The Annals of Applied Probability
2010, Vol. 20, No. 4, 1303–1340
DOI: 10.1214/09-AAP660

LAMPERTI-TYPE LAWS

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This paper explores various distributional aspects of random variables defined as the ratio of two independent positive random variables where one variable has an α-stable law, for $0 < \alpha < 1$, and the other variable has the law defined by polynomially tilting the density of an α-stable random variable by a factor $\theta > -\alpha$. When $\theta = 0$, these variables equate with the ratio investigated by Lamperti [*Trans. Amer. Math. Soc.* **88** (1958) 380–387] which, remarkably, was shown to have a simple density. This variable arises in a variety of areas and gains importance from a close connection to the stable laws. This rationale, and connection to the PD(α, θ) distribution, motivates the investigations of its generalizations which we refer to as Lamperti-type laws. We identify and exploit links to random variables that commonly appear in a variety of applications. Namely Linnik, generalized Pareto and z-distributions. In each case we obtain new results that are of potential interest. As some highlights, we then use these results to (i) obtain integral representations and other identities for a class of generalized Mittag–Leffler functions, (ii) identify explicitly the Lévy density of the semigroup of stable continuous state branching processes (CSBP) and hence corresponding limiting distributions derived in Slack and in Zolotarev [*Z. Wahrsch. Verw. Gebiete* **9** (1968) 139–145, *Teor. Veroyatn. Primen.* **2** (1957) 256–266], which are related to the recent work by Berestycki, Berestycki and Schweinsberg, and Bertoin and LeGall [*Ann. Inst. H. Poincaré Probab. Statist.* **44** (2008) 214–238, *Illinois J. Math.* **50** (2006) 147–181] on beta coalescents. (iii) We obtain explicit results for the occupation time of generalized Bessel bridges and some interesting stochastic equations for PD(α, θ)-bridges. In particular we obtain the best known results for the density of the time spent positive of a Bessel bridge of dimension $2 - 2\alpha$.

1. Introduction. Let S_α, for $0 < \alpha < 1$ denote a positive stable random variable, with density f_α, and having Laplace transform,

$$\mathbb{E}[e^{-\lambda S_\alpha}] = e^{-\lambda^\alpha}.$$

Additionally, for $\theta > -\alpha$ define variables $S_{\alpha,\theta}$ independent of S_α whose laws follow a polynomially tilted stable distribution having density proportional to

Received August 2007; revised November 2009.

[1]Supported in part by Grants HIA05/06.BM03, RGC-HKUST 6159/02P, DAG04/05.BM56 and RGC-HKUST 600907 of the HKSAR.

AMS 2000 subject classifications. Primary 60E07; secondary 60G09.

Key words and phrases. Bessel bridges, Galton Watson limits, hyperbolic characteristic function, Mittag–Leffler function, Poisson–Dirichlet distributions, stable continuous state branching processes.

$t^{-\theta} f_\alpha(t)$. When $\theta = 0$, $S_{\alpha,0} := S'_\alpha \stackrel{d}{=} S_\alpha$. In this case Lamperti [62] (see also Zolotarev [94] and Chaumont and Yor [20]) showed that, despite the general intractability of f_α, the ratio

$$X_\alpha \stackrel{d}{=} \frac{S_\alpha}{S'_\alpha}$$

has a remarkably simple density given as

$$f_{X_\alpha}(y) = \frac{\sin(\pi\alpha)}{\pi} \frac{y^{\alpha-1}}{y^{2\alpha} + 2y^\alpha \cos(\pi\alpha) + 1} \qquad \text{for } y > 0. \tag{1.1}$$

This variable arises in many important and often seemingly unrelated contexts. For instance, [13, 17, 25, 26, 67, 68, 73]. Inspired by these facts and connections to the (α, θ) Poisson Dirichlet family of distributions discussed in Pitman and Yor [80] leads us to investigate properties of variables defined as

$$X_{\alpha,\theta} \stackrel{d}{=} \frac{S_\alpha}{S_{\alpha,\theta}}.$$

We refer to these variables as being *Lamperti* variables or variables having *Lamperti-type* laws. Our purpose, from a broad perspective, is to demonstrate that these variables have strong connections to more familiar random variables that appear in a variety of applications in probability, statistics and related fields. In other words, the Lamperti variables, albeit often hidden, appear in many important contexts. Furthermore, we show how to utilize these links to both deduce properties of $X_{\alpha,\theta}$, and develop new nontrivial results related to the linked variables. These results can also be potentially used to expand modeling capabilities. Our results are suggestive of an *active* beta–gamma-stable calculus that extends the notion often associated with beta and gamma variables via Lukacs' [66] characterization.

1.1. *Outline*. We now present an outline of this paper highlighting specifics. More detailed references can be found in each section. Each section contains new results of a nontrivial nature that in some cases are generalizations of existing results. In addition, combined, they represent a nice partial survey of linked variables. Section 2 consists of essentially two parts. The first develops a series of pertinent distributional results for $X_{\alpha,\theta}$ and for a broader class defined by multiplying the Lamperti variables by beta variables. One shall notice the class of random variables we denote as $X^{(\sigma)}_{\alpha,1} \stackrel{d}{=} \beta_{\sigma,1-\sigma} X_{\alpha,\sigma}$ plays a major role throughout the sections. This multiplication is based on ideas we developed in [41]. The second constitutes a natural progression of ideas, each section building on the previous one. Specifically, Section 2.3 establishes links with positive Linnik variables. In particular, we obtain expressions for the density of Linnik variables and also establish an interesting gamma identity. Section 2.4, exploits this identity in connection with generalized Pareto distributions. Albeit brief, the main result is used to identify an unknown limiting distribution obtained by Zolotarev [95] and Slack [85] which we discuss in Section 4. Section 2.4 uses the characterization in the previous sections to demonstrate how one can develop a calculus involving z-distributions. In particular, we identify new classes of random variables, arising as solutions to stochastic equations involving z-distributions, having both hyperbolic characteristic functions and variables whose density can be computed explicitly. Section 3 obtains results for a generalization of Mittag–Leffler functions that can be expressed as Laplace transforms of $S_{\alpha,\theta}^{-\alpha}$ or $X_{\alpha,\theta}$ and can be represented in terms of densities of Linnik variables. Section 4 solves a fairly hard problem, identifying the explicit Lévy density of the semigroup of stable continuous state branching processes. Results in Sections 5 and 6, with the exception of $\alpha = 1/2$, present the best known results for occupation times of various quantities including times spent positive on certain random subsets. We also develop a series of interesting stochastic equations. As one highlight we obtain results for the otherwise elusive case of Bessel bridges of dimension $2 - 2\alpha$. Section 7 discusses aspects of Brownian time changed models where we close by exploiting an interesting, yet not well known, representation of symmetric stable variables of index $0 < 2\alpha \leq 1$, found in [24].

1.2. *Some notation and background.* Here we briefly recount some notation and background related to Bessel processes and the Poisson Dirichlet family of laws. See Pitman [76, 77] for a more precise exposition. Let $\mathcal{B} := (B_t, t > 0)$ denote a strong Markov process on $\mathbb{R}$ whose normalized ranked lengths of excursions, $(P_i) \in \mathcal{P} = \{\mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} s_i = 1\}$, follow a Poisson Dirichlet law with parameters $(\alpha, 0)$ for $0 < \alpha < 1$, as discussed in Pitman and Yor [80]. Denote this law as PD$(\alpha, 0)$. Let $(L_t; t > 0)$ denote its local time starting at 0, and let $\tau_\ell = \inf\{t : L_t > \ell\}$, $\ell \geq 0$ denote its inverse local time. In this case τ is an α-stable subordinator where we choose $\tau_1 \stackrel{d}{=} S_\alpha$. There is the scaling identity (see [79]),

$$L_1 \stackrel{d}{=} \frac{L_t}{t^\alpha} \stackrel{d}{=} \frac{s}{\tau_s^\alpha} \stackrel{d}{=} S_\alpha^{-\alpha},$$

where the local time up to time 1, L_1, satisfies

$$L_1 := \Gamma(1-\alpha)^{-1} \lim_{\epsilon \to 0} \epsilon^\alpha |\{i : P_i \geq \epsilon\}| \qquad \text{a.s.,} \tag{1.2}$$

and is said to follow a Mittag–Leffler distribution. This shows that (L_t, τ_t) have distributions determined by PD$(\alpha, 0)$. Furthermore, independent of (P_i), we suppose that for a fixed $0 < p < 1$, $\mathcal{B}$ is symmetrized so that $\mathbb{P}(B_t > 0) = p$. Under these specifications $\mathcal{B}$ could be a p-skewed Bessel process of dimension $2 - 2\alpha$. In particular when $p = 1/2, \alpha = 1/2$ then $\mathcal{B}$ behaves like a Brownian motion. An interesting aspect of $\mathcal{B}$ is the time its spends on certain subsets of $\mathbb{R}$. Let

$$A_t^+ = \int_0^t \mathbb{I}_{(B_s > 0)}\, ds \quad \text{and} \quad A_t^- = \int_0^t \mathbb{I}_{(B_s < 0)}\, ds,$$

such that $t = A_t^+ + A_t^-$, denote the time $\mathcal{B}$ spends positive and negative, respectively, up till time t. Remarkably, by excursion theory, the time changed-processes $(A_{\tau_\ell}^+; \ell > 0)$ and $(A_{\tau_\ell}^-; \ell > 0)$ are independent α-stable subordinators such that $A_{\tau_1}^+ \stackrel{d}{=} p^{1/\alpha} S_\alpha$ and $A_{\tau_1}^- \stackrel{d}{=} (1-p)^{1/\alpha} S'_\alpha$, for $S'_\alpha \stackrel{d}{=} S_\alpha$. This leads to

$$X_\alpha \stackrel{d}{=} \frac{A_{\tau_\ell}^+}{A_{\tau_\ell}^-} \stackrel{d}{=} c \frac{S_\alpha}{S'_\alpha} \quad \text{and} \quad \frac{cX_\alpha}{cX_\alpha + 1} \stackrel{d}{=} \frac{A_{\tau_\ell}^+}{\tau_\ell} \stackrel{d}{=} A_1^+. \tag{1.3}$$

Hereafter, denote the law that governs $\mathcal{B}$ and related functionals under the above specifications as $\mathbb{P}_{\alpha,0}^{(p)}$. Denote the corresponding expectation operator as $\mathbb{E}_{\alpha,0}^{(p)}$. Thus writing $\mathbb{P}_{\alpha,0}^{(p)}(A_1^+ \in dx)/dx$ equates with the density of the time spent positive on [0, 1] of a p-skewed Bessel process. As noticed by Barlow, Pitman and Yor [3] and Pitman and Yor [79], this law was originally obtained by Lamperti [62], and from (1.3) it equates with

$$\mathbb{P}_{\alpha,0}^{(p)}(A_1^+ \in dx)/dx = \mathbb{P}\big(cX_\alpha/(cX_\alpha + 1) \in dx\big)/dx.$$

Now for $\theta > -\alpha$ let $\mathbb{P}_{\alpha,\theta}^{(p)}$ and $\mathbb{E}_{\alpha,\theta}^{(p)}$ denote the law and expectation operator of functionals connected to a p-skewed process whose excursion lengths follow $\mathrm{PD}(\alpha, \theta)$. In particular if (P_i) is distributed according to $\mathrm{PD}(\alpha, \theta)$, then it satisfies, for measurable H,

$$\mathbb{E}_{\alpha,\theta}^{(p)}[H((P_i))] = \frac{\Gamma(\theta + 1)}{\Gamma(\theta/\alpha + 1)} \mathbb{E}_{\alpha,0}^{(p)}[H((P_i))\tau_1^{-\theta}].$$

When $\theta = \alpha$, this corresponds to the case of a Bessel bridge of dimension $2 - 2\alpha$. We use the notation $A_1^{(br)}$ for the variable that satisfies

$$\mathbb{P}_{\alpha,\theta}^{(p)}\big(A_1^{(br)} \in dx\big) = \mathbb{P}_{\alpha,\theta+\alpha}^{(p)}(A_1^+ \in dx).$$

Note also that under $\mathbb{P}_{\alpha,\theta}^{(p)}$, $L_1 \stackrel{d}{=} S_{\alpha,\theta}^{-\alpha}$, which is also equivalent in distribution to the α-diversity of a $\mathrm{PD}(\alpha, \theta)$ law.

Let now $U_1, U_2, \ldots,$ denote a sequence of i.i.d. uniform [0, 1] random variables for (P_i) distributed according to $\mathrm{PD}(\alpha, \theta)$, and $0 \le u \le 1$, the class of $\mathrm{PD}(\alpha, \theta)$ random cumulative distribution functions are defined as

$$P_{\alpha,\theta}(u) \stackrel{d}{=} \sum_{k=1}^{\infty} P_k \mathbb{I}_{(U_k \le u)}.$$

Furthermore under $\mathbb{P}_{\alpha,\theta}^{(u)}$,

$$A_1^+ \stackrel{d}{=} P_{\alpha,\theta}(u)$$

for a fixed u. See Bertoin [8] for applications to coagulation/fragmentation phenomena where it is called a $\mathrm{PD}(\alpha, \theta)$-bridge and Ishwaran and James [40] (see also Pitman [74]) for applications to Bayesian statistics where in particular $P_{\alpha,\theta}$ is referred to as a Pitman–Yor process. Under this name the process has also been applied to problems arising in natural language processing (see Teh [87]). When $\theta > 0$ and $\alpha = 0$ $P_{0,\theta}$ is a Dirichlet process which has, since the seminal work of Ferguson [32], played a fundamental role in Bayesian nonparametric statistics and related areas.

REMARK 1.1. The $P_{\alpha,\theta}$ processes can be defined more generally by replacing (U_k) with i.i.d. random variables (R_k) having common distribution F_R.

REMARK 1.2. For basic notation, we write γ_a and $\beta_{a,b}$ to denote a gamma random variable with shape a and scale 1, and a beta random variable with parameters (a, b). If $a > 0$, and $b = 0$, then we use $\beta_{a,0} := \lim_{b\to 0} \beta_{a,b} = 1$. Additionally ξ_σ denotes a Bernoulli variable with success parameter $0 < \sigma \leq 1$. If X and Y are random variables we will assume that XY is a product of independent random variables unless otherwise specified, if we write X, X' this will mean $X \stackrel{d}{=} X'$ but they are not equal. Last we always consider $c^\alpha = p/q = p/(1-p)$ where $q = 1 - p$ unless otherwise specified.

2. Distributional results for $X_{\alpha,\theta}$. In this section we shall derive various distributional properties of $X_{\alpha,\theta}$. For $\tau > 0$ and $0 < \sigma \leq 1$, we will sometimes work with the parametrization $\tau\sigma$, to accommodate values such as $\tau\sigma = \theta > 0$ and $\tau\sigma = \theta + \alpha$. First, we briefly discuss some pertinent properties of random variables referred to as Dirichlet means and the related class of infinitely divisible random variables whose distributions are generalized gamma convolutions (GGC), as they will play a significant role in our exposition. For more details and related notions, one may consult [16, 21–23, 30, 43, 44, 64] and, in particular for this exposition, [41].

For a generic positive random variable M, let

$$\mathcal{C}_{\tau\sigma}(\lambda; M) = \mathbb{E}[(1+\lambda M)^{-\tau\sigma}] = \mathbb{E}[e^{-\lambda\gamma_{\tau\sigma} M}]$$

denote its Cauchy–Stieltjes transform of order $\tau\sigma$. Similar to Laplace transforms, $\mathcal{C}_{\tau\sigma}(\lambda; M)$ uniquely characterizes the law of M. Let R denote a nonnegative random variable with distribution function F_R. A random variable M, depending on parameters $(\tau\sigma, R)$, is said to be a Dirichlet mean of order $\tau\sigma$ if

$$-\log \mathcal{C}_{\tau\sigma}(\lambda; M) = \tau\sigma\mathbb{E}[\log(1+\lambda R)] := \tau\sigma\psi_R(\lambda) < \infty. \tag{2.1}$$

Equivalently M satisfies the stochastic equation

$$M \stackrel{d}{=} \beta_{\tau\sigma,1} M + (1-\beta_{\tau\sigma,1})R.$$

We denote such variables as $M \stackrel{d}{=} M_{\tau\sigma}(F_R)$.

Importantly, Cifarelli and Regazzini [22] (see also [23]), apply an inversion formula to obtain an expression for the distribution of $M_{\tau\sigma}(F_R)$. In general these are expressed in terms of Abel-type transforms. An exception is the case of $\tau\sigma = 1$, where the density of $M_1(F_R)$ can be expressed as

$$\frac{1}{\pi}\sin(\pi F_R(x))e^{-\Phi_R(x)}, \tag{2.2}$$

where $\Phi_R(x) = \mathbb{E}[\log|x - R|\mathbb{I}_{(R\neq x)}]$. Additionally,

$$\tau\sigma\psi_R(\lambda) = \tau\sigma\int_0^\infty (1 - e^{s\lambda})s^{-1}\mathbb{E}[e^{-s/R}]\,ds$$

is also the Lévy exponent of an infinitely divisible random variable with Lévy density $\tau\sigma s^{-1}\mathbb{E}[e^{-s/R}]$. We say that such a random variable is GGC($\tau\sigma$, R) and may be represented in distribution as a gamma scale mixture

$$\gamma_{\tau\sigma}M_{\tau\sigma}(F_R) = \gamma_\tau\beta_{\tau\sigma,\tau(1-\sigma)}M_{\tau\sigma}(F_R). \tag{2.3}$$

Highly relevant to (2.3), and our exposition, is a result by James [41], that for each $0 < \sigma \le 1$,

$$\beta_{\tau\sigma,\tau(1-\sigma)}M_{\tau\sigma}(F_R) = M_\tau(F_{R\xi_\sigma}), \tag{2.4}$$

where ξ_σ is a Bernoulli variable with success probability σ. Note also that $\beta_{\tau\sigma,\tau(1-\sigma)} \overset{d}{=} M_\tau(F_{\xi_\sigma})$. One consequence is that a GGC($\tau\sigma$, R) variable is also a GGC(τ, $R\xi_\sigma$) variable. In other words, for a fixed $\theta > 0$, a GGC(θ, R) variable is a GGC(θ', $R\xi_{\theta/\theta'}$) variable for all $\theta' > \theta$. As pointed out in [41], one significant point about these multiple representations is that if $0 < \theta = \sigma \le 1$, then one can set $\theta' = 1$ and use the explicit density formula for Dirichlet means of order 1, (2.2), established by Cifarelli and Regazzini [22] to obtain an explicit representation of the density of such a GGC(σ, R) variable. See [41] for its precise form and further details.

REMARK 2.1. Letting F_R^{-1} denote a quantile function, variables $M_{\tau\sigma}(F_R)$ are called Dirichlet means since they can always be represented as

$$M_{\tau\sigma}(F_R) \overset{d}{=} \int_0^1 F_R^{-1}(u)P_{0,\tau\sigma}(du) \overset{d}{=} \int_0^\infty y D_{\tau\sigma}(dy),$$

where $D_{\tau\sigma}(y) \overset{d}{=} \sum_{k=1}^\infty P_k\mathbb{I}_{(R_k\le y)}$ is a Dirichlet process with $(P_k) \sim \mathrm{PD}(0, \tau\sigma)$ and where (R_k) are i.i.d. F_R.

2.1. *Identities.* For the case of $X_{\alpha,\theta}$, one can show that for $\theta > 0$,

$$\mathcal{C}_\theta(\lambda; X_{\alpha,\theta}) = (1 + \lambda^\alpha)^{-\theta/\alpha} = \mathbb{E}[e^{-\lambda\gamma_\theta X_{\alpha,\theta}}], \tag{2.5}$$

and for $\theta > -\alpha$,

$$\mathcal{C}_{1+\theta}(\lambda; X_{\alpha,\theta}) = \mathbb{E}[e^{-\lambda\gamma_{\theta+1}X_{\alpha,\theta}}] = (1 + \lambda^\alpha)^{-(\theta+\alpha)/\alpha} = \mathcal{C}_{\theta+\alpha}(\lambda; X_{\alpha,\theta+\alpha}). \tag{2.6}$$

We will use (2.5) and (2.6) to more easily establish the next series of results. However, we note that the expressions in (2.5) and (2.6) are not obvious. We will provide justification for (2.5) when we discuss Linnik variables in the next section. Assuming that (2.5) is true, (2.6) then follows from an identity due to Perman, Pitman and Yor [72],

$$\frac{1}{S_{\alpha,\theta}} \stackrel{d}{=} \frac{\beta_{\theta+\alpha,1-\alpha}}{S_{\alpha,\theta+\alpha}} \tag{2.7}$$

for $\theta > -\alpha$. (2.7) is another highly relevant component to our exposition and shows that $X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha} X_{\alpha,\theta+\alpha}$.

PROPOSITION 2.1. *The random variables $X_{\alpha,\theta}$ are Dirichlet means having the following properties*: *for $\theta > 0$*,

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta,1} X_{\alpha,\theta} + (1-\beta_{\theta,1}) X_\alpha \stackrel{d}{=} M_\theta(F_{X_\alpha}), \tag{2.8}$$

and for $\theta > -\alpha$ and $\sigma = (\theta+\alpha)/(1+\theta)$,

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \beta^{1/\alpha}_{((\theta+\alpha)/\alpha,(1-\alpha)/\alpha)} X_{\alpha,1+\theta} = M_{1+\theta}(F_{X_\alpha \xi_\sigma}) \tag{2.9}$$

with $X_{\alpha,\theta} \stackrel{d}{=} \beta_{1+\theta,1} X_{\alpha,\theta} + (1-\beta_{1+\theta,1}) X_\alpha \xi_\sigma$. As special cases of (2.9):

(i) $X_{\alpha,1} \stackrel{d}{=} \beta_{1+\alpha,1-\alpha} X_{\alpha,1+\alpha}$;

(ii) $X_{\alpha,1-\alpha} \stackrel{d}{=} \beta_{1,1-\alpha} X_{\alpha,1} \stackrel{d}{=} \beta^{1/\alpha}_{(1/\alpha,(1-\alpha)/\alpha)} X_{\alpha,2-\alpha}$;

(iii) $X_\alpha \stackrel{d}{=} \beta_{\alpha,1-\alpha} X_{\alpha,\alpha} \stackrel{d}{=} \beta^{1/\alpha}_{(1,(1-\alpha)/\alpha)} X_{\alpha,1} = M_1(F_{X_\alpha \xi_\alpha})$ *which yields the identity*,

$$X_\alpha \stackrel{d}{=} U X_\alpha + (1-U) X'_\alpha \xi_\alpha$$

for $X'_\alpha \stackrel{d}{=} X_\alpha$.

PROOF. In order to establish (2.8), we will calculate the Cauchy–Stieltjes transform of order $\theta+1$ of the variables appearing on the two sides of the first equality. This entails multiplication by an independent $\gamma_{\theta+1}$ variable. Hence (2.8) is true if

$$\gamma_{\theta+1} X_{\alpha,\theta} \stackrel{d}{=} \gamma_\theta X_{\alpha,\theta} + \gamma_1 X_\alpha.$$

Applications of (2.5) and (2.6) show that, $\mathcal{C}_{1+\theta}(\lambda; X_{\alpha,\theta}) = \mathcal{C}_\theta(\lambda; X_{\alpha,\theta}) \mathcal{C}_1(\lambda; X_\alpha)$, concluding the result. We will use similar arguments elsewhere but will omit such details. For (2.9), we again calculate $\mathcal{C}_{1+\theta}(\lambda; X_{\alpha,\theta})$. The first equality is easily checked. For the second we use

$$\mathcal{C}_{1+\theta}\big(\lambda; \beta^{1/\alpha}_{((\theta+\alpha)/\alpha,(1-\alpha)/\alpha)} X_{\alpha,1+\theta}\big) = \mathcal{C}_{(1+\theta)/\alpha}\big(\lambda^\alpha, \beta_{((\theta+\alpha)/\alpha,(1-\alpha)/\alpha)}\big). \tag{2.10}$$

In order to establish the equivalence to $M_{1+\theta}(F_{X_\alpha \xi_\sigma})$, first note that (2.8) establishes $X_{\alpha,\theta+\alpha} \stackrel{d}{=} M_{\theta+\alpha}(F_{X_\alpha})$. The result is then concluded by an application of (2.4) for $\tau = 1+\theta$, $\sigma = (\theta+\alpha)/(1+\theta)$, and $R = X_\alpha$. $\square$

The next result establishes results for the larger class of variables defined with (2.3) and (2.4) in mind, as

$$X_{\alpha,\tau}^{(\sigma)} \stackrel{d}{=} \beta_{\tau\sigma,\tau(1-\sigma)} X_{\alpha,\tau\sigma}.$$

Equation (2.9) of Proposition 2.1 is an important special case.

PROPOSITION 2.2. *For $\tau > 0$ and $0 < \sigma \le 1$, the random variables $X_{\alpha,\tau}^{(\sigma)} \stackrel{d}{=} \beta_{\tau\sigma,\tau(1-\sigma)} X_{\alpha,\tau\sigma}$ satisfy*

$$X_{\alpha,\tau}^{(\sigma)} \stackrel{d}{=} \beta_{\tau\sigma,\tau(1-\sigma)} X_{\alpha,\tau\sigma} \stackrel{d}{=} \beta_{(\tau\sigma/\alpha,(\tau(1-\sigma))/\alpha)}^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=} M_\tau(F_{X_\alpha \xi_\sigma}).$$

Which leads to the identity

$$\frac{\beta_{\tau\sigma,\tau(1-\sigma)}}{S_{\alpha,\tau\sigma}} \stackrel{d}{=} \frac{\beta_{(\tau\sigma/\alpha,(\tau(1-\sigma))/\alpha)}^{1/\alpha}}{S_{\alpha,\tau}}. \tag{2.11}$$

PROOF. The result is easily checked by following arguments similar to those used to establish (2.9). Hence we just note that one uses the calculation, $\mathcal{C}_\tau(\lambda; X_{\alpha,\tau}^{(\sigma)}) = \mathcal{C}_{\tau\sigma}(\lambda; X_{\alpha,\tau\sigma})$, in place of (2.10). The equality (2.11) follows immediately since stable random variables S_α are simplifiable (see [20], pages 11 and 12). $\square$

Note that Propositions 2.1 and 2.2 show that

$$-\log \mathcal{C}_\theta(\lambda; X_{\alpha,\theta}) = \frac{\theta}{\alpha} \log(1+\lambda^\alpha) = \theta \mathbb{E}[\log(1+\lambda X_\alpha)].$$

2.2. *Densities and explicit mixture representations.* We first describe some more pertinent features of X_α (see also [13, 41, 44]).

PROPOSITION 2.3. *Let $X_\alpha \stackrel{d}{=} S_\alpha/S'_\alpha$, having density* (1.1). *Then*:

(i) *the cdf of X_α can be represented explicitly as*

$$F_{X_\alpha}(x) = 1 - \frac{1}{\pi\alpha} \cot^{-1}\left(\cot(\pi\alpha) + \frac{x^\alpha}{\sin(\pi\alpha)}\right); \tag{2.12}$$

(ii) *its inverse is given by*

$$F_{X_\alpha}^{-1}(y) = \left[\frac{\sin(\pi\alpha(y))}{\sin(\pi\alpha(1-y))}\right]^{1/\alpha}; \tag{2.13}$$

(iii) *equations* (2.12) *and* (2.13) *yield the identity*

$$\begin{aligned}\sin(\pi\alpha F_{X_\alpha}(y)) &= y^\alpha \sin\bigl(\pi\alpha\bigl(1 - F_{X_\alpha}(y)\bigr)\bigr)\\ &= \frac{y^\alpha \sin(\pi\alpha)}{[y^{2\alpha} + 2y^\alpha\cos(\pi\alpha) + 1]^{1/2}};\end{aligned}\tag{2.14}$$

(iv) *additionally,*

$$\cos(\pi\alpha F_{X_\alpha}(y)) = \frac{y^\alpha\cos(\pi\alpha) + 1}{[y^{2\alpha} + 2y^\alpha\cos(\pi\alpha) + 1]^{1/2}}.$$

PROOF. This derivation of the cdf is influenced by arguments in Fujita and Yor [34] where it becomes clear that it is easier to work with the density of $(X_\alpha)^\alpha$. Specifically the density of $(X_\alpha)^\alpha$ is given by

$$\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{y^2 + 2y\cos(\pi\alpha) + 1} \qquad \text{for } y > 0.$$

It it then easy to obtain the form of the cdf of $(X_\alpha)^\alpha$ by direct integration. Now using the fact that this equates with $F_{X_\alpha}(y^{1/\alpha})$ yields statement (i). Statement (ii) then follows by using properties of the inverse cotangent. In order to establish (iii), use (2.13) which yields the identity

$$y = F_{X_\alpha}^{-1}(F_{X_\alpha}(y)) = \left[\frac{\sin(\pi\alpha(F_{X_\alpha}(y)))}{\sin(\pi\alpha(1 - F_{X_\alpha}(y)))}\right]^{1/\alpha}.\tag{2.15}$$

Hence statement (ii) follows. $\square$

We now focus on obtaining explicit distributional formulae for the pertinent random variables based on their representations as Dirichlet means. In relation to this, Proposition 2.3 gives precise details on the pertinent cdf F_{X_α}; it then remains to obtain a nice expression for the quantity

$$\Phi_\alpha(x) := \Phi_{X_\alpha}(x) = \mathbb{E}[\log|x - X_\alpha|]$$

for $x > 0$. The key to calculating $\Phi_\alpha(x)$ is the fact that we showed that X_α is a mean functional of the type $M_1(F_{\xi_\alpha X_\alpha})$, as described in Proposition 2.1. This sets up an equivalence between the form of the density of X_α obtained by Lamperti [62] and that of $M_1(F_{\xi_\alpha X_\alpha})$, obtained from (2.2). Hence we have the following calculation:

PROPOSITION 2.4. *For* $0 < \alpha < 1$, *and* $x > 0$,

$$\Phi_\alpha(x) = \frac{1}{2\alpha}\log\bigl(x^{2\alpha} + 2x^\alpha\cos(\alpha\pi) + 1\bigr).$$

PROOF. Since $X_\alpha \stackrel{d}{=} M_1(F_{X_\alpha \xi_\alpha})$, it follows by using (2.2) that the density of X_α satisfies the equivalence

$$f_{X_\alpha}(x) = \frac{1}{\pi} \sin\bigl(\pi\alpha[1 - F_{X_\alpha}(x)]\bigr) e^{-\alpha\Phi_\alpha(x)} x^{\alpha-1}.$$

Where on the left-hand side we use the expression in (1.1). Now applying the identity in (2.14) shows that

$$f_{X_\alpha}(x) = \frac{1}{\pi} \frac{x^{\alpha-1}\sin(\pi\alpha)}{[x^{2\alpha} + 2x^\alpha\cos(\pi\alpha) + 1]^{1/2}} e^{-\alpha\Phi_\alpha(x)}.$$

Solving this expression for $\Phi_\alpha(x)$ concludes the result. □

Set

$$\rho_{\alpha,\tau}(x^\alpha) = \frac{\tau}{\alpha}\arctan\left(\frac{\sin(\pi\alpha)}{\cos(\pi\alpha) + x^\alpha}\right) = \pi\tau[1 - F_{X_\alpha}(x)] \tag{2.16}$$

and define the function

$$\Delta_{\alpha,\tau}(x) = \frac{x^{\tau-1}}{\pi} \frac{\sin(\rho_{\alpha,\tau}(x^\alpha))}{[x^{2\alpha} + 2x^\alpha\cos(\alpha\pi) + 1]^{\tau/(2\alpha)}}. \tag{2.17}$$

We next obtain density formula for a key class of random variables that includes the case of X_α, and $X_{\alpha,1}$.

PROPOSITION 2.5. *For $0 < \sigma \le 1$, and $x > 0$, the densities of the random variables*

$$X_{\alpha,1}^{(\sigma)} \stackrel{d}{=} \beta_{\sigma,1-\sigma} X_{\alpha,\sigma} \stackrel{d}{=} [\beta_{(\sigma/\alpha,(1-\sigma)/\alpha)}]^{1/\alpha} X_{\alpha,1}$$

with $X_\alpha \stackrel{d}{=} X_{\alpha,1}^{(\alpha)}$ and $X_{\alpha,1} \stackrel{d}{=} X_{\alpha,1}^{(1)}$, can be expressed as $\Delta_{\alpha,\sigma}(x)$, given in (2.17). Furthermore,

(i) *$X_{\alpha,1}^{(\sigma)} \stackrel{d}{=} F_{X_\alpha}^{-1}(U_{\alpha,\sigma})$ where $U_{\alpha,\sigma} \stackrel{d}{=} F_{X_\alpha^\alpha}([X_{\alpha,1}^{(\sigma)}]^\alpha)$ has density*

$$\left[\frac{\sin(\pi\alpha)}{\sin(\pi\alpha u)}\right]^{(\alpha-\sigma)/\alpha} \frac{\sin(\pi\sigma(1-u))}{\sin(\pi\alpha(1-u))}, \qquad 0 < u < 1.$$

(ii) *If $0 < \sigma \le \alpha$, then $X_{\alpha,1}^{(\sigma)} \stackrel{d}{=} [\beta_{(\sigma/\alpha,(\alpha-\sigma)/\alpha)}]^{1/\alpha} X_\alpha$.*

PROOF. The representations of $X_{\alpha,1}^{(\sigma)}$ is just a special case of Proposition 2.2. The density is calculated based on Proposition 2.4 and the results discussed in [41] and [22], as mentioned previously. Statement (i) takes advantage of the properties of F_{X_α} and is otherwise straightforward to obtain. Statement (ii) is just a manipulation of the beta random variables. □

One important aspect of the previous result is that we can use it to obtain density/mixture representations for the following Lamperti random variables. This is facilitated by identity (2.9).

PROPOSITION 2.6. *Suppose that* $0 \le \theta \le 1-\alpha$, *then* $\alpha \le \sigma^* = \theta + \alpha \le 1$ *and there is the distributional identity*

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \beta_{1,\theta} X_{\alpha,1}^{(\sigma^*)}.$$

In particular, $X_{\alpha,1-\alpha} \stackrel{d}{=} \beta_{1,1-\alpha} X_{\alpha,1}$.

(i) *Hence for* $0 < \theta \le 1-\alpha$, *the density of* $X_{\alpha,\theta}$ *can be written as*

$$f_{X_{\alpha,\theta}}(x) = \theta \int_0^1 \frac{\Delta_{\alpha,\sigma^*}(x/u)}{u(1-u)^{1-\theta}}\, du, \qquad x > 0, \tag{2.18}$$

where $\Delta_{\alpha,\sigma^*}(x) \ge 0$ *is the density of* $X_{\alpha,1}^{(\sigma^*)}$. *When* $\theta = 0$, *the density is* $\Delta_{\alpha,\alpha}(x)$ *equating with* (1.1).

(ii) *As a special case, when* $\alpha \le 1/2$, $X_{\alpha,\alpha} \stackrel{d}{=} B_{1,\alpha} X_{\alpha,1}^{(2\alpha)}$, *where* $X_{\alpha,1}^{(2\alpha)}$ *has density*

$$\frac{\sin(\pi\alpha)}{\pi} \frac{2\alpha x^{2\alpha-1}[\cos(\pi\alpha) + x^\alpha]}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^2}. \tag{2.19}$$

PROOF. The result follows from Propositions 2.2 and 2.5 by writing

$$X_{\alpha,\theta} \stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \beta_{1,\theta} \beta_{\theta+\alpha,1-(\theta+\alpha)} X_{\alpha,\theta+\alpha}.$$

The simplification in (2.19) follows from

$$\sin\bigl(2\pi\alpha[1 - F_{X_\alpha}(x)]\bigr) = \frac{\sin(2\pi\alpha) + 2x^\alpha \sin(\pi\alpha)}{1 + 2x^\alpha \cos(\pi\alpha) + x^{2\alpha}}.$$

□

The previous results allow one to obtain simple mixture representations or densities for $X_{\alpha,\theta}$ in the range $0 \le \theta \le 1-\alpha$, and $\theta = 1$. The fact that we obtain such results for a continuous range of θ is significant, as shown in the next result.

PROPOSITION 2.7. *Set* $\theta = \sum_{j=1}^k \theta_j$ *where* $\theta_j > 0$. *Furthermore, let* $(D_1, \dots, D_k)$ *denote a Dirichlet random vector having density proportional to* $\prod_{i=1}^k x_i^{\theta_i}$. *That is each* $D_i \stackrel{d}{=} \beta_{\theta_i,\theta-\theta_i}$. *Then*,

$$X_{\alpha,\theta} \stackrel{d}{=} \sum_{j=1}^k D_j X_{\alpha,\theta_j},$$

where X_{α,θ_j} *are mutually independent and independent of* $(D_1, \dots, D_k)$. *When* θ_j *are chosen such that* $0 < \theta_j \le 1-\alpha$, *each* X_{α,θ_j} *has an explicit density* $f_{X_{\alpha,\theta_j}}$ *described in* (2.18). *When* $\theta = k$, *one can use* $\theta_j = 1$.

PROOF. Since we have shown that $X_{\alpha,\theta} \stackrel{d}{=} M_\theta(F_{X_\alpha})$, this result follows directly as a special case of Hjort and Ongaro ([37], Proposition 9). $\square$

2.3. *Positive Linnik variables.* For $\theta > 0$,

$$\chi_{\alpha,\theta} \stackrel{d}{=} \gamma_{\theta/\alpha}^{1/\alpha} S_\alpha \tag{2.20}$$

denotes the class of generalized Linnik variables as considered in [16, 26, 39, 46, 65, 73]. The results in the previous section depend on the validity of the transforms in (2.5) and (2.6). It is evident, and known, that $(1+\lambda^\alpha)^{-\theta/\alpha}$ appearing in (2.5) is the Laplace transform of $\chi_{\alpha,\theta}$ for $\theta > 0$. Hence (2.5) is verified if one shows that $\chi_{\alpha,\theta} \stackrel{d}{=} \gamma_\theta X_{\alpha,\theta}$, for $\theta > 0$. It is already known, using a result of Devroye [25] combined with (2.7), that

$$\chi_{\alpha,\alpha} \stackrel{d}{=} \gamma_1^{1/\alpha} S_\alpha = \gamma_1 X_\alpha \stackrel{d}{=} \gamma_\alpha X_{\alpha,\alpha}. \tag{2.21}$$

Furthermore, from Bondesson ([16], page 38), it follows that $\chi_{\alpha,\theta}$ are $\mathrm{GGC}(\theta, F_{X_\alpha})$. In the next result we will verify the usage of (2.5), and use the $X_{\alpha,1}^{(\sigma)}$ to obtain explicit density representations. In this regard, it is is important to note that we do not need explicit results for $X_{\alpha,\theta}$ to get corresponding results for $\chi_{\alpha,\theta}$. In addition we obtain some interesting identities.

PROPOSITION 2.8. *For all $\theta > 0$, $\chi_{\alpha,\theta}$ is a $\mathrm{GGC}(\theta, F_{X_\alpha})$ variable that satisfies*

$$\chi_{\alpha,\theta} \stackrel{d}{=} \gamma_\theta X_{\alpha,\theta}.$$

For $0 < \theta = \sigma \le 1$, $\chi_{\alpha,\sigma} \stackrel{d}{=} \gamma_1 X_{\alpha,1}^{(\sigma)}$ and hence has the density

$$f_{\chi_{\alpha,\sigma}}(x) = \int_0^\infty e^{x/y} y^{-1} \Delta_{\alpha,\sigma}(y)\,dy. \tag{2.22}$$

See also (2.25) for $\theta > 0$. Additionally:

(i) *For $\theta > -\alpha$,*

$$\chi_{\alpha,\theta+\alpha} \stackrel{d}{=} \gamma_{\theta+\alpha} X_{\alpha,\theta+\alpha} \stackrel{d}{=} \gamma_{1+\theta} X_{\alpha,\theta}. \tag{2.23}$$

(ii) *Hence, for $\theta > -\alpha$,*

$$\gamma_{(\theta+\alpha)/\alpha}^{1/\alpha} = \frac{\gamma_{\theta+\alpha}}{S_{\alpha,\theta+\alpha}} \stackrel{d}{=} \frac{\gamma_{1+\theta}}{S_{\alpha,\theta}}. \tag{2.24}$$

(iii) *For $-\alpha < \theta \le k$, $k = 0, 1, 2, \ldots,$*

$$\chi_{\alpha,\theta+\alpha} = \gamma_{k+1} X_{\alpha,k} \beta_{((\theta+\alpha)/\alpha,(k-\theta)/\alpha)}^{1/\alpha}.$$

(iv) *For $\theta = \sum_{i=1}^k \theta_i > 0$, $\chi_{\alpha,\theta} \stackrel{d}{=} \sum_{i=1}^k \chi_{\alpha,\theta_i}$, where χ_{α,θ_i} are independent.*

PROOF. From (2.20) and using the identity

$$e^{-x^\alpha/s^\alpha} = \mathbb{E}[e^{-x/s S_\alpha}],$$

it is easy to see that the density can be expressed as

$$\begin{aligned} f_{\chi_{\alpha,\theta}}(x) &\propto x^{\theta-1} \int_0^\infty e^{-x^\alpha/s^\alpha} s^{-\theta} f_\alpha(s)\,ds \\ &\propto x^{\theta-1} \int_0^\infty \int_0^\infty e^{-xv/s} (v/s)^\theta v^{-\theta} f_\alpha(v) f_\alpha(s)\,dv\,ds \end{aligned}$$

yielding the equivalence with $\gamma_\theta X_{\alpha,\theta}$. The expression in (2.22) is due to Proposition 2.5. Statement (i) follows from (2.7). Statement (ii) follows by removing S_α which is justified since it is a simplifiable variable. For (iii), apply Proposition 2.2(iv) follows from infinite divisibility. $\square$

REMARK 2.2. It is not difficult to show that a general expression for the density of $\chi_{\alpha,\theta}$, for all $\theta > 0$, is obtained by replacing $\Delta_{\alpha,\sigma}$ by $\Delta_{\alpha,\theta}$ as follows:

$$f_{\chi_{\alpha,\theta}}(z) = \frac{1}{\pi} \int_0^\infty \frac{e^{-zx} \sin(\pi\theta F_{X_\alpha}(x))\,dx}{[x^{2\alpha} + 2x^\alpha \cos(\alpha\pi) + 1]^{\theta/(2\alpha)}}. \tag{2.25}$$

However, $\Delta_{\alpha,\theta}$ can take negative values when $\theta > 1$, so this does not in general yield a mixture representation for $\chi_{\alpha,\theta}$. Nonetheless, it may not be difficult to evaluate numerically which is relevant for the Mittag–Leffler functions discussed in Section 3.

REMARK 2.3. The gamma identity in statement (2.24) of the previous proposition is quite remarkable, and, as we shall see below, has some interesting implications. We note that although not obvious, our result coincides with a variation of Bertoin and Yor ([11], Lemma 6). Checking moments one can see that, in their notation, $J_{s,s/\alpha} \overset{d}{=} S_{\alpha,s}^{-\alpha}$ for $s > 0$ and for $\theta > -\alpha$, $J^{(\alpha)}_{1+\theta,(\theta+/\alpha)/\alpha} \overset{d}{=} S_{\alpha,\theta}^{-\alpha}$. Our work provides some additional interpretation of their variables (see also [45]). See Kotlarski [56] for a general characterization of cases where products of variables result in gamma variables.

2.4. *Generalized Pareto laws.* Influenced in part by the gamma identity (2.24), we next look at relationships between the Lamperti laws and a class of generalized Pareto distributions. We note that the next result also plays an important role in Section 4 when discussing continuous state branching processes. Define random variables

$$W_{\alpha,\theta}^{1/\alpha} := \left(\frac{U^{\alpha/\theta}}{1 - U^{\alpha/\theta}}\right)^{1/\alpha}.$$

These represent a sub-class of generalized Pareto distributions with cdf and density given as

$$F_{W_{\alpha,\theta}^{1/\alpha}}(y) = \frac{y^\theta}{(1+y^\alpha)^{\theta/\alpha}}; \qquad f_{W_{\alpha,\theta}^{1/\alpha}}(y) = \frac{\theta y^{\theta-1}}{(1+y^\alpha)^{(\theta+\alpha)/\alpha}}.$$

PROPOSITION 2.9. *Let U denote a Uniform $[0,1]$ random variable, then for $\theta > 0$:*

(i) *There is the identity*

$$W_{\alpha,\theta}^{1/\alpha} := \left(\frac{U^{\alpha/\theta}}{1-U^{\alpha/\theta}}\right)^{1/\alpha} \stackrel{d}{=} \left(\frac{\gamma_{\theta/\alpha}}{\gamma_1}\right)^{1/\alpha} \stackrel{d}{=} \frac{\chi_{\alpha,\theta}}{\gamma_1} \stackrel{d}{=} \frac{\gamma_\theta}{\gamma_1} X_{\alpha,\theta}.$$

(ii) *For $0 < \sigma \le 1$, the random variable $\Sigma_{\alpha,\sigma} \stackrel{d}{=} \gamma_1 / X_{\alpha,1}^{(\sigma)}$ has Laplace transform*

$$\mathbb{E}[e^{-\lambda\Sigma_{\alpha,\sigma}}] = 1 - \lambda^\sigma (1+\lambda^\alpha)^{-\sigma/\alpha}. \tag{2.26}$$

PROOF. Statement (i) is an application of (2.24). For statement (ii) notice that

$$\mathbb{P}\left(\frac{\gamma_1}{\Sigma_{\alpha,\sigma}} > \lambda\right) = \mathbb{E}[e^{-\lambda\Sigma_{\alpha,\sigma}}],$$

but this is the survival function of the random variable

$$\frac{\gamma_1'}{\gamma_1} X_{\alpha,1}^{(\sigma)} \stackrel{d}{=} \frac{\gamma_\sigma}{S_{\alpha,\sigma}} \frac{S_\alpha}{\gamma_1} \stackrel{d}{=} W_{\alpha,\sigma}^{1/\alpha}. \qquad \square$$

REMARK 2.4. As we shall discuss in Section 4, Statement (ii), (2.26) serves to identify explicitly the (unknown) limiting distribution obtained by [95] and [85] corresponding to $\sigma = 1$. It is relevant also to note that the density of $W_{\alpha,1}^{1/\alpha}$ is the only case that corresponds to a Laplace transform. So here we see a distinguishing feature of $X_{\alpha,1}$.

2.5. *z-variables and hyperbolic laws*. Proposition 2.9, along with the works of [14, 38, 82, 93], motivate us to consider several questions related to z-distributions which are distributed as the logarithm of of the ratio of independent gamma variables. We also believe that some of the variables we discuss will be of interest in terms of applications along the lines discussed in [4] and [38]. In fact, [38] suggests the use of a class of variables that turn out to be equivalent in distribution to $\log(X_\alpha)$. Naturally we do this in the spirit of highlighting what one can do with Lamperti laws. We also obtain additional information about these variables.

In particular, for illustration, we consider the following generic type of problem. Suppose for generic variables X, Y, Z with Z and Y independent there is the following relation:

$$X \stackrel{d}{=} Y + Z.$$

One natural question is to ask, given explicit information about X and Y, what Z satisfies the above equation? In addition, does Z have an explicit density or mixture representation? Notice also that if the characteristic function of Z is not known then we can use X and Y to obtain this. We will consider Z that are variants of Lamperti laws.

We first give a brief discussion on z-distributions. Following [93], the class of z-distributions are defined as $\pi^{-1}\log(\gamma_{\theta_1}/\gamma_{\theta_2})$, having characteristic function

$$\mathbb{E}\big[e^{i\lambda/\pi \log(\gamma_{\theta_1}/\gamma_{\theta_2})}\big] = \frac{\Gamma(\theta_1 + i\lambda/\pi)\Gamma(\theta_2 - i\lambda/\pi)}{\Gamma(\theta_1)\Gamma(\theta_2)}.$$

As special cases, the variables, for $0 < \sigma < 1$, $M_\sigma \stackrel{d}{=} \pi^{-1}\log(\gamma_\sigma/\gamma_{1-\sigma})$, have Meixner distributions with characteristic function

$$\mathbb{E}[e^{-i\lambda M_\sigma}] = \frac{\cos(\varepsilon_\sigma)}{\cosh(\lambda - i\varepsilon_\sigma)}, \tag{2.27}$$

where $\varepsilon_\sigma = \pi(\sigma - 1/2)$. Note that a Meixner distributed random variable is usually defined as $(1/2)M_\sigma$. $\mathbb{S}_1 \stackrel{d}{=} \pi^{-1}\log(U/(1-U))$ has a *logistic distribution* with the characteristic function

$$\mathbb{E}[e^{i\lambda\mathbb{S}_1}] = \frac{\lambda}{\sinh(\lambda)}$$

and $\mathbb{C}_1 \stackrel{d}{=} \pi^{-1}\log(\gamma'_{1/2}/\gamma_{1/2})$ has the *hyperbolic distribution* with characteristic function

$$\mathbb{E}[e^{i\lambda\mathbb{C}_1}] = \frac{1}{\cosh(\lambda)}.$$

It is known that the variables $\mathbb{S}_1$ and $\mathbb{C}_1$ satisfy

$$\mathbb{C}_1 \stackrel{d}{=} \mathbb{S}_1 + \mathbb{T}_1, \tag{2.28}$$

where $\mathbb{T}_1$ is an independent variable having characteristic function

$$\mathbb{E}[e^{i\lambda\mathbb{T}_1}] = \frac{\tanh(\lambda)}{\lambda}.$$

Biane and Yor [15] showed that the density of $\mathbb{T}_1$ is

$$f_{\mathbb{T}_1}(x) = \frac{1}{\pi}\log\biggl(\coth\biggl(\frac{4}{\pi}|x|\biggr)\biggr), \qquad -\infty < x < \infty.$$

REMARK 2.5. Note that the characteristic function of $\alpha\pi^{-1}\log(X_\alpha)$ is equivalent to

$$\mathbb{E}\big[e^{i\alpha\lambda/\pi \log(X_\alpha)}\big] = \frac{\sinh(\alpha\lambda)}{\alpha\sinh(\lambda)}.$$

This expression can be found in Chaumont and Yor ([20], page 147). In addition we see that this characteristic function agrees with the class of generalized secant hyperbolic distributions discussed, for instance, in [38]. See also [78] for more on the variables $\mathbb{C}_1$, $\mathbb{T}_1$ and $\mathbb{S}_1$.

In the next result we obtain a description of the characteristic function of $\log(X_{\alpha,\theta})$ and $\log(X_{\alpha,1}^{(\sigma)})$.

PROPOSITION 2.10. *From Proposition 2.9, it follows that*:

(i) *For* $\theta > 0$,

$$\frac{1}{\alpha}\log\left(\frac{\gamma_{\theta/\alpha}}{\gamma_1}\right) \stackrel{d}{=} \log\left(\frac{\gamma_\theta}{\gamma_1}\right) + \log(X_{\alpha,\theta}).$$

(ii) *For* $\theta > -\alpha$

$$\mathbb{E}[e^{i\lambda\pi^{-1}\log(X_{\alpha,\theta})}] = \frac{\Gamma((\theta+\alpha)/\alpha + i\lambda/(\alpha\pi))\Gamma(1-i\lambda/(\alpha\pi))\Gamma(1+\theta)}{\Gamma(1+\theta+i\lambda/\pi)\Gamma(1-i\lambda/\pi)\Gamma((\theta+\alpha)/\alpha)}.$$

(iii) *For* $0 < \sigma \le 1$,

$$\frac{1}{\alpha}\log\left(\frac{\gamma_{\sigma/\alpha}}{\gamma_1}\right) \stackrel{d}{=} \log\left(\frac{\gamma_1'}{\gamma_1}\right) + \log(X_{\alpha,1}^{(\sigma)}),$$

where $\log(X_{\alpha,1}^{(\sigma)})$ *has density*

$$\frac{1}{\pi}\frac{\sin(\rho_{\alpha,\sigma}(e^{z\alpha}))}{[e^{-2z\alpha}+2e^{-z\alpha}\cos(\alpha\pi)+1]^{\sigma/(2\alpha)}}, \qquad -\infty < z < \infty,$$

and characteristic function

$$\mathbb{E}[e^{i\lambda\log(X_{\alpha,1}^{(\sigma)})}] = \frac{\sinh(\lambda\pi)}{\lambda\pi}\frac{\Gamma(\sigma/\alpha+i\lambda/\alpha)\Gamma(1-i\lambda/\alpha)}{\Gamma(\sigma/\alpha)}.$$

PROOF. This follows as a simple consequence of our previous results and the characteristic functions of z-distributions. $\square$

The next result identifies some variables that have characteristic functions based on hyperbolic functions and also have explicit densities. Define

$$H_{\alpha,\sigma} \stackrel{d}{=} \frac{X_{\alpha,1}^{(\alpha\sigma)}}{\beta_{1-\sigma,\sigma}^{1/\alpha}} \stackrel{d}{=} \frac{\beta_{\sigma,1-\sigma}^{1/\alpha}}{\beta_{1-\sigma,\sigma}^{1/\alpha}} X_\alpha,$$

where the equality follows from (2.11). In addition for $\alpha\delta \le \theta \le \alpha(1-\delta)$, for $\delta \le 1/2$, define

$$L_{\alpha,\theta}^{(\delta)} \stackrel{d}{=} \left(\frac{\beta_{(\delta,(\theta-\alpha\delta)/\alpha)}}{\beta_{((1-\alpha(1-\delta))/\alpha,(\alpha(1-\delta)-\theta)/\alpha)}}\right)^{1/\alpha}\frac{S_{\alpha,1-\theta}}{S_{\alpha,\theta}},$$

where one can easily check that the density of $S_{\alpha,1-\theta}/S_{\alpha,\theta}$, denoted as $f_{1-\theta,\theta}$, satisfies

$$f_{1-\theta,\theta}(x) = c_{\alpha,\theta} x^{-(1-\theta)} f_{X_{\alpha,1}}(x) = c_{\alpha,\theta} x^{-(1-\theta)} \Delta_{\alpha,1}(x)$$

for

$$c_{\alpha,\theta} = \frac{\Gamma(1/\alpha+1)\Gamma(\theta+1)\Gamma(2-\theta)}{\Gamma(\theta/\alpha+1)\Gamma((1-\theta)/\alpha+1)}.$$

PROPOSITION 2.11. *For* $0 < \sigma < 1$, $\delta \le 1/2$ *and* $\alpha\delta \le \theta \le \alpha(1-\delta)$, *there are the following relationships*:

(i) $\frac{1}{\alpha}\log(\frac{\gamma_\sigma}{\gamma_{1-\sigma}}) \stackrel{d}{=} \log(\frac{\gamma_1'}{\gamma_1}) + \log(H_{\alpha,\sigma})$. *Hence*,

$$\mathbb{E}\big[e^{i\alpha\lambda/\pi \log(H_{\alpha,\sigma})}\big] = \frac{\cos(\varepsilon_\sigma)\sinh(\lambda\alpha)}{\lambda\alpha\cosh(\lambda - i\varepsilon_\sigma)} = \frac{\sinh(\alpha\lambda)}{\alpha\sinh(\lambda)}\frac{\cos(\varepsilon_\sigma)\sinh(\lambda)}{\lambda\cosh(\lambda - i\varepsilon_\sigma)},$$

where $\varepsilon_\sigma = \pi(\sigma - 1/2)$.

(ii) $\frac{1}{\alpha}\log(\frac{\gamma_\delta}{\gamma_{1-\delta}}) \stackrel{d}{=} \log(\frac{\gamma_\theta}{\gamma_{1-\theta}}) + \log(L_{\alpha,\theta}^{(\delta)})$. *With*

$$\mathbb{E}\big[e^{i\alpha\lambda/\pi \log(L_{\alpha,\theta}^{(\delta)})}\big] = \frac{\cos(\varepsilon_\delta)\cosh(\lambda\alpha - i\varepsilon_\theta)}{\cos(\varepsilon_\theta)\cosh(\lambda - i\varepsilon_\delta)}.$$

(iii) *When* $\delta = 1/2$, *then* $\theta = \alpha/2$, *and* $L_{\alpha,\alpha/2}^{(1/2)} \stackrel{d}{=} S_{\alpha,1-\alpha/2}/S_{\alpha,\alpha/2}$ *where* $\log(L_{\alpha,\alpha/2}^{(1/2)})$, *has density*

$$\frac{c_{\alpha,\alpha/2}}{\pi}\frac{e^{z\alpha/2}\sin(\rho_{\alpha,1}(e^{z\alpha}))}{[e^{2z\alpha} + 2e^{z\alpha}\cos(\alpha\pi) + 1]^{1/(2\alpha)}}, \qquad -\infty < z < \infty,$$

and characteristic function

$$\mathbb{E}\big[e^{i\lambda/\pi \log(L_{\alpha,\alpha/2}^{(1/2)})}\big] = \frac{\cosh(\lambda - i\varepsilon_{\alpha/2})}{\cos(\varepsilon_{\alpha/2})\cosh(\lambda/\alpha)}.$$

PROOF. All the characteristic functions follow from that of $\mathbb{S}_1$ and the Meixner distributions (2.27). In order to establish (i) we use the identity $\gamma_1/S_\alpha = \gamma_1^{1/\alpha}$ and apply this to the second definition of $H_{\alpha,\sigma}$. Statement (ii) follows from a manipulation of (2.24) to force the form of the first two variables. The choice of $1-\theta$ and θ in the definition of $L_{\alpha,\theta}^{(\delta)}$ was deliberately made so that we could get an explicit expression of the density of the relevant ratios of stable variables. □

3. Mittag–Leffler functions. In this section we obtain integral representations and other identities for a generalization of the Mittag–Leffler function given by

$$\mathrm{E}_{\alpha,1+\theta}^{(\theta/\alpha+1)}(-\lambda) = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(\theta/\alpha+1)_k}{\Gamma(\alpha k+\theta+1)} \qquad \text{for } \theta > -\alpha, \tag{3.1}$$

where

$$(\theta/\alpha+1)_k = \frac{\Gamma(\theta/\alpha+1+k)}{\Gamma(\theta/\alpha+1)}.$$

So when $\theta = 0$, one recovers the Mittag–Leffler function as

$$\mathrm{E}_{\alpha,1}(-\lambda) = \mathrm{E}^{(1)}_{\alpha,1}(-\lambda) = \mathrm{E}^{(0)}_{\alpha,0}(-\lambda).$$

Note that using simple cancelations involving gamma functions it is easy to show that for $\theta > 0$,

$$\mathrm{E}^{(\theta/\alpha+1)}_{\alpha,1+\theta}(-z) = \frac{\Gamma(\theta)}{\Gamma(1+\theta)}\mathrm{E}^{(\theta/\alpha)}_{\alpha,\theta}(-z). \tag{3.2}$$

Recall that the Mittag–Leffler function can be expressed as the Laplace transform of $S_\alpha^{-\alpha}$. One can show that by taking a Taylor expansion and calculating moments that, for $\theta > -\alpha$,

$$\mathbb{E}[e^{-z/S^{\alpha}_{\alpha,\theta}}] = \mathbb{E}[e^{-z^{1/\alpha}X_{\alpha,\theta}}] = \Gamma(1+\theta)\mathrm{E}^{(\theta/\alpha+1)}_{\alpha,1+\theta}(-z). \tag{3.3}$$

One consequence of this observation is that one can use a Monte Carlo method based on $S_{\alpha,\theta}$ to evaluate this quantity. The next result develops more connections with $X_{\alpha,\theta}$ and $\chi_{\alpha,\theta}$.

PROPOSITION 3.1. *For $\theta > -\alpha$:*

(i) $f_{1/X_{\alpha,\theta}}(x) = x^{-\theta} f_{X_{\alpha,\theta}}(x)$;
(ii) $f_{\gamma_1/X_{\alpha,\theta}}(x) = \Gamma(\theta+1)x^{-\theta} f_{\chi_{\alpha,\theta+\alpha}}(x)$;
(iii) *for $\theta > 0$,*

$$\mathbb{P}\left(\frac{\gamma_1}{X_{\alpha,\theta}} > x\right) = \mathbb{E}[e^{-xX_{\alpha,\theta}}] = \Gamma(\theta)\mathrm{E}^{(\theta/\alpha)}_{\alpha,\theta}(-x^\alpha) = \Gamma(\theta)x^{1-\theta} f_{\chi_{\alpha,\theta}}(x).$$

Hence if $\theta = \sigma$ then these expressions are explicitly determined by (2.22), otherwise one might use (2.25).

(iv) *Applying Proposition 2.7, it follows that for $\sum_{i=1}^k \theta_i = \theta$, where $\theta_i > 0$,*

$$\mathrm{E}^{(\theta/\alpha)}_{\alpha,\theta}(-z) = \int_{\mathcal{S}_k} \prod_{i=1}^k \mathrm{E}^{(\theta_i/\alpha)}_{\alpha,\theta_i}(-zx_i^\alpha)x_i^{\theta_i-1}\,dx_i,$$

where $\mathcal{S}_k = \{(x_1,\dots,x_k) : 0 < \sum_{i=1}^k x_i \le 1\}$.

The result is fairly straightforward using the basic definitions as ratios of stable variables and the identities in Proposition 2.5. We omit the details.

REMARK 3.1. Recall that for $\alpha = 1/2$, $X_{1/2,\theta} \stackrel{d}{=} \gamma_{\theta+1/2}/\gamma_{1/2}$. Using the notation in Pitman [76], equations (88), (98) and Lemma 15, and noting [76], equation

(104), the conditional moments of the meander length $(1-G_1)$, of Brownian motion on [0, 1] conditioned on its local time L_1 is related to the generalized Mittag–Leffler function as follows:

$$\begin{aligned}\mathbb{E}_{1/2,0}\big[(1-G_1)^{\theta+1/2}|L_1=\sqrt{2}\lambda\big] &= \mathbb{E}[e^{-(\lambda^2/2)X_{1/2,\theta}}]\\ &= \mathbb{E}[|B_1|^{\theta+1/2}]h_{-(2\theta+1)}(\lambda)\\ &= \Gamma(1+\theta)\mathrm{E}^{(2\theta+1)}_{1/2,\theta+1}\biggl(-\frac{\lambda}{\sqrt{2}}\biggr)\\ &= \mu(\theta+1/2\|\lambda), \qquad \text{for } \theta>-1/2,\end{aligned}$$

which corresponds to the moments of the structural distribution of the Brownian excursion partition. $h_{-2q}(\lambda)$ is a Hermite function, here $q=\theta+1/2$.

REMARK 3.2. Equation (3.1) is a special case of the function introduced by Prabhakar [84],

$$\mathrm{E}^{\gamma}_{\rho,\mu}(-\lambda)=\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\frac{(\gamma)_k}{\Gamma(\rho k+\mu)}, \tag{3.4}$$

where $(\rho,\mu,\gamma\in\mathbb{C},\mathrm{Re}(\rho)>0)$. Equation (3.1) is the case where $\gamma=(\theta+\alpha)/\alpha$ and $\mu=\theta+1$. Additionally, quantity (3.1) represents a special sub-class of yet more general Mittag–Leffler-type functions which are discussed, for instance, in Kilbas, Saigo and Megumi [51]. See also [2, 5, 7, 19, 28, 36, 53, 54, 58].

4. The explicit Lévy density of Stable CSBPs and the Zolotarev–Slack distribution. We now show how our results lead to an explicit identification of the Lévy density of the semigroup of stable continuous state branching processes of index $1<\delta<2$, that is, $\delta=1+\alpha$ and the limiting distributions first obtained by Zolotarev [95] and Slack [85]. We also mention briefly its connection to the work of [6, 9] on beta coalescents. From Lamperti, [61] continuous state branching processes (CSBP) are Markov processes that can be characterized as limits of Galton–Watson branching processes when the population size grows to infinity. A $(1+\alpha)$-stable (CSBP) process $(Y_t, t>0)$ is a Markov process whose semigroup is specified by

$$\mathbb{E}_a[e^{-\lambda Y_t}]=\mathbb{E}[e^{-\lambda Y_t}|Y_0=a]=e^{-a\upsilon_t(\lambda)}, \tag{4.1}$$

where

$$\upsilon_t=\int_0^{\infty}(1-e^{-s\lambda})\nu_t(ds)=\lambda(\alpha t+\lambda^{\alpha})^{-1/\alpha}.$$

Furthermore (see, for instance, [6, 9]), there exists a process $(Y(t,a); t>0, a>0)$ such that for each t, $Y(t,\cdot)$ is a compound Poisson process with intensity ν_t. Related to this result, Kawazu and Watanabe [49] show that all continuous state branching processes with immigration arise as limits of Galton–Watson processes with immigration. Analogous to (4.1), Theorem 2.3 of their work yields the limiting $(1+\alpha)$-stable (CSBP) with immigration $(\hat{Y}(t), t > 0)$ satisfying

$$\mathbb{E}_a[e^{-\hat{Y}_t}] = (1+\alpha\lambda^\alpha t)^{-d/(\alpha c)}\mathbb{E}_a[e^{-Y_t}]. \tag{4.2}$$

It is evident from (4.2) that the entrance laws of $\hat{Y}$ are positive Linnik distributions. However, the intensity ν_t, which plays a fundamental role in [6, 9], is only known up to its Laplace transform. It is known that this Laplace transform coincides, up to some scaling factors, with the Laplace transforms of the limiting distributions obtained by Zolotarev [95] and Slack [85]. Before identifying these expressions we recount the limiting distributions obtained by Slack [85] and Berestycki, Berestycki and Schweinsberg [6]. As described, for instance, in [6], Slack's result describes the limiting distribution, say μ_α, of the number of offspring in generation n of a critical Galton–Watson process, re-scaled to have mean 1 and conditioned to be positive, when the offspring distribution is in the domain of attraction of a stable law of index $1 < \delta < 2$. This result complements Yaglom's [90] well-known result for the case where the offspring distribution has finite variance. In that case the limiting distribution is exponential with mean 1. Precisely, following the exposition in [69], we state a variation of Slack's result.

PROPOSITION 4.1 (Slack (1968) [85]). *Let $Z = (Z_n, n > 0)$ denote a supercritical Galton–Watson process initiated by a single process. Furthermore, suppose the nonextinction probability $Q_n = \mathbb{P}(Z_n > 0)$, satisfies*

$$Q_n = n^{-1/\alpha}L(n),$$

where $L(x)$ is a slowly varying function. Then,

$$\lim_{n\to\infty}\mathbb{P}(Q_n Z_n \le x | Z_n > 0) = \mu_\alpha([0, x]), \tag{4.3}$$

where for each $0 < \alpha < 1$, μ_α is the distribution of a random variable Σ_α satisfying

$$\int_0^\infty e^{-\lambda w}\mu_\alpha(dw) = \mathbb{E}[e^{-\lambda\Sigma_\alpha}] = 1 - \lambda(1+\lambda^\alpha)^{-1/\alpha}. \tag{4.4}$$

Zolotarev ([95], Theorem 7) also obtained this limit in the case of a class of continuous parameter regular branching processes. However, prior to our work, an explicit description of its density or corresponding random variable was not known. It is now evident from (4.4) that $\Sigma_\alpha \overset{d}{=} \Sigma_{\alpha,1}$ in Proposition 2.9, as we mentioned previously. These limits and or discussions related to (4.1), (4.2) appear more recently in, for instance, [6, 9, 31, 59, 60, 71]. Before we summarize our results we shall say a bit more about the context of [6]. Random variables with law μ_α arise in the work of Berestycki, Berestycki and Schweinsberg [6] (see also [9]) in connection with Beta$(2-\delta,\delta)$ coalescents for $1<\delta<2$. See in particular ([6], Theorem 1.2). Equivalently these are Beta$(1-\alpha,1+\alpha)$ coalescents.

In addition, there is a related result of Berestycki, Berestycki and Schweinsberg [6] that, with some work, would have otherwise allowed us describe the law μ_α. We quote their result below.

PROPOSITION 4.2 (Berestycki, Berestycki and Schweinsberg [6], Proposition 1.5). *Let* $(\Pi(t),t>0)$ *denote a* Beta$(1-\alpha,1+\alpha)$ *coalescent where* $0<\alpha<1$, *and let* $K(t)$ *denote the asymptotic frequency of the block of* $\Pi(t)$ *containing* 1. *Then*

$$\big(\Gamma(\alpha+2)t^{-1}\big)^{1/\alpha}K(t)\underset{d}{\to}\zeta_\alpha \qquad \text{as } t\downarrow 0, \tag{4.5}$$

where ζ_α *is a random variable satisfying*

$$\mathbb{E}[e^{-\lambda\zeta_\alpha}]=(1+\lambda^\alpha)^{-(1+\alpha)/\alpha}. \tag{4.6}$$

Furthermore, as noted in [6], ζ_α has the size biased distribution

$$\mathbb{P}(\zeta_\alpha\in dx)=x\mu_\alpha(dx). \tag{4.7}$$

We now summarize our result which again demonstrates the relevance of the random variable $X_{\alpha,1}\overset{d}{=}S_\alpha/S_{\alpha,1}$, which has explicit density

$$\Delta_{\alpha,1}(x)=\frac{1}{\pi}\frac{\sin(1/\alpha\arctan(\sin(\pi\alpha)/(\cos(\pi\alpha)+x^\alpha)))}{[x^{2\alpha}+2x^\alpha\cos(\alpha\pi)+1]^{1/(2\alpha)}}. \tag{4.8}$$

PROPOSITION 4.3. *For* $0<\alpha<1$, *and* $\Delta_{\alpha,1}(x)$ *defined in* (4.8), *there are the following results.*

(i) *The random variable described in* (4.5) *and* (4.6), ζ_α, *satisfies*

$$\zeta_\alpha\overset{d}{=}\gamma_2X_{\alpha,1}=\chi_{\alpha,1+\alpha}.$$

(ii) *Let* Σ_α *and* μ_α *be as in* (4.3) *and* (4.4), *then*

$$\Sigma_\alpha\overset{d}{=}\frac{\gamma_1}{X_{\alpha,1}}\overset{d}{=}\Sigma_{\alpha,1}.$$

(iii) *Furthermore, for each* $x>0$,

$$\mathbb{P}(\Sigma_\alpha>x)=\mu_\alpha([x,\infty))=\mathrm{E}_{\alpha,1}^{(1/\alpha)}(-x^\alpha)=f_{\gamma_1X_{\alpha,1}}(x).$$

(iv) *The Lévy density* ν_t *corresponding to the* $(1+\alpha)$ (*CSBP*) *specified by* (4.1) *is*

$$\begin{aligned}\nu_t(x)&=(\alpha t)^{-(1+\alpha)/\alpha}x^{\alpha-1}\mathrm{E}_{\alpha,1+\alpha}^{(1+\alpha)/\alpha}\left(-\frac{x^\alpha}{\alpha t}\right)\\&=(\alpha t)^{-2/\alpha}\int_0^\infty e^{-xy/(\alpha t)^{1/\alpha}}y\Delta_{\alpha,1}(y)\,dy.\end{aligned}$$

PROOF. Statements (i) and (ii) are now quite obvious from Propositions 2.8 and 2.9. Statement (iii) is deduced from Propositions 2.5 and 3.1. Statement (iv) follows from these facts and the specifications for ν_t, described in [6], Lemma 2.2, that is,

$$\nu_t(x) = (\alpha t)^{-1/\alpha}\mu_{\alpha,t}(x),$$

where $\mu_{\alpha,t}(x)$ is the density of the random variable $(\alpha t)^{1/\alpha}\Sigma_\alpha$. $\square$

5. Occupation times of generalized Bessel bridges. We now show how our results for $X_{\alpha,\theta}$ can be used to obtain new results related to A_1^+, which is equivalent in distribution to $P_{\alpha,\theta}(p)$, under $\mathbb{P}_{\alpha,\theta}^{(p)}$. This can be seen as a continuation of a subset of the work of James, Lijoi and Prünster [43], who looked at more general PD(α,θ) mean functionals, where, with the exception of $\alpha = 1/2$, the best results for describing the density of $P_{\alpha,\theta}(p)$ were obtained for $\theta = 1$, and $\theta = 1-\alpha$. The results for $\alpha = 1/2$ are classic. For $\alpha = 1/2$, and $p = 1/2$, Lévy [63] showed that A_1^+ under $(1/2, 0)$ and $(1/2, 1/2)$, follow the Arcsine and Uniform[0, 1] distributions, respectively. A general formula for $(1/2,\theta)$, for all $\theta > -1/2$ can be found in Carlton [18], equation (3.4) (see also [50]) and is given by

$$\mathbb{P}_{1/2,\theta}^{(p)}(A_1^+ \in dy)/dy = \frac{\Gamma(\theta+1)}{\Gamma(1/2)\Gamma(\theta+1/2)}\frac{pqy^{\theta-1/2}(1-y)^{\theta-1/2}}{(p^2(1-y)+q^2y)^{1+\theta}}$$

for $0 < y < 1$. We also obtain results for time spent positive on certain random subsets of [0, 1], and also develop some interesting stochastic equations. As a highlight, we obtain explicit results for the case of $\theta = \alpha$, corresponding to the the time spent positive of a Bessel bridge on [0, 1]. In this case the best previous expressions were obtained independently in [43, 91] (see also [64]). For some other related works see [12, 47, 48, 55, 88, 92].

Consider now the following stochastic equations and Cauchy–Stieltjes transforms that can be found in [43] with further references; for $\theta > 0$,

$$P_{\alpha,\theta}(p) = \beta_{\theta,1}P_{\alpha,\theta}(p) + (1-\beta_{\theta,1})P_{\alpha,0}(p) \tag{5.1}$$

and for $\theta > -\alpha$,

$$P_{\alpha,\theta}(p) = \beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(p) + (1-\beta_{\theta+\alpha,1-\alpha})\xi_p. \tag{5.2}$$

Additionally there are the Cauchy–Stieltjes transforms for $\theta > 0$,

$$\mathcal{C}_\theta(\lambda; P_{\alpha,\theta}(p)) = \bigl(q + (1+\lambda)^\alpha p\bigr)^{-\theta/\alpha} = e^{-\theta\psi_{\alpha,0}^{(p)}(\lambda)}, \tag{5.3}$$

where

$$\psi_{\alpha,0}^{(p)}(\lambda) = \mathbb{E}\bigl[\log\bigl(1+\lambda P_{\alpha,0}(p)\bigr)\bigr] = \mathbb{E}_{\alpha,0}^{(p)}[\log(1+\lambda A_1^+)]$$

and for $\theta > -\alpha$,

$$
\begin{aligned}
\mathcal{C}_{1+\theta}(\lambda; P_{\alpha,\theta}(p)) &= \frac{(1+\lambda)^{\alpha-1}p + (1-p)}{(q + (1+\lambda)^{\alpha}p)^{(\theta+\alpha)/\alpha}} \\
&= \mathcal{C}_{\theta+\alpha}(\lambda; P_{\alpha,\theta+\alpha}(p))\mathcal{C}_{1-\alpha}(\lambda; \xi_p).
\end{aligned}
\tag{5.4}
$$

The first equation (5.1) shows that $P_{\alpha,\theta}(p) \stackrel{d}{=} M_\theta(F_{P_{\alpha,0}(p)})$ for $\theta > 0$. The second equation (5.2) (see, for instance, [10, 27, 40, 77] for some other interpretations and applications) can be traced to Pitman and Yor ([79], Theorem 1.3.1) and Perman, Pitman and Yor ([72], Theorem 3.8, Lemma 3.11) as follows; Let $A^+_{G_1} = \int_0^{G_1} \mathbb{I}_{(B_s>0)}\, ds$ denote the time spent positive of $\mathcal{B}$ up till time G_1 which is the time of the last zero of $\mathcal{B}$ before time 1. Then under $\mathbb{P}^{(p)}_{\alpha,\theta}$, there is the equivalence

$$
(A^+_{G_1}, G_1) \stackrel{d}{=} \bigl(G_1 A_1^{(br)}, G_1\bigr) \stackrel{d}{=} (\beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(p), \beta_{\theta+\alpha,1-\alpha}).
$$

This shows that (5.2) can be rewritten in terms of the following decomposition:

$$
A_1^+ \stackrel{d}{=} A^+_{G_1} + (1-G_1)\xi_p \stackrel{d}{=} G_1 A_1^{(br)} + (1-G_1)\xi_p.
$$

See, for example, Enriquez, Lucas and Simenhaus [29] for an interesting recent application of this expression.

We now show that the density of $P_{\alpha,\theta}(p)$ can be expressed in terms of the density of $X_{\alpha,\theta}$. Hereafter, define

$$
r_p(y) = \frac{y}{(c(1-y))}
$$

for $c^\alpha = p/(1-p)$.

PROPOSITION 5.1. *For $\theta > -\alpha$, let $R_{\alpha,\theta} = cX_{\alpha,\theta}/(cX_{\alpha,\theta}+1)$. Then*

$$
\mathbb{P}^{(p)}_{\alpha,\theta}(A_1^+ \in dy) = \frac{(1-y)^\theta}{(1-p)^{\theta/\alpha}}\mathbb{P}(R_{\alpha,\theta} \in dy),
$$

where $A_1^+ \stackrel{d}{=} P_{\alpha,\theta}(p)$. Hence as special cases, using Propositions 2.5 and 2.6:

(i) $\mathbb{P}^{(p)}_{\alpha,1}(A_1^+ \in dy)/dy = (1-y)^{-1}p^{-1/\alpha}\Delta_{\alpha,1}(r_p(y)) = \Omega_{\alpha,1}(y)$.

(ii) *For $0 < \theta \le 1-\alpha$, and $\sigma^* = \theta+\alpha$,*

$$
\mathbb{P}^{(p)}_{\alpha,\theta}(A_1^+ \in dy)/dy = \frac{\theta(1-y)^{\theta-2}}{(1-p)^{(\theta-1)/\alpha}p^{1/\alpha}}\int_0^1 \frac{\Delta_{\alpha,\sigma^*}(r_p(y)/u)}{u(1-u)^{1-\theta}}\,du,
$$

where $\Delta_{\alpha,\sigma^*}(x) \ge 0$ is the density of $X^{(\sigma^*)}_{\alpha,1}$.

PROOF. From (1.3) it follows that, for measurable functions g,

$$\mathbb{E}[g(cX_{\alpha,\theta})] = \frac{(1-p)^{\theta/\alpha}}{\mathbb{E}[S_\alpha^{-\theta}]}\mathbb{E}^{(p)}_{\alpha,0}\left[g\left(\frac{A^+_{\tau_1}}{A^-_{\tau_1}}\right)(A^-_{\tau_1})^{-\theta}\right].$$

The result is concluded by showing that

$$\mathbb{E}^{(p)}_{\alpha,\theta}[g(A^+_1)] := \frac{1}{\mathbb{E}[S_\alpha^{-\theta}]}\mathbb{E}^{(p)}_{\alpha,0}[g(A^+_{\tau_1}/\tau_1)\tau_1^{-\theta}]$$

is equal to $(1-p)^{-\theta/\alpha}\mathbb{E}[g(R_{\alpha,\theta})(1+X_{\alpha,\theta})^{-\theta}]$. But this follows from $\tau_1 = (A^+_{\tau_1}/A^-_{\tau_1}+1)A^-_{\tau_1}$. $\square$

Pitman and Yor ([81], Proposition 15) establish an interesting relationship between the densities of A^+_1 and $A^+_{G_1}$ under the law $\mathbb{P}^{(p)}_{\alpha,0}$. Making no changes to the essence of their clever argument, one can easily extend this result to all (α,θ). Combining this with Proposition 5.1 yields the relationships, for $0 < p < 1$,

$$\begin{aligned}\mathbb{P}^{(p)}_{\alpha,\theta}(A^+_{G_1}\in dy) &= \frac{1-y}{(1-p)}\mathbb{P}^{(p)}_{\alpha,\theta}(A^+_1\in dy)\\ &= \frac{(1-y)^{1+\theta}}{(1-p)^{(\theta+\alpha)/\alpha}}\mathbb{P}(R_{\alpha,\theta}\in dy).\end{aligned}\tag{5.5}$$

Recall that under $P^{(p)}_{\alpha,\theta}$, $A^+_{G_1} \stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(p)$. The next result describes interesting properties of generalizations of this variable.

PROPOSITION 5.2. *For $\tau > 0$ and $0 < \sigma \le 1$, let $V \stackrel{d}{=} \beta_{(\tau\sigma/\alpha,\tau(1-\sigma)/\alpha)}$ and hence is a Dirichlet mean satisfying the stochastic equation*

$$V \stackrel{d}{=} \beta_{\tau/\alpha,1}V' + (1-\beta_{\tau/\alpha,1})\xi_\sigma$$

for $V \stackrel{d}{=} V'$. Then for $0 < p \le 1$,

$$\beta_{\tau\sigma,\tau(1-\sigma)}P_{\alpha,\tau\sigma}(p) \stackrel{d}{=} P_{\alpha,\tau}(pV) = M_\tau(F_{P_{\alpha,0}(p\xi_\sigma)}).\tag{5.6}$$

This leads to the stochastic equations, for $0 < p \le 1$,

$$\begin{aligned}P_{\alpha,\tau}(pV) &\stackrel{d}{=} \beta_{\tau,1}P'_{\alpha,\tau}(pV') + (1-\beta_{\tau,1})P_{\alpha,0}(p\xi_\sigma)\\ &\stackrel{d}{=} \beta_{\tau,1}P'_{\alpha,\tau}(pV) + (1-\beta_{\tau,1})P_{\alpha,0}(pV).\end{aligned}\tag{5.7}$$

In the first expression $P'_{\alpha,\tau}(pV')$ denotes a random variable equivalent only in distribution to $P_{\alpha,\tau}(pV)$. However in the second equation V is the same variable.

PROOF. First note that $\beta_{\tau\sigma,\tau(1-\sigma)}P_{\alpha,\tau\sigma}(p) \stackrel{d}{=} M_\tau(F_{P_{\alpha,0}(p\xi_\sigma)})$, follows from (2.4). Note also by the definition of $P_{\alpha,0}(p)$ it is easy to see that $P_{\alpha,0}(p)\xi_\sigma \stackrel{d}{=} P_{\alpha,0}(p\xi_\sigma)$. In order to establish the rest of (5.6) we can check Cauchy–Stieltjes transforms of order τ using (5.3). For the variable appearing on the left of (5.6) this is easy to calculate. Applying this to $P_{\alpha,\tau}(pV)$ conditioned on V, its final evaluation rests on the simple equality

$$\big(1 - pV + (1+\lambda)^\alpha pV\big) = 1 + [(1+\lambda)^\alpha - 1]pV.$$

Taking the Cauchy–Stieltjes transform of order τ/α for V yields the result. The second equality in (5.7) is then due to (5.1). □

Next is one of our main distributional results which is an analogue of Proposition 2.6 but also highlights the role of various randomly skewed processes.

PROPOSITION 5.3. *For* $0 < \sigma \le 1$, *set* $R^{(\sigma)}_{\alpha,1} = cX^{(\sigma)}_{\alpha,1}/(cX^{(\sigma)}_{\alpha,1} + 1)$, *then the density of*

$$\beta_{\sigma,(1-\sigma)}P_{\alpha,\sigma}(p) \stackrel{d}{=} P_{\alpha,1}\big(p\beta_{(\sigma/\alpha,(1-\sigma)/\alpha)}\big)$$

is for $0 < y < 1$, *equivalent to* $(1-p)^{-\sigma/\alpha}(1-y)\mathbb{P}(R^{(\sigma)}_{\alpha,1} \in dy)/dy$, *and is given explicitly as*

$$\Omega_{\alpha,\sigma}(y) = \frac{1}{\pi}\frac{y^{\sigma-1}\sin(\rho_{\alpha,\sigma}([r_p(y)]^\alpha))}{[y^{2\alpha}q^2 + 2qpy^\alpha(1-y)^\alpha\cos(\alpha\pi) + (1-y)^{2\alpha}p^2]^{\sigma/(2\alpha)}}, \tag{5.8}$$

where $\rho_{\alpha,\sigma}$ *is defined in* (2.16). *In particular* $\Omega_{\alpha,1}(y)$ *is the density of* $P_{\alpha,1}(p)$ *and* $\Omega_{\alpha,\alpha}(y)$ *is the density of* $\beta_{\alpha,1-\alpha}P_{\alpha,\alpha}(p)$. *In addition*:

(i) *if* $0 \le \theta \le 1-\alpha$, *then for* $\sigma^* = \theta + \alpha$,

$$\beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(p) \stackrel{d}{=} P_{\alpha,1+\theta}\big(p\beta_{(\sigma^*/\alpha,(1-\alpha)/\alpha)}\big) \stackrel{d}{=} \beta_{1,\theta}P_{\alpha,1}\big(p\beta_{(\sigma^*/\alpha,(1-\sigma^*)/\alpha)}\big)$$

and, using (5.5), *there is the explicit formula determining the densities of* A^+_1 *and* $A^+_{G_1}$,

$$\begin{aligned}\mathbb{P}^{(p)}_{\alpha,\theta}(A^+_{G_1} \in dy)/dy &= \frac{1-y}{(1-p)}\mathbb{P}^{(p)}_{\alpha,\theta}(A^+_1 \in dy)/dy \\ &= \int_0^1 \frac{\theta\Omega_{\alpha,\sigma^*}(y/u)}{u(1-u)^{1-\theta}}\,du.\end{aligned} \tag{5.9}$$

When $\theta = 0$, (5.9) *is* $\Omega_{\alpha,\alpha}(y)$ *which agrees with Pitman and Yor* ([81], *Proposition* 15).

(ii) *As in Proposition 2.7, for any $\theta > 0$, set $\theta = \sum_{j=1}^{k} \theta_j$ for some integer k and $\theta_j > 0$. This leads to the representation*

$$P_{\alpha,\theta}(p) \stackrel{d}{=} \sum_{j=1}^{k} D_j P_{\alpha,\theta_j}(p)$$

for independent variables $P_{\alpha,\theta_j}(p)$ and $(D_1, \dots, D_k)$ a Dirichlet vector as in Proposition 2.7. When θ_j are chosen such that $0 < \theta_j \le 1 - \alpha$ each variable has $\mathbb{P}^{(p)}_{\alpha,\theta_j}(A_1^+ \in dy)$ given by (5.9) with $\sigma^* = \theta_j + \alpha$. It suffices to choose $\theta_j = \theta/k$, for $0 < \theta \le k(1-\alpha)$. If $\theta = k$, one may set $\theta_j = 1$ and use $\Omega_{\alpha,1}$.

PROOF. The various representations of the random variables are due to Proposition 5.2 and otherwise an application of the beta/gamma calculus. The density $\Omega_{\alpha,\sigma}$ is obtained from $\Delta_{\alpha,\sigma}$, which is justified by the exponential tilting relationships discussed in James ([41], Section 3; see also [42]). $\square$

REMARK 5.1. The random variable $P_{\alpha,\tau}(pV)$ described in Proposition 5.2 has law,

$$\mathbb{P}\big(P_{\alpha,\tau}(pV) \in dx\big) = \mathbb{P}^{(pV)}_{\alpha,\tau}(A_1^+ \in dx) := \int_0^1 \mathbb{P}^{(pu)}_{\alpha,\tau}(A_1^+ \in dx) f_V(u)\, du.$$

That is, it may be read as the time spent positive up till one of a process $\mathcal{B}$ whose excursion lengths, conditional on V, follow a PD(α, τ) distribution and is otherwise randomly skewed by pV. See also Aldous and Pitman ([1], Section 5.1) for connections with T-partitions. This is made clear, as follows; For $(L_t; 0 \le t \le 1)$ governed by PD(α, θ), and letting $\bar{L}_t = L_t/L_1$, there is the equivalence

$$P_{\alpha,\theta}(u) \stackrel{d}{=} \inf\{t : \bar{L}_t \ge u\}, \qquad 0 \le u \le 1.$$

In other words, letting $P^{(-1)}_{\alpha,\theta}(\cdot)$ denote the random *quantile* function of $P_{\alpha,\theta}$, it follows that $\bar{L}_t \stackrel{d}{=} P^{(-1)}_{\alpha,\theta}(t), 0 \le t \le 1$. See the next section, Section 6, for more general V.

REMARK 5.2. In reference to Propositions 5.2 and 5.3, setting $Q_{\alpha,\tau}(\sigma, p) = \beta_{\tau\sigma,\tau(1-\sigma)} P_{\alpha,\tau\sigma}(p)$ leads to a well-defined bivariate process $(Q_{\alpha,\tau}(\sigma, p) : 0 \le p \le 1, 0 < \sigma \le 1)$, that has some natural connections to the coagulation operations discussed in Pitman [75]. This observation may be deduced from the subordinator representation given in Pitman and Yor ([80], Proposition 21). When $p = 1$, $Q_{\alpha,\tau}(\sigma, 1)$ is a Dirichlet process which corresponds to the operation of coagulating PD(α, τ) by PD$(0, \tau/\alpha)$. In general one may write

$$Q_{\alpha,\tau}(\sigma, p) \stackrel{d}{=} P_{0,\tau}(\sigma) P_{\alpha,\tau\sigma}(p) \stackrel{d}{=} P_{\alpha,\tau}(p P_{0,\frac{\tau}{\alpha}}(\sigma)).$$

We will not elaborate on this here except to note that in connection with results for the standard U-coalescent, setting $\tau=\alpha$, $p=1$ one recovers ([75], Corollary 33, and Proposition 32). In a distributional sense, that is, without the nice interpretation, one also recovers ([75], Corollary 16) by setting $\sigma=\alpha=e^{-t}$ and $\tau=1$, $p=1$. When $p\neq 1$, we expect that one can obtain new, but related, interpretations of $Q_{\alpha,\tau}(\sigma,p)$.

5.1. *Some special cases.* Note that, as in [43] (combined with Proposition 5.2), one can rewrite (5.2) as

$$\begin{aligned}P_{\alpha,\theta}(p) &\stackrel{d}{=} \beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(p)(1-\xi_p)+\big(1-\beta_{\theta+\alpha,1-\alpha}P_{\alpha,\theta+\alpha}(q)\big)\xi_p\\ &\stackrel{d}{=} P_{\alpha,1+\theta}\big(p\beta_{((\theta+\alpha)/\alpha,(1-\alpha)/\alpha)}\big)(1-\xi_p)\\ &\quad+\big(1-P_{\alpha,1+\theta}\big(q\beta_{((\theta+\alpha)/\alpha,(1-\alpha)/\alpha)}\big)\big)\xi_p.\end{aligned}$$

Besides giving an alternate mixture representation in terms of easily interpreted random variables, this also suggests that one can obtain the density of $P_{\alpha,\theta}(p)$ if one knows the density of $P_{\alpha,\theta+\alpha}(p)$. In [43], it was noted that this could be applied for $\theta+\alpha=1$, which yields an expression for the density of $P_{\alpha,1-\alpha}(p)$. In view of Proposition 5.3 we see that such a density representation can be extended to any $0\le\theta\le 1-\alpha$. Of course in terms of a density representation this is not as good as the expression one can obtain from (5.9), since it would have to be used twice. In this section we look at some specific cases of random variables that have either appeared in the literature or we anticipate might be of some interest.

EXAMPLE 5.1 [$(\alpha,1-\alpha)$ a distribution relevant to phylogenetic models]. As noted in [43] the case of $P_{\alpha,1-\alpha}(p)$ equates in distribution to the limit of a phylogenetic tree model appearing in ([35], Proposition 20). Here using Proposition 5.3 we obtain a slight improvement over the density given in ([43], Corollary 6.1). Since under $\mathbb{P}^{(p)}_{\alpha,1-\alpha}$, $A^+_{G_1}\stackrel{d}{=}\beta_{1,1-\alpha}P_{\alpha,1}(p)$, we have

$$\begin{aligned}\mathbb{P}^{(p)}_{\alpha,1-\alpha}(A^+_{G_1}\in dy)/dy &= \frac{1-y}{(1-p)}\mathbb{P}^{(p)}_{\alpha,1-\alpha}(A^+_1\in dy)/dy\\ &= \int_0^1\frac{(1-\alpha)\Omega_{\alpha,1}(y/u)}{u(1-u)^\alpha}\,du.\end{aligned}\tag{5.10}$$

EXAMPLE 5.2 [The case of $\beta_{1+\alpha,1-\alpha}P_{\alpha,1+\alpha}(p)$]. Under $\mathbb{P}^{(p)}_{\alpha,1}$,

$$A^+_{G_1}\stackrel{d}{=}\beta_{1+\alpha,1-\alpha}P_{\alpha,1+\alpha}(p)\stackrel{d}{=}P_{\alpha,2}\big(p\beta_{((1+\alpha)/\alpha,(1-\alpha)/\alpha)}\big).$$

Hence its density is given by

$$\mathbb{P}^{(p)}_{\alpha,1}(A^+_{G_1}\in dy)/dy=\frac{(1-y)}{(1-p)}\Omega_{\alpha,1}(y).$$

In view of the literature related to Section 4 we believe this variable will be of interest.

We now address some harder cases.

EXAMPLE 5.3 [(α,α) occupation time of a Bessel bridge]. Obtaining density expressions for the general case of A_1^+ when $\mathcal{B}$ is a Bessel bridge has been difficult, except for the case of $\alpha = 1/2$. Due to the importance of the PD(α,α) family this quantity arises in many contexts (see, for instance, Aldous and Pitman [1]). The best results were obtained independently by Yano [91] and James, Lijoi and Prünster [43], who give expressions in terms of Abel-type transforms, that is to say, integrals of possibly nonnegative functions. Hence this does not yield a mixture representation for $P_{\alpha,\alpha}(p) \stackrel{d}{=} A_1^+$. Here we show how our results in the previous section can be used to achieve this. Under $\mathbb{P}^{(p)}_{\alpha,\alpha}$,

$$A^+_{G_1} \stackrel{d}{=} \beta_{2\alpha,1-\alpha} P_{\alpha,2\alpha}(p) \stackrel{d}{=} P_{\alpha,1+\alpha}\bigl(p\beta_{(2,(1-\alpha)/\alpha)}\bigr).$$

Hence for $\alpha \le 1/2$, we can apply statement (i) of Proposition 5.3 writing

$$A^+_{G_1} \stackrel{d}{=} \beta_{1,\alpha} P_{\alpha,1}\bigl(p\beta_{(2,(1-2\alpha)/\alpha)}\bigr)$$

to get

$$\begin{aligned}\mathbb{P}^{(p)}_{\alpha,\alpha}(A^+_{G_1} \in dy)/dy &= \frac{1-y}{(1-p)}\mathbb{P}^{(p)}_{\alpha,\alpha}(A^+_1 \in dy)/dy \\ &= \int_0^1 \frac{\alpha\Omega_{\alpha,2\alpha}(y/u)}{u(1-u)^{1-\alpha}}\,du,\end{aligned}\tag{5.11}$$

where

$$\Omega_{\alpha,2\alpha}(y) = \frac{2\sin(\pi\alpha)}{\pi}\frac{py^{2\alpha-1}(1-y)^{\alpha}[qy^{\alpha} + \cos(\pi\alpha)p(1-y)^{\alpha}]}{[y^{2\alpha}q^2 + 2qpy^{\alpha}(1-y)^{\alpha}\cos(\pi\alpha) + (1-y)^{2\alpha}p^2]^2}$$

is the density of $P_{\alpha,1}(p\beta_{(2,(1-2\alpha)/\alpha)})$.

When $\alpha > 1/2$, we, at present, need to resort to statement (ii) of Proposition 5.3. So, for instance, for $\alpha \le 2/3$, it follows that

$$P_{\alpha,\alpha}(p) \stackrel{d}{=} \beta_{\alpha/2,\alpha/2} P_{\alpha,\alpha/2}(p) + (1-\beta_{\alpha/2,\alpha/2})P'_{\alpha,\alpha/2}(p),$$

where $P_{\alpha,\alpha/2}(p)$ and $P'_{\alpha,\alpha/2}(p)$ are i.i.d. variables having distribution $\mathbb{P}^{(p)}_{\alpha,\alpha/2}(A^+_1 \in dy)$ obtainable from (5.9).

EXAMPLE 5.4 [$(\alpha,\alpha-1)$, and fragmentation equations]. Suppose that we are interested in the case of $\theta = \alpha - 1$, that under $\mathbb{P}^{(p)}_{\alpha,\alpha-1}$, $P_{\alpha,\alpha-1}(p) \stackrel{d}{=} A^+_1$. Of course this only makes sense for $\alpha > 1/2$. Notice that

$$A^+_{G_1} \stackrel{d}{=} \beta_{2\alpha-1,1-\alpha} P_{\alpha,2\alpha-1}(p),$$

so we can apply (5.9) directly if $\alpha \le 2/3$. It is then interesting to note what other quantities we can obtain. We can use another stochastic equation that takes the form

$$P_{\alpha,\theta}(p) \stackrel{d}{=} \beta_{\theta+\alpha\delta,1-\alpha\delta} P_{\alpha,\theta+\alpha\delta}(p) + (1 - \beta_{\theta+\alpha\delta,1-\alpha\delta}) P_{\alpha,-\alpha\delta}(p)$$

for $\theta > -\alpha\delta$, $0 < \delta \le 1$. Note that this equation is not well known but it is simple to check. Furthermore, a close inspection shows that it is a nice way to code Pitman's [75] fragmentation. As as special case, set $\delta = (1-\alpha)/\alpha$ and $\theta = \alpha$ to obtain

$$P_{\alpha,\alpha}(p) \stackrel{d}{=} \beta_{1,\alpha} P_{\alpha,1}(p) + (1 - \beta_{1,\alpha}) P_{\alpha,\alpha-1}(p).$$

6. Power scaling property and randomly skewed processes. We saw that in the previous section the random quantity $P_{\alpha,\tau}(pV)$, where V is a beta variable, occurs naturally and plays an interesting role. An interpretation in terms of occupation times of randomly skewed processes is mentioned in Remark 5.1, and an interpretation via coagulation processes is hinted at in Remark 5.2. Also there is the surprising stochastic equation in (5.7). There is also a related result given in Proposition 2.2. One may wonder if properties of this sort only hold for beta random variables. We show in the next result, which was first obtained in [42], that there is a considerable generalization.

PROPOSITION 6.1. *Let $\mathcal{R} \stackrel{d}{=} M_{\tau/\alpha}(F_R)$ and $\mathcal{Q} \stackrel{d}{=} M_{\tau/\alpha}(F_Q)$ denote Dirichlet means with parameters $(\tau/\alpha, R)$ and $(\tau/\alpha, Q)$ where R is a nonnegative random variable, and Q is a random variable taking values in* $[0,1]$. *Equivalently,*

$$\begin{aligned} &\mathcal{R} \stackrel{d}{=} \beta_{(\tau/\alpha,1)}\mathcal{R} + \bigl(1-\beta_{(\tau/\alpha,1)}\bigr)R \quad \text{and} \\ &\mathcal{Q} \stackrel{d}{=} \beta_{(\tau/\alpha,1)}\mathcal{Q} + \bigl(1-\beta_{(\tau/\alpha,1)}\bigr)Q, \end{aligned} \tag{6.1}$$

which implies that $\mathcal{Q} \stackrel{d}{=} M_{\tau/\alpha}(F_Q)$ takes it values in $[0,1]$. *If Q is a constant, then $M_{\tau/\alpha}(F_Q) = Q$. Then the following results hold*:

(i) *$\mathcal{R}^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=} M_\tau(F_{X_\alpha R^{1/\alpha}})$, that is,*

$$\mathcal{R}^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=} \beta_{\tau,1}\mathcal{R}^{1/\alpha} X_{\alpha,\tau} + (1-\beta_{\tau,1}) R^{1/\alpha} X_\alpha.$$

(ii) *$P_{\alpha,\tau}(\mathcal{Q})$ is a Dirichlet mean with parameters $(\tau, P_{\alpha,0}(Q))$, and satisfies,*

$$\begin{aligned} P_{\alpha,\tau}(\mathcal{Q}) &\stackrel{d}{=} \beta_{\tau,1} P'_{\alpha,\tau}(\mathcal{Q}') + (1-\beta_{\tau,1}) P_{\alpha,0}(Q) \\ &\stackrel{d}{=} \beta_{\tau,1} P'_{\alpha,\tau}(\mathcal{Q}) + (1-\beta_{\tau,1}) P_{\alpha,0}(\mathcal{Q}), \end{aligned} \tag{6.2}$$

where $P'_{\alpha,\tau}(\mathcal{Q}')$ is equivalent only in distribution to $P_{\alpha,\tau}(\mathcal{Q})$, but $\mathcal{Q}$ is the same variable.

PROOF. The first result follows by noting

$$\tau\mathbb{E}[\log(1+\lambda X_\alpha R^{1/\alpha})] = (\tau/\alpha)\mathbb{E}[\log(1+\lambda^\alpha R)],$$

which gives the negative logarithm of $\mathcal{C}_\tau(\lambda; \mathcal{R}^{1/\alpha}X_{\alpha,\tau}) = \mathcal{C}_{\tau/\alpha}(\lambda^\alpha; \mathcal{R})$. For the second, evaluate $\mathcal{C}_\tau(\lambda, P_{\alpha,\tau}(\mathcal{Q}))$ conditional on $\mathcal{Q}$ and then notice similar to Proposition 5.2, that the transform of order τ coincides with the exponential of

$$\log \mathcal{C}_{\tau/\alpha}\big((1+\lambda)^\alpha - 1; \mathcal{Q}\big) = -\frac{\tau}{\alpha}\mathbb{E}\big[\psi^{(Q)}_{\alpha,0}\big].$$

It remains then to apply (5.1) to get the second equality in (6.2). $\square$

REMARK 6.1. Notice that setting $R = \xi_\sigma$ and $Q = p\xi_\sigma$ we recover Propositions 2.2 and 5.2. Setting $R = X_\delta$ for $0 < \delta < 1$ leads to the identity $X^{1/\alpha}_{\delta,\tau/\alpha} X_{\alpha,\tau} \stackrel{d}{=} X_{\alpha\delta,\tau}$, since it follows from known properties of stable random variables that $X^{1/\alpha}_\delta X_\alpha \stackrel{d}{=} X_{\alpha\delta}$. Furthermore, if one chooses $\mathcal{Q} := \mathcal{Q}(u)$ such for each fixed u it satisfies (6.1), and for $0 < u < 1$ it is an exchangeable bridge, that is a random cumulative distribution function, then $P_{\alpha,\tau}(\mathcal{Q}(u))$ identifies a coagulation operation as described in Pitman ([77], Lemma 5.18) (see also Bertoin [8]). In particular, one recovers Pitman's [75] coagulation as follows. Setting $\mathcal{Q} = P_{\beta,\tau/\alpha}(u)$, means that $Q = P_{\delta,0}(u)$, leading easily to, $P_{\alpha,0}(P_{\delta,0}(u)) \stackrel{d}{=} P_{\alpha\delta,0}(u)$, which implies

$$P_{\alpha,\tau}(P_{\delta,\tau/\alpha}(u)) \stackrel{d}{=} P_{\alpha\delta,\tau}(u).$$

We shall discuss other applications of Proposition 6.1 and related identities elsewhere.

7. Subordinators and symmetric generalized Linnik laws and processes. Using Proposition 6.1, we define processes $(T_\alpha(\tau), \tau \ge 0)$ and $(\hat{T}_\alpha(\tau), \tau \ge 0)$, such that for each fixed $\tau > 0$,

$$T_\alpha(\tau) \stackrel{d}{=} \gamma_\tau \mathcal{R}^{1/\alpha} X_{\alpha,\tau} \stackrel{d}{=} \chi_{\alpha,\tau}\mathcal{R}^{1/\alpha} \stackrel{d}{=} \gamma_\tau M_\tau(F_{X_\alpha R^{1/\alpha}})$$

and

$$\hat{T}_\alpha(\tau) \stackrel{d}{=} \gamma_\tau P_{\alpha,\tau}(\mathcal{Q})$$

are GGC$(\tau, X_\alpha R^{1/\alpha})$ and GGC$(\tau, P_{\alpha,0}(Q))$ variables, respectively. Where we are suppressing the fact that both $\mathcal{R}$ and $\mathcal{Q}$ depend on (α, τ). In fact $T_\alpha(\tau)$ and $\hat{T}_\alpha(\tau)$ are GGC subordinators varying in $\tau > 0$. Let $\mathcal{S}_\alpha(t)$ denote a positive stable subordinator such that $\mathcal{S}_\alpha(1) \stackrel{d}{=} S_\alpha$, and let $\hat{\mathcal{S}}_\alpha(t)$ denote the subordinator with

$$-\log \mathbb{E}\big[e^{-\lambda \hat{\mathcal{S}}_\alpha(t)}\big] = t[(1+\lambda)^\alpha - 1],$$

so that $\hat{\mathcal{S}}_\alpha(1) \stackrel{d}{=} \hat{S}_\alpha$ is a random variable with density $e^{-(t-1)} f_\alpha(t)$. It follows that

$$\mathcal{S}_\alpha(\gamma_{\tau/\alpha}\mathcal{R}) \stackrel{d}{=} T_\alpha(\tau) \quad \text{and} \quad \hat{\mathcal{S}}_\alpha(\gamma_{\tau/\alpha}\mathcal{Q}) \stackrel{d}{=} \hat{T}_\alpha(\tau).$$

However, an important aspect of our results in Proposition 6.1 is that usage of $T_\alpha(\tau)$ and $\hat{T}_\alpha(\tau)$ does not require working directly with the processes $\mathcal{S}_\alpha$ and $\hat{\mathcal{S}}_\alpha$. These Lévy processes are attractive in terms of potential applications arising for instance in finance, financial econometrics or Bayesian statistics. With applications to finance in mind, it is quite natural to use these processes as Brownian time changes creating process $\mathcal{B}(T_\alpha(\cdot))$ and $\mathcal{B}(\hat{T}_\alpha(\cdot))$, for $\mathcal{B}(\cdot)$ an independent Brownian motion, we take to have log characteristic function $-\lambda^2$. The characteristic functions of $\mathcal{B}(T_\alpha(\tau))$ and $\mathcal{B}(\hat{T}_\alpha(\tau))$ can be expressed as

$$\mathcal{C}_{\tau/\alpha}(\lambda^{2\alpha}; \mathcal{R}) = e^{-\tau/\alpha \psi_R(\lambda^{2\alpha})}$$

and

$$\mathcal{C}_{\tau/\alpha}\big((1+\lambda^2)^\alpha - 1; \mathcal{Q}\big) = e^{-\tau/\alpha \psi_Q((1+\lambda^2)^\alpha - 1)}.$$

Recall that $\mathcal{B}(\mathcal{S}_\alpha(\cdot))$ is a symmetric stable process of index $(0, 2]$ and $\mathcal{B}(\hat{\mathcal{S}}_\alpha(\cdot))$ is a process that includes the NIG process when $\alpha = 1/2$. When $\alpha = 0$ and $Q = p$, $\mathcal{B}(\hat{T}_0(\cdot))$, is a variance-gamma (VG) process. The case of $\mathcal{B}(\chi_{\alpha,\theta})$, corresponds to generalized Linnik processes considered by Pakes [70] (see also [25, 57]). We will focus on this case.

It suffices to examine the random variables

$$\mathcal{B}(\chi_{\alpha,\theta}) \stackrel{d}{=} N\sqrt{2\gamma_{\theta/\alpha}^{1/\alpha} S_\alpha} = N\sqrt{\chi_{\alpha,\theta}},$$

where N is a standard Normal random variable. For general α and $\theta > 0$ the extra randomization by N does not add much beyond our results for $\chi_{\alpha,\theta}$.

However, when $\alpha \le 1/2$ we are able to obtain some interesting results which we describe below. In this case, we will use a result of Devroye [24] which yields a tractable mixture representation for symmetric stable random variables of index between 0 and 1. Devroye's [24] result is not well known but as we shall show can be used to obtain a nice description of the density of $\mathcal{B}_\alpha(T_\alpha(\theta))$ for all fixed $\theta > 0$. We do, however, stress that there are many applications requiring $\alpha > 1/2$.

7.1. $\alpha \le 12$, *results based on Fejer-de la Vallee Poussin mixtures.* For symmetric stable random variables $N\sqrt{2S_\alpha}$ for $0 < \alpha \le 1/2$, Devroye [24] shows that

$$N\sqrt{2S_\alpha} \stackrel{d}{=} Y/Z^{1/(2\alpha)},$$

where Y has a Fejer-de la Vallee Poussin density

$$\omega(x) = \frac{1}{2\pi}\left(\frac{\sin(x/2)}{x/2}\right)^2, \qquad -\infty < x < \infty,$$

and $Z \stackrel{d}{=} \gamma_1(1 - \xi_{2\alpha}) + \gamma_2 \xi_{2\alpha}$. It follows that the density of a symmetric stable of index between $[0, 1]$ is

$$2\alpha \int_0^\infty \omega(xy) y^{2\alpha} e^{-y^{2\alpha}} [(1 - 2\alpha) + 2\alpha y^{2\alpha}]\, dy.$$

Hence as a mild extension of Devroye ([24], Example B), that is, the simple symmetric Linnik variable corresponding to $\theta = \alpha$, we have for all $\theta > 0$

$$N\sqrt{2\chi_{\alpha,\theta}} \stackrel{d}{=} Y/W^{1/(2\alpha)} \qquad \text{where } W \stackrel{d}{=} \frac{\gamma_1}{\gamma_{\theta/\alpha}}(1-\xi_{2\alpha}) + \frac{\gamma_2}{\gamma_{\theta/\alpha}}\xi_{2\alpha},$$

is a mixture of Pareto variables, having density for $0 < \alpha \le 1/2$

$$\frac{\theta[(1+2\theta)w + (1-2\alpha)]}{\alpha(1+w)^{2+\theta/\alpha}}, \qquad w > 0.$$

Naturally this representation extends to all $\mathcal{B}(T_\alpha(\cdot))$, provided that $\alpha \le 1/2$. In particular,

$$\mathcal{B}(T_\alpha(\theta)) \stackrel{d}{=} Y/\tilde{W}^{1/(2\alpha)} \qquad \text{where now } \tilde{W} \stackrel{d}{=} \frac{\gamma_1}{\gamma_{\theta/\alpha}\mathcal{R}}(1-\xi_{2\alpha}) + \frac{\gamma_2}{\gamma_{\theta/\alpha}\mathcal{R}}\xi_{2\alpha},$$

for $\mathcal{R} \stackrel{d}{=} M_{\theta/\alpha}(F_R)$. Quite interestingly the density of $\tilde{W}$ only requires information about the Laplace transform of $\gamma_{\theta/\alpha}\mathcal{R}$. Let $\psi_R^{(1)}(x)$ and $\psi_R^{(2)}(x)$ denote the first and second derivatives of $\psi_R(x)$. Then the density of $\tilde{W}$ is given by

$$\eta_{\alpha,\theta}(x) = \frac{\theta}{\alpha}e^{-\theta/\alpha\psi_R(x)}\left[\psi_R^{(1)}(x)(1-2\alpha) + x2\alpha\left[\left(\psi_R^{(1)}(x)\right)^2\frac{\theta}{\alpha} - \psi_R^{(2)}(x)\right]\right].$$

From this, we close with an interesting identity.

PROPOSITION 7.1. *For $0 < \alpha \le 1/2$, and $\theta > 0$, let $V \stackrel{d}{=} \beta_{1/2,1/2}$, then for $-\infty < x < \infty$*

$$\begin{aligned}\Phi_{\alpha,\theta}(x) &= \mathbb{E}\left[\frac{|x|}{2V\chi_{\alpha,\theta}}e^{-x^2/(4V\chi_{\alpha,\theta})}\right]\\ &= \sqrt{\frac{2}{\pi}}\mathbb{E}\left[\frac{1}{|\sqrt{2\chi_{\alpha,\theta}}|}e^{-x^2/(4\chi_{\alpha,\theta})}\right]\\ &= \int_0^\infty \frac{\omega(xy)2\theta y^{2\alpha}[(1+2\theta)y^{2\alpha} + (1-2\alpha)]}{(1+y^{2\alpha})^{2+\theta/\alpha}}\,dy,\end{aligned}$$

which is just the density of $N\sqrt{2\chi_{\alpha,\theta}}$. Additionally, for all fixed $\theta > 0$, the density of $\mathcal{B}(T_\alpha(\theta)) \stackrel{d}{=} N\sqrt{2\chi_{\alpha,\theta}}\mathcal{R}^{1/(2\alpha)}$, satisfies

$$\mathbb{E}\left[\Phi_{\alpha,\theta}\left(\frac{x}{\mathcal{R}^{1/(2\alpha)}}\right)\frac{1}{\mathcal{R}^{1/(2\alpha)}}\right] = 2\alpha\int_0^\infty \omega(xy)y^{2\alpha}\eta_{\alpha,\theta}(y^{2\alpha})\,dy$$

for $\mathcal{R} \stackrel{d}{=} M_{\theta/\alpha}(F_R)$.

PROOF. The result follows from the derivation of the density described above using ω, in combination with a derivation of the density based on $N^2 \stackrel{d}{=} 2\gamma_1 V$ and additionally Pitman and Yor ([83], equation (29)). $\square$

8. General remark about rational case. When α is rational it is known (see [20, 89, 94]) that S_α is equivalent in distribution to a product of independent beta and gamma variables. Extending an argument in Chaumont and Yor ([20], pages 143 and 144) using the gamma duplication formula, it follows that for $\alpha = m/n$ for integers, m, n, such that $m < n$, and all $\theta > -m/n$,

$$(X_{m/n,\theta})^m \stackrel{d}{=} \left(\frac{S_{m/n}}{S_{m/n,\theta}}\right)^m \stackrel{d}{=} \left(\prod_{k=1}^{m-1} \frac{\beta_{\theta/m+k/n,k(1/m-1/n)}}{\beta_{k/n,k(1/m-1/n)}}\right)\left(\prod_{k=m}^{n-1} \frac{\gamma_{\theta/m+k/n}}{\gamma_{k/n}}\right),$$

where all random variables are independent. Additionally,

$$\left(\frac{m}{S_{m/n,\theta}}\right)^m \stackrel{d}{=} n^n \left(\prod_{k=1}^{m-1} \beta_{\theta/m+k/n,k(1/m-1/n)}\right)\left(\prod_{k=m}^{n-1} \gamma_{\theta/m+k/n}\right).$$

An implication of these relationships is that one may use the result of Springer and Thompson [86] to express their densities in terms of Meijer-G functions. In many cases these are equivalent to expressions in terms of generalized Gauss hypergeometric functions. Furthermore, it is known that Laplace transforms of Meijer-G functions are also Meijer-G functions, with known arguments. Hence, Propositions 3.1 and 5.1 show that in the rational case of $\alpha = m/n$, one may express the generalized Mittag–Leffler functions and densities for $P_{\alpha,\theta}(p)$ in terms of Meijer-G functions. Such representations are not entirely appealing in many respects; for instance, the density $\Delta_{m/n,\sigma}$ is a much more desirable expression than its Meijer-G counterpart. However, from a computational viewpoint they are significant. This is due to the fact that Meijer-G functions, which constitute many special functions, are available as built-in functions in mathematical computational packages such as Mathematica or Maple. Naturally many of the quantities we discussed for general α can be expressed as the more general Fox-H functions [33, 51–54, 67]. However, in general, computations for these expressions are not yet available. Hence another contribution of our work is to give new explicit identities for a class of Meijer-G and Fox-H functions. That is to say quantities such as $\Delta_{\alpha,\sigma}$ give an explicit form to their corresponding Fox-H representation. We omit details of this representation, but it is not difficult to obtain.

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