# LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF A MARKOV CHAIN

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Consider a Markov chain $\{X_n\}_{n \geq 0}$ with an ergodic probability measure $\pi$. Let $\Psi$ be a function on the state space of the chain, with $\alpha$-tails with respect to $\pi$, $\alpha \in (0, 2)$. We find sufficient conditions on the probability transition to prove convergence in law of $N^{1/\alpha} \sum_n^N \Psi(X_n)$ to an $\alpha$-stable law. A "martingale approximation" approach and a "coupling" approach give two different sets of conditions. We extend these results to continuous time Markov jump processes $X_t$, whose skeleton chain satisfies our assumptions. If waiting times between jumps have finite expectation, we prove convergence of $N^{-1/\alpha} \int_0^{Nt} V(X_s)\, ds$ to a stable process. The result is applied to show that an appropriately scaled limit of solutions of a linear Boltzman equation is a solution of the fractional diffusion equation.

**1. Introduction.** Superdiffusive transport of energy is generically observed in a certain class of one-dimensional systems. This can be seen numerically in chains of anharmonic oscillators of the Fermi–Pasta–Ulam type and experimentally in carbon nanotubes (see [20] for a physical review). The nature of the stochastic processes describing these emerging macroscopic behaviors is a subject of a vivid debate in the physical literature and remarkably few mathematical results are present for deterministic microscopic models.

The macroscopic behavior of the energy in a chain of harmonic oscillators with the Hamiltonian dynamics perturbed by stochastic terms conserving energy and momentum has been studied in [2]. The density of energy distribution over spatial and momentum variables, obtained there in a proper kinetic limit, satisfies *a linear phonon Boltzmann equation*,

$$(1.1) \quad \partial_t u(t, x, k) + \omega'(k)\, \partial_x u(t, x, k) = \int R(k, k')\big(u(t, x, k') - u(t, x, k)\big)\, dk'.$$

As we have already mentioned, $u(t, x, k)$ is the density at time $t$ of energy of waves of Fourier's mode $k \in [0, 1]$, and the velocity $\omega'(k)$ is the derivative of the dispersion relation of the lattice.

We remark at this point that (1.1) appears also as a limit of scaled wave, or Schrödinger equations in a random medium with fast oscillating coefficients and initial data. It is sometimes called, in that context, *the radiative transport equation* (see, e.g., [1, 10, 12, 21, 27], or monography [13] for more details on this subject).

Since the kernel $R(k, k')$ appearing in (1.1) is positive, this equation has an easy probabilistic interpretation as a forward equation for the evolution of the density of a Markov process $(Y(t), K(t))$ on $\mathbb{R} \times [0, 1]$. In fact, here $K(t)$ is an autonomous jump process on $[0, 1]$ with jump rate $R(k, k')$, and $Y(t) = \int_0^t \omega'(K(s)) \, ds$ is an additive functional of $K(t)$. Momentum conservation in the microscopic model imposes a very slow jump rate for small $k$: $R(k, k') \sim k^2$ as $k \sim 0$, while velocity $\omega'(k)$ remains of order 1 even for small $k$. So when $K(t)$ has a small value, it may stay unchanged for a long time, as does the velocity of $Y(t)$. This is the mechanism that generates on a macroscopic scale the superdiffusive behavior of $Y(t)$.

The above example has motivated us to study the following general question. Consider a Markov chain $\{X_n, n \geq 0\}$ taking values in a general Polish metric space $(E, d)$. Suppose that $\pi$ is a stationary and ergodic probability Borel measure for this chain. Consider a function $\Psi : E \to \mathbb{R}$ and $S_N := \sum_{n=0}^{N-1} \Psi(X_n)$. If $\Psi$ is centered with respect to $\pi$, and possesses a second moment, one expects that the central limit theorem holds for $N^{-1/2} S_N$, as $N \to +\infty$. This, of course, requires some assumptions about the rate of the decay of correlations of the chain, as well as hypotheses about its dynamics. If $\Psi$ has an infinite second moment and its tails satisfy a power law, then one expects, again under some assumption on the transition probabilities, convergence of the laws of $N^{-1/\alpha} S_N$, for an appropriate $\alpha$ to the corresponding stable law.

In 1937 W. Doeblin himself looked at this natural question in his seminal article [7]. In the final lines of this paper, he observes that the method of dividing the sum into independent blocks, used in the paper to show the central limit theorem for countable Markov chains, can be used also in the infinite variance situation. A more complete proof, along the line of Doeblin's idea, can be found in an early paper of Nagaev [24], assuming a strong *Doeblin* condition.

Starting from the early sixties, another, more analytical approach, has been developed for proving central limit theorems for Markov chains, based on a martingale approximation of the additive functional. By solving (or by approximating the solution of) the Poisson equation $(I - P)u = \Psi$ where $P$ is the transition probability matrix, one can decompose the sum $S_N$ into a martingale plus a negligible term, thus reducing the problem to a central limit theorem for martingales. This is exploited by Gordin (see [15]) when $P$ has a spectral gap. In the following decades, much progress has been achieved using this approach. It has found applications in stochastic homogenization, random walks in random environments and interacting particle systems (i.e., infinite-dimensional problems, where renewal arguments

cannot be applied), culminating in the seminal paper of Kipnis and Varadhan [18] where reversibility of the chain is exploited in an optimal way (see also [5, 6, 14]). For nonreversible chains there are still open problems (see [22] and the review paper [25] for a more detailed list).

As far as we know, the martingale approximation approach has not been developed in the case of convergence to stable laws of functionals of Markov chains, even though corresponding theorems of martingale convergence have been available for a while (cf., e.g., [3, 9]). The present article is a first step in this direction.

More precisely, we are concerned with the limiting behavior of functionals formed over functions $\Psi$ with heavy tails that satisfy a power law decay, that is, $\pi(\Psi > \lambda) \sim c_*^+ \lambda^{-\alpha}$ and $\pi(\Psi < -\lambda) \sim c_*^- \lambda^{-\alpha}$ for $\lambda \gg 1$ with $\alpha \in (0, 2)$. We prove sufficient conditions under which the laws of the functionals of the form $N^{-1/\alpha} S_N$ converge weakly to $\alpha$-stable laws, as $N \to +\infty$. Theorem 2.4 is proven by martingale approximation, under a spectral gap condition.

We also give a proof by a more *classical* renewal method based on a coupling technique inspired by [4]. The coupling argument gives a simpler proof but under more restrictive assumptions on the form of the probability transition (cf. Condition 2.5). We point out, however, that such hypotheses are of local nature, in the sense that they involve only the behavior of the process around the singularity. In particular, the spectral gap condition (which is a global condition) can be relaxed in this coupling approach, to a moment bound for some regeneration times associated to the process (cf. Theorem 2.7).

Next, we apply these results to a continuous time Markov jump process $\{X_t, t \geq 0\}$ whose skeleton chain satisfies the assumptions made in the respective parts of Theorem 2.4. We prove that if the mean waiting time $t(x)$ has a finite moment with respect to the invariant measure $\pi$ and the tails of $V(x)t(x)$ obey the power laws, as above, then finite-dimensional distributions of the scaled functional of the form $N^{-1/\alpha} \int_0^{Nt} V(X_s) \, ds$ converge to the respective finite-dimensional distribution of a stable process (see Theorem 2.8).

Finally, these results are applied to deal with the limiting behavior of the solution $u(t, x, k)$ of the linear Boltzmann equation (1.1) in the spatial dimension $d = 1$. We prove that the long-time, large-scale limit of solutions of such an equation converges to the solution of the fractional heat equation

$$\partial_t \bar{u}(t, x) = -(-\partial_x^2)^{3/4} \bar{u}(t, x),$$

corresponding to a stable process with exponent $\alpha = 3/2$. Both approaches (i.e., martingale approximation and coupling) apply to this example.

*Note added to the second version*: After completing the first version of the present paper [16], we have received a preprint by Mellet, Mischler and Mouhot [23] that contains a completely analytical proof of the convergence of the solution of a linear Boltzmann equation to a fractional diffusion. The conditions assumed in [23] imply the same spectral gap condition as in our Theorem 2.4; consequently the corresponding result in [23] is related to our Theorem 2.8.

## 2. Preliminaries and statements of the main results.

2.1. *Some preliminaries on stable laws.* In this paper we shall consider three types of stable laws. When $\alpha \in (0, 1)$, we say that $X$ is distributed according to a stable law of type I if its characteristic function is of the form $\mathbb{E}e^{i\xi X} = e^{\psi(\xi)}$, where the Lévy exponent equals

$$(2.1) \qquad \psi(\xi) := \alpha \int_{\mathbb{R}} (e^{i\lambda\xi} - 1)|\lambda|^{-1-\alpha} c_*(\lambda)\, d\lambda$$

and

$$(2.2) \qquad c_*(\lambda) := \begin{cases} c_*^-, & \text{when } \lambda < 0, \\ c_*^+, & \text{when } \lambda > 0, \end{cases}$$

where $c_*^-, c_*^+ \geq 0$ and $c_*^- + c_*^+ > 0$. The stable law is of type II if $\alpha \in (1, 2)$ and its Lévy exponent equals

$$(2.3) \qquad \psi(\xi) := \alpha \int_{\mathbb{R}} (e^{i\lambda\xi} - 1 - i\lambda\xi)|\lambda|^{-1-\alpha} c_*(\lambda)\, d\lambda.$$

Finally, the stable law is of type III is $\alpha = 1$ and its Lévy exponent equals

$$(2.4) \qquad \psi(\xi) := \int_{\mathbb{R}} (e^{i\xi\lambda} - 1 - i\xi\lambda 1_{[-1,1]}(\lambda))|\lambda|^{-2} c_*(\lambda)\, d\lambda.$$

We say that $\{Z(t), t \geq 0\}$ is a stable process of type I (resp., II, or III) if $Z(0) = 0$ and it is a process with independent increments such that $Z(1)$ is distributed according to a stable law of type I (resp., II, or III).

2.2. *A Markov chain.* Let $(E, \mathrm{d})$ be a Polish metric space, $\mathcal{E}$ its Borel $\sigma$-algebra. Assume that $\{X_n, n \geq 0\}$ is a Markov chain with the state space $E$ and $\pi$—the law of $X_0$—is an *invariant* and *ergodic* measure for the chain. Denote by $P$ the transition operator corresponding to the chain. Since $\pi$ is invariant it can be defined, as a positivity preserving linear contraction, on any $L^p(\pi)$ space for $p \in [1, +\infty]$.

CONDITION 2.1. Suppose that $\Psi : E \to \mathbb{R}$ is Borel measurable such that there exist $\alpha \in (0, 2)$ and two constants $c_*^+, c_*^-$ satisfying $c_*^+ + c_*^- > 0$ and

$$(2.5) \qquad \begin{aligned} \lim_{\lambda \to +\infty} \lambda^\alpha \pi(\Psi \geq \lambda) &= c_*^+, \\ \lim_{\lambda \to +\infty} \lambda^\alpha \pi(\Psi \leq -\lambda) &= c_*^-. \end{aligned}$$

Condition (2.5) guarantees that $\Psi \in L^\beta(\pi)$ for any $\beta < \alpha$.

In the case of $\alpha \in (1, 2)$, we will always assume that $\int \Psi\, d\pi = 0$. We are interested in the asymptotic behavior of $S_N := \sum_{n=1}^N \Psi(X_n)$. We are looking for

sufficient conditions on the chain, which guarantee that the laws of $N^{-1/\alpha} S_N$ converge to a $\alpha$-stable law, as $N \to +\infty$.

We present two different approaches (by martingale approximation and by coupling) with two separate set of conditions.

2.3. *The martingale approach result.*   We suppose that the chain satisfies:

CONDITION 2.2 (Spectral Gap Condition).

$$(2.6) \qquad \sup[\|Pf\|_{L^2(\pi)} : f \perp 1, \|f\|_{L^2(\pi)} = 1] = a < 1.$$

Since $P$ is also a contraction in $L^1(\pi)$ and $L^\infty(\pi)$ we conclude, via the Riesz–Thorin interpolation theorem, that for any $p \in [1, +\infty)$,

$$(2.7) \qquad \|Pf\|_{L^p(\pi)} \leq a^{1 - |2/p - 1|} \|f\|_{L^p(\pi)},$$

for all $f \in L^p(\pi)$, such that $\int f \, d\pi = 0$.

In addition, we assume that the tails of $\Psi$ under the invariant measure do not differ very much from those with respect to the transition probabilities. Namely, we suppose:

CONDITION 2.3.   There exists a measurable family of Borel measures $Q(x, dy)$ and a measurable, nonnegative function $p(x, y)$ such that

$$(2.8) \qquad P(x, dy) = p(x, y)\pi(dy) + Q(x, dy) \qquad \text{for all } x \in E,$$

$$(2.9) \qquad C(2) := \sup_{y \in E} \int p^2(x, y)\pi(dx) < +\infty$$

and

$$(2.10) \quad Q(x, |\Psi| \geq \lambda) \leq C \int_{[|\Psi(y)| \geq \lambda]} p(x, y)\pi(dy) \qquad \forall x \in E, \lambda \geq 0.$$

A simple consequence of (2.8) and the fact that $\pi$ is invariant is that

$$(2.11) \quad \int p(x, y)\pi(dy) \leq 1 \quad \text{and} \quad \int p(y, x)\pi(dy) \leq 1 \qquad \forall x \in E.$$

If $\alpha \in (1, 2)$ then, in particular, $\Psi$ possesses the first absolute moment.

THEOREM 2.4.   *We assume here Conditions 2.1–2.3.*

   (i) *Suppose $\alpha \in (1, 2)$, $\Psi$ is centered. Furthermore, assume that for some $\alpha' > \alpha$, we have*

$$(2.12) \qquad \|P\Psi\|_{L^{\alpha'}(\pi)} < +\infty.$$

*Then the law of $N^{-1/\alpha} S_N$ converges weakly, as $N \to +\infty$, to a stable law of type* II.

(ii) *If $\alpha \in (0, 1)$, then the law of $N^{-1/\alpha} S_N$ converges weakly, as $N \to +\infty$, to a stable law of type* I.

(iii) *When $\alpha = 1$, assume that for some $\alpha' > 1$, we have*

$$(2.13) \qquad \sup_{N \geq 1} \| P\Psi_N \|_{L^{\alpha'}(\pi)} < +\infty,$$

*where $\Psi_N := \Psi 1[|\Psi| \leq N]$. Let $c_N := \int \Psi_N \, d\pi$. Then, the law of $N^{-1}(S_N - Nc_N)$ converges weakly, as $N \to +\infty$, to a stable law of type* III.

REMARK. A simple calculation shows that in case (iii) $c_N = (c + o(1)) \log N$ for some constant $c$.

2.4. *The coupling approach results.*

CONDITION 2.5. There exists a measurable function $\theta : E \to [0, 1]$, a probability $q$ and a transition probability $Q_1(x, dy)$, such that

$$P(x, dy) = \theta(x)q(dy) + (1 - \theta(x))Q_1(x, dy).$$

Furthermore, we assume that

$$(2.14) \qquad \bar{\theta} := \int \theta(x)\pi(dx) > 0$$

and that the tails of distribution of $\Psi$ with respect to $Q_1(x, dy)$ are uniformly lighter than its tails with respect to $q$,

$$(2.15) \qquad \lim_{\lambda \to \infty} \sup_{x \in E} \frac{Q_1(x, |\Psi| \geq \lambda)}{q(|\Psi| \geq \lambda)} = 0.$$

Clearly, because of (2.15), the function $\Psi$ satisfies condition (2.5) also with respect to the measure $q$, but with different constants.

$$(2.16) \qquad \begin{aligned} &\lim_{\lambda \to +\infty} \lambda^\alpha q(\Psi > \lambda) = c_*^+ \bar{\theta}^{-1}, \\ &\lim_{\lambda \to +\infty} \lambda^\alpha q(\Psi < -\lambda) = c_*^- \bar{\theta}^{-1}. \end{aligned}$$

The purpose of Condition 2.5 is that it permits to define a Markov chain $\{(X_n, \delta_n), n \geq 0\}$ on $E \times \{0, 1\}$ such that

$$(2.17) \qquad \begin{aligned} &\mathbb{P}(\delta_{n+1} = 0 | X_n = x, \delta_n = \epsilon) = \theta(x), \\ &\mathbb{P}(\delta_{n+1} = 1 | X_n = x, \delta_n = \epsilon) = 1 - \theta(x), \\ &\mathbb{P}(X_{n+1} \in A | \delta_{n+1} = 0, X_n = x, \delta_n = \epsilon) = q(A), \\ &\mathbb{P}(X_{n+1} \in A | \delta_{n+1} = 1, X_n = x, \delta_n = \epsilon) = Q_1(x, A) \end{aligned}$$

for $n \geq 0$. We call this Markov chain the *basic coupling*. It is clear that the marginal chain $\{X_n, n \geq 0\}$ has probability transition $P$. The dynamics of $\{(X_n, \delta_n), n \geq 0\}$ are easy to understand. When $X_n = x$, we choose $X_{n+1}$ according to the distribution $q(dy)$ with probability $\theta(x)$, and according to the distribution $Q_1(x, dy)$ with probability $1 - \theta(x)$.

Let $\kappa_n$ be the $n$th zero in the sequence $\{\delta_n, n \geq 0\}$. In a more precise way, define $\kappa_0 := 0$, and for $i \geq 1$,

$$\kappa_i := \inf\{n > \kappa_{i-1}, \delta_n = 0\}.$$

Notice that the sequence $\{\kappa_{i+1} - \kappa_i, i \geq 1\}$ is i.i.d., and $\mathbb{E}(\kappa_{i+1} - \kappa_i) = \bar{\theta}^{-1}$. We call the sequence $\{\kappa_n, n \geq 1\}$ the *regeneration times*.

Observe that, for any $i \geq 1$, the distribution of $X_{\kappa_i}$ is given by $q(dy)$. In particular, $X_{\kappa_i}$ *is independent of* $\{X_0, \ldots, X_{\kappa_i - 1}\}$. Therefore, the blocks

$$\{(X_{\kappa_i}, \delta_{\kappa_i}), \ldots, (X_{\kappa_{i+1}-1}, \delta_{\kappa_{i+1}-1})\}$$

are independent. The dynamics for each one of these blocks is easy to understand. Start a Markov chain $\{X_n^1, n \geq 0\}$ with initial distribution $q(dy)$ and transition probability $Q_1(x, dy)$. At each step $n$, we stop the chain with probability $\theta(X_n^1)$. We call the corresponding stopping time $\kappa_1$. Each one of the blocks, except for the first one, has a distribution $\{(X_0^1, 0), (X_1^1, 1), \ldots, (X_{\kappa_1-1}^1, 1)\}$. The first block is constructed in the same way, but starts from $X_0^1 = X_0$ instead of with the law $q(dy)$. Now we are ready to state:

CONDITION 2.6.

$$\sum_{n=1}^{\infty} n^{1+\alpha} \sup_x \mathbb{P}(\kappa_1 \geq n | X_0 = x) < +\infty.$$

THEOREM 2.7.   *Suppose that $\alpha \in (1, 2)$ and $\Psi$ is centered under $\pi$, or $\alpha \in (0, 1)$. Then under Conditions* 2.1, 2.5 *and* 2.6, *the law of* $N^{-1/\alpha} S_N$ *converges to an $\alpha$-stable law.*

2.5. *An additive functional of a continuous time jump process.*   Suppose that $\{\tau_n, n \geq 0\}$ is a sequence of i.i.d. random variables, independent of $\mathcal{F} := \sigma(X_0, X_1, \ldots)$ and such that $\tau_0$ has exponential distribution with parameter 1. Suppose that $t : E \to (0, +\infty)$ is a measurable function such that $t(x) \geq t_* > 0, x \in E$. Let

$$(2.18) \qquad\qquad t_N := \sum_{n=0}^{N} t(X_n) \tau_n.$$

One can define a compound Poisson process $X_t = X_n, t \in [t_N, t_{N+1})$. It is Markovian; see, for example, Section 2 of Appendix 1, pages 314–321, of [17] with the

generator

$$(2.19) \qquad Lf(x) = t^{-1}(x) \int [f(y) - f(x)] P(x, dy), \qquad f \in B_b(E).$$

Here $B_b(E)$ is the space of bounded and Borel measurable functions on $E$. Let

$$(2.20) \qquad \bar{t} := \int t \, d\pi < +\infty.$$

Suppose $V : E \to \mathbb{R}$ is measurable and $\Psi(x) := V(x)t(x)$ satisfies condition (2.5). We shall be concerned with the limit of the scaled processes,

$$(2.21) \qquad Y_N(t) := \frac{1}{N^{1/\alpha}} \int_0^{Nt} V(X(s)) \, ds, \qquad t \geq 0,$$

as $N \to +\infty$. Then $\bar{t}^{-1} t(x)\pi(dx)$ is an ergodic, invariant probability measure for $\{X_t, t \geq 0\}$. Our result can be formulated as follows.

THEOREM 2.8. (i) *Suppose that $\alpha \in (1, 2)$ and that the assumptions of either part* (i) *Theorem* 2.4, *or of Theorem* 2.7, *hold. Then, the convergence of finite-dimensional distributions takes place to a stable process of type* II.

(ii) *In case $\alpha \in (0, 1)$, we suppose that the assumptions of either part* (ii) *of Theorem* 2.4, *or of Theorem* 2.7 *hold. Then the finite distributions of processes $\{Y_N(t), t \geq 0\}$ converge, as $N \to +\infty$, to the respective distributions of a stable process of type* I.

(iii) *When $\alpha = 1$ and the assumptions of part* (iii) *of Theorem* 2.4 *hold, the finite distributions of processes $\{Y_N(t) - c_N t, t \geq 0\}$ converge, as $N \to +\infty$, to the respective distributions of a stable process of type* III. *Here $c_N := \int_{|\Psi| \leq N} \Psi \, d\pi$.*

## 3. An application: Superdiffusion of energy in a lattice dynamics.
In [2] it is proven that the Wigner distribution associated with the energy of a system of interacting oscillators with momentum and energy conserving noise converges, in an appropriate kinetic limit, to the solution $u(t, x, k)$ of the linear kinetic equation

$$(3.1) \qquad \begin{cases} \partial_t u(t, x, k) + \omega'(k) \, \partial_x u(t, x, k) = \mathcal{L}u(t, x, k), \\ u(0, x, k) = u_0(x, k), \end{cases}$$

where $(t, x, k) \in [0, +\infty) \times \mathbb{R}^d \times \mathbb{T}^d$ and the initial condition $u_0(\cdot, \cdot)$ is a function of class $C^{1,0}(\mathbb{R}^d \times \mathbb{T}^d)$. Here $\mathbb{T}$ is the one-dimensional circle, understood as the interval $[-1/2, 1/2]$ with identified endpoints, and $\mathbb{T}^d$ is the $d$-dimensional torus. The function $\omega(k)$ is the dispersion relation of the lattice and it is assumed that $\omega(-k) = \omega(k)$ and $\omega(k) \sim |k|$ for $|k| \sim 0$ (*acoustic dispersion*). The scattering operator $\mathcal{L}$, acting in (3.1) on variable $k$, is usually an integral operator that is a generator of a certain jump process.

In the case of $d = 1$, the scattering operator is given by

$$(3.2) \qquad \mathcal{L}f(k) = \int_{\mathbb{T}} R(k, k')[f(k') - f(k)] \, dk'$$

with the scattering kernel,

$$(3.3) \quad \begin{aligned} R(k, k') &= \tfrac{4}{3}[2\sin^2(2\pi k)\sin^2(\pi k') \\ &\quad + 2\sin^2(2\pi k')\sin^2(\pi k) - \sin^2(2\pi k)\sin^2(2\pi k')]. \end{aligned}$$

We shall assume that the dispersion relation is given by a function $\omega : \mathbb{T} \to [0, +\infty)$, that satisfies $\omega \in C^1(\mathbb{T} \setminus \{0\})$ and

$$(3.4) \quad c_l|\sin(\pi k)| \le \omega(k) \le c_u|\sin(\pi k)|, \qquad k \in \mathbb{T},$$

for some $0 < c_l \le c_u < +\infty$ while

$$(3.5) \quad \lim_{k \to \pm 0} \omega'(k) = \pm c_\omega.$$

In the case of a simple one-dimensional lattice, we have $\omega(k) = c|\sin(\pi k)|$.

The total scattering cross section is given by

$$(3.6) \quad R(k) = \int_{\mathbb{T}} R(k, k')\, dk' = \frac{4}{3}\sin^2(\pi k)\bigl(1 + 2\cos^2(\pi k)\bigr).$$

We define $t(k) := R(k)^{-1}$ since these are the expected waiting times of the scattering process.

Let $\{X_n, n \ge 0\}$ be a Markov chain on $\mathbb{T}$ whose transition probability equals

$$P(k, dk') := t(k)R(k, k')\, dk'.$$

Suppose that $\{\tau_n, n \ge 0\}$ is an i.i.d. sequence of random variables such that $\tau_0$ is exponentially distributed with intensity 1. Let $t_n := t(X_n)\tau_n$, $n \ge 0$. One can represent then the solution of (3.1) with the formula

$$(3.7) \quad u(t, x, k) = \mathbb{E}u_0(x(t), k(t)),$$

where

$$x(t) = x + \int_0^t \omega'(k(s))\, ds,$$

$$k(t) = X_n, \qquad t \in [t_n, t_{n+1}),$$

and $k(0) = X_0 = k$. We shall be concerned in determining the weak limit of the finite-dimensional distribution of the scaled process $\{N^{-1/\alpha}x(Nt), t \ge 0\}$, as $N \to +\infty$, for an appropriate scaling exponent $\alpha$.

It is straightforward to verify that

$$(3.8) \quad \pi(dk) = \frac{t^{-1}(k)}{\bar{R}}\, dk = \frac{R(k)}{\bar{R}}\, dk,$$

where $\bar{R} := \int_{\mathbb{T}} R(k)\, dk$ is a stationary and reversible measure for the chain. Then $P(k, dk') = p(k, k')\pi(dk')$ where

$$p(k, k') = \bar{R}t(k)R(k, k')t(k')$$

and after straightforward calculations, we obtain

$$p(k, k') = 6[\cos^2(\pi k) + \cos^2(\pi k') - 2\cos^2(\pi k)\cos^2(\pi k')]$$
$$\times \left[(1 + 2\cos^2(\pi k))(1 + 2\cos^2(\pi k'))\right]^{-1}$$

(3.9)
$$= 6\{[|\cos(\pi k)| - |\cos(\pi k')|]^2$$
$$+ 2|\cos(\pi k)\cos(\pi k')|[1 - |\cos(\pi k)\cos(\pi k')|]\}$$
$$\times \left[(1 + 2\cos^2(\pi k))(1 + 2\cos^2(\pi k'))\right]^{-1}.$$

We apply Theorem 2.8 and probabilistic representation (3.7) to describe the asymptotic behavior for long times and large spatial scales of solutions of the kinetic equation (3.1). The result is contained in the following.

THEOREM 3.1. *The finite-dimensional distributions of scaled processes* $\{N^{-2/3}x(Nt), t \geq 0\}$ *converge weakly to those of a stable process of type* II. *In addition, for any* $t > 0$, $x \in \mathbb{R}$, *we have*

(3.10)
$$\lim_{N \to +\infty} \int_{\mathbb{T}} |u(Nt, N^{2/3}x, k) - \bar{u}(t, x)|^2 \, dk = 0,$$

*where* $u(t, x, k)$ *satisfies* (3.1) *with the initial condition* $u_0(N^{-2/3}x, k)$, *such that* $u_0$ *is compactly supported, and* $\bar{u}(t, x)$ *is the solution of*

(3.11)
$$\begin{cases} \partial_t \bar{u}(t, x) = -(-\partial_x^2)^{3/4}\bar{u}(t, x), \\ \bar{u}(0, x) = \int_{\mathbb{T}} u_0(x, k) \, dk. \end{cases}$$

PROOF. We start verifying the hypotheses of Theorem 2.8 by finding the tails of

(3.12)
$$\Psi(k) = \omega'(k)t(k)$$

under measure $\pi$. Since $\omega'(k)$ is both bounded and bounded away from zero, the tails of $\Psi(k)$ under $\pi$ are the same as those of $t(k)$. Note that

(3.13)
$$\pi(k : t(k) \geq \lambda) = C_R \lambda^{-3/2}(1 + O(1)) \qquad \text{for } \lambda \gg 1,$$

and some $C_R > 0$. This verifies (2.5) with $\alpha = 3/2$. Since the density of $\pi$ with respect to the Lebesgue measure is even and $\Psi$ is odd, it has a null $\pi$-average. $\square$

*Verification of hypotheses of part* (i) *of Theorem* 2.4. Note that we can decompose $P(k, dk')$ as in (2.8) with $p(k, k')$ given by (3.9) and $Q(k, dk') \equiv 0$. Since $p(k, k')$ is bounded, Condition 2.3 and (2.12) are obviously satisfied. Operator $P$ is a contraction on $L^2(\pi)$, and by the Hilbert–Schmidt theorem (see, e.g., Theorem 4, page 247 of [19]) is symmetric and compact. In consequence, its spectrum is contained in $[-1, 1]$ and is discrete, except for a possible accumulation point at 0.

LEMMA 3.2. *Point* 1 *is a simple eigenvalue of both P and* $P^2$.

PROOF. Suppose

$$Pf = f. \tag{3.14}$$

We claim that $f$ is either everywhere positive, or everywhere negative. Let $f^+$, $f^-$ be the positive and negative parts of $f$. Suppose also that $f^+$ is nonzero on a set of positive $\pi$ measure. Then $f = f^+ - f^-$ and $Pf = Pf^+ - Pf^-$. Thus $f^+ = (Pf)^+ \le Pf^+$. Yet

$$\int f^+ \, d\pi \le \int Pf^+ \, d\pi = \int f^+ \, d\pi,$$

thus $Pf^+ = f^+$. Likewise, $Pf^- = f^-$. Since for each $k$ we have $p(k, k') > 0$, except for a set of $k'$ of measure $\pi$ zero, we conclude that $f^+ > 0 \, \pi$ a.e., hence $f^- \equiv 0$.

Now we know that $P1 = 1$. We claim that any other $f \not\equiv 0$ that satisfies (3.14) belongs to span$\{1\}$. Otherwise $f - c1$ for some $c$ would suffer change of sign. But this contradicts our conclusion reached above so the lemma holds for $P$. The argument for $P^2$ is analogous.

As a corollary of the above lemma we conclude that condition 2.2 holds. Applying part (i) of Theorem 2.8 to $N^{-2/3} \int_0^{Nt} \omega'(k(s)) \, ds$, we conclude that its finite-dimensional distributions converge in law to an $\alpha$-stable Lévy process for $\alpha = 3/2$.

We use the above result to prove (3.10). To abbreviate the notation denote $Y_N(t) := x + N^{-2/3} \int_0^{Nt} \omega'(k(s)) \, ds$. Using probabilistic representation for a solution of (3.1), we can write

$$u(Nt, N^{3/2}x, k) = \mathbb{E}_k u_0(Y_N(t), k(Nt)) \tag{3.15}$$
$$= \sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \hat{u}_0(\xi, \eta) \mathbb{E}_k \exp\{i\xi Y_N(t) + i\eta k(Nt)\} \, d\xi.$$

Here $\hat{u}_0(\xi, \eta)$ is the Fourier transform of $u(x, k)$, and $\mathbb{E}_k$ is the expectation with respect to the path measure corresponding to the momentum process $\{k(t), t \ge 0\}$ that satisfies $k(0) = k$. Since the dynamics of the momentum process are reversible with respect to the normalized Lebesgue measure $m$ on the torus and 0 is a simple eigenvalue for the generator $\mathcal{L}$, we have $\|P^t f\|_{L^2(m)} \to 0$, as $t \to +\infty$, provided $\int_{\mathbb{T}} f \, dk = 0$. Suppose that $\{a_N, N \ge 1\}$ is an increasing sequence of positive numbers tending to infinity and such that $a_N N^{-3/2} \to 0$. A simple calculation shows that for any $\xi, \eta \in \mathbb{R}$ and $e_\xi(x) := e^{ix\xi}$, we have

$$\left| \mathbb{E}_k[e_\xi(Y_N(t))e_\eta(k(Nt))] - \mathbb{E}_k[e_\xi(Y_N(t - ta_N/N))e_\eta(k(Nt))] \right| \to 0 \tag{3.16}$$
$$\text{as } N \to +\infty.$$

Using Markov property we can write that the second term under the absolute value in the formula above equals

$$\mathbb{E}_k\big[e_\xi\big(Y_N(t - ta_N/N)\big)P^{a_N t}e_\eta\big(k((N - a_N)t)\big)\big].$$

Let $\tilde{e}_\eta(k) := e_\eta(k) - \bar{e}_\eta$, where $\bar{e}_\eta := \int_{\mathbb{T}} e_\eta(k)\,dk$. By the Cauchy–Schwarz inequality, we obtain

(3.17)
$$\big|\mathbb{E}_k\big[e_\xi\big(Y_N(t - ta_N/N)\big)P^{a_N t}e_\eta\big(k((N - a_N)t)\big)\big]$$
$$- \mathbb{E}_k e_\xi\big(Y_N(t - ta_N/N)\big)\bar{e}_\eta\big|$$
$$\leq \big\{\mathbb{E}_k\big|P^{a_N t}\tilde{e}_\eta\big(k((N - a_N)t)\big)\big|^2\big\}^{1/2}.$$

The right-hand side of (3.17) tends to 0 in the $L^2$ sense with respect to $k \in \mathbb{T}$, as $N \to +\infty$.

From (3.17) we conclude that

(3.18)
$$\bigg|\sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{T}} \hat{u}_0(\xi, \eta)\mathbb{E}_k\big[e_\xi\big(Y_N(t - ta_N/N)\big)$$
$$\times P^{a_N t}e_\eta\big(k((N - a_N)t)\big)\big]\,d\xi\,dk$$
$$- \sum_{\eta \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{T}} \hat{u}_0(\xi, \eta)\mathbb{E}_k e_\xi\big(Y_N(t - ta_N/N)\big)\bar{e}_\eta\,d\xi\,dk\bigg| \to 0$$

as $N \to +\infty$. Combining this with (3.16), we complete the proof of the theorem. $\qquad\square$

*Verification of hypotheses of Theorem* 2.7. Here we show the convergence of $N^{-3/2}x(Nt)$ by using the coupling approach of Section 4. Define the functions

$$q_0(k) := \sin^2(2\pi k) = 4[\sin^2(\pi k) - \sin^4(\pi k)],$$
$$q_1(k) := \tfrac{4}{3}\sin^4(\pi k),$$

which are densities with respect to the Lebesgue measure in $\mathbb{T}$. A simple computation shows that $R(k, k') = 2^{-4}[q_0(k)q_1(k') + q_1(k)q_0(k')]$, and therefore, $R(k) = 2^{-4}[q_0(k) + q_1(k)]$. The transition probability $P(k, dk')$ can be written as

$$P(k, dk') = \frac{q_1(k)}{q_0(k) + q_1(k)}q_0(k')\,dk' + \frac{q_0(k)}{q_0(k) + q_1(k)}q_1(k')\,dk'.$$

In particular, in the notation of Section 4, this model satisfies Condition 2.5 with $q(dk') = q_0(k')\,dk'$, $\theta = q_1/(q_0 + q_1)$ and $Q_1(k, dk') = q_1(k')\,dk'$. Notice that the behavior around 0 of $\pi$ and $q$ is the same. Hence, $q(\Psi(k) \geq \lambda) \sim c\lambda^{-3/2}$ for $\lambda \gg 1$. We conclude, therefore, that the function $\Psi(k)$, given by (3.12), satisfies (2.16). Observe furthermore that $Q_1$ does not depend on $k$ and that

$Q_1(k', t(k) \geq \lambda) \sim c\lambda^{-5/2}$ for $\lambda \gg 1$. Due to this last observation, condition (2.15) is satisfied.

We are only left to check Condition 2.6. But this one is also simple, once we observe that the sequence $\{\delta_n, n \geq 0\}$ is a Markov chain with transition probabilities

$$P(\delta_{n+1} = 1|\delta_n = 0) = P(\delta_{n+1} = 0|\delta_n = 1)$$
$$= \int_{-1/2}^{1/2} \frac{q_0(k)q_1(k)\,dk}{q_0(k) + q_1(k)},$$

$$P(\delta_{n+1} = 1|\delta_n = 1) = \int_{-1/2}^{1/2} \frac{q_1^2(k)\,dk}{q_0(k) + q_1(k)},$$

$$P(\delta_{n+1} = 0|\delta_n = 0) = \int_{-1/2}^{1/2} \frac{q_0^2(k)\,dk}{q_0(k) + q_1(k)}.$$

We conclude that the regeneration time $\kappa_1$ satisfies $\mathbb{E}[\exp\{\gamma\kappa_1\}] < +\infty$ for $\gamma$ small enough. Condition 2.6 is therefore a consequence of the fact that the transition probability function $Q_1(k, dk')$ does not depend on $k$; therefore, we can write

$$\mathbb{P}[\kappa_1 \geq n|K_0 = k] = (1 - \theta(k))\mathbb{P}[\kappa_1 \geq n - 1].$$

**4. Proof of Theorem 2.7 by coupling.** Because of its simplicity, we present first the proof of Theorem 2.7 using a basic coupling argument. Let us define

$$\varphi_i = \sum_{j=\kappa_i}^{\kappa_{i+1}-1} \Psi(X_j),$$

$$\mathcal{M}(N) = \sup\{i \geq 0; \kappa_i \leq N\}.$$

Note that $\mathcal{M}(N) < +\infty$ a.s. An alternative way of defining $\mathcal{M}(N)$ is demanding the inequality $\kappa_{\mathcal{M}(N)} \leq N < \kappa_{\mathcal{M}(N)+1}$ to be satisfied. Then, we have

$$(4.1) \qquad S_N = \sum_{i=0}^{\mathcal{M}(N)} \varphi_i + R_N,$$

where

$$R_N := \sum_{j=\kappa_{\mathcal{M}(N)}+1}^{N} \Psi(X_j).$$

In (4.1) we have decomposed $S_N$ into a random sum of i.i.d. random variables, $\{\varphi_i, i \geq 1\}$, and two boundary terms: $\varphi_0$ and $R_N$. Notice also that $\kappa_N - \kappa_1$ is a sum of i.i.d. random variables. Consequently, the law of large numbers gives

$$(4.2) \qquad \frac{\kappa_N}{N} \to \bar{\kappa} = \mathbb{E}(\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\mathcal{M}(N)}{N} \to \bar{\kappa}^{-1} = \bar{\theta},$$

a.s., as $N \to +\infty$.

Observe also that when $\alpha \in (1, 2)$ and $\Psi$ is centered, random variable $\varphi_1$ is also centered. Indeed, by the ergodic theorem we have that a.s.

$$0 = \lim_{N \to +\infty} \frac{S_N}{N} = \lim_{N \to +\infty} \frac{1}{\mathcal{M}(N)} \sum_{i=1}^{\kappa_{\mathcal{M}(N)}} \varphi_i \times \frac{\mathcal{M}(N)}{N} = \mathbb{E}\varphi_1 \bar{\theta},$$

which proves that

$$(4.3) \qquad\qquad\qquad \mathbb{E}\varphi_1 = 0.$$

The idea now is that under Conditions 2.5 and 2.6, the random variable $\varphi_i$ is equal to $\Psi(X_{\kappa_i})$ plus a term with lighter tails. Before stating this result, we need a simple lemma.

LEMMA 4.1. *Let $\zeta$ be a random variable such that*

$$\lim_{x \to \infty} x^\alpha \mathbb{P}(\zeta > x) = c^+, \qquad \lim_{x \to \infty} x^\alpha \mathbb{P}(\zeta < -x) = c^-.$$

*Let $\xi$ be such that $\lim_{x \to \infty} \mathbb{P}(|\xi| > x)/\mathbb{P}(|\zeta| > x) = 0$. Then*

$$(4.4) \qquad \lim_{x \to \infty} x^\alpha \mathbb{P}(\zeta + \xi > x) = c^+, \qquad \lim_{x \to \infty} x^\alpha \mathbb{P}(\zeta + \xi < -x) = c^-.$$

PROOF. Without loss of generality, we just consider the first limit, the second one follows considering $-\zeta, -\xi$. We will prove that the $\liminf_{x \to \infty}$ of the previous expression is bigger than $c_+$ and the $\limsup$ is smaller than $c_+$. We start with the upper bound: for any $\epsilon > 0$ there exists $x_0$ so that for $x \geq x_0$, we have

$$x^\alpha \mathbb{P}(\zeta + \xi > x) \leq x^\alpha \mathbb{P}(\zeta > (1 - \epsilon)x) + x^\alpha \mathbb{P}(\xi > \epsilon x)$$

$$\leq \frac{c_+ + \epsilon}{(1 - 2\epsilon)^\alpha} + \frac{\mathbb{P}(|\xi| > \epsilon x)}{\mathbb{P}(|\zeta| > \epsilon x)} \times \frac{c_+ + c_-}{(\epsilon/2)^\alpha}.$$

Now take above the upper limit, as $x \to +\infty$, to get

$$\limsup_{x \to +\infty} x^\alpha \mathbb{P}(\zeta + \xi > x) \leq \frac{c_+ + \epsilon}{(1 - 2\epsilon)^\alpha}.$$

Since $\epsilon$ is arbitrary, we have proved the upper bound. The lower bound is very similar:

$$\mathbb{P}(\zeta + \xi > x) = \mathbb{P}(\zeta + \xi > x, \xi > -\epsilon x) + \mathbb{P}(\zeta + \xi > x, \xi \leq -\epsilon x)$$

$$\geq \mathbb{P}(\zeta > (1 + \epsilon)x, \xi > -\epsilon x)$$

$$\geq \mathbb{P}(\zeta > (1 + \epsilon)x) - \mathbb{P}(\zeta > (1 + \epsilon)x, \xi \leq -\epsilon x)$$

$$\geq \mathbb{P}(\zeta > (1 + \epsilon)x) - \mathbb{P}(\xi < -\epsilon x).$$

Starting from this last expression, the same computations done for the upper bound show that

$$\liminf_{x \to +\infty} x^\alpha \mathbb{P}(\zeta + \xi > x) \geq \frac{c_+}{(1 + 2\epsilon)^\alpha}.$$

Since $\epsilon > 0$ is arbitrary, the lemma is proved for the first expression in (4.4). The second case can be done in the same fashion. □

LEMMA 4.2. *Let $\Psi$ satisfy (2.5) with constants $c_*^+, c_*^-$ together with Conditions 2.5 and 2.6. Then the law of each $\varphi_i$ satisfies*

(4.5)
$$\lim_{\lambda \to +\infty} \lambda^\alpha \mathbb{P}(\varphi_i > \lambda) = c_*^+ \bar{\theta}^{-1},$$
$$\lim_{\lambda \to +\infty} \lambda^\alpha \mathbb{P}(\varphi_i < -\lambda) = c_*^- \bar{\theta}^{-1}.$$

PROOF. The idea of the proof is simple. Random variable $\varphi_i$ is the sum of a random variable with an $\alpha$-tail, $\Psi(X_{\kappa_i})$, and a finite (but random) number of random variables with lighter tails ($\Psi(X_{\kappa_i+1}), \ldots, \Psi(X_{\kappa_{i+1}-1})$). By Condition 2.6, the random number can be efficiently controlled. To simplify the notation, assume that $X_0$ is distributed according to $q$, so the first block is also distributed like the other ones. Then

(4.6)
$$\mathbb{P}\left(\sum_{j=1}^{\kappa_1-1} \Psi(X_j) \geq t\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{j=1}^{n-1} \Psi(X_j) \geq t, \kappa_1 = n\right)$$
$$\leq \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \mathbb{P}(\Psi(X_j) \geq t/(n-1), \kappa_1 = n).$$

The probability under the sum appearing in the last expression can be estimated by

(4.7)
$$\mathbb{P}(\Psi(X_j) \geq t/(n-1), \delta_p = 1, \forall p \leq j)$$
$$= \mathbb{E}[Q_1(X_{j-1}, \Psi \geq t/(n-1)), \delta_p = 1, \forall p \leq j].$$

When $j \geq n/2$ we can use (2.15) to bound the expression on the right-hand side of (4.7) from above by

(4.8)
$$\frac{n^\alpha g(t/n)}{t^\alpha} \mathbb{P}[\delta_p = 1, \forall p \leq j] \leq \frac{n^\alpha g(t/n)}{t^\alpha} \mathbb{P}[\kappa_1 \geq n/2].$$

Here $g(x)$ is a bounded function that goes to 0, as $x \to \infty$. On the other hand, when $j < n/2$, we rewrite the probability appearing under the sum on the right-hand side of (4.6) using the Markov property. It equals

$$\mathbb{E}[\mathbb{E}[\tilde{\kappa}_1 = n - j | \tilde{X}_0 = X_j], \Psi(X_j) \geq t/(n-1), \delta_0 = \cdots = \delta_j = 1].$$

Here $\{\tilde{X}_n, n \geq 0\}$ is another copy of the Markov chain $\{X_n, n \geq 0\}$, and $\tilde{\kappa}_1$ is the respective stopping time defined in correspondence to $\kappa_1$. We can estimate this expression by

$$\mathbb{P}[\Psi(X_j) \geq t/(n-1)] \sup_x \mathbb{P}[\kappa_1 \geq n - j | X_0 = x]$$

$$\leq \frac{n^\alpha g(t/n)}{t^\alpha} \sup_x \mathbb{P}[\kappa_1 \geq n/2 | X_0 = x].$$

Summarizing, we have shown that the utmost left-hand side of (4.6) can be estimated by

$$\sum_{n=1}^\infty \frac{g(t/n)n^{1+\alpha}}{t^\alpha} \sup_x \mathbb{P}_x(\kappa_1 \geq n/2).$$

We conclude that this expression is $o(t^{-\alpha})$ by invoking Lebesgue dominated convergence theorem and Condition 2.6. The negative tails are treated in the same way. Therefore, $\varphi_0 - \Psi(X_0)$ has lighter tails than $\Psi(X_0)$ itself. By Lemma 4.1, the sum of a random variable satisfying condition (2.16) and a random variable with lighter tails also satisfies condition (2.16) for the same constants $c^+ \bar{\theta}^{-1}$, $c^- \bar{\theta}^{-1}$.

At this point we are only left to recall the classical limit theorem for i.i.d. random variables. It follows that there exist

$$N^{-1/\alpha} S_N = \left(\frac{\mathcal{M}(N)}{N}\right)^{1/\alpha} \frac{1}{\mathcal{M}(N)^{1/\alpha}} \sum_{i=0}^{\mathcal{M}(N)} \varphi_i + \frac{1}{N^{1/\alpha}} \sum_{j=\kappa_{\mathcal{M}(N)}+1}^N \Psi(X_j)$$

$$= \left(\frac{\mathcal{M}(N)}{N\bar{\theta}}\right)^{1/\alpha} \frac{1}{\mathcal{M}(N)^{1/\alpha}} \sum_{i=0}^{\mathcal{M}(N)} \bar{\theta}^{1/\alpha} \varphi_i + \frac{1}{N^{1/\alpha}} \sum_{j=\kappa_{\mathcal{M}(N)}+1}^N \Psi(X_j).$$

Recall (4.2), and notice that by (4.5),

$$\lim_{\lambda \to +\infty} \lambda^\alpha \mathbb{P}(\bar{\theta}^{1/\alpha} \varphi_i > \lambda) = c_*^+,$$

$$\lim_{\lambda \to +\infty} \lambda^\alpha \mathbb{P}(\bar{\theta}^{1/\alpha} \varphi_i < -\lambda) = c_*^-.$$

Let $C_* := (c_*^- + c_*^+)\bar{\theta}^{-1}$. By virtue of the stable limit theorem for i.i.d. random variables (see, e.g., [8], Theorem 7.7, page 153), we know that for $c_N := N\mathbb{E}[\varphi_1, |\varphi_1| \leq (C_*N)^{1/\alpha}]$ such that the laws of $N^{-1/\alpha}(\sum_{i=0}^N \bar{\theta}^{1/\alpha} \varphi_i - c_N)$ converge to an $\alpha$-stable law. When $\alpha < 1$ constants $c_N \sim cN^{1/\alpha}$ and they can be discarded. Observe, however, that since $\mathbb{E}\varphi_1 = 0$, cf. (4.3), for $\alpha \in (1, 2)$, we have

$$c_N = -N\mathbb{E}[\varphi_1, |\varphi_1| > (C_*N)^{1/\alpha}] = N \int_{(C_*N)^{1/\alpha}}^{+\infty} \left[\mathbb{P}[\varphi_1 < -\lambda] - \mathbb{P}[\varphi_1 > \lambda]\right] d\lambda$$

$$= \bar{\theta}^{-1} N \int_{(C_*N)^{1/\alpha}}^{+\infty} (c_*^- c_*^+) \frac{d\lambda}{\lambda^\alpha} = C(1 + o(1))N^{1/\alpha}$$

for some constant $C$. The constants $c_N$ can again be discarded. We conclude therefore that the laws of

$$\mathcal{K}_N := N^{-1/\alpha} \left( \sum_{i=0}^{N} \bar{\theta}^{1/\alpha} \varphi_i \right)$$

weakly converge to some $\alpha$-stable law $\nu_*$. Since $\mathcal{L}_N := \bar{\theta}^{-1/\alpha} N^{-1} \mathcal{M}(N)$ converges a.s. to 1, the joint law of $(\mathcal{K}_N, \mathcal{L}_N)$ converges to $\nu_* \otimes \delta_1$, as $N \to +\infty$. According to the Skorochod representation theorem there exists a probability space and random variables $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N)$ such that $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N) \stackrel{d}{=} (\mathcal{K}_N, \mathcal{L}_N)$ for each $N$ and $(\bar{\mathcal{K}}_N, \bar{\mathcal{L}}_N) \to (Y_*, 1)$ a.s. The above in particular implies that $\bar{\mathcal{K}}_{N\bar{\mathcal{L}}_N}$ converges a.s. to $Y_*$. Since $\bar{\mathcal{K}}_{N\bar{\mathcal{L}}_N} \stackrel{d}{=} \mathcal{K}_{N\mathcal{L}_N}$, we conclude the convergence of the laws of $\mathcal{K}_{N\mathcal{L}_N}$ to $\nu_*$.  □

## 5. Proof of Theorem 2.4 by martingale approximation.

Below we formulate a stable limits law that shall be crucial during the course of the proof of the theorems.

Suppose that $\{Z_n, n \geq 1\}$ is a stationary sequence that is adapted with respect to the filtration $\{\mathcal{G}_n, n \geq 0\}$ and such that for any $f$ bounded and measurable, the sequence $\{\mathbb{E}[f(Z_n)|\mathcal{G}_{n-1}], n \geq 1\}$ is also stationary. We assume furthermore that there exist $\alpha \in (0, 2)$ and $c_*^+, c_*^- \geq 0$ such that $c_*^+ + c_*^- > 0$ and

(5.1)
$$\mathbb{P}[Z_1 > \lambda] = \lambda^{-\alpha}(c_*^+ + o(1)),$$
$$\mathbb{P}[Z_1 < -\lambda] = \lambda^{-\alpha}(c_*^- + o(1)) \qquad \text{as } \lambda \to +\infty.$$

In addition, for any $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have

(5.2)
$$\lim_{N \to +\infty} \mathbb{E} \left| \sum_{n=1}^{[Nt]} \mathbb{E}\left[ g\left( \frac{Z_n}{N^{1/\alpha}} \right) \Big| \mathcal{G}_{n-1} \right] - \alpha t \int_{\mathbb{R}} g(\lambda) \frac{c_*(\lambda)\, d\lambda}{|\lambda|^{1+\alpha}} \right| = 0$$

and

(5.3)
$$\lim_{N \to +\infty} N \mathbb{E} \left\{ \mathbb{E}\left[ g\left( \frac{Z_1}{N^{1/\alpha}} \right) \Big| \mathcal{G}_0 \right] \right\}^2 = 0.$$

Here $c_*(\cdot)$ appearing in (5.2) is given by (2.2). Let $M_N := \sum_{n=1}^{N} Z_n$, $N \geq 1$ and $M_0 := 0$.

When $\alpha = 1$ we shall also consider an array $\{Z_n^{(N)} : n \geq 1\}$, $N \geq 1$ of stationary sequences adapted with respect to the filtration $\{\mathcal{G}_n : n \geq 0\}$. Assume furthermore that for each $N \geq 1$ and any $f$ bounded and measurable sequence, $\{\mathbb{E}[f(Z_n^{(N)})|\mathcal{G}_{n-1}] : n \geq 1\}$ is stationary. We suppose that there exist nonnegative $c_*^+, c_*^-$ such that $c_*^+ + c_*^- > 0$ and

(5.4)
$$\lim_{\lambda \to +\infty} \sup_{N \geq 1} \left[ |\lambda \mathbb{P}[Z_1^{(N)} > \lambda] - c_*^+| + |\lambda \mathbb{P}[Z_1^{(N)} < -\lambda] - c_*^-| \right] = 0.$$

Let

$$\tilde{M}_N := \sum_{n=1}^{N} \{Z_n^{(N)} - \mathbb{E}[Z_n^{(N)} 1[|Z_n^{(N)}| \le N]|\mathcal{G}_{n-1}]\}, \qquad N \ge 1,$$

and $\tilde{M}_0 := 0$. The following result has been shown in Section 4 of [9].

THEOREM 5.1. (i) *Suppose that* $\alpha \in (1, 2)$, *conditions* (5.1)–(5.3) *hold, and*

$$\text{(5.5)} \qquad\qquad \mathbb{E}[Z_n|\mathcal{G}_{n-1}] = 0 \qquad \text{for } n \ge 1.$$

*Then* $N^{-1/\alpha} M_{[N\cdot]} \Rightarrow Z(\cdot)$, *as* $N \to +\infty$, *weakly in* $D[0, +\infty)$, *where* $\{Z(t), t \ge 0\}$ *is an* $\alpha$-*stable process of type* II.

(ii) *Suppose that* $\alpha \in (0, 1)$ *and conditions* (5.1)–(5.3) *hold. Then* $N^{-1/\alpha} \times M_{[N\cdot]} \Rightarrow Z(\cdot)$, *as* $N \to +\infty$, *weakly in* $D[0, +\infty)$, *where* $\{Z(t), t \ge 0\}$ *is an* $\alpha$-*stable process of type* I.

(iii) *For* $\alpha = 1$, *assume* (5.2) *and* (5.3) *with* $Z_n^{(N)}$ *replacing* $Z_n$ *and* (5.4). *Then* $N^{-1} \tilde{M}_{[N\cdot]} \Rightarrow Z(\cdot)$, *as* $N \to +\infty$, *weakly in* $D[0, +\infty)$ *to a Lévy process* $\{Z(t), t \ge 0\}$ *of type* III.

*Proof of part* (i) *of Theorem* 2.4. Let $\chi \in L^\beta(\pi)$, $\beta \in (1, \alpha)$ be the unique, zero-mean solution of the equation

$$\text{(5.6)} \qquad\qquad \chi - P\chi = \Psi.$$

Since $\Psi \in L^\beta(\pi)$ for $\beta \in (0, \alpha)$ is of zero mean, the solution to (5.6) exists in $L^\beta(\pi)$ and is given by $\chi = \sum_{n \ge 0} P^n \Psi$. This follows from the fact that $\|P^n \Psi\|_{L^\beta} \le a^{(2/\beta - 1)n} \|\Psi\|_{L^\beta}$, $n \ge 0$ [see (2.7)], so the series defining $\chi$ geometrically converges. Uniqueness is a consequence of (2.7). Indeed, if $\chi_1$ was another zero-mean solution to (5.6), then

$$\|\chi - \chi_1\|_{L^\beta} = \|P(\chi - \chi_1)\|_{L^\beta} \le a^{1 - |2/\beta - 1|} \|\chi - \chi_1\|_{L^\beta},$$

which clearly is possible only when $\chi - \chi_1 = 0$ (recall that $a < 1$). Note also that from (2.12) it follows that in fact $P\chi = (I - P)^{-1}(P\Psi) \in L^{\alpha'}(\pi)$. Thus in particular,

$$\text{(5.7)} \qquad\qquad \pi(|P\chi| > \lambda) \le \frac{\|P\chi\|_{L^{\alpha'}(\pi)}^{\alpha'}}{\lambda^{\alpha'}},$$

and consequently $\chi$ satisfies the same tail condition as $\Psi$ [cf. (2.5)].

Then by using (5.6), we can write

$$\text{(5.8)} \qquad\qquad S_N = \sum_{n=1}^{N} \Psi(X_n) = \sum_{n=1}^{N} Z_n + P\chi(X_0) - P\chi(X_N)$$

with $Z_n = \chi(X_n) - P\chi(X_{n-1})$.

In what follows, we denote by $C_0^\infty(\mathbb{R} \setminus \{0\})$ the space of all $C^\infty$ functions that are compactly supported in $\mathbb{R} \setminus \{0\}$. According to part (i) of Theorem 5.1, we only need to demonstrate the following.

PROPOSITION 5.2.    *For any $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$, equalities (5.2) and (5.3) hold.*

More explicitly, we have

$$E\left[g\left(\frac{Z_n}{N^{1/\alpha}}\right)\Big|\mathcal{G}_{n-1}\right] = \int g(N^{-1/\alpha}[\chi(y) - P\chi(X_{n-1})]) P(X_{n-1}, dy)$$

and using the stationarity of $\pi$, we can bound the left-hand side of (5.2) by

$$\mathbb{E}\left|\sum_{n=1}^{[Nt]} \int \left[g(N^{-1/\alpha}[\chi(y) - P\chi(X_{n-1})]) - g\left(\frac{\chi(y)}{N^{1/\alpha}}\right)\right] P(X_{n-1}, dy)\right|$$

(5.9)
$$+ \mathbb{E}\left|\sum_{n=1}^{[Nt]} \int g\left(\frac{\chi(y)}{N^{1/\alpha}}\right) P(X_{n-1}, dy) - [Nt] \int g\left(\frac{\chi(y)}{N^{1/\alpha}}\right)\pi(dy)\right|$$

$$+ \left|[Nt] \int g\left(\frac{\chi(y)}{N^{1/\alpha}}\right)\pi(dy) - \alpha t \int_{\mathbb{R}} g(\lambda)\frac{c_*(\lambda)\,d\lambda}{|\lambda|^{1+\alpha}}\right|,$$

so (5.2) is a consequence of the following three lemmas, each taking care of the respective term of (5.9):

LEMMA 5.3.

(5.10)
$$\lim_{N\to\infty} N \iint \big|g(N^{-1/\alpha}[\chi(y) - P\chi(x)]) $$
$$- g(N^{-1/\alpha}\chi(y))\big| P(x, dy)\pi(dx) = 0.$$

LEMMA 5.4.

(5.11)    $$\mathbb{E}\left|\sum_{n=1}^{N} \mathbb{E}[g(N^{-1/\alpha}\chi(X_n))|\mathcal{G}_{n-1}] - N\int g(N^{-1/\alpha}\chi(y))\pi(dy)\right| = 0.$$

LEMMA 5.5.

(5.12)    $$\lim_{N\to\infty}\left|N\int g(N^{-1/\alpha}\chi(y))\pi(dy) - \alpha\int_{\mathbb{R}} g(\lambda)\frac{c_*(\lambda)\,d\lambda}{|\lambda|^{1+\alpha}}\right| = 0,$$

*where $c_*(\cdot)$ is given by (2.2).*

*Proof of Lemma* 5.3. Suppose that $\operatorname{supp} g \subset [-M, M] \setminus [-m, m]$ for some $0 < m < M < +\infty$ and $\theta > 0$. Denote

$$A_{N,\theta} = \{(x, y) : |\chi(y) - \theta P\chi(x)| > N^{1/\alpha} m\}.$$

The left-hand side of (5.10) can be bounded from above by

$$N^{1-1/\alpha} \int_0^1 d\theta \iint |g'(N^{-1/\alpha}[\chi(y) - \theta P\chi(x)]) P\chi(x)| P(x, dy)\pi(dx)$$

$$\leq CN^{1-1/\alpha} \int_0^1 d\theta \iint_{A_{N,\theta}} |P\chi(x)| P(x, dy)\pi(dx)$$

$$\leq CN^{1-1/\alpha} \int_0^1 d\theta \left(\iint_{A_{N,\theta}} P(x, dy)\pi(dx)\right)^{1-1/\alpha'} \|P\chi\|_{L^{\alpha'}}.$$

From the tail behavior of $\chi$ and of $P\chi$, [see (5.7) and the remark below that estimates], it is easy to see that for any $\theta \in (0, 1)$,

$$\iint_{A_{N,\theta}} P(x, dy)\pi(dx) \leq \mathbb{P}[|\chi(X_1)| \geq (mN^{1/\alpha})/2]$$

$$+ \mathbb{P}[\theta|P\chi(X_0)| \geq (mN^{1/\alpha})/2]$$

$$\leq C[(Nm^\alpha)^{-1} + (Nm^\alpha)^{-\alpha'/\alpha}]$$

$$= \frac{C}{N}(1 + o(1))$$

as $N \gg 1$. Since $\alpha' > \alpha$ we obtain (5.10).

*Proof of Lemma* 5.4. To simplify the notation we assume that $\operatorname{supp} g \subset [m, M]$ for $0 < m < M < +\infty$. Denote $B_{N,\lambda} = \{y : \chi(y) \geq N^{1/\alpha}\lambda\}$. We can rewrite the left-hand side of (5.11) as

(5.13) $$\mathbb{E}\left|\int_0^\infty g'(\lambda) \sum_{n=1}^N G_N(X_{n-1}, \lambda) \, d\lambda\right|,$$

where

$$G_N(x, \lambda) = P(x, B_{N,\lambda}) - \pi(B_{N,\lambda}).$$

Notice that $\int G_N(y, \lambda)\pi(dy) = 0$ and

$$\int G_N^2(y, \lambda)\pi(dy) = \int P^2(y, B_{N,\lambda})\pi(dy) - \pi^2(B_{N,\lambda})$$

(5.14) $$\leq 2\int\left(\int_{B_{N,\lambda}} p(y, x)\pi(dx)\right)^2 \pi(dy)$$

$$+ 2\int Q^2(y, B_{N,\lambda})\pi(dy) - \pi^2(B_{N,\lambda}).$$

To estimate the first term on the utmost right-hand side, we use the Cauchy–Schwarz inequality, while for the second one we apply condition (2.10). For $\lambda \geq m$, we can bound the expression on the right-hand side of (5.14) by

$$
\frac{1}{2}\pi(B_{N,m}) \iint_{B_{N,m}} p^2(x, y)\pi(dx)\pi(dy) + C\pi^2(B_{N,m})
$$

(5.15)

$$
\leq \frac{1}{N}o(1) \qquad \text{as } N \to \infty,
$$

by virtue of (2.9) and the remark after (5.7). Thus we have shown that

$$
(5.16) \qquad N \sup_{\lambda \geq m} \int G_N^2(y, \lambda)\pi(dy) \to 0
$$

as $N \to \infty$. We will show now that (5.16) and the spectral gap together imply that

$$
(5.17) \qquad \sup_{\lambda \geq m} \mathbb{E}\left| \sum_{n=1}^{N} G_N(X_{n-1}, \lambda) \right|^2 \to 0
$$

as $N \to \infty$. Since $\operatorname{supp} g' \subset [m, M]$ expression in (5.13) can be then estimated by

$$
\sup_{\lambda \geq m} \mathbb{E}\left| \sum_{n=1}^{N} G_N(X_{n-1}, \lambda) \right| \times \int_0^{\infty} |g'(\lambda)|\, d\lambda \to 0
$$

as $N \to +\infty$ and the conclusion of the lemma follows.

To prove (5.17) let $u_N(\cdot, \lambda) = (I - P)^{-1}G_N(\cdot, \lambda)$. By the spectral gap condition (2.7), we have

$$
(5.18) \qquad \int u_N^2(y, \lambda)\pi(dy) \leq \frac{1}{1 - a^2} \int G_N^2(y, \lambda)\pi(dy).
$$

We can then rewrite

$$
\sum_{n=1}^{N} G_N(X_{n-1}, \lambda) = u_N(X_0) - u_N(X_N) + \sum_{n=1}^{N-1} U_n,
$$

where $U_n = u_N(X_n) - Pu_N(X_{n-1})$, $n \geq 1$ is a stationary sequence of martingale differences with respect to the natural filtration corresponding to $\{X_n, n \geq 0\}$. Consequently,

$$
\mathbb{E}\left| \sum_{n=1}^{N} G_N(X_{n-1}, \lambda) \right|^2 \leq CN \int u_N^2(y, \lambda)\pi(dy) \to 0
$$

and (5.17) follows from (5.16) and (5.18).

*Proof of Lemma* 5.5. To avoid long notation, we again assume that supp $g \subset [m, M]$ for $0 < m < M < +\infty$. The proof in the case of $g \subset [-M, -m]$ is virtually the same. Note that

$$N \int g\left(\frac{\chi(y)}{N^{1/\alpha}}\right) \pi(dy)$$

$$= N \iint_0^{+\infty} N^{-1/\alpha} g'\left(\frac{\lambda}{N^{1/\alpha}}\right) 1_{[0,\chi(y)]}(\lambda)\pi(dy)\,d\lambda$$

$$= N \int_0^{+\infty} N^{-1/\alpha} g'\left(\frac{\lambda}{N^{1/\alpha}}\right) \pi(\chi > \lambda)\,d\lambda$$

$$= N \int_0^{+\infty} g'(\lambda)\pi(\chi \geq N^{1/\alpha}\lambda)\,d\lambda.$$

Thanks to (2.5) the last expression tends, however, as $N \to +\infty$, to

$$\int_0^{+\infty} g'(\lambda)\frac{c_*^+\,d\lambda}{\lambda^\alpha} = \alpha \int_{\mathbb{R}} g(\lambda)\frac{c_*(\lambda)\,d\lambda}{|\lambda|^{\alpha+1}}.$$

*Proof of Proposition* 5.2. We have already shown (5.2), so only (5.3) requires a proof. To simplify the notation we assume $Q \equiv 0$. Suppose that supp $g \subset [m, M]$ for some $0 < m < M$. We can write

$$\mathbb{E}\left[g\left(\frac{Z_1}{N^{1/\alpha}}\right)\Big|\mathcal{G}_0\right]$$

(5.19)
$$= \int g(N^{-1/\alpha}\Psi(y))p(X_0, y)\pi(dy)$$

$$+ N^{-1/\alpha} \iint_0^1 h(X_0, y)g'\big(N^{-1/\alpha}\big(\Psi(y) + \theta h(X_0, y)\big)\big)$$

$$\times p(X_0, y)\pi(dy)\,d\theta,$$

where $h(x, y) := P\chi(y) - P\chi(x)$. Denote by $K_1$ and $K_2$ the first and the second terms appearing on the right-hand side above. By Cauchy–Schwarz inequality,

(5.20)
$$\mathbb{E}K_2^2 \leq \frac{\|g'\|_\infty^2}{N^{2/\alpha}}\left[\mathbb{E}\left(\int |P\chi(y)|p(X_0, y)\pi(dy)\right)^2\right.$$

$$\left. + \mathbb{E}\left(\int |P\chi(X_0)|p(X_0, y)\pi(dy)\right)^2\right]$$

$$\leq \frac{2\|g'\|_\infty^2\|P\chi\|_{L^2(\pi)}^2}{N^{2/\alpha}}.$$

Hence $\lim_{N\to+\infty} N\mathbb{E}K_2^2 = 0$.

On the other hand,

(5.21)
$$K_1 \leq \|g\|_\infty \int p(X_0, y)1[|\Psi(y)| > mN^{1/\alpha}/2]\pi(dy),$$

and in consequence, by Jensen's inequality,

$$(\mathbb{E}K_1^2)^{1/2} \le \|g\|_\infty \int (\mathbb{E}p^2(X_0, y))^{1/2} 1[|\Psi(y)| > aN^{1/\alpha}/2]\pi(dy)$$

$$\le \|g\|_\infty \left[\int\int p^2(x, y)1[|\Psi(y)| > aN^{1/\alpha}/2]\pi(dx)\pi(dy)\right]^{1/2}$$

$$\times \pi^{1/2}[|\Psi| > aN^{1/\alpha}/2].$$

Thus we have shown that

$$(5.22) \quad \begin{aligned} N\mathbb{E}K_1^2 &\le N\pi[|\Psi| > aN^{1/\alpha}/2] \\ &\times \int\int p^2(x, y)1[|\Psi(y)| > aN^{1/\alpha}/2]\pi(dx)\pi(dy) \to 0. \end{aligned}$$

Condition (5.3) is then a consequence of (5.20) and (5.22).

*Proof of part* (ii) *of Theorem* 2.4.   The proof of this part relies on part (ii) of Theorem 5.1. The following analogue of Proposition 5.2 can be established.

PROPOSITION 5.6.   *Suppose that* $\alpha \in (0, 1)$. *Then for any* $g \in C_0^\infty(\mathbb{R} \setminus \{0\})$,

$$(5.23) \quad \lim_{N\to+\infty} \mathbb{E}\left|\sum_{n=1}^{[Nt]} \mathbb{E}\left[g\left(\frac{\Psi(X_n)}{N^{1/\alpha}}\right)\Big|\mathcal{G}_{n-1}\right] - t\int_\mathbb{R} g(\lambda)\frac{C_*(\lambda)\,d\lambda}{|\lambda|^{1+\alpha}}\right| = 0$$

*and*

$$(5.24) \quad \lim_{N\to+\infty} N\mathbb{E}\left\{\mathbb{E}\left[g\left(\frac{\Psi(X_1)}{N^{1/\alpha}}\right)\Big|\mathcal{G}_0\right]\right\}^2 = 0.$$

PROOF.   The proof of this proposition is a simplified version of the argument used in the proof of Proposition 5.2. The expression in (5.23) can be estimated by

$$(5.25) \quad \begin{aligned} &\mathbb{E}\left|\sum_{n=1}^{[Nt]} \int g\left(\frac{\Psi(y)}{N^{1/\alpha}}\right)P(X_{n-1}, dy) - [Nt]\int g\left(\frac{\Psi(y)}{N^{1/\alpha}}\right)\pi(dy)\right| \\ &+ \left|[Nt]\int g\left(\frac{\Psi(y)}{N^{1/\alpha}}\right)\pi(dy) - \alpha t\int_\mathbb{R} g(\lambda)\frac{c_*(\lambda)\,d\lambda}{|\lambda|^{1+\alpha}}\right|. \end{aligned}$$

The proof that both the terms of the sum above vanish goes along the lines of the proofs of Lemmas 5.4 and 5.5. We can repeat word by word the argument used there, replacing this time $\chi$ by $\Psi$. As for the proof of (5.24) it is identical with the respective part of the proof of (5.3) (the one concerning term $K_1$).   $\square$

*Proof of part* (iii) *of Theorem* 2.4.    Recall that $\Psi_N := \Psi 1[|\Psi| \leq N]$. Let $\chi_N$ be the unique, zero mean solution of the equation

$$(5.26) \qquad\qquad\qquad \chi_N - P\chi_N = \Psi_N - c_N.$$

We can then write,

$$(5.27) \quad S_N - Nc_N = \sum_{n=1}^{N}(\Psi(X_n) - c_N) = \sum_{n=1}^{N} Z_n^{(N)} + P\chi_N(X_0) - P\chi_N(X_N)$$

with

$$Z_n^{(N)} = \chi_N(X_n) - P\chi_N(X_{n-1}) + \Psi(X_n)1[|\Psi(X_n)| > N].$$

We verify first assumptions (5.2), (5.3) and (5.4).

Condition (5.4) is an obvious consequence of the fact that

$$(5.28) \qquad\qquad Z_n^{(N)} = P\chi_N(X_n) - P\chi_N(X_{n-1}) + \Psi(X_n) - c_N$$

and assumption (2.13). To verify the remaining hypotheses, suppose that $\operatorname{supp} g \subset (m, M)$ and $m < 1 < M$. Let us fix $\delta > 0$, to be further chosen later on, such that $m < 1 - \delta < 1 + \delta < M$. We can then write $g = g_1 + g_2 + g_3$ where each $g_i \in C^\infty(\mathbb{R})$, $\|g_i\|_\infty \leq \|g\|_\infty$, and the supports of $g_1, g_2, g_3$ are correspondingly contained in $(m, 1 - \delta)$, $(1 - \delta, 1 + \delta)$, $(1 + \delta, M)$. We prove (5.2) and (5.3) for each of the function $g_i$s separately. Note that

$$\mathbb{E}\left[g_i\left(\frac{Z_n^{(N)}}{N}\right)\Big|\mathcal{G}_{n-1}\right] = \int g_i(w^{(N)}(X_{n-1}, y))P(X_{n-1}, dy),$$

where

$$w^{(N)}(x, y) := N^{-1}\Psi(y)1[|\Psi(y)| > N] + N^{-1}[\chi_N(y) - P\chi_N(x)].$$

For $i = 1$ and $i = 3$, we essentially estimate in the same way as in parts (i) and (ii) of the proof of the theorem, respectively. We shall only consider here the case $i = 2$.

Note that then $w^{(N)}(x, y) = w_1^{(N)}(x, y)$ where

$$(5.29) \qquad\qquad w_\theta^{(N)}(x, y) = N^{-1}[\Psi(y) - c_N] + N^{-1}\theta R_N(x, y)$$

with $R_N(x, y) := P\chi_N(y) - P\chi_N(x)$. However,

$$g_2(w^{(N)}(X_{n-1}, y)) = g_2(N^{-1}(\Psi(y) - c_N))$$
$$+ N^{-1}R_N(X_{n-1}, y)\int_0^1 g_2'(w_\theta^{(N)}(X_{n-1}, y))\, d\theta$$

and

$$\mathbb{E}\left|\sum_{n=1}^{N}\int g_2\big(w^{(N)}(X_{n-1},y)\big)P(X_{n-1},dy)\right|$$

$$\leq \mathbb{E}\left|\sum_{n=1}^{N}\int g_2\big(N^{-1}(\Psi(y)-c_N)\big)P(X_{n-1},dy)\right|$$

$$+\iint\int_0^1 |g_2'\big(w_\theta^{(N)}(x,y)\big)R_N(x,y)|P(x,dy)\pi(dy)\,d\theta.$$

Denote the first and the second term on the right-hand side by $J_1^{(N)}$ and $J_2^{(N)}$, respectively. Term $J_1^{(N)}$ can be now estimated as in the proof of part (ii) of the theorem. We conclude then, using the arguments contained in the proofs of Lemmas 5.4 and 5.5 that

$$\limsup_{N\to+\infty} J_1^{(N)} \leq \|g\|_\infty \int_{1-\delta}^{1+\delta}\frac{d\lambda}{\lambda^2}.$$

On the other hand, to estimate $\lim_{N\to+\infty} J_2^{(N)}=0$, since $g_2'(w_\theta^{(N)}(x,y))\to 0$ in measure $P(x,dy)\pi(dy)\,d\theta$ and the passage to the limit under he integral can be substantiated thanks to (2.13).

Choosing now sufficiently small $\delta > 0$ we can argue that the calculation of the limit can be reduced to the cases considered for $g_1$ and $g_3$ and that condition (5.2) can be established for $Z_n^{(N)}$. The proof of (5.3) can be repeated from the argument for part (i) of the theorem.

Finally, we show that

$$(5.30)\qquad \lim_{N\to+\infty}\frac{1}{N}\mathbb{E}\left|\sum_{n=1}^{N}\mathbb{E}[Z_n^{(N)}1[|Z_n^{(N)}|\leq N]|\mathcal{G}_{n-1}]\right|=0.$$

Denote the expression under the limit by $L^{(N)}$. Let $\Delta > 1$. We can write $L^{(N)} = L_1^{(N)} + L_2^{(N)} + L_3^{(N)}$ depending on whether $|\Psi(X_n)| > \Delta N$, $|\Psi(X_n)| \in (\Delta^{-1}N, \Delta N]$, or $|\Psi(X_n)| \leq (\Delta)^{-1}N$. Then

$$L_1^{(N)} \leq \sum_{n=1}^{N}\mathbb{P}[|\Psi(X_n)| > \Delta N, |Z_n^{(N)}| \leq N] = N\mathbb{P}[|\Psi(X_1)| > \Delta N, |Z_1^{(N)}| \leq N].$$

From formula (5.28) for $Z_n^{(N)}$, we conclude that the event under the conditional probability can take place only when $|P\chi_N(X_n)|$, or $|P\chi_N(X_{n-1})| > N(\Delta-1)/3$ for those $N$, for which $c_N/N \leq (\Delta-1)/3$. Using this observation, (2.13) and Chebyshev's inequality, one can easily see that

$$L_1^{(N)} \leq 2N[N(\Delta-1)/3]^{-\alpha'}\|P\chi_N\|_{L^{\alpha'}(\pi)}^{\alpha'} \to 0$$

as $N \to +\infty$. To deal with $L_2^{(N)}$ consider a nonnegative $g \in C^\infty(\mathbb{R})$ such that $\|g\|_\infty \leq 1$, $[\Delta^{-1}, \Delta] \subset \text{supp } g \subset [\Delta_1^{-1}, \Delta_1]$ for some $\Delta_1 > \Delta$. Repeating the foregoing argument for $g_2$, we conclude that

$$\limsup_{N \to +\infty} L_2^{(N)} \leq \|g\|_\infty \int_{\Delta_1^{-1}}^{\Delta_1} \frac{d\lambda}{\lambda^2},$$

which can be made as small as we wish by choosing $\Delta_1$ sufficiently close to 1. As for $L_3^{(N)}$, note that it equals

$$(5.31) \quad L_3^{(N)} = \frac{1}{N} \mathbb{E} \left| \sum_{n=1}^N \mathbb{E}[M_n^{(N)} \mathbb{1}[|M_n^{(N)}| \leq N, |\Psi(X_n)| \leq (\Delta)^{-1} N] | \mathcal{G}_{n-1}] \right|,$$

where

$$(5.32) \quad \begin{aligned} M_n^{(N)} &:= \chi_N(X_n) - P\chi_N(X_{n-1}) \\ &= \Psi_N(X_n) - c_N + P\chi_N(X_n) - P\chi_N(X_{n-1}). \end{aligned}$$

Thanks to the fact that $M_n^{(N)}$ are martingale differences, the expression in (5.31) can be written as $L_3^{(N)} = -(L_{31}^{(N)} + L_{32}^{(N)} + L_{33}^{(N)})$ where $L_{3i}^{(N)}$ correspond to taking the conditional expectation over the events $A_i$ for $i = 1, 2, 3$ given by

$$A_1 := [|M_n^{(N)}| > N, |\Psi(X_n)| \leq (\Delta)^{-1} N],$$

$$A_2 := [|M_n^{(N)}| > N, |\Psi(X_n)| > (\Delta)^{-1} N],$$

$$A_3 := [|M_n^{(N)}| \leq N, |\Psi(X_n)| > (\Delta)^{-1} N].$$

To estimate $L_{3i}^{(N)}$, $i = 1, 2$ we note from (5.32) that $|M_n^{(N)}| > N$ only when $\Psi_N(X_n) = \Psi(X_n)$ and $|\Psi(X_n)| \leq N$, or $P\chi_N(X_{n-1})$, $P\chi_N(X_n)$ are greater than $cN$ for some $c > 0$. In the latter two cases we can estimate similarly to $L_1^{(N)}$. In the first one, however, we end up with the limit

$$\limsup_{N \to +\infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^N \mathbb{E}[(|\Psi_N(X_n)| + |c_N| + |P\chi_N(X_n)| + |P\chi_N(X_{n-1})|),$$

$$N \geq |\Psi(X_n)| > \Delta^{-1} N | \mathcal{G}_{n-1}]$$

$$\leq \limsup_{N \to +\infty} N(1 + |c_N|/N) \pi[N \geq |\Psi| > (\Delta)^{-1} N]$$

$$+ \limsup_{N \to +\infty} \int (I + P)|P\chi_N| \mathbb{1}[N \geq |\Psi| > (\Delta)^{-1} N] d\pi.$$

The second term on the utmost right-hand side vanishes thanks to (2.13). The first one can be estimated as in the proof of Lemma 5.5, and we obtain that it is smaller

than $C \int_{\Delta^{-1}}^{1} \lambda^{-2} d\lambda$, which can be made as small as we wish upon choosing $\Delta$ sufficiently close to 1. We can estimate, therefore,

$$\limsup_{N \to +\infty} L_{33}^{(N)}$$

$$\leq \limsup_{N \to +\infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} \mathbb{E}\left[\left(|\Psi_N(X_n)| + |c_N|\right), N \geq |\Psi(X_n)| > \Delta^{-1} N | \mathcal{G}_{n-1}\right]$$

$$+ \limsup_{N \to +\infty} \frac{1}{N} \mathbb{E} \sum_{n=1}^{N} \mathbb{E}\left[\left(|P\chi_N(X_n)| + |P\chi_N(X_{n-1})|\right),\right.$$

$$\left. |\Psi(X_n)| > \Delta^{-1} N | \mathcal{G}_{n-1}\right]$$

$$= \limsup_{N \to +\infty} N\pi[N \geq |\Psi| > (\Delta)^{-1} N]$$

$$+ \limsup_{N \to +\infty} \int (I + P)|P\chi_N| \mathbb{1}[|\Psi| > (\Delta)^{-1} N] d\pi$$

$$\leq C \int_{\Delta^{-1}}^{1} \frac{d\lambda}{\lambda^2},$$

which again can be made arbitrarily small.

**6. Proof of Theorem 2.8.** Suppose that we are given a sequence of i.i.d. non-negative random variables $\{\rho_n, n \geq 0\}$ independent of $\{X_n, n \geq 0\}$ and such that $A_\alpha := \int_0^{+\infty} \rho^\alpha \varphi(d\rho) < +\infty$, where $\varphi(\cdot)$ is the distribuant of $\rho_0$ and $\alpha \in (0, 2)$. We consider a slightly more general situation than the one presented in Theorem 2.4 by allowing

$$(6.1) \qquad S_N(t) := \sum_{n=0}^{[Nt]} \Psi(X_n)\rho_n.$$

Observe that, if $\pi$ is the law of $X_n$, observable $\Psi$ satisfies the tail conditions (2.5), and $\rho_n$ is independent of $X_n$, then

$$\lambda^\alpha \mathbb{P}\left(\Psi(X_0)\rho_0 > \lambda\right) = \int_0^\infty \lambda^\alpha \pi(\Psi > \lambda\rho^{-1})\varphi(d\rho) \xrightarrow[\lambda \to +\infty]{} c_*^+ A_\alpha.$$

Define also

$$(6.2) \qquad C_N := \int_{|\Psi| \leq N} \Psi \, d\pi \, \mathbb{E}\rho_0.$$

Consider then the Markov chain $\{(X_n, \rho_n), n \geq 0\}$ on $E \times \mathbb{R}_+$. This Markov chain satifies all conditions used in the previous sections, with stationary ergodic measure given by $\pi(dy) \otimes \varphi(d\rho)$. Then with the same arguments as used in Section 5 we get the following.

THEOREM 6.1. (i) *Under the assumptions of the respective part* (i), *or* (ii) *of Theorem* 2.4, *we have* $N^{-1/\alpha} S_N(\cdot) \overset{\text{f.d.}}{\Rightarrow} Z(\cdot)$, *as* $N \to +\infty$ *where* $\{Z(t), t \geq 0\}$ *is an* $\alpha$-*stable process of type either type* I, *or* II *with the parameters of the corresponding Lévy measure* [*cf.* (2.2)] *given by*

$$(6.3) \qquad C_*(\lambda) := \begin{cases} \alpha A_\alpha c_*^-, & \text{when } \lambda < 0, \\ \alpha A_\alpha c_*^+, & \text{when } \lambda > 0. \end{cases}$$

*Here* $\overset{\text{f.d.}}{\Rightarrow}$ *denotes the convergence in the sense of finite-dimensional distributions.*

(ii) *In addition, under the assumptions of part* (iii) *of Theorem* 2.4 *finite-dimensional distributions of* $N^{-1} S_N(t) - C_N t$ *converge weakly to those of* $\{Z(t), t \geq 0\}$, *a stable process of type* III. *Here* $C_N$ *is given by* (6.2).

REMARK. The results of the first part of the above theorem follow under the conditions of Theorem 2.7, by using the coupling argument of Section 4.

Let us consider now the process $Y_N(t)$ defined by (2.21). We only show that one-dimensional distributions of $Y_N(t)$ converge weakly to the respective distribution of a suitable stable process $\{Z(t), t \geq 0\}$. The proof of convergence of finite-dimensional distributions can be done in the same way.

Given $t > 0$ define $n(t)$ as the positive integer, such that

$$t_{n(t)} \leq t < t_{n(t)+1},$$

where $t_N$ is given by (2.18). Let

$$s(t) := t/\bar{t},$$

$$B_N(t) := N^{-1/\alpha} \sum_{k=0}^{[Nt]} \Psi(X_k)\tau_k, \qquad t \geq 0,$$

where, as we recall, $\Psi(x) := V(x)t(x)$, $x \in E$ and $\{\tau_k, k \geq 0\}$ is a sequence of i.i.d. variables distributed according to an exponential distribution with parameter 1. Using the ergodic theorem one can easily conclude that

$$(6.4) \qquad s_N(t) := \frac{n(Nt)}{N} \to s(t) \qquad \text{as } N \to +\infty,$$

a.s. uniformly on intervals of the form $[t_0, T]$ where $0 < t_0 < T$. We have

$$Y_N(t) = \frac{1}{N^{1/\alpha}} \sum_{k=0}^{n(Nt)-1} \Psi(X_k)\tau_k + \frac{Nt - t_{n(Nt)}}{N^{1/\alpha}} V(X_k).$$

Note that

$$\frac{1}{N^{1/\alpha}} \sum_{k=0}^{n(Nt)} \Psi(X_k)\tau_k = B_N(s_N(t)).$$

LEMMA 6.2. *For any $t > 0$ and $\varepsilon > 0$ fixed, we have*

(6.5)
$$\lim_{N \to +\infty} \mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon] = 0.$$

PROOF. Let $\sigma > 0$ be arbitrary. We can write that

$$\mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon]$$

(6.6)
$$\leq \mathbb{P}[|s_N(t) - s(t)| > \sigma]$$
$$+ \mathbb{P}[|s_N(t) - s(t)| \leq \sigma, |Y_N(t) - B_N(s_N(t))| > \varepsilon].$$

The second term on the right-hand side can be estimated from above by

$$\mathbb{P}\big[|s_N(t) - s(t)| \leq \sigma, N^{-1/\alpha}|\Psi(X_{n(Nt)})|\tau_{n(Nt)} > \varepsilon\big]$$
$$\leq \mathbb{P}\big[\sup\{|\Psi(X_k)|\tau_k : k \in \big[(s(t) - \sigma)N, (s(t) + \sigma)N\big]\} > N^{1/\alpha}\varepsilon\big].$$

Using the stationarity of $\{|\Psi(X_k)|\tau_k, k \geq 0\}$ the term on the right-hand side equals

$$\mathbb{P}\big[\sup\{|\Psi(X_k)|\tau_k : k \in [0, 2\sigma N]\} > N^{1/\alpha}\varepsilon\big]$$
$$\leq 2\sigma N \int_0^{+\infty} e^{-\tau} \pi[|\Psi(x)| \geq \tau^{-1}N^{1/\alpha}\varepsilon] \, d\tau \leq \frac{C\sigma}{\varepsilon^\alpha}$$

for some constant $C > 0$, by virtue of (2.5). From (6.6) we obtain, therefore,

$$\limsup_{N \to +\infty} \mathbb{P}[|Y_N(t) - B_N(s_N(t))| > \varepsilon] \leq \frac{C\sigma}{\varepsilon^\alpha}$$

for an arbitrary $\sigma > 0$, which in turn implies (6.5). $\quad\square$

It suffices, therefore, to prove that the laws of $B_N(s_N(t))$ converge, as $N \to +\infty$, to the law of the respective stable process. According to Skorochod's embedding theorem, one can find pairs of random elements $(\tilde{B}_N(\cdot), \tilde{s}_N(t))$, $N \geq 1$, with values in $D[0, +\infty) \times [0, +\infty)$, such that the law of each pair is identical with that of $(B_N(\cdot), s_N(t))$, and $(\tilde{B}_N(\cdot), \tilde{s}_N(t))$ converges a.s., as $N \to +\infty$, in the Skorochod topology to $(Z(\cdot), s(t))$. Here $\{Z(t), t \geq 0\}$ is the stable process, as in Theorem 6.1. According to Proposition 3.5.3 page 119 of [11], the above means that for each $T > 0$ there exists a sequence of increasing homeomorphisms $\lambda_N : [0, T] \to [0, T]$ such that

(6.7)
$$\lim_{N \to +\infty} \gamma(\lambda_N) = 0,$$

where

$$\gamma(\lambda_N) := \sup_{0 < s < t < T} \left|\log \frac{\lambda_N(t) - \lambda_N(s)}{t - s}\right| = 0$$

and

(6.8)
$$\sup_{t \in [0,T]} |\tilde{B}_N \circ \lambda_N(t) - Z(t)| = 0.$$

As a consequence of (6.7) we have of course that

$$(6.9) \qquad \lim_{N \to +\infty} \sup_{t \in [0,T]} |\lambda_N(t) - t| = 0.$$

Note that the law of each $B_N(s_N(t))$ is identical with that of $\tilde{B}_N(\tilde{s}_N(t))$. We also have

$$|\tilde{B}_N(\tilde{s}_N(t)) - Z(s(t))| \le |\tilde{B}_N(\tilde{s}_N(t)) - Z \circ \lambda_N^{-1}(\tilde{s}_N(t))|$$
$$+ |Z \circ \lambda_N^{-1}(\tilde{s}_N(t)) - Z(s(t))|.$$

The right-hand side, however, vanishes a.s., as $N \to +\infty$, thanks to (6.8), (6.9) and the fact that for each fixed $s > 0$ one has $\mathbb{P}[Z(s-) = Z(s)] = 1$ (see, e.g., Theorem 11.1, page 59 of [26]). The above allows us to conclude that $|\tilde{B}_N(\tilde{s}_N(t)) - Z(s(t))| \to 0$ a.s., as $N \to +\infty$, thus the assertions of Theorem 2.8 follow.

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