

## MIXING TIME FOR THE SOLID-ON-SOLID MODEL<sup>1</sup>

BY FABIO MARTINELLI<sup>2</sup> AND ALISTAIR SINCLAIR<sup>3</sup>

*Università degli Studi Roma Tre and University of California at Berkeley*

We analyze the mixing time of a natural local Markov chain (the Glauber dynamics) on configurations of the solid-on-solid model of statistical physics. This model has been proposed, among other things, as an idealization of the behavior of contours in the Ising model at low temperatures. Our main result is an upper bound on the mixing time of  $\tilde{O}(n^{3.5})$ , which is tight within a factor of  $\tilde{O}(\sqrt{n})$ . The proof, which in addition gives some insight into the actual evolution of the contours, requires the introduction of a number of novel analytical techniques that we conjecture will have other applications.

**1. Introduction.** In the  $n \times n$  *solid-on-solid (SOS)* model [24, 25], a configuration is an assignment of an integer *height*  $\eta(i) \in [0, n]^4$  to each of  $n$  positions  $i \in [1, n]$ , with fixed boundary conditions  $\eta(0) = \eta(n+1) = 0$ . The probability of a configuration is given by the *Gibbs distribution*,

$$(1.1) \quad \mu(\eta) = Z_\beta^{-1} \exp \left\{ -\beta \sum_{i=1}^{n+1} |\eta(i-1) - \eta(i)| \right\}.$$

Here  $\beta$  is a positive parameter, and  $Z_\beta$  is a normalizing factor (the “partition function”). Thus a configuration  $\eta = \{\eta(i)\}$  of the SOS model may be pictured as an interface or contour with fixed endpoints  $(0, 0)$  and  $(n+1, 0)$ ; see Figure 1(a). Notice that the Gibbs distribution favors contours that are “smooth” (i.e., have no large jumps in height), this bias being more pronounced for larger values of  $\beta$ . Moreover, the contour can be thought of as the path of an  $n$ -step random walk with independent geometric increments, conditioned to be positive and smaller than  $n$  and to return to the origin at time  $n$ . Therefore, its typical maximum height will be of order  $\sqrt{n}$ .

In this paper we analyze the (discrete time) *Glauber dynamics* for the SOS model. This is a natural local Markov chain on configurations whose transitions

---

Received July 2010; revised March 2011.

<sup>1</sup>An extended abstract of this paper appeared in *Proceedings of the 41st ACM Symposium on Theory of Computing (STOC)*, 2009, pages 571–580.

<sup>2</sup>Supported in part by the European Research Council through AdG “PTRELSS” 228032.

<sup>3</sup>Supported in part by NSF Grants CCF-06-35153 and CCF-10-16896.

*MSC2010 subject classifications.* Primary 60K35; secondary 82C20.

*Key words and phrases.* Solid-on-solid model, Markov chain Monte Carlo, censoring, Glauber dynamics, mixing time, monotonicity, Ising model.

<sup>4</sup>Throughout the paper,  $[a, b]$  will denote the integer points in the interval  $[a, b]$ .

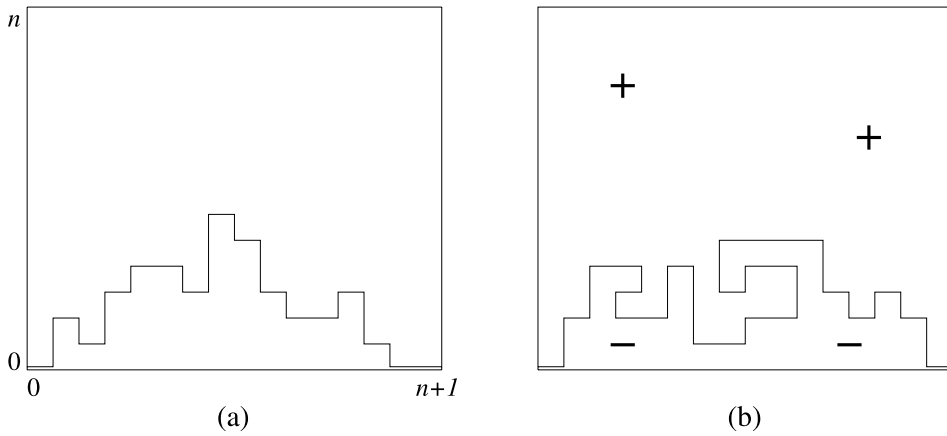


FIG. 1. (a) A contour in the SOS model. (b) A contour in the Ising model.

update the height at a randomly chosen position  $i$  from  $\eta(i)$  to  $\eta(i) \pm 1$ ; the transition probabilities are chosen so that the dynamics is reversible w.r.t. the Gibbs distribution  $\mu$  and thus converges to it from any initial configuration. Our goal is to determine the *mixing time*, that is, the number of steps until the dynamics is close to its equilibrium distribution  $\mu$  in variation distance.

Although Markovian dynamics for the SOS and related models have been studied extensively in many contexts connected with the behavior of random surfaces (see, e.g., [11–13, 23]), to the best of our knowledge the mixing time has not been rigorously analyzed. There are at least three motivations for studying this question, which we now describe.

The first motivation comes from the tight connection with the more familiar (two dimensional) Ising model, whose Glauber dynamics has been the focus of much attention in both statistical physics and computer science; see, for example, [2, 4, 7, 18–20, 29].

In the Ising model in an  $n \times n$  box  $\Lambda_n \subseteq \mathbb{Z}^2$ , the configurations are assignments  $\sigma$  of spin values  $\{+, -\}$  to the vertices of  $\Lambda_n$ . The Gibbs distribution is  $\mu(\sigma) = Z_\beta^{-1} \exp(-\beta D(\sigma))$ , where  $D(\sigma)$  is the number of neighbors in  $\Lambda_n$  whose spins differ, and  $\beta$  is inverse temperature. The (heat-bath) Glauber dynamics runs as follows: at each time step a random vertex  $i \in \Lambda_n$  is chosen, and its current spin value is replaced by a new value sampled from the equilibrium distribution at  $i$  given the neighboring spins. A variety of techniques have been introduced in order to analyze, at increasing levels of sophistication, the typical time scales of the relaxation process to the reversible Gibbs measure; see, for example, [7, 15, 18, 20]. These techniques have proved to be quite successful in the so-called “one-phase” region, corresponding to the case when the system has a unique Gibbs state. When instead the thermodynamic parameters of the system correspond to a point in the “phase coexistence” region, a whole class of new dynamical phenomena appear

(such as coarsening, phase nucleation and motion of interfaces between different phases) whose mathematical analysis at a microscopic level is still far from complete.

One of the most important and fundamental open problems is that of proving a polynomial (in  $n$ ) upper bound on the mixing time at low temperatures (large  $\beta$ ), when the boundary conditions around the edges of  $\Lambda_n$  are fixed to be  $+$  and hence force the system into the  $+$  phase.<sup>5</sup> (We remark that even the proof of a *lower bound*, usually a much simpler task, requires all the heavy technology of the Wulff construction [10] and the associated large deviation theory [4].) The above question is easily reduced to the following problem: if the box is initially filled with  $-$  spins, how long does it take until this large region of  $-$  is destroyed under the influence of the  $+$  boundary conditions and replaced by an equilibrium configuration? This in turn is equivalent to the question of how the outer contour of the  $-$  region contracts toward the center of the box. For large  $\beta$ , it is conjectured [14] that the contour evolves according to a mean curvature motion and therefore should disappear in polynomial time  $O(n^4)$  (independent of  $\beta$ );<sup>6</sup> however, until very recently only very weak upper bounds of the form  $\exp(O(n^{1/2+\varepsilon}))$  were known [18] (except in the qualitatively different zero temperature case, which is analyzed in [8]).

The SOS model has been proposed [25] as an idealized model of this Ising contour, in which we think of the sites above and below the SOS contour as being  $+$  and  $-$ , respectively. [Note that the sum  $\sum_i |\eta(i-1) - \eta(i)|$  in the Gibbs distribution (1.1) is, up to an additive constant, exactly  $D(\sigma)$  under this interpretation.] The mixing time is essentially the number of steps until the maximal contour (i.e., with  $\eta(i) = n$  for  $1 \leq i \leq n$ ) drops down close to the bottom of the box under the influence of the boundary conditions of height 0. The main simplification here is that, unlike the Ising model, the SOS contour has no “overhangs;” see Figure 1(b). However, for large  $\beta$  one may hope that overhangs are rare, so the approximation should give useful insight into the behavior of the true Ising contour; see [10] for much more on this point. One of our principal motivations in this paper is to introduce techniques that may find application to the Ising model. Indeed, this has already occurred, as the very recent paper [21] builds on some of the ideas and techniques of the present paper to obtain an upper bound of  $\exp(O(n^\varepsilon))$  (for arbitrary  $\varepsilon > 0$ ) on the mixing time of the Ising model at low temperatures with  $+$  boundaries, a substantial improvement on the  $\exp(O(n^{1/2+\varepsilon}))$  bound mentioned earlier though still quite far from polynomial.

The second motivation comes from general polymer models [13], and their natural Glauber-type evolutions; for example, [5, 17, 30]. In these models the “polymer” is just the  $n$ -step path of some type of random walk, starting and ending

---

<sup>5</sup>It is worth mentioning that when the underlying graph  $\mathbf{Z}^2$  is replaced by a regular tree or hyperbolic graph, then optimal  $O(n \log n)$  bounds on the mixing time [20] or on the spectral gap [3] have been established.

<sup>6</sup>In continuous time the corresponding scaling should be  $O(n^2)$ , apart from possible logarithmic corrections.

at zero and constrained to stay nonnegative. The associated Gibbs distribution is simply that induced by the probability distribution of the random walk. An additional interaction, or *pinning*, between the polymer and the line or “wall” at height zero can also be included, and the nature of this interaction (attractive, repulsive or even random) plays a crucial role. The Glauber dynamics can be defined in analogous fashion to the one studied in this paper. When the increments of the random walk are i.i.d.  $\pm 1$  random variables (with or without a bias), the mixing time of the associated Glauber chain has been analyzed quite precisely in various cases using the so called “Wilson method” [5]. When instead the increments are no longer uniformly bounded, as is the case for the SOS interface in this paper, a rigorous analysis of the associated Glauber dynamics apparently becomes much more challenging.

This brings us to our third motivation, which stems from the challenge that the SOS model poses to standard techniques. The two most natural approaches to estimating the mixing time seem to be the following:

(1) *Coupling*. One might hope that, under the natural monotone coupling of the SOS model (see Section 2), the expected Hamming distance between two coupled copies of the dynamics is nonincreasing. This would lead to a mixing time bound of  $\tilde{O}(n^5)$ ,<sup>7</sup> which as we shall see is rather weak and also gives little insight into the evolution of the contour. In fact even this is not always true (the distance may increase in expectation in some cases), and a direct approach based on monotone coupling remains elusive.

(2) *Comparison*. Another standard approach is to first analyze the “nonlocal” dynamics in which transitions are allowed to update the height  $\eta(i)$  to any value in  $[0, n]$ . Typically, nonlocal dynamics are easier to analyze precisely; see, for example, [17, 30]. One can then use the machinery of Diaconis and Saloff-Coste [9] to relate the mixing time of the local dynamics to that of the nonlocal one, as was done, for example, by Randall and Tetali [27] for the related “lozenge tilings” model. However, since such comparisons proceed via the spectral gap, they are usually quite wasteful; in particular, for the SOS model, this approach leads to a mixing time of  $\tilde{O}(n^8)$ .

In this paper we aim for a more refined analysis that gives, in addition to an almost tight bound, greater insight into the actual evolution of the contour in the SOS model. Our main result is the following:

**THEOREM 1.1.** *For the  $n \times n$  SOS model at any inverse temperature  $\beta > 0$ , the mixing time is  $\tilde{O}(n^{3.5})$ .*

---

<sup>7</sup>Throughout the paper the notation  $\tilde{O}(\cdot)$  hides factors of  $\text{polylog}(n)$ .

The bound on mixing time is tight up to a factor of  $\sqrt{n}$  (and logarithmic factors), as a lower bound of  $\Omega(n^3)$  follows from straightforward arguments; see Theorem 3.6 below.<sup>8</sup>

The high-level strategy of our analysis is as follows:

(a) We first prove (see Section 4.2) that in  $\tilde{O}(n^{3.5})$  steps the maximal configuration (i.e., the one in which the contour has height  $n$  everywhere) reaches equilibrium. This analysis in turn is split into  $O(\sqrt{n})$  repetitions of a basic *key result* which says that, starting in equilibrium but conditioned to be above height  $\tilde{O}(\sqrt{n})$ , in time  $\tilde{O}(n^3)$  the system reaches equilibrium; see Theorem 4.7. This result allows us to bring the original contour at height  $n$  down to equilibrium in a sequence of  $O(\sqrt{n})$  stages, each of which runs in  $\tilde{O}(n^3)$  steps and decreases the height by  $\Omega(\sqrt{n})$ .

(b) We then (see Section 4.3) analyze the time to reach equilibrium when the initial configuration is the minimal one (where the height is 0 everywhere), and show that  $\tilde{O}(n^3)$  steps suffice.

The results of (a) and (b) easily imply, by standard results on monotone coupling, that the mixing time is  $\tilde{O}(n^{3.5})$ .

Our analysis in the key intermediate result of part (a), and also in part (b), rests on the following four essential ingredients:

(i) First, we give a tight analysis of the nonlocal dynamics mentioned above, showing that its mixing time is  $O(n^3 \log n)$ ; see Theorem 3.1. This analysis, which we believe to be of independent interest, follows an idea of Wilson, developed in the context of the lozenge tilings model [30], in using an eigenvector of the discrete Laplacian to obtain a contraction in distance. However, to get this approach to work in our setting we need to bound a certain “entropy repulsion” effect due to the height barriers at 0 and  $n$ ; see Lemma 3.2.

(ii) We then relate the local to the nonlocal dynamics using a recent “censoring inequality” of Peres and Winkler [22], which says that *censoring* (i.e., not applying) some subset of updates in a monotone dynamics can only increase the distance from stationarity. This allows one to simulate a single move of the nonlocal dynamics, at position  $i$ , by censoring all local moves except those that update  $\eta(i)$ ; by the censoring inequality, this can only increase the mixing time. As a result, the mixing time of the local dynamics is bounded above by that of the nonlocal dynamics times a factor related to the mixing time of the one-dimensional local process within the  $i$ th “column.” Essentially, censoring allows us to “schedule” the updates and thus maintain detailed control of the shape of the contour.

---

<sup>8</sup>Very recently, after the acceptance of the present paper, a tight bound of  $\tilde{O}(n^3)$  on the mixing time has been obtained in [6]. A key ingredient in that result is our bound of  $\tilde{O}(n^3)$  on the mixing time starting from a contour of maximal height  $\tilde{O}(\sqrt{n})$  (see Theorem 4.7 below).

(iii) A naïve application of the censoring inequality would entail a substantial overhead of  $O(n^2)$  due to the mixing time within a column, which is essentially the square of the maximum height difference between the two neighboring columns. To overcome this, we need to control the height differences, or “gradients” along the contour. For this purpose, we work with a sequence of “bounding dynamics” with gradually decreasing boundary conditions [these correspond to the  $O(\sqrt{n})$  repetitions of the basic result mentioned earlier]; since the boundary conditions are—intuitively at least—the source of large gradients, this gives us control of the gradients. As a result, we are able to cut the simulation overhead between the local and nonlocal dynamics to  $\tilde{O}(\sqrt{n})$ . We note that this sequence of bounding dynamics captures some of the intuition about the actual evolution of the contour.

(iv) Making rigorous the above bound on gradients requires detailed information about the nonequilibrium shape of the contour, which is notoriously difficult to obtain. We get around this difficulty by starting the bounding dynamics *in equilibrium*, but *conditioned on a certain rare event*  $A$ . (The conditioning is necessary to ensure that the bounding property holds.) By choosing  $A$  such that its probability, though tiny, is nonetheless much larger than the probability of large gradients in equilibrium, we are able to argue that large gradients do not occur during the evolution. This technique is isolated in Lemma 4.2.

**2. Preliminaries.**

*Gibbs distribution.* We denote by  $\Omega_n = [0, n]^n$  the set of all configurations  $\eta = \{\eta(i)\}_{i=1}^n$  of the  $n \times n$  solid-on-solid model, as defined in the [Introduction](#). The probability of a configuration  $\eta$  is given by the Gibbs distribution defined in equation (1.1). This distribution induces a conditional distribution on the height  $\eta(i)$  at position  $i$ , given the heights  $\eta(i \pm 1)$  at its neighbors, as follows. Let  $a = \min\{\eta(i - 1), \eta(i + 1)\}$ ,  $b = \max\{\eta(i - 1), \eta(i + 1)\}$ . Then  $\mu_{ab}(j) := \Pr[\eta(i) = j | a, b]$  is given by

$$(2.1) \quad \mu_{ab}(j) = \begin{cases} e^{-\beta(b-a)-2\beta(a-j)} / Z, & \text{if } 0 \leq j < a, \\ e^{-\beta(b-a)} / Z, & \text{if } a \leq j \leq b, \\ e^{-\beta(b-a)-2\beta(j-b)} / Z, & \text{if } b < j \leq n, \end{cases}$$

where  $Z = Z_\beta$  is a normalizing factor. Note that  $\mu_{ab}$  is uniform on the interval  $[a, b]$  and decays exponentially (at a rate depending on  $\beta$ ) outside it.

*Single-site dynamics.* Our goal is to analyze the *single-site Glauber dynamics*,<sup>9</sup> which is a reversible Markov chain  $\mathcal{M}_n^{\text{ss}}$  on  $\Omega_n$  with transitions defined as follows, where  $\eta = \eta_t$  denotes the current configuration at time  $t$ :

---

<sup>9</sup>This dynamics has been chosen for concreteness; our arguments apply with minor modifications to any reversible, monotone local dynamics.

(1) Pick  $i \in [1, n]$  u.a.r.

(2) Let  $\eta^-, \eta^+$  be the configurations obtained from  $\eta$  by replacing  $\eta(i)$  by  $\max\{\eta(i) - 1, 0\}$  and  $\min\{\eta(i) + 1, n\}$ , respectively. Set  $\eta_{t+1}$  equal to  $\eta^-$  or  $\eta^+$  with probabilities  $p^-, p^+$ , respectively, determined as follows (where  $a, b$  are the minimum and maximum heights of the neighbors, as above): if  $\eta(i) \leq a$ , then  $p^- = \frac{1}{4}e^{-2\beta}$ , else  $p^- = \frac{1}{4}$ ; if  $\eta(i) \geq b$ , then  $p^+ = \frac{1}{4}e^{-2\beta}$ , else  $p^+ = \frac{1}{4}$ . With the remaining probability  $1 - (p^- + p^+)$ , set  $\eta_{t+1} = \eta$ .

It is standard that  $\mathcal{M}_n^{ss}$  is an ergodic, reversible Markov chain that converges to the stationary distribution  $\mu$  on  $\Omega_n$ . Our goal is to estimate its *mixing time*, that is, the number of steps required for the distribution to get close (in variation distance) to  $\mu$  from an arbitrary initial configuration.

*Column dynamics.* We will analyze  $\mathcal{M}_n^{ss}$  by first analyzing a related Glauber dynamics  $\mathcal{M}_n^{col}$  that makes *nonlocal* moves. (The term ‘‘column’’ refers to the set  $[0, n]$  of possible heights at  $i$ .) If the configuration at time  $t$  is  $\eta_t = \eta$ ,  $\mathcal{M}_n^{col}$  makes a transition as follows:

(1) Pick  $i \in [1, n]$  u.a.r.

(2) For each  $j \in [0, n]$ , let  $\eta^j$  denote the configuration obtained from  $\eta$  by replacing  $\eta(i)$  by  $j$ . Set  $\eta_{t+1} = \eta^j$  with probability proportional to  $\mu(\eta^j)$ .

$\mathcal{M}_n^{col}$  is again ergodic and reversible with stationary distribution  $\mu$ . Note that both  $\mathcal{M}_n^{ss}$  and  $\mathcal{M}_n^{col}$  update the height at a randomly chosen position  $i$  in a manner, that is, reversible w.r.t. the conditional distribution (2.1). The difference is that  $\mathcal{M}_n^{ss}$  considers only local moves (changing the height by  $\pm 1$ ), while  $\mathcal{M}_n^{col}$  allows the height at  $i$  to be set to any value. Accordingly, we call  $\mathcal{M}_n^{col}$  the ‘‘column dynamics’’ and  $\mathcal{M}_n^{ss}$  the ‘‘single-site dynamics.’’

2.1. *Mixing time.* Let  $\mathcal{M}$  be any reversible Markov chain on  $\Omega_n$  with stationary distribution  $\mu$ . Following standard practice, we measure the convergence rate of  $\mathcal{M}$  via the quantity

$$\tau_{\mathcal{M}}(\varepsilon) = \min\{t : \|v_t^\xi - \mu\| \leq \varepsilon \ \forall \xi \in \Omega_n\},$$

where  $v_t^\xi$  denotes the distribution of the configuration at time  $t$  starting from configuration  $\xi$  at time 0, and  $\|\cdot\|$  denotes variation distance. Thus  $\tau_{\mathcal{M}}(\varepsilon)$  is the number of steps until the variation distance from  $\mu$  drops to  $\varepsilon$ , for an arbitrary initial configuration. For definiteness we define the *mixing time* as  $\tau_{\mathcal{M}}^{mix} = \tau_{\mathcal{M}}(1/2e)$ ; it is well known (see, e.g., [1]), that  $\tau_{\mathcal{M}}(\varepsilon) \leq \lceil \ln \varepsilon^{-1} \rceil \times \tau_{\mathcal{M}}^{mix}$  for all  $\varepsilon > 0$ .

*Monotonicity and coupling.* We define a natural partial order on  $\Omega_n$  as follows: for configurations  $\eta, \xi \in \Omega_n$ , we say that  $\eta \leq \xi$  iff  $\eta(i) \leq \xi(i)$  for all  $i \in [1, n]$ . Note that  $\leq$  has unique maximal and minimal elements  $\eta^{max}$  and  $\eta^{min}$  given by  $\eta^{max}(i) = n$  and  $\eta^{min}(i) = 0$  for  $1 \leq i \leq n$ . We can naturally extend this ordering

to probability distributions as follows: for two distributions  $\nu, \mu$  on  $\Omega_n$ , we write  $\nu \preceq \mu$  if for any increasing<sup>10</sup> function  $f$  the average of  $f$  w.r.t.  $\nu$  is less than or equal to its average w.r.t.  $\mu$ .

A key fact we shall exploit throughout is the existence of a *complete coupling* of the Glauber dynamics (single-site or column), that is, monotone w.r.t.  $\preceq$ . A complete coupling of a Markov chain  $\mathcal{M}$  on  $\Omega_n$  is a random function  $f : \Omega_n \rightarrow \Omega_n$  that preserves the transition probabilities of  $\mathcal{M}$ , that is,  $\Pr_f[f(\eta) = \eta'] = \Pr_{\mathcal{M}}(\eta \rightarrow \eta')$  for all  $\eta, \eta' \in \Omega_n$ . Note that  $f$  simultaneously couples the evolution of the Markov chain at all configurations. For the column dynamics, we define  $f$  as follows. Suppose the current configuration is  $\eta$ :

- (1) Pick  $i \in [1, n]$  and a real number  $r \in [0, 1]$  independently and u.a.r.
- (2) Let  $g(k) = \sum_{j=0}^k \mu_{ab}(j)$  be the cumulative distribution function of the height at position  $i$ , given neighboring heights  $a, b$ . Set  $\eta'(i) = \min\{k : g(k) \leq r\}$ .

An analogous definition holds for the single-site dynamics. It is simple to check that these couplings are *monotone* w.r.t. the partial order  $\preceq$ , in the sense that if  $\eta_t \preceq \xi_t$ , and  $\eta_{t+1}, \xi_{t+1}$  are the corresponding configurations at the next time step under the coupling, then  $\eta_{t+1} \preceq \xi_{t+1}$ .

A further standard fact we will need is that the mixing time of the Glauber dynamics is bounded above by the time until the coupled evolutions started in the two extremal configurations,  $\eta^{\max}$  and  $\eta^{\min}$ , coincide with constant probability. More precisely:

PROPOSITION 2.1. *Let  $(\eta_t^{\max}), (\eta_t^{\min})$  denote the coupled evolutions of two copies of a monotone Glauber dynamics  $\mathcal{M}$  on  $\Omega_n$  started in configurations  $\eta^{\max}, \eta^{\min}$ , respectively, and let  $\nu_t^{\max}, \nu_t^{\min}$  denote the corresponding marginals. Then:*

- (a)  $\tau_{\mathcal{M}}(\varepsilon) \leq \min\{t : \Pr[\eta_t^{\max} \neq \eta_t^{\min}] \leq \varepsilon\}$ ;
- (b)  $\Pr[\eta_t^{\max} \neq \eta_t^{\min}] \leq n^2 \|\nu_t^{\max} - \nu_t^{\min}\|$ .

PROOF. Part (a) is standard; see [26]. As far as part (b) is concerned, monotonicity and the Markov inequality give

$$\begin{aligned} \Pr[\eta_t^{\max} \neq \eta_t^{\min}] &\leq \sum_{i=1}^n \Pr[\eta_t^{\max}(i) > \eta_t^{\min}(i)] \\ &\leq \sum_{i=1}^n \sum_{j=1}^n j (\Pr[\eta_t^{\max}(i) = j] - \Pr[\eta_t^{\min}(i) = j]) \\ &\leq n^2 \|\nu_t^{\max} - \nu_t^{\min}\|. \end{aligned} \quad \square$$

<sup>10</sup>A real-valued function  $f$  on  $\Omega_n$  is *increasing* w.r.t.  $\preceq$  if  $\eta \preceq \xi$  implies  $f(\eta) \leq f(\xi)$ .



*Censoring.* In our analysis of the single-site dynamics, we shall also need a useful tool from recent work of Peres and Winkler, which says that *censoring* (i.e., not applying) any subset of updates in the dynamics can only increase the distance from stationarity. This so-called “censoring inequality” applies to any monotone single-site dynamics.

LEMMA 2.2 ([22]). *Suppose a monotone single-site dynamics is started in a random initial configuration with distribution  $\nu_0$  such that  $\nu_0/\mu$  is increasing w.r.t.  $\leq$ . Let  $\nu$  denote the distribution after updates at positions  $i_1, i_2, \dots, i_m$ , and  $\nu'$  the distribution after updates at a subsequence of these positions  $i_{j_1}, i_{j_2}, \dots, i_{j_{m'}}$  (chosen a priori). Then  $\nu/\mu$  is increasing and  $\|\nu - \mu\| \leq \|\nu' - \mu\|$ .*

REMARK 2.3. [22, Theorem 16.5] states this result for the special case in which  $\nu_0$  is concentrated on the maximal state  $\eta^{\max}$ . However, it is easy to see that the proof requires only the weaker assumption that  $\nu_0/\mu$  is increasing. Moreover, by symmetry the lemma clearly also holds with “increasing” replaced by “decreasing.”

The censoring inequality can be used to relate the single-site and column dynamics via the following observation. If we censor all moves of the single-site dynamics except for those that update a certain position  $i$ , then after some fixed number of steps  $T$  (which depends on the mixing time of the single-site dynamics just within the  $i$ th column, with its neighbors fixed) we will, up to small error, have simulated one move of the column dynamics. By Lemma 2.2 the censoring can only slow down convergence of the single-site dynamics, so the mixing time of  $\mathcal{M}_n^{\text{ss}}$  is bounded above by roughly  $T$  times that of  $\mathcal{M}_n^{\text{col}}$ . We shall use a more sophisticated version of this argument in Section 4.

**3. The column dynamics.** Our goal in this section is to provide a tight analysis of the column dynamics  $\mathcal{M}_n^{\text{col}}$ . Specifically, we will prove the following theorem.

THEOREM 3.1. *For any  $\beta > 0$ , the mixing time of the column dynamics  $\mathcal{M}_n^{\text{col}}$  is  $O(n^3 \log n)$ .*

We believe this result, which we show is tight up to the  $\log n$  factor (see Theorem 3.6 below), is interesting in its own right. It will also be a key ingredient in our analysis of the single-site dynamics later.

Recall that, if the current configuration of  $\mathcal{M}_n^{\text{col}}$  is  $\eta_t$ , and we choose position  $i \in [1, n]$  at the next step, then the new height  $\eta_{t+1}(i)$  is drawn from the conditional distribution (2.1), where  $a, b$  are the minimum and maximum heights, respectively, of the neighbors  $\eta_t(i \pm 1)$ . A key observation is that, under such a move, the expected value of the new height  $\eta_{t+1}(i)$  is close to the average  $\frac{a+b}{2}$

of its two neighbors; moreover, the error term satisfies a natural ordering property w.r.t.  $a, b$ .

LEMMA 3.2. *In the above situation, and assuming  $a + b \leq n$ , the expected value of the new height  $\eta_{t+1}(i)$  satisfies*

$$(3.1) \quad E[\eta_{t+1}(i)|a, b] = \frac{a + b}{2} + \varepsilon(a, b),$$

where  $\varepsilon(a, b) \geq 0$ . Moreover,  $\varepsilon(a, b) \leq \varepsilon(c, d)$  for any pair  $c, d$  with  $c \leq \min\{a, d\} \leq \max\{a, d\} \leq b$ .

We defer the proof of the lemma, which is somewhat technical, to the [Appendix](#). However, the intuition is as follows. Note that the distribution of  $\eta_{t+1}(i)$  is uniform on the interval  $[a, b]$ , and decays symmetrically on either side except for the effects of the barriers at heights 0 and  $n$ . Thus we would expect its mean to be close to  $\frac{a+b}{2}$ . The term  $\varepsilon(a, b)$  captures the “entropy repulsion” effect of the barriers. This effect is more pronounced for pairs  $(a, b)$  that are closer to 0, as is the case for the pair  $(c, d)$  in the lemma.

We can derive from Lemma 3.2 the following more symmetrical form that allows us to compare the heights of two ordered configurations under the monotone coupling (again, see the [Appendix](#) for a proof).

COROLLARY 3.3. *Suppose  $\eta_t$  and  $\xi_t$  are two configurations satisfying  $\eta_t \preceq \xi_t$ , and let  $a = \min\{\xi_t(i - 1), \xi_t(i + 1)\}$ ,  $b = \max\{\xi_t(i - 1), \xi_t(i + 1)\}$ ,  $c = \min\{\eta_t(i - 1), \eta_t(i + 1)\}$ ,  $d = \max\{\eta_t(i - 1), \eta_t(i + 1)\}$ . Then*

$$(3.2) \quad 0 \leq E[\xi_{t+1}(i)|a, b] - E[\eta_{t+1}(i)|c, d] \leq \frac{a + b}{2} - \frac{c + d}{2}.$$

Armed with Corollary 3.3, we can now proceed to our analysis of  $\mathcal{M}_n^{\text{col}}$ .

PROOF OF THEOREM 3.1. Following Proposition 2.1, it suffices to show that two coupled copies of  $\mathcal{M}_n^{\text{col}}$ , started in configurations  $\eta^{\text{max}}$  and  $\eta^{\text{min}}$ , will coincide with constant probability after  $O(n^3 \log n)$  steps. Call these two copies  $(\eta_t^{\text{max}})$ ,  $(\eta_t^{\text{min}})$ , respectively.

We will measure the distance between  $\eta_t^{\text{max}}$  and  $\eta_t^{\text{min}}$  using the quantity

$$(3.3) \quad D(t) = \sum_{i=1}^n w(i)(\eta_t^{\text{max}}(i) - \eta_t^{\text{min}}(i)),$$

where  $w(i)$  is a suitably chosen, strictly positive weight function. Note that  $\eta_t^{\text{max}}(i) \geq \eta_t^{\text{min}}(i)$  for all  $i, t$  by monotonicity, so all terms in the sum are nonnegative; and  $D(t) = 0$  iff  $\eta_t^{\text{max}} = \eta_t^{\text{min}}$ . Following an idea of Wilson [30], we choose  $w$  as the second eigenvector of the discrete Laplacian operator  $\Delta$  on  $[1, n]$  with zero

boundary conditions, defined by  $\Delta g(i) = -\frac{1}{2}(g(i+1) + g(i-1)) + g(i)$ ,  $g(0) = g(n+1) = 0$ . It is well known (and easy to verify) that  $w(i) = \cos(-\frac{\pi}{2} + \frac{\pi i}{n+1})$  with corresponding eigenvalue  $\lambda = 1 - \cos(\frac{\pi}{n+1}) = \Theta(\frac{1}{n^2})$ .

The reason for this choice is that, by Corollary 3.3, one step of the dynamics behaves very like the Laplacian, so choosing  $w$  as an eigenvector of  $\Delta$  should give us a contraction of  $(1 - \frac{\lambda}{n})$  in  $D$  at every step. The argument proceeds as follows:

$$\begin{aligned}
 & \mathbb{E}[D(t+1) - D(t) | \eta_t^{\max}, \eta_t^{\min}] \\
 &= \frac{1}{n} \sum_{i=1}^n w(i) \{ \mathbb{E}[\eta_{t+1}^{\max}(i) | \eta_t^{\max}(i-1), \eta_t^{\max}(i+1)] \\
 & \quad - \mathbb{E}[\eta_{t+1}^{\min}(i) | \eta_t^{\min}(i-1), \eta_t^{\min}(i+1)] \\
 & \quad - (\eta_t^{\max}(i) - \eta_t^{\min}(i)) \} \\
 &\leq -\frac{1}{n} \sum_i w(i) (\Delta \eta_t^{\max}(i) - \Delta \eta_t^{\min}(i)) \\
 (3.4) \quad &= -\frac{1}{n} \sum_i \Delta w(i) (\eta_t^{\max}(i) - \eta_t^{\min}(i)) = -\frac{\lambda}{n} D(t),
 \end{aligned}$$

where in the inequality we have used Corollary 3.3.

Thus after  $t$  steps of the dynamics we have  $\mathbb{E}[D(t)] \leq (1 - \frac{\lambda}{n})^t D(0) \leq (1 - \frac{c}{n^3})^t n^2$  for a constant  $c > 0$ . Taking  $t = t^* = c' n^3 \log(\frac{n}{\epsilon})$  for a sufficiently large constant  $c'$  ensures that  $\mathbb{E}[D(t^*)] \ll \frac{\epsilon}{n^2}$ . Finally, we may bound the coupling probability at time  $t^*$  as follows:

$$\begin{aligned}
 \Pr[\eta_{t^*}^{\max} \neq \eta_{t^*}^{\min}] &\leq \sum_i \Pr[\eta_{t^*}^{\max}(i) - \eta_{t^*}^{\min}(i) \geq 1] \\
 &\leq \left( \min_i w(i) \right)^{-1} \sum_i w(i) \mathbb{E}[\eta_{t^*}^{\max}(i) - \eta_{t^*}^{\min}(i)] \\
 &= \left( \min_i w(i) \right)^{-1} \mathbb{E}[D(t^*)] \leq \epsilon,
 \end{aligned}$$

where in the second line we used Markov's inequality, and in the last line the fact that  $\min_i w(i) = \cos(-\frac{\pi}{2} + \frac{\pi}{n+1}) = \Theta(\frac{1}{n^2})$ . Thus, by Proposition 2.1,  $\tau_{\mathcal{M}_n^{\text{col}}}(\epsilon) \leq t^* = O(n^3 \log(n/\epsilon))$ .  $\square$

For our analysis of the single-site dynamics, it will be convenient to introduce a ‘‘parallel’’ version  $\mathcal{M}_n^{\text{par}}$  of the column dynamics in which all odd-numbered (or all even-numbered) positions are updated simultaneously at each step. Moreover, since repeated updates of odd or even positions have no effect, we may as well assume that odd and even updates alternate. This leads to the following definition

of  $\mathcal{M}_n^{\text{par}}$ , in which  $O, E$  denote updates of all odd and even positions, respectively, and the update at any given position is performed as in the column dynamics:

- (1) Flip a single fair coin.
- (2) If heads, perform  $t$  pairs of odd–even updates [i.e.,  $(OE)^t$ ], else if tails perform  $t$  pairs of even–odd updates [i.e.,  $(EO)^t$ ].

Note that  $\mathcal{M}_n^{\text{par}}$  is a convex combination of two reversible Markov chains, one performing the update sequence  $(OE)^t$  and the other  $(EO)^t$ . We will call these chains  $\mathcal{M}_n^{\text{OE}}$  and  $\mathcal{M}_n^{\text{EO}}$ , respectively.

Following our analysis of  $\mathcal{M}_n^{\text{col}}$ , it is straightforward to see that  $\mathcal{M}_n^{\text{par}}$  inherits a similar bound on the mixing time, with a factor  $n$  speedup coming from the parallelization of the updates.

**THEOREM 3.4.** *The mixing time of  $\mathcal{M}_n^{\text{par}}$  is  $O(n^2 \log n)$ .*

**PROOF.** We use the same distance measure (3.3) as in the proof of Theorem 3.1. From equation (3.4) of that proof, we conclude that under the  $t$ -step evolution of the column dynamics this distance satisfies  $E[D(t)] \leq \frac{1}{n} \sum_i (I - \Delta)^t w(i) (\eta_0^{\text{max}}(i) - \eta_0^{\text{min}}(i))$ , where  $I$  is the identity operator  $Ig = g$ . An analogous calculation for  $\mathcal{M}_n^{\text{par}}$  leads to

$$\begin{aligned} E[D(t)] &\leq \frac{1}{2} \sum_i ((I - \Delta)^{2t} w + (I - \Delta)^{2t-1} w)(i) (\eta_0^{\text{max}}(i) - \eta_0^{\text{min}}(i)) \\ &\leq \frac{1}{2} ((1 - \lambda)^{2t} + (1 - \lambda)^{2t-1}) D(0) \\ &\leq (1 - \lambda)^{2t-1} D(0). \end{aligned}$$

Using the facts that  $\lambda = \Theta(\frac{1}{n^2})$  and  $D(0) \leq n^2$ , and arguing as in the previous proof, gives  $\tau_{\mathcal{M}_n^{\text{par}}}(\varepsilon) = O(n^2 \log(n/\varepsilon))$ , as claimed.  $\square$

**REMARK 3.5.** The proofs of Theorems 3.1 and 3.4 show the stronger results that  $\tau_{\mathcal{M}_n^{\text{col}}}(\varepsilon) = O(n^3 \log(n/\varepsilon))$  and  $\tau_{\mathcal{M}_n^{\text{par}}}(\varepsilon) = O(n^2 \log(n/\varepsilon))$ . We shall use this result for  $\tau_{\mathcal{M}_n^{\text{par}}}(\varepsilon)$  in the next section.

We close this section with a lower bound which shows that the above bound on the mixing time of the column dynamics is tight up to the  $\log n$  factor. This lower bound also applies to the single-site dynamics, which will imply that our upper bound on its mixing time derived in the next section is tight within a factor of  $\tilde{O}(\sqrt{n})$ , as claimed in the [Introduction](#).

**THEOREM 3.6.** *The mixing times of both  $\mathcal{M}_n^{\text{col}}$  and  $\mathcal{M}_n^{\text{ss}}$  are at least  $\Omega(n^3)$ .*

PROOF. Recall that the *spectral gap* of a reversible dynamics  $\mathcal{M}$  is given by

$$(3.5) \quad \text{gap}_{\mathcal{M}} = \frac{1}{2} \inf_f \frac{\sum_{\eta, \eta'} \mu(\eta) \Pr_{\mathcal{M}}[\eta \rightarrow \eta'] (f(\eta) - f(\eta'))^2}{\text{Var}_{\mu}(f)},$$

where the infimum is over all nonconstant functions  $f : \Omega_n \rightarrow \mathbb{R}$  (see, e.g., [16]). As is well known, the mixing time is bounded below by  $\text{gap}_{\mathcal{M}}^{-1}$ , so it suffices to show that  $\text{gap}_{\mathcal{M}} \leq n^{-3}$ .

Take the test function  $f(\eta) = \sum_i w(i)(\eta(i+1) - \eta(i))$ , where  $w$  is as in the proof of Theorem 3.1. Following step by step the proof of the analogous result in the context of polymer dynamics (see Proposition 5.2 in [5]) we get that for both  $\mathcal{M}_n^{\text{col}}$  and  $\mathcal{M}_n^{\text{ss}}$ , the numerator of (3.5) is at most  $c_1/n^2$ , and the denominator is at least  $c_2n$ , for constants  $c_1, c_2 > 0$ . This completes the proof.  $\square$

**4. The single-site dynamics.** In this section we prove our main result, Theorem 1.1 of the Introduction, which we restate here for convenience.

**THEOREM 4.1.** *The mixing time of the single-site dynamics  $\mathcal{M}_n^{\text{ss}}$  at any inverse temperature  $\beta > 0$  is  $\tilde{O}(n^{3.5})$ .*

As indicated in the Introduction, we analyze separately the time required for maximal and minimal contours to reach equilibrium under a monotone complete coupling; by Proposition 2.1 this will suffice to bound the mixing time. We handle the more challenging case of the maximal contour in Section 4.2 and the minimal contour in Section 4.3. We begin with a basic analytical tool that we will use in both parts, which allows us to relate the single-site dynamics to the column dynamics analyzed previously.

**4.1. Basic building block.** As explained in the Introduction, our main tool for analyzing the evolution of the single-site dynamics is to relate it to the column dynamics, for which we obtained a tight mixing time analysis in Section 3. To do this we will use the censoring idea explained in Section 2. As indicated in the Introduction, the overhead in the mixing time introduced by censoring depends crucially on the *maximum gradient* (or height difference) that arises in the dynamics. In this subsection, we show that this overhead can be kept very low (polylogarithmic in  $n$ ) provided we start the dynamics in the equilibrium distribution conditioned on a monotone event  $A$  whose probability is not extremely small [at least  $\exp(-\text{polylog}(n))$ ]. In our subsequent analysis, we will use this basic building block repeatedly by conditioning on various suitable events  $A$ .

**LEMMA 4.2.** *Let  $A$  be any increasing or decreasing event, and consider the single-site dynamics started from  $\nu_0 := \mu(\cdot | A)$ . Denote by  $\nu_t$  its distribution after  $t$  steps. Let  $D := \lceil \log(\frac{1}{\mu(A)}) \rceil$ , and  $t_{n,D} := 2n^3 D^2 \log^8 n$ . Then for any  $t \geq t_{n,D}$  and any fixed  $b > 0$  we have*

$$\|\nu_t - \mu\| = o(1/n^b).$$

REMARK 4.3. Here and elsewhere in this section, in the interests of clarity of exposition we make no attempt to minimize the number of log factors in our bounds. In particular, we frequently use a log factor in place of a sufficiently large constant. Also, we generally ignore issues of rounding throughout.

PROOF. We consider only the case of an increasing event  $A$ ; the decreasing case is entirely symmetrical. To bound the mixing time of the single-site dynamics, we relate it to the corresponding parallel column dynamics using the censoring inequality (Lemma 2.2). Note that this is valid because the initial distribution  $\nu_0$  satisfies the requirement that  $\nu_0/\mu = \chi_A/\mu(A)$  is increasing w.r.t.  $\preceq$ .

To do this, we split the time  $t_{n,D}$  into  $M := n^2 \log^2 n$  epochs each of length  $m := 2nD^2 \log^6 n$ . Given  $t_{n,D}$  random positions  $\mathbf{i} = (i_1, i_2, \dots, i_{t_{n,D}})$  in  $[1, n]$ , the distribution  $\nu_{t_{n,D}}$  can be written as the average over  $\mathbf{i}$  of the distribution  $\nu_{\mathbf{i}}$  obtained by applying, in the given order,  $t_{n,D}$  single-site updates at positions  $i_1, i_2, \dots, i_{t_{n,D}}$ . Next we write  $\mathbf{w}(\mathbf{i}) = (\mathbf{w}_1, \dots, \mathbf{w}_M)$  by grouping together positions in the same epoch. Finally, we define two censored versions of the dynamics as follows. In the first version, we delete all even positions from the odd epochs and all odd positions from the even epochs; denote the resulting censored vector  $\mathbf{OE}(\mathbf{i}) = (\mathbf{OE}_1, \dots, \mathbf{OE}_M)$  and the associated distribution  $\nu_{\mathbf{OE}(\mathbf{i})}$ . In the second version, we reverse the roles of odd and even and denote the resulting censored vector  $\mathbf{EO}(\mathbf{i}) = (\mathbf{EO}_1, \dots, \mathbf{EO}_M)$  and the associated distribution  $\nu_{\mathbf{EO}(\mathbf{i})}$ .

This construction gives us

$$(4.1) \quad \|\nu_{t_{n,D}} - \mu\| = \|\text{Av}_{\mathbf{i}}\nu_{\mathbf{i}} - \mu\| \leq \text{Av}_{\mathbf{i}}\|\nu_{\mathbf{i}} - \mu\| \leq \text{Av}_{\mathbf{i}}\left\|\frac{1}{2}(\nu_{\mathbf{OE}(\mathbf{i})} + \nu_{\mathbf{EO}(\mathbf{i})}) - \mu\right\|,$$

where  $\text{Av}_{\mathbf{i}}$  denotes the spatial average over the column index  $i$ , and the last step relies on the censoring inequality. Note that the expected number of times any position  $i$  appears in  $\mathbf{i}$  is  $m/n = 2D^2 \log^6 n$ . Hence a standard Chernoff bound guarantees that, apart from an error, that is, exponentially small in  $\log^6 n$  (and hence certainly  $o(1/n^b)$ ), the r.h.s. of (4.1) is bounded above by

$$(4.2) \quad \max_{\mathbf{i} \in \Sigma} \left\| \frac{1}{2}(\nu_{\mathbf{OE}(\mathbf{i})} + \nu_{\mathbf{EO}(\mathbf{i})}) - \mu \right\|,$$

where  $\Sigma$  consists of all  $\mathbf{i}$  such that the censored vectors  $\mathbf{OE}(\mathbf{i})$  and  $\mathbf{EO}(\mathbf{i})$  contain at least  $D^2 \log^6 n$  updates of every position  $i \in [1, n]$  in every epoch  $k \in \{1, \dots, M\}$ .

Now we claim that, for  $\mathbf{i} \in \Sigma$ , the distribution  $\frac{1}{2}(\nu_{\mathbf{OE}(\mathbf{i})} + \nu_{\mathbf{EO}(\mathbf{i})})$  is very close to the distribution at time  $M = n^2 \log^2 n$  of the parallel column dynamics  $\mathcal{M}_n^{\text{par}}$ , with the same initial distribution. To establish this, we need to show that  $D^2 \log^6 n$  single-site updates at position  $i$ , with its neighboring heights fixed, are enough to simulate (with small error) one column update at  $i$ . This relies crucially on the fact that  $\mathcal{M}_n^{\text{par}}$  is unlikely to produce configurations with large gradients, which we define to be at least  $D \log^2 n$ . Accordingly, define the set of “bad” configurations

$$(4.3) \quad B = \{\eta : |\eta(i+1) - \eta(i)| \geq D \log^2 n \text{ for some } i \in [0, n]\}.$$

CLAIM 4.4. For  $\mathbf{i} \in \Sigma$ , we have

$$\left\| \frac{1}{2}(\nu_{\mathbf{OE}(\mathbf{i})} + \nu_{\mathbf{EO}(\mathbf{i})}) - \nu_M^{\text{par}} \right\| \leq M \left( \max_s \{ \nu_s^{\text{OE}}(B) + \nu_s^{\text{EO}}(B) \} + e^{-\Omega(\log^2 n)} \right),$$

where  $\nu_s^{\text{OE}}$  and  $\nu_s^{\text{EO}}$  denote the distributions of  $\mathcal{M}_n^{\text{OE}}$  and  $\mathcal{M}_n^{\text{EO}}$ , respectively, after  $s$  steps, starting from  $\nu_0$ .

The intuition for this claim, which is proved formally in the [Appendix](#), is the following. The first term on the r.h.s. bounds the probability of seeing a bad configuration in  $\mathcal{M}_n^{\text{par}}$ , so we may assume that  $\eta \notin B$ . A sequence of single-site updates at position  $i$  (with its neighboring heights  $a, b$  fixed) can be viewed as a nearest-neighbor random walk on column  $i$  with stationary distribution equal to the distribution of a column update. This distribution [see (2.1)] is uniform on the interval  $[a, b]$  and decays exponentially outside it. Hence the mixing time of this random walk, starting from a position at distance  $\ell$  from the interval  $[a, b]$ , is  $O((b - a)^2 + \ell)$ . But since  $\eta \notin B$ , both  $(b - a)$  and  $\ell$  are bounded by  $2D \log^2 n$ , so the mixing time is  $O((D \log^2 n)^2)$ . Thus  $D^2 \log^6 n$  single-site updates at position  $i$  suffice to simulate a single column update with very small error  $e^{-\Omega(\log^2 n)}$ , which is the second term in the bound. The factor  $M$  comes from a union bound over steps of the column dynamics.

In order to use Claim 4.4, we need to bound  $\nu_s^{\text{OE}}(B)$  [and, symmetrically,  $\nu_s^{\text{EO}}(B)$ ], the probability of the dynamics creating a large gradient. This is in general a nontrivial task because it requires detailed nonequilibrium information about the contours. However, it is here that our choice of the initial distribution  $\nu_0 = \mu(\cdot | A)$  becomes crucial. Since  $\mu$  remains invariant under any number of steps of  $\mathcal{M}_n^{\text{OE}}$  (and of  $\mathcal{M}_n^{\text{EO}}$ ), we can write, for any  $s$ ,

$$(4.4) \quad \nu_s^{\text{OE}}(B) \leq \mu(B) / \mu(A),$$

with an identical bound for  $\nu_s^{\text{EO}}(B)$ . But the right-hand side here is easy to evaluate as it is the ratio of the probabilities of two events *in equilibrium!* In particular, the following straightforward bound is proved in part (c) of Lemma C.1 in the [Appendix](#):

$$\mu(B) \leq n^a e^{-(D \log^2 n)/c}$$

for some constants  $a, c > 0$ . Hence, thanks to the definition of  $D$ ,

$$(4.5) \quad \mu(B) / \mu(A) \leq e^{-\Omega(\log^2 n)}.$$

We can now put everything together. For each  $\mathbf{i} \in \Sigma$ , the quantity in (4.2) is bounded by

$$(4.6) \quad \left\| \frac{1}{2}(\nu_{\mathbf{OE}(\mathbf{i})} + \nu_{\mathbf{EO}(\mathbf{i})}) - \nu_M^{\text{par}} \right\| + \left\| \nu_M^{\text{par}} - \mu \right\|,$$

where  $\nu_3^{\text{par}}$  denotes the distribution obtained from  $\nu_0$  after  $s$  steps of the parallel column dynamics. By Claim 4.4 and inequalities (4.4) and (4.5), the first term in (4.6) is bounded by  $e^{-\Omega(\log^2 n)}$ , which is certainly  $o(1/n^b)$  for any fixed  $b$ , while the second term is  $o(1/n^b)$  by Theorem 3.4 and Remark 3.5 and the fact that  $M \gg n^2 \log n$  (the mixing time of  $\mathcal{M}_h^{\text{par}}$ ). Hence the variation distance of the dynamics is  $o(1/n^b)$ , as claimed in the lemma.  $\square$

4.2. *From maximal height to equilibrium.* In this subsection we show that, after at most  $\tilde{O}(n^{3.5})$  steps, the single-site dynamics starting from the maximal configuration (in which the contour has height  $n$  everywhere) reaches equilibrium. For convenience we will work throughout this subsection with a slightly modified model in which the set of allowed heights is  $\mathbb{N}$  rather than  $[0, n]$ . The equilibrium distribution  $\mu$  for this model is defined exactly as in (1.1), where the partition function  $Z_\beta$  is appropriately defined. (Note that  $Z_\beta$  remains bounded for any  $\beta > 0$ .) We will show that the variation distance between the contour at height  $n$  and the equilibrium contour in this model becomes very small in  $\tilde{O}(n^{3.5})$  steps. This immediately implies the same result for our original model with height set  $[0, n]$  because of monotonicity and the fact that the variation distance between the two equilibrium distributions is exponentially small in  $n$  (see Remark C.2 in the Appendix). We will use  $\Omega_n^\infty$  to denote the set of configurations with height set  $\mathbb{N}$ . We note also that our basic building block, Lemma 4.2, is easily seen to hold in this setting also.

The main ingredient in this subsection is the following lemma, which says roughly that an initial contour at height  $h \leq \sqrt{n}$  drops to height approximately  $h/2$  after  $\tilde{O}(n^3)$  steps.

LEMMA 4.5. *Let  $C(\beta) \log n \leq h \leq \sqrt{n}$ , where  $C(\beta)$  is a specific constant depending only on  $\beta$ . Let  $\nu_t$  be the distribution at time  $t$  of the single-site dynamics started from  $\mu$  conditioned on the event  $A_h := \{\eta(i) \geq h \ \forall i \in [1, n]\}$ . Then there exists a time  $t_n = \tilde{O}(n^3)$  such that, for any increasing function  $f : \Omega_n^\infty \mapsto \mathbf{R}$  with  $\|f\|_\infty \leq 1$ ,*

$$(4.7) \quad \nu_{t_n}(f) \leq \mu(f|_{A_{h/2}}) + o(1/n),$$

where the term  $o(1/n)$  is independent of  $f$ .

PROOF. In the proof, we will make use of the single-site dynamics on the enlarged interval  $[-\ell + 1, n + \ell]$ , with boundary conditions at positions  $-\ell$  and  $n + \ell + 1$ . The parameter  $\ell \leq n$  will be chosen later in such a way that the equilibrium probability of the event  $A_h$  in the enlarged interval is  $\Omega(1/\text{poly}(n))$ . We may construct a monotone coupling of this dynamics with our original one by choosing the position  $i$  to be updated from the enlarged interval  $[-\ell + 1, n + \ell]$ , and doing nothing in the original dynamics if  $i \notin [1, n]$ . Plainly this slows down the original dynamics by at most a factor of 3 and so does not affect our results.



Now consider the enlarged dynamics started in its equilibrium distribution  $\mu^{(\ell)}$  conditioned on the event  $A_h$ . Denote its distribution at time  $t$  by  $\nu_t^{(\ell)}$ . By part (b) of Lemma C.1 in the Appendix, we have

$$\mu^{(\ell)}(A_h) \geq \frac{1}{cn^a} e^{-ch^2/\ell}$$

for constants  $a, c > 0$ . Moreover the event  $A_h$  is clearly increasing, and therefore Lemma 4.2 applied to the enlarged dynamics implies that for a suitable time  $t = \tilde{O}(n^3 \frac{h^4}{\ell^2})$  the variation distance between  $\nu_t^{(\ell)}$  and  $\mu^{(\ell)}$  is  $o(1/n)$ . If we now set the free parameter  $\ell$  equal to  $\lfloor \frac{\delta h^2}{\log n} \rfloor$ , where  $\delta \ll 1$  will be fixed later, we have that  $t = \tilde{O}(n^3)$ . (Note also that  $\ell \leq n$ , as stipulated earlier; this is why we require the upper bound on  $h$ .) Thus we may take  $t_n := t = \tilde{O}(n^3)$  and get that, for any  $f$  as in the statement of the lemma,

$$(4.8) \quad \nu_{t_n}(f) \leq \nu_{t_n}^{(\ell)}(f) \leq \mu^{(\ell)}(f) + o(1/n).$$

We now bound  $\mu^{(\ell)}(f)$ . Let  $E_h = \{\eta \in \Omega_n^\infty : \max(\eta(1), \eta(n)) \leq h\}$ . Then

$$(4.9) \quad \begin{aligned} \mu^{(\ell)}(f) &\leq \mu^{(\ell)}(f|E_{h/2}) + \mu^{(\ell)}(E_{h/2}^c) \\ &\leq \mu(f|A_{h/2}) + \mu^{(\ell)}(E_{h/2}^c). \end{aligned}$$

Now by part (a) of Lemma C.1 in the Appendix, provided  $h/2\ell \leq \beta/2$ , we have

$$\mu^{(\ell)}(E_{h/2}^c) \leq n^a e^{-h^2/c\ell}$$

for constants  $a, c > 0$ . By choosing the constant  $\delta$  in our definition of  $\ell$  small enough, we can make this latter quantity  $o(1/n)$ , which via (4.9) and (4.8) yields the desired bound (4.7). Finally, the condition  $h/2\ell \leq \beta/2$  translates to  $h \geq \frac{1}{\delta\beta} \log n$ , that is,  $h \geq C(\beta) \log n$ .  $\square$

A simple iterative application of Lemma 4.5 yields the following:

**COROLLARY 4.6.** *In the setting of Lemma 4.5, for any integer  $j$  such that  $2^{-j}h \geq C(\beta) \log n$  we have*

$$(4.10) \quad \nu_{jt_n}(f) \leq \mu(f|A_{h/2^j}) + o(j/n).$$

**PROOF.** Write  $\nu_t^{(h)}$  for the distribution of the single-site dynamics at time  $t$ , started in  $\mu(\cdot|A_h)$ . Let  $f^{(j-1)}$  be the function on  $\Omega_n^\infty$  obtained by applying the transition matrix of the single-site dynamics  $(j-1)t_n$  times to the original function  $f$ . Clearly  $f^{(j-1)}$  is still increasing with  $\|f^{(j-1)}\| \leq 1$ , so Lemma 4.5 yields

$$\begin{aligned} \nu_{jt_n}^{(h)}(f) &= \nu_{t_n}^{(h)}(f^{(j-1)}) \leq \mu(f^{(j-1)}|A_{h/2}) + o(1/n) \\ &= \nu_{(j-1)t_n}^{(h/2)}(f) + o(1/n). \end{aligned}$$

Iterating over  $j$  completes the proof.  $\square$

We are now in a position to prove our first main result, which says that the mixing time of the single-site dynamics starting at height  $\sqrt{n}$  is  $\tilde{O}(n^3)$ .

**THEOREM 4.7.** *Let  $A = \{\eta \in \Omega_n^\infty : \eta(i) \geq \sqrt{n} \ \forall i \in [1, n]\}$ , and let  $\nu_t$  be the distribution at time  $t$  of the single-site dynamics started in the distribution  $\mu(\cdot | A)$ . Then for some time  $t_n = \tilde{O}(n^3)$  we have*

$$\|\nu_{t_n} - \mu\| = o(1/\sqrt{n}).$$

**PROOF.** The event  $A$  is increasing, so the relative density between the initial distribution and the equilibrium one given by  $g(\eta) := \frac{\mu(\eta|A)}{\mu(\eta)} = \frac{\chi(\eta \in A)}{\mu(A)}$  is also increasing. As shown in [22] the same holds for  $g_t(\eta) = \nu_t(\eta)/\mu(\eta)$ . In particular the event  $U_t = \{\eta : \nu_t(\eta) \geq \mu(\eta)\}$  is increasing.

By Corollary 4.6 with  $j = O(\log n)$  there exists a time  $s = \tilde{O}(n^3)$  such that

$$\nu_s(f) \leq \mu(f | A_{\lceil C(\beta) \log n \rceil}) + o((\log n)/n)$$

for any increasing function  $f$  with  $\|f\|_\infty \leq 1$ , where  $A_h$  is defined as in Lemma 4.5. Thus, for any  $t > 0$ , we can bound the probability  $\nu_{s+t}(U_{s+t})$  by

$$\nu_{s+t}(U_{s+t}) \leq \tilde{\nu}_t(U_{s+t}) + o(1/\sqrt{n}),$$

where  $\tilde{\nu}_t$  is the distribution of the Glauber chain at time  $t$  started from the equilibrium distribution  $\mu$  conditioned on the event  $A_{\lceil C(\beta) \log n \rceil}$ . It follows from the proof of part (b) of Lemma C.1 that  $\log\left(\frac{1}{\mu(A_{\lceil C(\beta) \log n \rceil})}\right) = O(\log n)$ . Therefore, by Lemma 4.2 we have that, for some  $t = \tilde{O}(n^3)$ ,

$$\tilde{\nu}_t(U_{s+t}) = \mu(U_{s+t}) + o(1/\sqrt{n}).$$

In conclusion, setting  $t_n := s + t = \tilde{O}(n^3)$  we get

$$\|\nu_{t_n} - \mu\| = \nu_{t_n}(U_{t_n}) - \mu(U_{t_n}) = o(1/\sqrt{n}),$$

which completes the proof.  $\square$

A simple iterative application of the above theorem shows that, starting at height  $n$ , the single-site dynamics reaches equilibrium with a further  $O(\sqrt{n})$  factor overhead, that is, in total time  $\tilde{O}(n^{3.5})$ .

**THEOREM 4.8.** *Let  $B = \{\eta \in \Omega_n^\infty : \eta(i) \geq n \ \forall i \in [1, n]\}$ , and let  $\nu_t$  be the distribution at time  $t$  of the single-site dynamics started from  $\mu(\cdot | B)$ . Then for some time  $t_n = \tilde{O}(n^{3.5})$  we have*

$$\|\nu_{t_n} - \mu\| = o(1).$$

**REMARK 4.9.** By monotonicity, the theorem immediately implies the same conclusion for the single-site dynamics started in the (maximal) configuration  $\eta(i) = n$  for all  $i \in [1, n]$ .

PROOF. Let us define a new height set  $H^{(1)} := [n - \sqrt{n}, \infty]$ , let  $\Omega_n^{(1)} := \{\eta \in \Omega_n^\infty : \eta(i) \in H^{(1)} \forall i \in [1, n]\}$ , and let  $\mu^{(1)}$  be the equilibrium distribution on  $\Omega_n^{(1)}$  given by  $\mu(\cdot | \Omega_n^{(1)})$ . Let  $v_t^{(1)}$  be the distribution at time  $t$  of the single-site dynamics on  $\Omega_n^{(1)}$  started from  $\mu^{(1)}(\cdot | B)$  and with boundary conditions  $\eta(0) = \eta(n + 1) = n - \sqrt{n}$ .

By monotonicity and Theorem 4.7 applied to  $v_t^{(1)}$  we have that, after time  $t = \tilde{O}(n^3)$ , for any increasing function  $f$  with  $\|f\|_\infty \leq 1$ ,

$$v_t(f) \leq v_t^{(1)}(f) \leq \mu^{(1)}(f) + o(1/\sqrt{n}).$$

If we now define  $H^{(2)} := [n - 2\sqrt{n}, \infty]$  and  $\Omega_n^{(2)}, \mu^{(2)}$  analogously, we get that after a further  $t$  steps the distribution  $v_{2t}$  of the original chain satisfies

$$v_{2t}(f) \leq \mu^{(2)}(f) + o(1/\sqrt{n}) + o(1/\sqrt{n}).$$

Iterating  $\sqrt{n}$  times shows that at time  $t_n := \sqrt{nt} = \tilde{O}(n^{3.5})$  we have

$$v_{t_n}(f) \leq \mu(f) + o(1).$$

The proof is complete once we apply the above inequality to the indicator of the increasing set  $U = \{\eta \in \Omega_n^\infty : v_{t_n}(\eta) \geq \mu(\eta)\}$ .  $\square$

4.3. *From zero height to equilibrium.* In this subsection we prove the complementary fact that the single-site dynamics starting from the *minimal* configuration (in which all heights are zero) reaches equilibrium in  $\tilde{O}(n^3)$  steps. In our argument we will make use of auxiliary versions of the dynamics in which certain heights are fixed to be zero. Specifically, for any integer  $m$ , let  $\mathcal{M}_{n,m}^{ss}$  denote the single-site dynamics defined as before, except that  $\eta(i)$  is constrained always to be zero for  $i \in \{jm : j = 1, 2, \dots, \lfloor n/m \rfloor\}$ . Let  $\mu^{(m)}$  denote its stationary distribution. Clearly  $\mu^{(m)}$  is equivalent to  $\mu(\cdot | A_m)$ , where  $A_m$  is the event that  $\eta(i) = 0$  at the above positions  $i$ ; moreover,  $\mu^{(m)}$  is a product of  $\lfloor n/m \rfloor + 1$  unconditioned SOS Gibbs distributions each on an interval of length (at most)  $m - 1$ .

The idea in the proof is to control the evolution of the contour by coupling it with the sequence of dynamics  $\mathcal{M}_{n,m}^{ss}$  for  $m = 2, 4, 8, \dots$ , so that the number of positions with height fixed to zero is successively halved. At each stage in the sequence, we will allow  $\mathcal{M}_{n,m}^{ss}$  to reach its equilibrium distribution  $\mu^{(m)}$ . Initially, in the minimum configuration, all heights are zero; ultimately we will reach equilibrium with no heights fixed to zero, which is our desired SOS equilibrium distribution  $\mu$ .

The main ingredient in our proof is the following lemma, which says that if we start in the equilibrium distribution  $\mu^{(m)}$  (with every  $m$ th height fixed to zero), then after  $\tilde{O}(n^3)$  steps of  $\mathcal{M}_{n,2m}^{ss}$  we will reach the equilibrium distribution  $\mu^{(2m)}$  [with every  $(2m)$ th height fixed to zero].

LEMMA 4.10. *Let  $v_t$  denote the distribution at time  $t$  of  $\mathcal{M}_{n,2m}^{ss}$  started in the distribution  $\mu^{(m)} = \mu^{(2m)}(\cdot | A_m)$ . Then for some time  $s_n = \tilde{O}(n^3)$  we have*

$$\|v_{s_n} - \mu^{(2m)}\| = o(1/n).$$

PROOF. Note that we can view  $\mathcal{M}_{n,2m}^{ss}$  as a collection of  $r = \lfloor n/(2m) \rfloor$  independent standard dynamics on intervals of length  $2m - 1$ , with zero boundary conditions at positions  $2mj, 2m(j + 1)$ . Let us focus on the dynamics restricted to one such interval  $I_j$ . Let  $\tilde{\mu}$  denote the stationary distribution within  $I_j$ , and  $\tilde{A}$  the event  $A_m$  restricted to  $I_j$  [i.e.,  $\tilde{A}$  is just the event  $\eta(2mj + m) = 0$ ]. Clearly  $\tilde{A}$  is a decreasing event, and standard random walk arguments [28] imply that  $\tilde{\mu}(\tilde{A}) \geq \frac{1}{c}m^{-c}$  for a constant  $c > 0$ . Hence a slight modification of Lemma 4.2 applied to the dynamics within  $I_j$  implies that, after  $t = O(m^3 \log^{10} n)$  steps of this dynamics, the variation distance from  $\tilde{\mu}$  is  $o(1/n^2)$ . (The presence of  $\log n$  rather than  $\log m$  here is to ensure a variation distance that depends on  $n$ .)

Returning now to the full dynamics  $\mathcal{M}_{n,m}^{ss}$ , suppose we execute sufficiently many steps  $T$  that at least the above number  $t$  updates are performed within each interval  $I_j$  for  $j = 1, 2, \dots, r$ . Since  $\mu^{(2m)}$  is a product distribution, this will ensure that the variation distance  $\|v_T - \mu^{(2m)}\|$  is  $o(r/n^2) = o(1/n)$ . But by a Chernoff bound it suffices to take  $T = 2tr = \tilde{O}(nm^2) = \tilde{O}(n^3)$  in order to ensure the above condition with probability  $1 - \exp(-\Omega(\log^{10} n)) = 1 - o(1/n)$ . Taking  $s_n = T$  completes the proof.  $\square$

An iterative application of the above lemma now proves the main result of this subsection, which is the analog of Theorem 4.8 starting from the minimal configuration.

THEOREM 4.11. *Let  $v_t$  be the distribution at time  $t$  of the single-site dynamics started from the minimal configuration. Then for some time  $t_n = \tilde{O}(n^3)$  we have*

$$\|v_{t_n} - \mu\| = o(1).$$

PROOF. Let  $\tilde{v}_t$  be the distribution at time  $t$  of the following dynamics, starting from the minimal configuration. For the first  $s_n$  steps [where  $s_n = \tilde{O}(n^3)$  is as defined in Lemma 4.10], run the dynamics  $\mathcal{M}_{n,2}^{ss}$ ; for the next  $s_n$  steps run the dynamics  $\mathcal{M}_{n,4}^{ss}$ ; and so on (i.e., run  $s_n$  steps of each dynamics  $\mathcal{M}_{n,2j}^{ss}$  for  $j = 1, 2, \dots$ ). Note that the distribution of the initial configuration is exactly  $\mu^{(1)}$ . Thus by Lemma 4.10 we have  $\|\tilde{v}_{s_n} - \mu^{(2)}\| = o(1/n)$ . Similarly, applying Lemma 4.10 iteratively implies that

$$\|\tilde{v}_{js_n} - \mu^{(2^j)}\| = o(j/n)$$

for  $j = 1, 2, \dots$ . Since  $\mu^{(m)} = \mu$  for  $m > n$ , we may set  $j = \lceil \log n \rceil$  and  $t_n := \lceil \log n \rceil s_n = \tilde{O}(n^3)$  to conclude

$$(4.11) \quad \|\tilde{v}_{t_n} - \mu\| = o((\log n)/n) = o(1).$$

Finally, note that by monotonicity we have  $\tilde{v}_t \leq v_t \leq \mu$  for all  $t$ . Hence (4.11) implies  $\|v_{t_n} - \mu\| = o(1)$ , as required.  $\square$

4.4. *Proof of Theorem 4.1.* The proof of Theorem 4.1 is now easily deduced from Theorems 4.8 and 4.11. Let  $v_t^{\max}$  and  $v_t^{\min}$  denote the distributions of the single-site dynamics at time  $t$  starting in the maximal and minimal configurations, respectively. By Proposition 2.1, to prove the theorem it suffices to show that for some time  $t = \tilde{O}(n^{3.5})$  we have  $\|v_t^{\max} - v_t^{\min}\| = o(1/n^2)$ . But Theorem 4.8 shows that, for some  $t' = \tilde{O}(n^{3.5})$ ,  $\|v_{t'}^{\max} - \mu\| = o(1)$ , and Theorem 4.11 that  $\|v_{t'}^{\min} - \mu\| = o(1)$ . Hence  $\|v_{t'}^{\max} - v_{t'}^{\min}\| = o(1)$ . It is now enough to observe that  $\gamma(t) := \max\{\|v_{t'}^{\max} - \mu\|, \|v_{t'}^{\min} - \mu\|\}$  satisfies  $\gamma(t + s) \leq 4\gamma(t)\gamma(s)$  (see, e.g., Corollary 2.7 in [21]) to conclude that, for  $t = c \log(n)t'$  with  $c$  large enough,  $\|v_t^{\max} - v_t^{\min}\| = o(1/n^2)$ .

REMARK 4.12. It should be clear that our result generalizes to the SOS dynamics on a region  $[1, n] \times [0, h]$ , in which the allowed height set is  $[0, h]$ . The mixing time for the single-site dynamics is then  $\tilde{O}(\max\{n^3, n^{2.5}h\})$ . The only difference from the  $h = n$  case is in the analysis of the maximal contour: by an argument analogous to that in Theorem 4.7, we achieve a height reduction of  $\sqrt{n}$  in  $\tilde{O}(n^3)$  steps, yielding the above bound.

### APPENDIX A: PROOFS OF LEMMA 3.2 AND COROLLARY 3.3

PROOF OF LEMMA 3.2. Abusing notation, we write  $\eta$  for  $\eta_{t+1}(i)$  and abbreviate  $E[\cdot | a, b]$  and  $\Pr[\cdot | a, b]$  to  $E[\cdot]$  and  $\Pr[\cdot]$ . We also introduce the notation  $S_m$  for the sum  $\sum_{j=1}^m e^{-2\beta j} = S_\infty(1 - e^{-2\beta m})$ , where  $S_\infty := e^{-2\beta}/(1 - e^{-2\beta})$ , and  $S_0 = 0$ . Note that the normalizing factor in (2.1) can then be written as

$$\begin{aligned} Z &= \sum_{j=0}^{a-1} e^{-\beta(b-a)-2\beta(a-j)} + \sum_{j=a}^b e^{-\beta(b-a)} + \sum_{j=b+1}^n e^{-\beta(b-a)-2\beta(j-b)} \\ &= e^{-\beta(b-a)}(S_a + (b - a + 1) + S_{n-b}). \end{aligned}$$

Our goal is to evaluate  $E[\eta_{t+1}(i) | a, b] = E[\eta]$ , which we may write as

$$E[\eta] = \Pr[\eta \leq a + b]E[\eta | \eta \leq a + b] + \Pr[\eta > a + b]E[\eta | \eta > a + b].$$

Since the conditional distribution  $\Pr[\cdot | \eta \leq a + b]$  is symmetric w.r.t. to the point  $\frac{a+b}{2}$  then  $E[\eta | \eta \leq a + b] = (a + b)/2$ , while

$$\begin{aligned} E[\eta | \eta > a + b] &= \left( \sum_{j=a+b+1}^n j e^{-2\beta j + \beta(a+b)} \right) / \left( \sum_{j=a+b+1}^n e^{-2\beta j + \beta(a+b)} \right) \\ &= \left( \sum_{k=1}^{n-(a+b)} (a + b + k) e^{-2\beta k} \right) / \left( \sum_{k=1}^{n-(a+b)} e^{-2\beta k} \right) \\ &= a + b + T_{n-(a+b)}, \end{aligned}$$

where  $T_m := (\sum_{j=1}^m j e^{-2\beta j}) / S_m = 1/(1 - e^{-2\beta}) - m e^{-2\beta m} / (1 - e^{-2\beta m})$ . Therefore, we have

$$E[\eta] = \frac{a + b}{2} + \varepsilon(a, b),$$

where

$$(A.1) \quad \varepsilon(a, b) = \Pr[\eta > a + b]((a + b)/2 + T_{n-(a+b)}).$$

Since plainly  $\varepsilon(a, b) \geq 0$ , this gives the first part of the lemma.

To prove the second part, we claim it suffices to show that  $\varepsilon(a, b)$  is a decreasing function of  $a$  and of  $b$  (subject to  $a \leq b$  and  $a + b \leq n$ ). To see this, consider any pair  $(c, d)$  such that  $c \leq \min\{a, d\} \leq \max\{a, d\} \leq b$ . (Note that this also implies  $c + d \leq n$ .) If  $a \leq d$  then using monotonicity in  $b$  and then in  $a$  we have  $\varepsilon(a, b) \leq \varepsilon(a, d) \leq \varepsilon(c, d)$ ; if on the other hand  $d \leq a$  then we have similarly  $\varepsilon(a, b) \leq \varepsilon(c, b) \leq \varepsilon(c, d)$ .

To show that  $\varepsilon$  is decreasing with  $b$ , note first that

$$\Pr[\eta > a + b] = \frac{e^{-2\beta a} S_{n-(a+b)}}{S_a + (b - a + 1) + S_{n-b}}.$$

Plugging this into (A.1) and differentiating w.r.t.  $b$  gives

$$(A.2) \quad \begin{aligned} \frac{\partial}{\partial b} \varepsilon(a, b) &= \left( \frac{1}{2} - T'_{n-(a+b)} \right) \Pr[\eta > a + b] \\ &\quad + ((a + b)/2 + T_{n-(a+b)}) \frac{\partial}{\partial b} \Pr[\eta > a + b], \end{aligned}$$

where  $\frac{\partial}{\partial b} \Pr[\eta > a + b]$  is given by the expression

$$e^{-2\beta a} \frac{-S'_{n-(a+b)}(S_a + (b - a + 1) + S_{n-b}) - (1 - S'_{n-b})S_{n-(a+b)}}{[S_a + (b - a + 1) + S_{n-b}]^2}.$$

(Here we are viewing  $S_m$  and  $T_m$  as continuous functions of the parameter  $m$ .) Note that  $S'$  is nonnegative, and that

$$(A.3) \quad S'_{n-(a+b)} S_{n-b} \geq S'_{n-b} S_{n-(a+b)}$$

because

$$\begin{aligned} S'_{n-(a+b)} S_{n-b} &= 2\beta S_\infty^2 e^{-2\beta(n-(a+b))} (1 - e^{-2\beta(n-b)}), \\ S'_{n-b} S_{n-(a+b)} &= 2\beta S_\infty^2 e^{-2\beta(n-b)} (1 - e^{-2\beta(n-(a+b))}). \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial b} \Pr[\eta > a + b] \leq -\frac{e^{-2\beta a} S_{n-(a+b)}}{[S_a + (b - a + 1) + S_{n-b}]^2} = -\frac{\Pr[\eta > a + b]}{S_a + (b - a + 1) + S_{n-b}}.$$

Plugging this into (A.2), and noting that  $T'$  is also nonnegative, we get

$$(A.4) \quad \frac{\partial}{\partial b} \varepsilon(a, b) \leq \frac{1}{2} \Pr[\eta > a + b] \left( 1 - \frac{a + b + 2T_{n-(a+b)}}{S_a + (b - a + 1) + S_{n-b}} \right).$$

We will therefore be done if we can show

$$a + b + 2T_{n-(a+b)} \geq S_a + (b - a + 1) + S_{n-b}.$$

We do this in two steps, as follows:

$$(A.5) \quad a + b + 2T_{n-(a+b)} \geq a + b + 1 + S_{n-(a+b)} \geq S_a + (b - a + 1) + S_{n-b}.$$

Note that we may assume  $n - (a + b) \geq 1$  since if  $a + b = n$ , then  $\varepsilon(a, b) = 0$ .

*Proof of first inequality in (A.5):* Setting  $m = n - (a + b)$ , we need to show that

$$(A.6) \quad S_m + 1 \leq 2T_m \quad \text{for } 1 \leq m \leq n.$$

Recall that  $S_m = S_\infty(1 - e^{-2\beta m})$ , where  $S_\infty = e^{-2\beta}/(1 - e^{-2\beta})$ , and that  $T_m = \sum_{j=1}^m j e^{-2\beta j}/S_m$ . A simple calculation shows that

$$T_m = \frac{1}{1 - e^{-2\beta}} - \frac{m e^{-2\beta m}}{1 - e^{-2\beta m}}.$$

Thus, writing  $x = e^{-2\beta}$ , the desired inequality in (A.6) becomes

$$\frac{x}{1 - x} (1 - x^m) + 1 \leq \frac{2}{1 - x} - \frac{2mx^m}{1 - x^m} \quad \text{for } 0 < x < 1.$$

Rearranging yields the equivalent expression

$$(A.7) \quad 1 - (2m + 1)x^m + (2m + 1)x^{m+1} - x^{2m+1} \geq 0.$$

Differentiating the left-hand side w.r.t.  $x$  gives

$$(A.8) \quad \begin{aligned} & -m(2m + 1)x^{m-1} + (m + 1)(2m + 1)x^m - (2m + 1)x^{2m} \\ & = -(2m + 1)x^{m-1} [m - (m + 1)x + x^{m+1}]. \end{aligned}$$

But the function  $f(x) := m - (m + 1)x + x^{m+1}$  is zero at  $x = 1$  and monotonically decreasing for  $0 \leq x < 1$ , so is always nonnegative on this interval. Therefore, the

derivative in (A.8) is nonpositive. Thus the left-hand side of (A.7) is a nonincreasing function of  $x$ , and is zero at  $x = 1$ ; hence the inequality holds, and (A.6) is proved.

*Proof of second inequality in (A.5):* We need to show

$$(A.9) \quad S_a + S_{n-b} - S_{n-(a+b)} \leq 2a.$$

The left-hand side here is equal to

$$S_\infty(1 - e^{-2\beta a} + e^{-2\beta(n-b)}(e^{2\beta a} - 1)).$$

For any fixed  $a \geq 0$ , this expression is maximized by taking  $n - b$  as small as possible, which means  $n - b = a + 1$  (since we are assuming  $a + b \leq n - 1$ ). Thus the expression is bounded above by

$$S_\infty(1 - e^{-2\beta a} + e^{-2\beta a}(e^{2\beta a} - 1)) = 2S_\infty(1 - e^{-2\beta a}).$$

Plugging this back into (A.9) means we need to prove  $2S_\infty(1 - e^{-2\beta a}) \leq 2a$ , or equivalently,

$$\frac{e^{-2\beta}}{1 - e^{-2\beta}}(1 - e^{-2\beta a}) \leq a.$$

Writing  $x = e^{-2\beta}$  and rearranging gives the equivalent inequality

$$a - (a + 1)x + x^{a+1} \geq 0 \quad \text{for } 0 < x < 1.$$

But the function on the LHS here is precisely the function  $f(x)$  in the previous part of the proof (with  $m$  replaced by  $a$ ), so we know that it is nonnegative throughout the desired interval. This completes the proof of (A.9), and also the proof that  $\frac{\partial}{\partial b} \varepsilon(a, b) \leq 0$ .

We now carry out a similar computation for  $\frac{\partial}{\partial a} \varepsilon(a, b)$  to get

$$(A.10) \quad \begin{aligned} \frac{\partial}{\partial a} \varepsilon(a, b) &= \left( \frac{1}{2} - T'_{n-(a+b)} \right) \Pr[\eta > a + b] \\ &\quad + ((a + b)/2 + T_{n-(a+b)}) \frac{\partial}{\partial a} \Pr[\eta > a + b], \end{aligned}$$

where

$$\begin{aligned} &\frac{\partial}{\partial a} \Pr[\eta > a + b] \\ &= -2\beta \Pr[\eta > a + b] \\ &\quad + e^{-2\beta a} \frac{-S'_{n-(a+b)}(S_a + (b - a + 1) + S_{n-b}) - (S'_a - 1)S_{n-(a+b)}}{[S_a + (b - a + 1) + S_{n-b}]^2}. \end{aligned}$$

Using (A.3) and the fact that  $S'$  is nonnegative, we get

$$(A.11) \quad \frac{\partial}{\partial a} \Pr[\eta > a + b] \leq \Pr[\eta > a + b] \left( -2\beta + \frac{1 - S'_a - S'_{n-b}}{S_a + (b - a + 1) + S_{n-b}} \right).$$



We claim that the term in parentheses here is bounded above by  $-\beta$ . To see this, note first that  $S'_m = 2\beta(S_\infty - S_m)$  for all  $m > 0$ . Thus the term in parentheses can be written

$$\frac{-4\beta S_\infty - 2\beta(b - a + 1) + 1}{S_a + S_{n-b} + (b - a + 1)}.$$

This is bounded by  $-\beta$  provided

$$4\beta S_\infty + 2\beta(b - a + 1) - \beta(S_a + S_{n-b} + (b - a + 1)) \geq 1.$$

Using the facts that  $S_\infty \geq S_a, S_b$ , and that  $b - a + 1 \geq 1$ , it is sufficient to show

$$\beta(2S_\infty + 1) \geq 0.$$

Plugging in  $S_\infty = e^{-2\beta}/(1 - e^{-2\beta})$  and rearranging, this is equivalent to

$$f(\beta) := (\beta - 1)e^{2\beta} + \beta + 1 \geq 0.$$

The function  $f$  is 0 at  $\beta = 0$ , and its first and second derivatives are  $e^{2\beta}(2\beta - 1) + 1$  and  $4\beta e^{2\beta}$ , respectively. Since the first derivative is 0 at  $\beta = 0$ , and the second derivative is nonnegative for all  $\beta \geq 0$ ,  $f$  must indeed be  $\geq 0$  throughout this range.

Replacing the parenthesis in (A.11) by the upper bound  $-\beta$ , and plugging this into (A.10), we get

$$\begin{aligned} \frac{\partial}{\partial a} \varepsilon(a, b) &\leq \frac{1}{2} \Pr[\eta > a + b](1 - 2T'_{n-(a+b)} - \beta(a + b + 2T_{n-(a+b)})) \\ &\leq \frac{1}{2} \Pr[\eta > a + b](1 - 2T'_m - 2\beta T_m), \end{aligned}$$

where  $m = n - (a + b)$ . We will thus be done if we can show

$$T'_m + \beta T_m \geq \frac{1}{2}.$$

Substituting for  $T_m$  and  $T'_m$ , writing  $y = 2\beta m$  and rearranging, we see that this is equivalent to

$$\frac{e^{-y}}{2(1 - e^{-y})^2} (y - 2 + (2 + y)e^{-y}) + \frac{\beta}{1 - e^{-2\beta}} \geq \frac{1}{2}.$$

But since  $\beta/(1 - e^{-2\beta}) \geq 1/2$  for all  $\beta > 0$ , it is sufficient to show that the function  $g(y) := y - 2 + (2 + y)e^{-y}$  is nonnegative for all  $y \geq 0$ . But  $g(y)$  is zero at  $y = 0$ , and its first and second derivatives are  $1 - e^{-y} - ye^{-y}$  and  $ye^{-y}$ , respectively. Since the first derivative is 0 at  $y = 0$ , and the second derivative is nonnegative for all  $y \geq 0$ ,  $g$  must indeed be nonnegative for all  $y \geq 0$ .

This completes the proof that  $\frac{\partial}{\partial a} \varepsilon(a, b) \leq 0$ , and hence also the proof of the lemma.  $\square$

PROOF OF COROLLARY 3.3. Note first that, by symmetry, if  $a + b \geq n$  then we get a complementary statement to Lemma 3.2 with  $\varepsilon(a, b)$  replaced by  $-\varepsilon(b', a')$ , where  $a' = n - a$ ,  $b' = n - b$ . (This corresponds to exchanging the roles of the top and bottom barriers.) We shall use both versions below.

Consider first the case  $a + b \leq n$  and  $c + d \leq n$ . Note that  $\eta_t \leq \xi_t$  implies that  $c \leq \min\{a, d\} \leq \max\{a, d\} \leq b$ . Then we have

$$E[\xi_{t+1}(i)|a, b] - E[\eta_{t+1}(i)|c, d] = \frac{a + b}{2} - \frac{c + d}{2} + \varepsilon(a, b) - \varepsilon(c, d),$$

and by Lemma 3.2 we can conclude that  $\varepsilon(a, b) \leq \varepsilon(c, d)$ , as required.

Now consider the case  $a + b \geq n$  and  $c + d \geq n$ . In this case we can change variables to  $\eta'(i) = n - \eta(i)$ ,  $\xi'(i) = n - \xi(i)$ ,  $a' = n - b$ ,  $b' = n - a$ ,  $c' = n - d$ ,  $d' = n - c$ . Note that  $a' \leq \min\{b', c'\} \leq \max\{b', c'\} \leq d'$ . Then we have

$$\begin{aligned} E[\xi_{t+1}(i)|a, b] - E[\eta_{t+1}(i)|c, d] &= E[\eta'_{t+1}(i)|c', d'] - E[\xi'_{t+1}(i)|a', b'] \\ &= \frac{a + b}{2} - \frac{c + d}{2} + \varepsilon(c', d') - \varepsilon(a', b'), \end{aligned}$$

and by Lemma 3.2 we can conclude that  $\varepsilon(c', d') \leq \varepsilon(a', b')$ , as required.

Finally, if  $a + b \geq n$  but  $c + d \leq n$  then we apply the above change of variables only to  $\eta$ ,  $a$  and  $b$  to get

$$\begin{aligned} E[\xi_{t+1}(i)|a, b] - E[\eta_{t+1}(i)|c, d] &= n - E[\xi'_{t+1}(i)|b', a'] - E[\eta_{t+1}(i)|c, d] \\ &= \frac{a + b}{2} - \frac{c + d}{2} - \varepsilon(a', b') - \varepsilon(c, d), \end{aligned}$$

and use the fact that both  $\varepsilon(a', b')$  and  $\varepsilon(c, d)$  are nonnegative.  $\square$

### APPENDIX B: PROOF OF CLAIM 4.4

Let  $\tilde{\mu}$  be any probability distribution on  $\Omega_n$  or  $\Omega_n^{(\infty)}$ , and let  $\mathbf{w} = \{i_1, \dots, i_k\}$  be a sequence of even positions in which every even position  $i \in [1, n]$  appears at least  $D^2 \log^6 n$  times. Let  $\tilde{\mu}_{\text{even}}$  be the new distribution obtained by applying to  $\tilde{\mu}$  one parallel update  $E$  of the even positions, and  $\tilde{\mu}_{\mathbf{w}}$  the distribution obtained by applying to  $\tilde{\mu}$  the single-site updates at the sequence of positions  $\mathbf{w}$ .

We claim that

$$(B.1) \quad \|\tilde{\mu}_{\text{even}} - \tilde{\mu}_{\mathbf{w}}\| \leq \tilde{\mu}(B) + e^{-\Omega(\log^2 n)},$$

where the  $\Omega(\log^2 n)$  term is independent of  $\tilde{\mu}$ . An entirely analogous statement will clearly also hold for a parallel update  $O$  of the odd positions. This will complete the proof by a simple induction over the number of epochs  $M$ , since it ensures that every step of  $\mathcal{M}_n^{\text{OE}}$  and  $\mathcal{M}_n^{\text{EO}}$  (the constituent chains of  $\mathcal{M}_n^{\text{par}}$ ) is simulated correctly with error at most  $\tilde{\mu}(B) + e^{-\Omega(\log^2 n)}$ , where  $\tilde{\mu}$  is the current distribution of the respective chain and clearly satisfies  $\tilde{\mu}(B) \leq \max_s \{v_s^{\text{OE}}(B) + v_s^{\text{EO}}(B)\}$ .

To prove (B.1), let  $\eta \notin B$  be a configuration without large gradients, which accounts for the term  $\tilde{\mu}(B)$  on the right-hand side. We need to show that, for each even position  $i \in [1, n]$ , applying at least  $D^2 \log^6 n$  single-site updates to  $i$  is equivalent to applying one column update at  $i$ , except for an error  $e^{-\Omega(\log^2 n)}$ . Recall that a column update at  $i$  replaces the height  $\eta(i)$  by a random height drawn from the distribution (2.1). Note that this distribution is uniform on the interval  $[a, b]$ , where  $a = \min\{\eta(i-1), \eta(i+1)\}$  and  $b = \max\{\eta(i-1), \eta(i+1)\}$ , and decays exponentially outside this interval (at a rate that depends on  $\beta$ ). On the other hand, under the single-site updates the height at  $i$  performs a nearest neighbor random walk on  $\mathbb{N}$ , that is, reversible w.r.t. this distribution, and of course converges to it. If we can show that the random walk starting at  $\eta(i)$  is very close to equilibrium after  $D^2 \log^6 n$  steps, we will have proved the Claim.

Now we claim that the mixing time of the above random walk, starting at a position at distance  $\ell$  from the interval  $[a, b]$ , is  $O((b-a)^2 + \ell)$ . To see this, note that the random walk is symmetric on  $[a, b]$  and has a uniform drift toward this interval from everywhere outside it. The term  $O(\ell)$  in the mixing time reflects the time to reach  $[a, b]$ , which is linear because of the drift, and the term  $O((b-a)^2)$  is the mixing time starting inside the interval, which is essentially the same as that of symmetric random walk on  $[a, b]$ . This can be verified formally by showing that the walk started at distance  $\ell$  from  $[a, b]$  can be coupled with a stationary walk so that the two walks coalesce with constant probability in the interval  $[a, b]$  after  $O((b-a)^2 + \ell)$  steps. But the fact that  $\eta \notin B$  implies that  $(b-a) \leq 2D \log^2 n$  and  $\ell \leq D \log^2 n$ , so the above coupling time is  $O((D \log^2 n)^2)$ . Hence  $D^2 \log^6 n$  steps suffice to simulate a column update with error  $e^{-\Omega(\log^2 n)}$ . This completes the proof of (B.1) and hence of the Claim.

### APPENDIX C: EQUILIBRIUM BOUNDS

The lemma below contains bounds on the probabilities of various events under the equilibrium distribution  $\mu$  that are used extensively in Section 4. We state and prove the lemma for the case where  $\mu$  is the equilibrium distribution of the SOS model with height set  $\mathbb{N}$  (which we use in Section 4.2). However, essentially the same bounds hold for the SOS model with height set  $[0, n]$  once we observe that the variation distance between the two equilibrium distributions is exponentially small in  $n$  (see Remark C.2 below for details).

LEMMA C.1. *Let  $\mu$  be the equilibrium distribution for the SOS model on  $[1, n]$  with height set  $\mathbb{N}$ . There exist positive constants  $a, c$  such that:*

- (a) *for any  $h \geq 0$  and  $\ell \in [1, n]$  with  $h/\ell \leq \beta/2$ ,*

$$\mu(\eta(\ell) \geq h) \leq n^a e^{-h^2/c\ell};$$

(b) for any  $h \geq 0$ ,  $0 \leq \ell \leq \lfloor n/2 \rfloor$ , and  $A = \{\eta \in \Omega_n : \eta(i) \geq h \ \forall i \in [\ell + 1, n - \ell]\}$ ,

$$\mu(A) \geq \frac{1}{cn^a} e^{-ch^2/\ell};$$

(c) if  $B = \{\eta : |\eta(i + 1) - \eta(i)| \geq d \text{ for some } i \in [0, n]\}$ , then

$$\mu(B) \leq cn^a e^{-d/c};$$

(d) if  $C = \{\forall i : \eta(i) \leq n\}$ , then

$$\mu(C^c) \leq cn^{a+1} e^{-n/c}.$$

REMARK C.2. It should be clear that the equilibrium distribution with height set  $[0, n]$  is nothing other than the distribution  $\mu$  conditioned on the event  $C$  above. Thus the variation distance between these two distributions is no larger than  $2\mu(C^c)$ , that is, it is exponentially small in  $n$ . In particular the bounds (a) and (c) above hold (possibly with different constants  $a', c'$ ) for height set  $[0, n]$ . The lower bound (b) also holds for height set  $[0, n]$  under the additional restriction that  $h^2/\ell \ll n$  [so that the variation distance between the two distributions is negligible compared to the r.h.s. in (b)].

PROOF. (a) Let  $Z_0 = 0$ , and let  $\{Z_j\}_{j \geq 1}$  be i.i.d. geometric random variables with random signs, so that  $\Pr[Z_j = z] = \Pr[Z_j = -z] \propto e^{-\beta z}$ . Define  $X(i) := \sum_{j=0}^i Z_j$  to be the symmetric random walk on  $\mathbb{Z}$  started at the origin whose increments are the  $Z_j$ .

Then the SOS equilibrium distribution  $\mu$  on  $[1, n]$  with height set  $\mathbb{N}$  can be written as

$$\mu(\eta) = \Pr[\{X(i)\}_{i=0}^{n+1} = \eta | X(n+1) = 0; X(i) \geq 0 \ \forall i \in [1, n]].$$

Standard random walk bounds show that

$$\Pr[X(n+1) = 0, X(i) \geq 0 \ \forall i \in [1, n]] \geq 1/n^a$$

for some constant  $a$ . Therefore, by Markov's inequality applied to the random variable  $\exp(\lambda \sum_i Z_i)$ , for any  $\lambda \in [0, \beta)$  we have

$$\begin{aligned} \mu(\eta(\ell) \geq h) &\leq n^a \Pr[X(\ell) \geq h] \\ &\leq n^a e^{-\lambda h} \mathbf{E}(e^{\lambda Z_1})^\ell \\ (C.1) \qquad &\leq n^a e^{-\lambda h} (1 + \frac{1}{2} \lambda^2 \mathbf{E}(e^{\lambda |Z_1|} Z_1^2))^\ell, \end{aligned}$$

where we used the inequality  $e^{\lambda x} \leq 1 + \lambda x + \frac{1}{2} \lambda^2 x^2 e^{\max(\lambda x, 0)}$  together with the fact that  $\mathbf{E}(Z_1) = 0$ .



(c) It is enough to write

$$\mu(B) \leq n \max_i \frac{\Pr(|Z_i| \geq d)}{\Pr[X(i) \geq 0 \forall i \in [1, n]; X(n) = 0 | X(0) = 0]} \leq e^{-\beta d} n^{\gamma+1/2},$$

again by standard random walk bounds.

(d) After a union bound over the index  $i \in [1, n]$  it is enough to repeat the arguments used in the proof of part (a) up to (C.1), and then choose the free parameter  $\lambda$  small enough (independent of  $n$ ).  $\square$

## REFERENCES

- [1] ALDOUS, D. (1983). Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on Probability, XVII. Lecture Notes in Math.* **986** 243–297. Springer, Berlin. [MR0770418](#)
- [2] BERGER, N., KENYON, C., MOSSEL, E. and PERES, Y. (2005). Glauber dynamics on trees and hyperbolic graphs. *Probab. Theory Related Fields* **131** 311–340. [MR2123248](#)
- [3] BIANCHI, A. (2008). Glauber dynamics on nonamenable graphs: Boundary conditions and mixing time. *Electron. J. Probab.* **13** 1980–2013. [MR2453553](#)
- [4] BODINEAU, T. and MARTINELLI, F. (2002). Some new results on the kinetic Ising model in a pure phase. *J. Stat. Phys.* **109** 207–235. [MR1927919](#)
- [5] CAPUTO, P., MARTINELLI, F. and TONINELLI, F. L. (2008). On the approach to equilibrium for a polymer with adsorption and repulsion. *Electron. J. Probab.* **13** 213–258. [MR2386733](#)
- [6] CAPUTO, P., MARTINELLI, F. and TONINELLI, F. L. (2011). Mixing times of monotone surfaces and SOS interfaces: A mean curvature approach. Preprint. Available at <http://arxiv.org/abs/1101.4190>.
- [7] CESI, F. (2001). Quasi-factorization of the entropy and logarithmic Sobolev inequalities for Gibbs random fields. *Probab. Theory Related Fields* **120** 569–584. [MR1853483](#)
- [8] CHAYES, L., SCHONMANN, R. H. and SWINDLE, G. (1995). Lifshitz’ law for the volume of a two-dimensional droplet at zero temperature. *J. Stat. Phys.* **79** 821–831. [MR1330364](#)
- [9] DIACONIS, P. and SALOFF-COSTE, L. (1993). Comparison theorems for reversible Markov chains. *Ann. Appl. Probab.* **3** 696–730. [MR1233621](#)
- [10] DOBRUSHIN, R., KOTECKÝ, R. and SHLOSMAN, S. (1992). *Wulff Construction: A Global Shape from Local Interaction. Translations of Mathematical Monographs* **104**. Amer. Math. Soc., Providence, RI. [MR1181197](#)
- [11] DUNLOP, F. M., FERRARI, P. A. and FONTES, L. R. G. (2002). A dynamic one-dimensional interface interacting with a wall. *J. Stat. Phys.* **107** 705–727. [MR1898854](#)
- [12] FUNAKI, T. (2005). Stochastic interface models. In *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1869** 103–274. Springer, Berlin. [MR2228384](#)
- [13] GIACOMIN, G. (2007). *Random Polymer Models*. Imperial College Press, London. [MR2380992](#)
- [14] HUSE, D. and FISHER, D. (1987). Dynamics of droplet fluctuations in pure and random Ising systems. *Phys. Rev. B* **35** 6841–6846.
- [15] LEVIN, D. A., LUCZAK, M. J. and PERES, Y. (2010). Glauber dynamics for the mean-field Ising model: Cut-off, critical power law, and metastability. *Probab. Theory Related Fields* **146** 223–265. [MR2550363](#)
- [16] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. [MR2466937](#)

- [17] LUBY, M., RANDALL, D. and SINCLAIR, A. (2001). Markov chain algorithms for planar lattice structures. *SIAM J. Comput.* **31** 167–192 (electronic). [MR1857394](#)
- [18] MARTINELLI, F. (1999). Lectures on Glauber dynamics for discrete spin models. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1997)*. *Lecture Notes in Math.* **1717** 93–191. Springer, Berlin. [MR1746301](#)
- [19] MARTINELLI, F. and OLIVIERI, E. (1994). Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case. *Comm. Math. Phys.* **161** 447–486. [MR1269387](#)
- [20] MARTINELLI, F., SINCLAIR, A. and WEITZ, D. (2004). Glauber dynamics on trees: Boundary conditions and mixing time. *Comm. Math. Phys.* **250** 301–334. [MR2094519](#)
- [21] MARTINELLI, F. and TONINELLI, F. L. (2010). On the mixing time of the 2D stochastic Ising model with “plus” boundary conditions at low temperature. *Comm. Math. Phys.* **296** 175–213. [MR2606632](#)
- [22] PERES, Y. (2005). Mixing for Markov chains and spin systems. Lecture Notes for PIMS Summer School at UBC, August 2005. Available at [www.stat.berkeley.edu/~peres/ubc.pdf](http://www.stat.berkeley.edu/~peres/ubc.pdf).
- [23] POSTA, G. (1995/97). Spectral gap for an unrestricted Kawasaki type dynamics. *ESAIM Probab. Stat.* **1** 145–181 (electronic). [MR1444246](#)
- [24] PRIVMAN, V. and ŠVRAKIĆ, N. M. (1988). Difference equations in statistical mechanics. II. Solid-on-solid models in two dimensions. *J. Stat. Phys.* **51** 1111–1126. [MR0971048](#)
- [25] PRIVMAN, V. and ŠVRAKIĆ, N. M. (1989). Line interfaces in two dimensions: Solid-on-solid models. In *Directed Models of Polymers, Interfaces, and Clusters: Scaling and Finite-Size Properties*. *Lecture Notes in Physics* **338** 32–60. Springer, Berlin.
- [26] PROPP, J. G. and WILSON, D. B. (1996). Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures Algorithms* **9** 223–252. [MR1611693](#)
- [27] RANDALL, D. and TETALI, P. (2000). Analyzing Glauber dynamics by comparison of Markov chains. *J. Math. Phys.* **41** 1598–1615. [MR1757972](#)
- [28] SPITZER, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton, NJ. [MR0171290](#)
- [29] STROOCK, D. W. and ZEGARLIŃSKI, B. (1992). The logarithmic Sobolev inequality for discrete spin systems on a lattice. *Comm. Math. Phys.* **149** 175–193. [MR1182416](#)
- [30] WILSON, D. B. (2004). Mixing times of Lozenge tiling and card shuffling Markov chains. *Ann. Appl. Probab.* **14** 274–325. [MR2023023](#)

DEPARTMENT OF MATHEMATICS  
 UNIVERSITY OF ROMA TRE  
 LG. SAN MURIALDO 1  
 00146 ROMA  
 ITALY  
 E-MAIL: [martin@mat.uniroma3.it](mailto:martin@mat.uniroma3.it)

COMPUTER SCIENCE DIVISION  
 UNIVERSITY OF CALIFORNIA  
 BERKELEY, CALIFORNIA 94720-1776  
 USA  
 E-MAIL: [sinclair@cs.berkeley.edu](mailto:sinclair@cs.berkeley.edu)