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MALYSHEV'S THEORY AND JS-QUEUES. ASYMPTOTICS OF STATIONARY PROBABILITIES[1]

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Malyshev's theory of asymptotics of stationary probabilities for a random walk in a quarter-plane is extended to cover the case of join-the-shorter-queues.

1. Preliminaries.

1.1. *An informal introduction.* In the early 1970s Malyshev [13–15] proposed a pioneering approach to asymptotical studies of stationary probabilities of a random walk (RW) on the lattice quarter-plane $\mathbf{Z}_+^2 = \{(m,n): m,n = 0,1,\dots\}$. Traditionally, main applications of this approach are associated with queueing theory (see [5] and references therein, in particular, [13–15, 2, 8, 10]), although it is not the only domain where it is useful (see [12]). An essential condition used in [5, 13–15] (at least to obtain treatable formulas) is that the jump probabilities are invariant (or homogeneous) under space-shifts, separately, for the interior $\widehat{\mathbf{Z}}_+^2 = \{(m,n): m,n = 1,2,\dots\}$ of $\mathbf{Z}_+^2$ and the "positive" parts of its "boundary," $\mathbf{Z}_+^{(1)} = \{(m,0): m = 1,2,\dots\}$ and $\mathbf{Z}_+^{(2)} = \{(0,n): n = 1,2,\dots\}$. We call this a universal homogeneity condition (UHC). Unfortunately, the UHC is restrictive from the point of view of queueing theory. For example, the simplest model of the so-called join-the-shorter-queue (briefly, JS-queue) does not satisfy the UHC.

A JS-queue, with two exponential servers, is as follows. Tasks (or customers) arrive within three independent Poisson processes Ξ_1, Ξ_2 and Ξ'. Processes Ξ_1 and Ξ_2 are of rate λ and Ξ' of rate λ'. Tasks from Ξ_1 go to server 1, tasks from Ξ_2 to server 2 and tasks from Ξ' choose the shortest queue, breaking ties at random. In other words, Ξ_1 and Ξ_2 generate dedicated and Ξ' opportunistic traffic. The service rate at each server equals 1, and tasks are served according to a conservative discipline (say, FCFS), without interruption. Such a system is described by a continuous-time Markov process on $\mathbf{Z}_+^2$; in a state (m,n) there are m tasks in the queue at server 1 and n at server 2. It is easy to check that the corresponding continuous-time Markov process (or the embedded jump RW) is positive recurrent iff

$$\lambda + \lambda'/2 < 1 \tag{1.1}$$

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(see, e.g., [9]). The jump probabilities in the JS-queue model do not possess the above invariance property. Instead, they satisfy a more tricky homogeneity condition involving reflection about the diagonal $\mathbf{Z}_+^{\mathrm{d}} = \{(m, m) : m \geq 1\}$. Nevertheless, many ideas and technical tools used under the UHC can be used in this situation as well.

The problem we address in this paper is on asymptotics of the stationary probabilities for a JS-queue. When $\lambda' = 0$, we have a pair of isolated exponential servers: This is the only case where the stationary probabilities can be precisely calculated. The opposite case, where $\lambda = 0$ was studied in [8] by using a special modification of methods from [13–15]. In this paper we consider the case where $\lambda, \lambda' > 0$. In fact, we introduce a class of nearest-neighbor RWs on $\mathbf{Z}_+^2$ whose jump probabilities satisfy (1.2) called asymmetric homogeneity condition (SHC); this class includes the above JS-queue models.

Essentially the SHC means that we consider a RW in an *eighth-plane*. It is convenient to organize the jump probabilities specifying the RW in three stochastic vectors: a four-dimensional vector **p** (for jumps from the interior of the eighth-plane), a three-dimensional vector **b** (for jumps from the positive half of the x-axis) and a four-dimensional vector **d** (for jumps from the bisecting diagonal). See Figure 1.

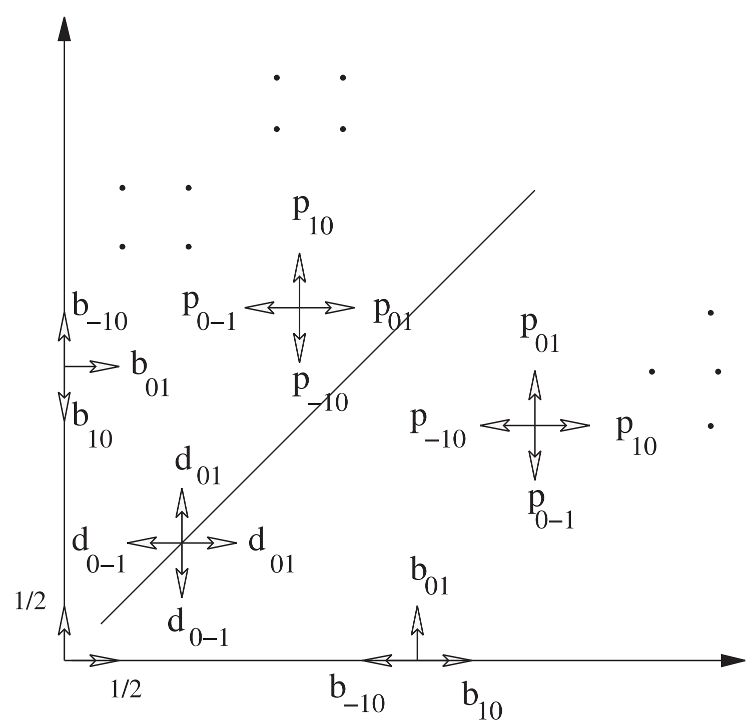


FIG. 1.

The first novel fact about SHC random walks established in this paper is a criterion (1.3) for its positive recurrence. See Theorem 1.1. It is written in terms of two quadratic and two linear expressions in components of the vectors **p**, **b** and **d**, and has a form adapted for subsequent analysis. For the subclass of the JS-queues it coincides with previously established bounds; see [11, 16, 17]. Then, under bounds (1.3), we analyze geometric asymptotics of the *stationary probabilities* $\rho_{m+n,m}$ of the RW as $m, n \to \infty$, $(m+n)/m \sim \operatorname{ctg}\gamma$. Here $\gamma \in [0, \pi/4]$ is a direction in the eighth-plane. (Similar asymptotics hold for $\rho_{n,m+n}$.) Throughout the paper, the symbol $\sim$ means that the ratio of two expressions tends to 1. As in [13–15], the asymptotical analysis $\rho_{m+n,m}$ is based on complex integration on a Riemannian surface (which is a two-dimensional torus). Our analysis is restricted to a "generic" situation where the components of vectors **p**, **b** and **d** [obeying bounds (1.3)] do *not* satisfy certain (nonlinear) equations. These equations [there are three of them; see (4.3)–(4.5)] describe various "degeneracies" that may occur in the asymptotical behavior of the complex integrals under consideration. The degenerate cases are not discussed in this paper, although it is clear that they may be formally analyzed by using similar methods. From the practical point of view, these cases are rather exotic; we prove that two of them, related to (4.4) and (4.5), do not occur for a large class of SHC RWs including the JS-queues. We also believe that the first of these cases related to (4.3) does not occur either; an argument supporting this conjecture, although falling short of a complete proof, is given in Section 4 (in fact, if completed, this argument would exclude all three cases).

Assuming (1.3) and the above nondegeneracy condition, we are able to identify a "principal" geometric term $q^m r^n$ defining the log-asymptotics of $\rho_{m+n,m}$. It is the second novel fact on SHC RWs established in this paper. Here, the values $q, r \in [0, 1]$ are determined by the direction γ. More precisely, for given **p**, **b** and **d** satisfying (1.3) and (4.3)–(4.5), we define explicitly two angles, $\gamma_0, \widetilde{\gamma}_0 \in [0, \pi/4]$, dividing the segment $[0, \pi/4]$ into at most three smaller segments where q and r have particular properties. Specifically, (i) q, r are the same for all $\gamma \in [0, \min\{\gamma_0, \widetilde{\gamma}_0\}]$, (ii) q, r are the same for all $\gamma \in [\max\{\gamma_0, \widetilde{\gamma}_0\}, \pi/4]$, (iiia) if $\widetilde{\gamma}_0 < \gamma_0$, then for all $\gamma \in [\widetilde{\gamma}_0, \gamma_0]$ the pair q, r is one of two pairs q, r found in cases (i) and (ii) with the largest value of $qr^{\operatorname{ctg}\gamma - 1}$. [There exists an angle $\widehat{\gamma}_0 \in (\widetilde{\gamma}_0, \gamma_0)$ such that for all $\gamma \in [\widetilde{\gamma}_0, \widehat{\gamma}_0]$, this pair is the one of case (i) and for all $\gamma \in [\widehat{\gamma}_0, \gamma_0]$ this pair is the one of case (ii).] (iiib) If $\gamma_0 < \widetilde{\gamma}_0$ then for $\gamma \in (\gamma_0, \widetilde{\gamma}_0)$ q, r vary with γ. Globally, q and r are continuous functions of γ, smooth (and even analytic) for $\gamma \neq \gamma_0, \widetilde{\gamma}_0$, but their derivatives may jump at $\gamma = \gamma_0$ and $\widetilde{\gamma}_0$.

Furthermore, it is possible to find the factor in front of the leading term $q^m r^n$; this factor is either a constant C [cases (i) and (ii) and (iiia) in the previous paragraph], or has the form $C/\sqrt{n}$ or $C/(n\sqrt{n})$ [case (iiib)]. The constant C (which may vary with $\gamma \in [0, \pi/4]$ in the same manner as above) can be calculated through values of generating functions of stationary probabilities ρ_{m0} and ρ_{0n} (more precisely their meromorphic continuations). At the moment, the following methods have been proposed for how to specify these generating functions: (a) with the help of Abelian differentials on the *uniformizing* complex plane [13, 14]; (b) by a direct analysis on the *original* complex plane ([3] and [5], Chapter 5). Both methods use a complex boundary-value problem (of a Riemann–Hilbert or Riemann–Carleman type). In this paper, specification of constant C follows the second approach and is based on a remark [4] produced during independent exchanges. In our view, specification of constant C is the third novel fact on SHC RWs; it should be attributed to [4].

In the case of JS-queues, the general description can be made much more precise. This is an important outcome of the present paper as it demonstrates applicability of the general theory in its present state. As a result, most of the facts obtained in [8] for the case $\lambda = 0$ are extended to the case where $\lambda > 0$.

We must stress that this paper emerged as a result of an attempt to understand, clarify and unite facts scattered across the existent literature and folklore, not always accessible to the public. From this point of view, SHC random walks and JS-queues are a good example of where one can check the validity of general constructions and their relevance to practical calculations.

The paper is organized as follows: In the rest of Section 1 we introduce a SHC RW and state necessary and sufficient conditions for its positive recurrence in Theorem 1.1. Sections 2–5 deal with further aspects of the theory of general SHC RWs. In particular, in Sections 2 and 3 we discuss, respectively, geometric and analytic aspects of the theory. Formulas for geometric asymptotics of stationary probabilities for a general SHC random walk are given in Section 4 (see Theorems 4.1–4.4, Corollary 4.1). Then Section 5 gives the Fayolle–Iasnogorodski argument for calculating constants in front of the geometric terms. In our opinion, the formulas from Sections 4 and 5 are rather involved, but still manageable.

The reader interested in the application to the JS-queues can concentrate on Section 6 where all related results are stated in a self-contained fashion (see Theorems 6.1 and 6.2 and Corollary 6.1). We also discuss the border cases where $\lambda = 0$ or $\lambda' = 0$.

The proof of Theorem 1.1 is given in Section 7.

Throughout the paper we repeatedly use ideas and technical tools from papers [13–15] and the book in [5].

1.2. *Random walk $\mathcal{L}$: an eighth-plane symmetry.* We study a discrete-time nearest-neighbor RW $\mathcal{L}$ in $\mathbf{Z}_+^2$, whose jump probabilities $\mathbb{P}(\mathcal{L}(s+1) = (m', n') | \mathcal{L}(s) = (m, n))$ are homogeneous in time variable $s = 0, 1, \dots$ and possess some homogeneity and symmetry properties in space variables (m, n), $(m', n') \in \mathbf{Z}_+^2$ (homogeneity relative to space shifts and symmetry relative to the reflection about the diagonal). More precisely, $\forall s, s', m, m', n, n' = 0, 1, \dots,$

$$
\begin{aligned}
&\mathbb{P}\big(\mathcal{L}(s'+1) = (m \pm 1, n) | \mathcal{L}(s') = (m, n)\big) \\
&\quad = \mathbb{P}\big(\mathcal{L}(s+1) = (m', n' \pm 1) | \mathcal{L}(s) = (m', n')\big) \\
&\qquad \text{if } m > n \geq 1 \text{ and } n' > m' \geq 1, \text{ or } 1 \leq m < n \text{ and } 1 \leq n' < m', \\
&\mathbb{P}\big(\mathcal{L}(s'+1) = (m \pm 1, 0) | \mathcal{L}(s') = (m, 0)\big) \\
&\quad = \mathbb{P}\big(\mathcal{L}(s+1) = (0, n \pm 1) | \mathcal{L}(s) = (0, n)\big) \qquad \text{if } \min[m, n] \geq 1, \\
&\mathbb{P}\big(\mathcal{L}(s'+1) = (m, 1) | \mathcal{L}(s') = (m, 0)\big) \\
&\quad = \mathbb{P}\big(\mathcal{L}(s+1) = (1, n) | \mathcal{L}(s) = (0, n)\big) \qquad \text{if } \min[m, n] \geq 1, \\
&\mathbb{P}\big(\mathcal{L}(s'+1) = (m \pm 1, m) | \mathcal{L}(s') = (m, m)\big) \\
&\quad = \mathbb{P}\big(\mathcal{L}(s+1) = (n, n \pm 1) | \mathcal{L}(s) = (n, n)\big) \qquad \text{if } \min[m, n] \geq 1, \\
&\mathbb{P}\big(\mathcal{L}(s'+1) = (1, 0) | \mathcal{L}(s') = (0, 0)\big) \\
&\quad = \mathbb{P}\big(\mathcal{L}(s+1) = (0, 1) | \mathcal{L}(s) = (0, 0)\big) = 1/2.
\end{aligned} \tag{1.2}
$$

In other words, under condition (1.2), the jump probabilities are described by a six-dimensional array formed by four stochastic vectors, $\mathbf{p} = (p_{10}, p_{01}, p_{-10}, p_{0-1})$ (probabilities of jumps from the interior of the lower $\pi/4$-angle), $\mathbf{b} = (b_{10}, b_{01}, b_{-10})$ (probabilities of jumps from the positive horizontal half-line), $\mathbf{d} = (d_{01}, d_{01}, d_{0-1}, d_{0-1})$ (probabilities of jumps from the positive diagonal half-line) and $(1/2, 1/2)$ (probabilities of jumps from the origin). See Figure 1 (the components of the above vectors are listed anticlockwise). Condition (1.2) represents the SHC. Throughout the paper we assume that stochastic vector $\mathbf{p}$ has all components greater than 0.

The ergodicity condition for RW $\mathcal{L}$ is stated in terms of vector $\mathbf{d}$ and the drift vectors $E = (E_1, E_2)$ and $E^b = (E_1^b, E_2^b)$, where

$$E_1 = p_{10} - p_{-10}, \quad E_2 = p_{01} - p_{0-1}, \quad E_1^b = b_{10} - b_{-10}, \quad E_2^b = b_{01}.$$

THEOREM 1.1. *Assume that* $(E_1, E_2) \neq (0, 0)$. *Then the RW* $\mathcal{L}$ *is positive recurrent iff one of the following pairs of inequalities holds true*:

$$\begin{cases} E_2 < 0, \\ E_1 E_2^b - E_2 E_1^b < 0, \end{cases} \quad \textit{or} \quad \begin{cases} E_2 \geq 0, \\ d_{01} E_2 + d_{0-1} E_1 < 0. \end{cases} \tag{1.3}$$

The proof of this theorem is given in Section 6. In the above example of the JS-queue we have $p_{10} = \lambda/(2\lambda + \lambda' + 2)$, $p_{01} = (\lambda + \lambda')/(2\lambda + \lambda' + 2)$, $p_{-10} = p_{0-1} = 1/(2\lambda + \lambda' + 2)$, $b_{10} = \lambda/(2\lambda + \lambda' + 1)$, $b_{01} = (\lambda + \lambda')/(2\lambda + \lambda' + 1)$, $b_{-10} = 1/(2\lambda + \lambda' + 1)$, $d_{01} = (\lambda + \lambda'/2)/(2\lambda + \lambda' + 2)$, $d_{0-1} = 1/(2\lambda + \lambda' + 2)$ and $p_{10}^0 = p_{01}^0 = 1/2$, with $E_1 = (\lambda - 1)/(2\lambda + \lambda' + 2)$, $E_2 = (\lambda + \lambda' - 1)/(2\lambda + \lambda' + 2)$. Condition (1.3) becomes (1.1), and the positivity of $\mathbf{p}$ means that $\lambda > 0$.

Under condition (1.3), RW $\mathcal{L}$ is positive recurrent and possesses a unique stationary probability distribution $\rho_{m,n}$, $m, n = 0, 1, \dots$. The stationary probabilities satisfy the symmetry condition $\rho_{m,n} = \rho_{n,m}$. The moment-generating functions Π, β and δ defined by

$$\Pi(x,t) = \sum_{m,n\geq 1} \rho_{m+n,m} x^{n-1} t^{m-1},$$
$$\beta(x) = \sum_{n\geq 1} \rho_{n,0} x^{n-1}, \qquad \delta(t) = \sum_{m\geq 1} \rho_{m,m} t^{m-1}, \tag{1.4}$$

as functions of complex variables, are analytic at least in open discs $|x|, |t| < 1$. Technically, we use that $\mathbf{p} > \mathbf{0}$ to ensure that the Riemannian surface $\mathbf{T}$ (see Section 2.1) is a torus. Condition (1.3) is used in the proof of Lemma 3.5 (see Section 3.3), in the same fashion as in [12] and [15].

2. Geometric aspects of the theory.

2.1. *The Riemannian surface of a random walk.* In this section we construct the Riemannian surface of RW $\mathcal{L}$ and list useful facts about it.

A. *The functional equation.* Consider the functional equation

$$Q(x,t)\Pi(x,t) = B(x,t)\beta(x) + D(x,t)\delta(t) + A(x)\rho_{00}, \qquad |x|, |t| \leq 1. \tag{2.1}$$

Here, functions Π, β and δ are treated as unknowns. The functions Q, B, D, A are related to the generating functions of jump probabilities of $\mathcal{L}$ in the corresponding parts of $\mathbf{Z}_+^2$,

$$Q(x,t) = xt - p_{01}t^2 - p_{10}x^2 t - p_{-10}t - p_{0-1}x^2,$$
$$A(x) = (x-1)/2,$$
$$B(x,t) = b_{01}t + b_{10}x^2 + b_{-10} - x, \tag{2.2}$$
$$D(x,t) = (2d_{01}xt + 2d_{-10}x - t)/2.$$

Equation (2.1) follows directly from the stationarity equations for $\rho_{m+n,m}$.

Note that functions Q, B, D, A defined by (2.2) give conformal homeomorphisms of the complex Riemannian sphere $\mathbf{S}$. Our aim is to construct meromorphic continuations of functions Π, β and δ from the unit discs to the Riemannian surface defined by the equation $Q(x,t) = 0$ (for that reason Q will be called a kernel of the RW). Our approach is based on a series of observations.

B. *Random walk* $\bar{\mathcal{L}}$. A RW $\bar{\mathcal{L}}$ satisfying UHC emerges when we set, $\forall s \geq 0$ and $m, n \geq 1$ [i.e., $(m,n) \in \widehat{\mathbf{Z}}_+^2$],

$$\bar{p}_{\pm 10} := \mathbb{P}(\bar{\mathcal{L}}(s+1) = (m \pm 1, n) | \bar{\mathcal{L}}(s) = (m,n)) = p_{\pm 10},$$
$$\bar{p}_{-11} := \mathbb{P}(\bar{\mathcal{L}}(s+1) = (m-1, n+1) | \bar{\mathcal{L}}(s) = (m,n)) = p_{01},$$
$$\bar{p}_{1-1} := \mathbb{P}(\bar{\mathcal{L}}(s+1) = (m+1, n-1) | \bar{\mathcal{L}}(s) = (m,n)) = p_{0-1}$$

and all other probabilities $\bar{p}_{\cdot\cdot}$ of jumps from (m,n) to be 0. [The probabilities of jumps from the boundary $\mathbf{Z}_+^{(1)} \cup \mathbf{Z}_+^{(2)} \cup \{(0,0)\}$ in $\bar{\mathcal{L}}$ do not play any role in this observation; e.g., they may be same as in $\mathcal{L}$.] The drift vectors $\bar{E} = (\bar{E}_1, \bar{E}_2)$ and $E = (E_1, E_2)$ for $\bar{\mathcal{L}}$ and $\mathcal{L}$ in $\widehat{\mathbf{Z}}_+^2$ obey $\bar{E}_2 = E_2$, $\bar{E}_1 = E_1 - E_2$. The point is that function Q from (2.2) coincides with the kernel of the RW $\bar{\mathcal{L}}$: $Q(x,t) = xt(\sum_{i,j=\pm 1} \bar{p}_{ij} x^i t^j - 1)$. This fact allows us to quote a number of properties inherited from the case where a UHC is met.

C. *Solutions to the equation* $Q(x,t) = 0$. When x (resp. t) is fixed, $Q(x,t)$ is a polynomial of the second degree in t (resp. x),

$$Q(x,t) \equiv U(x)t^2 + V(x)t + W(x) \equiv \widetilde{U}(t)x^2 + \widetilde{V}(t)x + \widetilde{W}(t), \tag{2.3a}$$

where

$$\begin{aligned} U(x) &:= -p_{01}, & \widetilde{U}(t) &:= -p_{10}t - p_{0-1}, \\ V(x) &:= x - p_{10}x^2 - p_{-10}, & \widetilde{V}(t) &:= t, \\ W(x) &:= -p_{0-1}x^2, & \widetilde{W}(t) &:= -p_{01}t^2 - p_{-10}t. \end{aligned} \tag{2.3b}$$

Thus, $\forall$ fixed x (resp. t), the quadratic equation $Q(x,t) = 0$ has solutions $T_1(x)$, $T_2(x)$ [resp. $X_1(t)$, $X_2(t)$],

$$\begin{aligned} T_{1,2}(x) &= \frac{x - p_{10}x^2 - p_{-10} \pm \sqrt{(x - p_{10}x^2 - p_{-10})^2 - 4p_{01}p_{0-1}x^2}}{2p_{01}}, \\ X_{1,2}(t) &= \frac{t \pm \sqrt{t^2 - 4(p_{0-1} + p_{10}t)(p_{01}t^2 + p_{-10}t)}}{2(p_{0-1} + p_{10}t)}. \end{aligned} \tag{2.4}$$

Functions T_1, T_2 (resp. X_1, X_2) are branches of an algebraic function T (resp. X) defined by the equation $Q(x,t) \equiv 0$. T (resp. X) has four branching points, x_i (resp. t_i), $i = 1,2,3,4$, where $T_1(x) = T_2(x)$ [resp. $X_1(t) = X_2(t)$]. They can be found explicitly by equating the square roots in (2.4) with zero:

$$\begin{aligned} x_1 &= \Big(1 + 2\sqrt{p_{01}p_{0-1}} - \sqrt{(1 + 2\sqrt{p_{01}p_{0-1}})^2 - 4p_{10}p_{-10}}\Big)/(2p_{10}), \\ x_2 &= \Big(1 - 2\sqrt{p_{01}p_{0-1}} - \sqrt{(1 - 2\sqrt{p_{01}p_{0-1}})^2 - 4p_{10}p_{-10}}\Big)/(2p_{10}), \\ x_3 &= \Big(1 - 2\sqrt{p_{01}p_{0-1}} + \sqrt{(1 - 2\sqrt{p_{01}p_{0-1}})^2 - 4p_{10}p_{-10}}\Big)/(2p_{10}), \\ x_4 &= \Big(1 + 2\sqrt{p_{01}p_{0-1}} + \sqrt{(1 + 2\sqrt{p_{01}p_{0-1}})^2 - 4p_{10}p_{-10}}\Big)/(2p_{10}); \\ t_1 &= 0, \\ t_{2,3} &= \Big(1/4 - p_{0-1}p_{01} - p_{10}p_{-10} \\ &\qquad \pm \sqrt{(1/4 - p_{0-1}p_{01} - p_{10}p_{-10})^2 - 4p_{01}p_{10}p_{0-1}p_{-10}}\Big)/(2p_{10}p_{01}), \\ t_4 &= \infty. \end{aligned} \tag{2.5}$$

Under the condition that $\mathbf{p} > \mathbf{0}$, if $E_2 \neq 0$ then $0 < x_1 < x_2 < 1 < x_3 < x_4$, and if $E_2 = 0$ and $E_1 < 0$ then $0 < x_1 < x_2 = 1 < x_3 < x_4$. If $E_1 \neq E_2$, then $t_1 < t_2 < 1 < t_3 < t_4$ and if $E_1 = E_2$ and $E_2 < 0$ then $t_1 < 1 = t_2 < t_3 < t_4$. For the proof, see [5], pages 23–26, together with observations from part B (and use RW $\bar{\mathcal{L}}$).

D. *Construction of surface* $\mathbf{T}$. To construct the Riemannian surface of T, we take two copies, $\mathbf{S}_x^1$ and $\mathbf{S}_x^2$, of the Riemannian sphere $\mathbf{S}$ and cut each of $\mathbf{S}_x^1$, $\mathbf{S}_x^2$ along the segments $[x_1, x_2]$ and $[x_3, x_4]$. Then we "glue" together these spheres along the borders of these cuts joining the "lower" border of segment $[x_1, x_2]$ (resp. $[x_3, x_4]$) on $\mathbf{S}_x^1$ to the "upper" border of the same segment on $\mathbf{S}_x^2$ and vice versa. The resulting surface $\mathbf{T}$ is homeomorphic to a torus and is projected to $\mathbf{S}$ by a canonical covering map $h_x : \mathbf{T} \to \mathbf{S}$. Conversely, $\mathbf{T}$ may be considered as a disjoint union of the "incised" spheres $\widehat{\mathbf{S}}_x^j = \mathbf{S}_x^j \setminus ([x_1, x_2] \cup [x_3, x_4])$, $j = 1, 2$, (the two branches of $\mathbf{T}$) plus the borders of the cuts (the latter form four "cycles": $\Sigma_x^{1,2}$, $\Sigma_x^{3,4}$, $\Sigma_t^{1,2}$, $\Sigma_t^{3,4}$).

In a standard way, we can "lift" function T to $\mathbf{T}$, by setting $T(s) := T_j(h_x(s))$ if $s \in \widehat{\mathbf{S}}_x^j \subset \mathbf{T}$; on the remaining part of $\mathbf{T}$ the function T assumes the corresponding limiting values. Thus, T is single-valued and continuous on $\mathbf{T}$; furthermore, $Q(h_x(s), T(s)) \equiv 0$, $s \in \mathbf{T}$. We call $\mathbf{T}$ the Riemannian surface of T.

In a similar fashion, one constructs the Riemannian surface of function X, by gluing together two copies $\mathbf{S}_t^1$ and $\mathbf{S}_t^2$ of sphere $\mathbf{S}$ cut along segments $[t_1, t_2]$ and $[t_3, t_4]$. It is again homeomorphic to a torus and contains the "incised" spheres $\widehat{\mathbf{S}}_t^j = \mathbf{S}_t^j \setminus ([t_1, t_2] \cup [t_3, t_4])$, $j = 1, 2$.

Since the Riemannian surfaces for X and T are equivalent, we can work on a *single* Riemannian surface $\mathbf{T}$, but with two different covering maps $h_x, h_t : \mathbf{T} \to \mathbf{S}$. We set $x(s) := h_x(s)$ and $t(s) := h_t(s)$, $s \in \mathbf{T}$, and will often represent a point $s \in \mathbf{T}$ by the pair of its "coordinates" $(x(s), t(s))$. [These coordinates are not independent because of the equation $Q(x(s), t(s)) \equiv 0$, $s \in \mathbf{T}$.] See Figure 2.

E. "*Real*" *points of* $\mathbf{T}$. The set Φ of "real" points of $\mathbf{T}$ is where $x(s)$ and $t(s)$ are both real or equal to infinity. Note that for t real, $X(t)$ is real when $t \leq t_1$ or $t_2 \leq t \leq t_3$ and complex when $0 = t_1 < t < t_2$ or $t_3 < t < t_4 = \infty$. Likewise, for x real, $T(x)$ is real when $x \leq x_1$ or $x_2 \leq x \leq x_3$ or $x \geq x_4$; $t(x)$ is complex when $x_1 < x < x_2$ and $x_3 < x < x_4$. Thus $X(t)$ and $T(x)$ are complex precisely on the segments of the real line where the cuts were made to construct the Riemannian surface. Therefore (see Figure 2), set Φ consists of two nonintersecting closed analytic curves Φ_0 and Φ_1 homotopically equivalent (briefly, homotopic) to a basic cycle on $\mathbf{T}$ (the equivalence class containing Φ_0 and Φ_1 is disjoint from that containing the basic cycle $h_x^{-1}\{x : |x| = 1\}$),

$$\begin{aligned} \Phi_0 &= \{s : x_2 \leq x(s) \leq x_3\} = \{s : t_2 \leq t(s) \leq t_3\}, \\ \Phi_1 &= \{s : x(s) \leq x_1 \text{ or } x(s) \geq x_4\} = \{s : t(s) \leq t_1 = 0\}. \end{aligned} \tag{2.6}$$

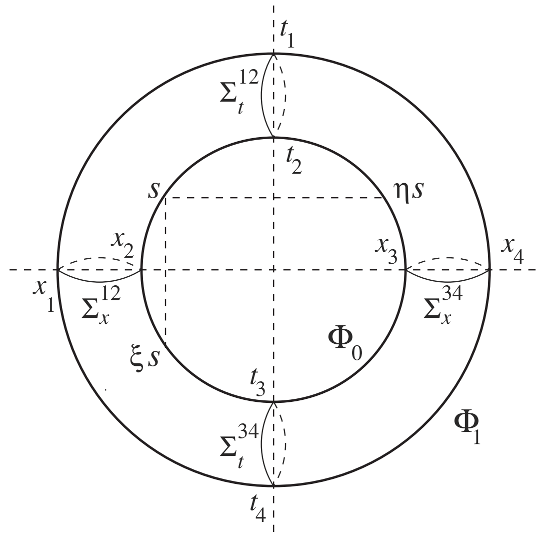


FIG. 2.

Cycle Φ_0 will play a special role in our analysis.

F. *Galois automorphisms.* $\forall s \in \mathbf{T}$ with $x(s) \neq x_1, x_2, x_3, x_4$ there exists a unique point $s' \in \mathbf{T}$ different from s such that $x(s) = x(s')$. Furthermore, if $s \in \widehat{\mathbf{S}}_x^1$ then $s' \in \widehat{\mathbf{S}}_x^2$ and vice versa. When $x(s) = x_1, x_2, x_3$ or x_4, we have $s = s'$. Moreover, as $Q(x(s), t(s)) = 0$, $t(s)$ and $t(s')$ give two values of function T at $x(s) = x(s')$. By Vieta's theorem, $t(s)t(s') = p_{0-1}x^2(s)/p_{01}$.

Similarly, $\forall s \in \mathbf{T}$ with $t(s) \neq t_1, t_2, t_3, t_4$ there exists a unique point $s'' \in \mathbf{T}$ different from s such that $t(s) = t(s'')$, and if $t \in \widehat{\mathbf{S}}_t^1$ then $s'' \in \widehat{\mathbf{S}}_t^2$ and vice versa. Again, if $t(s) = t_1, t_2, t_3$ or t_4, we have $s = s''$. Moreover, as $Q(x(s), t(s)) = 0$, $x(s)$ and $x(s'')$ give two values of function X at $t(s) = t(s'')$. By Vieta's theorem, $x(s)x(s'') = (p_{01}t^2(s) + p_{-10}t(s))/(p_{10}t(s) + p_{0-1})$.

Define mappings $\xi : \mathbf{T} \to \mathbf{T}$ and $\eta : \mathbf{T} \to \mathbf{T}$ by

$$\begin{aligned} \xi s &= s' \qquad \text{iff } x(s) = x(s'), \\ \eta s &= s'' \qquad \text{iff } t(s) = t(s''). \end{aligned} \tag{2.7}$$

Following [13], we call them *Galois automorphisms* of $\mathbf{T}$. Then $\xi^2 = \mathrm{Id}$, $\eta^2 = \mathrm{Id}$ and

$$\begin{aligned} t(\xi s) &= \frac{p_{0-1}x^2(s)}{p_{01}t(s)}, \\ x(\eta s) &= \frac{p_{01}t^2(s) + p_{-10}t(s)}{(p_{10}t(s) + p_{0-1})x(s)}. \end{aligned} \tag{2.8}$$

Any $s \in \mathbf{T}$ with $x(s) = x_1, x_2, x_3$ or x_4 [resp. $t(s) = t_1, t_2, t_3$ or t_4] is a fixed point for ξ (resp. η). Also $\xi(\Phi_0 \cap \widehat{\mathbf{S}}^1_x) = \Phi_0 \cap \widehat{\mathbf{S}}^2_x$ and $\eta(\Phi_0 \cap \widehat{\mathbf{S}}^1_t) = \Phi_0 \cap \widehat{\mathbf{S}}^2_t$. It is helpful to draw the straight line through the pair of points of Φ_0, where $x(s) = x_2$ or x_3 [resp. $t(s) = t_2$ or t_3]. Then, for $s \in \Phi_0$, points s and ξs (resp. ηs) are "symmetric" about this straight line; see Figure 2.

G. *Basic cycles on* $\mathbf{T}$. The domain $\mathbf{G}_x = \{s : |x(s)| < 1\}$ [resp. $\mathbf{G}_t = \{s : |t(s)| < 1\}$] on $\mathbf{T}$ is bordered by two closed curves, Γ_0 and Γ_1 (resp. $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$).

Here,

$$\begin{aligned} \Gamma_0 &:= \{s : |x(s)| = 1\} \cap \{s : |t(s)| \le 1\}, \\ \Gamma_1 &:= \{s : |x(s)| = 1\} \cap \{s : |t(s)| \ge 1\}, \\ \widetilde{\Gamma}_0 &:= \{s : |t(s)| = 1\} \cap \{s : |x(s)| \le 1\}, \\ \widetilde{\Gamma}_1 &:= \{s : |t(s)| = 1\} \cap \{s : |x(s)| \ge 1\}. \end{aligned} \tag{2.9}$$

Owing to the construction of $\mathbf{T}$ and the fact that $x_1, x_2, t_1, t_2 \in [-1, 1]$ and $x_3, x_4, t_3, t_4 \notin [-1, 1]$, cycles Γ_0, Γ_1, $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$, are homotopic to a "basic" cycle $h_x^{-1}\{x : |x| = 1\}$; the corresponding homotopy class is disjoint from the class containing Φ_0 and Φ_1. In addition, $\xi\Gamma_0 = \Gamma_1$ and $\eta\widetilde{\Gamma}_0 = \widetilde{\Gamma}_1$. In the following Proposition 2.1 we relate vector $E = (E_1, E_2)$ to the location of Γ_0, Γ_1, $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$. See Figure 3 [the point of Φ_0 where two or three cycles meet has $x(s) = t(s) = 1$].

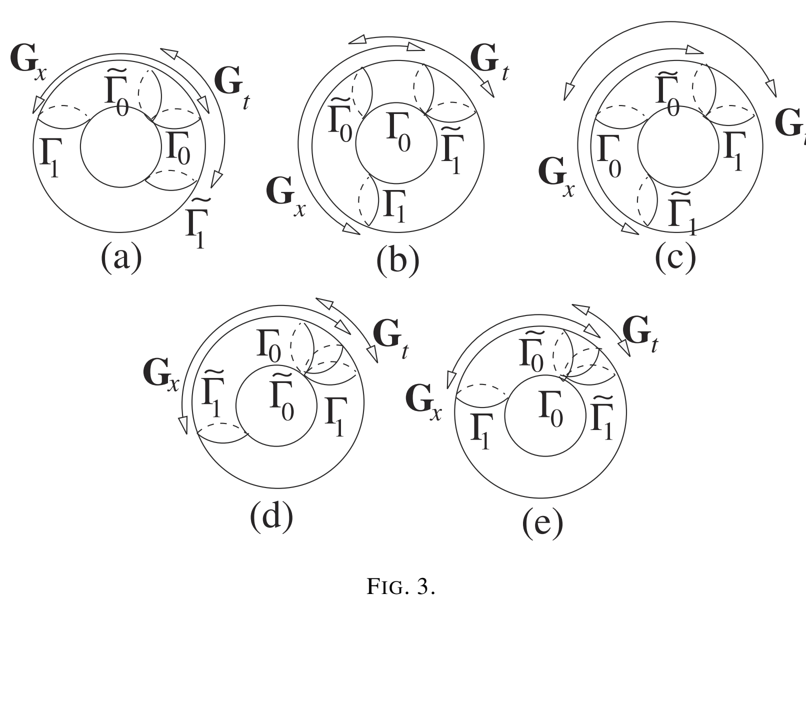


FIG. 3.

PROPOSITION 2.1. *Cycles* $\Gamma_0, \Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ *defined in* (2.9) *are located on* $\mathbf{T}$ *in the following fashion*:

(a) *If* $E_2 < 0$, $E_1 - E_2 < 0$, *then* $\Gamma_0 \cap \widetilde{\Gamma}_0 = (1,1)$, $\Gamma_0 \subset \bar{\mathbf{G}}_y$, $\widetilde{\Gamma}_0 \subset \bar{\mathbf{G}}_x$, $\Gamma_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_t$, $\widetilde{\Gamma}_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_x$; *see Figure* 3(a).

(b) *If* $E_2 < 0$, $E_1 - E_2 > 0$, *then* $\Gamma_0 \cap \widetilde{\Gamma}_1 = (1,1)$, $\Gamma_0 \subset \bar{\mathbf{G}}_y$, $\widetilde{\Gamma}_0 \subset \mathbf{G}_x$, $\Gamma_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_y$, $\widetilde{\Gamma}_1 \subset \mathbf{T} \setminus \mathbf{G}_x$; *see Figure* 3(b).

(c) *If* $E_2 > 0$, $E_1 - E_2 < 0$, *then* $\widetilde{\Gamma}_0 \cap \Gamma_1 = (1,1)$, $\Gamma_0 \subset \mathbf{G}_t$, $\widetilde{\Gamma}_0 \subset \bar{\mathbf{G}}_x$, $\Gamma_1 \subset \mathbf{T} \setminus \mathbf{G}_t$, $\widetilde{\Gamma}_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_x$; *see Figure* 3(c).

(d) *If* $E_2 = 0$, $E_1 - E_2 < 0$, *then* $\widetilde{\Gamma}_0 \cap \Gamma_1 \cap \Gamma_0 = (1,1)$, $\Gamma_0 \subset \bar{\mathbf{G}}_y$, $\widetilde{\Gamma}_0 \subset \bar{\mathbf{G}}_x$, $\Gamma_1 \subset \mathbf{T} \setminus \mathbf{G}_t$, $\widetilde{\Gamma}_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_x$; *see Figure* 3(d).

(e) *If* $E_2 < 0$, $E_1 - E_2 = 0$, *then* $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_1 \cap \Gamma_0 = (1,1)$, $\Gamma_0 \subset \bar{\mathbf{G}}_t$, $\widetilde{\Gamma}_0 \subset \bar{\mathbf{G}}_x$, $\Gamma_1 \subset \mathbf{T} \setminus \bar{\mathbf{G}}_y$, $\widetilde{\Gamma}_1 \subset \mathbf{T} \setminus \mathbf{G}_x$; *see Figure* 3(e).

For the proof, simply combine the arguments from [5], pages 30–32, with the argument from paragraph in this section.

H. *Lifting functions from* $\mathbf{S}$ *on* $\mathbf{T}$. To begin with, we lift functions B, D, A by setting

$$B(s) := B(x(s), t(s)), \quad D(s) := D(x(s), t(s)), \quad A(s) := A(x(s)), \qquad s \in \mathbf{T}.$$

Thus we have β and δ defined in closures $\bar{\mathbf{G}}_x$ and $\bar{\mathbf{G}}_t$,

$$\beta(s) := \beta(x(s)), \qquad s \in \bar{\mathbf{G}}_x, \qquad \delta(s) := \delta(t(s)), \qquad s \in \bar{\mathbf{G}}_t. \tag{2.10}$$

Since $Q(x(s), t(s)) \equiv 0$ on $\mathbf{T}$, (2.1) implies

$$B(s)\beta(s) + D(s)\delta(s) + A(s)\rho_{00} = 0, \qquad s \in \bar{\mathbf{G}}_x \cap \bar{\mathbf{G}}_t. \tag{2.11}$$

As β (resp. δ) is analytic in $\mathbf{G}_x$ (resp. $\mathbf{G}_t$) and bounded and continuous in $\bar{\mathbf{G}}_x$ (resp. $\bar{\mathbf{G}}_t$), (2.11) allows us to extend it meromorphically to $\mathbf{G}_t$ (resp. $\mathbf{G}_x$). Namely,

$$\beta(s) := -\frac{D(s)\delta(s) + A(s)\rho_{00}}{B(s)}, \qquad \delta(s) := -\frac{B(s)\beta(s) + A(s)\rho_{00}}{D(s)}. \tag{2.12}$$

Then (2.11) holds in $\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t \subset \mathbf{T}$. Functions β and δ are meromorphic inside this domain and continuous on its boundary $\Gamma_1 \cup \widetilde{\Gamma}_1$. They may have poles only at zeros of B in $\mathbf{G}_t$ and of D in $\mathbf{G}_x$, respectively.

2.2. *Meromorphic continuation of* β *and* δ. The cut cycles $\Sigma_x^{1,2}$, $\Sigma_x^{3,4}$, $\Sigma_t^{1,2}$, $\Sigma_t^{3,4}$ are homotopic to $\Gamma_0, \widetilde{\Gamma}_0, \Gamma_1, \widetilde{\Gamma}_1$. The pair Σ_x^{34}, Σ_t^{34} partitions $\mathbf{T}$ into two open domains. One of them, which does not contain $\mathbf{G}_x \cup \mathbf{G}_t$, is denoted by $\mathbf{D}$; it contains the interval of Φ_0 where $X(t_3) < x(s) < x_3$ and $T(x_3) < t(s) < t_3$. We also make an agreement that $\mathbf{D}$ lies in the incised spheres $\widehat{\mathbf{S}}_t^2$ and $\widehat{\mathbf{S}}_x^2$ (which is merely a matter of notation). See Figure 4.

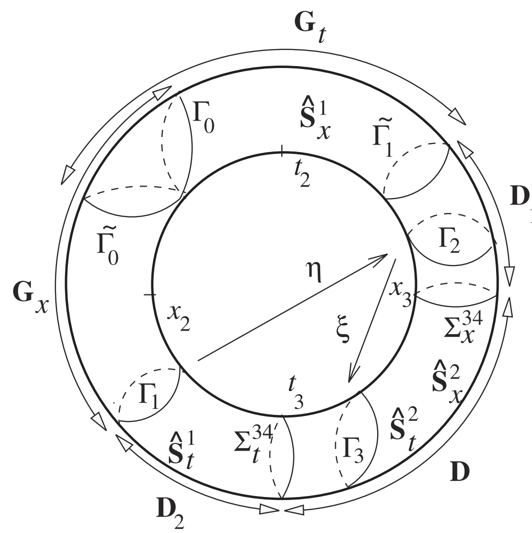


FIG. 4.

THEOREM 2.2. *Functions β and δ can be meromorphically continued on the whole of $\mathbf{T}$, except for cycle Σ_x^{34} for β and Σ_t^{34} for δ. Furthermore,*

$$\beta(s) = \beta(\xi s), \quad \delta(s) = \delta(\eta s), \qquad s \in \mathbf{T} \tag{2.13}$$

and

$$B(s)\beta(s) + D(s)\delta(s) + A(s)\rho_{00} = 0, \qquad s \in \mathbf{S} \setminus \mathbf{D}. \tag{2.14}$$

PROOF. We give the detailed proof in the case $E_2 < 0$ and $E_1 - E_2 < 0$ [see Proposition 2.1a]. Other cases are treated similarly. Denote by $\mathbf{D}_1$ (resp. $\mathbf{D}_2$) the open domain of $\mathbf{T}$ bordered by Σ_x^{34} and $\tilde{\Gamma}_1$ (resp. Σ_t^{34} and Γ_1). Then $\mathbf{T}$ is a disjoint union $(\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t) \cup \mathbf{D}_1 \cup \Sigma_x^{34} \cup \mathbf{D} \cup \Sigma_t^{34} \cup \mathbf{D}_2$; see Figure 4.

In Section 2.1, paragraph H the functions β and δ have been continued meromorphically to $\mathbf{G}_x \cup \mathbf{G}_t$. We shall extend them as meromorphic functions to $\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t \cup \mathbf{D}_1 \cup \mathbf{D}_2$, preserving (2.13) and (2.14). Then their continuation to $\bar{\mathbf{D}}$ is obvious:

$$\beta(s) := \beta(\xi s), \qquad \delta(s) := \delta(\eta s), \qquad s \in \bar{\mathbf{D}}. \tag{2.15}$$

In fact, β (resp. δ) is already meromorphic in $\xi\mathbf{D}$ (resp. $\eta\mathbf{D}$) and in a neighborhood of $\xi\Sigma_t^{34}$ (resp. $\eta\Sigma_x^{34}$). Due to the fact that $\xi\mathbf{D}$, $\xi\Sigma_t^{34}$, $\eta\mathbf{D}$, $\eta\Sigma_x^{34}$ are subsets of $\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t \cup \mathbf{D}_1 \cup \mathbf{D}_2$, equations (2.15) allow us to continue β (resp. δ) meromorphically on $\mathbf{D}$ through Σ_t^{34} (resp. Σ_x^{34}).

The continuation of β and δ on $\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t \cup \mathbf{D}_1 \cup \mathbf{D}_2$ will be performed according to the following recursive procedure.

Step 1. Set $\mathbf{G}^{(0)} := \mathbf{G}_x \setminus \bar{\mathbf{G}}_t$. This domain is bordered by $\widetilde{\Gamma}_0$ and Γ_1. Apply the automorphism η to $\mathbf{G}^{(0)}$. Then $\eta\mathbf{G}^{(0)} \subset \mathbf{S}_t^2$ and $\eta\mathbf{G}^{(0)}$ is bordered by $\eta\widetilde{\Gamma}_0 = \widetilde{\Gamma}_1$ (this border is common with $\mathbf{G}_t$) and a closed curve $\Gamma_2 := \eta\Gamma_1$ homotopic to Γ_1 which crosses cycle Φ_0 at a point s with $t(s) = p_{0-1}/p_{01} > 1$. Observe that $\eta\mathbf{G}^{(0)} \cap \bar{\mathbf{G}}_t = \eta\mathbf{G}^{(0)} \cap \eta\mathbf{G}_t = \eta(\mathbf{G}^{(0)} \cap \mathbf{G}_t) = \eta\varnothing = \varnothing$ and that $\Gamma_2 \cap \bar{\mathbf{G}}_t = \eta(\Gamma_1 \cap \bar{\mathbf{G}}_t) = \varnothing$. Hence $\eta\mathbf{G}^{(0)} \cap \mathbf{D}_1 \neq \varnothing$. Put

$$\delta(s) := \delta(\eta s) \qquad \text{for } s \in \eta\mathbf{G}^{(0)} \cap \mathbf{D}_1;$$
$$\beta(s) := -\frac{D(s)\delta(s) + A(s)\rho_{00}}{B(s)} \qquad \text{for } s \in \eta\mathbf{G}^{(0)} \cap \mathbf{D}_1. \tag{2.16}$$

Since $\eta(\eta\mathbf{G}^{(0)}) = \mathbf{G}^{(0)}$, β and δ are well defined in $\eta\mathbf{G}^{(0)} \cap \mathbf{D}_1$ by (2.16). Furthermore, δ can be meromorphically continued on this domain through $\widetilde{\Gamma}_1$, as it is already meromorphic in $\mathbf{G}^{(0)}$ and in a neighborhood of its border $\eta\widetilde{\Gamma}_1 = \widetilde{\Gamma}_0$. Also, β can be meromorphically continued through $\widetilde{\Gamma}_1$, as (2.14) holds in $\mathbf{G}_x \cup \mathbf{G}_t$. Finally the invariance of these functions w.r.t. the automorphisms (2.13) is preserved.

Set $\mathbf{G}^{(1)} := \eta\mathbf{G}^{(0)} \cup \widetilde{\Gamma}_1 \cup (\mathbf{G}_t \setminus \bar{\mathbf{G}}_x)$. Domain $\mathbf{G}^{(1)}$ is bordered by Γ_0 and $\Gamma_2 = \eta\Gamma_1$.

Step 2. The domain $\mathbf{G}^{(2)} := \xi\mathbf{G}^{(1)}$ is bordered by $\Gamma_1 = \xi\Gamma_0$ (this border is common with $\mathbf{G}_x$) and a cycle $\Gamma_3 := \xi\Gamma_2 = \xi\eta\Gamma_1$ homotopic to Γ_1. (The exact location of Γ_3 depends of course on Γ_2; Γ_3 crosses Γ_x^{34} or lies in $\widehat{\mathbf{S}}_x^1$ or $\widehat{\mathbf{S}}_x^2$.) Note that $\mathbf{G}^{(2)} \cap \mathbf{G}^{(0)} = \varnothing$ and $\Gamma_3 \cap \bar{\mathbf{G}}_x = \varnothing$. Thus $\mathbf{G}^{(2)} \cap \mathbf{D}_2 \neq \varnothing$ and we put

$$\beta(s) := \beta(\xi s) \qquad \text{for } s \in \xi\mathbf{G}^{(1)} \cap \mathbf{D}_2;$$
$$\delta(s) := -\frac{B(s)\beta(s) + A(s)\rho_{00}}{D(s)} \qquad \text{for } s \in \xi\mathbf{G}^{(1)} \cap \mathbf{D}_2. \tag{2.17}$$

Equations (2.17) yield the meromorphic continuation of β and δ on $\xi\mathbf{G}^{(1)} \cap \mathbf{D}_2$ through Γ_1. This follows from the fact that (a) $\xi(\xi\mathbf{G}^{(1)} \cap \mathbf{D}_2) \subset \mathbf{G}^{(1)} \cap \mathbf{S}_x^1 = (\eta\mathbf{G}^{(0)} \cap \mathbf{D}_1) \cup \widetilde{\Gamma}_1 \cup (\mathbf{G}_t \setminus \bar{\mathbf{G}}_x)$, (b) β is meromorphic in the domain $(\eta\mathbf{G}^{(0)} \cap \mathbf{D}_1) \cup \widetilde{\Gamma}_1 \cup (\mathbf{G}_t \setminus \bar{\mathbf{G}}_x)$ and in a neighborhood of its border $\xi\Gamma_1 = \Gamma_0$ and (c) (2.11) is valid in $\mathbf{G}_x \cup \mathbf{G}_t$. Also note that (2.13) and (2.14) continue to hold. See Figure 4.

If after Steps 1 and 2 β and δ are extended to the whole of $\bar{\mathbf{G}}_x \cup \bar{\mathbf{G}}_t \cup \mathbf{D}_1 \cup \mathbf{D}_2$, we stop the procedure. Otherwise, we continue with Step 3.

Step 3. Consider the domain $\mathbf{G}^{(3)} = \eta\mathbf{G}^{(2)}$ bordered by $\Gamma_2 = \eta\Gamma_1$ (this border is common with $\mathbf{G}^{(1)}$) and a cycle $\Gamma_4 := \eta\Gamma_3$ homotopic to Γ_2. We have $\mathbf{G}^{(3)} \cap \mathbf{G}^{(1)} = \varnothing$, because $\eta(\mathbf{G}^{(3)} \cap \mathbf{G}^{(1)}) = \mathbf{G}^{(2)} \cap \mathbf{G}^{(0)} = \varnothing$. As the procedure has not been stopped after Step 2, the domain $\mathbf{D}_1$ has not been covered by $\mathbf{G}^{(1)}$ after Step 1. (Otherwise $\Gamma_2 \subset \mathbf{S}_x^2$, consequently $\xi\Gamma_2 \subset \mathbf{S}_x^1$, domain $\mathbf{D}_2$ would have been covered after Step 2, and the whole procedure would have been stopped.) Thus $\mathbf{G}^{(3)} \cap \mathbf{D}_1 \neq \varnothing$. Taking into account the fact that $\eta(\mathbf{G}^{(3)} \cap \mathbf{D}_1) \subset \mathbf{G}^{(2)} \cap \mathbf{S}_t^1 = \mathbf{G}^{(2)} \cap \mathbf{D}_2$, we continue β and δ to $\mathbf{G}^{(3)} \cap \mathbf{D}_1$ through Γ_2 meromorphically, by (2.16).

Next, we construct the domain $\mathbf{G}^{(4)} = \xi \mathbf{G}^{(3)}$ bordered by $\Gamma_3 = \xi\Gamma_2$ and $\Gamma_5 := \xi\Gamma_4$, and define β and δ on $\mathbf{G}^{(4)} \cap \mathbf{D}_2$ by (2.17); see the diagram below:

$$\begin{matrix} \mathbf{G}^{(0)} & & \mathbf{G}^{(1)} & & & \mathbf{G}^{(2n-2)} & & \mathbf{G}^{(2n-1)} & & \mathbf{G}^{(2n)} & & \mathbf{G}^{(2n+1)} \\ \Gamma_1 & \to & \Gamma_2 & \to \cdots \to & & \Gamma_{2n-1} & \to & \Gamma_{2n} & \to & \Gamma_{2n+1} & \to & \Gamma_{2n+2} \end{matrix}$$

After $2n$ steps we successively construct $2n$ domains $\mathbf{G}^{(1)}, \dots, \mathbf{G}^{(2n)}$, where $\mathbf{G}^{(2n)} = \xi \mathbf{G}^{(2n-1)} = \xi(\eta \mathbf{G}^{(2n-2)}) = \cdots = \xi(\eta\xi)^{n-1}\mathbf{G}^{(1)}$. Each of domains $\mathbf{G}^{(2k)}$, $1 \le k \le n$, (resp. $\mathbf{G}^{(2k+1)}, 0 \le k < n$) is bordered by cycles $\Gamma_{2k-1} = (\xi\eta)^{k-1}\Gamma_1$ and $\Gamma_{2k+1} = (\xi\eta)^k\Gamma_1$ [resp. $\Gamma_{2k} = \eta(\xi\eta)^{k-1}\Gamma_1$ and $\Gamma_{2k+2} = \eta(\xi\eta)^k\Gamma_1$] homotopic to Γ_1. Domains $\mathbf{G}^{(0)}, \mathbf{G}^{(2)}, \dots, \mathbf{G}^{(2n)}$ (resp. $\mathbf{G}^{(1)}, \mathbf{G}^{(3)}, \dots, \mathbf{G}^{(2n-1)}$) are pairwise disjoint, but share borders; cycle Γ_l is the common border of $\mathbf{G}^{(l-1)}$ and $\mathbf{G}^{(l+1)}$. Functions β and δ are meromorphically continued by recursion to $\bigcup_{l=0}^{2n} \mathbf{G}^{(l)} \cap (\bar{\mathbf{G}}_t \cup \bar{\mathbf{G}}_x \cup \mathbf{D}_1 \cup \mathbf{D}_2)$ and (2.13) and (2.14) hold. Suppose that after $2n$ steps the domains $\mathbf{D}_1$ and $\mathbf{D}_2$ are not yet completely covered.

Step $2n+1$. Define the domain $\mathbf{G}^{(2n+1)} = \eta \mathbf{G}^{(2n)} = (\eta\xi)^n \mathbf{G}^{(1)}$ bordered by $\Gamma_{2n} = \eta(\xi\eta)^{n-1}\Gamma_1$ and $\Gamma_{2n+2} = \eta(\xi\eta)^n\Gamma_1$. The domain $\mathbf{G}^{(2n+1)} \cap \mathbf{D}_1$ is not empty; otherwise Γ_{2n} would have lain in $\widehat{\mathbf{S}}_x^2$ and $\mathbf{D}_1$ would have been covered after Step $2n-1$. Moreover, $\Gamma_{2n+1} = \xi\Gamma_{2n}$ would have lain in $\widehat{\mathbf{S}}_x^1$ and $\mathbf{D}_2$ would have been covered after Step $2n$. Thus the procedure would have been stopped. Note that $\eta(\mathbf{G}^{(2n+1)} \cap \mathbf{D}_1) \subset \mathbf{G}^{(2n)} \cap \widehat{\mathbf{S}}_t^1 = \mathbf{G}^{(2n)} \cap \mathbf{D}_2$. In domain $\mathbf{G}^{(2n)} \cap \mathbf{D}_2$, functions β and δ are already defined by recursion. Then β and δ are continued by (2.16) to $\mathbf{G}^{(2n+1)} \cap \mathbf{D}_1$. The continuation of δ through Γ_{2n} to this domain is again meromorphic because δ is meromorphic in a neighborhood of $\eta(\Gamma_{2n} \cap \mathbf{D}_1) \subset \Gamma_{2n-1} \cap \mathbf{D}_2$ and in $\mathbf{G}^{(2n)} \cap \mathbf{D}_2$ by induction. Also function β continued by (2.14) is meromorphic, and equations (2.13) are preserved.

Step $2n+2$. At this step we construct the domain $\mathbf{G}^{(2n)} = \xi \mathbf{G}^{(2n+1)}$ and extend β and δ to $\mathbf{G}^{(2n+2)} \cap \mathbf{D}_2$ by using (2.17).

The procedure stops whenever $\bigcup_{k=1}^{n} \mathbf{G}^{(k)}$ covers $\mathbf{D}_1 \cup \mathbf{D}_2$. It stops after finitely many steps, because for some $\varepsilon > 0$ the distance between the points s and $\xi\eta s$ is $\ge \varepsilon$ $\forall s \in \mathbf{T}$. In fact, assume the opposite. Then there would be a sequence of points $s_n \in \mathbf{T}$ such that $x(s_n) - x(\xi\eta s_n) \to 0$ and $t(s_n) - t(\xi\eta s_n) \to 0$. Then, by the definition of ξ, $x(s_n) - x(\eta s_n) \to 0$. But $x(s_n)$ and $x(\eta s_n)$ are values of X at $t = t(s_n)$. Then $t(s_n)$ must tend to one of branching points t_1, t_2, t_3, t_4. Then $t(\xi\eta s_n)$ tends to the value of T at $x = X(t_i)$, $i = 1, 2, 3, 4$, which is different from t_i. But $X(t_i)$ are not branching points for $T(x)$. Thus $t(s_n) - t(\xi\eta s_n) \not\to 0$. □

COROLLARY 2.3. *β (resp. δ) can be meromorphically continued to sphere* $\mathbf{S}$ *cut along* $[x_3, x_4]$ (*resp.* $[t_3, t_4]$).

For the proof, define $\beta(x) := \beta(s)$ for s with $x(s) = x$, and $\delta(t) := \delta(s)$ for s with $t(s) = t$. Owing to (2.13), these formulas provide the meromorphic continuation.

3. Analytic aspects of the theory.

3.1. *Contour integrals on* **T**. From now on we will concentrate on the case $E_2 < 0$, $E_1 - E_2 < 0$ [see Proposition 2.1(a)]. The analysis of other cases is carried out in a similar fashion.

We represent the stationary probabilities $\rho_{m+n,m}$ as sums of contour integrals along cycles Γ_1 and $\widetilde{\Gamma}_1$. This part of the argument is close to [12]. We orient cycles Γ_1 and $\widetilde{\Gamma}_1$ in such a way that moving along Γ_1 (resp. $\widetilde{\Gamma}_1$) implies negative rotation along $\{x : |x| = 1\}$ [resp. positive rotation along $\{t : |t| = 1\}$] in the corresponding complex plane. We also introduce a differential form on **T**, $d\omega = (2U(x)t + V(x))^{-1}\,dx = -(2\widetilde{U}(t)x + \widetilde{V}(t))^{-1}\,dt$; see (2.3).

LEMMA 3.1. *For all sufficiently large* m, n,

$$\rho_{m+n,m} = \frac{1}{2\pi i}\biggl(\int_{\Gamma_1} \frac{B(s)\beta(s)}{x^n(s)t^m(s)}\,d\omega(s) + \int_{\widetilde{\Gamma}_1} \frac{D(s)\delta(s) + A(s)\rho_{00}}{x^n(s)t^m(s)}\,d\omega(s)\biggr). \tag{3.1}$$

The proof of Lemma 3.1 is a straightforward exercise based on Cauchy's formula. We only observe that the residues of the integrand, the zeros $T_1(x)$ and $T_2(x)$ of function $t \to Q(x,t)$ and the zeros of $X_1(t)$, $X_2(t)$ of function $x \to Q(x,t)$, can be separated by the circles $|t| = 1 + \varepsilon$ and $|x| = 1 + \varepsilon$, respectively, which is true according to Proposition 2.1(a).

3.2. *Asymptotics of stationary probabilities.* Our objective is now to evaluate the asymptotics of $\rho_{m+n,m}$ as $m, n \to \infty$ with $(m+n)/m \sim \operatorname{ctg}\gamma$, $0 \le \gamma \le \pi/4$. The denominator of integrands in (3.1) behaves as $(x^{\operatorname{ctg}\gamma - 1}(s)t(s))^m$ for $0 < \gamma \le \pi/4$ and $(x(s))^n$ for $\gamma = 0$. Set

$$\chi_0(s) = |x(s)|, \qquad \chi_\gamma(s) = \bigl|x^{\operatorname{ctg}\gamma - 1}(s)t(s)\bigr|, \qquad 0 < \gamma \le \pi/4. \tag{3.2}$$

The asymptotics of $\rho_{m+n,m}$ for $0 < \gamma < \pi/4$ is established by using the saddlepoint approximation. It is convenient to introduce a new covering map $y : \mathbf{T} \to \mathbf{S}$ with $y(s) = t(s)/x(s)$. After the substitution $t = xy$ in $Q(x,t)$, coordinates $x(s)$ and $y(s)$ on **T** are connected by

$$Q\bigl(x(s), y(s)\bigr) = p_{10}x^2(s)y(s) + p_{-10}y(s) + p_{01}x(s)y^2(s) + p_{0-1}x(s) - x(s)y(s) \equiv 0. \tag{3.3}$$

Now Lemma 2.4 from [12] asserts that for $0 < \gamma < \pi/4$ χ_γ has on **T** four critical points $s(\gamma, i)$, $i = 1, 2, 3, 4$, of Morse index 1, with

$$\chi_\gamma(s(\gamma, 1)) < \chi_\gamma(s(\gamma, 2)) < \chi_\gamma(s(\gamma, 3)) < \chi_\gamma(s(\gamma, 4)),$$

where $s(\gamma, 2), s(\gamma, 3) \in \Phi_0$ and $s(\gamma, 1), s(\gamma, 4) \in \Phi_1$. For $0 < \gamma < \pi/4$ the saddlepoint $S(\gamma)$ coincides with $s(\gamma, 3)$. As $\gamma \to 0$, $S(\gamma) \to (x_3, T(x_3))$ and as $\gamma \to \pi/4$, $s(\gamma) \to (X(t_3), t_3)$. Furthermore, the function $\gamma \mapsto S(\gamma)$ determines a homeomorphism between the segment $[0, \pi/4]$ and a segment J of Φ_0 lying inside $\mathbf{D}$. Coordinates $x(S(\gamma))$ and $t(S(\gamma))$ can be found as a unique solution to the system

$$\operatorname{tg} \gamma = \frac{p_{01} t/x - p_{0-1} x/t}{p_{10} x - p_{-10}/x}, \qquad Q(x, t) = 0, \tag{3.4}$$
$$X(t_3) < x(\gamma) < x_3, \qquad T(x_3) < t(\gamma) < t_3.$$

We set $x(\gamma) := x(S(\gamma))$, $t(\gamma) := t(S(\gamma))$ and

$$H(\gamma) := \chi_\gamma(S(\gamma)) = \left|x(\gamma)^{\operatorname{ctg} \gamma - 1} t(\gamma)\right|. \tag{3.5}$$

Observe that $X(t_3), T(x_3) > 1$ and hence $x(\gamma), t(\gamma) > 1$ for $\gamma \in [0, \pi/4]$.

It should be said that the homeomorphism $\gamma \to S(\gamma)$ can be continued to the whole segment $[-\pi, \pi]$; for any point $s = (x, t) \in \Phi_0 \cap \hat{\mathbf{S}}_x^1$ there corresponds an angle $\gamma \in [-\pi, 0]$ and for any point $(x, t) \in \Phi_0 \cap \hat{\mathbf{S}}_x^2$ there corresponds an angle $\gamma \in [0, \pi]$ with the tangent defined by (3.4). Thus, whenever γ runs $[-\pi, \pi]$, $S(\gamma)$ runs Φ_0 from $(x_2, T(x_2))$, passing through $(X(t_2), t_2)$ for $\gamma = -3\pi/4$, $(x_3, T(x_3))$ for $\gamma = 0$, $(t_3, X(t_3))$ for $\gamma = \pi/4$ and coming back to $(x_2, T(x_2))$ for $\gamma = \pi$.

DEFINITION 3.1. In what follows, we call a point $s' \in \mathbf{T}$ *inferior* to $s'' \in \mathbf{T}$ (and s'' *superior* to s') for a given γ if $\chi_\gamma(s')$ and $\chi_\gamma(s'')$ are real and $\chi_\gamma(s'') < \chi_\gamma(s'')$; we use the same terminology when comparing a point and a line.

DEFINITION 3.2. The most inferior of a (finite) collection of points of $\mathbf{T}$ is called *lowest*, as opposite to *highest*.

The locus

$$\left\{s : \operatorname{Im} \ln x^{\operatorname{ctg} \gamma - 1}(s) t(s) = \operatorname{Im} \ln x^{\operatorname{ctg} \gamma - 1}(S(\gamma)) t(S(\gamma)) = 0\right\}$$

represents two curves of steepest descent of χ_γ through $S(\gamma)$. These curves are orthogonal at $S(\gamma)$, and in a neighborhood $\mathbf{U}$ of $S(\gamma)$ one of them is a part of Φ_0. Technically it is convenient to fix two closed contours Θ_γ and $\widetilde{\Theta}_\gamma$ so that in $\mathbf{U}$ they both coincide with the curve of steepest descent orthogonal to Φ_0 and outside $\mathbf{U}$ are superior to $S(\gamma)$. See Figure 5.

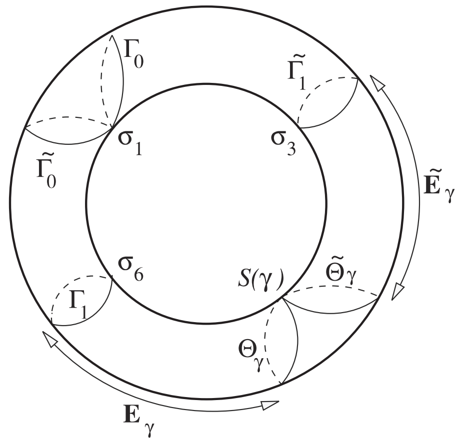


FIG. 5.

Denote by $\mathbf{E}_\gamma$ (resp. $\widetilde{\mathbf{E}}_\gamma$) the domain in $\mathbf{U}$ bordered by Γ_1 (resp. $\widetilde{\Gamma}_1$) and Θ_γ (resp. $\widetilde{\Theta}_\gamma$) and disjoint from $\mathbf{G}_x \cup \mathbf{G}_t$; see Figure 5. We are now prepared to specify the asymptotics of $\rho_{m+n,m}$ as $m, n \to \infty$, with $(m+n)/m \sim \operatorname{ctg}\gamma$, where $0 < \gamma < \pi/4$.

LEMMA 3.2. *Assume that $m, n \to \infty$, with $(m+n)/m \sim \operatorname{ctg}\gamma$, where $0 < \gamma < \pi/4$. Suppose that $\beta(s)$ have k poles $s_1, \dots, s_k$ in $\mathbf{E}_\gamma$ and $\delta(s)$ have l poles $\widetilde{s}_1, \dots, \widetilde{s}_l$ in $\widetilde{\mathbf{E}}_\gamma$. Then, $\forall$ integer $K \ge 1$,*

$$
\begin{aligned}
\rho_{m+n,m} - \Bigg(& \sum_{1\le i\le k} \frac{B(s_i)\operatorname{res}_{x(s_i)}\beta}{x^n(s_i)t^m(s_i)[2U(x(s_i))t(s_i) + V(x(s_i))]} \\
& + \sum_{1\le i\le l} \frac{D(\widetilde{s}_i)\operatorname{res}_{t(\widetilde{s}_i)}\delta}{x^n(\widetilde{s}_i)t^m(\widetilde{s}_i)[2\widetilde{U}(t(\widetilde{s}_i))x(\widetilde{s}_i) + \widetilde{V}(t(\widetilde{s}_i))]}\Bigg) \\
\approx & \frac{1}{x^n(\gamma)t^m(\gamma)} \sum_{0\le j\le K} c_j(\gamma) m^{-j-1/2},
\end{aligned}
\tag{3.6}
$$

where $c_j(\gamma)$ are constants.

The symbol $\approx$ indicates a standard asymptotic expansion (see, e.g., [7]): $G(m,n) \approx G^K(m,n)$ if the difference $G(m,n) - G^K(m,n)$ decreases as $m, n \to \infty$ faster than any term from $G^K(m,n)$.

To prove Lemma 3.2, one simply moves the integration contours in (3.1) to Θ_γ and $\widetilde{\Theta}_\gamma$ and uses Cauchy's theorem, together with the standard saddle-point approximation of the integrals along shifted contours. (See, e.g., Theorem 1.7 in [7], Chapter 4.)

Explicit formulas for coefficients $c_j(\gamma)$ are rather cumbersome. For example,

$$\begin{aligned} c_0(\gamma) = {} & \big[B(S(\gamma))\beta(S(\gamma)) + D(S(\gamma))\delta(S(\gamma)) + A(S(\gamma))\rho_{00}\big] \\ & \times H(\gamma)^{1/2}\big[2U(x(\gamma))t(\gamma) + V(x(\gamma))\big]^{-1} \\ & \times \left|\frac{d^2 x^{\operatorname{ctg}\gamma - 1}(\gamma)T(x(\gamma))}{dx^2}\right|^{-1/2}, \end{aligned} \tag{3.7}$$

with $c_0(0) = \lim_{\gamma\to 0} c_0(\gamma) = 0$, $c_0(\pi/4) = \lim_{\gamma\to\pi/4} c_0(\gamma) = 0$. We also have

$$\begin{aligned} c_1(0) &= \lim_{\gamma\to 0} c_1(\gamma) \\ &= \frac{\delta(x_3)[b_{01}D(x_3, T(x_3)) - (2d_{01}x_3 - 1)B(x_3, T(x_3))] + A(x_3)b_{01}\rho_{00}}{(B(x_3, T(x_3)))^2} \\ &\quad \times \frac{\sqrt{x_3(x_3 - x_1)(x_3 - x_2)(x_4 - x_3)}}{2p_{01}x_3^2 + p_{-10} - x_3}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} c_1(\pi/4) &= \lim_{\gamma\to\pi/4} c_1(\gamma) \\ &= \left[\frac{\beta(t_3)[(d_{01}t_3 + d_{0-1})B(X(t_3), t_3) - (2b_{01}X(t_3) - 1)D(X(t_3), t_3)]}{D^2(X(t_3), t_3)}\right. \\ &\qquad \left. + \frac{[(d_{01}t_3 + d_{0-1})A(X(t_3)) - D(X(t_3), t_3)/2]\rho_{00}}{(D(X(t_3), t_3))^2}\right] \\ &\quad \times \frac{\sqrt{t_3(t_1 - t_3)(t_2 - t_3)(t_4 - t_3)}}{2(p_{0-1} + p_{01}t_3)}. \end{aligned} \tag{3.9}$$

Lemma 3.2 implies that, for $0 < \gamma < \pi/4$, the asymptotics of $\rho_{m+n,m}$ are determined either by the saddle point or by the lowest (see Definition 3.2) among the poles from functions β and δ in $\mathbf{E}_\gamma$ and $\widetilde{\mathbf{E}}_\gamma$, respectively. To compare the contributions of these points we need a more detailed analysis. This is done in the remaining parts of Section 3.

3.3. *Poles of β and δ on* $\mathbf{T}$. Consider a selection of reference points on Φ_0,

$$\begin{aligned} &\sigma_1 := (1, 1), \quad \sigma_2 := \big(X(t_2), t_2\big), \\ &\sigma_3 := \left(\frac{p_{01} + p_{-10}}{p_{0-1} + p_{10}}, 1\right), \quad \sigma_4 := \big(x_3, T(x_3)\big), \\ &\sigma_5 := \big(X(t_3), t_3\big), \quad \sigma_6 := \left(1, \frac{p_{0-1}}{p_{01}}\right), \quad \sigma_7 := \big(x_2, T(x_2)\big). \end{aligned} \tag{3.10}$$

(Points σ_2 and σ_7 are not essential for the forthcoming argument and are only included for definiteness.) We also choose on Φ_0 the orientation agreeing with the order of the σ_i's and treat connected closed (resp. open or semiopen) pieces

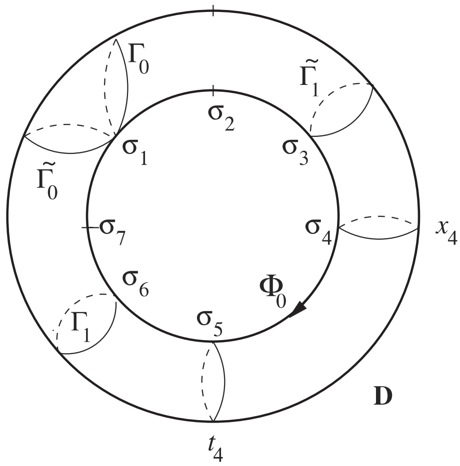


FIG. 6.

of this curve as "segments" $[s', s''] \subset \Phi_0$ (resp. "intervals" or "semiintervals" (s', s''), $[s', s'')$, $(s', s''] \subset \Phi_0$) directed according to this orientation. See Figure 6. Then $S(\gamma) \in (\sigma_4, \sigma_5)$, with $\lim_{\gamma\to 0} S(\gamma) = \sigma_4$, $\lim_{\gamma\to\pi/4}(\gamma) = \sigma_5$ and $\mathbf{E}_\gamma \cap \Phi_0 = (S(\gamma), \sigma_6)$ and $\widetilde{\mathbf{E}}_\gamma \cap \Phi_0 = (\sigma_3, S(\gamma))$.

Dealing with the level curves of χ_γ, we will use the property of *structural stability*. See Lemma 3.3 from [12] and Lemma 4 from [15]. As in these papers, the structural stability in the context of this work means that the topological picture of the level curves of χ_γ for $0 < \gamma < \pi/4$ is the same as that for $\gamma = 0$. [The latter can be established in a straightforward fashion. See Figures 7(a) and (b).] Formally, there exists a conformal homeomorphism of $\mathbf{T}$ transforming the foliation by the level curves for $\gamma \in (0, \pi/4)$ to that for $\gamma = 0$ and preserving the order of the level curves (so that for any pair of level curves, if one of them is inferior to another then their images are ordered in the same way). Observe that the structural stability is not extended to $\gamma = \pi/4$; see Figure 7(c).

Note that the level curves of χ_γ through $S(\gamma)$ are orthogonal to each other and partition the neighborhood of $S(\gamma)$ into four connected domains. Each of these curves is homotopic to Γ_1 and the two intersect only at $S(\gamma)$. The same is true of the level curves of χ_γ through point $s(\gamma; 2)$. (The above facts are straightforward for $\gamma = 0$ and hold for $0 < \gamma < \pi/4$, owing to the structural stability.)

On the other hand, for any point $s^* \in \Phi_0$ different from $s(\gamma; 2)$ and $S(\gamma)$, the level set $\{s : \chi_\gamma(s) = \chi_\gamma(s^*)\}$ has two disjoint connected components (one passing through s^* and the other not). Both these components are homotopic to Γ_1. (Again, these facts are straightforward for $\gamma = 0$ and hold for $0 < \gamma < \pi/4$, owing to the structural stability.)

In Lemmas 3.3 and 3.4 we assume that $0 < \gamma < \pi/4$.

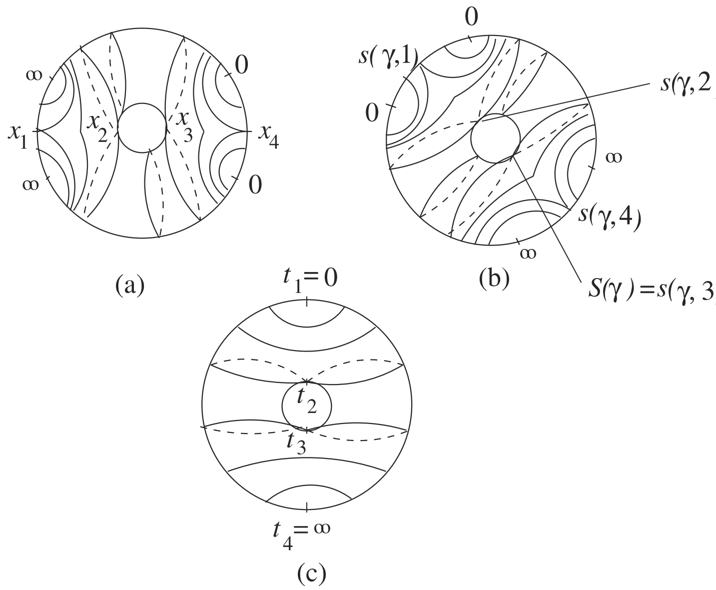


FIG. 7.

LEMMA 3.3. *Assume that function β (resp. δ) has poles in $\mathbf{E}_\gamma$ (resp. $\widetilde{\mathbf{E}}_\gamma$). Then the lowest among them lies on Φ_0. Moreover, on the level curve of the function χ_γ through this pole, there are no other poles of β (resp. δ).*

PROOF. Consider first function δ in domain $\widetilde{\mathbf{E}}_\gamma$. Given $s^* \in [\sigma_3, S(\gamma))$, denote by $\Lambda_\gamma(s^*)$ the connected component of the level set $\{s : \chi_\gamma(s) = \chi_\gamma(s^*)\}$ passing through s^*. As s^* tends to $S(\gamma)$, cycle $\Lambda_\gamma(s^*)$ approaches one of the level curves through $S(\gamma)$; we denote this limiting curve again by $\Lambda_\gamma(S(\gamma))$. Thus, $\Lambda_\gamma(s^*)$ is defined for $s^* \in [\sigma_3, S(\gamma)]$. See Figure 8.

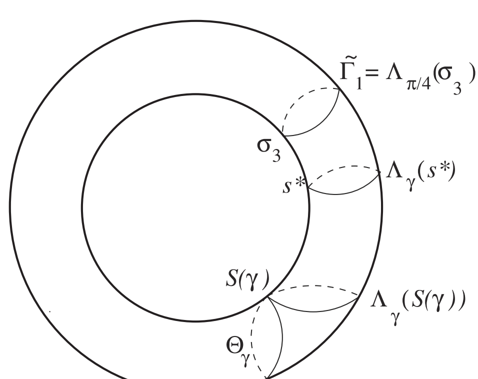


FIG. 8.

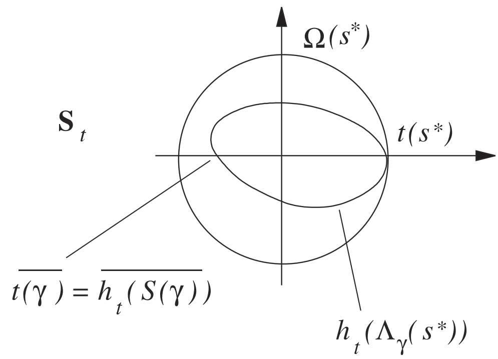


FIG. 9.

We will prove that the image $h_t\Lambda_\gamma(s^*)$ of $\Lambda_\gamma(s^*)$ on $\mathbf{S}$ lies inside the circle $\Omega(s^*) = \{z : |z| = t(s^*)\}$, apart from the point $z = t(s^*)$ where $h_t\Lambda_\gamma(s^*)$ and $\Omega(s^*)$ touch each other. See Figure 9.

Clearly, the real point $t(s^*)$ is a common point for $\Omega(s^*)$ and $h_t\Lambda_\gamma(s^*)$. We want to show that (a) $t(s^*)$ is a unique common point, and (b) all other points of $h_t\Lambda_\gamma(s^*)$ lie strictly inside circle $\Omega(s^*)$.

To prove (a), we simply check that $\forall s^*$ from $[\sigma_3, S(\gamma)]$, curve $h_t\Lambda_\gamma(s^*)$ and circle $\Omega(s^*)$ intersect only at $t(s^*)$. Assume the opposite. Then there would exist a nonreal pair (x,t) with $|x^{\operatorname{ctg}\gamma-1}t| = x^{\operatorname{ctg}\gamma-1}(s^*)t(s^*)$ and $|t| = t(s^*)$. Then $|x| = x(s^*)$. But (x,t) and $(x(s^*), t(s^*))$ must both satisfy equation $Q(x,t) = 0$. Then the point of the unit circle, $(\widetilde{x}, \widetilde{t}) = (x/x(s^*), t/t(s^*))$, satisfies $\widetilde{p}_{01}\widetilde{t}/\widetilde{x} + \widetilde{p}_{10}\widetilde{x} + \widetilde{p}_{-10}/\widetilde{x} + \widetilde{p}_{0-1}\widetilde{x}/\widetilde{t} = 1$, where $\widetilde{p}_{i,j} = p_{i,j}x^{i-j}(s^*)t^j(s^*)$ are real positive coefficients with $\sum_{i,j}\widetilde{p}_{ij} = 1$. By a simple argument about sums of complex numbers, this is impossible if $(\widetilde{x}, \widetilde{t}) \neq (1,1)$.

To prove (b), it suffices to check that for some $\gamma \in (0, \pi/4)$ close to $\pi/4$, there is a point $\bar{s}(\gamma) \in \Lambda_\gamma(s^*)$ such that the projected point $h_t\bar{s}(\gamma)$ lies inside circle $\Omega(s^*)$. We choose $\bar{s}(\gamma) = \Lambda_\gamma(s^*) \cap \Phi_1$. But from the construction of $\mathbf{T}$ and Φ_1, it is easy to see that $\bar{s}(\gamma)$ always has $t(\bar{s}(\gamma)) < 0$ (i.e., is projected on the negative part of the real line by map h_t) and that $\lim_{\gamma\to\pi/4-} x(\bar{s}(\gamma)) > x_4$. Set $\vartheta := \operatorname{ctg}\gamma - 1$ and $(\bar{x}(\vartheta), \bar{t}(\vartheta)) := (x(\bar{s}(\gamma)), -t(\bar{s}(\gamma)))$, $\vartheta \in (0,\infty)$. Then $\bar{x}^{\vartheta}(\vartheta)\bar{t}(\vartheta) = x^{\vartheta}(s^*)t(s^*)$. Take the derivative of $\bar{t}(\vartheta)$ at $\vartheta = 0+$,

$$\frac{d\bar{t}(\vartheta)}{d\vartheta}\bigg|_{\vartheta=0+} = t(s^*)\ln x(s^*) - \bar{t}(0+)\ln \bar{x}(0+).$$

This derivative is less than 0, as $t(s^*) = -t(\bar{s}(\pi/4-)) = \bar{t}(0+)$ and $x_2 \le x(s^*) \le x_3 < x_4 \le x(\bar{s}(\pi/4-)) = \bar{x}(0+)$. Thus for γ sufficiently close to $\pi/4$, the projection $h_t\bar{s}(\gamma) \in \Omega(s^*)$. As point $\bar{s}(\gamma)$ depends continuously on $\gamma \in (0,\pi/4)$, the same is true for all $0 < \gamma < \pi/4$.

We now can complete the proof of the assertion of Lemma 3.3 concerning function δ by quoting the Viventi–Pringsheim theorem (see, e.g., [1], Section 17E, pages 143 and 144). This theorem says that, for a function analytic in a neighborhood of zero and with nonnegative Taylor coefficients, the so-called nearest singularity (i.e., the singular point with the smallest absolute value) occurs on the real axis. A similar argument is used for function β. □

LEMMA 3.4. (a) *Let s^0 be the lowest among the poles of β in $\mathbf{E}_\gamma$ and of δ in $\widetilde{\mathbf{E}}_\gamma$. If s^0 is a pole of function β then $B(\xi s^0) = 0$. In this case its multiplicity equals that of the corresponding zero ξs^0 of B, provided that $\delta(\xi s^0)D(\xi s^0) + A(\xi s^0)\rho_{00} \neq 0$. If s^0 is a pole of δ then $B(\eta s^0) = 0$. In this case its multiplicity equals that of the corresponding zero ηs^0 of D, provided that $\beta(\eta s^0)B(\eta s^0) + A(\eta s^0)\rho_{00} \neq 0$.*

(b) *Let $s \in (\sigma_1, \sigma_4) \in \Phi_0$, $B(s) = 0$ and $\delta(s)D(s) + A(s)\rho_{00} \neq 0$. Then $\xi s \in (\sigma_4, \sigma_6)$ is a pole of β and $B(\xi s) \neq 0$. Similarly, let $s \in (\sigma_5, \sigma_1) \in \Phi_0$, $D(s) = 0$ and $B(s)\beta(s) + A(s)\rho_{00} \neq 0$. Then $\eta s \in (\sigma_3, \sigma_5)$ is a pole of δ and $B(\eta s) \neq 0$.*

PROOF. (a) Let s^0 be the lowest pole. Suppose, for example, that s^0 is the pole of β in $\mathbf{E}_\gamma$. Then by Lemma 3.4, $s^0 \in \mathbf{E}_\gamma \cap \Phi_0 = (S(\gamma), \sigma_6)$. We have $\beta(s^0) = \beta(\xi s^0)$, where $\xi s^0 \in (\sigma_1, \xi S(\gamma))$. Then ξs^0 is also a pole of β. Note that (2.14) is valid at ξs^0. It implies that either $B(\xi s^0) = 0$ or ξs^0 is a pole of δ. We will show that the last case is impossible.

In fact, in this case $\xi s^0 \in (\sigma_3, \xi S(\gamma))$, as δ is analytic on $[\sigma_1, \sigma_3]$. Note that $t(\xi s^0) < t(s^0)$; this fact holds also for s close to σ_4 and then, by continuity, for all $s \in (\sigma_4, \sigma_6)$ [as $t(\xi s) \neq t(s)$ for any $s \in \mathbf{T}$, except for $x(s) = x_1, x_2, x_3$ or x_4]. Thus $x^{\operatorname{ctg}\gamma - 1}(\xi s^0)t(\xi s^0) < x^{\operatorname{ctg}\gamma - 1}(s^0)t(s^0)$. It then follows that ξs^0 is a pole of δ in $\widetilde{\mathbf{E}}_\gamma$ inferior to s^0, which is impossible. Hence, $B(\xi s^0) = 0$. By (2.14) the multiplicity of ξs^0 equals that of the corresponding zero of B. Analogously, if s^0 is the pole of δ in $\widetilde{\mathbf{E}}_\gamma$ then $D(\eta s^0) = 0$.

(b) Let $s \in (\sigma_1, \sigma_4) \in \Phi_0$. As $B(s) = 0$; $\delta(s)D(s) + A(s)\rho_{00} \neq 0$. From (2.14) it follows that s is the pole for β. Then ξs is also the pole of β, as $\beta(s) = \beta(\xi s)$. It is easy to check that $B(s) - B(\xi s) = p'_{01}(t(s) - t(\xi s)) \neq 0$ for all $s \in \mathbf{T}$, except

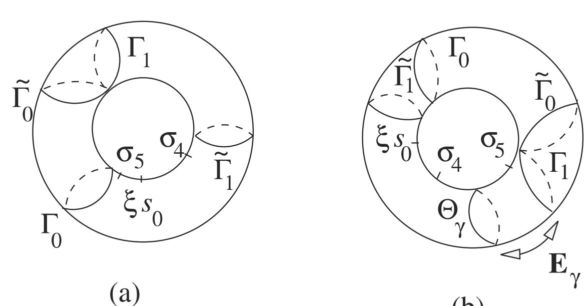


FIG. 10.

for $x(s) = x_1, x_2, x_3$ or x_4. The case of $s \in (\sigma_5, \sigma_1)$ is similar. This completes the proof. □

LEMMA 3.5. *For any $s \in (\sigma_1, \sigma_4]$ (resp. $s \in [\sigma_5, \sigma_1)$) function B (resp. D) has a zero on the interval (σ_1, s) [resp. (s, σ_1)] iff $B(s) > 0$ [resp. $D(s) > 0$]. Moreover, this zero, whenever it exists, is unique and of the first order.*

PROOF. The proof follows the same argument as Lemma 9 in [15] or Lemma 2.5 in [12] and we only sketch it briefly. The ergodicity condition (1.3) plays the crucial role; observe that $B(\sigma_1) = 0$ and the derivative $B'_x(x, T_1(x))|_{x=1,T_1(1)=1} = B'_x(\sigma_1) = E_2^{-1}(E_2 E_1^b - E_1 E_2^b) < 0$. Hence, by continuity there is an odd number of zeros of B on the interval in question if $B(s) > 0$ and even (may be zero) if $B(s) < 0$. We can show in the same way as in Lemma 2.5 of [12] that $B(s)$ does not have zeros of order more than 1 for any jumps p_{ij}^b. Then the number of zeros of $B(s)$ should be the same for all parameters from the set $\{p_{ij}^b \ge 0 : \sum_{i,j} p_{ij}^b = 1, B(s) > 0\}$ and for all parameters from the set $\{p_{ij}^b \ge 0 : \sum_{i,j} p_{ij}^b = 1, B(s) < 0\}$. Thus, it suffices to check explicitly a special simple case from these sets where, say, for example, at most two values p_{ij}^b are not zeros. The reasoning is the same for $D(s)$; note only that $D(\sigma_1) = 0$ and the derivative $D'_t(\sigma_1) = E_2/(E_2 - E_1) - 2d_{0-1} = 2(d_{01}E_2 + d_{0-1}E_1)/(E_2 - E_1) < 0$. □

4. Results for random walk $\mathcal{L}$.

4.1. *Analysis for a fixed γ.* We are now in a position to offer our results for RW $\mathcal{L}$. We consider four cases, depending on what singularity is "dominant" in terms of the contribution into the right-hand side of (3.6). In Case 1 we have a saddlepoint domination, in Case 2 the domination of a pole of β, in Case 3 of a pole of δ, and in Case 4 a competition between two poles.

Case 1.

$$B\left(x(\gamma), \frac{p_{0-1}x^2(\gamma)}{p_{01}t(\gamma)}\right) < 0, \qquad D\left(\frac{p_{01}t^2(\gamma) + p_{-10}t(\gamma)}{(p_{10}t(\gamma) + p_{0-1})x(\gamma)}, t(\gamma)\right) < 0.$$

In this case we rely on the information about the solution $(x(\gamma), t(\gamma))$ to (3.4).

Case 2.

$$B\left(x(\gamma), \frac{p_{0-1}x^2(\gamma)}{p_{01}t(\gamma)}\right) > 0, \qquad D\left(\frac{p_{01}t^2(\gamma) + p_{-10}t(\gamma)}{(p_{10}t(\gamma) + p_{0-1})x(\gamma)}, t(\gamma)\right) < 0.$$

In this case, the system of equations

$$B(x, t) = 0, \qquad Q(x, t) = 0 \tag{4.1}$$

has a unique solution (x_0, t_0) satisfying the inequalities $1 < x_0 < x(\gamma)$ and $t_0 = T_1(x_0) < T_2(x_0)$ [recall $Q(x, T_1(x)) \equiv Q(x, T_2(x)) \equiv 0$]. [Under $E_2 < 0$,

$E_1 - E_2 < 0$ this means that $s_0 = (x_0, t_0)$ is a unique zero of $B(s)$ on $(\sigma_1, \xi s(\gamma))$ according to Lemma 3.5; see Figure 6.] Furthermore, owing to Lemma 3.4, β has a pole of the first order at x_0.

Case 3.

$$B\left(x(\gamma), \frac{p_{0-1}x^2(\gamma)}{p_{01}t(\gamma)}\right) < 0, \qquad D\left(\frac{p_{01}t^2(\gamma) + p_{-10}t(\gamma)}{(p_{10}t(\gamma) + p_{0-1})x(\gamma)}, t(\gamma)\right) > 0.$$

In this case the system of equations

$$D(x,t) = 0, \qquad Q(x,t) = 0 \tag{4.2}$$

has a unique solution $(\widetilde{x}_0, \widetilde{t}_0)$ satisfying the inequalities $1 < \widetilde{t}_0 < t(\gamma)$ and $\widetilde{x}_0 = X_1(\widetilde{t}_0) < X_2(\widetilde{t}_0)$ $(Q(X_1(t), t) \equiv Q(X_2(t), t) \equiv 0)$. [Under $E_2 < 0$, $E_1 - E_2 < 0$ this means that $\widetilde{s}_0 = (\widetilde{x}_0, \widetilde{t}_0)$ is the unique zero of $D(s)$ on $(\eta s(\gamma), \sigma_1)$ according to Lemma 3.5.] Furthermore, according to Lemma 3.4, δ has a pole of the first order at $\widetilde{t}_0$.

Case 4.

$$B\left(x(\gamma), \frac{p_{0-1}x^2(\gamma)}{p_{01}t(\gamma)}\right) > 0, \qquad D\left(\frac{p_{01}t^2(\gamma) + p_{-10}t(\gamma)}{(p_{10}t(\gamma) + p_{0-1})x(\gamma)}, t(\gamma)\right) > 0.$$

Here (4.1) have a unique solution (x_0, t_0) satisfying the inequalities of the Case 2, and (4.2) have a unique solution $(\widetilde{x}_0, \widetilde{t}_0)$ satisfying the inequalities of Case 3. Thus, either x_0 is a pole of the first order for β or $\widetilde{t}_0$ a pole of the first order for δ.

However, we also have to avoid certain additional equalities. More precisely, in Case 1 we assume that for $0 < \gamma < \pi/4$ constant $c_0(\gamma) \neq 0$ in (3.7), which is equivalent to

$$\beta(S(\gamma))B(S(\gamma)) + \delta(S(\gamma))D(S(\gamma)) + \rho_{00}A(S(\gamma)) \neq 0, \tag{4.3}$$

while for $\gamma = 0$ or $\pi/4$ constant $c_1(\gamma) \neq 0$ in (3.8) or (3.9). In Case 2 we assume that

$$\delta(s_0)D(s_0) + A(s_0)\rho_{00} \neq 0, \tag{4.4}$$

in Case 3,

$$\beta(\widetilde{s}_0)B(\widetilde{s}_0) + A(\widetilde{s}_0)\rho_{00} \neq 0, \tag{4.5}$$

and in Case 4 that one of (4.4) and (4.5) holds true.

Under these assumptions we are able to specify the behavior of $\rho_{m+n,m}$ as $m, n \to \infty$, $(m+n)/n \to \operatorname{ctg}\gamma$.

THEOREM 4.1. (i) *Suppose that* $0 < \gamma < \pi/4$ *and Case* 1 *holds. Then*

$$\rho_{m+n,m} \sim \frac{c_0(\gamma)}{\sqrt{m}} x^{-n}(\gamma) t^{-m}(\gamma), \tag{4.6}$$

where c_0 *is determined in* (3.7).

(ii) *Suppose that* $\gamma = 0$ *or* $\gamma = \pi/4$ *and Case* 1 *holds. Then*

$$\rho_{n,0} \sim \frac{c_1(0)}{n\sqrt{n}} x_3^{-n} \quad or \quad \rho_{m,m} \sim \frac{c_1(\pi/4)}{m\sqrt{m}} t_3^{-m}, \tag{4.7}$$

respectively, where $c_1(0)$, $c_1(\pi/4)$ *are constants from* (3.8) *and* (3.9) (*which therefore are positive*).

THEOREM 4.2. (i) *Suppose that* $0 < \gamma < \pi/4$ *and Case* 2 *holds. Then*

$$\rho_{m+n,m} \sim \left[B\left(x_0, \frac{p_{0-1}x_0^2}{p_{01}t_0}\right) \operatorname{res}_{x_0} \beta\right] x_0^{-n} \left(\frac{p_{0-1}x_0^2}{p_{01}t_0}\right)^{-m}. \tag{4.8}$$

(ii) *Suppose that* $\gamma = 0$ *or* $\gamma = \pi/4$ *and Case* 2 *holds. Then*

$$\rho_{n,0} \sim [\operatorname{res}_{x_0} \beta] x_0^{-n+1} \quad or \quad \rho_{m,m} \sim [\operatorname{res}_{\xi t_0} \delta] \left(\frac{p_{0-1}x_0^2}{p_{01}t_0}\right)^{-m+1}, \tag{4.9}$$

respectively. Furthermore, if $D(s_0)$ *and* $A(s_0)$ *have the same sign then* (4.4) *holds true and can be omitted from the assumptions.*

THEOREM 4.3. (i) *Suppose that* $0 < \gamma < \pi/4$ *and Case* 3 *holds. Then*

$$\begin{aligned} \rho_{m+n,m} \sim {} & \left[D\left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}, \widetilde{t}_0\right) \operatorname{res}_{\widetilde{t}_0} \delta\right] \\ & \times \left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}\right)^{-n} \widetilde{t}_0^{-m}. \end{aligned} \tag{4.10}$$

(ii) *Suppose that* $\gamma = 0$ *or* $\gamma = \pi/4$ *and Case* 3 *holds. Then*

$$\begin{aligned} & \rho_{n,0} \sim [\operatorname{res}_{\eta\widetilde{x}_0} \beta] \left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}\right)^{-n+1} \quad or \\ & \rho_{m,m} \sim [\operatorname{res}_{\widetilde{t}_0} \delta] \widetilde{t}_0^{-n+1}, \end{aligned} \tag{4.11}$$

respectively. Furthermore if $B(\widetilde{s}_0)$ *and* $A(\widetilde{s}_0)$ *have the same sign then* (4.5) *holds true and can be omitted from the assumptions.*

THEOREM 4.4. (i) *Suppose that* $0 < \gamma < \pi/4$ *and Case* 4 *holds. Then*

$$\begin{aligned} \rho_{n+m,m} \sim {} & \left[B\left(x_0, \frac{p_{0-1}x_0^2}{p_{01}t_0}\right) \operatorname{res}_{x_0} \beta\right] x_0^{-n} \left(\frac{p_{0-1}x_0^2}{p_{01}t_0}\right)^{-m} \\ & + \left[D\left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}, \widetilde{t}_0\right) \operatorname{res}_{\widetilde{t}_0} \delta\right] \\ & \times \left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}\right)^{-n} \widetilde{t}_0^{-m}. \end{aligned} \tag{4.12}$$

(ii) *Suppose that* $\gamma = 0$ *or* $\gamma = \pi/4$ *and Case* 4 *holds. Then*

$$\rho_{n,0} \sim [\mathrm{res}_{x_0}\beta]x_0^{-n+1} + [\mathrm{res}_{\eta\widetilde{x}_0}\beta]\left(\frac{p_{01}\widetilde{t}_0^2 + p_{-10}\widetilde{t}_0}{(p_{10}\widetilde{t}_0 + p_{0-1})\widetilde{x}_0}\right)^{-n+1}, \tag{4.13}$$

$$\rho_{m,m} \sim [\mathrm{res}_{\xi t_0}\delta]\left(\frac{p_{0-1}x_0^2}{p_{01}t_0}\right)^{-m+1} + [\mathrm{res}_{\widetilde{t}_0}\delta]\widetilde{t}_0^{-m+1}, \tag{4.14}$$

respectively. Furthermore, if $D(s_0)$ *and* $A(s_0)$ *have the same sign or* $B(\widetilde{s}_0)$ *and* $A(\widetilde{s}_0)$ *have the same sign then one of* (4.4) *or* (4.5) *holds true and can be omitted from the assumptions.*

In the remaining part of Section 4.1 we discuss the proof of Theorems 4.1–4.4 in the situation where $E_2 < 0$, $E_1 - E_2 < 0$; see Proposition 2.1(a). The proof in other situations is similar and based on Figures 3(b)–(e). Recall, $S(\gamma) = (x(\gamma), t(\gamma))$ is the saddlepoint of χ_γ on $\mathbf{T}$.

PROOF OF THEOREMS 4.1–4.4. Suppose first that $0 < \gamma < \pi/4$.

Case 1. Here $B(\xi S(\gamma)) < 0$ and $D(\eta S(\gamma)) < 0$. By Lemma 3.5, there are no zeros of B on $(\sigma_1, \xi S(\gamma))$ and of D on $(\eta S(\gamma), \sigma_1)$. By Lemmas 3.3 and 3.4, there are no poles of β and δ in $\mathbf{E}_\gamma$ and $\widetilde{\mathbf{E}}_\gamma$, respectively. Then by Lemma 3.2, the asymptotics of $\rho_{m+n,n}$ is determined by the saddle point $S(\gamma)$. Hence, we get (4.6), provided that (4.3) holds true.

Case 2. In this case $B(\xi S(\gamma)) > 0$ and $D(\eta S(\gamma)) < 0$. By Lemma 3.5, there is a unique zero $s_0 = (x_0, t_0)$ of B on $(\sigma_1, \xi S(\gamma))$, which is of the first order, and there are no zeros of D on $(\eta S(\gamma), \sigma_1)$. Then by Lemma 3.4(b), ξs_0 is the pole of β provided that (4.4) holds true. By Lemma 3.4(a), this pole is the lowest among the poles of β in $\mathbf{E}_\gamma$ and of δ in $\widetilde{\mathbf{E}}_\gamma$, and its multiplicity equals 1. Then the asymptotics of $\rho_{m+n,n}$ is determined by the pole ξs_0, and (4.8) follows.

Assume now that $D(s_0)$ and $A(s_0)$ have the same sign but (4.4) is not true. Then $\delta(s_0) < 0$. Since $\delta(s) = \sum_{m=1}^{\infty} \rho_{m,m} t(s)^{m-1}$ in $[\sigma_1, \sigma_3]$, then $s_0 \in (\sigma_3, \xi S(\gamma))$ and there exists a pole s' of $\delta(s)$ on (σ_3, s_0). Then the set of poles of $\beta(s)$ in $\mathbf{E}_\gamma$ or of $\delta(s)$ in $\widetilde{\mathbf{E}}_\gamma$ is not empty and by Lemma 3.4(a), the lowest of them, s'', should satisfy $B(\xi s'') = 0$ or $D(\xi s'') = 0$. Then by Lemma 3.5 it is a pole of $\beta(s)$ and $s'' = \xi s_0$. But then $\xi s'' = s_0$ is a pole of $\beta(s)$ as well. The last case is impossible, as $\beta(s) = (-D(s)\delta(s) - A(s)\rho_{00})/B(s)$, where the numerator is analytic and equals zero at $s = s_0$ and the denominator has zero of the first order at s_0 by Lemma 3.5. Hence, (4.4) is satisfied.

Case 3. Here $B(\xi S(\gamma)) < 0$ and $D(\eta S(\gamma)) > 0$. The analysis is completely similar to Case 2 provided that (4.5) holds true.

Case 4. In this case $B(\xi S(\gamma)) > 0$ and $D(\eta S(\gamma)) > 0$. By Lemma 3.5, there is a unique zero of the first order of B on $(\sigma_1, \xi S(\gamma))$ and a unique zero of the first order of D on $(\eta S(\gamma), \sigma_1)$. Then by Lemma 3.4(b), at least one of the points ξs or ηs is the pole of β or δ in $\mathbf{E}_\gamma$ or $\widetilde{\mathbf{E}}_\gamma$, respectively, provided that at least one of

(4.4) or (4.5) hold. By Lemma 3.4(a), the lowest pole is either ξs or ηs, and it is of the first order. Thus by Lemma 3.2, we have the asymptotics (4.12).

Now assume that $D(s_0)$ and $A(s_0)$ have the same sign or $B(\widetilde{s}_0)$ and $A(\widetilde{s}_0)$ have the same sign but none of (4.4) or (4.5) holds true. Then by the same reasoning as in Case 1 there exists the lowest pole s'' and by Lemma 3.4(a) either $\xi s'' = s_0$ is a pole of $\beta(s)$ or $\eta s'' = \widetilde{s}_0$ is a pole of $\delta(s)$. Neither case is possible, as these are zeros of $B(s)$ and $D(s)$ of the first order. Thus one of (4.4) or (4.5) is satisfied.

For $\gamma = 0$ or $\gamma = \pi/4$, the proof is different. We have to find the asymptotics of Taylor's coefficients of functions β and δ at zero. Our proof relies on Theorem A.1 from the Appendix. We will give a detailed argument for $\gamma = 0$ only, as the changes for $\gamma = \pi/4$ are purely technical.

Case 1. By Lemma 3.5, there are no zeros of B and D in (σ_1, σ_4) and $(\eta\sigma_4, \sigma_1)$, respectively. Then there are no poles of β in $[\sigma_4, \sigma_6]$. In fact, assume the contrary. Let s be the lowest pole for $\gamma = 0$. As $\beta(s) = \beta(\xi s)$, $\xi s \in (\sigma_1, \sigma_4)$ is also a pole of β. From (2.3) at ξs it follows that either $B(\xi s) = 0$, which is impossible by Lemma 3.5, or ξs is a pole of δ. In the latter case $\xi s \in (\sigma_3, \sigma_4)$, and $\eta\xi s$ is a pole for δ. Note that $\eta\xi s \in (\eta\sigma_4, \sigma_1) \subset (\sigma_5, \sigma_1)$. Moreover, $\eta\xi s \in (s, \sigma_1)$, as a consequence of the fact that $t(\eta\xi s) = t(\xi s) < t(s)$. [For s close to σ_4, obviously $t(\xi s) < t(s)$. Then, since $t(\xi s) \neq t(s)$ for no one $s \in (\sigma_4, \sigma_6)$, this remains true for all $s \in (\sigma_4, \sigma_6)$.] From (2.3) at $\eta\xi s$ it follows that either $D(\eta\xi s) = 0$, which is impossible by Lemma 3.5, or $\eta\xi s$ is a pole of β. But $\eta\xi s \in (s, \sigma_1)$ is inferior to s. So, the first singularity of β on $\mathbf{C}_x$ is the point $x(\sigma_4) = x_3$. In a neighborhood of this point,

$$\beta(x) = -\frac{D(x, T(x))\delta(x) + A(x)\rho_{00}}{B(x, T(x))}.$$

Then the main term in the expansion of $\beta(x)$ at x_3 is of order $(x - x_3)^{1/2}$. By Theorem 6.1 we get (4.7), provided that $c_1(0) \neq 0$.

Case 2. In this case there is a unique zero $s_0 = (x_0, t_0)$ of B on (σ_1, σ_4), which is of the first order, and there are no zeros of D on $(\eta\sigma_4, \sigma_1)$. Then the lowest pole of β is ξs_0. To show this, note first that it is indeed a pole because $\beta(s) = \beta$ and by (2.14) at ξs_0, provided that (4.4) holds true. Assume that a point s is the lowest pole. Then as in Case 1, by (2.14), either $B(\xi s) = 0$ or δ has a pole at ξs. The last fact is again impossible: otherwise either $D(\eta\xi s) = 0$, contradicting Lemma 3.5, or $\eta\xi s$ is a pole of β inferior to s. Thus $\xi s = s_0$, and by (2.14) and Lemma 3.5, ξs_0 is a pole of the first order. By Theorem 6.1, the asymptotics of the coefficients of $\beta(x)$ is determined by $x(s_0)$.

Case 3. In Case 3, Lemma 3.5 implies that there are no zeros of B in (σ_4, σ_1), and there is a unique zero of the first order, $\widetilde{s}_0 = (\widetilde{x}_0, \widetilde{t}_0)$ of D on $(\eta\sigma_4, \sigma_1)$. Then $\xi\eta\widetilde{s}_0$ is the pole of β on (σ_1, σ_4), provided (4.5) holds true. In fact, δ has a pole at $\widetilde{s}_0$ by (2.14). Then it has a pole at $\eta\widetilde{s}_0 \in (\sigma_1, \sigma_4)$. Then by (2.14), $\eta\widetilde{s}_0$ is a pole of β, and so is $\xi\eta\widetilde{s}_0$. Moreover, by the same arguments as in Case 2 , $\xi\eta\widetilde{s}_0$ is the lowest pole and is of the first order. By Theorem 6.1, the asymptotics of the coefficients of function β is determined by $x(\xi\eta\widetilde{s}_0) = x(\eta\widetilde{s}_0)$.

Case 4. In this case the lowest pole of β is one of ξs_0 and $\xi\eta\widetilde{s}_0$, provided that at least one of (4.3) or (4.4) hold. Therefore, the asymptotics of $\beta(x)$ is determined by one of these poles. □

REMARK 4.1. We managed to eliminate assumptions (4.4) and (4.5) for a certain subclass of RWs. It includes JS-queues considered above. Note also that even if (4.3) is not true, we know the next term of the asymptotics from the saddlepoint approximation as in (3.6).

However, we believe that (4.3)–(4.5) always hold. The reason is the beautiful argument suggested in [15], which, however, we cannot complete in our situation. Suppose that one of these assumptions fails. Then the correspondng asymptotics of stationary probabilities is of smaller order. More precisely, if (4.3) fails, then the factor $1/(n\sqrt{n})$ is added according to (3.6) and if (4.4) or (4.5) fails, then the asymptotics is determined by the saddle point, which is inferior to ξs_0 or $\eta\widetilde{s}_0$, respectively. Consider a slightly more general RW $\widetilde{\mathcal{L}}$, where jumps from the origin can be performed to a finite number of points $(k,l) \in \mathbf{Z}_+^2$, $k \geq l$, with probabilities $p_{k,l}^0$ and to symmetric points (l,k) with equal probabilities $p_{l,k}^0 = p_{k,l}^0$. Then only the function $A(x,t) = (\sum_{k,l} p_{k,l}^0 x^{k-l} t^l - 1)/2$ is modified while the rest of the analysis remains the same. Suppose that one could find the collection of $p_{k,l}^0$ such that the corresponding inequality among (4.3)–(4.5) is valid. Then the asymptotics of stationary probabilities $\widetilde{\rho}_{m+n,m}$ for $\widetilde{\mathcal{L}}$ is determined by Theorems 4.1–4.4. Thus $\rho_{m+n,m}/\widetilde{\rho}_{m+n,m} \to 0$. But the stationary probability of state $(m+n,m)$ for a Markov chain is equal to the stationary probability of $(0,0)$ times the mean number of visits to $(m+n,m)$, starting from $(0,0)$ and prior coming back to $(0,0)$. As the jumps of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are the same from all points of $\mathbf{Z}_2^+$ except for $(0,0)$, then $\rho_{m+n,m}/\widetilde{\rho}_{m+n,m} \geq C > 0$ for some C, which contradicts $\rho_{m+n,m}/\widetilde{\rho}_{m+n,m} \to 0$.

The open question is: Could one find probabilities $p_{k,l}^0$ such that, say, (4.4) holds true? Since at least one of $x(s_0)$ or $t(s_0)$ is > 1, one could take, for example, $p_{k,0}^0 = 1/2$ or $p_{k,k}^0 = 1$ for k large enough, to make $A(s_0)$ large. Then it would suffice to show that $\delta(s_0)/\rho_{00}$ remains bounded when k grows. In [15] an integral representation established in [13] is used; unfortunately, we do not have such a representation in the general SHC case.

4.2. *Analysis for fixed jump probabilities.* Theorems 4.1–4.4 analyze the situation for a given γ depending on which among Cases 1–4 occurs. If we fix vectors $\mathbf{p}$, $\mathbf{b}$ and $\mathbf{d}$ and vary γ then the asymptotics of $\rho_{m+n,m}$ is determined by the signs of $B(x_3, T(x_3))$ and $D(X(t_3), t_3)$. Let us now specify angles γ_0 and $\widetilde{\gamma}_0$ announced in the Introduction.

If $B(x_3, T(x_3)) < 0$ [or $B(x_3, T(x_3)) = 0$, $B'_x(x, T_1(x))|_{x=x_3-} > 0$], then by Lemmas 3.4 and 3.5 , $\forall\gamma \in [0, \pi/4]$ there is no pole of $\beta(s)$ in $\mathbf{E}_\gamma \cap \Phi_0$. Then we set $\gamma_0 = 0$.

Now assume that $B(x_3, T(x_3)) > 0$ [or $B(x_3, T(x_3)) = 0$, $B'_x(x, T_1(x))|_{x=x_3-} < 0$]. Then take the unique solution (x_0, t_0) of the system of equations $Q(x,t) = B(x,t) = 0$ satisfying the inequalities $1 < x_0 < x_3$ and $t_0 = T_1(x_0) < T_2(x_0)$. In other words, $s_0 = (x_0, t_0)$ is the unique zero of $B(s)$ on (σ_1, σ_4) (it exists by Lemma 3.5). Then $\xi s_0 \in (\sigma_4, \sigma_6)$ is the pole of $\beta(s)$. Let $\gamma_0^+ \in (0, \pi)$ be the angle corresponding to the pole ξs_0 by the homeomorphism $\gamma \to S(\gamma)$. In view of (3.4), it suffices to specify $\operatorname{tg} \gamma_0^+$,

$$\operatorname{tg} \gamma_0^+ = \frac{p_{0-1} x_0 / t_0 - p_{01} t_0 / x_0}{p_{10} x_0 - p_{-10} / x_0}.$$

If γ_0^+ happens to be greater than $\pi/4$, then $\xi s_0 \in (\sigma_5, \sigma_6)$, in which case ξs_0 for all $\gamma \in [0, \pi/4]$ lies in $\mathbf{E}_\gamma \cap \Phi_0$ and contributes to the asymptotics of $\rho_{m+n,m}$. Otherwise, that is, if $\gamma_0 \le \pi/4$, this is true only for $\gamma \in [0, \gamma_0^+]$. Hence, in this case we set $\gamma_0 = \min\{\pi/4, \gamma_0^+\}$.

Similarly, if $D(X(t_3), t_3) < 0$ [or $D(X(t_3), t_3) = 0$, $D'_t(X_1(t), t)|_{t=t_3-} > 0$] then $\forall \gamma \in [0, \pi/4]$, there is no pole of $\delta(s)$ in $\widetilde{\mathbf{E}}_\gamma \cap \Phi_0$. Then we set $\widetilde{\gamma}_0 = \pi/4$.

On the other hand, if $D(X(t_3), t_3) > 0$ [or $D(X(t_3), t_3) = 0$, $D'_t(X_1(t), t)|_{t=t_3-} < 0$] then we take the unique solution $(\widetilde{x}_0, \widetilde{t}_0)$ of the system of equations $Q(x,t) = B(x,t) = 0$ satisfying the inequalities $1 < \widetilde{t}_0 < t_3$ and $\widetilde{t}_0 = T_1(\widetilde{x}_0) < T_2(\widetilde{x}_0)$. In other words, $\widetilde{s}_0 = (\widetilde{x}_0, \widetilde{t}_0)$ is the unique zero of $D(s)$ on (σ_5, σ_1) (it exists by Lemma 3.5). Let $\widetilde{\gamma}_0^- \in (-3\pi/4, \pi/4)$ be the angle corresponding to the pole $\eta \widetilde{s}_0 = (\eta \widetilde{x}_0, \widetilde{t}_0) \in (\sigma_3, \sigma_5)$ of $\delta(s)$ by the homeomorphism $\gamma \to S(\gamma)$. Again, in view of (3.4) it suffices to specify $\operatorname{tg} \widetilde{\gamma}_0^-$,

$$\operatorname{tg} \widetilde{\gamma}_0^- = \frac{\widetilde{t}_0 (p_{01} (p_{10} \widetilde{t}_0 + p_{0-1})^2 \widetilde{x}_0^2 - p_{0-1} (p_{01} \widetilde{t}_0 + p_{-10})^2)}{p_{10} (p_{01} \widetilde{t}_0^2 + p_{-10} \widetilde{t}_0)^2 - p_{-10} (p_{10} \widetilde{t}_0 + p_{0-1})^2 \widetilde{x}_0^2}.$$

If $\widetilde{\gamma}_0^-$ happens to be < 0 then $\eta \widetilde{s}_0 \in (\sigma_3, \sigma_4)$, in which case $\eta \widetilde{s}_0$ for all $\gamma \in [0, \pi/4]$ lies in $\widetilde{\mathbf{E}}_\gamma \cap \Phi_0$ and contributes to the asymptotics of $\rho_{m+n,m}$. Otherwise, that is, if $\widetilde{\gamma}_0^- \ge 0$, this is true only for $\gamma > \widetilde{\gamma}_0^-$. Hence, in this case we set $\widetilde{\gamma}_0 = \max\{0, \widetilde{\gamma}_0^-\}$.

We are now able to formulate the following consequence of Theorems 4.1–4.4.

COROLLARY 4.5. (i) *For* $\gamma \in [0, \min\{\gamma_0, \widetilde{\gamma}_0\})$, *the asymptotics of* $\rho_{m+n,m}$ *is determined by the pole of* β; *see* (4.8) *and* (4.9). *Furthermore, if* $\gamma_0 = \widetilde{\gamma}_0 = \pi/4$ *and* $\gamma_0^+ > \pi/4$, *then* (4.9) *holds true for* $\gamma = \gamma_0 = \widetilde{\gamma}_0$.

(ii) *For* $\gamma \in (\max\{\gamma_0, \widetilde{\gamma}_0\}, \pi/4]$ *the asymptotics of* $\rho_{m+n,m}$ *is determined by the pole of* δ, *see* (4.10) *and* (4.11). *Furthermore, if* $\gamma_0 = \widetilde{\gamma}_0 = 0$ *and* $\widetilde{\gamma}_0^- < 0$, *then* (4.11) *holds true for* $\gamma = \gamma_0 = \widetilde{\gamma}_0$.

(iiia) *If* $\widetilde{\gamma}_0 < \gamma_0$, *then for* $\gamma \in (\widetilde{\gamma}_0, \gamma_0)$ *the asymptotics of* $\rho_{m+n,m}$ *is determined by the lowest of two poles of* β *and* δ; *see* (4.12). *Furthermore, if* $\widetilde{\gamma}_0 = 0$ *and* $\widetilde{\gamma}_0^- < 0$ *or* $\gamma_0 = \pi/4$ *and* $\gamma_0^+ > \pi/4$, *then* (4.13) *holds true for* $\gamma = \widetilde{\gamma}_0$ *or* $\gamma = \gamma_0$, *respectively.*

(iiib) *If* $\gamma_0 < \widetilde{\gamma}_0$, *then for* $\gamma \in (\gamma_0, \widetilde{\gamma}_0)$ *the asymptotics of* $\rho_{m+n,m}$ *is determined by the saddlepoint*; *see* (4.6). *Furthermore*, *if* $\gamma_0 = 0$ *or* $\widetilde{\gamma}_0 = \pi/4$, *then* (4.7) *holds true for* $\gamma = \gamma_0$ *or* $\gamma = \widetilde{\gamma}_0$, *respectively.*

5. The Fayolle–Iasnogorodski argument. In this section we give a recipe for specifying constants $c_0(\gamma)$ in (4.6), $c_1(0)$, $c_1(\pi/4)$ in (4.7) and the coefficients $\mathrm{res}_{x_0}\,\beta$, $\mathrm{res}_{\xi t_0}\,\delta$, $\mathrm{res}_{\widetilde{t}_0}\,\delta$, $\mathrm{res}_{\eta\widetilde{x}_0}\,\beta$ in (4.8)–(4.14) in terms of the probabilities p_{ij}, b_{ij}, d_{ij} of random walk $\mathcal{L}$. More precisely, in (3.8) determining value $c_0(\gamma)$, we specify $\beta(S(\gamma))$, $\delta(S(\gamma))$ and ρ_{00}.

In fact, as was mentioned in the Introduction, the ratios $\beta(s)/\rho_{00}$ and $\delta(s)/\rho_{00}$ are first specified for s within an explicitly described domain $\mathbf{O}$ on torus $\mathbf{T}$. Next, one uses the meromorphic continuation of β/ρ_{00} and δ/ρ_{00} to the whole of $\mathbf{T}$. To find ρ_{00}, one then uses an additional normalizing equation [see (5.12)], combined again with the meromorphic continuation. As a result, $\beta(s)$ and $\delta(s)$ become determined on the whole of $\mathbf{T}$.

To find $\beta(s)/\rho_{00}$ and $\delta(s)/\rho_{00}$ in $\mathbf{O}$, one uses a Riemann–Carleman (R–C) type boundary problem in a domain $\mathbf{M}$ of a standard complex plane and then lifts its unique solution from $\mathbf{M} \subset \mathbf{C}$ to $\mathbf{O} \subset \mathbf{T}$: $\beta(s) := \beta(x(s))$, $\delta(s) = \delta(t(s))$.

The analysis below of a R–C problem in a standard complex plane $\mathbf{C}$ belongs to Iasnogorodski and Fayolle. Define the closed contour $\partial\mathbf{M}$ in a complex x-plane $\mathbf{C}$ $(= \mathbf{C}_x)$,

$$\partial\mathbf{M} = X_1[t_1, t_2] \cup X_2[t_1, t_2] \tag{5.1}$$

and let $\mathbf{M}$ be in its interior. The boundary problem arises from an equation,

$$\frac{\beta(z)}{\rho_{00}} V(z) - \frac{\beta(\bar{z})}{\rho_{00}} V(\bar{z}) = U(z), \qquad z \in \partial\mathbf{M}, \tag{5.2}$$

or a similar equation for $\frac{\delta(z)}{\rho_{00}}$. The functions V and U are determined on the whole of $\mathbf{C}$ by

$$V(x) = \frac{B(x, T_1(x))}{D(x, T_1(x))}, \qquad U(x) = \frac{A(X_2(T_1(x)), T_1(x))}{D(X_2(T_1(x)), T_1(x))} - \frac{A(x, T_1(x))}{D(x, T_1(x))}. \tag{5.3}$$

One wants a solution β/ρ_{00} to (5.1) meromorphic in $\mathbf{M}$.

Equation (5.1) is derived as in [5], page 95; in fact it is a direct consequence of (2.1).

It is convenient to work with analytic, rather than meromorphic, functions in $\mathbf{M}$; to this end consider the (complex) polynomial

$$Z(x) = \prod_{r \in \mathbf{M} \cap \{|x|>1\}: B(r, T_1(r))=0} (x - r), \qquad x \in \mathbf{C} \tag{5.4}$$

[cf. [5], equation (5.4.9)]. Multiply β and divide V by Z; you get an equation similar to (5.1) and look for a solution $(\beta Z)/\rho_{00}$ that is analytic in $\mathbf{M}$. One uses here the fact that the poles of $\beta(x)$ in $\mathbf{M}$ coincide with the zeros of function $x \to B(x, T_1(x))$ in $\mathbf{M} \cap \{|x| > 1\}$.

A particular feature of problem (5.1) is that its index $\Delta\arg_{\mathbf{M}} V(z)/Z(z)$ equals

$$2\pi - 2\pi \sum_{u_j \in \{|u|<1\} \cap \mathbf{M}} \mathbf{1}\big(B(u_j, T_1(u_j) = 0\big) - 2\pi \sum_{v_j \in \{|v|<1\}} \mathbf{1}\big(D(X_1(v_j), v_j = 0\big).$$

In order to apply standard methods of solution, the index should be reduced to 2π. This is achieved by introducing the function

$$\psi(x) = \sum_{i=1}^{k} \frac{\beta(x) - \beta(X_1(u_k))}{(x - X_1(u_k))\rho_{00}} a_i + \sum_{j=1}^{m} \frac{\beta(x) - \beta(X_1(T_1(v_j)))}{x - X_1(T_1(v_j))\rho_{00}} b_j. \tag{5.5}$$

Here, $u_1, \dots, u_k$ are the zeros of function $t \mapsto D(X_1(t), t)$ in the complex t-plane $\mathbf{C}$ $(= \mathbf{C}_t)$ inside the unit disc and $v_1, \dots, v_m$ the zeros of function $x \mapsto B(x, T_1(x))$ in the x-plane $\mathbf{C}$ $(= \mathbf{C}_x)$ inside the intersection of the unit disc and $\mathbf{M}^c$, the complement of $\mathbf{M}$ (cf. equation (5.4.16) from [5] where a similar function is written without coefficients a_i, b_j). As in (5.4.13) and (5.4.14) from [5], values $\beta(X_1(u_k))$ and $\beta(X_1(T)\mathbf{1}(v_j))$ can be found explicitly:

$$\begin{aligned} \frac{\beta(X_1(u_i))}{\rho_{00}} &= -\frac{A(X_1(u_i), u_i)}{B(X_1(u_i), u_i)}, \\ \frac{\beta(X_1(T_1(v_j)))}{\rho_{00}} &= \frac{U(v_k) D(X_1(Y_1(v_j)), T_1[X_1(T_1(v_j))])}{B(X_1(T_1(v_j)), T_1[X_1(T_1(v_j))])}. \end{aligned} \tag{5.6}$$

Coefficients a_i, b_j are now treated as unknowns: they are found from a system of linear equations which arises from the condition

$$\begin{aligned} &\sum_{i=1}^{k} \frac{a_i}{x - X_1(u_i)} + \sum_{j=1}^{m} \frac{b_j}{x - X_1(T_1(v_j))} \\ &\qquad = \left(\prod_{i=1}^{k} \big(x - X_1(u_i)\big) \prod_{j=1}^{m} \big(x - X_1(T_1(v_j))\big) \right)^{-1}. \end{aligned} \tag{5.7}$$

Any solution to this linear system (and there always exists one) determines $\psi(x)$ from (5.5). Then one writes for β/ρ_{00} the representation

$$\frac{\beta(x)}{\rho_{00}} = \psi(x) R_0(x) + R_1(x). \tag{5.8}$$

Here,

$$R_0(x) = \prod_{i=1}^{k} (x - X_1(u_i)) \prod_{j=1}^{m} \big(x - X_1(T_1(v_j))\big) \tag{5.9}$$

and

$$(5.10)\quad \begin{aligned} R_1(x) = &-\sum_{i=1}^{k} a_i \frac{A(X_1(u_i), u_i)}{B(X_1(u_i), u_i)} \frac{R_0(x)}{x - X_1(u_i)} \\ &+ \sum_{j=1}^{m} b_j \frac{U(v_k) D(X_1(Y_1(v_j)), T_1[X_1(T_1(v_j))])}{B(X_1(T_1(v_j)), T_1[X_1(T_1(v_j))])} \frac{R_0(x)}{x - X_1(T_1(v_j))}. \end{aligned}$$

The product ψZ [see (5.4)] is a function analytic in $\mathbf{M}$ and satisfying the equation

$$(5.11)\quad \begin{aligned} &\psi(z) Z(z) \frac{V(z) R_0(z)}{Z(z)} - \psi(\bar{z}) Z(\bar{z}) \frac{V(\bar{z}) R_0(\bar{z})}{Z(\bar{z})} \\ &\quad = U(z) - R_1(z) V(z) + R_1(\bar{z}) V(\bar{z}), \qquad z \in \partial\mathbf{M}. \end{aligned}$$

Equation (5.11) has a unique solution which allows us to determine the original ratio β/ρ_{00} in $\mathbf{M}$. The recipe for producing an explicit formula for the solution to (5.11) can be found in [5], pages 125–127 and (5.2.44).

Similarly, one finds the ratio δ/ρ_{00} in $\mathbf{M}$. Then we lift β/ρ_{00} and δ/ρ_{00} on a domain $\mathbf{O} \subset \mathbf{T}$ bounded by $\Sigma_x^{1,2}$ and $\Sigma_t^{1,2}$ and make the meromorphic continuation to the whole torus $\mathbf{T}$ described in Theorem 2.2. (After that we can go back to the complex plane and consider the meromorphic continuation of β/ρ_{00} and δ/ρ_{00} to the whole of $\mathbf{C}$ cut along $[x_3, x_4]$ or $[t_3, t_4]$, respectively.) Observe that in both cases the point 1 is reached by an analytic continuation.

The final task is to specify ρ_{00}. This is now easy; from (2.1) we find $\Pi(x,t)/\rho_{00}$ and then use the straightforward normalization relation

$$(5.12)\quad \Pi(1,1)/\rho_{00} + \beta(1)/\rho_{00} + \delta(1)/\rho_{00} + 1 = 1/\rho_{00}.$$

6. Applications to JS-queues. In the case of JS-queues we always have $E_1 < E_2$. Thus we have to consider only two cases, $E_2 > 0$ and $E_2 < 0$, corresponding to Figures 10(a) and 10(b). Functions B and D have the form

$$(6.1)\quad B(x,t) = \frac{\lambda + \lambda'}{2\lambda + \lambda' + 1} t + \frac{\lambda}{2\lambda + \lambda' + 1} x^2 + \frac{1}{2\lambda + \lambda' + 1} - x,$$

$$(6.2)\quad D(x,t) = \frac{2\lambda + \lambda'}{2\lambda + \lambda' + 2} xt + \frac{2}{2\lambda + \lambda' + 2} x - t,$$

and the points of branching are

$$(6.3)\quad \begin{aligned} x_{1,2,3,4} = \Big[&2 + 2\lambda + \lambda' \pm 2\sqrt{\lambda + \lambda'} \\ &\pm \big((2 + 2\lambda + \lambda')^2 \pm 4\sqrt{\lambda + \lambda'}(2 + 2\lambda + \lambda') + 4\lambda'\big)^{1/2}\Big]/(2\lambda), \end{aligned}$$

$$t_1 = 0,$$

$$(6.4)\quad t_{2,3} = \Big[(2 + 2\lambda + \lambda')^2/4 - (2\lambda + \lambda')$$

$$\pm\Big(\big((2+2\lambda+\lambda')^2/4-(2\lambda+\lambda')\big)^2 - 4\lambda(\lambda+\lambda')\Big)^{1/2}\Big]\big/\big(2\lambda(\lambda+\lambda')\big),$$

$$t_4 = \infty$$

with

$$T(x_i) = \frac{(2\lambda+\lambda'+2)x_i - \lambda x_3^2 - 1}{2(\lambda+\lambda')}, \tag{6.5}$$

$$X(t_i) = \frac{(2\lambda+\lambda'+2)t_i}{2(1+\lambda t_i)}, \qquad i = 1,2,3,4. \tag{6.6}$$

Recall that $X(t_3), T(x_3) > 1$ and $X(t_3) < x_3$, $T(x_3) < t_3$. For $0 < \gamma < \pi/4$ the saddlepoint $S(\gamma) = (x(\gamma), t(\gamma))$ is the unique solution to

$$\begin{aligned} &(\lambda x - 1/x)\,\mathrm{tg}\,\gamma = (\lambda+\lambda')t/x - x/t, \\ &(2\lambda+\lambda'+2)xt - (\lambda+\lambda')t^2 - \lambda x^2 t - t - x^2 = 0, \\ &X(t_3) < x(\gamma) < x_3, \qquad T(x_3) < t(\gamma) < t_3. \end{aligned} \tag{6.7}$$

Recall that $x(\gamma)$ strictly decreases and $t(\gamma)$ strictly increases with γ, with $x(0+) = x_3$, $t(0+) = T(x_3)$, $x(\pi/4-) = X(t_3)$, $t(\pi/4-) = t_3$. Unfortunately, we know no explicit formulas available for $x(\gamma), t(\gamma)$.

The solutions to (4.1) are

$$\sigma_1 = (1,1), \qquad s_0 = \left(\frac{1}{\lambda}, \frac{1}{\lambda}\right), \tag{6.8}$$

and to (4.2),

$$\sigma_1 = (1,1), \qquad \widetilde{s}_0 = \left(\frac{2}{2\lambda+\lambda'}, \left(\frac{2}{2\lambda+\lambda'}\right)^2\right), \tag{6.9}$$

and zeros s_0 and $\widetilde{s}_0$ are of the first order.

The lowest of all poles of β and γ can be either ξs_0 or $\eta\widetilde{s}_0$ and it must be of the first order. More precisely, it is a pole of β, if it occurs at ξs_0 and of δ, if it occurs at $\eta\widetilde{s}_0$. We can also calculate ξs_0 and $\eta\widetilde{s}_0$:

$$\begin{aligned} \xi s_0 &= \big(x(s_0), t(\xi s_0)\big) = \left(\frac{1}{\lambda}, \frac{1}{(\lambda+\lambda')\lambda}\right), \\ \eta\widetilde{s}_0 &= \big(x(\eta\widetilde{s}_0), t(\widetilde{s}_0)\big) \\ &= \left(\frac{2(4(\lambda+\lambda')+(2\lambda+\lambda')^2)}{(2\lambda+\lambda')(4\lambda+(2\lambda+\lambda')^2)}, \left(\frac{2}{2\lambda+\lambda'}\right)^2\right). \end{aligned} \tag{6.10}$$

The point ξs_0 corresponds to the angle γ with $\mathrm{tg}\,\gamma = E_2/E_1 = (\lambda+\lambda'-1)/(\lambda-1)$ by the homeomorphism $\gamma \to S(\gamma)$; see (6.7). If $E_2 < 0$, that is,

$\lambda + \lambda' - 1 < 0$, then $0 < \operatorname{tg}\gamma = E_2/E_1 < 1$ and (since both coordiantes of ξs_0 are greater than 1) $\xi s_0 \in (\sigma_4, \sigma_5)$. Thus, if $\operatorname{tg}\gamma < E_2/E_1$, $\xi s_0 \in \mathbf{E}_\gamma$ and ξs_0 dominates the contribution of the saddlepoint to the asymptotics of $\rho_{m+n,m}$. If $\operatorname{tg}\gamma > E_2/E_1$, $\xi s_0 \notin \mathbf{E}_\gamma$ and ξs_0 does not influence these asymptotics. See Figure 10(a). If $E_2 > 0$, the ergodicity condition (1.1) gives $E_1 < 0$ and $\operatorname{tg}\gamma = E_2/E_1 < 0$. Note that the angles corresponding to the points $(1, 1)$ and ξs_0 are different by plus or minus π, as they have the same tangent by (6.7). Thus these two points should belong to different spheres $\hat{\mathbf{S}}_x^i$, $i = 1, 2$, on $\mathbf{T}$. Therefore, as $(1, 1) \in \Gamma_1$, then $\xi s_0 \notin \mathbf{E}_\gamma$ for any $0 \le \gamma \le \pi/4$, and ξs_0 has no impact on the asymptotics of $\rho_{m+n,m}$. See Figure 10(b). This argument allows us to replace the inequalities

$$B\left(x(\gamma), \frac{p_{0-1}x(\gamma)^2}{p_{01}t(\gamma)}\right) \gtrless 0$$

figuring Cases 1–4 in Section 4 by the more transparent condition $\lambda + \lambda' \gtrless 1$. However, we cannot get rid of similar inequalities involving function D.

Theorems 4.1–4.4 for the JS-queues take the form of Theorems 6.1 and 6.2 below. Here, we use definitions (6.1)–(6.7), assuming condition (1.1). As before, we consider the limit $m, n \to \infty$, $(m + n)/m \to \operatorname{ctg}\gamma$, where $0 \le \gamma \le \pi/4$. We then analyze the cases $\gamma = 0$, $0 < \gamma < \pi/4$ and $\gamma = \pi/4$. More precisely, when $\lambda + \lambda' < 1$, we distinguish the cases $0 < \operatorname{tg}\gamma < (\lambda + \lambda' - 1)/(\lambda - 1)$ and $(\lambda + \lambda' - 1)/(\lambda - 1) < \operatorname{tg}\gamma < 1$. Constants $c_0(\gamma)$, $c_1(0)$ and $c_1(\pi/4)$ from (3.7)–(3.9) can be specified in terms of λ, λ' and meromorphic continuations of functions β and δ. A straightforward calculation gives that $A(s_0) > 0$, $A(\widetilde{s}_0) > 0$, $D(s_0) > 0$ and $B(\widetilde{s}_0) > 0$, owing to the ergodicity condition (1.3). Therefore, for the JS-queues nondegeneracy assumptions (4.4) and (4.5) can be omitted.

THEOREM 6.1. *Assume that*

$$D\left(\frac{(\lambda + \lambda')(t(\gamma))^2 + t(\gamma)}{(\lambda t(\gamma) + 1)x(\gamma)}, t(\gamma)\right) < 0.$$

(i) *Suppose that* $\lambda + \lambda' < 1$.
If $\gamma = 0$ *then*

$$\rho_{m+n,m} \sim \left[\operatorname{res}_{x(s_0)} \beta\right]\lambda^{n+1}. \tag{6.11}$$

If $0 < \operatorname{tg}\gamma < (\lambda + \lambda' - 1)/(\lambda - 1)$ *then*

$$\rho_{m+n,m} \sim \left[B\left(\frac{1}{\lambda}, \frac{1}{(\lambda + \lambda')\lambda}\right) \operatorname{res}_{x(s_0)} \beta\right]\lambda^n\left((\lambda + \lambda')\lambda\right)^m. \tag{6.12}$$

If $(\lambda + \lambda' - 1)/(\lambda - 1) < \operatorname{tg}\gamma < \pi/4$ *and the constant* $c_0(\gamma) \neq 0$ *in* (3.7), *that is*,

$$\begin{aligned} &\beta\big(x(\gamma), t(\gamma)\big)B\big(x(\gamma), t(\gamma)\big) \\ &\quad + \delta\big(x(\gamma), t(\gamma)\big)D\big(x(\gamma), t(\gamma)\big) + \rho_{00}A(x(\gamma)) \neq 0, \end{aligned} \tag{6.13}$$

then

$$\rho_{m+n,m} \sim \frac{c_0(\gamma)}{\sqrt{m}}(x(\gamma))^{-n}(t(\gamma))^{-m}. \tag{6.14}$$

Finally, if $\gamma = \pi/4$ *and* $c_1(\pi/4) \neq 0$ *in* (3.9) *then*

$$\rho_{m+n,m} \sim \frac{c_1(\pi/4)}{m\sqrt{m}} t_3^{-m}. \tag{6.15}$$

(ii) *Suppose that* $\lambda + \lambda' > 1$. *If for a given* $\gamma \in (0, \pi/4)$, (6.13) *holds then the asymptotics of* $\rho_{m+n,m}$ *is determined by* (6.14). *Finally, if* $\gamma = 0$ (*resp.* $\gamma = \pi/4$) *and the constant* $c_1(0) \neq 0$ *in* (3.8) [*resp.* $c_1(\pi/4) \neq 0$ *in* (3.9)] *then*

$$\rho_{m+n,m} \sim \frac{c_1(0)}{n\sqrt{n}} x_3^{-n} \qquad \left[\textit{resp. } \rho_{m+n,m} \sim \frac{c_1(\pi/4)}{m\sqrt{m}} t_3^{-m}\right]. \tag{6.16}$$

THEOREM 6.2. *Assume that*

$$D\left(\frac{(\lambda + \lambda')(t(\gamma))^2 + t(\gamma)}{(\lambda t(\gamma) + 1)x(\gamma)}, t(\gamma)\right) > 0.$$

(i) *Suppose that* $\lambda + \lambda' < 1$.
If $\gamma = 0$ *then*

$$\begin{aligned} \rho_{m+n,m} &\sim [\operatorname{res}_{x(s_0)} \beta]\lambda^{n-1} \\ &\quad + [\operatorname{res}_{x(\eta\widetilde{s}_0)} \beta]\left(\frac{(2\lambda + \lambda')(4\lambda + (2\lambda + \lambda')^2)}{2(4(\lambda + \lambda') + (2\lambda + \lambda')^2)}\right)^{n-1}. \end{aligned} \tag{6.17}$$

If $0 < \operatorname{tg}\gamma \le (\lambda + \lambda' - 1)/(\lambda - 1)$ *then*

$$\begin{aligned} \rho_{m+n,m} &\sim \left[B\left(\frac{1}{\lambda}, \frac{1}{(\lambda + \lambda')\lambda}\right) \operatorname{res}_{x(s_0)} \beta\right]\lambda^n((\lambda + \lambda')\lambda)^m \\ &\quad + \left[D\left(\frac{2(4(\lambda + \lambda') + (2\lambda + \lambda')^2)}{(2\lambda + \lambda')(4\lambda + (2\lambda + \lambda')^2)}, \left(\frac{2}{2\lambda + \lambda'}\right)^2\right) \operatorname{res}_{t(\widetilde{s}_0)} \delta\right] \\ &\quad \times \left(\frac{(2\lambda + \lambda')(4\lambda + (2\lambda + \lambda')^2)}{2(4(\lambda + \lambda') + (2\lambda + \lambda')^2)}\right)^n \left(\frac{2\lambda + \lambda'}{2}\right)^{2m}. \end{aligned} \tag{6.18}$$

If $(\lambda + \lambda' - 1)/(\lambda - 1) \le \operatorname{tg}\gamma < 1$ *then*

$$\begin{aligned} \rho_{m+n,m} &\sim \left[D\left(\frac{2(4(\lambda + \lambda') + (2\lambda + \lambda')^2)}{(2\lambda + \lambda')(4\lambda + (2\lambda + \lambda')^2)}, \left(\frac{2}{2\lambda + \lambda'}\right)^2\right) \operatorname{res}_{t(\widetilde{s}_0)} \delta\right] \\ &\quad \times \left(\frac{(2\lambda + \lambda')(4\lambda + (2\lambda + \lambda')^2)}{2(4(\lambda + \lambda') + (2\lambda + \lambda')^2)}\right)^n \left(\frac{2\lambda + \lambda'}{2}\right)^{2m}. \end{aligned} \tag{6.19}$$

Finally if $\gamma = \pi/4$ *then*

$$\rho_{m+n,m} \sim [\mathrm{res}_{t(\widetilde{s}_0)}\,\delta]\left(\frac{2\lambda+\lambda'}{2}\right)^{2(m-1)}. \tag{6.20}$$

(ii) *Suppose that* $\lambda+\lambda' > 1$. *If* $\gamma = 0$ *then*

$$\rho_{m+n,m} \sim [\mathrm{res}_{x(\eta\widetilde{s}_0)}\,\beta]\left(\frac{(2\lambda+\lambda')(4\lambda+(2\lambda+\lambda')^2)}{2(4(\lambda+\lambda')+(2\lambda+\lambda')^2)}\right)^{n-1}. \tag{6.21}$$

If $0 < \gamma < \pi/4$ *then the asymptotics of* $\rho_{m+n,n}$ *is determined by* (6.19) *and if* $\gamma = \pi/4$ *by* (6.20).

It is not difficult to find the angles γ_0 and $\widetilde{\gamma}_0$. Namely, we have $\gamma_0 = \mathrm{arctg}\,(\lambda+\lambda'-1)/(\lambda-1)$ if $\lambda+\lambda' < 1$ and $\gamma_0 = 0$ if $\lambda+\lambda' > 1$.

Furthermore, if $D(X(t_3),t_3) < 0$ [or $D(X(t_3),t_3) = 0$, $D'_t(X_1(t),t)|_{t=t_3-} > 0$] then $\eta\widetilde{s}_0 \notin \widetilde{\mathbf{E}}_\gamma$ for all $\gamma \in [0,\pi/4]$ and we put $\widetilde{\gamma}_0 = \pi/4$. Otherwise, that is, if $D(X(t_3),t_3) > 0$ (or $D(X(t_3),t_3) = 0$, $D'_t(X_1(t),t)|_{t=t_3-} < 0$), $\eta\widetilde{s}_0 \in \widetilde{\mathbf{E}}_\gamma$ for some $\gamma \in [0,\pi/4]$. To find the angle $\widetilde{\gamma}_0^- \in (-3\pi/4,\pi/4)$ corresponding to this pole, we use (6.7),

$$\mathrm{tg}\,\widetilde{\gamma}_0^- = \frac{4(\lambda+\lambda')(4\lambda+(2\lambda+\lambda')^2)^2 - (2\lambda+\lambda')^2(4(\lambda+\lambda')+(2\lambda+\lambda')^2)^2}{4\lambda(4(\lambda+\lambda')+(2\lambda+\lambda')^2)^2 - (2\lambda+\lambda')^2(4\lambda+(2\lambda+\lambda')^2)^2}.$$

If $\widetilde{\gamma}_0^- < 0$ then $\eta\widetilde{s}_0$ lies in $\widetilde{\mathbf{E}}_\gamma \cap \Phi_0$ and contributes to the asymptotics of $\rho_{m+n,m}$ for all $\gamma \in [0,\pi/4]$. Otherwise, this is true only for $\gamma \ge \widetilde{\gamma}_0^-$. Hence, we have $\widetilde{\gamma}_0 = \max\{0,\widetilde{\gamma}_0^-\}$.

COROLLARY 6.1. (i) *For* $\gamma \in [0,\min\{\gamma_0,\widetilde{\gamma}_0\})$, *the asymptotics of* $\rho_{m+n,m}$ *is determined by the pole of* β; *see* (6.11) *and* (6.12).

(ii) *For* $\gamma \in (\max\{\gamma_0,\widetilde{\gamma}_0\},\pi/4]$, *the asymptotics of* $\rho_{m+n,m}$ *is determined by the pole* δ; *see* (6.19) *and* (6.20). *Furthermore, if* $\gamma_0 = \widetilde{\gamma}_0 = 0$ *and* $\widetilde{\gamma}_0^- < 0$, *then* (6.21) *holds true for* $\gamma = \gamma_0 = \widetilde{\gamma}_0$.

(iiia) *If* $\widetilde{\gamma}_0 < \gamma_0$ *then for* $\gamma \in (\widetilde{\gamma}_0,\gamma_0)$ *the asymptotics of* $\rho_{m+n,m}$ *is determined by the lowest of two poles of* β *and* δ; *see* (6.18). *Furthermore, if* $\widetilde{\gamma}_0 = 0$ *and* $\widetilde{\gamma}_0^- < 0$, *then* (6.17) *holds true for* $\gamma = \widetilde{\gamma}_0$.

(iiib) *If* $\gamma_0 < \widetilde{\gamma}$ *then for* $\gamma \in (\gamma_0,\widetilde{\gamma}_0)$ *the asymptotics of* $\rho_{m+n,m}$ *is determined by the saddlepoint*; *see* (6.14). *Furthermore, if* $\gamma_0 = 0$ *or* $\widetilde{\gamma}_0 = \pi/4$, *then* (6.15) *and* (6.16) *hold true for* $\gamma = \gamma_0$ *or* $\gamma = \widetilde{\gamma}_0$, *respectively.*

REMARK 6.1 (Discussion of border cases $\lambda' = 0$ and $\lambda = 0$). If $\lambda' = 0$, then $E_1 = E_2$. This is the case of Figure 3(e) [see Proposition 2.1(e)], which can be studied similarly to the cases of Figures 3(a)–(c). Then the queuing system consists of two independent $M/M/1$ servers and the stationary probabilities $\rho_{m+n,m}$ are known explicitly. The formal substitution $\lambda' = 0$ into Theorems 6.1 and 6.2

gives the correct result: $\rho_{m+n,m} \sim \text{const}\,\lambda^{2m+n}$ [in fact $\rho_{m+n,m} = (1-\lambda^2)\lambda^{2m+n}$]. Furthermore, in this case $\gamma_0 = \text{arctg}\,1 = \pi/4$ and since $\xi s_0 = \eta\widetilde{s}_0$, we have that $\widetilde{\gamma}_0 = \pi/4$. So we are in case (i) of Corollary 6.1: the asymptotics here is determined by the pole ξs_0 of β for all $\gamma \in [0, \pi/4]$ and $\lambda \in (0, 1)$.

On the other hand, if $\lambda = 0$, functions $X_i(t)$ and $T_i(x)$, $i = 1, 2$, have only two branching points, $x_2 = (\lambda' + 2 - 2\sqrt{\lambda'})^{-1}$, $x_3 = (\lambda' + 2 + 2\sqrt{\lambda'})^{-1}$, $t_2 = 0$, $t_3 = 4(\lambda'^2 + 4)^{-1}$. The Riemannian surface is homeomorphic to a sphere, not a torus. Thus, our analysis has to be modified. This case was studied in detail in [8] where the functions $\beta(s)$ and $\delta(s)$ were found explicitly. In turn, this leads to explicit asymptotics of stationary probabilities $\rho_{m+n,m}$.

If we substitute $\lambda = 0$ directly into Theorems 6.1 and 6.2 and Corollary 6.1, we again obtain correct asymptotics. Namely, we have $\gamma_0 = 0$ if $\lambda' > 1$ and $\gamma_0 = \text{arctg}\,(1-\lambda')$ if $\lambda' \le 1$. One can readily check that for $\lambda = 0$, $D(X(t_3), t_3) > 0$ and $\text{tg}\,\widetilde{\gamma}_0^- = ((4+\lambda')^2 - 4\lambda')/\lambda'^2 > 1$. Therefore, $\widetilde{\gamma}_0^- \in (-3\pi/4, -\pi/2)$ and $\widetilde{\gamma}_0 = 0$. This means that we are in case (ii) or (iiia) of Corollary 6.1. Thus, owing to Lemma 3.5, the assumptions of Theorem 6.2 hold $\forall\,\gamma \in [0, \pi/4]$ and $\lambda' \in (0, 2)$. By Corollary 6.1, the asymptotics of $\rho_{m+n,m}$ is determined by the pole $\eta\widetilde{s}_0$ of $\delta(s)$,

$$\rho_{m+n,m} \sim \text{const}\left(\frac{\lambda'^2}{2(4+\lambda')}\right)^n \left(\frac{\lambda'}{2}\right)^{2m} \tag{6.22}$$

$\forall\,\gamma \in [0, \pi/4]$ and $\lambda' \in (0, 2)$. This is precisely the answer given in Section 3 of [8], if one takes $i = m$, $j = m+n$ and $\alpha = 1/\lambda'$. (As was noted before, the formulas from [8] in addition specify the constant in (6.22).)

We conclude this section by showing how to specify, in the case where $\lambda, \lambda' > 0$, the constants in front of geometrically decreasing terms in Theorems 6.1 and 6.2. In this discussion we will repeat, for the case of JS-queues, some parts of the argument given in Section 5 for a general SHC random walk. Observe that when $\lambda + \lambda' > 1$, the function $B(x, T_1(x))$ does not have zeros with $|x| > 1$. Otherwise, its unique zero is $x = 1/\lambda$ where $T_1(x) = 1/\lambda$. It is a trivial computation then to check that $1/\lambda \notin \mathbf{M}$ as $1/\lambda \notin [X(t_1), X(t_2)]$. In fact, $X(t_1) = 0$, $X(t_2) = (2\lambda + \lambda' + 2)t_2/(2(1 + \lambda t_2) > 1/\lambda$ due to the ergodicity condition (1.1) and the fact that $t_2 \le 1$. It follows that β is analytic in $\mathbf{M}$ and $Z(x) \equiv 1$, $x \in \mathbf{C}$. Furthermore, the zeros of functions B and D on $\mathbf{S}$ are at $(1, 1)$, $(1/\lambda, 1/\lambda)$, $(2/(2\lambda + \lambda'), 4/(2\lambda + \lambda')^2)$. All these points are outside the open unit disc again by (1.1). Hence, the index of problem (5.1) equals 2π and function β/ρ_{00} can be determined directly from (5.1) without introducing function ψ and performing the index reduction.

The situation with function δ/ρ_{00} is only slightly different. Function $D(X_1(t), t)$ has a unique zero outisde the unit disc, at $t = 4/(2\lambda + \lambda')^2$. Under the condition $T(x_1) < 4/(2\lambda + \lambda')^2 < T(x_2)$ [see (6.5) and (6.6)], this zero belongs to $\mathbf{M}$; otherwise it doesnot. In the first case, $Z(x) = x - 4/(2\lambda + \lambda')^2$, in the second $Z(x) \equiv 1$,

$x \in \mathbf{C}$. In the first case, the original equation for δ/ρ_{00} should be multiplied by Z and solved for $\delta Z/\rho_{00}$. As before, the index of problem (5.1) equals 2π and we do not need function ψ.

So, the explicit formulas for β/ρ_{00} and δ/ρ_{00} (or $\delta Z/\rho_{00}$) can be obtained from the solution to general problem (5.2) by substituting the concrete form of functions $A(X_2(T_1(x)), T_1(x))$, $A(x, T_1(x))$, $B(x, T_1(x))$, $D(x, T_1(x))$ and $D(X_2(T_1(x)), T_1(x))$ [see (6.1), (6.2) and (2.4)], with jump probabilities specified in Section 1.2 in terms of λ, λ'. The resulting expressions are still cumbersome, but simpler than in a general SHC case as function ψ is not needed. It means that the recipe provided in [5], equations (5.2.44) and arguments on pages 125–127, is applicable without a change.

Summarizing, the algorithm of finding constants $c_0(\gamma)$ in (6.14), $c_1(0)$ and $c_1(\pi/4)$ in (6.15), (6.16), $\mathrm{res}_{x(s_0)}\beta$ in (6.11), (6.12), (6.17) and (6.18), $\mathrm{res}_{x(\eta\tilde{s}_0)}\beta$ in (6.17), (6.21) and $\mathrm{res}_{t(\tilde{s}_0)}\delta$ in (6.18), (6.19), (6.20) is as follows. (i) Calculate functions V and U in (5.3) by using formulas (6.1), (6.2) and (2.5), with the jump probabilities specified in terms of λ, λ'. (ii) Solve problem (5.2) in the interior domain $\mathbf{M}$ of the contour $\partial\mathbf{M}$ [see (5.1)], taking into account simplifications indicated in the previous paragraph. (iii) "Lift" domain $\mathbf{M} \subset \mathbf{C}$ to the Riemannian surface $\mathbf{T}$. Obtain the domain $\mathbf{O} \subset \mathbf{T}$ bounded by cycles Σ_x^{12} and Σ_t^{12}. Substitute $\beta(s) := \beta(x(s))$, $\delta(s) = \delta(\eta(s))$ for $s \in \mathbf{O}$. (iv) By using Galois automorphisms ξ and η (2.8), prolong β/ρ_{00}, δ/ρ_{00} meromophically to the whole of $\mathbf{T}$; it will require finitely many steps. (This procedure is described in detail in Theorem 2.2.) (v) Calculate ρ_{00} from (5.12). (vi) Calculate residues $\mathrm{res}_{x(s_0)}\beta$ $\mathrm{res}_{x(\eta\tilde{s}_0)}\beta$ $\mathrm{res}_{t(\tilde{s}_0)}\delta$ from the meromorphic continuations constructed in (iv). (vii) Calculate constants $c_0(\gamma)$ by substituting values of $\beta(S(\gamma))$ and $\delta(S(\gamma))$ with $x(\gamma), t(\gamma))$ found from (6.7), (3.5), (2.3b) into (3.7). (viii) Calculate constants $c_1(0)$, $c_1(\pi/4)$ by substituting values $\delta(x_3, T(x_3))$ and $\beta(X(t_3), t_3)$ into (3.8) and (3.9).

7. The ergodicity criterion.

PROOF OF THEOREM 1.1. To prove the positive recurrency, we use Foster's criterion (Theorem 2.2.3 from [6]). Accordingly, $\mathcal{L}$ is positive recurrent iff there exists a positive function $f(x, y)$ on $\mathbf{Z}_+^2$, a number $\varepsilon_0 > 0$ and a finite set $\mathbf{A} \subset \mathbf{Z}_+^2$ such that

$$\mathbb{E}f(x + \theta_x, y + \theta_y) - f(x, y) \le -\varepsilon_0 \qquad \text{for all } (x, y) \in \mathbf{Z}_+^2 \setminus \mathbf{A}, \tag{7.1}$$

where (θ_x, θ_y) is a random vector distributed as the jump of the RW from the state (x, y) and $\mathbb{E}$ stands for the corresponding expectation.

To start with, consider the first pair of inequalities in (1.3) with $E_2 < 0$. Set $f(x, y) := \sqrt{ux^2 + vy^2 + wxy}$ for $x \ge y$ and $f(x, y) := f(y, x)$ for $x \le y$. The coefficients u, v, w are chosen in the following way. First, we choose $u > 0$ and w such that

$$2uE_1 + wE_2 < 0, \qquad 2uE_1^b + wE_2^b < 0.$$

(Note that this is possible under condition $E_1E_2^b - E_2E_1^b < 0$.) Then we fix $v > w^2/(4u)$ such that $(2u+w)E_1 + (2v+w)E_2 < 0$ and $(2u+w)d_{01} - (2v+w)d_{0-1} < 0$. By Lemma 3.3.3 from [6], because $f(x,y)$ is defined as the square root of a positive quadratic form in a neighborhood of (x,y), we have

$$\begin{aligned}&\mathbb{E}f(x+\theta_x, y+\theta_y) - f(x,y)\\ &\qquad = \frac{(2uE\theta_x + wE\theta_y)x + (2vE\theta_y + wE\theta_x)y}{2f(x,y)} + o(1)\end{aligned} \tag{7.2}$$

for $x \neq y$, as $x^2+y^2 \to \infty$. Furthermore, it is elementary to show that for $x = y$,

$$\begin{aligned}&\mathbb{E}f(x+\theta_x, x+\theta_y) - f(x,x)\\ &\qquad = \frac{(2ud_{01} - wd_{0-1})x + (-2vd_{0-1} + wd_{01})x}{2f(x,x)} + o(1)\end{aligned} \tag{7.3}$$

as $x \to \infty$. So property (7.1) is directly verified. The level curves of f look as in Figure 11(a).

The construction is more involved in the case of the second pair of inequalities (1.3) with $E_2 \geq 0$. Here we first fix numbers u, v, w where $u > 0$, $-2u < w < 0$, $v > w^2/(4u)$, so that

$$(2u+w)E_1 + (2v+w)E_2 < 0, \qquad (2u+w)d_{01} - (2v+w)d_{0-1} < 0.$$

(Such a choice is possible under the condition $E_1d_{0-1} + E_2d_{01} < 0$.) Next, we draw a level curve L_1 defined by $ux^2 + vy^2 + wxy \equiv 1$ in the domain $\{(x,y) \in \mathbf{R}_+^2 : y \leq x \leq -2uy/w\}$. At the end point $x/y = -2u/w$, the tangent to this curve is parallel to the y-axis. Let us continue L_1 *smoothly* to the domain $\{(x,y) \in \mathbf{R}_+^2 : x/y > -2u/w\}$, until the x-axis, by a *convex* curve L_2, which will be tangent to the x-axis at the end. (L_2 may be chosen, e.g., as a quarter of a circle.)

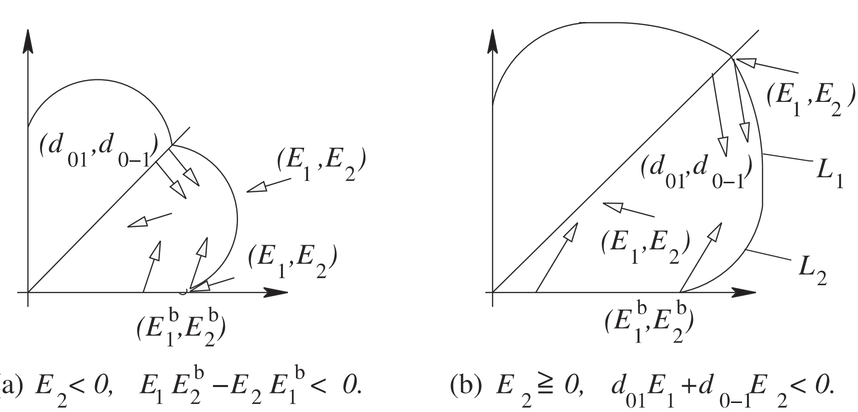


FIG. 11. *Level curves of Lyapunov functions*: *positive recurrence*.

See Figure 11(b) . Set $L_1 \cup L_2 = L$. Then $\forall r > 0$ define

$$f(rx, ry) = r \qquad \text{if } x \geq y, \tag{7.4}$$
$$f(rx, ry) = r \qquad \text{if } x \leq y, (y, x) \in L.$$

Hence, for some $\varepsilon > 0$ and any $(x, y) \in \mathbf{Z}_+^2$ with $x^2 + y^2$ large enough, if $x \neq y$, then

$$f(x + \mathbb{E}\theta_x, y + \mathbb{E}\theta_y) - f(x, y) < -5\varepsilon. \tag{7.5}$$

Had $f(x, y)$ been linear, the proof would have been complete. Although this is not the case, $f(x, y)$ satisfies the so-called principle of local linearity (cf. [6], page 46): for any $(x, y) \in \mathbf{R}_+^2$ with $|x - y| > 1$ and $x^2 + y^2$ large enough,

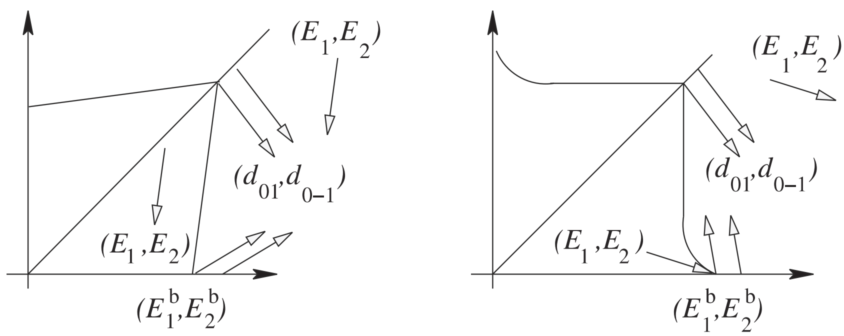
$$\inf_{\phi} \sup_{\|(\tilde{x}, \tilde{y}) - (x, y)\| \leq 1} |f(\tilde{x}, \tilde{y}) - \phi(\tilde{x}, \tilde{y})| < \varepsilon. \tag{7.6}$$

Here the infimum is taken over all linear functions ϕ on $\mathbf{R}^2$. [Obviously, the optimal ϕ has at (x, y) the level curves tangent to those of f and $\phi(x, y) = f(x, y)$.] Then by (7.5) and Lemmas 3.3.4, 3.3.5 from [6] Foster's criterion applies for all sufficiently large (x, y), except for $|x - y| \leq 1$. But if $|x - y| \leq 1$, (7.2) and (7.3) are valid and Foster's criterion holds by the choice of the coefficients u, v, w.

To show that the Markov chain is not positive recurrent, we apply Theorem 2.2.6 from [6]. Namely, for $\mathcal{L}$ to be not positive recurrent, it suffices to find a function $f(x, y)$ on $\mathbf{Z}_+^2$ and a constant $C > 0$ such that

$$\mathbb{E}f(x + \theta_x, y + \theta_y) - f(x, y) \geq 0 \qquad \text{for all } (x, y) \in \{(x, y) : f(x, y) > C\} \tag{7.7}$$

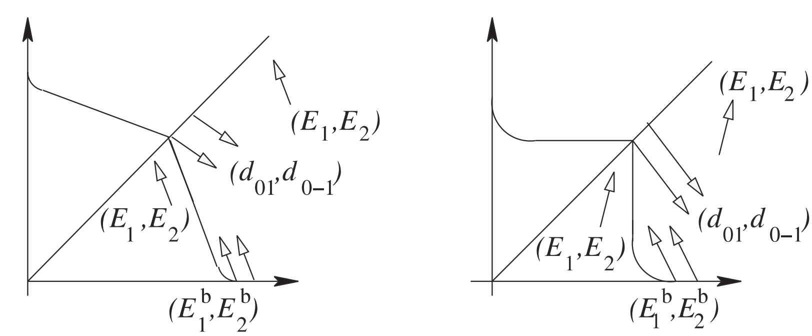
and the sets $\{(x, y) : f(x, y) > C\}$ and $\{(x, y) : f(x, y) \leq C\}$ are both not empty. If $E_2 < 0$, $E_1 E_2^b - E_2 E_1^b \geq 0$ and $E_1 < 0$, we set $f(x, y) = -E_2 x + E_1 y$ for $x \geq y$ and $f(x, y) = -E_2 y + E_1 x$ for $x \leq y$. See Figure 12A(a). In other cases we use construction (7.4) and the curve L in $\{(x, y) \in \mathbf{R}_2^+ : x \geq y\}$ is drawn as follows. In the case $E_2 < 0$, $E_1 E_2^b - E_2 E_1^b \geq 0$ and $E_1 \geq 0$, we continue the line $x \equiv 1$ from some point (say $y = 1/2$) down to the x-axis smoothly by a concave curve which will have the tangent vector between (E_1, E_2) and (E_1^b, E_2^b) at the end. See Figure 12A(b). In the case $E_2 \geq 0$, $d_{01} E_2 + d_{0-1} E_1 \geq 0$ and $E_1 < 0$, we continue smoothly the line $E_2 x - E_1 y \equiv 1$ from some point down to the x-axis by a concave curve tangent to the x-axis at the end. See Figure 12B(a). Finally, if $E_2 \geq 0$, $d_{01} E_2 + d_{0-1} E_1 \geq 0$ and $E_1 \geq 0$, we smooth line $x = 1$ from some point down to the x-axis, by a concave curve that is tangent to the x-axis at the meeting point. See Figure 12B(b). In all these cases we have the inverse inequality to (7.5) and the principle of local linearity (7.6) for $|x - y| > 1$ is valid, which implies (7.7). For $|x - y| \leq 1$, (7.7) is verified explicitly. This completes the proof of Theorem 1.1. $\square$



(a) $E_2 < 0$, $E_1E_2^b - E_2E_1^b \geqq 0$, $E_1 < 0$.

(b) $E_2 < 0$, $d_{01}E_2 + d_{0-1}E_2 \geqq 0$, $E_1 \geqq 0$.

FIG. 12A. *Level curves of Lyapunov functions*: *no positive recurrence*.



(a) $E_2 \geqq 0$, $d_{01}E_1 + d_{0-1}E_2 \geqq 0$, $E_1 < 0$.

(b) $E_2 \geqq 0$, $d_{01}E_1 + d_{0-1}E_2 \geqq 0$, $E_1 \geqq 0$.

FIG. 12B. *Level curves of Lyapunov functions*: *no positive recurrence*.

APPENDIX

The following theorem is straightforward. (It was suggested to one of the authors by Malyshev.)

THEOREM A.1. *Let h be a complex function analytic at $z = 0$. Suppose that in a neighborhood of* 0, $h(z) = \sum_{n=0}^{\infty} h_n z^n$, *where coefficients* $h_n \geq 0$. *Let* $x_0 > 0$ *be the first singular point of* h. *Assume that in the neighborhood of* x_0,

$$h(z) = \sum_{i=1}^{d} g_i(z)\left(1 - \frac{z}{x_0}\right)^{\vartheta_i} + g_0(z),$$

where g_i, $i = 0, 1, \dots, d$, *are analytic functions not vanishing at* $z = x_0$, *and* $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_d$ *are rational but* $\vartheta_i \notin \{0, 1, 2, \dots\}$. *Then*

$$h_n \sim \frac{g_1(x_0)}{\Gamma(-\vartheta_1)} n^{-\vartheta_1 - 1} x_0^{-n}.$$

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