shifted, but more complicated rules such as the product rule remained resistant to analysis. By 2004 at the latest (see [15]), extensive simulations of D'Souza and others strongly suggested that the product rule in particular shows much more interesting behavior than simply a slightly shifted critical point; it exhibits a phenomenon known as "explosive percolation."

As usual, we say that an event E (formally a sequence of events E_n) holds *with high probability* (*whp*) if $\mathbb{P}(E) \to 1$ as $n \to \infty$. Explosive percolation is said to occur if there is a critical t_{c} and a positive δ such that for any fixed $\varepsilon > 0$, whp L_1 jumps from $o(n)$ to at least δn in fewer than εn steps around $m = t_{\mathrm{c}} n$. Recently, Achlioptas, D'Souza and Spencer [1] presented "conclusive numerical evidence" for the conjecture that the product rule exhibits explosive percolation, suggesting indeed that the largest component grows from size at most $\sqrt{n}$ to size at least $n/2$ in at most $2n^{2/3}$ steps. Bohman [2] describes this explosive percolation conjecture as an important and intriguing mathematical question.

Our main result disproves this conjecture. The result applies to all Achlioptas processes as defined at the start of the section (including the product rule) and, in fact, to a more general class of processes (ℓ-vertex rules) defined in Section 2. A form of this result first appeared in [13], with more restrictive assumptions, and without full technical details.

THEOREM 1. *Let $\mathcal{R}$ be an ℓ-vertex rule for some $\ell \geq 2$. For each n, let $(G(m))_{m \geq 0} = (G_n^{\mathcal{R}}(m))_{m \geq 0}$ be the random sequence of graphs on $\{1, 2, \dots, n\}$ associated to $\mathcal{R}$. Given any functions $h_L(n)$ and $h_m(n)$ that are $o(n)$, and any constant $\delta > 0$, the probability that there exist m_1 and m_2 with $L_1(G(m_1)) \leq h_L(n)$, $L_1(G(m_2)) \geq \delta n$ and $m_2 \leq m_1 + h_m(n)$ tends to 0 as $n \to \infty$.*

Let $N_k(G)$ denote the number of vertices of a graph G in components with k vertices, so $N_k(G)$ is k times the number of k-vertex components. Similarly, $N_{\leq k}(G)$ and $N_{\geq k}(G)$ denote the number of vertices in components with at most (at least) k vertices. Having a rule $\mathcal{R}$ in mind, and suppressing the dependence on n, we write $N_k(m)$ for the random quantity $N_k(G(m))$, and similarly $L_1(m)$ for $L_1(G(m))$.

Under a mild additional condition (which holds for all Achlioptas processes), a slight modification of the proof of Theorem 1 shows, roughly speaking, that the giant component is unique. In fact, we obtain much more; whp there is no time at which there are "many" vertices in "large" components but not in the single largest component. For the precise definition of a "merging" rule see Section 3; any Achlioptas process is merging.

THEOREM 2. *Let $\mathcal{R}$ be a merging ℓ-vertex rule for some $\ell \geq 2$. For each n, let $(G(m))_{m\geq 0} = (G_n^{\mathcal{R}}(m))_{m\geq 0}$ be the random sequence of graphs on $\{1, 2, \dots, n\}$ associated to $\mathcal{R}$. For each $\varepsilon > 0$ there is a $K = K(\varepsilon, \ell)$ such that*

$$\mathbb{P}\bigl(\forall m : N_{\geq K}(m) < L_1(m) + \varepsilon n\bigr) \to 1$$

as $n \to \infty$.

With ℓ fixed, our proof gives a value for K of the form $\exp(\exp(c\varepsilon^{-(\ell-1)}))$ for some positive $c = c(\ell)$. Furthermore, we can allow ε to depend on n, as long as $\varepsilon = \varepsilon(n) \geq d/(\log\log n)^{1/(\ell-1)}$, where $d = d(\ell) > 0$.

For the classical random graph process it is well known that at any fixed time, whp there will be at most one "giant" component. Indeed, the maximum size of the second largest component throughout the evolution of the process is whp $o(n)$; this can be read out of the original results of Erdős and Rényi [8] or (more easily) the more precise results of Bollobás [5]. Spencer's "no two giants" conjecture (personal communication) states that this should also hold for Achlioptas processes. Theorem 2 proves this conjecture for the larger class of merging ℓ-vertex rules; indeed, it readily implies that, with high probability, the second largest component has size at most $\max\{K, \varepsilon n\} = \varepsilon n$. Allowing ε to vary with n as noted above, the bound we obtain is of the form $d(\ell)n/(\log\log n)^{1/(\ell-1)}$.

Before turning to the proofs of Theorems 1 and 2, let us discuss some related questions of convergence.

We say that the rule $\mathcal{R}$ is *locally convergent* if there exist functions $\rho_k = \rho_k^{\mathcal{R}} : [0, \infty) \to [0, 1]$ such that, for each fixed $k \geq 1$ and $t \geq 0$, we have

$$\frac{N_k(\lfloor tn \rfloor)}{n} \xrightarrow{\text{p}} \rho_k(t) \tag{1}$$

as $n \to \infty$. The rule $\mathcal{R}$ is *globally convergent* if there exists an increasing function $\rho = \rho^{\mathcal{R}} : [0, \infty) \to [0, 1]$ such that for any t at which ρ is continuous we have

$$\frac{L_1(\lfloor tn \rfloor)}{n} \xrightarrow{\text{p}} \rho(t)$$

as $n \to \infty$.

Theorem 1 clearly implies that if a rule $\mathcal{R}$ is globally convergent, then the limiting function ρ is continuous at the critical point $t_{\text{c}} = \inf\{t : \rho(t) > 0\}$. Using Theorem 2, it is not hard to establish continuity elsewhere for merging rules; see Theorem 7 and Corollary 8 in Section 3. Unfortunately, we cannot show that the product rule *is* globally convergent. However, as we shall see in Section 4, Theorem 2 implies the following result.

THEOREM 3. *Let $\mathcal{R}$ be a merging ℓ-vertex rule for some $\ell \geq 2$. If $\mathcal{R}$ is locally convergent, then $\mathcal{R}$ is globally convergent, and the limiting function $\rho^{\mathcal{R}}$ is continuous and satisfies $\rho^{\mathcal{R}}(t) = 1 - \sum_{k\geq 1} \rho_k^{\mathcal{R}}(t)$.*

The conditional result above is, of course, rather unsatisfactory. However, for many Achlioptas processes, local convergence is well known; global convergence had not previously been established for any nontrivial rule. In particular, Theorem 3 settles two conjectures of Spencer and Wormald [15] concerning so-called "bounded size Achlioptas processes" (see Section 5).

Recently, in a paper in the physics literature, da Costa, Dorogovtsev, Goltsev and Mendes [6] announced a version of Theorem 1. However, their actual analysis concerned only one specific rule (not the product rule, though they claim that "clearly" the product rule is less likely to have a discontinuous transition). More importantly, even the "analytic" part of it is heuristic, and of a type that seems to us very hard (if at all possible) to make precise. Crucially, the starting point for their analysis is not only to assume convergence, but also to assume that the phase transition is continuous! From this, and some further assumptions, by solving approximations to certain equations they deduce certain "self-consistent behavior," which apparently justifies the assumption of continuity. The argument (which is considerably more involved than the simple proof presented here) is certainly very interesting, and the conclusion is (as we now know) correct, but it seems to be very far from a mathematical proof.

In the next section we prove Theorem 1. In Section 3, restricting the class of rules slightly, we prove Theorem 2 and deduce that jumps in L_1 are also impossible after a giant component first emerges. Next, in Section 4, we prove Theorem 3. Finally, in Section 5 we consider more restrictive rules such as bounded size rules, and discuss the relationship of our results to earlier work.

2. Definitions and proof of Theorem 1.

Throughout, we fix an integer $\ell \geq 2$. For each n, let $(\underline{v}_1, \underline{v}_2, \ldots)$ be an i.i.d. sequence where each $\underline{v}_m$ is a sequence $(v_{m,1}, \ldots, v_{m,\ell})$ of ℓ vertices from $[n] = \{1, 2, \ldots, n\}$ chosen independently and uniformly at random. Suppressing the dependence on n, informally, an *ℓ-vertex rule* is a random sequence $(G(m))_{m\geq 0}$ of graphs on $[n]$ satisfying (i) $G(0)$ is the empty graph, (ii) for $m \geq 1$ $G(m)$ is formed from $G(m-1)$ by adding a (possibly empty) set E_m of edges, with all edges in E_m between vertices in $\underline{v}_m$ and (iii) if all ℓ vertices in $\underline{v}_m$ are in distinct components of $G(m-1)$, then $E_m \neq \varnothing$. The set E_m may be chosen according to any deterministic or random online rule.

Formally, we assume the existence of a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ such that $\underline{v}_m$ is $\mathcal{F}_m$-measurable and independent of $\mathcal{F}_{m-1}$, and require E_m [and hence, $G(m)$] to be $\mathcal{F}_m$-measurable.

In other words, the agent is presented with the random list (set) $\underline{v}_m$ of vertices, and, unless two or more are already in the same component, must add one or more edges between them, according to any deterministic or random rule that depends only on the history. In the original examples of Achlioptas, the rule always adds either the edge $\{v_{m,1}, v_{m,2}\}$ or the edge $\{v_{m,3}, v_{m,4}\}$. Note that (for now) no connection between the algorithms used for different n (or indeed at different steps m) is assumed.

The arguments that follow are robust to small changes in the definition, since they can be written to rely only on deterministic properties of $(G(m))$, plus bounds on the probabilities of certain events at each step. The latter always have $\Theta(1)$ elbow room. It follows that we may weaken the conditions on $(\underline{v}_m)$; it suffices if, for $m = O(n)$, say, the conditional distribution of $\underline{v}_m$ given the history (i.e., given $\mathcal{F}_{m-1}$) is close to [at total variation distance $o(1)$ from, as $n \to \infty$] that described above. This covers variations such as picking an ℓ-tuple of *distinct* vertices, or picking (the ends of) $\ell/2$ randomly selected (distinct) edges not already present in $G(m-1)$.

The proof of Theorem 1 is based on two observations, which we first present in heuristic form.

Observation 1: If at some time t (i.e., when $m \sim tn$) there are αn vertices in components of order at least k, then within time $\gamma = O(1/(\alpha^{\ell-1}k))$ a component of order at least $\alpha n/\ell^2 = \beta n$ will emerge. Indeed, fix a set W with $|W| \geq \alpha n$ consisting of components of order at least k. At every subsequent step we have probability at least α^ℓ of choosing only vertices in W, and if no component has order more than βn, it is likely that all these vertices are in different components, so the rule is *forced* to join two components meeting W. This cannot happen more than $|W|/k$ times.

(A form of Observation 1 appears in a paper of Friedman and Landsberg [9] as a key part of a heuristic argument *for* explosive percolation. It is not quite stated correctly, although this does not seem to be why the heuristic fails.)

Observation 2: Components of order k have a half-life that may be bounded in terms of k; in an individual step, such a component disappears (by joining another component) with probability at most $k\ell/n$. Assuming (which we shall not assume in the actual proof) that the rule $\mathcal{R}$ is locally convergent, it follows easily that for all t_1, t_2 and k we have $\rho_k(t_1+t_2) \geq \rho_k(t_1)e^{-k\ell t_2}$.

We place vertices into "bins" corresponding to component sizes between 2^j and $2^{j+1}-1$, writing $\sigma_j(t)$ for $\sum_{2^j \leq k < 2^{j+1}} \rho_k(t)$. Let $\alpha > 0$ be constant and suppose that $\sigma_j(t) \geq \alpha$ for some $t < t_c$. Writing $k = 2^j$, by Observation 1 we have $t_c - t = O(1/k)$, with the implicit constant depending on α, since the $\geq \alpha n$ vertices in components of size at least k will quickly form a giant component. Using Observation 2, it follows that $\sigma_j(t_c) \geq g(\alpha) > 0$, for some (explicit but irrelevant) function $g(\alpha)$.

Let $\sigma_j = \sup_{t \le t_c} \sigma_j(t)$. If $\sigma_j > \alpha$, then $\sigma_j(t_c) \ge g(\alpha)$. Counting vertices, we have $\sum_j \sigma_j(t_c) \le 1$. Hence, for each $\alpha > 0$, only a finite number of σ_j can exceed α. Thus $\sigma_j \to 0$ as $j \to \infty$. It follows that for any constant $B \ge 2$ and any $k = k(n) \to \infty$, at no $t = t(n) < t_c$ can there be $\Theta(n)$ vertices in components of size between k and Bk.

Using Observation 1, it is easy to deduce that there cannot be a discontinuous transition. Indeed, if $\lim_{t \to t_c^+} \rho(t) \ge \delta > 0$, then for any k, at time $t_k = t_c - \delta/(\ell^2 k)$, there must be at least $\delta n/2$ vertices in components of order at least k, so $\rho_{\ge k}(t_k) \ge \delta/2$, where $\rho_{\ge k} = 1 - \sum_{k' < k} \rho_{k'}$. For any constant $B \ge 2$, if k is large it follows that $\rho_{\ge Bk}(t_k) \ge \delta/3$. Taking B large enough, Observation 1 then implies that $t_c - t_k$ is much smaller than $\delta/(\ell^2 k)$.

We now make the above argument precise, without assuming convergence. This introduces some minor additional complications, but they are easily handled. We start with two lemmas corresponding to the two observations above.

LEMMA 4. *Given $0 < \alpha \le 1$, let $\mathcal{C}(\alpha)$ denote the event that for all $0 \le m \le n^2$ and $1 \le k \le \frac{\alpha}{16} \frac{n}{\log n}$ the following holds*: *$N_{\ge k}(m) \ge \alpha n$ implies $L_1(m + \Delta) > \frac{\alpha}{\ell^2} n$ for $\Delta = \lceil \frac{4}{\alpha^{\ell-1}} \frac{n}{k} \rceil$. Then $\mathbb{P}(\mathcal{C}(\alpha)) \ge 1 - n^{-1}$.*

PROOF. It suffices to consider fixed m and k and show that, conditional on $\mathcal{F}_m$, if $G(m)$ satisfies $N_{\ge k}(m) \ge \alpha n$, then we have $L_1(m + \Delta) > \frac{\alpha}{\ell^2} n$ with probability at least $1 - n^{-4}$.

Condition on $\mathcal{F}_m$. Let W be the union of all components with size at least k in $G(m)$, set $\tilde{\alpha} = |W|/n \ge \alpha$ and let $\beta = \tilde{\alpha}/\ell^2$. We now consider the next Δ steps.

We say that a step is *good* if (a) all ℓ randomly chosen vertices are in W and (b) all these vertices are in different components. Let X_j denote the indicator function of the event that step $m + j$ is good. Set $X = \sum_{1 \le j \le \Delta} X_j$ and $Y = \sum_{1 \le j \le \Delta} Y_j$, where

$$Y_j = \begin{cases} X_j, & \text{if } L_1(m + j - 1) \le \beta n, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, in each step (a) holds with probability $\tilde{\alpha}^\ell$. Furthermore, whenever $L_1(m + j - 1) \le \beta n$ holds, in step $m + j$ the probability that (a) holds and (b) fails is at most $\binom{\ell}{2} \tilde{\alpha}^{\ell-1} \beta < \tilde{\alpha}^\ell/2$ (there must be v_a and v_b with $1 \le a < b \le \ell$ such that v_b lies in the same component as v_a; all v_c must also be in W) and so in this case step $m + j$ is good with probability at least $\tilde{\alpha}^\ell/2$. Since, otherwise, $Y_j = 1$ by definition, we deduce that Y stochastically dominates a binomial random variable with mean $\Delta \tilde{\alpha}^\ell/2 \ge 2\tilde{\alpha} n/k$. Standard Chernoff bounds now imply that $\mathbb{P}(Y \le \tilde{\alpha} n/k) \le e^{-\tilde{\alpha} n/(4k)} \le e^{-\alpha n/(4k)} \le n^{-4}$.

Assume that $L_1(m + \Delta) \le \beta n$. Then by monotonicity $L_1(m + j - 1) \le \beta n$ for every $1 \le j \le \Delta$, so $X = Y$. Note that W contains at most $|W|/k = \tilde{\alpha} n/k$ components in $G(m)$. Since every good step joins two components meeting W

[at least one such edge must be added since by (a) all endpoints are in W and by (b) all endpoints are in distinct components] we deduce that $Y \le \tilde{\alpha} n/k$. Hence, $\mathbb{P}(L_1(m+\Delta) \le \beta n) \le \mathbb{P}(Y \le \tilde{\alpha} n/k) \le n^{-4}$, as required. $\square$

Applying Lemma 4 with $m = 0$, $k = 1$ and $\alpha = 1$, we readily deduce that whp a giant component exists after at most $4n$ steps. In fact, it is easy to see that for any $\varepsilon > 0$, whp there is a giant component after at most $(1+\varepsilon)n$ steps (see the proof of Lemma 6).

LEMMA 5. *Fix $0 < \alpha \le 1$, $D > 0$ and an integer $B \ge 2$. Define $M_k^B(m) = N_{\ge k}(m) - N_{\ge Bk}(m)$. Let $\mathcal{L}(\alpha, B, D)$ denote the event that for all $0 \le m \le n^2$ and $1 \le k \le \min\{\frac{\alpha^2 e^{-4\ell BD}}{8\ell^2 B^2 D}\frac{n}{\log n}, \frac{n}{2B}\}$ the following holds: $M_k^B(m) \ge \alpha n$ implies $M_k^B(m+\Delta) > \frac{\alpha}{2B} e^{-2\ell BD} n$ for every $0 \le \Delta \le D\frac{n}{k}$. Then $\mathbb{P}(\mathcal{L}(\alpha, B, D)) \ge 1 - n^{-1}$.*

PROOF. As in the proof of Lemma 4, it suffices to consider fixed m and k, and show that conditional on $\mathcal{F}_m$, if $G(m)$ satisfies $M_k^B(m) \ge \alpha n$, then with probability at least $1 - n^{-4}$ we have $M_k^B(m+\Delta) > \frac{\alpha}{2B} e^{-2\ell BD} n$ for every $0 \le \Delta \le \tilde{\Delta}$, where $\tilde{\Delta} = \lfloor Dn/k \rfloor$.

Condition on $\mathcal{F}_m$, and let $C_1, \dots, C_r$ be the components of $G(m)$ with sizes between k and $Bk - 1$. Note that $r \ge M_k^B(m)/(Bk) \ge \alpha n/(Bk)$.

Starting from $G(m)$, we now analyze the next $\tilde{\Delta}$ steps. We say that C_i is *safe* if in each of these steps none of the ℓ randomly chosen vertices is contained in C_i, and we denote by X the number of safe components. Using $|C_i| \le Bk \le n/2$, note that C_i is safe with probability

$$(1 - |C_i|/n)^{\ell\tilde{\Delta}} > e^{-2\ell\tilde{\Delta}|C_i|/n} \ge e^{-2\ell BD},$$

which gives $\mathbb{E}X \ge re^{-2\ell BD}$. Clearly, the random variable X can be written as $X = f(\underline{v}_{m+1}, \dots, \underline{v}_{m+\Delta})$, where the $\underline{v}_j$ denote the ℓ-tuples generated by the ℓ-vertex process in each step (uniformly and independently). The function f satisfies $|f(\omega) - f(\tilde{\omega})| \le \ell$ whenever ω and $\tilde{\omega}$ differ in one coordinate. So, using $r \ge \alpha n/(Bk)$, McDiarmid's inequality [11] implies that $\mathbb{P}(X \le re^{-2\ell BD}/2)$ is at most

$$\exp\left(-\frac{2[re^{-2\ell BD}/2]^2}{\tilde{\Delta}\ell^2}\right) \le \exp\left(-\frac{\alpha^2 e^{-4\ell BD}}{2\ell^2 B^2 D}\frac{n}{k}\right) \le n^{-4}.$$

Suppose that $X > re^{-2\ell BD}/2$. Since every safe component contributes at least k vertices to every $M_k^B(m+\Delta)$ with $0 \le \Delta \le \tilde{\Delta}$ (in each step all edges which can be added are disjoint from safe components), using $r \ge \alpha n/(Bk)$ we deduce that for all such Δ we have $M_k^B(m+\Delta) \ge kX > \alpha e^{-2\ell BD} n/(2B)$, and the proof is complete. $\square$

Note that by considering instead the number Y of vertices in safe components one can prove the slightly stronger bound $M_k^B(m+\Delta) > (1-\varepsilon)\alpha e^{-2\ell BD} n$, for k not too large.

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1. Let $h_L(n)$ and $h_m(n)$ be nonnegative functions satisfying $h_L(n) = o(n)$ and $h_m(n) = o(n)$, and let $\delta > 0$ be constant. Let $\mathcal{X} = \mathcal{X}_n(\delta, h_L, h_m)$ denote the event that there exist m_1 and m_2 satisfying $L_1(m_1) \le h_L(n)$, $L_1(m_2) \ge \delta n$, and $m_2 \le m_1 + h_m(n)$, so our aim is to show that $\mathbb{P}(\mathcal{X}) \to 0$ as $n \to \infty$. We shall define a "good" event $\mathcal{G} = \mathcal{G}_n(\delta)$ such that $\mathbb{P}(\mathcal{G}) \to 1$ as $n \to \infty$ and show *deterministically* that there is some n_0 such that for $n \ge n_0$, when $\mathcal{G}$ holds, $\mathcal{X}$ does not.

To be totally explicit, set $\alpha = \delta/4$, $A = 5/\alpha^{\ell-1}$ and $D = 1$. Set $B = \lceil 2A\ell^2/\delta \rceil$, and let $\beta = \alpha e^{-2\ell B}/(2B) > 0$. Finally, let $K = B^{1+\lceil 1/\beta \rceil}$, noting that K does not depend on n.

Let $\mathcal{G}$ be the event that $\mathcal{C}(1)$, $\mathcal{C}(\delta/4)$ and $\mathcal{L}(\delta/4, B, D)$ all hold simultaneously. By Lemmas 4 and 5, $\mathbb{P}(\mathcal{G}) \ge 1 - 3n^{-1} = 1 - o(1)$. The definition of $\mathcal{G}$ ensures that if n is large enough (larger than some constant depending only on δ and ℓ), then for all $m \le 5n$ and $k \le K$ the following hold:

$$\text{(i) } N_{\ge k}(m) \ge \delta n/4 \quad \text{implies} \quad \text{(ii) } L_1(m + \lfloor An/k \rfloor) \ge \delta n/(4\ell^2)$$

and

$$\text{(iii) } M_k^B(m) \ge \delta n/4 \quad \text{implies} \quad \text{(iv) } M_k^B(m') \ge \beta n \qquad \text{for all } m \le m' \le m + n/k.$$

Suppose that $\mathcal{G}$ holds, and that $m^- = \max\{m : L_1(m) \le h_L(n)\}$ and $m^+ = \min\{m : L_1(m) \ge \delta n\}$ differ by at most $h_m(n)$. It suffices to show deterministically that if n is large enough, then this leads to a contradiction.

Since $N_1(0) = n$ and $\mathcal{C}(1)$ holds, we have $L_1(4n) \ge n/\ell^2$. If n is large enough, it follows that $m^- \le 4n$, so $m^+ \le 5n$.

For $k \le K/B$ set $m_k = m^+ - \delta n/(\ell^2 k)$, which is easily seen to be positive; we ignore the irrelevant rounding to integers. Since at most $\binom{\ell}{2}(m^+ - m_k) < \ell^2(m^+ - m_k)/2$ edges are added passing from $G(m_k)$ to $G(m^+)$, the components of $G(m_k)$ with size at most k together contribute at most $k\ell^2(m^+ - m_k)/2 \le \delta n/2$ vertices to any one component of $G(m^+)$. It follows that

$$N_{\ge k}(m_k) \ge L_1(m^+) - \delta n/2 \ge \delta n/2.$$

Suppose that $N_{\ge Bk}(m_k) \ge \delta n/4$. Then (i) holds at step m_k with $Bk \le K$ in place of k, so (ii) tells us that by step

$$m^* = m_k + \lfloor An/(Bk) \rfloor \le m_k + \delta n/(2\ell^2 k) = m^+ - \delta n/(2\ell^2 k) = m^+ - \Theta(n),$$

we have $L_1(m^*) > \delta n/(4\ell^2)$, which is larger than $h_L(n)$ if n is large enough. Since $m^+ - m^- \le h_m(n) = o(n)$, if n is large enough we have $m^* < m^-$, contradicting the definition of m^-.

It follows that $M_k^B(m_k) = N_{\geq k}(m_k) - N_{\geq Bk}(m_k) \geq \delta n/4$. Using (iii) implies (iv), this gives $M_k^B(m^+) \geq \beta n$. Applying this for $k = 1, B, B^2, \dots, B^{\lceil 1/\beta \rceil}$ shows that $G(m^+)$ has more than n vertices, a contradiction. $\square$

Setting $D = 2\delta/\ell^2$ (instead of $D = 1$), the proof above shows that the number of steps between $m^- = \max\{m : L_1(m) \leq \delta/(4\ell^2)n\}$ and $m^+ = \min\{m : L_1(m) \geq \delta n\}$ is at least $\delta n/(2\ell^2 B^{\lceil 1/\beta \rceil}) = f(\delta)n$, where $f(\delta)$ essentially grows like the inverse of a double exponential in $\delta^{-(\ell-1)}$ for $\delta \to 0$.

3. Results for merging rules. Although Theorem 1 applies to any ℓ-vertex rule, for many questions, this class is too broad. Indeed, consider a rule which only joins two components when forced to (i.e., when presented with ℓ vertices from distinct components) and then joins the two smallest components presented. Such a rule will *never* join two of the $\ell - 1$ largest components, and it is not hard to see that during the process $\ell - 1$ giant components [with order $\Theta(n)$] will emerge and grow simultaneously, with their sizes keeping roughly in step. In what follows we could replace "the largest component" by "the union of the $\ell - 1$ largest components" and work with arbitrary ℓ-vertex rules, but this seems rather unnatural.

By an *r-Achlioptas rule* we mean an ℓ-vertex rule with $\ell = 2r$ that always joins (at least) one of the pairs $\{v_1, v_2\}, \{v_3, v_4\}, \dots, \{v_{\ell-1}, v_\ell\}$. (How we treat the case where one or more of these pairs is in fact a single vertex will not be relevant.) An *Achlioptas rule* is an r-Achlioptas rule for any $r \geq 1$. Taking $r = 2$ and insisting that only one edge is added gives the original class of rules suggested by Achlioptas.

Let us say that an ℓ-vertex rule is *merging* if, whenever C, C' are distinct components with $|C|, |C'| \geq \varepsilon n$, then in the next step we have probability at least ε^ℓ of joining C to C'. This implies that the probability that they are *not* united after m further steps is at most $e^{-\varepsilon^\ell m}$. [We could replace ε^ℓ by any $f(\varepsilon) > 0$, and it suffices if the chance of merging in one of the next few steps, rather than the next step, is not too small.] Clearly, any Achlioptas rule is merging; with probability at least ε^ℓ all $r = \ell/2$ potential edges join C to C'. There are other interesting examples of merging rules (see Section 5).

For merging rules we have the following variant of Lemma 4. We write $V_{\geq k}(m)$ for the union of all components with size at least k in $G(m)$, so $|V_{\geq k}(m)| = N_{\geq k}(m)$.

LEMMA 6. *Let $\mathcal{R}$ be a merging ℓ-vertex rule, let $\varepsilon > 0$, let $k \geq 1$ and m be integers, and set $\Delta = 2\lceil \frac{2^\ell}{\varepsilon^{\ell-1}} \frac{n}{k} \rceil$. Conditioned on $\mathcal{F}_m$, with probability at least $1 - \ell \exp(-cn/k)$ there is a component of $G(m + \Delta)$ containing at least $N_{\geq k}(m) - \varepsilon n$ vertices from $V_{\geq k}(m)$, where $c = c(\varepsilon, \ell) > 0$.*

PROOF. Let $W = V_{\geq k}(m)$, so $|W| = N_{\geq k}(m)$. We may assume that $|W| - \varepsilon n \geq 0$. Let $\alpha = |W|/n \geq \varepsilon$. Until the point that there are $\ell - 1$ components between them containing at least $(\alpha - \varepsilon/2)n$ vertices from W, at each step we have probability at least $\alpha(\varepsilon/2)^{\ell-1}$ of choosing ℓ vertices of W in distinct components to form $\underline{v}_j$, in which case the number of components meeting W must decrease by (at least) one. As in the proof of Lemma 4, it follows that off an event whose probability is exponentially small in n/k, after $\Delta/2$ steps we do have $\ell-1$ components $C_1, \dots, C_{\ell-1}$ together containing at least $(\alpha-\varepsilon/2)n$ vertices of W. Ignoring any containing fewer than $\varepsilon n/(2\ell)$ vertices of W, using the property of merging rules noted above, the probability that some pair of the remaining C_i are not joined in the next $\Delta/2$ steps is exponentially small in n/k. $\square$

It is easy to check that we may take $c(\varepsilon,\ell)=\varepsilon/\ell^{\ell}$. With this technical result in hand, we now prove Theorem 2.

PROOF OF THEOREM 2. We outline the argument, much of which is very similar to the proof of Theorem 1 given in the previous section.

Let $\varepsilon>0$ be given and set $\delta=\varepsilon/5$. Lemma 6 implies that there is some $A=A(\delta,\ell)$ such that for any fixed k, it is very likely that (i) there is a component of $G(m+\lfloor An/k\rfloor)$ containing at least $N_{\geq k}(m)-\delta n$ vertices. By Lemma 5, for every fixed B there is some $\beta=\beta(\delta,\ell,B)>0$ such that if (ii) $M_k^B(m)=N_{\geq k}(m)-N_{\geq Bk}(m)\geq \delta n$, then it is very likely that (iii) $M_k^B(m')\geq \beta n$ for all $m\leq m'\leq m+n/k$, say.

To be more precise, let $B=\lceil A\ell^2/\delta\rceil$ and $K=B^{1+\lceil 1/\beta\rceil}$. Then it follows easily from Lemma 5, Lemma 6 and the union bound that for n large enough there is a good event $\mathcal{G}=\mathcal{G}_n(\delta)$ such that $\mathbb{P}(\mathcal{G})\to 1$ and such that whenever $\mathcal{G}$ holds, then for all $m\leq n^2$ and $k\leq K$, (i) holds and (ii) implies (iii).

Suppose that $\mathcal{G}$ holds and that $m^+=\min\{m : N_{\geq K}(m)\geq L_1(m)+\varepsilon n\}$ exists. It suffices to show deterministically that if n is large enough, then this leads to a contradiction. Since $\mathcal{G}$ holds, considering (i) with $m=0$ and $k=1$ shows that for some $C=C(\delta,\ell)$ we have $L_1(Cn)\geq(1-\delta)n>(1-\varepsilon)n$, so $m^+\leq Cn$.

For $k\leq K/B$, set $m_k=m^+-2\delta n/(\ell^2 k)$. Recall that $V_{\geq k}(m)$ denotes the the union of all components with size at least k in $G(m)$. Since at most $\binom{\ell}{2}(m^+-m_k)<\delta n/k$ edges are added passing from $G(m_k)$ to $G(m^+)$, vertices outside of $V_{\geq k}(m_k)$ contribute at most $2\delta n$ vertices to $V_{\geq k}(m^+)$. Hence,

$$N_{\geq k}(m_k)\geq N_{\geq k}(m^+)-2\delta n\geq N_{\geq K}(m^+)-2\delta n\geq L_1(m^+)+(\varepsilon-2\delta)n.$$

Suppose that $N_{\geq Bk}(m_k)\geq N_{\geq k}(m_k)-\delta n$. Then (i) (with Bk in place of k) tells us that by step

$$m=m_k+\lfloor An/(Bk)\rfloor\leq m_k+\delta n/(\ell^2 k)=m^+-\delta n/(\ell^2 k)<m^+$$

there exists a component of $G(m)$ containing at least

$$N_{\geq Bk}(m_k)-\delta n\geq N_{\geq k}(m_k)-2\delta n\geq L_1(m^+)+(\varepsilon-4\delta)n>L_1(m^+)$$

vertices, which contradicts $G(m^+) \supseteq G(m)$. It follows that $M_k^B(m_k) \geq \delta n$. Using (ii) implies (iii) we deduce that $M_k^B(m^+) \geq \beta n$. Applying this for $k = 1, B, B^2, \dots, B^{\lceil 1/\beta \rceil}$ and counting vertices in $G(m^+)$ gives a contradiction. □

Working through the conditions on the constants in the proof above, and using $D = 3\delta/\ell^2$ instead of $D = 1$ when applying Lemma 5, one can check that for some positive constants c and d depending only on ℓ the result holds for any $\varepsilon = \varepsilon(n) \geq d/(\log \log n)^{1/(\ell-1)}$, with $K = K(\varepsilon) \leq \exp(\exp(c\varepsilon^{-(\ell-1)}))$.

THEOREM 7. *Let $\mathcal{R}$ be a merging ℓ-vertex rule. For each n, let $(G(m))_{m \geq 0} = (G_n^{\mathcal{R}}(m))_{m \geq 0}$ be the random sequence of graphs on $\{1, 2, \dots, n\}$ associated to $\mathcal{R}$. Given any function $h_m(n)$ that is $o(n)$, and any constants $0 \leq a < b$, the probability that there exist m_1 and m_2 with $L_1(G(m_1)) \leq an$, $L_1(G(m_2)) \geq bn$ and $m_2 \leq m_1 + h_m(n)$ tends to 0 as $n \to \infty$.*

Note that for merging rules, Theorem 7 implies the conclusion of Theorem 1; a "jump" from $o(n)$ to $\geq \delta n$ implies a "jump" from $\leq \delta n/2$ to $\geq \delta n$.

PROOF OF THEOREM 7. Let $a < b$ be given, and set $\varepsilon = (b - a)/2$. Using Theorem 2 we may assume that there exists $K = K(\varepsilon, \ell)$ such that $N_{\geq K}(m) < L_1(m) + \varepsilon n$ for all m. Suppose that $m^- = \max\{m : L_1(m) \leq an\}$ and $m^+ = \min\{m : L_1(m) \geq bn\}$ differ by at most $h_m(n)$. Set $m^* = m^+ - \varepsilon n/(2\ell^2 K)$. As before, we have

$$N_{\geq K}(m^*) \geq L_1(m^+) - \ell^2 K(m^+ - m^*) > (b - \varepsilon)n = (a + \varepsilon)n. \tag{2}$$

If n is large enough, which we assume, then $m^+ \leq m^- + h_m(n)$ implies $m^* < m^-$. This gives $N_{\geq K}(m^*) \leq N_{\geq K}(m^-) < L_1(m^-) + \varepsilon n \leq (a + \varepsilon)n$, contradicting (2). □

Let us remark that Theorem 7 (which can be proved without first proving Theorem 2) gives an alternative proof of Spencer's "no two giants" conjecture; if at any time there are two components with at least εn vertices, then in the step after the last such time, L_1 must increase by at least εn in a single step. Hence, Theorem 7 implies that if $\mathcal{R}$ is merging, then for any $\varepsilon > 0$ we have $\max_m L_2(m) \leq \varepsilon n$ whp.

COROLLARY 8. *Let $\mathcal{R}$ be a merging ℓ-vertex rule. If $\mathcal{R}$ is globally convergent, then $\rho^{\mathcal{R}}$ is continuous on $[0, \infty)$.*

PROOF. Let $\rho(t) = \rho^{\mathcal{R}}(t)$. We have $0 \leq \rho(t) \leq \binom{\ell}{2} t$, so ρ is continuous at 0. Suppose ρ is discontinuous at some $t > 0$. Since ρ is increasing, $\sup_{t' < t} \rho(t') < \inf_{t' > t} \rho(t')$, so we may pick $a < b$ with $\sup_{t' < t} \rho(t') < a < b < \inf_{t' > t} \rho(t')$. By definition of global convergence, for any fixed $\varepsilon > 0$,

$$\mathbb{P}\big(L_1\big(\lfloor (t - \varepsilon)n \rfloor\big) \leq an \text{ and } L_1\big(\lfloor (t + \varepsilon)n \rfloor\big) \geq bn\big) \geq 1 - \varepsilon, \tag{3}$$

if n is large enough. It follows as usual that there is some $\varepsilon(n) \to 0$ such that (3) holds with $\varepsilon = \varepsilon(n)$. But this contradicts Theorem 7. $\square$

4. Convergence considerations. From the beginning, a key question about Achlioptas processes has been which rules are globally convergent. In some cases, local convergence has been established, but as far as we are aware, global convergence has not been shown for any nontrivial rules.

We now turn to the proof of Theorem 3, that local convergence implies global convergence for merging rules (in particular, for Achlioptas rules). We comment further on local convergence below. Theorem 3 is easy to deduce from Theorem 2; we shall give a more direct proof that seems more informative.

PROOF OF THEOREM 3. Suppose $\mathcal{R}$ is locally convergent. Then there exist functions $\rho_k : [0, \infty) \to [0, 1]$ such that (1) holds for any fixed $k \geq 1$ and $t \geq 0$. Since N_k changes by at most $2k$ when an edge is added to a graph, it follows easily that each ρ_k is continuous (indeed Lipschitz). From monotonicity of the underlying process, it is easy to see that for each k, the function $\rho_{\leq k}(t) = \sum_{j \leq k} \rho_j(t)$ is decreasing.

Define $\rho = \rho^{\mathcal{R}}$ by

$$\rho(t) = 1 - \sum_{k=1}^{\infty} \rho_k(t) = 1 - \lim_{k \to \infty} \rho_{\leq k}(t),$$

so $\rho : [0, \infty) \to [0, 1]$ is increasing. We claim that for any fixed $t > 0$ and $\varepsilon > 0$, the probability that

$$\sup_{0 \leq t' < t} \rho(t') - \varepsilon \leq \frac{L_1(\lfloor tn \rfloor)}{n} \leq \rho(t) + \varepsilon \tag{4}$$

tends to 1 as $n \to \infty$. This clearly implies that $L_1(\lfloor tn \rfloor)/n \xrightarrow{\mathrm{p}} \rho(t)$ whenever ρ is continuous at t, which is the definition of global convergence. Corollary 8 then implies that ρ is continuous.

The upper bound in (4) is immediate; by definition of ρ there is some K such that $\rho_{\leq K}(t) \geq 1 - \rho(t) - \varepsilon/4$. Summing (1) up to K gives $N_{\leq K}(\lfloor tn \rfloor)/n \geq 1 - \rho(t) - \varepsilon/2$ whp. When n is large enough, this bound implies $L_1(\lfloor tn \rfloor)/n \leq \rho(t) + \varepsilon$.

For the lower bound, we combine the "sprinkling" argument of Erdős and Rényi [8] with Lemma 6. Choose $t' < t$ such that $\rho(t')$ is within $\varepsilon/2$ of the supremum, and let $m_1 = \lfloor t'n \rfloor$ and $m_2 = \lfloor tn \rfloor$, so $m_2 - m_1 = \Theta(n)$. It suffices to show that $L_1(m_2)/n \geq \rho(t') - \varepsilon/2$ holds whp. In doing so we may assume that $\rho(t') - \varepsilon/2 \geq 0$. For any constant K, whp we have $N_{\leq K}(m_1)/n \leq \rho_{\leq K}(t') + \varepsilon/4 \leq 1 - \rho(t') + \varepsilon/4$, so $N_{\geq K}(m_1)/n \geq \rho(t') - \varepsilon/4$ whp. If K is large enough (depending only on t' and ε), Lemma 6 then gives $L_1(m_2)/n \geq \rho(t') - \varepsilon/2$ whp, as required. $\square$

REMARK 9. Since nonmerging ℓ-vertex rules have received some attention (see, e.g., [12]), let us spell out what our method gives for such rules. Lemma 6 applies in this case provided "there is a component containing" is changed to "there are $\ell - 1$ components together containing." Let $L(m)$ denote the sum of the sizes of the $\ell - 1$ largest components in $G(m)$. With this modified Lemma 6, the proof of Theorem 2 goes through with L_1 replaced by L. The same is true of Theorem 7 [with an extra $-(\ell - 2)K$ in (2), since the largest $\ell - 1$ components may not all be large]. Finally, Corollary 8 and Theorem 3 similarly go through, now with ρ defined using L rather than L_1.

5. Size rules. So far, even in the Achlioptas-rule case our rules have been very general, making choices between the given edges using any information about the current graph. There is a natural much smaller class (of vertex or Achlioptas rules) called *size rules*, where only the sequence $c_1, \dots, c_\ell$ of the orders of the components containing the presented vertices $v_1, \dots, v_\ell$ may be used to decide which edge(s) to add. (Here we suppress the dependence on the step m in the notation.) Note that the product rule is a size rule.

In fact, most past results concern *bounded size rules*; here there is a constant B such that all sizes $c_i > B$ are treated the same way by the rule, so the rule only "sees" the data $(\min\{c_i, B+1\})_{i=1}^{\ell}$. Perhaps the simplest example is the Bohman–Frieze process, the bounded size rule with $B = 1$ in which the edge v_1v_2 is added if $c_1 = c_2 = 1$, and otherwise v_3v_4 is added. Bohman and Frieze [3] showed that for a closely related rule there is no giant component when $m \sim 0.535n$. [The actual rule they used considered whether v_1 and v_2 are isolated in the graph formed by all pairs *offered* to the rule, rather than the graph $G(m)$ formed by the pairs *accepted* so far.]

Considering, for simplicity, rules in which one edge is added at each step, a key property of bounded size rules is that at each step, the expected change in N_k can be expressed as a simple function of $N_1, N_2, \dots, N_{\max\{k,B\}}$. (It is clear that the rate of formation of k-vertex components can be so expressed; for the rate of destruction, consider separately the cases k joins to k' for each $k' \le B$ and the case k joins to some $k' > B$.) Spencer and Wormald [15], who considered bounded size Achlioptas rules, and Bohman and Kravitz [4], who considered a large subset of such rules, noted that in this case one can easily use Wormald's "differential equation method" [16] to show that the rule is locally convergent, and that the $\rho_k(t)$ satisfy certain differential equations. This remark applies to all bounded size ℓ-vertex rules.

Resolving a conjecture of Spencer [14], Spencer and Wormald [15] proved that any bounded size 2-Achlioptas rule exhibits a phase transition: there is some t_c, depending on the rule, such that for $t < t_c$, whp $L_1(\lfloor tn \rfloor) = o(n)$ [in fact $O(\log n)$], while for $t > t_c$, $L_1(\lfloor tn \rfloor) = \Omega(n)$ whp. They conjectured that any bounded size 2-Achlioptas rule is globally convergent, and that the phase transition is second order (continuous). Theorem 3 establishes both these conjectures.

Very recently, Janson and Spencer [10] established bounds on the size of the giant component in the Bohman–Frieze process just above the (known) critical point t_c. They deduce that *if* it is globally convergent, then the right derivative of ρ at t_c has a certain specific value. The required "if" part is established by Theorem 3.

Informally, let us call a size rule *nice* if there is some K such that, for each k, the expected change in N_k is a function of $N_1, N_2, \dots, N_{\max\{k,K\}}$. [More precisely, the individual decisions whether to create or destroy a component of size k depend only on the data $(\min\{c_i, k'+1\})_{i=1}^{\ell}$ where $k' = \max\{k, K\}$ and c_i is the size of the component containing v_i.] Just as in the bounded size case, using the differential equation method, it is easy to show that any nice rule is locally convergent. Hence, by Theorem 3, any nice *merging* rule is globally convergent with continuous phase transition; this applies to all *nice* Achlioptas rules.

The simplest examples of nice rules have $K = 1$, that is, only compare component sizes. One example is "join the two smallest." For $\ell = 3$ this rule is mentioned briefly by Friedman and Landsberg [9] as another example of a rule that should be explosive, and discussed by D'Souza and Mitzenmacher [7], who "established" the explosive nature of the transition for this and a related nice rule numerically; Theorem 1 contradicts these predictions.

Another nice rule is the following: join the smaller of C_1 and C_2 to the smaller of C_3 and C_4, where C_i is the component containing v_i. We call this the "dCDGM" rule since it was introduced by da Costa, Dorogovtsev, Goltsev and Mendes [6]. Note that this is *not* an Achlioptas rule, but it *is* merging; if $|C|, |C'| \geq \varepsilon n$ then with probability at least ε^4 we choose $v_1, v_2 \in C$ and $v_3, v_4 \in C'$ and so join C to C'. Hence, the dCDGM rule, which is locally convergent by the differential equation method, is globally convergent and has a continuous phase transition. Da Costa, Dorogovtsev, Goltsev and Mendes [6] proposed this rule as simpler to analyze than the product rule, but at least as likely to have a discontinuous phase transition. For a brief discussion of their arguments, see the end of the Introduction.

There are many open questions concerning the precise nature of the phase transitions in various Achlioptas and related processes. One of the most intriguing is the following: Is the product rule globally convergent?

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