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UNIFORM CONVERGENCE FOR COMPLEX [0,1]-MARTINGALES

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Positive T-martingales were developed as a general framework that extends the positive measure-valued martingales and are meant to model intermittent turbulence. We extend their scope by allowing the martingale to take complex values. We focus on martingales constructed on the interval $T = [0, 1]$ and replace random measures by random functions. We specify a large class of such martingales for which we provide a general sufficient condition for almost sure uniform convergence to a nontrivial limit. Such a limit yields new examples of naturally generated multifractal processes that may be of use in multifractal signals modeling.

1. Introduction.

1.1. *Foreword about multifractal functions.* Multifractal analysis is a natural framework to describe the heterogeneity that is reflected in the distribution at small scales of the Hölder singularities of a given locally bounded function or signal $F : I \mapsto \mathbb{C}$ where I is an interval. The Hölder singularity of F can be defined, at every point t, by

$$h_F(t) = \liminf_{r \to 0^+} \frac{\log \operatorname{Osc}_F([t-r, t+r])}{\log(r)}$$

or

$$h_F(t) = \liminf_{n \to \infty} \frac{\log_2 \operatorname{Osc}_F(I_n(t))}{-n},$$

where $I_n(t)$ is the dyadic interval of length 2^{-n} containing t and $\operatorname{Osc}_F(J) = \sup_{s,t \in J} |F(t) - F(s)|$. The multifractal analysis of F classifies points according to h_F. It may compute the singularity spectrum of F, that is, the Hausdorff dimension of the sets $h_F^{-1}(\{h\})$ for $h \geq 0$ or, more roughly, measure the asymptotic number of dyadic intervals of generation n needed to cover the sets $h_F^{-1}(\{h\})$ by estimating the large deviation spectrum

$$L_F(h) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log_2 \#\{J \in \mathcal{G}_n, 2^{-n(h+\varepsilon)} \leq \operatorname{Osc}_F(J) \leq 2^{-n(h-\varepsilon)}\}}{n},$$

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where $\mathcal{G}_n$ is the set of dyadic intervals of generation n. One says that F is monofractal if there exists a unique $h \geq 0$ such that $E_F(h) \neq \varnothing$ or $L_F(h) \neq -\infty$. Otherwise, F is multifractal (see [13, 25] for more details).

1.2. *Motivations and methods to build multifractal processes.* The main motivation for constructing and studying multifractal functions or stochastic processes comes from the need to model empirical signals for which the estimation of L_F and related quantities reveals striking scaling invariance properties. These signals concern physical or social intermittent phenomena like energy dissipation in turbulence [11, 19, 20], spatial rainfall [12], human heart rate [29], internet traffic [26] and stock exchange prices [22]. Models of these phenomena are the statistically self-similar measures constructed in [2, 5, 17, 20]. These objects are special examples of limit of "T-martingales," which consist in a class of random measures developed in [14, 15] after the seminal work [19] about Gaussian multiplicative chaos (see also [10, 30]). When $T = [0, 1]$, these martingales and their limit are also used to build models of nonmonotonic scaling invariant signals as follows: By performing a multifractal time change in Fractional Brownian motions or stable Lévy processes [2, 22, 25], by integrating a positive $[0, 1]$-martingale with respect to the Brownian motion or using such a martingale to specify the covariance of some Gaussian processes to get new types of multifractal random walks [2, 18], or by considering random wavelet series whose coefficients are built from a multifractal measure [1, 8].

1.3. *A natural alternative construction.* This paper considers the natural alternative to these constructions which allows the multiplicative processes involved in $[0, 1]$-martingales to take complex values.

Let us now recall what are $[0, 1]$-martingales. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, endow the interval $[0, 1]$ with the Borel σ-algebra $\mathcal{B}([0, 1])$ and the product space $[0, 1] \times \Omega$ with the product σ-algebra $\mathcal{B}([0, 1]) \otimes \mathcal{B}$. Let $(\mathcal{B}_n)_{n\geq 1}$ be a nondecreasing sequence of σ-algebras in $\mathcal{B}$. Also let $(Q_n)_{n\geq 1}$ be a sequence of complex-valued measurable functions defined on $[0, 1] \times \Omega$ such that for each $t \in [0, 1]$, $\{Q_n(t, \cdot), \mathcal{B}_n\}_{n\geq 1}$ is a martingale of expectation 1. Such a sequence of functions is called a $[0, 1]$-martingale. Given a Radon measure λ on $[0, 1]$, for every $n \geq 1$ we can define the random complex measure μ_n whose density with respect to λ is equal to Q_n.

If the functions Q_n take nonnegative values, then, with probability 1, the sequence of Radon measures $(\mu_n)_{n\geq 1}$ weakly converges to a measure μ ([14, 15]). This property is an almost straightforward consequence of the positive martingale convergence theorem and Riesz's representation theorem. When the random functions Q_n cease to be nonnegative, the martingales $Q_n(t)$ need not be bounded in L^1 norm; hence the total variations of the complex measures μ_n may diverge, and $(\mu_n)_{n\geq 1}$ need not converge almost surely weakly to an element of the dual

of $\mathcal{C}([0,1])$, the space of continuous complex-valued functions over $[0,1]$. In this paper we rather consider the sequence of random continuous functions

$$F_n : t \in [0,1] \mapsto \mu_n([0,t]) = \int_0^t Q_n(u)\,\mathrm{d}\lambda(u).$$

Then the following questions arise naturally:

QUESTION 1. Does there exist a general necessary and sufficient condition under which $(F_n)_{n\geq 1}$ converges almost surely uniformly to a limit which is nontrivial (i.e., different from 0) with positive probability?

QUESTION 2. When the sequence $(F_n)_{n\geq 1}$ diverges, or converges to 0 in $\mathcal{C}([0,1])$, can a natural normalization of F_n make it converge to a nontrivial multifractal limit $\widetilde{F}$, at least in distribution?

QUESTION 3. Consider the case of strong or weak convergence to a limit process F or $\widetilde{F}$ having scaling invariance properties. What is the multifractal nature of F (or $\widetilde{F}$), and does F or $\widetilde{F}$ possess the remarkable property to be naturally decomposed as a monofractal function in multifractal time, like for some other classes of multifractal functions [2, 21, 22, 27]?

We will introduce a subclass of complex [0, 1]-martingales, namely $\mathcal{M}$, such that for $(Q_n)_{n\geq 1} \in \mathcal{M}$, we have a general sufficient condition for the almost sure uniform convergence of $(F_n)_{n\geq 1}$ to a nontrivial limit, as well as a result of global Hölder regularity for the limit function (Theorem 2.1). Our result makes it possible to construct the complex extensions of some fundamental examples of statistically self-similar positive multiplicative cascades mentioned above (see Section 2.3 and an illustration in Figure 1).

Companion papers [3] and [4] provide further results and answers to the previous questions in the particular case of complex b-adic independent cascades (it is worth noting that these objects also play a role in the study of directed polymers in a random medium [9]).

Section 2 introduces the class $\mathcal{M}$, states Theorem 2.1 and provides fundamental examples in $\mathcal{M}$. Section 3 provides the proof of Theorem 2.1. We end this section with some definitions.

1.4. *Definitions.* Given an integer $b \geq 2$, we denote by $\mathscr{A}$ the alphabet $\{0, \ldots, b-1\}$ and define $\mathscr{A}^* = \bigcup_{n\geq 0} \mathscr{A}^n$ (by convention $\mathscr{A}^0$ is the set reduced to the empty word denoted $\varnothing$). For every $n \geq 0$, the length of an element of $\mathscr{A}^n$ is by definition equal to n, and we denote it by $|w|$. For $w \in \mathscr{A}^*$, we define $t_w = \sum_{i=1}^{|w|} w_i b^{-i}$ and $I_w = [t_w, t_w + b^{-|w|}[$. For $n \geq 1$ we define $T_n = \{t_w : w \in \mathscr{A}^n\} \cup \{1\}$ and then $T_* = \bigcup_{n\geq 1} T_n$.

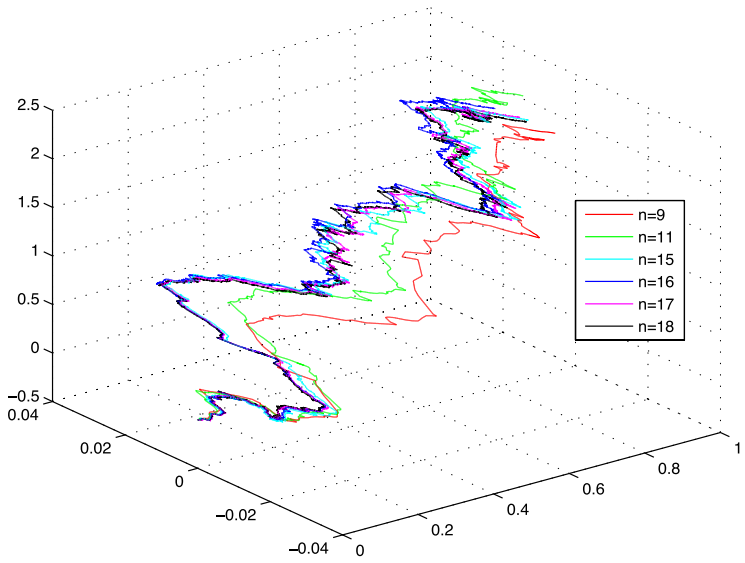

FIG. 1. *A complex valued canonical dyadic cascade* F_n *for* $n = 9, 11, 15, 16, 17, 18$.

For any $t \in [0, 1)$ and $n \geq 1$, we denote by $t|n$ the unique word in $\mathscr{A}^n$ such that $t \in I_{t|n}$. We also denote by $t|0$ the empty word.

If $f \in \mathcal{C}([0, 1])$ we denote by $\|f\|_\infty$ the norm $\sup_{t\in[0,1]} |f(t)|$.

We denote by $(\Omega, \mathcal{B}, \mathbb{P})$ the probability space on which the random variables considered in this paper are defined.

2. A class of complex [0, 1]-martingales.

2.1. *Definition*. Consider a sequence of measurable complex functions

$$P_n : ([0, 1] \times \Omega, \mathcal{B}([0, 1]) \otimes \mathcal{B}) \mapsto (\mathbb{C}, \mathcal{B}(\mathbb{C})), \qquad n \geq 1.$$

For $n \geq 1$ and I, a subinterval of $[0, 1]$, let $\mathcal{F}_n^I$ be the σ-field generated in $\mathcal{B}$ by the family of random variables $\{P_m(t, \cdot)\}_{t\in I, 1\leq m\leq n}$. Also let $\overline{\mathcal{F}}_n^I$ be the σ-field generated in $\mathcal{B}$ by the family of random variables $\{P_m(t, \cdot)\}_{t\in I, m>n}$. The σ-fields $\mathcal{F}_n^{[0,1]}$ and $\overline{\mathcal{F}}_n^{[0,1]}$ are simply denoted by $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$.

(P1) For all $t \in [0, 1]$, $P_n(t, \cdot)$ is integrable, and $\mathbb{E}(P_n(t, \cdot)) = 1$.

(P2) For every $n \geq 1$, $\mathcal{F}_n$ and $\overline{\mathcal{F}}_n$ are independent.

(P3) There exist two integers $b \geq 2$ and $N \geq 1$ such that for every $n \geq 1$ and every family $\mathcal{G}$ of b-adic subintervals of $[0, 1]$ of generation n such that $d(I, J) \geq Nb^{-n}$ for every $I \neq J \in \mathcal{G}$, the σ-algebra's $\overline{\mathcal{F}}_n^I$, $I \in \mathcal{G}$, are mutually independent, where $d(I, J) = \inf\{|t - s| : s \in I, t \in J\}$.

Under the properties (P1) and (P2), for each $t \in (0, 1)$ the sequence

$$Q_n(t, \cdot) = \prod_{k=1}^{n} P_k(t, \cdot)$$

is a martingale of expectation 1 with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$.

We denote by $\mathcal{M}$ the class of martingales $(Q_n)_{n\geq 1}$ obtained as above and which satisfy properties (P1)–(P3).

We denote by $\mathcal{M}'$ the subclass of $\mathcal{M}$ of those $(Q_n)_{n\geq 1}$ which, in addition to (P1)–(P3), satisfy the statistical self-similarity property:

(P4) Let b be as in (P3). For every closed b-adic subinterval I of $[0, 1]$, let $n(I)$ and S_I, respectively, stand for the generation of I and the canonical affine map from $[0, 1]$ onto I. The processes $(P_{n(I)+n} \circ S_I)_{n\geq 1}$ and $(P_n)_{n\geq 1}$ have the same distributions.

Let λ be a Radon measure on $[0, 1]$. If $(Q_n)_{n\geq 1} \in \mathcal{M}$, for $n \geq 1$, we define

$$F_n(t) = \int_0^t Q_n(u)\, d\lambda(u). \tag{2.1}$$

2.2. *Convergence theorem for* $(F_n)_{n\geq 1}$. Theorem 2.1 provides a sufficient condition for the almost sure uniform convergence of F_n, as n tends to ∞, to a limit F such that $\mathbb{P}(F \neq 0) > 0$. This condition is the extension of the condition introduced in Part II of [6] to show that when $(Q_n)_{n\geq 1}$ is nonnegative, the sequence of measures F_n' converge almost surely weakly to a random measure μ such that $\mathbb{P}(\mu \neq 0) > 0$. When $(Q_n)_{n\geq 1}$ is not nonnegative, the uniform convergence of F_n is a more delicate issue.

For $p \in \mathbb{R}_+$ and $n \geq 1$ we define

$$S(n, p) = \sum_{w\in\mathscr{A}^n} \lambda(I_w)^{p-1} \int_{I_w} \mathbb{E}(|Q_n(t)|^p)\, d\lambda(t) \quad \text{and} \tag{2.2}$$

$$\varphi(p) = \liminf_{n\to\infty} \left(-\frac{1}{n} \log_b S(n, p) \right). \tag{2.3}$$

We notice that φ is a concave function of p, $\varphi(0) \leq 0$ by construction, and that due to our assumption that $\mathbb{E}(Q_n(t)) = 1$, we also have $\varphi(1) \leq 0$.

THEOREM 2.1.

(1) *Suppose that* $\varphi(p) > 0$ *for some* $p \in (0, 1)$, *and that there exists a function* $\psi : \mathbb{N}_+ \to \mathbb{R}_+$ *such that* $\psi(n) = o(n)$ *and* $\mathbb{E}(\sup_{t\in I_w} |Q_n(t)|^p) \leq \exp(\psi(n)) \times \mathbb{E}(|Q_n(t)|^p)$ *for all* $n \geq 1$, $w \in \mathscr{A}^n$ *and* $t \in I_w$. *Then, with probability* 1, F_n *converge uniformly to* 0 *as* $n \to \infty$.

(2) *Let* $p \in (1, 2]$. *Suppose that* $\varphi(p) > 0$. *The functions* F_n *converge uniformly, almost surely and in* L^1 *norm, to a limit* F, *as* $n \to \infty$. *The function* F *is* γ*-Hölder continuous for all* $\gamma \in (0, \max_{q\in(1,p]} \varphi(q)/q)$. *Moreover,* $\mathbb{E}(\|F\|_\infty^p) < \infty$.

REMARK 2.1. (1) The proof of Theorem 2.1(1) will show that this result does not require (P1), (P2) or (P3). The existence of the function ψ corresponds to a kind of bounded distortion principle.

(2) Under the assumptions of Theorem 2.1(2), let $\beta = \min\{p \in [1,2) : \varphi(p) = 0\}$. The nonnegative sequence $(Q_n^{(\beta)})_{n\geq 1} = (|Q_n|^\beta)_{n\geq 1}$ is an element of $\mathcal{M}$, and by construction, the corresponding function φ is positive near 1^+. Consequently, the sequence $F_n^{(\beta)}$ defined by $\int_0^\cdot Q_n^{(\beta)}(u)\,du$ converges uniformly to a nondecreasing function $F^{(\beta)}$. Inspired by the results obtained in [4], it is natural to ask under which additional assumptions it is possible to write $F = B_{1/\beta} \circ F^{(\beta)}$ where $B_{1/\beta}$ is a monofractal function of exponent $1/\beta$.

(3) Suppose that φ is not positive over $[0,2]$. In the case where the martingale $(F_n(1))_{n\geq 1}$ is not bounded in L^2 norm, inspired again by what is done in [4], it is natural to look at the process $F_n/\sqrt{\mathbb{E}(F_n(1)^2)}$ and seek for conditions under which it converges in distribution, as $n \to \infty$.

2.3. *Examples*.

Homogeneous b-adic independent cascades. We consider the complex extension of the nonnegative $[0,1]$-martingales introduced in [20]. Let b be an integer ≥ 2 and for every $k \geq 0$ let $W^{(k)} = (W_0^{(k)}, \ldots, W_{b-1}^{(k)})$ be a vector such that each of its components is complex, integrable and has an expectation equal to 1. Then, consider $\{W^{(|w|)}(w)\}_{w\in\mathscr{A}^*}$, a family of independent vectors such that for each $k \geq 0$ and $w \in \Sigma_k$ the vector $W^{(k)}(w)$ is a copy of $W^{(k)}$.

An element of $\mathcal{M}$ is obtained as follows. For $t \in [0,1)$ and $n \geq 1$ let $P_n(t) = W_{t_n}^{(n-1)}(t|n-1)$ and then $Q_n(t) = \prod_{k=1}^n P_k(t)$. If λ is the inhomogeneous Bernoulli measure associated with a sequence of probability vectors $(\lambda^{(k)} = \lambda_0^{(k)}, \ldots, \lambda_{b-1}^{(k)})_{k\geq 0}$, then

$$\varphi(p) = \liminf_{n\to\infty} \left(-\frac{1}{n} \sum_{k=0}^{n-1} \log_b \mathbb{E}\left(\sum_{i=0}^{b-1} (\lambda_i^{(k)} |W_i^{(k)}|)^p \right) \right).$$

If all the vectors $W^{(k)}$ have the same distribution as a vector W, then $(Q_n)_{n\geq 1}$ belongs to $\mathcal{M}'$. *Canonical cascades* correspond to W whose components are i.i.d. and λ equal to the Lebesgue measure. Then a necessary and sufficient condition for the almost sure uniform convergence of F_n to a nontrivial limit is $\varphi'(1^-) > 0$ if $W \geq 0$ [16, 17] and $\varphi(p) > 0$ for some $p \in (1,2]$ for the special "monofractal" examples considered in [7].

Compound Poisson cascades. We consider the complex extension of the nonnegative $[0,1]$-martingales introduced in [5]. Let ν be a positive Radon measure over $(0,1]$ and denote by Λ the measure Leb $\otimes\, \nu$ where Leb stands for the Lebesgue measure over $\mathbb{R}$. We consider a Poisson point process S of intensity Λ.

To each point M of S, we associate a random variable W_M picked in a collection of random variables that are independent, independent of S, and are identically distributed with an integrable complex random variable W. We fix $\beta > 0$, and for $n \geq 1$ and $t \in [0, 1]$ we define the truncated cone

$$\Delta\mathcal{C}_n(t) = \{(t', r) : b^{-n} < r \leq b^{1-n}, t - \beta r/2 \leq t' < t + \beta r/2\}.$$

We obtain an element of $\mathcal{M}$ as follows. For $t \in [0, 1)$ and $n \geq 1$ we define

$$P_n(t) = e^{-\Lambda(\Delta\mathcal{C}_n(t))(\mathbb{E}(W)-1)} \prod_{M \in S \cap \Delta\mathcal{C}_n(t)} W_M,$$

and then

$$Q_n(t) = \prod_{k=1}^{n} P_k(t).$$

If λ is the Lebesgue measure and $\widetilde{\beta}$ stands for $\limsup_{n\to\infty} n^{-1} \log_b \Lambda(\bigcup_{k=1}^{n} \Delta\mathcal{C}_n)$,

$$\varphi(p) = p - 1 + \widetilde{\beta}\big(p\big(\mathbb{E}(\Re W) - 1\big) - \big(\mathbb{E}(|W|^p) - 1\big)\big).$$

If, moreover, there exists $\delta > 0$ such that $\nu(\mathrm{d}r) = \delta\, \mathrm{d}r/r^2$, that is, if Λ possesses scaling invariance properties, we have $\widetilde{\beta} = \beta\delta$, and $(Q_n)_{n\geq 1}$ belongs to $\mathcal{M}'$.

Log-infinitely divisible cascades. This example is an extension of compound Poisson cascades when the weights W_M take the form $\exp(L_M)$, and, in particular, the W_M do not vanish. We use the notations of the previous section and take $\beta = \delta = 1$. Let ψ be a characteristic Lévy exponent ψ defined on $\mathbb{R}^2$, that is,

$$\psi : \xi \in \mathbb{R}^2 \mapsto i\langle \xi | a\rangle - Q(\xi)/2 + \int_{\mathbb{R}^2} \big(1 - e^{i\langle \xi | x\rangle} + i\langle \xi | x\rangle \mathbf{1}_{|x|\leq 1}\big)\pi(\mathrm{d}x), \tag{2.4}$$

where $a \in \mathbb{R}^2$, Q is a nonnegative quadratic form and π is a Radon measure on $\mathbb{R}^2 \setminus \{0\}$ such that $\int (1 \wedge |x|^2)\pi(\mathrm{d}x) < \infty$.

Then let $\rho = (\rho_1, \rho_2)$ be an independently scattered infinitely divisible random $\mathbb{R}^2$-valued measure on $\mathbb{R} \times \mathbb{R}_+^*$ with Λ as control measure and ψ as Lévy exponent (see [24] for the definition). In particular, for every Borel set $B \in \mathbb{R} \times \mathbb{R}_+^*$ and $\xi \in \mathbb{R}^2$ we have

$$\mathbb{E}\big(e^{i\langle \xi | \rho(B)\rangle}\big) = \exp(\psi(\xi)\Lambda(B)),$$

and for every finite family $\{B_i\}$ of pairwise disjoint Borel subsets of $\mathbb{R} \times \mathbb{R}_+^*$ such that $\Lambda(B_i) < \infty$, the random variables $\rho(B_i)$ are independent.

Let I_1 be the interval of those $\xi_1 \in \mathbb{R}$ such that $\int_{|x|\geq 1} e^{\xi_1 x_1}\pi(\mathrm{d}x) < \infty$. The function ψ has a natural extension $\widetilde{\psi}$ to $\mathcal{D} = \mathbb{R}^2 \cup (-iI_1 \times \mathbb{R})$ given by the same expression as in (2.4) if we extend Q to an Hermitian form on $\mathbb{C}^2$. Then for every $\xi \in \mathcal{D}$ and every Borel subset of $\mathbb{R} \times \mathbb{R}_+^*$ we have $\mathbb{E}(e^{i\langle \xi | \rho(B)\rangle}) = \exp(\widetilde{\psi}(\xi)\Lambda(B))$.

Now, we assume that $\xi_0 = (-i, 1) \in \mathcal{D}$, and without loss of generality we set

$$\widetilde{\psi} := \widetilde{\psi} - \widetilde{\psi}(\xi_0).$$

Then, with the same definition of cones as in the previous section, if $n \geq 1$ and $t \in [0, 1]$, we define

$$P_n(t) = \exp[\langle \xi_0 | \rho(\Delta \mathcal{C}_n(t)) \rangle] = \exp[\rho_1(\Delta \mathcal{C}_n(t)) + i\rho_2(\Delta \mathcal{C}_n(t))]$$

and $Q_n(t) = \prod_{k=1}^n P_n(t)$. If we take λ equal to the Lebesgue measure, and if $p \in \mathbb{R}$ is such that $(-ip, 0) \in \mathcal{D}$, then

$$\varphi(p) = p - 1 - \widetilde{\beta}\widetilde{\psi}(-ip, 0). \tag{2.5}$$

In the positive case, this construction that has been proposed has an extension of compound Poisson cascades in [2]. If $\nu(\mathrm{d}r) = \mathrm{d}r/r^2$, then $(Q_n)_{n\geq 1}$ belongs to $\mathcal{M}'$. In [2], a modification of $P_1(t)$ is introduced, which yields a nice exact statistical scaling invariance property for the increments of the limit measure. It can be easily checked that this property, which is different from the statistical self-similarity imposed by (P4), also holds for the complex extension.

3. Proof of Theorem 2.1.

PROOF OF THEOREM 2.1(1). For any $w \in \mathscr{A}^*$ and $n \geq 1$, define

$$\Delta F_n(I_w) = F_n(t_w + b^{-n}) - F_n(t_w) = \int_{I_w} Q_n(t)\,\mathrm{d}\lambda(t). \tag{3.1}$$

We have $\mathbb{E}(\|F_n\|_\infty^p) \leq \mathbb{E}((\sum_{w\in\mathscr{A}^n} |\Delta F_n(I_w)|)^p) \leq \mathbb{E}(\sum_{w\in\mathscr{A}^n} |\Delta F_n(I_w)|^p)$, where we have used the subadditivity of $x \geq 0 \mapsto x^p$ ($p \in (0, 1]$). Thus

$$\begin{aligned}
\mathbb{E}(\|F_n\|_\infty^p) &\leq \sum_{w\in\mathscr{A}^n} \mathbb{E}\Big(\Big|\int_{I_w} Q_n(t)\,\mathrm{d}\lambda(t)\Big|^p\Big) \\
&\leq \sum_{w\in\mathscr{A}^n} \lambda(I_w)^p \mathbb{E}\Big(\sup_{t\in I_w} |Q_n(t)|^p\Big) \\
&\leq \sum_{w\in\mathscr{A}^n} \exp(\psi(n))\lambda(I_w)^{p-1} \int_{I_w} \mathbb{E}(|Q_n(t)|^p)\,\mathrm{d}\lambda(t) \\
&= \exp(\psi(n))S(n, p).
\end{aligned}$$

Due to the property of $\psi(n)$, we have $\limsup_{n\to\infty} \log_b(\mathbb{E}(\|F_n\|_\infty^p))/n \leq -\varphi(p) < 0$. This implies the result. □

PROOF OF THEOREM 2.1(2). The two following crucial statements, which take natural and classical forms, will be proved at the end of the section.

PROPOSITION 3.1. *There exists a constant $C_p > 0$ such that*

$$(\forall n \geq 2) \qquad \mathbb{E}\Big(\max_{t\in T_n} |F_n(t) - F_{n-1}(t)|^p\Big) \leq C_p S(n, p). \tag{3.2}$$

Consequently, for every b-adic number $t \in T_*$, $F_n(t)$ converges almost surely and in L^p norm as $n \to \infty$.

PROPOSITION 3.2. *Let $\gamma \in (0, \max_{q\in(1,p]} \varphi(q)/q)$. With probability* 1, *there exists $\eta_\gamma > 0$ such that for any $t, s \in T_*$ such that $|t - s| < \eta_\gamma$ we have*

$$\sup_{n\geq 1} |F_n(t) - F_n(s)| \leq C_\gamma |t - s|^\gamma, \tag{3.3}$$

where C_γ is a constant depending on γ only.

Since $F_n(0) = 0$ almost surely for all $n \geq 1$, it follows from Propositions 3.2 and Ascoli–Arzela's theorem that, with probability 1, the sequence of continuous functions $(F_n)_{n\geq 1}$ is relatively compact, and all the limits of subsequences of F_n are γ-Hölder continuous for all $0 < \gamma < \max_{q\in(1,p]} \varphi(q)/q$. Moreover, Proposition 3.1 tells us that, with probability 1, F_n is convergent over the dense countable subset T_* of $[0, 1]$. This yields the uniform convergence of F_n and the Hölder regularity of the limit F.

We then prove that $\|\|F(t)\|_\infty\|_p < \infty$. For $n \geq 1$, let $M_n = \max_{t\in T_n} |F_n(t)|$. We have

$$M_{n+1} \leq M_n + \max_{t\in T^n} |F_{n+1}(t) - F_n(t)| + b \cdot \max_{w\in\mathscr{A}^{n+1}} |\Delta F_{n+1}(w)|. \tag{3.4}$$

Then Minkowski's inequality yields

$$\|M_{n+1}\|_p \leq \|M_n\|_p + \Big\| \max_{t\in T_n} |F_{n+1}(t) - F_n(t)| \Big\|_p + b \cdot \Big\| \max_{w\in\mathscr{A}^{n+1}} |\Delta F_{n+1}(w)| \Big\|_p.$$

Also, due to Proposition 3.1 we have $\sum_{n\geq 1} \| \max_{t\in T_n} |F_{n+1}(t) - F_n(t)| \|_p < \infty$. Moreover,

$$\Big\| \max_{w\in\mathscr{A}^{n+1}} |\Delta F_{n+1}(w)| \Big\|_p \leq \Big(\sum_{w\in\mathscr{A}^{n+1}} \mathbb{E}(|\Delta F_{n+1}(w)|^p) \Big)^{1/p} \leq S(n + 1, p)^{1/p},$$

so $\sum_{n\geq 1} \| \max_{w\in\mathscr{A}^{n+1}} |\Delta F_{n+1}(w)| \|_p < \infty$. This implies $\sup_{n\geq 1} \|M_n\|_p < \infty$, and since, with probability 1, F_n converges uniformly to F_∞, and T_* is dense in $[0, 1]$, we get $\| \sup_{t\in[0,1]} |F(t)| \|_p \leq \liminf_{n\to\infty} \|M_n\|_p < \infty$. In particular, F belongs to L^1 and for every $n \geq 1$, the conditional expectation of F with respect to $\mathcal{F}_n$ is well defined, and it converges almost surely and in L^1 norm to F (see Proposition V-2-6 in [23]). It remains to prove that $F_n = \mathbb{E}(F|\mathcal{F}_n)$ almost surely. For every $t \in T_*$, we have shown that the martingale $(F_n(t), \mathcal{F}_n)_{n\geq 1}$ is uniformly integrable, so $F_n(t) = \mathbb{E}(F(t)|\mathcal{F}_n)$ almost surely. Consequently, since T_* is countable, with probability 1, the restriction of $\mathbb{E}(F|\mathcal{F}_n)$ coincides with the function F_n over T_*. Moreover, these two random functions are continuous and T_* is dense in $[0, 1]$, so, with probability 1, they are equal.

PROOF OF PROPOSITION 3.1. Fix $n \geq 2$ and denote the elements of T_n by t_j, $0 \leq j \leq b^n$, where $0 = t_0 < t_1 < \cdots < t_{b^n} = 1$. Also define $J_j = [t_j, t_{j+1}]$ for $0 \leq j < b^n$. We can write

$$F_n(t_j) - F_{n-1}(t_j) = \sum_{k=0}^{j-1} \int_{J_k} U(t)V(t)\,\mathrm{d}\lambda(t)$$

with $U(t) = Q_{n-1}(t)$ and $V(t) = P_n(t) - 1$. Then we divide the family $\{J_j\}_{0\leq j<b^n}$ into bN sub-families, namely the $\{J_{bNk+i}\}_{k\geq 0, 0\leq bNk+i<b^n}$, for $0 \leq i \leq bN-1$. Also we define $M_n = \max_{0\leq j\leq b^n} |F_n(t_j) - F_{n-1}(t_j)|$ and remark that

$$M_n \leq bN \max_{\substack{0\leq j<b^n \\ 0\leq i\leq bN-1}} \left| \sum_{\substack{k\geq 0 \\ 0\leq bNk+i\leq j}} \int_{J_{bNk+i}} U(t)V(t)\,\mathrm{d}\lambda(t) \right|.$$

By raising both sides of the previous inequality to the power p we can get

$$\begin{aligned} M_n^p &\leq (bN)^p \max_{\substack{0\leq j<b^n \\ 0\leq i\leq bN-1}} \left| \int_{J_{bNk+i}} U(t)V(t)\,\mathrm{d}\lambda(t) \right|^p \\ &\leq (bN)^p \sum_{i=0}^{bN-1} \max_{0\leq j\leq b^n} \left| \sum_{\substack{k\geq 0 \\ 0\leq bNk+i\leq j}} \int_{J_{bNk+i}} U(t)V(t)\,\mathrm{d}\lambda(t) \right|^p. \end{aligned} \tag{3.5}$$

We are going to use the following lemma. It is proved for real valued random variables in [28], and its extension to the complex case is immediate.

LEMMA 3.1. *Let $p \in (1, 2]$. There exists a constant $C_p > 0$ such that for every $n \geq 1$ and every sequence $\{V_j\}_{1\leq j\leq n}$ of independent and centered complex random variables we have*

$$\mathbb{E}\left(\max_{1\leq k\leq n} \left| \sum_{j=1}^{k} V_j \right|^p \right) \leq C_p \sum_{j=1}^{n} \mathbb{E}(|V_j|^p).$$

Due to (P3), for each $0 \leq i \leq bN-1$, the restrictions of the function $V(t)$ to the intervals J_{bNk+i}, $0 \leq bNk+i < b^n$, are centered and independent. Also, due to (P2), the functions $U(t)$ and $V(t)$ are independent. Consequently, by taking the conditional expectation with respect to $\mathcal{F}_{n-1}$ in (3.5) and using Lemma 3.1 we get for each $0 \leq i \leq bN-1$

$$\begin{aligned} &\mathbb{E}\left(\max_{0\leq j\leq b^n} \left| \sum_{\substack{k\geq 0 \\ 0\leq bNk+i\leq j}} \int_{J_{bNk+i}} U(t)V(t)\,\mathrm{d}\lambda(t) \right|^p \Big| \mathcal{F}_{n-1} \right) \\ &\quad \leq C_p \sum_{\substack{k\geq 0 \\ 0\leq bNk+i<b^n}} \mathbb{E}\left(\left| \int_{J_{bNk+i}} U(t)V(t)\,\mathrm{d}\lambda(t) \right|^p \Big| \mathcal{F}_{n-1} \right). \end{aligned}$$

This implies

$$\mathbb{E}(M_n^p|\mathcal{F}_{n-1}) \le \widetilde{C}_p \sum_{0\le j\le b^n} \mathbb{E}\Big(\Big|\int_{J_j} U(t)V(t)\,\mathrm{d}\lambda(t)\Big|^p \Big|\mathcal{F}_{n-1}\Big) \tag{3.6}$$

with $\widetilde{C}_p = C_p(bN)^{p+1}$. Now, since $p > 1$, the Jensen inequality yields

$$\Big|\int_{I_j} U(t)V(t)\,\mathrm{d}\lambda(t)\Big|^p \le \lambda(I_j)^{p-1}\int_{I_j} |U(t)V(t)|^p\,\mathrm{d}\lambda(t).$$

Moreover, since $\mathbb{E}(|P_n(t)|) \ge 1$ and $p \ge 1$, we have

$$\mathbb{E}(|V(t)|^p) \le 2^{p-1}\big(1 + \mathbb{E}(|P_n(t)|^p)\big) \le 2^p\mathbb{E}(|P_n(t)|^p). \tag{3.7}$$

Thus, taking the expectation in (3.6) yields

$$\mathbb{E}\Big(\max_{t\in T_n} |F_n(t) - F_{n-1}(t)|^p\Big) \le 2^p\widetilde{C}_p S(n,p),$$

that is, (3.2). If $\varphi(p) > 0$, by definition of φ, $S(n,p)$ converges exponentially fast to 0; hence the series $\sum_{n\ge1} S(n,p)^{1/p}$ converge and, due to (3.2) and the fact that $T_* = \bigcup_{n\ge0} T_n$, $F_n(t)$ converges almost surely and in L^p norm as $n\to\infty$ for all $t \in T_*$. □

PROOF OF PROPOSITION 3.2. Recall (3.1). Let $q \in (1,p]$ such that $\varphi(q) > 0$. It follows from (P1) that $(\Delta F_n(I_w))_{n\ge1}$ is a martingale, so Doob's and then Jensen's inequalities yield a constant C_q such that for $n \ge 1$

$$\mathbb{E}\Big(\max_{1\le k\le n} |\Delta F_k(I_w)|^q\Big) \le C_q\mathbb{E}(|\Delta F_n(I_w)|^q) \le C_q\lambda(I_w)^{q-1}\int_{I_w} \mathbb{E}(|Q_n(t)|^q)\,\mathrm{d}\lambda(t).$$

Consequently

$$\sum_{w\in A^n} \mathbb{E}\Big(\max_{1\le k\le n} |\Delta F_k(I_w)|^q\Big) \le C_q S(n,q). \tag{3.8}$$

By using Markov's inequality as well as (3.8) and Proposition 3.1, we get

$$\begin{aligned}
&\mathbb{P}\Big(\max_{w\in A^n}\max_{0\le k\le n} |\Delta F_k(I_w)| > b^{-n\gamma} \text{ or } \max_{t\in T_n} |F_n(t) - F_{n-1}(t)| > b^{-n\gamma}\Big)\\
&\quad\le \sum_{w\in A^n} \mathbb{P}\Big(\max_{0\le k\le n} |\Delta F_k(I_w)| > b^{-n\gamma}\Big) + \mathbb{P}\Big(\max_{t\in T_n} |F_n(t) - F_{n-1}(t)| > b^{-n\gamma}\Big)\\
&\quad\le \sum_{w\in A^n} b^{n\gamma q}\cdot\mathbb{E}\Big(\max_{0\le k\le n} |\Delta F_k(I_w)|^q\Big) + b^{n\gamma q}\cdot\mathbb{E}\Big(\max_{t\in T_n} |F_n(t) - F_{n-1}(t)|^q\Big)\\
&\quad\le C_q b^{n\gamma q} S(n,q),
\end{aligned}$$

where C_q is another constant depending only on q. Since $\gamma \in (0, \varphi(q)/q)$, by definition of $\varphi(q)$ the series $\sum_{n\geq 1} b^{n\gamma q} S(n,q)$ converges, and by the Borel–Cantelli lemma, with probability 1, there exists n_1 such that for all $n \geq n_1$,

$$\max_{w\in A^n} \max_{0\leq k\leq n} |\Delta F_k(I_w)| \leq b^{-n\gamma} \quad \text{and} \quad \max_{t\in T_n} |F_n(t) - F_{n-1}(t)| \leq b^{-n\gamma}. \tag{3.9}$$

Now fix $n \geq n_1$. We are going to prove by induction that for all $M \geq n+1$ and $t, s \in T_M$ such that $0 < t - s < b^{-n}$ we have

$$\begin{aligned} &\Delta_M(t,s) \leq 2b \sum_{m=n+1}^{M} b^{-m\gamma} \qquad \text{where} \\ &\Delta_M(t,s) = \max_{0\leq k\leq M} |F_k(t) - F_k(s)|. \end{aligned} \tag{3.10}$$

If $M = n+1$, then there exist i and i' with $0 < i - i' < 2b$ such that $t = ib^{-(n+1)}$ and $s = i'b^{-(n+1)}$, so due to (3.9) applied at generation $n+1$,

$$\Delta_{n+1}(t,s) \leq (i - i')b^{-(n+1)\gamma} \leq 2b \cdot b^{-(n+1)\gamma}.$$

Now let $M \geq n+1$ and suppose that (3.10) holds for all $n+1 \leq m \leq M$. Let $t, s \in T_{M+1}$ such that $0 < t - s < b^{-n}$. If there is no element of T_M between s and t, then (3.9) yields $\Delta_{M+1}(t,s) \leq (b-1)b^{-(M+1)\gamma}$. Otherwise, consider $\bar{t} = \max\{u \in T_M : u \leq t\}$ and $\bar{s} = \min\{u \in T_M : u \geq s\}$. We have

$$\begin{aligned} s \leq \bar{s} \leq \bar{t} \leq t, \qquad & t - \bar{t} \leq (b-1)b^{-(M+1)}, \\ \bar{s} - s \leq (b-1)b^{-(M+1)}, \qquad & \bar{t} - \bar{s} < b^{-n}. \end{aligned}$$

Since $\bar{s}$ and $\bar{t}$ belong to $T_M \subset T_{M+1}$, we deduce from (3.9) that

$$\begin{cases} \max\{\Delta_{M+1}(t,\bar{t}), \Delta_{M+1}(\bar{s},s)\} \leq (b-1)b^{-(M+1)\gamma}, \\ \max\{|F_{M+1}(\bar{s}) - F_M(\bar{s})|, |F_{M+1}(\bar{t}) - F_M(\bar{t})|\} \leq b^{-(M+1)\gamma}. \end{cases}$$

Also, due to (3.10) we have $\Delta_M(\bar{t},\bar{s}) \leq 2b\sum_{m=n+1}^{M} b^{-m\gamma}$. Consequently,

$$\begin{aligned} \Delta_{M+1}(t,s) &\leq \Delta_{M+1}(t,\bar{t}) + \Delta_{M+1}(\bar{s},s) + \Delta_M(\bar{t},\bar{s}) \\ &\quad + |F_{M+1}(\bar{s}) - F_M(\bar{s})| + |F_{M+1}(\bar{t}) - F_M(\bar{t})| \\ &\leq 2(b-1)b^{-(M+1)\gamma} + 2b \sum_{m=n+1}^{M} b^{-j\gamma} + 2b^{-(M+1)\gamma}, \end{aligned}$$

so (3.10) holds for $m = M+1$. Let $C_\gamma = 2b/(1 - b^{-\gamma})$. Letting M tend to infinity in (3.10) yields that $\max_{k\geq 1} |F_k(t) - F_k(s)| \leq C_\gamma b^{-(n+1)\gamma}$ for all $n \geq n_1$ and $t, s \in T_*$ such that $|t - s| \leq b^{-n}$. Now, for $t, s \in T_*$ with $|t - s| \leq b^{-n_1}$, there is a unique $n \geq n_1$ such that $b^{-(n+1)} \leq |t - s| < b^{-n}$ and $\max_{k\geq 1} |F_k(t) - F_k(s)| \leq C_\gamma b^{-(n+1)\gamma} \leq C_\gamma |t - s|^\gamma$. The conclusion comes from the density of T_* in $[0,1]$ and the continuity of the F_k. $\square$

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