The Annals of Applied Probability
2010, Vol. 20, No. 3, 951–977
DOI: 10.1214/09-AAP641

ON COLLISIONS OF BROWNIAN PARTICLES[1]

BY TOMOYUKI ICHIBA AND IOANNIS KARATZAS

University of California, Santa Barbara and INTECH

We examine the behavior of n Brownian particles diffusing on the real line with bounded, measurable drift and bounded, piecewise continuous diffusion coefficients that depend on the current configuration of particles. Sufficient conditions are established for the absence and for the presence of triple collisions among the particles. As an application to the Atlas model for equity markets, we study a special construction of such systems of diffusing particles using Brownian motions with reflection on polyhedral domains.

1. Introduction. It is well known that, with probability one, the n-dimensional Brownian motion started away from the origin will hit the origin infinitely often for $n = 1$ while it will never hit the origin for $n \geq 2$. This is also true for the n-dimensional Brownian motion with constant drift and diffusion coefficients, by Girsanov's theorem and re-orientation of coordinates. The next step of generalization is the case of bounded drift and diffusion coefficients. The existence of weak solutions for the stochastic equations that describe such processes was discussed by Krylov [16] and Stroock and Varadhan [24] through the study of appropriate martingale problems.

Now let us suppose that $\mathbb{R}^n$ is partitioned as a finite union of disjoint polyhedra. Bass and Pardoux [3] established the existence and uniqueness of a weak solution to the stochastic integral equation,

$$X(t) = x_0 + \int_0^t \mu(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \qquad 0 \leq t < \infty, \tag{1.1}$$

with initial condition $x_0 \in \mathbb{R}^n$ where the measurable functions $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are bounded, and, moreover, σ is everywhere nonsingular and piecewise constant (i.e., constant on each polyhedron). The continuous process $\{W(t), 0 \leq t < \infty\}$ is an n-dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Here uniqueness is understood in the sense of the probability distribution.

Received October 2008; revised April 2009.

[1]Supported in part by NSF Grant NSF-DMS-06-01774 at Columbia University; the second author is on leave from Columbia University's Department of Mathematics.

AMS 2000 subject classifications. Primary 60G17, 60G44; secondary 60G85.

Key words and phrases. Martingale problem, triple collision, effective dimension, Bessel process, reflected Brownian motion, comparison theorem, Atlas model.

Bass and Pardoux also discovered an interesting phenomenon, namely, that the weak solution to (1.1) may satisfy

$$\mathbb{P}_{x_0}\big(X(t)=0,\ \text{i.o.}\big)=1;\qquad x_0\in\mathbb{R}^n, \tag{1.2}$$

for a diffusion matrix $\sigma(\cdot)$ with special structure and without drift $\mu(\cdot)\equiv 0$. Here $\mathbb{P}_{x_0}$ is the solution to the martingale problem corresponding to (1.1). In the Bass and Pardoux [3] example, the whole space $\mathbb{R}^n$ is partitioned into a finite number of polyhedral domains with common vertex at the origin, carefully chosen *small* apertures and $\sigma(\cdot)$ constant in each domain. We review this example in Remark 2.4.

In the present paper we find conditions sufficient for ruling (1.2) out. More specifically, we are interested in the case of a bounded, measurable drift vector $\mu(\cdot)$ and of a bounded, piecewise continuous diffusion matrix,

$$\sigma(x)=\sum_{\nu=1}^{m}\sigma_\nu(x)\mathbf{1}_{\mathcal{R}_\nu}(x)\equiv\sigma_{\mathfrak{p}(x)}(x);\qquad x\in\mathbb{R}^n, \tag{1.3}$$

under the assumption of *well-posedness* (existence and uniqueness of solution) when $n\ge 3$. Here $\mathbf{1}_{\{\cdot\}}$ is the indicator function; the sets $\{\mathcal{R}_\nu\}_{\nu=1}^m$ form a partition of $\mathbb{R}^n$ for some $m\in\mathbb{N}$, namely, $\mathcal{R}_\nu\cap\mathcal{R}_\kappa=\varnothing$ for $\nu\ne\kappa$ and $\bigcup_{\nu=1}^m\mathcal{R}_\nu=\mathbb{R}^n$ and the mapping $\mathfrak{p}:\mathbb{R}^n\to\{1,\dots,m\}$ satisfies $x\in\mathcal{R}_{\mathfrak{p}(x)}$ for every $x\in\mathbb{R}^n$. Throughout this paper we shall assume that $\mathcal{R}_\nu$ is an n-dimensional polyhedron for each $\nu=1,\dots,m$, and that the $(n\times n)$ matrix-valued functions $\{\sigma_\nu(\cdot)\sigma_\nu'(\cdot)\}_{\nu=1}^m$ are positive-definite everywhere.

We shall also assume throughout that there exists a unique weak solution for equation (1.1). Existence is guaranteed by the measurability and boundedness of the functions $\mu(\cdot)$ and $\sigma(\cdot)\sigma'(\cdot)$ as well as the uniform strong nondegeneracy of $\sigma(\cdot)\sigma'(\cdot)$ (e.g., Krylov [17], Remark 2.1) where the superscript $'$ represents the transposition. Uniqueness holds when $n=1$ or $n=2$; for $n\ge 3$, the argument of Chapter 7 of Stroock and Varadhan [24] implies uniqueness if the function $\sigma(\cdot)$ in (1.3) is continuous on $\mathbb{R}^n$ (Theorem 7.2.1 of [24]) or close to constant (Theorem 7.1.6 of [24]), namely, if there exists a constant $(n\times n)$ matrix α and a sufficiently small $\delta>0$, depending on the dimension n and the bounds of eigenvalues of $\sigma(\cdot)$ such that $\sup_{x\in\mathbb{R}^n}\|\sigma(x)\sigma'(x)-\alpha\|\le\delta$. Bass and Pardoux [3] showed uniqueness for piecewise-constant coefficients, that is, $\sigma_\nu(\cdot)\equiv\sigma_\nu$, $\nu=1,\dots,m$. For further discussion on uniqueness and non-uniqueness, we refer to the paper by Krylov [17] and the references therein. The structural assumption (1.3) may be weakened to more general bounded cases, under modified conditions.

Our main concern is to obtain sufficient conditions on $\mu(\cdot)$ and on $\sigma(\cdot)$ of the form (1.3) so that with $n\ge 3$ we have

$$\begin{aligned}&\mathbb{P}_{x_0}\big(X_i(t)=X_j(t)=X_k(t),\ \text{for some } t\ge 0\big)=0\quad\text{or}\\ &\mathbb{P}_{x_0}\big(X_i(t)=X_j(t)=X_k(t),\ \text{for some } t\ge 0\big)=1;\qquad x_0\in\mathbb{R}^n,\end{aligned} \tag{1.4}$$

for some $1 \le i < j < k \le n$. Put differently, we study conditions on their drift and diffusion coefficients, under which three Brownian particles moving on the real line can collide at the same time, and conditions under which such "*triple collisions*" can never occur. Propositions 1 and 2 provide answers to these questions in Section 2.

In Section 3 we study a class of the weak solutions to the stochastic differential equation (1.1), clarifying the relationship between the rank of process coordinates and the reflected Brownian motion on $(n-1)$-dimensional polyhedral domain. Proposition 3 shows that the process has no triple collisions under some parametric conditions.

The results have consequences in the computations of local times for the differences $\{X_i(t) - X_j(t), X_j(t) - X_k(t)\}$. We discuss such local times with application to the analysis of a so-called "Atlas model" for equity markets in Section 4. Proofs of selected results are presented in Appendix.

Recent work related to this problem was done by Cépa and Lépingle [5]. These authors consider a system of mutually repelling Brownian particles and show the absence of triple collisions. The electrostatic repulsion they consider comes from unbounded drift coefficients; in our setting, all drifts are bounded.

2. A first approach.

2.1. *The setting*. Consider the stochastic integral equation (1.1) with coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ as in (1.3), and assume that the matrix-valued functions $\sigma_\nu(\cdot)$, $\nu = 1, \dots, m$, are uniformly positive-definite. Then the inverse $\sigma^{-1}(\cdot)$ of the diffusion coefficient $\sigma(\cdot)$ exists in the sense $\sigma^{-1}(\cdot) = \sum_{\nu=1}^m \sigma_\nu^{-1}(\cdot)\mathbf{1}_{\mathcal{R}_\nu}(\cdot)$. As usual, a *weak solution* of this equation consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; a filtration $\{\mathcal{F}_t, 0 \le t < \infty\}$ of sub-σ-fields of $\mathcal{F}$ which satisfies the "usual conditions" of right-continuity and augmentation by the $\mathbb{P}$-negligible sets in $\mathcal{F}$; and two adapted, n-dimensional processes on this space $X(\cdot)$, $W(\cdot)$ on this space, such that $W(\cdot)$ is Brownian motion and (1.1) is satisfied $\mathbb{P}$-almost surely. The concept of uniqueness associated with this notion of solvability, is *uniqueness in distribution* for $X(\cdot)$.

2.2. *Removal of drift*. We start by observing that the bounded drift has no effect on the probability of absence of triple collisions. Indeed, if we define an n-dimensional process $\xi(t) := \sigma^{-1}(X(t))\mu(X(t)), 0 \le t < \infty$, then the nature of the functions $\mu(\cdot)$ and $\sigma(\cdot)$ in (1.3) guarantees that the mapping $t \mapsto \xi(t)$ is right-continuous or left-continuous on each boundary $\partial\mathcal{R}_{\mathfrak{p}(X(t))}$ at each time t, deterministically, according to the position $\mathcal{R}_{\mathfrak{p}(X(t-))}$ of $X(t-)$. Thus, although the sample path of n-dimensional process $\xi(\cdot)$ is not entirely right-continuous or left-continuous, it *is* progressively measurable. Moreover, $\xi(\cdot)$ is bounded, so the exponential process

$$(2.1) \qquad \eta(t) = \exp\left[-\int_0^t \langle \xi(u), dW(u)\rangle - \frac{1}{2}\int_0^t \|\xi(u)\|^2\, du\right]; \qquad 0 \le t < \infty,$$

is a continuous martingale where $\|x\|^2 := \sum_{j=1}^n x_j^2, x \in \mathbb{R}^n$, stands for n-dimensional Euclidean norm, and the bracket $\langle x, y\rangle := \sum_{j=1}^n x_j y_j$ is the inner product of two vectors $x, y \in \mathbb{R}^n$. By Girsanov's theorem,

$$\widetilde{W}(t) := W(t) + \int_0^t \sigma^{-1}(X(u))\mu(X(u))\,du, \qquad \mathcal{F}_t; 0 \le t < \infty, \tag{2.2}$$

is an n-dimensional standard Brownian motion under the new probability measure $\mathbb{Q}$, *locally* equivalent to $\mathbb{P}$, that satisfies

$$\mathbb{Q}_{x_0}(C) = \mathbb{E}^{\mathbb{P}_{x_0}}(\eta(T)1_C); \qquad C \in \mathcal{F}_T, 0 \le T < \infty. \tag{2.3}$$

Let us define an increasing family of events $C_T := \{X_i(t) = X_j(t) = X_k(t)$, for some $t \in [0, T]\}$, $T \ge 0$. If we know a priori that

$$\mathbb{Q}_{x_0}\big(X_i(t) = X_j(t) = X_k(t), \text{ for some } t \ge 0\big) = 0, \tag{2.4}$$

then we obtain $0 = \mathbb{Q}_{x_0}(C_\ell) = \mathbb{P}_{x_0}(C_\ell)$ for $\ell \ge 1$, and so

$$\begin{aligned}\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t), \text{ for some } t \ge 0\big) &= \mathbb{P}_{x_0}\Bigg(\bigcup_{\ell=1}^\infty C_\ell\Bigg)\\ &= \lim_{\ell\to\infty} \mathbb{P}_{x_0}(C_\ell) = 0.\end{aligned} \tag{2.5}$$

Thus, in order to evaluate the probability of absence of triple collisions in (1.4), it is enough to consider the case of $\mu(\cdot) \equiv 0$ in (1.1), namely

$$X(t) = x_0 + \int_0^t \sigma(X(s))\,d\widetilde{W}(s), \qquad 0 \le t < \infty, \tag{2.6}$$

under the new probability measure $\mathbb{Q}_{x_0}$. The infinitesimal generator $\mathcal{A}$ of this process, defined on the space $C^2(\mathbb{R}^n; \mathbb{R})$ of twice continuously differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, is given as

$$\mathcal{A}\varphi(x) := \frac{1}{2}\sum_{i,k=1}^n a_{ik}(x)\frac{\partial^2}{\partial x_i\,\partial x_k}[\varphi(x)]; \qquad \varphi \in C^2(\mathbb{R}^n; \mathbb{R}), \tag{2.7}$$

where $\sigma_{ij}(\cdot)$ is the (i, j)th element of the matrix-valued function $\sigma(\cdot)$, and

$$a_{ik}(x) := \sum_{j=1}^n \sigma_{ij}(x)\sigma_{kj}(x), \qquad A(x) := \{a_{ij}(x)\}_{1\le i,j\le n}; \qquad x \in \mathbb{R}^n. \tag{2.8}$$

The uniform positive-definiteness of the matrices $\{\sigma_\nu\sigma_\nu'\}(\cdot)$, $\nu = 1, \dots, m$, in (1.3) implies that the operator $\mathcal{A}$ is uniformly elliptic. As is well known from [24], existence (respectively, uniqueness) of a weak solution to the stochastic integral equation (2.6), is equivalent to the solvability (respectively, well-posedness) of the martingale problem associated with the operator $\mathcal{A}$.

2.3. *Comparison with Bessel processes.* Without loss of generality we start from the case $i = 1$, $j = 2, k = 3$ in (1.4). Let us define $(n \times 1)$ vectors d_1, d_2, d_3 to extract the information of the diffusion matrix $\sigma(\cdot)$ on (X_1, X_2, X_3), namely

$$d_1 := (1, -1, 0, \dots, 0)', \qquad d_2 := (0, 1, -1, 0, \dots, 0)',$$
$$d_3 := (-1, 0, 1, 0, \dots, 0)',$$

where the superscript $'$ stands for transposition. Define the $(n \times 3)$-matrix $D = (d_1, d_2, d_3)$ for notational simplicity. The cases we consider in (1.4) for $i = 1$, $j = 2, k = 3$ are equivalent to

$$\mathbb{P}_{x_0}\big(s^2(X(t)) = 0,\ \text{for some } t \geq 0\big) = 0 \quad \text{and}$$
$$\mathbb{P}_{x_0}\big(s^2(X(t)) = 0,\ \text{for some } t \geq 0\big) = 1; \qquad x_0 \in \mathbb{R}^n$$

where the continuous function

$$\begin{aligned} s^2(x) &:= (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \\ &= d_1' x x' d_1 + d_2' x x' d_2 + d_3' x x' d_3 = x' D D' x; \qquad x \in \mathbb{R}^n \end{aligned} \tag{2.9}$$

measures the sum of squared distances for the three particles of interest. Thus, it suffices to study the behavior of the continuous, nonnegative process $\{s^2(X(t)); 0 \leq t < \infty\}$ around its zero set

$$\mathcal{Z} := \{x \in \mathbb{R}^n : s(x) = 0\}. \tag{2.10}$$

Let us define the following positive, piecewise continuous functions $Q(\cdot)$, $\widetilde{R}(\cdot)$ computed from the variance–covariance matrix $A(\cdot) = \sigma(\cdot)\sigma'(\cdot)$:

$$\begin{aligned} \widetilde{R}(x) &:= \frac{\operatorname{trace}(D'A(x)D) \cdot x'DD'x}{x'DD'A(x)DD'x} = \frac{\operatorname{trace}(D'A(x)D)}{Q(x)}, \qquad \text{where} \\ Q(x) &:= \frac{x'DD'A(x)DD'x}{x'DD'x}; \qquad x \in \mathbb{R}^n \setminus \mathcal{Z}. \end{aligned} \tag{2.11}$$

Under the new probability measure $\mathbb{Q}_{x_0}$ of (2.3) the process $s(X(\cdot))$ is a semimartingale with decomposition $ds(X(t)) = \widetilde{h}(X(t))\,dt + d\widetilde{\Theta}(t)$ where

$$\begin{aligned} \widetilde{h}(x) &:= \frac{1}{2s^3(x)} \left(s^2(x) \sum_{i=1}^{3} d_i' \sigma(x)\sigma(x)' d_i - \left\| \sum_{i=1}^{3} \sigma(x)' d_i d_i' x \right\|^2 \right) \\ &= \frac{x'DD'x \cdot \operatorname{trace}(D'A(x)D) - x'DD'A(x)DD'x}{2(x'DD'x)^{3/2}} \\ &= \frac{(\widetilde{R}(x) - 1)Q(x)}{2s(x)}; \qquad x \in \mathbb{R}^n \setminus \mathcal{Z}, \end{aligned} \tag{2.12}$$

and

$$\widetilde{\Theta}(t) := \int_0^t \left(\sum_{i=1}^{3} \frac{\sigma'(X(\tau)) d_i d_i' X(\tau)}{s(X(\tau))} \right) d\widetilde{W}(\tau),$$

$$\langle \widetilde{\Theta} \rangle(t) = \int_0^t \frac{x' D D' A(x) D D' x}{x' D D' x} \bigg|_{x = X(\tau)} d\tau = \int_0^t Q(X(\tau))\, d\tau; \qquad 0 \le t < \infty,$$

respectively. Here, as we shall see (2.27) in Remark 2.1, we have

$$Q(\cdot) \ge c_0 := 3 \min_{1 \le i \le n, x \in \mathbb{R}^n \setminus \mathcal{Z}} \lambda_i(x) > 0 \qquad \text{in } \mathbb{R}^n \setminus \mathcal{Z} \tag{2.13}$$

for the eigenvalues $\{\lambda_i(\cdot), 1 \le i \le n\}$ of $A(\cdot)$, and so $\langle \widetilde{\Theta} \rangle(\cdot)$ is strictly increasing, when $X(\cdot) \in \mathbb{R}^n \setminus \mathcal{Z}$. Now we define the increasing family of stopping times $\Lambda_u := \inf\{t \ge 0 : \langle \widetilde{\Theta} \rangle(t) \ge u\}$, $0 \le u < \infty$, and note that we have

$$\mathfrak{s}(u) := s(X(\Lambda_u)) = s(x_0) + \int_0^{\Lambda_u} \widetilde{h}(X(t))\, dt + \widetilde{B}(u); \qquad 0 \le u < \infty,$$

where $\widetilde{B}(u) := \widetilde{\Theta}(\Lambda_u), 0 \le u < \infty$, is a standard Brownian motion, by the Dambis–Dubins–Schwarz theorem on time-change for martingales. Thus, with $\mathfrak{d}(u) := \widetilde{R}(X(\Lambda_u))$ we can write

$$d\mathfrak{s}(u) = \frac{\mathfrak{d}(u) - 1}{2\mathfrak{s}(u)}\, du + d\widetilde{B}(u); \qquad 0 \le u < \infty, \tag{2.14}$$

because

$$\widetilde{h}(X(\Lambda_u))\Lambda_u' = \frac{[\widetilde{R}(X(\Lambda_u)) - 1] Q(X(\Lambda_u))}{2s(X(\Lambda_u))} \cdot \frac{1}{Q(X(\Lambda_u))} = \frac{\mathfrak{d}(u) - 1}{2\mathfrak{s}(u)}.$$

The dynamics of the process $\mathfrak{s}(\cdot)$ are therefore comparable to those of the δ-dimensional Bessel process, namely

$$d\mathfrak{r}(u) = \frac{\delta - 1}{2\mathfrak{r}(u)}\, du + d\widetilde{B}(u); \qquad 0 \le u < \infty.$$

By a comparison argument similar to Ikeda and Watanabe [14] and Exercise 5.2.19 in [15], we prove in Section A.1 the following result.

LEMMA 2.1. *Suppose* $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$. *If* $\underline{\mathfrak{d}} := \operatorname{essinf} \inf_{0 \le t < \infty} \mathfrak{d}(t) \ge 2$,

$$\mathbb{Q}_{x_0}\big(\mathfrak{s}(t) > 0, \text{ for some } t \ge 0\big) = 0. \tag{2.15}$$

If, on the other hand, $\overline{\mathfrak{d}} := \operatorname{essup} \sup_{0 \le t < \infty} \mathfrak{d}(t) < 2$, *then*

$$\mathbb{Q}_{x_0}\big(\mathfrak{s}(t) = 0, \text{ for infinitely many } t \ge 0\big) = 1; \tag{2.16}$$

and we have the following estimate:

$$\mathbb{Q}_{x_0}\big(\mathfrak{s}(t) = 0, \text{ for some } t \in [0, T]\big) \ge 1 - \kappa(T; s(x_0), \overline{\mathfrak{d}}), \tag{2.17}$$

where $\kappa(\cdot; y, \delta)$ is the tail distribution of the first hitting-time at the origin for Bessel process in dimension $\delta \in (0, 2)$, starting at $y > 0$,

$$\kappa(T; y, \delta) := \int_T^\infty \frac{1}{t\Gamma(\delta)}\left(\frac{y^2}{2t}\right)^\delta e^{-y^2/2t}\,dt; \qquad 0 \le T < \infty, y > 0. \tag{2.18}$$

This function decreases as $T^{-\delta}$ with $T \uparrow \infty$. Combining Lemma 2.1 with the reasoning in Section 2.2 and the definition $\mathfrak{d}(\cdot) = \widetilde{R}(X(\Lambda.))$, we obtain the following result on the *absence* of triple collisions:

PROPOSITION 1. *Suppose that the matrices $\sigma_\nu(\cdot)$, $\nu = 1, \dots, m$, in (1.3) are uniformly bounded and positive-definite and satisfy the following condition*:

$$\inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} \widetilde{R}(x) \ge 2 \tag{2.19}$$

for $\widetilde{R}(\cdot)$ in (2.11). Then for the weak solution $X(\cdot)$ to (2.6) we have

$$\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t \ge 0\big) = 0 \qquad \forall x_0 \in \mathbb{R}^n \setminus \mathcal{Z}.$$

Reasoning as in (2.4)–(2.5) for the weak solution $X(\cdot)$ to (1.1), we get

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t \ge 0\big) = 0 \qquad \forall x_0 \in \mathbb{R}^n \setminus \mathcal{Z}. \tag{2.20}$$

A class of examples satisfying (2.19) is given in Remarks 2.2–2.3 and Section A.3 below. On the other hand, regarding the *presence* of triple collisions, we have the following result; its proof is in Section A.2.

PROPOSITION 2. *Suppose that the matrices $\sigma_\nu(\cdot)$, $\nu = 1, \dots, m$, in (1.3) are uniformly bounded and positive-definite, and*

$$\delta_0 := \sup_{x\in\mathbb{R}^n\setminus\mathcal{Z}} \widetilde{R}(x) < 2. \tag{2.21}$$

Then the weak solution $X(\cdot)$ to (2.6) starting at any $x_0 \in \mathbb{R}^n$ satisfies

$$\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t \ge 0\big) = 1,$$

and we have an estimate similar to (2.17),

$$\begin{aligned}&\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t \in [0, T]\big)\\ &\quad \ge 1 - \kappa(c_0 T; s(x_0), \delta_0).\end{aligned} \tag{2.22}$$

Here the distance function $s(\cdot)$ and the tail probability $\kappa(\cdot; y, \delta_0)$ are given by (2.9) and (2.18), now with dimension $\delta_0 \in (0, 2)$ as in (2.21), and the positive constant c_0 is given by (2.13).

Moreover, if $\delta_* := \sup_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) < 2$ *holds for the modification*

$$
\begin{aligned}
R(x) &:= \frac{[\mathrm{trace}(D'A(x)D) + 2x'DD'\mu(x)]\cdot x'DD'x}{x'DD'A(x)DD'x} \\
&= \widetilde{R}(x) + \frac{2x'DD'\mu(x)}{Q(x)}; \qquad x\in\mathbb{R}^n\setminus\mathcal{Z},
\end{aligned}
\tag{2.23}
$$

of the function $\widetilde{R}(\cdot)$ *in* (2.11), *then*

$$
\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t\ge 0\big) = 1,
\tag{2.24}
$$

and we have an estimate similar to (2.17), (2.22),

$$
\begin{aligned}
&\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t),\ \text{for some } t\in[0,T]\big) \\
&\qquad \ge 1 - \kappa(c_0T; s(x_0), \delta_*).
\end{aligned}
\tag{2.25}
$$

REMARK 2.1. Since $A(\cdot)$ is positive-definite and $\mathrm{rank}(D) = 2$, the matrix $D'A(\cdot)D$ is nonnegative-definite and the number of its nonzero eigenvalues is equal to $\mathrm{rank}(D'A(\cdot)D) = 2$. This implies

$$
\widetilde{R}(x) \ge \frac{\sum_{i=1}^{3}\lambda_i^D(x)}{\max_{1\le i\le 3}\lambda_i^D(x)} > 1; \qquad x\in\mathbb{R}^n\setminus\mathcal{Z},
$$

where $\{\lambda_i^D(\cdot), i = 1,2,3\}$ are the eigenvalues of the (3×3) matrix $D'A(\cdot)D$. On the other hand, an upper bound for $\widetilde{R}(\cdot)$ is given by

$$
\widetilde{R}(x) \le \frac{\mathrm{trace}(D'A(x)D)}{3\min_{1\le i\le n}\lambda_i(x)}; \qquad x\in\mathbb{R}^n\setminus\mathcal{Z},
\tag{2.26}
$$

where $\{\lambda_i(\cdot), 1\le i\le n\}$ are the eigenvalues of $A(\cdot)$. In fact, we can verify $DD'DD' = 3DD'$, $\{x\in\mathbb{R}^n : DD'x = 0\} = \mathcal{Z}$, and so if $DD'x \ne 0 \in \mathbb{R}^n$, we obtain the upper bound (2.26) for $\widetilde{R}(\cdot)$ from

$$
\min_{1\le i\le n}\lambda_i(x) \le \frac{x'DD'A(x)DD'x}{x'DD'DD'x} = \frac{Q(x)}{3} = \frac{\mathrm{trace}(D'A(x)D)}{3\widetilde{R}(x)}.
\tag{2.27}
$$

REMARK 2.2. For the standard, n-dimensional Brownian motion, that is, $\sigma(\cdot)\equiv I_n$, $n\ge 3$, the quantity $\widetilde{R}(\cdot)$ of (2.11) is computed easily; $\widetilde{R}(\cdot)\equiv 2$. More generally, suppose that the variance covariance rate $A(\cdot)$ is

$$
A(x) := \sum_{\nu=1}^{m}\big(\alpha_\nu I_n + \beta_\nu DD' + \mathbb{I}\mathbb{I}'\,\mathrm{diag}(\gamma_\nu)\big)\cdot \mathbf{1}_{\mathcal{R}_\nu}(x); \qquad x\in\mathbb{R}^n,
$$

for some scalar constants α_ν, β_ν and $(n\times 1)$ constant vectors γ_ν, $\nu = 1,\dots,m$. Here $\mathrm{diag}(x)$ is the $(n\times n)$ diagonal matrix whose diagonal entries are the elements of $x\in\mathbb{R}^n$, and $\mathbb{I}$ is the $(n\times 1)$ vector with all entries equal to one. Then $\widetilde{R}(\cdot)\equiv 2$

in $\mathbb{R}^n \setminus \mathcal{Z}$ because $\mathbb{I}'D = (0,0,0) \in \mathbb{R}^{1\times 3}$ and

$$DD' = \frac{1}{3}DD'DD' = \begin{pmatrix} 2 & -1 & -1 & \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & \\ & 0 & 0 & \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Hence, if the coefficients α_ν, β_ν and γ_ν, $\nu = 1,\dots,m$, are chosen above so that $A(\cdot)$ is positive-definite, we have (2.20).

REMARK 2.3. The condition (2.19) in Proposition 1 holds under several circumstances. For example, take $n = 3$ and fix the elements $a_{11}(\cdot) = a_{22}(\cdot) = a_{33}(\cdot) \equiv 1$ of the symmetric matrix $A(\cdot) = \sigma\sigma'(\cdot)$ in (2.8) and choose the other parameters by

$$\begin{aligned} &a_{12}(x) = a_{21}(x) := \alpha_{1+}\mathbf{1}_{\mathcal{R}_{1+}}(x) + \alpha_{1-}\mathbf{1}_{\mathcal{R}_{1-}}(x), \\ &a_{23}(x) = a_{32}(x) := \alpha_{2+}\mathbf{1}_{\mathcal{R}_{2+}}(x) + \alpha_{2-}\mathbf{1}_{\mathcal{R}_{2-}}(x), \\ &a_{31}(x) = a_{13}(x) := \alpha_{3+}\mathbf{1}_{\mathcal{R}_{3+}}(x) + \alpha_{3-}\mathbf{1}_{\mathcal{R}_{3-}}(x); \qquad x \in \mathbb{R}^3, \end{aligned} \tag{2.28}$$

where $\mathcal{R}_{i\pm}$, $i = 1,2,3$, are subsets of $\mathbb{R}^3$ defined by

$$\begin{aligned} &\mathcal{R}_{1+} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) > 0\}, \qquad \mathcal{R}_{2+} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) = 0, \mathfrak{f}_2(x) > 0\}, \\ &\mathcal{R}_{1-} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) < 0\}, \qquad \mathcal{R}_{2-} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) = 0, \mathfrak{f}_2(x) < 0\}, \\ &\mathcal{R}_{3+} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0, \mathfrak{f}_3(x) > 0\}, \\ &\mathcal{R}_{3-} := \{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0, \mathfrak{f}_3(x) < 0\}, \\ &\mathfrak{f}_1(x) := [x_3 - x_1 - (-2+\sqrt{3})(x_2 - x_3)] \cdot [x_3 - x_1 - (-2-\sqrt{3})(x_2 - x_3)], \\ &\mathfrak{f}_2(x) := [x_2 - x_3 - (-2+\sqrt{3})(x_1 - x_2)] \cdot [x_2 - x_3 - (-2-\sqrt{3})(x_1 - x_2)], \\ &\mathfrak{f}_3(x) := [x_1 - x_2 - (-2+\sqrt{3})(x_3 - x_1)] \cdot [x_1 - x_2 - (-2-\sqrt{3})(x_3 - x_1)] \end{aligned}$$

for $x \in \mathbb{R}^3$ with the six constants $\alpha_{i\pm}$ satisfying $0 < \alpha_{i+} \le 1/2$, $-1/2 \le \alpha_{i-} < 0$, for $i = 1,2,3$. Note that the zero set $\mathcal{Z}$ defined in (2.10) is $\{x \in \mathbb{R}^3 : \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = \mathfrak{f}_3(x) = 0\}$. Thus we split the region $\mathbb{R}^3 \setminus \mathcal{Z}$ into six disjoint polyhedral regions $\mathcal{R}_{i\pm}$, $i = 1,2,3$. See Figure 1, and Section A.3 for the details of this example.

REMARK 2.4. In the example of Bass and Pardoux [3], mentioned briefly in the Introduction, the diffusion matrix $\sigma(\cdot) = \sum_{\nu=1}^m \sigma_\nu(\cdot)\mathbf{1}_{\mathcal{R}_\nu}(\cdot)$ in (1.3) has a special characteristic in the allocation of its eigenvalues: *All eigenvalues but the largest are small*; namely, they are of the form $(1, \varepsilon, \dots, \varepsilon)$ where $0 < \varepsilon < 1/2$ satisfies, for some $0 < \delta < 1/2$,

$$\left| \frac{x'\sigma(x)\sigma'(x)x}{\|x\|^2} - 1 \right| \le \delta \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1.$$

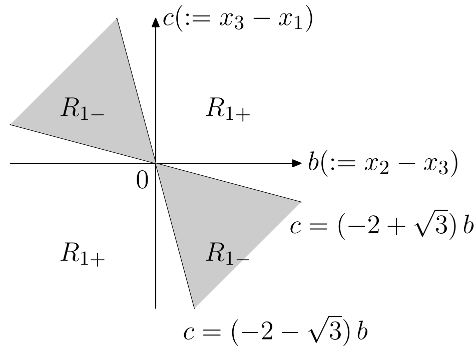

FIG. 1. *Polyhedral regions in Remark 2.3.*

This is the case when the diffusion matrix $\sigma(\cdot)$ can be written as a piecewise constant function $\sum_{\nu=1}^{m} \sigma_\nu \mathbf{1}_{\mathcal{R}_\nu}(\cdot)$ where the constant $(n \times n)$ matrices $\{\sigma_\nu, \nu = 1, \dots, m\}$ have the decomposition,

$$\sigma_\nu \sigma_\nu' := (y_\nu, B_\nu)\, \mathrm{diag}(1, \varepsilon^2, \dots, \varepsilon^2) \begin{pmatrix} y_\nu' \\ B_\nu' \end{pmatrix},$$

the fixed $(n \times 1)$ vector $y_\nu \in \mathbb{R}_\nu$ satisfies

$$\|y_\nu\| = 1, \qquad \frac{|\langle x, y_\nu \rangle|^2}{\|x\|^2} \geq 1 - \varepsilon; \qquad x \in \mathcal{R}_\nu \setminus \{0\},$$

and the $(n \times (n-1))$ matrix B_ν consists of $(n-1)$ orthonormal n-dimensional vectors orthogonal to each other and orthogonal to y_ν, for $\nu = 1, \dots, m$. Then for all $x \in \mathbb{R}^n$, we have

$$\frac{\|x\|^2 \,\mathrm{trace}(\sigma(x)\sigma'(x))}{x'\sigma(x)\sigma'(x)x} - 1 \leq \frac{(n-1)\varepsilon^2 + \delta}{1 - \delta} < 1.$$

This is sufficient for the process $X(\cdot)$ to hit the origin in finite time.

To exclude this situation, we introduce the *effective dimension* $\mathrm{ED}_{\mathcal{A}}(\cdot)$ of the elliptic second-order operator $\mathcal{A}$ defined in (2.7), namely

$$\mathrm{ED}_{\mathcal{A}}(x) := \frac{\|x\|^2 \,\mathrm{trace}(\sigma(x)\sigma'(x))}{x'\sigma(x)\sigma'(x)x} = \frac{\|x\|^2 \,\mathrm{trace}(A(x))}{x'A(x)x} \tag{2.29}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. This function comes from the theory of the so-called *exterior Dirichlet problem* for second-order elliptic partial differential equations, pioneered by Meyers and Serrin [18]. These authors showed that

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \mathrm{ED}_{\mathcal{A}}(x) > 2 \tag{2.30}$$

is a sufficient condition for the existence of solution to an exterior Dirichlet problem. In a manner similar to the proof of Proposition 1, it is possible to show that (2.30) is sufficient for $\mathbb{P}_{x_0}(X_1(t) = \cdots = X_n(t) = 0 \text{ for some } t \geq 0) = 0$ since $\widetilde{R}(\cdot)$ becomes $\mathrm{ED}_{\mathcal{A}}(\cdot)$ when the matrix D is replaced by the identity matrix. [In this manner, the function $\widetilde{R}(\cdot)$ of (2.12) is interpreted as a "local" version of the effective dimension.]

With $\sigma(\cdot)$ as in (1.3), the effective dimension $\mathrm{ED}_{\mathcal{A}}(\cdot)$ satisfies

$$\mathrm{ED}_{\mathcal{A}}(x) \geq \min_{\nu=1,\dots,m}\left(\frac{\|x\|^2 \operatorname{trace}(\sigma_\nu(x)\sigma_\nu'(x))}{x'\sigma_\nu(x)\sigma_\nu'(x)x}\right) \geq \min_{\nu=1,\dots,m}\left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\dots,n}\lambda_{i\nu}(x)}\right)$$

for $x \in \mathbb{R}^n \setminus \{0\}$ where $\{\lambda_{i\nu}(\cdot), i=1,\dots,n\}$ are the eigenvalues of the matrix-valued functions $\sigma_\nu(\cdot)\sigma_\nu'(\cdot)$, for $\nu = 1,\dots,m$. Thus $\mathrm{ED}_{\mathcal{A}}(\cdot) > 2$ if

$$\inf_{x\in\mathbb{R}^n\setminus\{0\}} \min_{\nu=1,\dots,m}\left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\dots,n}\lambda_{i\nu}(x)}\right) > 2;$$

this can be interpreted as mandating that the relative size of the maximum eigenvalue is not too large when compared to all the other eigenvalues.

REMARK 2.5. Friedman [9] established theorems on the nonattainability of lower-dimensional manifolds by nondegenerate diffusions. Let $\mathcal{M}$ be a closed k-dimensional C^2-manifold in $\mathbb{R}^n$ with $k \leq n-1$. At each point $x \in \mathcal{M}$, let $N_{k+i}(x)$ form a set of linearly independent vectors in $\mathbb{R}^n$ which are *normal to* $\mathcal{M}$ at x. Consider the matrix $\alpha(x) := (\alpha_{ij}(x))$ with

$$\alpha_{ij}(x) = \langle A(x)N_{k+i}(x), N_{k+j}(x)\rangle; \qquad 1 \leq i, j \leq n-k, x \in \mathcal{M}.$$

Roughly speaking, the strong solution of (1.1) under a linear growth condition and a Lipschitz condition on the coefficients cannot attain $\mathcal{M}$ if $\operatorname{rank}(\alpha(x)) \geq 2$ holds for all $x \in \mathcal{M}$. The rank indicates how wide the orthogonal complement of $\mathcal{M}$ is. If the rank is large, the manifold $\mathcal{M}$ is *too thin* to be attained. The fundamental lemma there is based on the solution $u(\cdot)$ of partial differential inequality $\mathcal{A}u(\cdot) \leq \mu u(\cdot)$ for some $\mu \geq 0$, outside but near $\mathcal{M}$ with $\lim_{\operatorname{dist}(x,\mathcal{M})\to\infty} u(x) = \infty$ which is different from our treatment in the previous sections.

Ramasubramanian [20, 21] examined the recurrence and transience of projections of weak solution to (1.1) for *continuous* diffusion coefficient $\sigma(\cdot)$ showing that any $(n-2)$-dimensional C^2-manifold is not hit. The integral test developed there has an integrand similar to the effective dimension studied in [18] as pointed out by M. Cranston in *Mathematical Reviews.*

Propositions 1 and 2 are complementary to these previous general results since the coefficients here are allowed to be *piecewise continuous*; however, they depend on the typical geometric characteristic on the manifold $\mathcal{Z}$ we are interested in. Since the manifold of interest in this work is the zero set $\mathcal{Z}$ of the function $s(\cdot)$, the projection $s(X(\cdot))$ of the process and the corresponding effective dimensions $\mathrm{ED}_{\mathcal{A}}(\cdot)$ and $\widetilde{R}(\cdot)$ are studied.

REMARK 2.6. As V. Papathanakos first pointed out, the conditions (2.19), (2.21) in Propositions 1 and 2 are disjoint, and there is a "gray" zone of sets of coefficients which satisfy neither of the conditions. This is because we compare with Bessel processes, replacing the n-dimensional problem by a solvable *one*-dimensional problem. In order to look at a finer structure, we discuss a special case in the next section by reducing it to a *two*-dimensional problem. This follows a suggestion of A. Banner.

3. A second approach. In this section we discuss a class of weak solutions to equation (1.1) with the structure (1.3) which exhibits "*no triple collisions*" using the n-dimensional ranked process and the $(n-1)$-dimensional reflected Brownian motion on polyhedral domains.

3.1. *Ranked process.* Given a vector process $X(\cdot) := \{(X_1(t), \dots, X_n(t)); 0 \le t < \infty\}$, we define the vector $X_{(\cdot)} := \{(X_{(1)}(t), \dots, X_{(n)}(t)); 0 \le t < \infty\}$ of *ranked processes* ordered from largest to smallest by

$$X_{(k)}(t) := \max_{1 \le i_1 < \cdots < i_k \le n} (\min(X_{i_1}(t), \dots, X_{i_k}(t))); \qquad 0 \le t < \infty, \tag{3.1}$$

for $k = 1, \dots, n$. If, for every $j = 1, \dots, n-2$, the two-dimensional process

$$(Y_j(\cdot), Y_{j+1}(\cdot))' := \big(X_{(j)}(\cdot) - X_{(j+1)}(\cdot), X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\big)' \tag{3.2}$$

obtained by looking at the "gaps" among the three adjacent ranked processes $X_{(j)}(\cdot), X_{(j+1)}(\cdot), X_{(j+2)}(\cdot)$, never reaches the corner $(0,0)'$ of $\mathbb{R}^2$, almost surely, then the process $X(\cdot)$ satisfies

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t), \text{ for some } (i,j,k), t > 0\big) = 0 \tag{3.3}$$

for $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$. On the other hand, if for some $j = 1, \dots, n-2$ the vector of gaps $(X_{(j)}(\cdot) - X_{(j+1)}(\cdot), X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot))'$ does reach the corner $(0,0)'$ of $\mathbb{R}^2$ almost surely, then we have

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t), \text{ for some } (i,j,k), t > 0\big) = 1; \qquad x_0 \in \mathbb{R}^n.$$

Thus, we are led to study the ranked process $X_{(\cdot)}$ and its adjacent differences. In the following we use the parametric result of Varadhan and Williams [25] on Brownian motion in a two-dimensional wedge with oblique reflection at the boundary, and the result of Williams [26] on Brownian motion with reflection along the faces of a polyhedral domain.

There is a long list of contributions to the study of attainability of the origin for the Brownian motion with reflection. Recently Delarue [6] considered the hitting time of a corner by a reflected diffusion in the square. Rogers [22, 23] and Burdzy and Marshall [4] considered Brownian motion in a half-space with variable angle of reflection. Here we consider oblique constant reflection on each face of the polyhedral region.

3.2. *Reflected Brownian motion.* Let $e_1, \dots, e_{n-1}$ be unit vectors in $\mathbb{R}^{n-1}$, $n \geq 3$, and consider the nonnegative orthant

$$\mathfrak{S} := \mathbb{R}_+^{n-1} = \left\{ \sum_{k=1}^{n-1} x_k e_k : x_1 \geq 0, \dots, x_{n-1} \geq 0 \right\},$$

whose $(n-2)$-dimensional faces $\mathfrak{F}_1, \dots, \mathfrak{F}_{n-1}$ are given as

$$\mathfrak{F}_i := \left\{ \sum_{k=1}^{n-1} x_k e_k : x_k \geq 0 \text{ for } k = 1, \dots, n-1, x_i = 0 \right\}; \qquad 1 \leq i \leq n-1.$$

Let us denote the $(n-3)$-dimensional faces of intersection by $\mathfrak{F}_{ij}^o := \mathfrak{F}_i \cap \mathfrak{F}_j$ for $1 \leq i < j \leq n-1$ and their union by $\mathfrak{F}^o := \bigcup_{1 \leq i < j \leq n-1} \mathfrak{F}_{ij}^o$.

We define the $(n-1)$-dimensional reflected Brownian motion $Y(\cdot) := \{(Y_1(t), \dots, Y_{n-1}(t)); t \geq 0\}$ on the orthant $\mathbb{R}_+^{n-1}$ with zero drift, constant $((n-1) \times (n-1))$ constant variance/covariance matrix $\mathfrak{A} := \Sigma\Sigma'$ and reflection along the faces of the boundary along constant directions by

$$Y(t) = Y(0) + \Sigma B(t) + \mathfrak{R} L(t); \qquad 0 \leq t < \infty, Y(0) \in \mathbb{R}_+^{n-1} \setminus \mathfrak{F}^o. \tag{3.4}$$

Here, $\{B(t); 0 \leq t < \infty\}$ is $(n-1)$-dimensional standard Brownian motion starting at the origin of $\mathbb{R}^{n-1}$. The $((n-1) \times (n-1))$ reflection matrix $\mathfrak{R}$ has all its diagonal elements equal to one, and a spectral radius strictly smaller than one. Finally, the components of the $(n-1)$-dimensional process $L(t) := (L_1(t), \dots, L_{n-1}(t)); 0 \leq t < \infty$, are adapted, nondecreasing, continuous and satisfy $\int_0^\infty Y_i(t)\, dL_i(t) = 0$ [i.e., $L_i(\cdot)$ is flat off the set $\{t \geq 0 : Y_i(t) = 0\}$] almost surely, for each $i = 1, \dots, n-1$. Note that, if $Y(t)$ lies on $\mathfrak{F}_{ij}^o = \mathfrak{F}_i \cap \mathfrak{F}_j$, then $Y_i(t) = Y_j(t) = 0$ for $1 \leq i \neq j \leq n-1$.

Harrison and Reiman [11] introduced and constructed this process pathwise through the multi-dimensional Skorohod reflection problem.

3.2.1. *Rotation and rescaling.* Assume that the constant covariance matrix $\mathfrak{A} = \Sigma\Sigma'$ is positive-definite; let U be a unitary matrix whose columns are the orthonormal eigenvectors of $\mathfrak{A}$; and let $\mathfrak{L}$ be the corresponding diagonal matrix of (positive) eigenvalues such that $\mathfrak{L} = U'\mathfrak{A}U$. Define $\widetilde{Y}(\cdot) := \mathfrak{L}^{-1/2} U Y(\cdot)$ and note that, by this rotation and rescaling, we obtain

$$\widetilde{Y}(t) = \widetilde{Y}(0) + \widetilde{B}(t) + \mathfrak{L}^{-1/2} U \mathfrak{R} L(t); \qquad 0 \leq t < \infty,$$

from (3.4) where $\widetilde{B}(t) := \mathfrak{L}^{-1/2} U \Sigma B(t)$, $0 \leq t < \infty$, is another standard $(n-1)$-dimensional Brownian motion. We may regard $\widetilde{Y}(\cdot)$ as reflected Brownian motion in a new state space $\widetilde{\mathfrak{S}} := \mathfrak{L}^{-1/2} U \mathbb{R}_+^{n-1}$. The transformed reflection matrix $\widetilde{\mathfrak{R}} := \mathfrak{L}^{-1/2} U \mathfrak{R}$ can be written as

$$\begin{aligned} &\widetilde{\mathfrak{R}} = \mathfrak{L}^{-1/2} U \mathfrak{R} = (\widetilde{\mathfrak{N}} + \widetilde{\mathfrak{Q}})\mathfrak{C} = (\widetilde{\mathfrak{r}}_1, \dots, \widetilde{\mathfrak{r}}_{n-1}), \qquad \text{where} \\ &\mathfrak{C} := \mathfrak{D}^{-1/2}, \qquad \mathfrak{D} := \operatorname{diag}(\mathfrak{A}), \qquad \widetilde{\mathfrak{N}} := \mathfrak{L}^{1/2} U \mathfrak{C} \equiv (\widetilde{\mathfrak{n}}_1, \dots, \widetilde{\mathfrak{n}}_{n-1}), \\ &\widetilde{\mathfrak{Q}} := \mathfrak{L}^{-1/2} U \mathfrak{R} \mathfrak{C}^{-1} - \widetilde{\mathfrak{N}} \equiv (\widetilde{\mathfrak{q}}_1, \dots, \widetilde{\mathfrak{q}}_{n-1}). \end{aligned} \tag{3.5}$$

Here $\mathfrak{D} = \operatorname{diag}(\mathfrak{A})$ is the $((n-1)\times(n-1))$ diagonal matrix with the same diagonal elements as those of $\mathfrak{A} = \Sigma\Sigma'$ (the variances). The constant vectors $\widetilde{\mathfrak{r}}_i, \widetilde{\mathfrak{q}}_i, \widetilde{\mathfrak{n}}_i$, $i = 1, \dots, n-1$, are $((n-1)\times 1)$ column vectors.

Since U is an orthonormal matrix that rotates the state space $\mathfrak{S} = \mathbb{R}_+^{n-1}$, and $\mathfrak{L}^{1/2}$ is a diagonal matrix which changes the scale in the positive direction, the new state space $\widetilde{\mathfrak{S}}$ is an $(n-1)$-dimensional polyhedron whose ith face $\widetilde{\mathfrak{F}}_i := \mathfrak{L}^{-1/2}U\mathfrak{F}_i$ has dimension $(n-2)$, for $i = 1, \dots, n-1$.

Note that $\operatorname{diag}(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}) = 0$ and $\operatorname{diag}(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}}) = I$, that is, $\widetilde{\mathfrak{n}}_i$ and $\widetilde{\mathfrak{q}}_i$ are orthogonal and $\widetilde{\mathfrak{n}}_i$ is a unit vector, that is, $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_i = 0$ and $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{n}}_i = 1$ for $i = 1, \dots, n-1$. Also note that $\widetilde{\mathfrak{n}}_i$ is the inward unit normal to the ith face $\widetilde{\mathfrak{F}}_i$ of the new state space $\widetilde{\mathfrak{S}}$ on which the continuous, nondecreasing process $L_i(\cdot)$ actually increases, for $i = 1, \dots, n-1$. The ith face $\widetilde{\mathfrak{F}}_i$ can be written as $\{x \in \widetilde{\mathfrak{S}} : \widetilde{\mathfrak{n}}_i' x = \mathfrak{b}_i\}$ for some $\mathfrak{b}_i \in \mathbb{R}$, for $i = 1, \dots, n-1$.

Moreover, the ith column $\widetilde{\mathfrak{r}}_i$ of the new reflection matrix $\widetilde{\mathfrak{R}}$ is decomposed into components that are normal and tangential to $\widetilde{\mathfrak{F}}_i$, that is, $\widetilde{\mathfrak{r}}_i = \mathfrak{C}_{ii}(\widetilde{\mathfrak{n}}_i + \widetilde{\mathfrak{q}}_i)$ for $i = 1, \dots, n-1$ where $\mathfrak{C}_{ii}$ is the (i,i)-element of the diagonal matrix $\mathfrak{C}$. Since the matrix $\mathfrak{L}^{-1/2}U$ of the transformation is invertible, we obtain

$$\text{(3.6)} \quad \widetilde{Y}(\cdot) \in \widetilde{\mathfrak{F}}_{ij}^o := \widetilde{\mathfrak{F}}_i \cap \widetilde{\mathfrak{F}}_j \quad \Longleftrightarrow \quad Y(\cdot) \in \mathfrak{F}_{ij}^o; \qquad 1 \le i < j \le n-1.$$

Thus, in order to decide whether the process $Y(\cdot)$ in (3.4) attains $\mathfrak{F}^o$, it is enough to decide whether the transformed process $\widetilde{Y}(\cdot)$ attains the set $\widetilde{\mathfrak{F}}^o := \mathfrak{L}^{-1/2}U\mathfrak{F}^o = \bigcup_{1\le i<j\le n-1} \widetilde{\mathfrak{F}}_{ij}^o$.

3.3. *Attainability.* With (3.6) we consider, for $n = 3$ and $n > 3$ separately, the hitting times for $1 \le i \ne j \le n-1$:

$$\tau_{ij} := \inf\{t > 0 : Y(t) \in \mathfrak{F}_{ij}^o\} = \inf\{t > 0 : \widetilde{Y}(t) \in \widetilde{\mathfrak{F}}_{ij}^o\}.$$

First we look at the case $n = 3$, that is, *two-dimensional reflected Brownian motion and the hitting time* τ_{12} *of the origin.* The directions of reflection $\widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{r}}_2$ can be written in terms of angles. Note that the angle ξ of the two-dimensional wedge $\widetilde{\mathfrak{S}}$ is positive and smaller than π since all the eigenvalues of $\mathfrak{A}$ are positive. Let θ_1 and θ_2 with $-\pi/2 < \theta_1, \theta_2 < \pi/2$ be the angles between $\widetilde{\mathfrak{n}}_1$ and $\widetilde{\mathfrak{r}}_1$, and between $\widetilde{\mathfrak{n}}_2$ and $\widetilde{\mathfrak{r}}_2$, respectively, measured so that θ_1 is positive if and only if $\widetilde{\mathfrak{r}}_1$ points toward the corner with local coordinate $(0,0)'$; similarly for θ_2. See Figure 2.

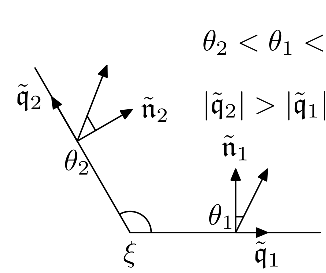

FIG. 2. *Directions of reflection*: $\theta_1 + \theta_2 < 0$.

Paraphrasing the result of Varadhan and Williams [25] for Brownian motion reflected on the *two*-dimensional wedge, we obtain the following result on the relationship between the stopping time and the sum $\theta_i + \theta_j$ of angles of reflection directions when $n - 1 = 2$.

LEMMA 3.1 (Theorem 2.2 of [25]). *Suppose that $\widetilde{Y}(0) = \tilde{y}_0 \in \widetilde{\mathfrak{S}} \setminus \widetilde{\mathfrak{F}}^o$, and consider the ratio $\beta := (\theta_1 + \theta_2)/\xi$.*

The submartingale problem for the reflected Brownian motion on the two-dimensional wedge is well-posed for $\beta < 2$ whereas it has no solution for $\beta \ge 2$. If $0 < \beta < 2$, we have $\mathbb{P}(\tau_{12} < \infty) = 1$; if, on the other hand, $\beta \le 0$, then we have $\mathbb{P}(\tau_{12} < \infty) = 0$.

In terms of the reflection vectors $\widetilde{\mathfrak{n}}_1, \widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{n}}_2, \widetilde{\mathfrak{r}}_2$, and with the aid of (3.6), we can cast this result as follows; the proof is in Section A.4.

LEMMA 3.2. *Suppose that $Y(0) = y_0 \in \mathbb{R}^2 \setminus \mathfrak{F}^o$. If $\widetilde{\mathfrak{n}}_1'\widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2'\widetilde{\mathfrak{q}}_1 > 0$, then we have $\mathbb{P}(\tau_{12} < \infty) = 1$. If, on the other hand, $\widetilde{\mathfrak{n}}_1'\widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2'\widetilde{\mathfrak{q}}_1 \le 0$, then $\mathbb{P}(\tau_{12} < \infty) = 0$.*

We consider the general case $n > 3$ next. From (3.6) and Theorem 1.1 of Williams [26] we obtain the following result, valid for $n \ge 3$.

LEMMA 3.3. *Suppose that $Y(0) = y_0 \in \mathbb{R}_+^{n-1} \setminus \mathfrak{F}^o$ and $n \ge 3$ and that the so-called* skew-symmetry condition

$$\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j'\widetilde{\mathfrak{q}}_i = 0; \qquad 1 \le i < j \le n-1, \tag{3.7}$$

holds. Then we have $\mathbb{P}(\tau < \infty) = 0$ where $\tau := \inf\{t > 0 : Y(t) \in \mathfrak{F}^o\}$.

Moreover, the components of the adapted, continuous and nondecreasing process $L(\cdot)$ defined in (3.4) are identified then as the local times at the origin of the one-dimensional component processes

$$2L_i(t) = Y_i(t) - Y_i(0) - \int_0^t \operatorname{sgn}(Y_i(s))\, dY_i(s); \qquad 0 \le t < \infty,\ i = 1, \dots, n.$$

REMARK 3.1. In the planar (two-dimensional) setting of Lemma 3.2, the skew-symmetry condition (3.7) takes a weaker form, that of an inequality. In the next section we shall discuss some details of the resulting model as an application of Lemma 3.3.

Lemmata 3.2 and 3.3 lead to the following result, proved in Section 4.2.2, on the absence of triple-collisions for a system of n one-dimensional Brownian particles interacting through their ranks. Let us introduce a collection $\{Q_k^{(i)}\}_{1 \le i,k \le n}$ of polyhedral domains in $\mathbb{R}^n$, such that $\{Q_k^{(i)}\}_{1 \le i \le n}$ is partition $\mathbb{R}^n$ for each fixed k, and $\{Q_k^{(i)}\}_{1\le k\le n}$ is partition $\mathbb{R}^n$ for each fixed i. By analogy with (3.1), the interpretation is as follows:

$$y = (y_1, \dots, y_n)' \in Q_k^{(i)} \qquad \text{means that } y_i \text{ is ranked } k\text{th among } y_1, \dots, y_n$$

with ties resolved by resorting to the smallest index for the highest rank.

PROPOSITION 3. *For $n \ge 3$, consider the weak solution of the equation (2.6) with diffusion coefficient (1.3) where $\sigma(\cdot)$ is the diagonal matrix*

$$\sigma(x) := \operatorname{diag}\left(\sum_{k=1}^n \widetilde{\sigma}_k 1_{Q_k^{(1)}}(x), \dots, \sum_{k=1}^n \widetilde{\sigma}_k 1_{Q_k^{(n)}}(x)\right); \qquad x \in \mathbb{R}^n. \tag{3.8}$$

If the positive constants $\{\widetilde{\sigma}_k; 1 \le k \le n\}$ satisfy the linear growth condition

$$\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 = \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2 = \cdots = \widetilde{\sigma}_n^2 - \widetilde{\sigma}_{n-1}^2, \tag{3.9}$$

then (3.3) holds: there are no triple-collisions among the n particles.

If $n = 3$, the weaker condition $\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 \ge \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2$ is sufficient for the absence of triple collisions.

REMARK 3.2. The special structure (3.8) has been studied in the context of Mathematical Finance. Recent work on interacting particle systems by Pal and Pitman [19] clarifies the long-range behavior of the spacings between the arranged Brownian particles under the equal variance condition: $\widetilde{\sigma}_1 = \cdots = \widetilde{\sigma}_n$; the setting of systems with countably many particle is also studied there, and related work from Mathematical Physics on competing tagged particle systems is surveyed. The "linear growth" condition (3.9) should be seen in the light of Figure 5.5, page 109 in Fernholz [8].

4. Application.

4.1. *Atlas model for an Equity Market.* Let us recall the Atlas model

$$\begin{aligned} dX_i(t) &= \left(\sum_{k=1}^n g_k 1_{Q_k^{(i)}}(X(t)) + \gamma\right) dt \\ &\quad + \sum_{k=1}^n \widetilde{\sigma}_k 1_{Q_k^{(i)}}(X(t))\, dW_i(t); \\ &\quad \text{for } 1 \le i \le n, 0 \le t < \infty, (X_1(0), \dots, X_n(0))' = x_0 \in \mathbb{R}^n, \end{aligned} \tag{4.1}$$

introduced by Fernholz [8] and studied by Banner, Fernholz and Karatzas [1]. Here $X(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))'$ represents the vector the logarithms of asset capitalizations in an equity market, and we are using the notation of Proposition 3.

We assume that the constants $\tilde{\sigma}_k > 0$ and g_k, $k = 1, \dots, n$ satisfy the following conditions which ensure that $X(\cdot)$ is ergodic:

$$g_1 < 0, \qquad g_1 + g_2 < 0, \qquad \dots,$$
$$g_1 + \dots + g_{n-1} < 0, \qquad g_1 + \dots + g_n = 0.$$

The dynamics of (4.1) induce corresponding dynamics for the ranked processes $X_{(1)}(\cdot) \ge X_{(2)}(\cdot) \ge \dots \ge X_{(n)}(\cdot)$ of (3.1). These involve the local times $\Lambda^{k,\ell}(\cdot) \equiv L^{X_{(k)} - X_{(\ell)}}(\cdot)$ for $1 \le k < \ell \le n$, where $L^Y(\cdot)$ denotes the local time at the origin of a continuous semimartingale $Y(\cdot) \ge 0$. An increase in $\Lambda^{k,\ell}(\cdot)$ is due to a simultaneous collision of $\ell - k + 1$ particles in the ranks k through ℓ. In general, when multiple collisions can occur, there are $(n-1)n/2$ such possible local times; all these appear then in the dynamics of the ranked processes, as in Banner and Ghomrasni [2].

Let $S_k(t) := \{i : X_i(t) = X_{(k)}(t)\}$ be the set of indices of processes which are kth ranked, and denote its cardinality by $N_k(t) := |S_k(t)|$ for $0 \le t < \infty$. Banner and Ghomrasni show in Theorem 2.3 of [2] that for any n-dimensional continuous semimartingale $X(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))$, its ranked process $X_{(\cdot)}(\cdot)$ with components $X_{(k)}(t) = X_{p_t(k)}(t)$, $k = 1, \dots, n$, is

$$\begin{aligned} dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbf{1}_{\{X_{(k)}(t) = X_i(t)\}}\, dX_i(t) \\ &\quad + \frac{1}{N_k(t)} \Biggl[\sum_{j=k+1}^{n} d\Lambda^{k,j}(t) - \sum_{j=1}^{k-1} d\Lambda^{j,k}(t) \Biggr]. \end{aligned} \tag{4.2}$$

Here $p_t := \{(p_t(1), \dots, p_t(n))\}$ is the random permutation of $\{1, \dots, n\}$ which describes the relation between the indices of $X(t)$ and the ranks of $X_{(\cdot)}(t)$ such that $p_t(k) < p_t(k+1)$ if $X_{(k)}(t) = X_{(k+1)}(t)$ for $0 \le t < \infty$.

Let Π_n be the symmetric group of permutations of $\{1, \dots, n\}$. The map $p_t : \Omega \times [0, \infty) \to \Pi_n$ is measurable with respect to σ-field generated by the adapted continuous process $\{X(s), 0 \le s \le t\}$, so is predictable. Consider the inverse map $p_t^{-1} := (p_t^{-1}(1), \dots, p_t^{-1}(n)) : \Omega \times [0, \infty) \to \Pi_n$, also predictable, indicating the rank of $X_i(t)$ in the n-dimensional vector $X(t)$;

$$X_{(p_t^{-1}(i))}(t) = X_i(t); \qquad i = 1, \dots, n,\ 0 \le t < \infty. \tag{4.3}$$

Under the assumption of "*no triple collisions*" [that is, when the only nonzero change-of-rank local times are those of the form $\Lambda^{k,k+1}(\cdot)$, $1 \le k \le n-1$], Fernholz [8] considered the stochastic differential equation of the vector of ranked process $X_{(\cdot)}$ in a general framework; Banner, Fernholz and Karatzas [1] obtained a rather complete analysis of the Atlas model (4.1).

In this section we apply the main results of the previous sections to the Atlas model. There are some cases of piecewise constant diffusion coefficients which satisfy the conditions in Proposition 1 or 3. Obviously, if the $\{\widetilde{\sigma}_k^2\}$ are all equal, we are in the case of standard Brownian motion. A bit more interestingly, if $\{\widetilde{\sigma}_k^2\}$ are linearly growing in the sense of (3.9), we can construct a weak solution to (4.1) with no collision of three or more particles.

REMARK 4.1. On page 2305, the paper by Banner, Fernholz and Karatzas [1] contains the erroneous statement that the "uniform nondegeneracy of the variance structure and boundedness of the drift coefficients" preclude triple collisions. Part of our motivation in undertaking the present work was a desire to correct this error.

4.2. *Construction of weak solution.*

4.2.1. *Reflected Brownian motion.* Let us start by writing the dynamics of the sum (total log-capitalization) $\mathfrak{X}(t) := X_1(\cdot) + \cdots + X_n(\cdot)$ as

$$d\mathfrak{X}(t) = n\gamma\, dt + \sum_{i=1}^{n}\sum_{k=1}^{n} \widetilde{\sigma}_k 1_{Q_k^{(i)}}(X(t))\, dW_i(t) = n\gamma\, dt + \sum_{k=1}^{n} \widetilde{\sigma}_k\, dB_k(t), \tag{4.4}$$

where $B(\cdot) := \{(B_1(t), \dots, B_n(t))', 0 \le t < \infty\}$ is given by $B_k(t) := \sum_{i=1}^{n} \int_0^t 1_{Q_k^{(i)}}(X(s))\, dW_i(s)$ for $1 \le k \le n$, $0 \le t < \infty$. By the F. Knight theorem (e.g., Chapter 3 in Karatzas and Shreve [15]), this process $B(\cdot)$ is an n-dimensional Brownian motion started at the origin.

Next, let h and $\widetilde{\Sigma}$ be the $(n-1) \times 1$ vector and the $(n-1) \times n$ triangular matrix with entries

$$h := (g_1 - g_2, \dots, g_{n-1} - g_n)', \qquad \widetilde{\Sigma} := \begin{pmatrix} \widetilde{\sigma}_1 & -\widetilde{\sigma}_2 & & & \\ & \widetilde{\sigma}_2 & -\widetilde{\sigma}_3 & & \\ & & \ddots & \ddots & \\ & & & \widetilde{\sigma}_{n-1} & -\widetilde{\sigma}_n \end{pmatrix},$$

where the elements in the lower-triangular part and the upper-triangular part, except the first diagonal above the main diagonal, are zeros. Then the process $\{ht + \widetilde{\Sigma} B(t), 0 \le t < \infty\}$ is an $(n-1)$-dimensional Brownian motion starting at the origin of $\mathbb{R}^{n-1}$ with constant drift h and the covariance matrix

$$\mathfrak{A} := \widetilde{\Sigma}\widetilde{\Sigma}' := \begin{pmatrix} \widetilde{\sigma}_1^2 + \widetilde{\sigma}_2^2 & -\widetilde{\sigma}_2^2 & & \\ -\widetilde{\sigma}_2^2 & \widetilde{\sigma}_2^2 + \widetilde{\sigma}_3^2 & \ddots & \\ & \ddots & \ddots & -\widetilde{\sigma}_{n-1}^2 \\ & & -\widetilde{\sigma}_{n-1}^2 & \widetilde{\sigma}_{n-1}^2 + \widetilde{\sigma}_n^2 \end{pmatrix}. \tag{4.5}$$

Now we construct as in Section 3.2 an $(n-1)$-dimensional process $Z(\cdot) := \{(Z_1(t), \dots, Z_{n-1}(t))', 0 \le t < \infty\}$ on $\mathbb{R}_+^{n-1}$ by

$$\begin{aligned} Z_k(t) &:= (g_k - g_{k+1})t + \widetilde{\sigma}_k B_k(t) - \widetilde{\sigma}_{k+1} B_{k+1}(t) \\ &\quad + \Lambda^{k,k+1}(t) - \tfrac{1}{2}\bigl(\Lambda^{k-1,k}(t) + \Lambda^{k+1,k+2}(t)\bigr); \qquad 0 \le t < \infty, \end{aligned} \tag{4.6}$$

for $k = 1, \dots, n-1$. Here $\Lambda^{k,k+1}(\cdot)$ is a continuous, adapted and nondecreasing process with $\Lambda^{k,k+1}(0) = 0$ and $\int_0^\infty Z_k(t)\, d\Lambda^{k,k+1}(t) = 0$ almost surely. Setting $\Lambda^{0,1}(\cdot) \equiv \Lambda^{n,n+1}(\cdot) \equiv 0$, we write in matrix form

$$Z(t) = ht + \widetilde{\Sigma} B(t) + \mathfrak{R}\Lambda(t); \qquad 0 \le t < \infty.$$

Here $\Lambda(\cdot) = (\Lambda^{1,2}(\cdot), \dots, \Lambda^{k-1,k}(\cdot))'$ and the reflection matrix $\mathfrak{R} = I - \mathfrak{Q}$ is

$$\mathfrak{R} = I - \mathfrak{Q} := \begin{pmatrix} 1 & -1/2 & & & \\ -1/2 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1/2 \\ & & & -1/2 & 1 \end{pmatrix}. \tag{4.7}$$

If the process $X(\cdot)$ has no "triple collisions," then from (4.2) we get

$$\begin{aligned} dX_{(k)}(t) = {} & \sum_{i=1}^n \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}}\, dX_i(t) \\ & + \frac{1}{2}\big(d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t)\big), \qquad 0 \le t < \infty. \end{aligned}$$

Substituting (4.1) into this equation and subtracting, we obtain that

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t); \qquad 1 \le k \le n-1, 0 \le t < \infty, \tag{4.8}$$

and that $\Lambda^{k,k+1}(\cdot)$ is the local time at the origin of the one-dimensional process $Z_k(\cdot) \ge 0$ for $k = 1, \dots, n-1$. In general, the process $X(\cdot)$ may have triple (or higher-multiplicity) collisions, so that we have additional terms in (4.8):

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t) + \zeta_k(t), \qquad 1 \le k \le n-1, 0 \le t < \infty. \tag{4.9}$$

The contribution $\zeta(\cdot) := (\zeta_1(\cdot), \dots, \zeta_{n-1}(\cdot))$ from triple or higher-multiplicity collisions can be written for $1 \le k \le n-1, 0 \le t < \infty$ as $\zeta_k(0) = 0$ and

$$\begin{aligned} d\zeta_k(t) = {} & \sum_{\ell=3}^n \ell^{-1} \mathbf{1}_{\{N_k(t)=\ell\}} \Bigg[\sum_{j=k+2}^n d\Lambda^{k,j}(t) - \sum_{j=1}^{k-2} d\Lambda^{j,k}(t) \Bigg] \\ & - \sum_{\ell=3}^n \ell^{-1} \mathbf{1}_{\{N_k(t)=\ell\}} \Bigg[\sum_{j=k+3}^n d\Lambda^{k+1,j}(t) - \sum_{j=1}^{k-1} d\Lambda^{j,k+1}(t) \Bigg]. \end{aligned}$$

REMARK 4.2. Note that $\zeta(\cdot)$ consists of (random) linear combinations of local times from collisions of three or more particles. It is *flat*, unless there are triple collisions; that is, $\int_0^\infty \mathbf{1}_{\mathfrak{G}^c}\, d\zeta(s) = 0$, where $\mathfrak{G} := \{s \ge 0 : X_i(t) = X_j(t) = X_k(t)$ for some $1 \le i < j < k \le n\}$. We use this fact with Lemma 4.1 in the next subsection.

4.2.2. *Proof of Proposition* 3. Under the assumption of Proposition 3, we can apply Lemma 3.3 to obtain

$$\mathbb{P}\big(Z_i(t) = Z_j(t) = 0, \exists t > 0, \exists (i, j), 1 \le i \ne j \le n\big) = 0; \tag{4.10}$$

see Section A.5. Thus $Z(\cdot)$ is a special case of multi-dimensional reflected Brownian motion for which each continuous, nondecreasing process $\Lambda^{k,k+1}(\cdot)$ is exactly the local time at the origin of $Z_k(\cdot)$.

Now let us state the following lemma to examine the local times from collisions of three or more particles. Its proof is in Section A.6.

LEMMA 4.1. *Let $\alpha(\cdot) = \{\alpha(t); 0 \le t < \infty\}$ be a nonnegative continuous function with decomposition $\alpha(t) = \beta(t) + \gamma(t)$ where $\beta(\cdot)$ is strictly positive and continuous, and $\gamma(\cdot)$ is of finite variation and flat off $\{t \ge 0 : \alpha(t) = 0\}$, that is, $\int_0^\infty \mathbf{1}_{\{\alpha(t)>0\}}\, d\gamma(t) = 0$. Assume $\gamma(0) = 0$ and $\alpha(0) = \beta(0) > 0$; then, $\gamma(t) = 0$ and $\alpha(t) = \beta(t)$ for all $0 \le t < \infty$.*

Under the assumption of Proposition 3, applying the above Lemma 4.1 with (4.9), (4.10) and $\alpha(\cdot) = X_{(k)}(\cdot,\omega) - X_{(k+2)}(\cdot,\omega)$, $\beta(\cdot) = Z_k(\cdot,\omega) + Z_{k+1}(\cdot,\omega)$ and $\gamma(\cdot) = \zeta_k(\cdot,\omega) + \zeta_{k+1}(\cdot,\omega)$ for $\omega \in \Omega$, we obtain $\alpha(\cdot) = \beta(\cdot)$:

$$X_{(k)}(\cdot) - X_{(k+2)}(\cdot) = Z_k(\cdot) + Z_{k+1}(\cdot), \qquad k = 1, \dots, n-2. \tag{4.11}$$

Combining (4.11) with (4.10), we obtain $X_{(k)}(\cdot) - X_{(k+2)}(\cdot) > 0$ or

$$\mathbb{P}\big(X_{(k)}(t) = X_{(k+1)}(t) = X_{(k+2)}(t), \exists t > 0, \exists k, 1 \le k \le n-2\big) = 0.$$

Therefore, there are "*no triple collisions*" under the assumption of Proposition 3, whose proof is now complete.

4.2.3. *Recovery.* In conclusion, we *recover* the n-dimensional ranked process $X_{(\cdot)}$ of X by considering a linear transformation. Specifically, we construct the n-dimensional "ranked" process,

$$\Psi_{(\cdot)}(t) := \big(\Psi_{(1)}(t), \dots, \Psi_{(n)}(t)\big); \qquad 0 \le t < \infty,$$

from the sum $\mathfrak{X}(t)$, $0 \le t < \infty$, defined in (4.4) and the reflected Brownian motion $Z(\cdot)$, so that the differences (gaps) satisfy

$$\Psi_{(k)}(t) - \Psi_{(k+1)}(t) = Z_k(t), \qquad k = 1, \dots, n-1, \tag{4.12}$$

and the sum satisfies

$$\sum_{k=1}^{n} \Psi_{(k)}(t) = \mathfrak{X}(t); \qquad 0 \le t < \infty. \tag{4.13}$$

In particular, each component of $\Psi_{(\cdot)}(t)$ is uniquely determined by

$$\begin{pmatrix} \Psi_{(1)}(t) \\ \Psi_{(2)}(t) \\ \vdots \\ \Psi_{(n)}(t) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \mathfrak{X}(t) + Z_{n-1}(t) + (n-2)Z_{n-2}(t) + \cdots + (n-1)Z_1(t) \\ \mathfrak{X}(t) + Z_{n-1}(\cdot) + (n-2)Z_{n-2}(t) + \cdots - Z_1(t) \\ \vdots \\ \mathfrak{X}(t) - (n-1)Z_{n-1}(t) - (n-2)Z_{n-2}(t) - \cdots - Z_1(t) \end{pmatrix}$$

for $0 \le t < \infty$. Under the assumption of Proposition 3, we obtain (4.10) and hence with (4.12) we arrive, in the same way as discussed in (3.3), at

$$\mathbb{P}\big(\Psi_{(k)}(t) = \Psi_{(k+1)}(t) = \Psi_{(k+2)}(t), \exists t > 0, 1 \le \exists k \le n-2\big) = 0.$$

Thus, the ranked process $\{X_{(\cdot)}(t), 0 \le t < \infty\}$ of the original process $X(\cdot)$ without collision of three or more particles, and the ranked process $\Psi_{(\cdot)}(\cdot)$ defined in the above, are equivalent, since both of them have the same sum (4.13) and the same nonnegative difference processes $Z(\cdot)$ identified in (4.8) and (4.12). We may thus view $\Psi_{(\cdot)}(\cdot)$ as the weak solution to the SDE for the ranked process $X_{(\cdot)}(\cdot)$. Finally, we define $\Psi(\cdot) := (\Psi_1(\cdot), \dots, \Psi_n(\cdot))$ where $\Psi_i(\cdot) = \Psi_{(p_t^{-1}(i))}(\cdot)$ for $i = 1, \dots, n$, and $p_t^{-1}(i)$ is defined in (4.3). Then, $\Psi(\cdot)$ is the weak solution of SDE (4.1). This construction of solution leads us to the invariance properties of the Atlas model given in [1] and [13].

APPENDIX

A.1. Proof of Lemma 2.1. From the assumption $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$, where the zero set $\mathcal{Z}$ is defined in (2.10), it follows that $\mathfrak{s}(0) = s(X(\Lambda_0)) > 0$ and there exists an integer m_0 such that $m_0^{-1} < \mathfrak{s}(0) < m_0$. Recall that with $\widetilde{R}(X(\Lambda.)) = \mathfrak{d}(\cdot)$ and $s(X(\Lambda.)) = \mathfrak{s}(\cdot)$ we obtained (2.14); namely,

$$\mathfrak{s}(t) = \mathfrak{s}(0) + \int_0^t \frac{\mathfrak{d}(u) - 1}{2\mathfrak{s}(u)}\, du + \widetilde{B}(t); \qquad 0 \le t < \infty.$$

Let us consider first the case $\overline{\mathfrak{d}} := \operatorname{essup} \sup_{0 \le t < \infty} \mathfrak{d}(t) < 2$ for (2.16). Define two continuous functions $b_1(x) := (\overline{\mathfrak{d}} - 1)/(2x)$ and $b_2(x) := \overline{\mathfrak{d}}/(4x)$ for $x \in (0, \infty)$. If $\overline{\mathfrak{d}} < 2$, then $b_1(\cdot) < b_2(\cdot)$ in $(0, \infty)$. For each integer $m \ge m_0$, there exists a nonincreasing Lipschitz continuous function $f_m(\cdot) := (b_1(\cdot) + b_2(\cdot))/2$ with Lipschitz coefficient $K_m := \max_{x \in [m^{-1}, m]} |b_2'(x)|$, such that $b_1(\cdot) \le f_m(\cdot) \le b_2(\cdot)$ in $[m^{-1}, m]$.

Define an auxiliary Bessel process $\mathfrak{r}(\cdot)$ of dimension $(\overline{\mathfrak{d}} + 2)/2$ (< 2):

$$\mathfrak{r}(t) := \mathfrak{s}(0) + \int_0^t b_2(\mathfrak{r}(u))\, du + \widetilde{B}(t); \qquad 0 \le t < \infty.$$

Consider also the increasing sequence of stopping times

$$\tau_m := \inf\{t \ge 0 : \max[\mathfrak{s}(t), \mathfrak{r}(t)] \ge m \text{ or } \min[\mathfrak{s}(t), \mathfrak{r}(t)] \le m^{-1}\} \tag{A.1}$$

for $m_0 \le m < \infty$, and $\tau := \inf\{t \ge 0 : \mathfrak{r}(t) = 0\}$. From the property of the Bessel process with dimension strictly less than 2, the process $\mathfrak{r}(\cdot)$ attains the origin within finite time; $\tau_* := \lim_{m\to\infty} \tau_m \le \tau < \infty$ holds a.s.

Now take a strictly decreasing sequence $\{a_n\}_{n=0}^{\infty} \subset (0, 1]$ with $a_0 = 1$, $\lim_{n\to\infty} a_n = 0$ and $\int_{(a_n,a_{n-1})} u^{-2}\,du = n$ for every $n \ge 1$. For each $n \ge 1$, there exists a continuous function $\rho_n(\cdot)$ on $\mathbb{R}$ with support in (a_n, a_{n-1}), so that $0 \le \rho_n(x) \le 2(nx^2)^{-1}$ holds for every $x > 0$ and $\int_{(a_n,a_{n-1})} \rho_n(x)\,dx = 1$. Then the function $\psi_n(x) := \int_0^{|x|}(\int_0^y \rho_n(u)\,du)\,dy$; $x \in \mathbb{R}$, is even and twice continuous differentiable with $|\psi_n'(x)| \le 1$ and $\lim_{n\to\infty} \psi_n(x) = |x|$ for $x \in \mathbb{R}$. Define $\varphi_n(\cdot) := \psi_n(\cdot)\mathbf{1}_{(0,\infty)}(\cdot)$.

By combining the properties of $\varphi_n(\cdot)$, $b_1(\cdot), b_2(\cdot)$ and $f_m(\cdot)$, we see that the difference $\Delta(\cdot) := \mathfrak{s}(\cdot) - \mathfrak{r}(\cdot)$ is a continuous process with

$$\begin{aligned}
\varphi_n(\Delta(t)) &\le \int_0^t \varphi_n'(\Delta(u))\big(b_1(\mathfrak{s}(u)) - b_2(\mathfrak{r}(u))\big)\,du \\
&\le \int_0^t \varphi_n'(\Delta(u))\big(f_m(\mathfrak{s}(u)) - f_m(\mathfrak{r}(u))\big)\,du \\
&\le K_m \int_0^t \varphi_n'(\Delta(u))\big(\mathfrak{s}(u) - \mathfrak{r}(u)\big)^+\,du \\
&\le K_m \int_0^t (\Delta(u))^+\,du; \qquad 0 \le t \le \tau_m.
\end{aligned}$$

Letting $n \to \infty$ we obtain $(\Delta(t))^+ \le K_m \int_0^t (\Delta(u))^+\,du$ for $0 \le t \le \tau_m$. From the Gronwall inequality and the sample-path continuity of $\mathfrak{s}(\cdot), \mathfrak{r}(\cdot)$ in $[0, \infty)$, we obtain $\Delta(\cdot) = \mathfrak{s}(\cdot) - \mathfrak{r}(\cdot) \le 0$ on $[0, \tau_m]$ for $m \ge m_0$ and

$$\text{(A.2)} \qquad \mathfrak{s}(\tau_*) = \lim_{t\to\tau_*} \mathfrak{s}(t) \le \lim_{t\to\tau_*} \mathfrak{r}(t) = \mathfrak{r}(\tau_*) \quad \text{and} \quad \max[\mathfrak{s}(\tau_*), \mathfrak{r}(\tau_*)] < \infty,$$

almost surely. On the other hand, from the definition of $\{\tau_m\}$ we obtain $0 = \mathfrak{r}(\tau_*) \ge \mathfrak{s}(\tau_*)$, thus $\mathfrak{s}(\tau_*) = 0$ and $\mathfrak{s}(t) \le \mathfrak{r}(t)$ for $0 \le t \le \tau_*$, a.s., so for $\overline{\mathfrak{d}} = \operatorname{essup\,sup}_{0\le t<\infty} \mathfrak{d}(t) < 2$ we conclude

$$\mathbb{Q}_{x_0}\big(s(X(t)) = 0 \text{ for some } t > 0\big) = \mathbb{Q}_{x_0}\big(\mathfrak{s}(t) = 0 \text{ for some } t \ge 0\big) = 1.$$

By the strong Markov property of the process $X(\cdot)$ under $\mathbb{Q}$, we obtain

$$1 = \mathbb{Q}_{x_0}\big(s(X(t)) = 0,\ \text{inf. many } t \ge 0\big) = \mathbb{Q}_{x_0}\big(\mathfrak{s}(t) = 0,\ \text{inf. many } t \ge 0\big).$$

This gives (2.16) of Lemma 2.1. Moreover, by the formula of the first hitting-time probability density function for the Bessel process with dimension $\overline{\mathfrak{d}}$ in Elworthy, Li and Yor [7] and Göing–Jaeschke and Yor [10], we obtain

$$\begin{aligned}
\mathbb{Q}_{x_0}\big(\mathfrak{s}(t) = 0,\ \text{for some } t \in (0, T]\big) &\ge \mathbb{Q}_{x_0}\big(\mathfrak{r}(t) = 0,\ \text{for some } t \in (0, T]\big) \\
&= 1 - \kappa(T; s(x_0), \overline{\mathfrak{d}}),
\end{aligned}$$

where the tail probability distribution function $\kappa(\cdot; \cdot, \cdot)$ is defined in (2.18).

We consider next the case of $\underline{\mathfrak{d}} := \operatorname{essinf} \inf_{0 \le t < \infty} \mathfrak{d}(t) \ge 2$. Define $b_3(x) := (\underline{\mathfrak{d}} - 1)/(2x)$ and $b_4(x) := \underline{\mathfrak{d}}/(4x)$ for $x \in (0, \infty)$. Following a course similar to the previous case, using $b_3(\cdot), b_4(\cdot)$ and defining a nonincreasing Lipschitz continuous function $g_m(\cdot) := (b_3(\cdot) + b_4(\cdot))/2$ with the Lipschitz coefficient $L_m := \max_{x \in [m^{-1}, m]} |b_3'(x)|$ [rather than using $b_1(\cdot), b_2(\cdot)$, $f_m(\cdot)$ and K_m], we obtain the reverse inequality $\mathfrak{q}(\cdot) \le \mathfrak{s}(\cdot)$ in $[0, \widetilde{\tau}_m]$ a.s. Here $\mathfrak{q}(\cdot)$ is the Bessel process in dimension $(\underline{\mathfrak{d}} + 2)/2$ (≥ 2); namely

$$\mathfrak{q}(t) = \mathfrak{s}(0) + \int_0^t b_4(\mathfrak{q}(u))\, du + \widetilde{B}(t); \qquad 0 \le t < \infty,$$

and the stopping times $\{\widetilde{\tau}_m\}$ are defined as in (A.1) but with $\mathfrak{r}(\cdot)$ replaced by $\mathfrak{q}(\cdot)$. By a well-known property for Bessel processes of dimension at least 2, the process $\mathfrak{q}(\cdot)$ never attains the origin; that is, $\mathfrak{q}(\cdot) > 0$ on $[0, \infty)$, a.s.

If $\widetilde{\tau}_* := \lim_{m \to \infty} \widetilde{\tau}_m < \infty$, then by analogy with (A.2), we obtain $\mathfrak{s}(\widetilde{\tau}_*) \ge \mathfrak{q}(\widetilde{\tau}_*) > 0$ and $\max[\mathfrak{s}(\widetilde{\tau}_*), \mathfrak{q}(\widetilde{\tau}_*)] < \infty$ a.s., and from the construction of $\{\widetilde{\tau}_m\}$ a contradiction follows: $0 = \mathfrak{s}(\widetilde{\tau}_*) > 0$. Therefore, $\mathbb{Q}_{x_0}(\mathfrak{s}(t) > 0 \text{ for } 0 \le t < \infty) = 1$. This gives (2.15) of Lemma 2.1 for $\underline{\mathfrak{d}} \ge 2$.

A.2. Proof of Propositions 1 and 2. Proposition 1 and the first half of Proposition 2 are direct consequences of Lemma 2.1 and of the reasoning developed in Section 2.2. Note that $\langle \widetilde{\Theta} \rangle(t) \ge c_0 t$, $t \ge 0$, in this uniformly nondegenerate case. We obtain (2.22), because $\mathbb{Q}_{x_0}(s(X(t)) = 0 \text{ for some } t \in [0, T]) \ge \mathbb{Q}_{x_0}(\mathfrak{s}(u) = 0,$ for some $u \in [0, c_0 T])$. Under the original probability measure $\mathbb{P}_{x_0}$, because of the drift $\mu(\cdot)$, the process $s(X(\cdot))$ is a semimartingale with the decomposition

$$\begin{aligned} ds(X(t)) &= \left(\frac{(R(x) - 1) Q(x)}{2 s(x)} + \frac{x' D D' \mu(x)}{s(x)} \right)\bigg|_{x = X(t)} dt + d\Theta(t) \\ &= h(X(t))\, dt + d\Theta(t); \qquad 0 \le t < \infty, \end{aligned}$$

where $h(\cdot)$, $\Theta(\cdot)$ are obtained from $\widetilde{h}(\cdot)$, $\widetilde{\Theta}(\cdot)$ in (2.12) upon replacing $\widetilde{R}(\cdot)$ in (2.11) by $R(\cdot)$ in (2.23) and $\widetilde{W}(\cdot)$ in (2.2) by $W(\cdot)$. The comparison with Bessel processes is then repeated in a similar manner. When $\sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) < 2$, we get (2.24) and (2.25).

A.3. Example in Remark 2.3. With some computations we obtain the following simplification of the effective dimension given in (2.29):

$$\mathrm{ED}_{\mathcal{A}}(x) = 2 + \frac{\begin{bmatrix} \|x\|^2 - 4a_{12}(x) \cdot x_1 x_2 \mathbf{1}_{\mathcal{R}_{1+} \cup \mathcal{R}_{1-}} \\ -4a_{23}(x) \cdot x_2 x_3 \mathbf{1}_{\mathcal{R}_{2+} \cup \mathcal{R}_{2-}} \\ -4a_{31}(x) \cdot x_3 x_1 \mathbf{1}_{\mathcal{R}_{3+} \cup \mathcal{R}_{3-}} \end{bmatrix}}{x' A(x) x} \qquad \text{for } x \in \mathbb{R}^3 \setminus \{0\}$$

and

$$R(x) = 2 + \frac{\begin{pmatrix} 4a_{12}(x) \cdot [(x_1 - x_2)^2 + 2(x_2 - x_3)(x_3 - x_1)]\mathbf{1}_{\mathcal{R}_{1+} \cup \mathcal{R}_{1-}} \\ + 4a_{23}(x) \cdot [(x_2 - x_3)^2 + 2(x_3 - x_1)(x_1 - x_2)]\mathbf{1}_{\mathcal{R}_{2+} \cup \mathcal{R}_{2-}} \\ + 4a_{31}(x) \cdot [(x_3 - x_1)^2 + 2(x_2 - x_3)(x_1 - x_2)]\mathbf{1}_{\mathcal{R}_{3+} \cup \mathcal{R}_{3-}} \end{pmatrix}}{x' DD' A(x) DD' x}$$

for $x \in \mathbb{R}^3 \setminus \mathcal{Z}$ where $R(\cdot)$ is defined in (2.29) and $\mathcal{Z}$ is defined in (2.10). Under the specification (2.28), we verify $\mathrm{ED}(\cdot) > 2$ and $R(\cdot) > 2$, since the denominators of the fractions on the right-hand sides are positive quadratic forms and their numerators can be written as

$$\begin{aligned} &\|x\|^2 - 4a_{12}(x)x_1x_2 \\ &\qquad = (1 - 4a_{12}^2)x_2^2 + x_3^2 + (x_1 - 2a_{12}x_2)^2 > 0; \qquad x \in \mathcal{R}_{1+} \cup \mathcal{R}_{1-}, \\ &4a_{12}(x)[(x_1 - x_2)^2 + 2(x_2 - x_3)(x_3 - x_1)] \\ &\qquad = 4a_{12}(x)\mathfrak{f}_1(x) > 0; \qquad x \in \mathcal{R}_{1+} \cup \mathcal{R}_{1-}, \end{aligned}$$

with similar formulas for $x \in \mathcal{R}_{i+} \cup \mathcal{R}_{i-}$, $i = 2, 3$.

A.4. Proof of Lemma 3.2. We recall the special geometric structure of orthogonality $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_i = 0$ and $\|\widetilde{\mathfrak{n}}_i\| = 1$, and observe that

$$(\text{A.3}) \qquad (\widetilde{\mathfrak{N}}' \widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}' \widetilde{\mathfrak{N}})_{ij} \underset{<}{\overset{\geq}{}} 0 \quad \iff \quad \widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i \underset{<}{\overset{\geq}{}} 0 \qquad \forall (i, j).$$

Note that if $n = 3$, that is, $n - 1 = 2$, then $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j = \|\widehat{\mathfrak{q}}_j\| \operatorname{sgn}(-\theta_j) \sin(\xi)$ for $1 \leq i \neq j \leq 2$ where $\operatorname{sgn}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$. The length $\|\widetilde{\mathfrak{q}}_2\|$ of $\widetilde{\mathfrak{q}}_2$ determines the angle θ_2 and vice versa, that is,

$$\|\widetilde{\mathfrak{q}}_i\| \underset{<}{\overset{\geq}{}} \|\widetilde{\mathfrak{q}}_j\| \quad \iff \quad |\theta_i| \underset{<}{\overset{\geq}{}} |\theta_j|.$$

With this observation and $0 < \xi < \pi$, $\sin(\xi) > 0$, we obtain

$$\begin{aligned} \widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i = \sin(\xi)\big(\|\widetilde{\mathfrak{q}}_j\| \operatorname{sgn}(-\theta_j) + \|\widetilde{\mathfrak{q}}_i\| \operatorname{sgn}(-\theta_i)\big) \underset{<}{\overset{\geq}{}} 0 \\ \iff \quad \beta = (\theta_i + \theta_j)/\xi \underset{>}{\overset{\leq}{}} 0; \qquad 1 \leq i \neq j \leq 2. \end{aligned}$$

Thus, we apply Lemma 3.1 and obtain Lemma 3.2.

A.5. Coefficient structure, and proof of (4.10). Next, we consider the case of linearly growing variance coefficients defined in (3.9), and recall the tri-diagonal matrices $\mathfrak{A} = \widetilde{\Sigma}\widetilde{\Sigma}'$ as in (4.5) and $\mathfrak{R}$ as in (4.7). Consider the $(n-1)$-dimensional reflected Brownian motion $Y(\cdot)$ defined in (3.4) with $\Sigma = \widetilde{\Sigma}$ and $\mathfrak{R}$ as in (4.7). Such a pair $(\widetilde{\Sigma}, \mathfrak{R})$ satisfies

$$(\text{A.4}) \qquad (2\mathfrak{D} - \mathfrak{Q}\mathfrak{D} - \mathfrak{D}\mathfrak{Q} - 2\mathfrak{A})_{ij} = 0; \qquad 1 \leq i, j \leq n - 1,$$

where $\mathfrak{D}$ is the diagonal matrix with the same diagonal elements as $\mathfrak{A}$ of (3.5), and $\mathfrak{Q}$ is the $((n-1)\times(n-1))$ matrix whose first-diagonal elements above and below the main diagonal are all $1/2$ and other elements are zeros as in (4.5). In fact, it suffices to consider $j=i+1, i=2,\dots,n-1$, for which the equalities (A.4) are

$$0-(\widetilde{\sigma}_i^2+\widetilde{\sigma}_{i+1}^2)-(\widetilde{\sigma}_{i-1}^2+\widetilde{\sigma}_i^2)+4\widetilde{\sigma}_i^2=0,$$

or equivalently, (3.9): $\widetilde{\sigma}_i^2-\widetilde{\sigma}_{i-1}^2=\widetilde{\sigma}_{i+1}^2-\widetilde{\sigma}_i^2$ for $2\le i\le n-1$. Moreover, the equalities (A.4) are equivalent to $(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}+\widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})_{ij}=0$ in (A.3). In fact, from (3.5) with $\mathfrak{D}^{1/2}=\mathfrak{C}^{-1}$ we compute

$$\begin{aligned}\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}&=\mathfrak{D}^{-1/2}U'\mathfrak{L}^{1/2}\mathfrak{L}^{-1/2}U\mathfrak{R}\mathfrak{D}^{1/2}-\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}}\\&=\mathfrak{D}^{-1/2}(I-\mathfrak{Q})\mathfrak{D}^{1/2}-\mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2},\\ \widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}+\widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}}&=2I-\mathfrak{D}^{-1/2}\mathfrak{Q}\mathfrak{D}^{1/2}-\mathfrak{D}^{1/2}\mathfrak{Q}\mathfrak{D}^{-1/2}-2\mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2}\end{aligned}$$

and multiply both from the left and the right by the diagonal matrix $\mathfrak{D}^{1/2}$ whose diagonal elements are all positive:

$$\mathfrak{D}^{1/2}(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}+\widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})\mathfrak{D}^{1/2}=2\mathfrak{D}-\mathfrak{Q}\mathfrak{D}-\mathfrak{D}\mathfrak{Q}-2\mathfrak{A}. \tag{A.5}$$

The equality in the relation (A.4) is equivalent to the so-called *skew-symmetry condition* $\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}+\widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}}=0$ introduced and studied by Harrison and Williams in [12, 26]. It follows from (A.3), (A.4) and (A.5) that the reflected Brownian motion $Z(\cdot)$ defined in (4.6), under the assumption of Proposition 3, is such that any two dimensional process (Z_i,Z_j) *never* attains the corner $(0,0)'$ for $1\le i<j\le n-1$, that is, (4.10) *holds*. Using this fact, we construct a weak solution to (4.1) from the reflected Brownian motion. This final step is explained as an application in the last part of Section 4.2.2.

A.6. Proof of Lemma 4.1. We fix an arbitrary $T\in[0,\infty)$. Since $\beta(\cdot)$ is strictly positive, we cannot have simultaneously $\alpha(t)=\beta(t)+\gamma(t)=0$ and $\gamma(t)\ge 0$. The continuous function $\beta(\cdot)$ attains its minimum on $[0,T]$, so

$$\begin{aligned}\{t\in[0,T]:\alpha(t)=0\}&=\{t\in[0,T]:\alpha(t)=0,\gamma(t)<0\}\\&\subset\Big\{t\in[0,T]:\gamma(t)\le-\min_{0\le s\le T}\beta(s)<0\Big\}.\end{aligned} \tag{A.6}$$

Let us define $t_0:=\inf\{t\in[0,T]:\alpha(t)=0\}$ with $t_0=\infty$ if the set is empty. If $t_0=\infty$, then $\alpha(t)>0$ for $0\le t<\infty$; thus, it follows from the assumptions $\gamma(0)=0$ and $\int_0^T\mathbf{1}_{\{\alpha(t)>0\}}\,d\gamma(t)=0$ for $0\le T<\infty$ that $\gamma(\cdot)\equiv 0$. On the other hand, if $t_0<\infty$, then it follows from the same argument as in (A.6) that $\gamma(t_0)<-\min_{0\le s\le t_0}\beta(s)<0$. This is impossible, however, since $\alpha(s)>0$ for $0\le s<t_0$ by the definition of t_0, and hence the continuous function $\gamma(\cdot)$ is flat on $[0,t_0)$, that is, $0=\gamma(0)=\gamma(t_0-)=\gamma(t_0)$. Thus, $t_0=\infty$ and $\gamma(\cdot)\equiv 0$. Therefore, the conclusions of Lemma 4.1 hold.

Acknowledgments. We are grateful to Drs. Robert Fernholz, Adrian Banner and Vasileios Papathanakos for many discussions on this subject and for providing the motivation to study this problem. We wish to thank Professors Michel Emery, Marc Yor and Robert Neel for very helpful discussions, and Dr. Adrian Banner for his very careful reading of the manuscript. We also give many thanks to the referees for suggestions and comments that resulted in major improvements to the manuscript, especially the first two propositions.

REFERENCES

[1] BANNER, A. D., FERNHOLZ, R. and KARATZAS, I. (2005). Atlas models of equity markets. *Ann. Appl. Probab.* **15** 2296–2330. MR2187296

[2] BANNER, A. D. and GHOMRASNI, R. (2008). Local times of ranked continuous semimartingales. *Stochastic Process. Appl.* **118** 1244–1253. MR2428716

[3] BASS, R. F. and PARDOUX, É. (1987). Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields* **76** 557–572. MR917679

[4] BURDZY, K. and MARSHALL, D. (1992). Hitting a boundary point with reflected Brownian motion. In *Séminaire de Probabilités, XXVI. Lecture Notes in Mathematics* **1526** 81–94. Springer, Berlin. MR1231985

[5] CÉPA, E. and LÉPINGLE, D. (2007). No multiple collisions for mutually repelling Brownian particles. In *Séminaire de Probabilités XL. Lecture Notes in Mathematics* **1899** 241–246. Springer, Berlin. MR2409009

[6] DELARUE, F. (2008). Hitting time of a corner for a reflected diffusion in the square. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 946–961. MR2453777

[7] ELWORTHY, K. D., LI, X.-M. and YOR, M. (1999). The importance of strictly local martingales; applications to radial Ornstein–Uhlenbeck processes. *Probab. Theory Related Fields* **115** 325–355. MR1725406

[8] FERNHOLZ, E. R. (2002). *Stochastic Portfolio Theory. Stochastic Modelling and Applied Probability* **48**. Springer, New York. MR1894767

[9] FRIEDMAN, A. (1974). Nonattainability of a set by a diffusion process. *Trans. Amer. Math. Soc.* **197** 245–271. MR0346903

[10] GÖING-JAESCHKE, A. and YOR, M. (2003). A survey and some generalizations of Bessel processes. *Bernoulli* **9** 313–349. MR1997032

[11] HARRISON, J. M. and REIMAN, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Probab.* **9** 302–308. MR606992

[12] HARRISON, J. M. and WILLIAMS, R. J. (1987). Brownian models of open queueing networks with homogeneous customer populations. *Stochastics* **22** 77–115. MR912049

[13] ICHIBA, T. (2006). Note on Atlas model of equity markets. Mimeo.

[14] IKEDA, N. and WATANABE, S. (1977). A comparison theorem for solutions of stochastic differential equations and its applications. *Osaka J. Math.* **14** 619–633. MR0471082

[15] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[16] KRYLOV, N. V. (1980). *Controlled Diffusion Processes. Applications of Mathematics* **14**. Springer, New York. MR601776

[17] KRYLOV, N. V. (2004). On weak uniqueness for some diffusions with discontinuous coefficients. *Stochastic Process. Appl.* **113** 37–64. MR2078536

[18] MEYERS, N. and SERRIN, J. (1960). The exterior Dirichlet problem for second order elliptic partial differential equations. *J. Math. Mech.* **9** 513–538. MR0117421

[19] PAL, S. and PITMAN, J. (2008). One-dimensional Brownian particle systems with rank-dependent drifts. *Ann. Appl. Probab.* **18** 2179–2207. MR2473654

[20] RAMASUBRAMANIAN, S. (1983). Recurrence of projections of diffusions. *Sankhyā Ser. A* **45** 20–31. MR749350

[21] RAMASUBRAMANIAN, S. (1988). Hitting of submanifolds by diffusions. *Probab. Theory Related Fields* **78** 149–163. MR940875

[22] ROGERS, L. C. G. (1990). Brownian motion in a wedge with variable skew reflection. II. In *Diffusion Processes and Related Problems in Analysis* **I**. *Progress in Probability* **22** 95–115. Birkhäuser, Boston, MA. MR1110159

[23] ROGERS, L. C. G. (1991). Brownian motion in a wedge with variable skew reflection. *Trans. Amer. Math. Soc.* **326** 227–236. MR1008701

[24] STROOCK, D. W. and VARADHAN, S. R. S. (2006). *Multidimensional Diffusion Processes*. Springer, Berlin. MR2190038

[25] VARADHAN, S. R. S. and WILLIAMS, R. J. (1985). Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.* **38** 405–443. MR792398

[26] WILLIAMS, R. J. (1987). Reflected Brownian motion with skew symmetric data in a polyhedral domain. *Probab. Theory Related Fields* **75** 459–485. MR894900

DEPARTMENT OF STATISTICS & APPLIED PROBABILITY
SOUTH HALL
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
SANTA BARBARA, CALIFORNIA 93106
USA
E-MAIL: ichiba@pstat.ucsb.edu

INTECH INVESTMENT MANAGEMENT
ONE PALMER SQUARE, SUITE 441
PRINCETON, NEW JERSEY 08542
USA
E-MAIL: ik@enhanced.com