

WEAK LAWS OF LARGE NUMBERS IN GEOMETRIC PROBABILITY

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Using a coupling argument, we establish a general weak law of large numbers for functionals of binomial point processes in d -dimensional space, with a limit that depends explicitly on the (possibly nonuniform) density of the point process. The general result is applied to the minimal spanning tree, the k -nearest neighbors graph, the Voronoi graph and the sphere of influence graph. Functionals of interest include total edge length with arbitrary weighting, number of vertices of specified degree and number of components. We also obtain weak laws of large numbers for functionals of marked point processes, including statistics of Boolean models.

1. Introduction. Establishing laws of large numbers (LLN) for functionals of random Euclidean point sets is of considerable interest. When the point set forms the vertex set of a graph, functionals of interest include total edge length with arbitrary weighting, number of edges and number of components. Relevant graphs include those in computational geometry, such as the minimal spanning tree, k -nearest neighbors graph, Voronoi graph and sphere of influence graph (these graphs are formally defined in Section 2.3). When the random Euclidean point set is a marked point set, then functionals of interest include those arising in the stochastic geometry of Boolean models.

For many functionals, subadditivity works well as a basic tool; see [27] and [29] for surveys. This is the case with power-weighted edge length functionals. For example, if $G(n, f)$ is the minimal spanning tree (MST) on n i.i.d. random d -vectors with common density f on \mathbb{R}^d , and if $|e|$ denotes the length of the edge e , then the following asymptotics hold for the sum $\sum_{e \in G(n, f)} |e|^\alpha$ of the power-weighted edge lengths:

THEOREM 1.1. *If either (i) $1 \leq \alpha < d$, $\int_{\mathbb{R}^d} f(x)^{(d-\alpha)/d} dx < \infty$, and $\int_{\mathbb{R}^d} |x|^r f(x) dx < \infty$ for some $r > d/(d - \alpha)$ or (ii) $\alpha \geq d$ and f has support on $[0, 1]^d$, and is bounded away from zero, then as $n \rightarrow \infty$,*

$$(1.1) \quad n^{(\alpha-d)/d} \sum_{e \in G(n, f)} |e|^\alpha \rightarrow C(\alpha, d) \int_{\mathbb{R}^d} f(x)^{(d-\alpha)/d} dx \quad a.s.,$$

where $C(\alpha, d)$ is a positive constant.

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The two cases (i) and (ii) are proved in [29] and [30], respectively.

However, many functionals are not amenable to subadditive methods. For example, if the edge lengths $|e|$ in a Euclidean graph G are weighted by a general function ϕ , giving a sum of the form $\sum_{e \in G} \phi(|e|)$, then subadditive methods break down, and the LLN behavior is much less well understood. Functionals involving the Voronoi, Delaunay and sphere of influence graphs are also generally not amenable to subadditive methods. Functionals of marked point processes are usually not subadditive either. Moreover, even when subadditive methods are applicable, they provide little information on the numerical values of limiting constants such as $C(\alpha, d)$ appearing in (1.1).

An alternative approach is the so-called “objective method.” Steele [27] coined this term for a philosophy whereby, loosely speaking, one uses the locally Poisson nature of a binomial point process to describe the limiting behavior of functionals on finite point sets in terms of related functionals defined on infinite Poisson point sets. Aldous and Steele [1] used this idea to analyze certain functionals associated with the MST on uniform points, but one might expect it to be applicable to any functional, including those defined over nonuniform point samples, consisting of contributions which are locally determined in some sense. As noted in [1, 27], making formal sense of this intuition is not always quite so simple as one might imagine.

In an attempt to formulate in general terms the idea of locally determined contributions, Penrose and Yukich [23, 24] introduce a concept of “stabilizing” functionals and essentially use the objective method to establish a strong law of large numbers (Theorem 3.2 of [24]) for stabilizing functionals on uniform point sets in \mathbb{R}^d . Jimenez and Yukich [15] obtain sufficient conditions yielding laws of large numbers for sums involving general edge weights and nonuniform point sets, but their conditions are rather strong and are limited to increasing weight functions ϕ .

The goal of this paper is to use the objective method to provide relatively simple conditions guaranteeing a general weak LLN for stabilizing functionals on possibly nonuniform point samples of size n . We illustrate the diverse applications of the general LLN by obtaining weak laws for functionals of spatial point processes in computational geometry as well as functionals of marked point processes, including those arising in packing processes and the stochastic geometry of Boolean models.

Many stabilizing functionals are defined in terms of graphs G , which are themselves stabilizing, that is, locally determined in a sense to be made precise below; stabilizing graphs include the MST, k -nearest neighbor, Voronoi and sphere of influence graph. Given a stabilizing graph G , the theory applies to functionals such as the number of leaves, the number of components and the sum of weighted edge lengths $\sum_{e \in G} \phi(|e|)$. In these graphs, edges are between “nearby” points, and since the density of points grows in proportion to n , the typical distance between nearby points can be thought of as decreasing in proportion to $n^{-1/d}$. Therefore

we consider sums of the form $\sum_{e \in G} \phi(n^{1/d}|e|)$, establishing weak LLN behavior. The limiting constants, such as those arising in (1.1), are defined explicitly in terms of the weight ϕ , the density f and certain graphs on Poisson processes, thereby providing extra information that is not given by subadditive methods alone, even in the classic case when ϕ is the identity function.

2. Main results.

2.1. *Terminology.* In Section 2.2 we shall formulate a collection of general LLN results. Before doing so we need some terminology.

Given $\mathcal{X} \subset \mathbb{R}^d$ and a positive scalar a , let $a\mathcal{X} := \{ax : x \in \mathcal{X}\}$. Given $y \in \mathbb{R}^d$, set $y + \mathcal{X} := \{y + x : x \in \mathcal{X}\}$. For $x \in \mathbb{R}^d$, let $|x|$ be its Euclidean modulus and for $r > 0$, let $B(x; r)$ denote the Euclidean ball $\{y \in \mathbb{R}^d : |y - x| \leq r\}$. Let $\mathbf{0}$ denote the origin of \mathbb{R}^d .

Suppose $\xi(x; \mathcal{X})$ is a measurable \mathbb{R}^+ -valued function defined for all pairs (x, \mathcal{X}) , where $\mathcal{X} \subset \mathbb{R}^d$ is finite and x is an element of \mathcal{X} . Suppose ξ is translation invariant; that is, $\xi(y + x; y + \mathcal{X}) = \xi(y; \mathcal{X})$ for all $y \in \mathbb{R}^d$ and all x, \mathcal{X} . Then ξ induces a translation-invariant functional H_ξ defined on finite point sets $\mathcal{X} \subset \mathbb{R}^d$ by

$$(2.1) \quad H_\xi(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x; \mathcal{X}).$$

Functionals that admit the representation (2.1) include the total edge length, the total number of edges, the total number of components and the total number of vertices of fixed degree of Euclidean graphs. In cases with $x \notin \mathcal{X}$, we abbreviate the notation $\xi(x; \mathcal{X} \cup \{x\})$ to $\xi(x; \mathcal{X})$.

We probe the behavior of the functional H_ξ by inserting an extra point into its domain. For “typical” point sets \mathcal{X} , it is conceivable that the contribution $\xi(x; \mathcal{X})$ is not affected by changes in \mathcal{X} which are far from x . We formalize this notion as follows. A set $\mathcal{S} \subset \mathbb{R}^d$ is *locally finite* if it contains only finitely many points in any bounded region. For any locally finite point set $\mathcal{S} \subset \mathbb{R}^d$ and any integer $m \in \mathbb{N}$, define

$$(2.2) \quad \bar{\xi}(\mathcal{S}; m) := \sup_{n \in \mathbb{N}} \left(\operatorname{ess\,sup}_{m,n} \{ \xi(\mathbf{0}; (\mathcal{S} \cap B(\mathbf{0}; m)) \cup \mathcal{A}) \} \right)$$

and

$$(2.3) \quad \underline{\xi}(\mathcal{S}; m) := \inf_{n \in \mathbb{N}} \left(\operatorname{ess\,inf}_{m,n} \{ \xi(\mathbf{0}; (\mathcal{S} \cap B(\mathbf{0}; m)) \cup \mathcal{A}) \} \right),$$

where $\operatorname{ess\,sup}_{m,n}$ (respectively, $\operatorname{ess\,inf}_{m,n}$) is essential supremum (infimum) with respect to Lebesgue measure on \mathbb{R}^{dn} , over sets $\mathcal{A} \subset \mathbb{R}^d \setminus B(\mathbf{0}; m)$ of cardinality n . Define $\xi_\infty(\mathcal{S})$, called the *limit* of ξ on \mathcal{S} , by

$$\xi_\infty(\mathcal{S}) := \limsup_{m \rightarrow \infty} \bar{\xi}(\mathcal{S}; m).$$

We shall say the functional ξ *stabilizes* on \mathcal{S} if

$$(2.4) \quad \lim_{m \rightarrow \infty} \bar{\xi}(\mathcal{S}; m) = \lim_{m \rightarrow \infty} \underline{\xi}(\mathcal{S}; m) = \xi_\infty(\mathcal{S}).$$

For $\tau \in (0, \infty)$, let \mathcal{P}_τ be a homogeneous Poisson point process of intensity τ on \mathbb{R}^d . We are interested particularly in functionals that stabilize almost surely on \mathcal{P}_τ . Note that with probability 1, $\bar{\xi}(\mathcal{P}_\tau; m)$ is nonincreasing in m and $\underline{\xi}(\mathcal{P}_\tau; m)$ is nondecreasing in m , so they both converge. Stabilization means they converge to the same limit almost surely. The present formulation of stabilization is weaker than that of [24]. Any functional $\xi(x; \mathcal{X})$ which depends only on the points of \mathcal{X} within a fixed distance of x is stabilizing on \mathcal{P}_τ .

We are interested in functionals on spatial point processes that involve nonuniform points, defined as follows. Let X_1, X_2, \dots be i.i.d. d -dimensional random variables with common density f , which is fixed but arbitrary. Define the induced *binomial point processes*

$$(2.5) \quad \mathcal{X}_n := \mathcal{X}_n(f) := \{X_1, \dots, X_n\}, \quad n \in \mathbb{N}.$$

Our general limit theory is not for $H_\xi(\mathcal{X}_n)$, but for $H_{\xi_n}(\mathcal{X}_n)$, where we define

$$(2.6) \quad \xi_n(x; \mathcal{X}) := \xi(n^{1/d}x; n^{1/d}\mathcal{X}).$$

To obtain a LLN for $H_{\xi_n}(\mathcal{X}_n)$ we use the following approach. By coupling $n^{1/d}\mathcal{X}_n$ to a Poisson process of varying intensity, we show that the local behavior of $\xi(n^{1/d}X_1; n^{1/d}\mathcal{X}_n)$ is approximated by the local behavior of the coupled Poisson process. If the functional ξ stabilizes on homogeneous Poisson point processes, then a conditioning argument shows distributional convergence of $\xi(n^{1/d}X_i; n^{1/d}\mathcal{X}_n)$ for each $X_i \in \mathcal{X}_n$. Under appropriate moment conditions on ξ , this gives a weak LLN for H_{ξ_n} . This formalizes the intuitive notion that the limiting behavior of H_{ξ_n} on finite sets is related to the behavior of ξ_∞ on the infinite set \mathcal{P}_τ .

Many of the applications that we consider are concerned with functionals of graphs of the form $G := G(\mathcal{X})$ defined for each locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, where either $G(\mathcal{X})$ or (in the case of the Voronoi graph) its planar dual has vertex set \mathcal{X} .

We shall say G is *translation invariant* if translation by y is a graph isomorphism from $G(\mathcal{X})$ to $G(y + \mathcal{X})$ for all $y \in \mathbb{R}^d$ and all locally finite point sets \mathcal{X} . We shall say G is *scale invariant* if scalar multiplication by a induces a graph isomorphism from $G(\mathcal{X})$ to $G(a\mathcal{X})$ for all \mathcal{X} and all $a > 0$.

It is useful to have a notion of stabilization for these graphs. Given G and given a vertex $x \in \mathcal{X}$, let $\mathcal{E}(x; G(\mathcal{X}))$ be the set of edges of $G(\mathcal{X})$ incident to x (or for the Voronoi graph, the set of edges whose planar duals are incident to x). Let $\mathcal{P}_{\tau,0} := \mathcal{P}_\tau \cup \{\mathbf{0}\}$. We shall say that G *stabilizes* on \mathcal{P}_τ if there exists a random but almost surely finite variable R such that

$$\mathcal{E}(\mathbf{0}; G(\mathcal{P}_{\tau,0})) = \mathcal{E}(\mathbf{0}; G(\mathcal{P}_{\tau,0} \cap B(\mathbf{0}; R)) \cup \mathcal{A})$$

for all finite $\mathcal{A} \subset \mathbb{R}^d \setminus B(\mathbf{0}; R)$.

Stabilization of the graph G says that the local behavior of the graph in a bounded region is unaffected by points beyond a finite (but random) distance from that region. As we shall see, the minimal spanning tree (with the definition suitably extended from finite to locally finite point sets) and the k -nearest neighbors, Voronoi, Delaunay and sphere of influence graphs are all stabilizing on \mathcal{P}_τ , $\tau \in (0, \infty)$.

2.2. General LLN results. The following theorem places the objective method in a general context, shows that the asymptotic behavior of $H_{\xi_n}(\mathcal{X}_n)$ is sensitive to the underlying density f and explicitly identifies the asymptotic constants in terms of f and the limit functional ξ_∞ . It will be proved in Section 3.

THEOREM 2.1 (General LLN). *Suppose $q = 1$ or $q = 2$. Suppose ξ is almost surely stabilizing on \mathcal{P}_τ , with limit $\xi_\infty(\mathcal{P}_\tau)$, for all $\tau \in (0, \infty)$. If ξ satisfies the moments condition*

$$(2.7) \quad \sup_{n \in \mathbb{N}} E[\xi(n^{1/d} X_1; n^{1/d} \mathcal{X}_n)^p] < \infty$$

for some $p > q$, then as $n \rightarrow \infty$,

$$(2.8) \quad n^{-1} H_{\xi_n}(\mathcal{X}_n) \rightarrow \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})] f(x) dx \quad \text{in } L^q$$

and the limit in (2.8) is finite.

Obviously, for there to be any possibility at all for the mean of the left-hand side of (2.8) to converge to a finite limit, the moments condition (2.7) must hold for $p = 1$. In this sense, when $q = 1$, the moments condition (2.7) is close to being the best possible.

A simplification arises in the case where there is a constant $\gamma > 0$ such that ξ satisfies the relation

$$\xi(ax; a\mathcal{X}) = a^\gamma \xi(x; \mathcal{X})$$

for all positive scalars a and all finite point sets \mathcal{X} and $x \in \mathcal{X}$. In this case we say ξ is *homogeneous of order γ* . Homogeneity of order γ implies that $\xi_n(x; \mathcal{X}) = n^{\gamma/d} \xi(x; \mathcal{X})$. Moreover, almost sure stabilization on \mathcal{P}_1 with limit $\xi_\infty(\mathcal{P}_1)$, together with homogeneity of order γ , implies stabilization on \mathcal{P}_τ with limit $\tau^{-\gamma/d} \xi_\infty(\mathcal{P}_1)$, for any $\tau > 0$. Therefore the L^q limit in (2.8) simplifies to

$$(2.9) \quad E[\xi_\infty(\mathcal{P}_1)] \int_{\mathbb{R}^d} f(x)^{(d-\gamma)/d} dx.$$

Even simpler is the special case where the function ξ is *scale invariant*, or, in other words, is homogeneous of order 0. In this case the expression (2.9) simplifies to $E[\xi_\infty(\mathcal{P}_1)]$ and thus the large- n behavior of scale-invariant functionals is not sensitive to the density of the underlying point set.

Theorem 2.1 admits the following extension to functionals defined on marked point sets. Let $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, P_{\mathcal{K}})$ be a probability space. A *marked point set* is a subset of $\mathbb{R}^d \times \mathcal{K}$, to be denoted $\tilde{\mathcal{X}}$, where \mathcal{X} is an (unmarked) subset of \mathbb{R}^d and the tilde indicates that each element x of \mathcal{X} carries a mark in \mathcal{K} (x , together with its mark, is denoted \tilde{x}). In this context, a functional $\xi(\tilde{x}; \tilde{\mathcal{X}})$ is said to be translation invariant if for all $y \in \mathbb{R}^d$ and for any element \tilde{x} of any marked point set $\tilde{\mathcal{X}}$, we have $\xi(\tau_y(\tilde{x}); \tau_y(\tilde{\mathcal{X}})) = \xi(\tilde{x}; \tilde{\mathcal{X}})$, where τ_y is the translation operator sending any element $(x, t) \in \mathbb{R}^d \times \mathcal{K}$ to $(y + x, t)$ (i.e., leaving the mark unchanged).

In the random setting, assume the marks are i.i.d. with distribution $P_{\mathcal{K}}$. We are interested in the cases where \mathcal{X} is the point process \mathcal{X}_n or \mathcal{P}_{τ} ; in both cases assume the mark values are independent of the set \mathcal{X} . We say that ξ stabilizes on the marked Poisson point process $\tilde{\mathcal{P}}_{\tau}$ if (2.4) holds with \mathcal{S} replaced by $\tilde{\mathcal{P}}_{\tau}$. In this context, the essential supremum and essential infimum in (2.2) and (2.3), respectively, refer to an n -fold product measure of Lebesgue measure on \mathbb{R}^d with $P_{\mathcal{K}}$.

If ξ is translation invariant and almost surely stabilizing on $\tilde{\mathcal{P}}_{\tau}$, with limit $\xi_{\infty}(\tilde{\mathcal{P}}_{\tau})$, $\tau \in (0, \infty)$, and if ξ satisfies the moments condition (2.7) for some $p > q$, then as $n \rightarrow \infty$, we obtain a version of (2.8) for marked processes, namely,

$$(2.10) \quad n^{-1} H_{\xi_n}(\tilde{\mathcal{X}}_n) \rightarrow \int_{\mathbb{R}^d} E[\xi_{\infty}(\tilde{\mathcal{P}}_{f(x)})] f(x) dx \quad \text{in } L^q.$$

In applications of (2.10), it will be clear that we are considering marked point processes and for simplicity we will thus suppress the tilde.

Many applications of Theorem 2.1 are defined in terms of functionals of graphs arising in computational geometry. Suppose the graph $G := G(\mathcal{X})$ is defined for all locally finite \mathcal{X} . Given G , functionals such as total length, number of edges or number of edges less than some specified length are of interest. These and other functionals may be interpreted as a total of ϕ -weighted edge lengths, that is, as a sum

$$(2.11) \quad L_{\phi}^G(\mathcal{X}) := \sum_{e \in G(\mathcal{X})} \phi(|e|)$$

with $\phi : [0, \infty] \rightarrow [0, \infty)$ a specified function. Also of interest are the number of components, which we denote $K^G(\mathcal{X})$, and, for any specified finite connected unlabeled graph Γ , the number of vertices $x \in \mathcal{X}$ for which $G(\mathcal{X})$ contains a subgraph isomorphic to Γ with a vertex at x , which we denote $V_{\Gamma}^G(\mathcal{X})$. Let σ_G be the order of the component (i.e., the number of vertices in the component) containing the origin of $G(\mathcal{P}_{1,0})$ and let E_{Γ} be the event that $G(\mathcal{P}_{1,0})$ contains a subgraph isomorphic to Γ with a vertex at the origin.

Note that if G is scale invariant, then both $K^G(\mathcal{X})$ and $V_{\Gamma}^G(\mathcal{X})$, as well as $L_{\phi}^G(\mathcal{X})$ in the case $\phi \equiv 1$ (the total number of edges), are scale-invariant functionals of \mathcal{X} .

The following general result, proved in Section 3, is a consequence of Theorem 2.1.

THEOREM 2.2. *Suppose G is translation and scale invariant and stabilizes on \mathcal{P}_1 . Then*

$$(2.12) \quad n^{-1} K^G(\mathcal{X}_n) \rightarrow E[\sigma_G^{-1}] \quad \text{in } L^2$$

and for any finite connected graph Γ ,

$$(2.13) \quad n^{-1} V_\Gamma^G(\mathcal{X}_n) \rightarrow P[E_\Gamma] \quad \text{in } L^2.$$

Moreover, if $q = 1$ or $q = 2$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a specified function with

$$(2.14) \quad \sup_{n \in \mathbb{N}} E \left[\left(\sum_{e \in \mathcal{E}(X_1; G(\mathcal{X}_n))} \phi(n^{1/d}|e|) \right)^p \right] < \infty$$

for some $p > q$, then

$$(2.15) \quad n^{-1} L_\phi^G(n^{1/d} \mathcal{X}_n) \rightarrow \frac{1}{2} \int_{\mathbb{R}^d} E \sum_{e \in \mathcal{E}(\mathbf{0}; G(\mathcal{P}_{1,0}))} \phi\left(\frac{|e|}{f(x)^{1/d}}\right) f(x) dx \quad \text{in } L^q$$

and the limit in (2.15) is finite.

The conclusions of Theorems 2.1 and 2.2 can be strengthened in more than one way. If, for example, $q > 2$ is an integer and if ξ satisfies the moments condition (2.7) for some $p > q$, then a modification of the coupling arguments given in Section 3 yields convergence in L^q . Moreover, for many functionals, convergence of the means $E[n^{-1} H_{\xi_n}(\mathcal{X}_n)]$, as given by Theorems 2.1 and 2.2, implies almost sure and even complete convergence using concentration inequalities involving either isoperimetric methods or sums of martingale differences. We will not pursue this, but refer to [15, 24, 29] for details. We will also not seek to establish rates of convergence in our general results.

REMARKS. (a) In the special case where $\phi(x) = x^\alpha$ (i.e., power-weighted edges), the left-hand side of (2.15) equals $n^{(\alpha-d)/d} L_\phi^G(\mathcal{X}_n)$, while the right-hand side of (2.15) simplifies to $C_G(d, \alpha) \int_{\mathbb{R}^d} f(x)^{(d-\alpha)/d} dx$, with $C_G(d, \alpha) := \frac{1}{2} E \sum_e \phi(|e|)$, the sum being over all $e \in \mathcal{E}(\mathbf{0}; G(\mathcal{P}_{1,0}))$. This provides extra information about limiting constants, including, for example, those in (1.1).

(b) The stabilizing hypothesis of Theorem 2.1 can be weakened to one requiring that ξ be almost surely stabilizing on $\mathcal{P}_{f(x)}$, with limit $\xi_\infty(\mathcal{P}_{f(x)})$, for almost all x in the support of the density f . These weakened hypotheses are used in Theorem 2.7 below. The moments condition (2.7) is not always easy to check, but is obviously true for any p when the functional ξ is uniformly bounded. In Section 2.3 we shall verify the moments condition (2.14) for various graphs.

(c) Even without the moments condition (2.7), the stabilization assumption in Theorem 2.1 is enough to guarantee that the right-hand side of (2.8) is a lower bound for $\liminf E[n^{-1} H_{\xi_n}(\mathcal{X}_n)]$; this is proved by following the proof of

Lemma 3.2 and applying Fatou's lemma. Likewise, even without the moments condition (2.14), a weaker version of (2.15) holds in which the right-hand side is a lower bound for the \liminf of the expectation of the left-hand side.

(d) Limit laws such as (2.12) and (2.13) for scale-invariant functionals are of interest in multidimensional scaling [4]. In this context, given the matrix of interpoint distances between pairs of points in a data set \mathcal{X} , one seeks to identify the dimension in which a data set lives. To identify the underlying dimension, it is useful to study scale-invariant functionals of the interpoint distances, since these are precisely the functionals whose asymptotics are sensitive only to the dimension of the support of the distribution of the data points and not on the underlying density. This approach to dimension identification may be relatively inexpensive from a computational point of view [5].

(e) Without further conditions on ξ and H_{ξ_n} , we are unable to obtain asymptotics for H_{ξ_n} over point sets consisting of random d -vectors having a law with a singular component.

(f) Suppose the conditions of Theorem 2.1 are satisfied with $q = 2$. Since ξ^2 is almost surely stabilizing on \mathcal{P}_τ , with limit $\xi_\infty^2(\mathcal{P}_\tau)$, and since ξ^2 satisfies the moments condition (2.7) for some $p > 1$, it follows that the sample variance of $\{\xi_n(X_i; \mathcal{X}_n), 1 \leq i \leq n\}$, namely, the quantity

$$\frac{\sum_{i=1}^n (\xi_n(X_i; \mathcal{X}_n) - n^{-1} \sum_{i=1}^n \xi_n(X_i; \mathcal{X}_n))^2}{n}$$

converges in L^1 to $\text{Var}(\xi_\infty(\mathcal{P}_{f(X_1)}))$.

(g) The limit (2.13) says, loosely speaking, that the number of vertices of $G(\mathcal{X}_n)$ satisfying any property determined by the local graph structure within a bounded graph distance exhibits LLN behavior. For example, it yields a LLN for the proportionate number of vertices of any fixed degree, among other things.

(h) A version of Theorem 2.2 also holds for *directed* graphs. In this context, we say that distinct vertices x and y lie in the same component if and only if there is both a directed path from x to y and a directed path from y to x ; also, the definition of V_Γ^G should be in terms of a specified directed graph Γ , and the limit (2.15) holds without the factor of $1/2$ and with $\mathcal{E}(x; G(\mathcal{X}))$ defined to be the set of directed edges going into x .

2.3. Applications. The applications of Theorems 2.1 and 2.2 range from the treatment of functionals in computational geometry to the statistics of Boolean models as well as to an analysis of packing processes. The following discussion is not meant to exhaustively treat applications, but is merely meant to indicate the variety of uses of the main theorems. To uniformly bound moments of edge lengths in the graph $G(\mathcal{X})$, we shall in some cases require that the support of the underlying density f be a convex set and that f be bounded away from zero on its support.

(a) *Minimal spanning tree.* Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, $d \geq 2$, and given $a > 0$, let $\mathcal{G}_a(\mathcal{X})$ be the graph with vertex set \mathcal{X} and with edge set $\{\{x, y\} : |x - y| < a\}$. Let $MST(\mathcal{X})$ be the graph with vertex set \mathcal{X} obtained by including each edge $\{x, y\}$ such that x and y lie in different components of $\mathcal{G}_{|x-y|}(\mathcal{X})$ and at least one of these components is finite. If \mathcal{X} is finite with distinct interpoint distances, then $MST(\mathcal{X})$ is the *minimal spanning tree* on \mathcal{X} , that is, the connected graph with vertex set \mathcal{X} of minimal total edge length; see Aldous and Steele [1], in particular Lemma 12. Clearly $MST(\mathcal{X})$ is translation and scale invariant.

The following theorem considerably expands upon Theorem 1.1. We say that ϕ has *polynomial growth of order* $a < \infty$ if $\phi(x) \leq \text{const} \cdot (1 + x^a)$ for all x .

THEOREM 2.3. *Let $G(\mathcal{X}) = MST(\mathcal{X})$. Then (2.13) holds for any finite connected Γ . Also, (2.15) holds with $q = 2$ if either (i) ϕ is bounded or (ii) ϕ has polynomial growth of order $a < \infty$, the support of f is a convex polyhedron and f is bounded away from infinity and zero on its support. Finally, (2.15) holds with $q = 1$ if (iii) ϕ has polynomial growth of order $a < d$ and $\int_{\mathbb{R}^d} |x|^r f(x) dx < \infty$ for some $r > \max\{ad/(d - a), d/(d - a)\}$ and $\int_{\mathbb{R}^d} f(x)^{(d-a)/d} dx < \infty$.*

Special cases of Theorem 2.3 include an L^2 version of Theorem 1.1 (take ϕ to be a power function) and the L^2 LLN for the empirical distribution function of rescaled edge lengths in the MST, first obtained by Penrose [21] [take $\phi(x) := \mathbb{1}_{[0,t]}(x)$]. For uniformly distributed points, Bezuidenhout, Grimmett and Löffler [3] obtained an almost sure version of the latter result. The first part (2.13) of the conclusion in Theorem 2.3 yields an L^2 version of the results of [28] and Theorem 5.2.2 of [27] concerning the number of vertices of fixed degree.

LEMMA 2.1. *The set of edges of $MST(\mathcal{P}_{1,0})$ incident to the origin is unaffected by any additions or deletions of points outside a ball of random but almost surely finite radius, that is, $G(\mathcal{X}) = MST(\mathcal{X})$ is stabilizing on \mathcal{P}_1 .*

PROOF. If $\{\mathbf{0}, X\}$ is an edge of $MST(\mathcal{P}_{1,0})$, then $\mathbf{0}$ and X lie in distinct components of $\mathcal{G}_{|X|}(\mathcal{P}_{1,0})$ and at least one of these components, denoted C say, is finite. In this case, let $\text{Rad}(C)$ be the smallest r such that all vertices of C lie in $B(\mathbf{0}; r)$ and let $S_X := \text{Rad}(C) + |X|$. Then C is unaffected by changes to $\mathcal{P}_{1,0}$ outside $B(\mathbf{0}; S_X)$ (in particular, any added points outside the ball are distant greater than $|X|$ from C) and $\{\mathbf{0}, X\}$ remains in the MST after any such changes.

Let $S := \max\{S_X : \{\mathbf{0}, X\} \in MST(\mathcal{P}_{1,0})\}$. By Lemma 4 of [1], the degree of $\mathbf{0}$ in $MST(\mathcal{P}_{1,0})$ is finite, so $S < \infty$. Thus, for any finite point set \mathcal{Y} , we have the implication

$$(2.16) \quad \begin{aligned} & [\mathcal{Y} \cap B(\mathbf{0}; S) = \mathcal{P}_{1,0} \cap B(\mathbf{0}; S)] \\ & \implies [\mathcal{E}(\mathbf{0}; MST(\mathcal{P}_{1,0})) \subseteq \mathcal{E}(\mathbf{0}; MST(\mathcal{Y}))]. \end{aligned}$$

We need an inclusion the other way, which is harder to show. First, we show that there exists R such that for any locally finite point set \mathcal{Y} with $\mathcal{Y} \cap B(\mathbf{0}; R) = \mathcal{P}_{1,0} \cap B(\mathbf{0}; R)$, there is no edge incident to $\mathbf{0}$ in $MST(\mathcal{Y})$ of length greater than R . This actually follows from Lemmas 3 and 5 of Kesten and Lee [16], but we sketch an elementary argument to aid the uninitiated reader.

By standard methods in continuum percolation, namely discretization along with a Peierls (enumeration) argument (see Grimmett [9]), there exists a finite λ such that the complement of the union of closed unit balls centered at points of \mathcal{P}_λ has no unbounded component, almost surely. Hence by a rescaling argument, there is a finite constant ρ such that the complement of the union of closed balls of radius ρ centered at points of \mathcal{P}_1 has no unbounded component, almost surely.

Let V be the union of $B(\mathbf{0}; 2\rho)$ with all components of $\mathbb{R}^d \setminus \bigcup_{X \in \mathcal{P}_1} B(X; \rho)$ that have nonempty intersection with $B(\mathbf{0}; 2\rho)$ (see Figure 1). Then V is bounded almost surely. Set

$$R := \inf\{t > 0 : V \subseteq B(\mathbf{0}; t)\} + 2\rho.$$

Given $x \in \mathbb{R}^d \setminus B(\mathbf{0}; R)$, let x' be the last point of exit from the “vacant” region V as one travels along the ray from $\mathbf{0}$ to x , that is, let $x' := \beta x$ with $\beta := \sup\{a > 0 : ax \in V\}$. Then $2\rho \leq |x'| \leq R - 2\rho$ and there exists a point $X \in \mathcal{P}_1$ such that $|X - x'| \leq \rho$. It follows by the triangle inequality (see Figure 1) that

$$\max(|X - x|, |X|) \leq \rho + \max(|x' - x|, |x'|) \leq |x| - \rho,$$

which implies that $\mathbf{0}$ and x lie in the same component of $\mathcal{G}_{|x|}(\{\mathbf{0}, X, x\})$. By the definition of the MST, this implies that $\{\mathbf{0}, x\}$ is not an edge of the MST for any finite set containing $\{\mathbf{0}, X, x\}$. Also, $X \in B(\mathbf{0}; R)$. Hence for any finite point set \mathcal{Y} and any $x \in \mathcal{Y} \setminus B(\mathbf{0}; R)$,

$$(2.17) \quad [\mathcal{Y} \cap B(\mathbf{0}; R) = \mathcal{P}_{1,0} \cap B(\mathbf{0}; R)] \implies [\{\mathbf{0}, x\} \notin MST(\mathcal{Y})].$$

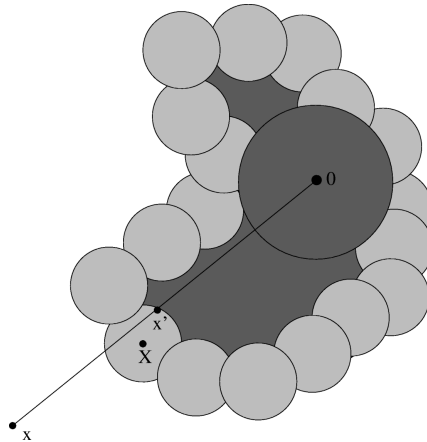


FIG. 1. The smaller disks have radius ρ . The more heavily shaded region is V .

Let us say that \mathcal{P}_1 has the *uniqueness property* if for every point $X \in \mathcal{P}_1$, the graph $\mathcal{G}_{|X|}(\mathcal{P}_{1,0})$ has at most one infinite component. Then \mathcal{P}_1 has the uniqueness property almost surely, by uniqueness of the infinite component in continuum percolation; see Alexander [2]. In fact, since \mathcal{P} is locally finite, one can show that \mathcal{P}_1 has the uniqueness property a.s. without appealing to the *simultaneous uniqueness* result in [2].

For $X \in \mathcal{P}_1$, define the number T_X as follows. If $\mathbf{0}$ and X lie in the same component of $\mathcal{G}_{|X|}(\mathcal{P}_{1,0})$, let T_X be the smallest t such that there exists a path in $\mathcal{G}_{|X|}(\mathcal{P}_{1,0})$ from $\mathbf{0}$ to X with all vertices in $B(\mathbf{0}; t)$. Otherwise, set $T_X := 0$.

Suppose \mathcal{P}_1 has the uniqueness property and suppose $X \in \mathcal{P}_1$ with $\{\mathbf{0}, X\} \notin MST(\mathcal{P}_{1,0})$. Then $\mathbf{0}$ and X will be in the same component of $\mathcal{G}_{|X|}(\mathcal{P}_{1,0})$, so by definition of T_X , $\{\mathbf{0}, X\}$ will not be an edge of $MST(\mathcal{Y})$ for any point set \mathcal{Y} with $\mathcal{Y} \cap B(\mathbf{0}; T_X) = \mathcal{P}_{1,0} \cap B(\mathbf{0}; T_X)$.

Let $R' := \max(R, \max_{X \in \mathcal{P}_1 \cap B(\mathbf{0}; R)} T_X)$ and note that R' is finite a.s. Suppose \mathcal{Y} is a finite point set with $\mathcal{Y} \cap B(\mathbf{0}; R') = \mathcal{P}_{1,0} \cap B(\mathbf{0}; R')$ and suppose $\{\mathbf{0}, Y\}$ is an edge of $MST(\mathcal{Y})$. Then $Y \in B(\mathbf{0}; R)$ by (2.17) and hence $\{\mathbf{0}, Y\} \in MST(\mathcal{P}_{1,0})$ by definition of R' . Together with (2.16), this completes the proof. \square

PROOF OF THEOREM 2.3. By Lemma 2.1, $G(\mathcal{X}) = MST(\mathcal{X})$ is stabilizing on \mathcal{P}_1 . Theorem 2.2 then gives us (2.13). To prove (2.15) it remains only to verify the moments condition (2.14). Note first that points in the MST have a degree which is uniformly bounded by some finite constant $C(d)$ (Lemma 4 of [1]). Hence under condition (i), that is, if ϕ is bounded, then $\sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi(|e|)$ is also bounded for any point set \mathcal{X} and $x \in \mathcal{X}$, and thus (2.14) is clearly satisfied, showing that (2.15) holds in this case.

The uniform bound on vertex degrees implies that for any $p > 1$, there is a second constant $C(p, d)$ such that for any \mathcal{X} and any $x \in \mathcal{X}$,

$$(2.18) \quad \left(\sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi(n^{1/d}|e|) \right)^p \leq C(p, d) \sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi^p(n^{1/d}|e|).$$

Suppose condition (ii) holds, that is, ϕ has polynomial growth of order $a < \infty$ and the support of f is a convex polyhedron with the restriction of f to its support bounded away from zero and infinity. Then by the uniform bound on vertex degrees, the right-hand side of (2.18) is bounded by a constant plus a constant multiple of

$$(2.19) \quad n^{ap/d} \max_{e \in \mathcal{E}(x; G(\mathcal{X}))} |e|^{ap}.$$

Lemma 2.1 of [30], which makes heavy use of $f > 0$, shows that

$$E[\max\{|e|^{ap} : e \in \mathcal{E}(X_1; G(\mathcal{X}_n))\}] = O(n^{-ap/d})$$

and therefore the left side of (2.14) is uniformly bounded by a constant. Therefore, under condition (ii), the conclusion (2.15) (with $q = 2$) holds.

Finally consider case (iii). Put $\xi_n(x; \mathcal{X}) := \sum_{e \in \mathcal{E}(x, G(\mathcal{X}))} \phi(n^{1/d}|e|)$. If ϕ has polynomial growth of order a , then by exchangeability and (2.18),

$$\begin{aligned}
 E[(\xi_n(X_1; \mathcal{X}_n))^p] &= n^{-1} E \sum_{i=1}^n (\xi_n(X_i; \mathcal{X}_n))^p \\
 &\leq C(p, d)n^{-1} E \sum_{i=1}^n \sum_{e \in \mathcal{E}(X_i; G(\mathcal{X}_n))} \phi^p(n^{1/d}|e|) \\
 (2.20) \quad &= 2C(p, d)n^{-1} E \sum_{e \in G(\mathcal{X}_n)} \phi^p(n^{1/d}|e|) \\
 &\leq C_1(p, d) + C_2(p, d)n^{(ap-d)/d} E \sum_{e \in G(\mathcal{X}_n)} |e|^{ap}.
 \end{aligned}$$

We now bound the right-hand side of (2.20). Let $L^{ap}(\mathcal{X}) := \sum_{e \in G(\mathcal{X})} |e|^{ap}$ and for all $k \geq 1$, let A_k denote the annular shell centered around the origin of \mathbb{R}^d with inner radius 2^k and outer radius 2^{k+1} . (Let A_0 be the ball centered at the origin with radius 2.) Note that, as in (7.21) of [29],

$$(2.21) \quad L^{ap}(\mathcal{X}_n) \leq \sum_{0 \leq k \leq s(n)} L^{ap}(\mathcal{X}_n \cap A_k) + C(p) \max_{1 \leq i \leq n} |X_i|^{ap},$$

where $s(n)$ is the largest k such that $A_k \cap \mathcal{X}_n$ is not empty. We need to show, after taking expectations and dividing by $n^{(d-ap)/d}$, that the two terms on the right are bounded uniformly in n . By Jensen's inequality and the growth bounds $L^{ap}(\mathcal{X}) \leq C(\text{diam } \mathcal{X})^{ap} (\text{card}(\mathcal{X}))^{(d-ap)/d}$ (see Lemma 3.3 of [29]), the first term is upper bounded by

$$C \sum_{k \geq 1} 2^{k ap} (P[X_1 \in A_k])^{(d-ap)/d},$$

which is finite by the integrability hypothesis (see page 85 of [21]). The second term is bounded by

$$(2.22) \quad C(p) \int_0^\infty P \left[\max_{1 \leq i \leq n} |X_i|^{ap} \geq tn^{(d-ap)/d} \right] dt$$

$$(2.23) \quad \leq nC(p) \int_0^\infty P[|X_1|^{apd/(d-ap)} \geq t^{d/(d-ap)}n] dt.$$

By Markov's inequality, together with the moment condition $\int_{\mathbb{R}^d} |x|^r f(x) dx < \infty$, for some $r > ad/(d-a)$, this last integral is finite. Thus under condition (iii), we can choose $p > 1$ so that (2.20) is uniformly bounded and hence the moments condition (2.14) is satisfied, showing the validity of the conclusion under condition (iii). \square

(b) *k*-nearest neighbors graphs. Let k be a positive integer. Given a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, the k -nearest neighbors (undirected) graph on \mathcal{X} , denoted $NG(\mathcal{X})$, is the graph with vertex set \mathcal{X} obtained by including $\{x, y\}$ as an edge whenever y is one of the k nearest neighbors of x and/or x is one of the k nearest neighbors of y . If the k th nearest neighbor of x is not well-defined (i.e., if there is a “tie” in the ordering of interpoint distances involving x), use the lexicographic ordering as a “tie-breaker” to determine the k nearest neighbors. Such a tie has zero probability for the random point sets under consideration here.

The k -nearest neighbors (directed) graph on \mathcal{X} , denoted $NG'(\mathcal{X})$, is the directed graph with vertex set \mathcal{X} obtained by including a directed edge from each point to each of its k nearest neighbors. In the case of $NG'(\mathcal{X})$, Theorem 2.2 should be interpreted with reference to Remark (h) about directed graphs at the end of Section 2.2.

It is clear that the k -nearest neighbors graphs are translation and scale invariant. The following generalizes and extends the asymptotics for the sum of power-weighted edge lengths by McGivney [18], Yukich (Theorem 8.3 of [29]) and Jimenez and Yukich [15], who limited attention to continuous densities and increasing ϕ . It also extends Theorem 2 of Eppstein, Paterson and Yao [7], who proved convergence of the mean number of components for uniform samples.

THEOREM 2.4. *Let $G(\mathcal{X})$ denote either $NG(\mathcal{X})$ or $NG'(\mathcal{X})$. Then (2.12) and (2.13) hold. Moreover, (2.15) holds with $q = 2$ if either (i) ϕ is bounded or (ii) ϕ has polynomial growth of order $a < \infty$, the support of f is a convex polyhedron and f is bounded away from infinity and zero. Finally the directed graph version of (2.15) holds for $NG'(\mathcal{X})$ with $q = 1$ if ϕ has polynomial growth of order $a < d$, $\int_{\mathbb{R}^d} f(x)^{(d-a)/d} dx < \infty$ and $\int_{\mathbb{R}^d} |x|^r f(x) dx < \infty$ for some $r > d/(d-a)$.*

PROOF. We apply Theorem 2.2. As shown in Lemma 6.1 of [23] (even though the definition of stabilization there is slightly different), the set of edges incident to the origin in $NG(\mathcal{P}_{1,0})$ is unaffected by the addition or removal of points outside a ball of random but almost surely finite radius, that is, the graph $G(\mathcal{X}) = NG(\mathcal{X})$ is stabilizing on \mathcal{P}_1 . Similar arguments show that $NG'(\mathcal{X})$ is stabilizing on \mathcal{P}_1 . Therefore Theorem 2.2 yields (2.12) and (2.13) for $NG(\mathcal{X})$ and $NG'(\mathcal{X})$.

To prove (2.15) it remains only to verify the moments condition (2.14). This condition is checked, under any of the conditions (i), (ii) or (iii), in very much the same manner as for the MST of Theorem 2.3. The existence of a uniform bound on the degree of vertices in the k -nearest neighbors graph is Lemma 8.4 of [29]. Under condition (iii), we note that since NG' is subadditive without any error term, the last term in (2.21) is not needed, eliminating the need for the condition $r > ad/(d-a)$. \square

Henze [12] considered the fraction of points in \mathcal{X}_n which are the nearest neighbors of exactly j other points; he also considered the fraction of points in \mathcal{X}_n

that are the l th nearest neighbors to their own k th nearest neighbor. He showed that the limits of these fractions converge to explicit but rather complicated limiting constants. A directed-graph version of (2.13) in Theorem 2.2 yields Henze's results and shows that one can interpret his limiting constants in terms of a functional evaluated at a point in the origin of the homogeneous Poisson point process.

(c) *Voronoi and Delaunay graphs.* Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$ and given $x \in \mathcal{X}$, the locus of points closer to x than to any other point $\in \mathcal{X}$ is called the *Voronoi cell* centered at x . The graph on vertex set \mathcal{X} in which each pair of adjacent cell centers is connected by an edge is called the *Delaunay graph* on \mathcal{X} ; if $d = 2$, then the planar dual graph consisting of all boundaries of Voronoi cells is called *Voronoi graph* generated by \mathcal{X} . Edges of the Voronoi graph can be finite or infinite. Let $DEL(\mathcal{X})$ [respectively, $VOR(\mathcal{X})$] denote the collection of edges in the Delaunay graph (respectively, the Voronoi graph) on \mathcal{X} . The Voronoi and Delaunay graphs are clearly scale and translation invariant.

THEOREM 2.5. *Let $d = 2$ and let $G(\mathcal{X}) = VOR(\mathcal{X})$. Then (2.13) holds. Also, if ϕ has polynomial growth with $\phi(\infty) = 0$, if the support of f is a convex polygonal region and if f is bounded away from infinity and zero on its support, then (2.15) holds with $q = 2$.*

This result adds to the results of McGivney and Yukich [19] and Jimenez and Yukich [15], which require continuous f and functions ϕ which are either power functions or increasing. If ϕ is the identity and the support of f is the unit square, then the right-hand side of (2.15) simply reduces to $2 \int_{[0,1]^2} f(x)^{1/2} dx$ (see Theorem 1.2 of [19]).

PROOF OF THEOREM 2.5. Once again we apply Theorem 2.2. As in [23], we can verify $G(\mathcal{X})$ is stabilizing, that is, the Voronoi cell centered at the origin for $\mathcal{P}_{1,0}$ is unaffected by changes beyond a random but almost surely finite distance from the origin. Let $\mathcal{E}(X; VOR(\mathcal{X}_n))$ denote the edges of the cell around X_1 in $VOR(\mathcal{X}_n)$. Concerning the moments condition (2.14), a modification of Lemma 8.1 of [23] shows that under the prescribed conditions on f and its support,

$$(2.24) \quad \sup_{n \in \mathbb{N}} E \left[\left(\sum_{e \in \mathcal{E}(X_1; VOR(\mathcal{X}_n))} (n^{1/d} |e|)^a \right)^p \right] < \infty$$

for all $a > 0$ and $p > 2$. A modification of the proof of Lemma 2.6 of [19], pages 286–287, shows that the cardinality of $\mathcal{E}(X_1; VOR(\mathcal{X}_n))$ has a finite p th moment for all $p > 2$. Combining this with (2.24) thus shows that for all ϕ with polynomial growth,

$$(2.25) \quad \sup_{n \in \mathbb{N}} E \left[\left(\sum_{e \in \mathcal{E}(X_1; VOR(\mathcal{X}_n))} \phi(n^{1/d} |e|) \right)^p \right] < \infty$$

for all $p > 2$. Thus (2.14) holds and the conclusion (2.15) follows. \square

A similar result to Theorem 2.5 holds if $G(\mathcal{X})$ is taken to be the Delaunay graph and $\mathcal{E}(x; DEL(\mathcal{X}))$ denotes the edges in $DEL(\mathcal{X})$ incident to x . In this case the conclusion (2.13) yields a LLN for the number of vertices of the Delaunay graph of fixed degree m , for any $m = 3, 4, 5, \dots$. Since this quantity is the same as the total number of cells in the Voronoi graph on \mathcal{X}_n which are m -gons, this adds to results of Hayen and Quine [11], who determined the proportion of triangles in the Voronoi graph on \mathcal{P}_1 .

Finally, for each $t > 0$ consider the case where $\xi(x; \mathcal{X})$ equals 1 or 0 according to whether the area of the Voronoi cell around x is bounded by t or not. This is one case where it is natural to use Theorem 2.1 rather than Theorem 2.2, and that result yields a LLN for $H_{\xi_n}(\mathcal{X}_n) := \sum_{i=1}^n \xi(n^{1/d} X_i; n^{1/d} \mathcal{X}_n)$, that is, a LLN for the empirical distribution function of the rescaled areas of the Voronoi cells on \mathcal{X}_n .

(d) *Sphere of influence graph.* Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, the sphere of influence graph $SIG(\mathcal{X})$ is a graph with vertex set \mathcal{X} , constructed as follows: for each $x \in \mathcal{X}$, let B_x be a ball around x with radius equal to $\min_{y \in \mathcal{X} \setminus \{x\}} \{|y - x|\}$. Then B_x is called the sphere of influence of x . Draw an edge between x and y iff the balls B_x and B_y overlap. The collection of such edges is the sphere of influence graph (SIG) on \mathcal{X} . It is clearly translation and scale invariant.

The following LLN is apparently new, even for the identity function $\phi(x) = x$. In the case $\phi(x) \equiv 1$ it extends a result of Füredi [8] on the mean number of edges of the SIG on uniform point sets (Füredi identified the limiting constant in this case).

THEOREM 2.6. *Let $G(\mathcal{X}) = SIG(\mathcal{X})$. Then (2.12) and (2.13) both hold. If ϕ has polynomial growth and if the support of f is a convex polyhedron and if f is bounded away from infinity and zero on its support, then (2.15) holds with $q = 2$.*

PROOF. We apply Theorem 2.2. As in [23], we can check that the edges incident to the origin in $SIG(\mathcal{P}_{1,0})$ are unaffected by changes beyond a random but almost surely finite distance of the origin, that is, the graph $G(\mathcal{X}) = SIG(\mathcal{X})$ is stabilizing. Concerning the moments condition (2.14), the arguments of [23] (Theorem 7.2) show that for any $a > 0$, $p > 1$, we have under the prescribed conditions on f ,

$$(2.26) \quad \sup_{n \in \mathbb{N}} E \left[\left(\sum_{e \in \mathcal{E}(X_1; SIG(\mathcal{X}_n))} (n^{1/d} |e|)^a \right)^p \right] < \infty.$$

Moreover, since the third moment of the degree of vertices in the SIG on \mathcal{X}_n is uniformly bounded over all vertices (see, e.g., pages 142–143 of [13]) we obtain for any ϕ with polynomial growth,

$$(2.27) \quad \sup_{n \in \mathbb{N}} E \left[\left(\sum_{e \in \mathcal{E}(X_1; SIG(\mathcal{X}_n))} \phi(n^{1/d} |e|) \right)^p \right] < \infty.$$

Thus (2.14) holds. \square

(e) *Proximity graphs.* Devroye [6] defined a proximity graph on \mathcal{X} to be one in which each $\{x, y\}$ is included as an edge if a specified set $S(x, y)$ does not contain any points of \mathcal{X} . If $S(x, y)$ is the ball with opposite poles at x, y , then the associated proximity graph is the *Gabriel graph*. If $S(x, y)$ is the intersection of $B(x; |y - x|)$ and $B(y; |y - x|)$, then it is the *relative neighborhood graph*. For a survey of applications of proximity graphs such as these and also of the sphere of influence graph, see Jaromczyk and Toussaint [14].

Both the Gabriel graph and the relative neighborhood graph are translation and scale invariant, and stabilize on \mathcal{P}_1 , and also satisfy (2.14) subject to conditions on ϕ and f similar to those already given for the sphere of influence graph; see remarks in [23], Section 9. Therefore Theorem 2.2 yields information about these graphs that adds to results in Devroye [6] on the expected number of edges. Numerical estimates for values of mean edge length and mean degree of various proximity graphs over homogeneous Poisson point sets were given by Smith (Chapter III C of [26]).

(f) *Boolean models.* As already indicated, Theorem 2.1 extends to marked processes via the limit (2.10). An example of application is to Boolean models, whose importance in stochastic geometry and spatial statistics can be seen from, for example, Hall [10] and Molchanov [20]. Let μ_S be a *shape distribution*, that is, a probability distribution on the space \mathcal{S} of all compact sets in \mathbb{R}^d . For measure-theoretic details, see Matheron ([17], page 27). Assume that μ_S is concentrated on connected sets contained in $B(\mathbf{0}; K)$ for some fixed finite K (i.e., uniformly bounded connected sets). On a suitable probability space let $(S_i, i \geq 1)$ be a family of random closed sets each with distribution μ_S , independent of each other and of (X_1, X_2, \dots) (as usual, X_1, X_2, \dots are i.i.d. d -vectors with common density f). Let $\Xi_n := \bigcup_{i=1}^n (X_i + n^{-1/d} S_i)$. We refer to $X_i + n^{-1/d} S_i$ as a *random shape centered at X_i* . The random set Ξ_n is a scale-changed Boolean model in the sense of Hall ([10], pages 141 and 233).

A connected component of Ξ_n is often called a *clump*. A clump of order k is one which comprises precisely k random shapes. Let $U_k(\Xi_n)$ be the number of clumps of order k and let $U(\Xi_n) := \sum_k U_k(\Xi_n)$ be the total number of clumps. Let $V(\Xi_n)$ denote the volume of the set Ξ_n . In the case $d = 2$, when the random shapes are all convex with finitely many corners, we consider the *total curvature functional* of the set Ξ_n (the product of 2π and the Euler characteristic of Ξ_n ; see Hall [10], Chapter 4.3), which we shall denote $W(\Xi_n)$.

Another statistic of interest is the *off-line packing functional* for the collection $X_i + n^{-1/d} S_i, 1 \leq i \leq n$. This functional, denoted by $M(\Xi_n)$, is the maximal number of nonintersecting random shapes in the collection $X_i + n^{-1/d} S_i, 1 \leq i \leq n$. Additional statistics associated with Boolean models are found in Molchanov [20], for example.

Let $\Xi_{\infty, \lambda}$ be the infinite Boolean model $\bigcup_{X \in \mathcal{P}_\lambda} (X + S_X)$, where \mathcal{P}_λ is a homogeneous Poisson process of intensity λ on \mathbb{R}^d , each point carrying an

independent \mathcal{S} -valued mark S_X with distribution μ_S . Let $\lambda_c := \lambda_c(d, \mu_S)$ denote the continuum percolation threshold, that is, let λ_c be the supremum of the set of values of λ such that $\Xi_{\infty, \lambda}$ almost surely has no infinite connected component (note $\lambda_c = \infty$ if $d = 1$).

THEOREM 2.7. (i) *There exist constants $u_{k, \infty}$ ($k \in \mathbb{N}$) and u_∞ (dependent on μ_S), such that $n^{-1}U_k(\Xi_n) \rightarrow u_{k, \infty}$ in L^2 (for each $k \in \mathbb{N}$) and such that $n^{-1}U(\Xi_n) \rightarrow u_\infty$ in L^2 .*

(ii) *There exists a constant v_∞ (dependent on μ_S) such that $V(\Xi_n) \rightarrow v_\infty$ in L^2 .*

(iii) *If $d = 2$ and the measure μ_S is concentrated on convex sets with finitely many corners, then there exists a constant w_∞ (dependent on μ_S) such that $W(\Xi_n)$ converges in L^2 to w_∞ .*

(iv) *If $\sup_{x \in \mathbb{R}^d} f(x) < \lambda_c$, then there is a constant m_∞ such that $n^{-1}M(\Xi_n)$ converges in L^2 to m_∞ .*

Part (i) of this result adds to existing results in the literature such as Hall's result ([10], Theorem 4.7) on the number of clumps of order 1, especially since we do not restrict attention to a uniform underlying density f for the points X_i . As usual, the value of the limiting constant u_∞ is of the form $\int E[\xi_\infty(\mathcal{P}_{f(x)})]f(x) dx$, where $\xi(x; \mathcal{X})$ is described in the proof below.

Part (ii) shows that the volume functional satisfies a weak LLN over nonuniform point sets, adding to results of Hall ([10], Chapter 3.4) involving the vacancy functional of Ξ_n . Part (iii) is a weak LLN for the total curvature functional over nonuniform samples and also adds to results of Hall ([10], Chapter 4.3).

In part (iii), the constant w_∞ is again of the form $\int E[\xi_\infty(\mathcal{P}_{f(x)})]f(x) dx$ and if μ_S is isotropic, there exist analytic formulae for $E[\xi_\infty(\mathcal{P}_\lambda)]$; see (2.14) of [20], or (4.28) of [10].

Concerning part (iv), the off-line packing functional can be shown to be subadditive, and methods based on this fact show that a weak LLN also holds, at least for the uniform distribution on a cube of volume $1/\lambda$, even in the supercritical case $\lambda \geq \lambda_c$.

SKETCH OF PROOF OF THEOREM 2.7. Define the rescaled Boolean model Ξ'_n by

$$\Xi'_n := n^{1/d} \Xi_n = \bigcup_{i=1}^n (n^{1/d} X_i + S_i).$$

Let the mark space be \mathcal{S} with mark distribution μ_S .

(i) Given a marked point set $\mathcal{X} \subset \mathbb{R}^d$ with marks $S_x, x \in \mathcal{X}$, let $\xi(x; \mathcal{X})$ be the reciprocal of the order of the clump of $\bigcup_{y \in \mathcal{X}} (y + S_y)$ containing $x + S_x$. Then $H_\xi(\mathcal{X})$ is the total number of clumps of $\bigcup_{y \in \mathcal{X}} (y + S_y)$; hence

$$H_{\xi_n}(\mathcal{X}_n) = U(\Xi'_n) = U(\Xi_n).$$

Stabilization of ξ follows from the fact that μ_S is concentrated on uniformly bounded sets. The moments condition (2.7) follows from the uniform bound $\xi(x; \mathcal{X}) \leq 1$. Therefore the LLN for $U(\Xi_n)$ follows from (2.10). The LLN for $U_k(\Xi_n)$ is proved similarly.

(ii) This time let $\xi(x; \mathcal{X})$ be the volume of the intersection of $\bigcup_{y \in \mathcal{X}} S_y$ with the Voronoi cell around x for \mathcal{X} . Then

$$H_{\xi_n}(\mathcal{X}_n) = V(\Xi'_n) = nV(\Xi_n).$$

Since μ_S is concentrated on sets contained in $B(\mathbf{0}; K)$, for any $x \in \mathcal{X}$, the intersection of $\bigcup_{y \in \mathcal{X}} S_y$ with the Voronoi cell around x is contained in $B(x; 2K)$, since any point lying outside $B(x; 2K)$ but in $\bigcup_{y \in \mathcal{X}} S_y$ must be closer to some point $y \in \mathcal{X} \setminus \{x\}$ than it is to x . Both the stabilization and the moments conditions in the marked point process version of Theorem 2.1 hold as a consequence of this, and (2.10) yields the LLN for $V(\Xi_n)$.

(iii) This time let $\xi(x; \mathcal{X})$ be the contribution of the random set S_x to the total curvature of the union $\bigcup_{y \in \mathcal{X}} \Xi_y$.

This gives us $H_{\xi}(\mathcal{X}_n) = W(\Xi_n)$. By convexity, $\xi(x; \mathcal{X})$ is uniformly bounded, which gives us the moments condition (2.7).

By the assumption of uniform boundedness of the random sets, if \mathfrak{d}_0 is an independent random shape inserted at the origin, changes outside $B(\mathbf{0}; 2K)$ do not affect the contribution of \mathfrak{d}_0 to the total curvature. Thus, for all τ , ξ is almost surely stabilizing on \mathcal{P}_τ . This enables us to deduce the stabilization condition in the marked point process version of Theorem 2.1, and, we therefore deduce the LLN behavior.

(iv) Let $\xi(x; \mathcal{X})$ be either 1 or 0, depending upon whether or not $x + S_x$ is included in the maximal subset of nonintersecting shapes. If there are several such maximal subsets, choose one in an arbitrary deterministic but translation-invariant manner. Then $H_{\xi}(\mathcal{X})$ is the maximal number of disjoint shapes and $H_{\xi_n}(\mathcal{X}_n) = M(\Xi'_n)$. Since $\xi \leq 1$, this gives the moments condition (2.7).

If $f < \lambda_c$, then there is almost surely no infinite cluster in $\mathcal{P}_{f(x)}$, for we are in the subcritical phase of continuum percolation. Since there is no infinite cluster, inserting a random shape at the origin thus almost surely changes the order of only finite clusters and thus the packing functional M stabilizes. \square

(g) *Packing processes.* Consider the following prototypical random sequential packing model. Let $B_{n,1}, B_{n,2}, \dots, B_{n,n}$ be a sequence of d -dimensional balls of volume n^{-1} whose centers are independent random d -vectors with common probability density function $f: \mathbb{R}^d \rightarrow [0, \infty)$. Let the first ball $B_{n,1}$ be *packed* and, recursively for $i = 2, 3, \dots, n$, let the i th ball $B_{n,i}$ be packed iff $B_{n,i}$ does not overlap any ball in $B_{n,1}, \dots, B_{n,i-1}$ which has already been packed. If not packed, the i th ball is discarded. Let $N_f(n)$ be the number of packed balls out of the first n to arrive. This is sometimes called on-line packing, in contrast with the off-line scheme described earlier.

We may use our general result (2.10) for marked processes to obtain LLN for random sequential packing, as follows. For any finite point set $\mathcal{X} \subset \mathbb{R}^d$, assume the points have marks which are independent and uniformly distributed over $[0, 1]$. Assume unit volume balls centered at the points of \mathcal{X} arrive sequentially in an order determined by the associated marks and assume as before that each ball is packed or discarded according to whether or not it overlaps a previously packed ball. Let $\xi(x; \mathcal{X})$ be either 1 or 0 depending on whether the ball centered at x is packed or discarded. Then with the binomial point process \mathcal{X}_n defined at (2.5), and ξ_n defined at (2.6), it can be seen that $H_{\xi_n}(\mathcal{X}_n)$ has the same distribution as $N_f(n)$. Following [24], we can show that ξ is almost surely stabilizing on \mathcal{P}_τ , $\tau \in (0, \infty)$, with limit ξ_∞ . Since ξ is bounded it satisfies the moments condition (2.7) and, therefore, using the limit (2.10), we get the following LLN for $N_f(n)$.

THEOREM 2.8. *Let $f: \mathbb{R}^d \rightarrow [0, \infty)$ be an arbitrary density. As $n \rightarrow \infty$,*

$$(2.28) \quad n^{-1}N_f(n) \rightarrow \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})]f(x) dx \quad \text{in } L^2.$$

This result represents a finite input version of Theorem 5.1 of Penrose [22], but with a more general class of densities f .

Theorem 2.8 extends to more general versions of the prototypical packing model. For example, by following the general stabilization analysis of [24], we can develop asymptotics in the finite input setting for the number of packed balls in the following general models: (i) models with balls replaced by particles of random size/shape/charge, (ii) time dependent, dynamic models, (iii) cooperative sequential adsorption models and (iv) ballistic deposition models. In each case, we obtain a LLN for the number of packed balls from among the first n to arrive when the distribution of the balls has a density $f: \mathbb{R}^d \rightarrow [0, \infty)$. See [24] for a discussion of these models and for laws of large numbers in the special case where the particles arrive *uniformly* at random over \mathbb{R}^d .

(h) *Combinatorial optimization.* Both the shortest traveling salesman tour and the minimal matching are translation- and scale-invariant graphs on finite point sets. However, it is not known whether their definition can be extended to infinite sets in a manner that makes them stabilizing; therefore, at present we are unable to apply Theorem 2.1 or 2.2 in these cases.

3. Proofs. The proof of Theorem 2.1 centers around suitably coupling a version of the binomial process \mathcal{X}_n to a Cox process, that is, a Poisson process whose intensity measure is itself random. We do this as follows. On a suitable probability space suppose we have, independently, a d -dimensional variable X with density f and a homogeneous Poisson processes \mathcal{P} of rate 1 on $\mathbb{R}^d \times [0, \infty)$.

Define coupled point processes $\mathcal{P}(n)$, \mathcal{X}'_{n-1} and \mathcal{H}_n , and a random variable ζ_n , all in terms of \mathcal{P} and X , as follows. Let $\mathcal{P}(n)$ be the image of the restriction of \mathcal{P} to the set

$$\{(x, t) \in \mathbb{R}^d \times [0, \infty) : t \leq nf(x)\}$$

under the projection $(x, t) \mapsto x$. Then $\mathcal{P}(n)$ is a Poisson process in \mathbb{R}^d with intensity function $nf(\cdot)$, consisting of $N(n)$ random points with common density f . Discard $(N(n) - (n - 1))^+$ of these points chosen at random and add $(n - 1 - N(n))^+$ extra independent points with common density f . The resulting set of points is denoted \mathcal{X}'_{n-1} , and has the same distribution as \mathcal{X}_{n-1} defined earlier.

To define \mathcal{H}_n , let \mathcal{P}^n be the restriction of \mathcal{P} to the set

$$\{(x, t) : t \leq nf(X)\}.$$

Let \mathcal{H}_n be the image of the point set \mathcal{P}^n under the mapping

$$(x, t) \mapsto n^{1/d}(x - X).$$

Given $X = x$, the point process \mathcal{P}^n is a homogeneous Poisson process of intensity 1 on $\mathbb{R}^d \times [0, nf(x)]$ and, therefore, given $X = x$, \mathcal{H}_n is a homogeneous Poisson process on \mathbb{R}^d of intensity $f(x)$. Define ζ_n to be the limit $\xi_\infty(\mathcal{H}_n)$.

Then \mathcal{H}_n is a Cox process, where the randomness of the intensity measure comes from the value of $f(X)$. Note that the distribution of \mathcal{H}_n , and hence that of ζ_n , does not depend on n .

LEMMA 3.1. *Given $K > 0$, we have*

$$(3.1) \quad \lim_{n \rightarrow \infty} P[n^{1/d}(\mathcal{X}'_{n-1} - X) \cap B(\mathbf{0}; K) = \mathcal{H}_n \cap B(\mathbf{0}; K)] = 1.$$

PROOF. Suppose X lies at a Lebesgue point of f (see, e.g., [25]). Given $X = x$, the expected number of points of \mathcal{P} in $B(x; Kn^{-1/d}) \times [0, \infty)$ that contribute to \mathcal{H}_n but not to $\mathcal{P}(n)$ is

$$n \int_{B(x; Kn^{-1/d})} (f(x) - f(y))^+ dy,$$

which tends to zero because x is a Lebesgue point of f . The expected number of points of \mathcal{P} in $B(x; Kn^{-1/d}) \times [0, \infty)$ that contribute to $\mathcal{P}(n)$ but not to \mathcal{H}_n is

$$n \int_{B(x; Kn^{-1/d})} (f(y) - f(x))^+ dy,$$

which also tends to zero for the same reason. Finally the probability that $\mathcal{P}(n) \cap B(x; Kn^{-1/d}) \neq \mathcal{X}'_{n-1} \cap B(x; Kn^{-1/d})$ tends to zero as $n \rightarrow \infty$, since $|N(n) - (n - 1)|$ is $o(n)$ in probability. Integrating over possible values of X and using dominated convergence, we obtain (3.1). \square

LEMMA 3.2. *Suppose that ξ is almost surely stabilizing on \mathcal{P}_τ , with limit $\xi_\infty(\mathcal{P}_\tau)$, for all $\tau \in (0, \infty)$ and that ξ satisfies the moments condition (2.7) for some $p > 1$. Then*

$$(3.2) \quad \lim_{n \rightarrow \infty} E[n^{-1} H_{\xi_n}(\mathcal{X}_n)] = \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})] f(x) dx$$

and the limit in (3.2) is finite.

PROOF. Let $\varepsilon > 0$. Then since

$$\xi_n(X; \mathcal{X}'_{n-1}) = \xi(\mathbf{0}; n^{1/d}(\mathcal{X}'_{n-1} - X))$$

for any $K \in \mathbb{N}$, we have

$$(3.3) \quad P[|\xi_n(X; \mathcal{X}'_{n-1}) - \zeta_n| > \varepsilon] \\ \leq P[n^{1/d}(\mathcal{X}'_{n-1} - X) \cap B(\mathbf{0}; K) \neq \mathcal{H}_n \cap B(\mathbf{0}; K)]$$

$$(3.4) \quad + P[\bar{\xi}(\mathcal{H}_n; K) - \underline{\xi}(\mathcal{H}_n; K) > \varepsilon].$$

By the stabilization assumption, we can choose $K > 0$ so that (3.4) is less than $\varepsilon/2$ and then, by Lemma 3.1, the expression (3.3) is also less than $\varepsilon/2$ for n large. Since ζ_n have the same distribution for all n , it follows that

$$(3.5) \quad \xi_n(X; \mathcal{X}'_{n-1}) \xrightarrow{\mathcal{D}} \zeta_1.$$

Since the bounded p th moments condition (2.7) is assumed to hold for some $p > 1$, the variables $\xi_n(X; \mathcal{X}'_{n-1})$ are uniformly integrable; hence their expectations converge to that of ζ_1 and $E[\zeta_1] < \infty$. By conditioning on X , we obtain

$$(3.6) \quad E[\zeta_1] = \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})] f(x) dx := \mu,$$

which is the right-hand side of (3.2). Since $E[H_{\xi_n}(\mathcal{X}_n)] = nE[\xi_n(X; \mathcal{X}'_{n-1})]$, this gives us (3.2). \square

Next we consider $E[\xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n)]$ and use a refinement of the coupling argument. Assume on a suitable probability space that we have, independently, two d -dimensional variables X, Y with density f and two homogeneous Poisson processes \mathcal{P}, \mathcal{Q} , both of unit intensity on $\mathbb{R}^d \times [0, \infty)$.

Define coupled point processes \mathcal{X}'_{n-2} (a binomial process), and $\mathcal{H}_n^X, \mathcal{H}_n^Y$ (both Cox processes), and variables ζ_n^X and ζ_n^Y , all in terms of $\mathcal{P}, \mathcal{Q}, X$ and Y , as follows. Let \mathcal{X}'_{n-2} be obtained just as \mathcal{X}'_{n-1} was before, that is, let $\mathcal{P}(n)$ be the image of the restriction of \mathcal{P} to the set $\{(x, t) \in \mathbb{R}^d \times [0, \infty) : t \leq nf(x)\}$ under the projection $(x, t) \mapsto x$, and let $N(n)$ be the number of points of $\mathcal{P}(n)$. Discard $(N(n) - (n - 2))^+$ of the points of $\mathcal{P}(n)$, chosen at random, and add $(n - 2 - N(n))^+$ extra independent points with common density f . The resulting set of points is denoted \mathcal{X}'_{n-2} and has the same distribution as \mathcal{X}_{n-2} .

Let F_X be the half-space of points in \mathbb{R}^d closer to X than to Y and let F_Y be the half-space of points in \mathbb{R}^d closer to Y than to X . Construct \mathcal{H}_n^X as follows. Let \mathcal{P}_X^n be the restriction of \mathcal{P} to the set $F_X \times [0, nf(X)]$. Let \mathcal{Q}_X^n be the restriction of \mathcal{Q} to the set $F_Y \times [0, nf(X)]$. Let \mathcal{H}_n^X be the image of the point process $\mathcal{P}_X^n \cup \mathcal{Q}_X^n$ under the mapping

$$(x, t) \mapsto n^{1/d}(x - X).$$

Given $X = x$, the point process $\mathcal{P}_X^n \cup \mathcal{Q}_X^n$ is a homogeneous Poisson process of intensity 1 on $\mathbb{R}^d \times [0, nf(x)]$. Hence, given $X = x$, \mathcal{H}_n^X is a homogeneous Poisson process on \mathbb{R}^d of intensity $f(x)$. Let ζ_n^X be the associated limit $\xi_\infty(\mathcal{H}_n^X)$.

Construct \mathcal{H}_n^Y in the following analogous manner. Let \mathcal{P}_Y^n be the restriction of \mathcal{P} to the set $F_Y \times [0, nf(Y)]$. Let \mathcal{Q}_Y^n be the restriction of \mathcal{Q} to the set $F_X \times [0, nf(Y)]$. Let \mathcal{H}_n^Y be the image of the point process $\mathcal{P}_Y^n \cup \mathcal{Q}_Y^n$ under the mapping

$$(x, t) \mapsto n^{1/d}(x - Y).$$

By an argument similar to that used for \mathcal{H}_n^X , the point process \mathcal{H}_n^Y , given $Y = y$, is a homogeneous Poisson process on \mathbb{R}^d of intensity $f(y)$. We set $\zeta_n^Y := \xi_\infty(\mathcal{H}_n^Y)$.

We can now prove the following result, which is the case $q = 2$ of Theorem 2.1.

PROPOSITION 3.1. *If ξ is almost surely stabilizing on \mathcal{P}_τ with limit $\xi_\infty(\mathcal{P}_\tau)$ for all $\tau \in (0, \infty)$ and if ξ satisfies the moments condition (2.7) for some $p > 2$, then as $n \rightarrow \infty$,*

$$(3.7) \quad n^{-1} H_{\xi_n}(\mathcal{X}_n) \rightarrow \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})] f(x) dx \quad \text{in } L^2.$$

PROOF. For any $K > 0$, we have

$$(3.8) \quad \lim_{n \rightarrow \infty} P[n^{1/d}((\mathcal{X}'_{n-2} \cup \{Y\}) - X) \cap B(\mathbf{0}; K) = \mathcal{H}_n^X \cap B(\mathbf{0}; K)] = 1,$$

$$(3.9) \quad \lim_{n \rightarrow \infty} P[n^{1/d}((\mathcal{X}'_{n-2} \cup \{X\}) - Y) \cap B(\mathbf{0}; K) = \mathcal{H}_n^Y \cap B(\mathbf{0}; K)] = 1.$$

The proof of these facts is just the same as that of (3.1) using the additional observation that

$$\lim_{n \rightarrow \infty} P[B(X; Kn^{-1/d}) \subset F_X] = \lim_{n \rightarrow \infty} P[B(Y; Kn^{-1/d}) \subset F_Y] = 1.$$

Note that \mathcal{H}_n^X and \mathcal{H}_n^Y are *independent* identically distributed Cox processes. Independence follows by conditioning on the values of X, Y . Given these values, the point processes \mathcal{H}_n^X and \mathcal{H}_n^Y are constructed from Poisson processes on disjoint regions of space. Therefore, for each n , the variables ζ_n^X and ζ_n^Y are independent. Also the joint distribution of ζ_n^X, ζ_n^Y is independent of n .

Since by translation invariance,

$$\xi_n(X; \mathcal{X}'_{n-2} \cup \{X, Y\}) = \xi(\mathbf{0}; n^{1/d}((\mathcal{X}'_{n-2} \cup \{Y\}) - X))$$

and

$$\xi_n(Y; \mathcal{X}'_{n-2} \cup \{X, Y\}) = \xi(\mathbf{0}; n^{1/d}((\mathcal{X}'_{n-2} \cup \{X\}) - Y)),$$

it follows from (3.8) and (3.9), by a similar argument to that which yielded (3.5), that as $n \rightarrow \infty$,

$$(3.10) \quad \xi_n(X; \mathcal{X}'_{n-2} \cup \{X, Y\})\xi_n(Y; \mathcal{X}'_{n-2} \cup \{X, Y\}) \xrightarrow{\mathcal{D}} \zeta_1^X \zeta_1^Y$$

and hence

$$(3.11) \quad \xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n) \xrightarrow{\mathcal{D}} \zeta_1^X \zeta_1^Y.$$

By assumption, the bounded p th moments condition (2.7) holds for some $p > 2$. Then by Cauchy–Schwarz,

$$\sup_{n \in \mathbb{N}} E[(\xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n))^{p/2}] < \infty$$

and therefore the variables $\xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n)$, defined for each $n \geq 2$, are uniformly integrable, so that the convergence (3.11) also holds in the sense of convergence of means, that is,

$$(3.12) \quad \lim_{n \rightarrow \infty} E[\xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n)] = E[\zeta_1^X \zeta_1^Y] = \mu^2$$

with μ defined at (3.6). Here we have used independence of ζ_n^X and ζ_n^Y .

To complete the proof, observe that

$$E[(n^{-1}H_{\xi_n}(\mathcal{X}_n))^2] = n^{-1}E[\xi_n(X_1; \mathcal{X}_n)^2] + \left(1 - \frac{1}{n}\right)E[\xi_n(X_1; \mathcal{X}_n)\xi_n(X_2; \mathcal{X}_n)]$$

and in the right-hand side the first term tends to zero by the bounded p th moments condition, while the second term tends to μ^2 by (3.12). Therefore $E[n^{-1}H_{\xi_n}(\mathcal{X}_n)] \rightarrow \mu$ and $E[(n^{-1}H_{\xi_n}(\mathcal{X}_n))^2] \rightarrow \mu^2$, so that $n^{-1}H_{\xi_n}(\mathcal{X}_n)$ tends to μ in mean square. \square

Finally we prove L^1 convergence, that is, the case $q = 1$ of Theorem 2.1, completing the proof of that result.

PROPOSITION 3.2. *If ξ is almost surely stabilizing on \mathcal{P}_τ with limit $\xi_\infty(\mathcal{P}_\tau)$ for all $\tau \in (0, \infty)$ and if ξ satisfies the moments condition (2.7) for some $p > 1$, then as $n \rightarrow \infty$,*

$$(3.13) \quad n^{-1}H_{\xi_n}(\mathcal{X}_n) \rightarrow \int_{\mathbb{R}^d} E[\xi_\infty(\mathcal{P}_{f(x)})]f(x) dx \quad \text{in } L^1.$$

PROOF. Given $K > 0$, define the functional

$$\xi^K(x, \mathcal{X}) := \min(\xi(x; \mathcal{X}), K).$$

Then by the stabilization condition for ξ , the truncated functional ξ^K also stabilizes on \mathcal{P}_τ with limit $\xi_\infty^K(\mathcal{P}_\tau) := \min(\xi_\infty(\mathcal{P}_\tau), K)$. We leave it to the reader to verify this assertion. Since ξ^K is uniformly bounded, by Proposition 3.1 and the fact that L^2 convergence implies L^1 convergence, we have

$$(3.14) \quad n^{-1}H_{\xi_n^K}(\mathcal{X}_n) \rightarrow \int_{\mathbb{R}^d} E[\xi_\infty^K(\mathcal{P}_{f(x)})]f(x) dx \quad \text{in } L^1.$$

Moreover,

$$\begin{aligned} 0 &\leq E[n^{-1}H_{\xi_n}(\mathcal{X}) - n^{-1}H_{\xi_n^K}(\mathcal{X})] \\ &= E[\xi(n^{1/d}X_1; n^{1/d}\mathcal{X}_n) - \xi^K(n^{1/d}X_1; n^{1/d}\mathcal{X}_n)], \end{aligned}$$

which tends to zero as $K \rightarrow \infty$, uniformly in n , because the assumed moments condition (2.7), $p > 1$, implies uniform integrability of the family of variables $\xi(n^{1/d}X_1; n^{1/d}\mathcal{X}_n)$, $n \geq 1$. Also, by monotone convergence the right-hand side of (3.14) converges to the right-hand side of (3.13) as $K \rightarrow \infty$. Combining these facts and taking K to infinity in (3.14), we obtain (3.13). \square

We now work toward a proof of Theorem 2.2. First we show that stabilization of the graph G as defined in Section 2.1 implies an apparently stronger form of the stabilizing property.

LEMMA 3.3. *Suppose the graph $G(\mathcal{X})$ is translation invariant and stabilizes on \mathcal{P}_1 . Then with probability 1, for all points $X \in \mathcal{P}_{1,0}$ there exists $R(X) < \infty$ such that the set of edges of $G(\mathcal{P}_{1,0})$ incident to X is unchanged if points are added and/or deleted outside $B(\mathbf{0}; R(X))$.*

PROOF. Let us say that a point x in a locally finite set \mathcal{X} is *unstable* for $G(\mathcal{X})$ if there does *not* exist $r > 0$ such that

$$\mathcal{E}(x; G(\mathcal{X})) = \mathcal{E}(x; G(\mathcal{X} \cap B(\mathbf{0}; r)) \cup \mathcal{A})$$

for almost all finite $A \subset \mathbb{R}^d \setminus B(\mathbf{0}; r)$.

Let $t > 1$. By translation invariance and the fact that the Poisson point process is its own Palm distribution, the mean number of points of \mathcal{P}_1 in $B(\mathbf{0}; t)$ that are unstable for $G(\mathcal{P}_1)$ is equal to

$$\int_{B(\mathbf{0}; t)} P[x \text{ is unstable for } \mathcal{P}_1 \cup \{x\}] dx,$$

which is zero by the assumption that G stabilizes on \mathcal{P}_1 . Therefore, since t can be arbitrarily large, with probability 1 the Poisson process \mathcal{P}_1 has no unstable points.

A further application of Palm theory for Poisson processes shows that the mean number of pairs of distinct points X, Y of \mathcal{P}_1 in $B(\mathbf{0}; t)$ such that Y is an unstable point of \mathcal{P}_1 is equal to

$$\int_{B(\mathbf{0}; t)} \int_{B(\mathbf{0}; t)} P[y \text{ is unstable for } \mathcal{P}_1 \cup \{x, y\}] dx dy$$

and since the mean number of such pairs is zero, the above integral is zero. Therefore, for almost all x in $B(\mathbf{0}; t)$,

$$(3.15) \quad \text{Leb}\{y \in B(\mathbf{0}; t) : P[y \text{ is unstable for } \mathcal{P}_1 \cup \{x, y\}] > 0\} = 0.$$

Choose x_0 in $B(\mathbf{0}; 1)$ such that (3.15) holds with $x = x_0$. Then the set of z in $B(-x_0; t)$ such that

$$P[z \text{ is unstable for } \mathcal{P}_1 \cup \{\mathbf{0}, z\}] > 0$$

has zero measure. Integrating over $z \in B(-x_0; t)$ and using Palm theory yet again, we find that the mean number of Poisson points $X \in \mathcal{P}_1$ in $B(-x_0; t)$ such that X is unstable for $G(\mathcal{P}_1 \cup \{\mathbf{0}\})$ is zero. Since t can be arbitrarily large, this gives us the result. \square

PROOF OF THEOREM 2.2. To prove (2.12) let $\xi(x; \mathcal{X})$ be the reciprocal of the order of the component containing x in $G(\mathcal{X})$. If the component containing the origin in $G(\mathcal{P}_{1,0})$ is finite, then by stabilization of G (Lemma 3.3) there exists R such that alterations to \mathcal{P}_1 outside $B(\mathbf{0}; R)$ will not cause any change in this component. If the component containing the origin in $G(\mathcal{P}_{1,0})$ is infinite, then given $\varepsilon > 0$ we can find a connected subgraph of $G(\mathcal{P}_{1,0})$ of order greater than ε^{-1} . Then by Lemma 3.3 there exists R such that alterations to \mathcal{P}_1 outside $B(\mathbf{0}; R)$ will not cause any removal of edges in this subgraph and, hence, the functional ξ stabilizes on \mathcal{P}_1 with limit σ_G^{-1} . By scale invariance, ξ also stabilizes on \mathcal{P}_τ with limit having the same mean as σ_G^{-1} . Since $\xi(x; \mathcal{X})$ is uniformly bounded by 1, the moments condition (2.7) is trivially satisfied and by Theorem 2.1 we have (2.12).

To prove (2.13) let $\xi(x; \mathcal{X})$ be equal to 1 if $G(\mathcal{X})$ contains a subgraph isomorphic to Γ with a vertex at x and equal to zero if not. This is bounded by 1 and, by Lemma 3.3, ξ stabilizes on \mathcal{P}_1 with limit equal to 1 if E_G occurs and equal to 0 if not. Also ξ stabilizes on \mathcal{P}_τ with limit having the same distribution and Theorem 2.1 applies to yield (2.13).

To prove (2.15), observe that the functional $L_\phi^G(\mathcal{X}) = \sum_{e \in G(\mathcal{X})} \phi(|e|)$ has the representation $L_\phi^G(\mathcal{X}) = L_\xi(\mathcal{X})$ with

$$(3.16) \quad \xi(x; \mathcal{X}) = \frac{1}{2} \sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi(|e|).$$

If G is scale invariant, then $G(\mathcal{P}_\tau)$ has the same distribution as $G(\tau^{-1/d} \mathcal{P}_1)$ and, therefore, ξ stabilizes on \mathcal{P}_τ with

$$(3.17) \quad E[\xi_\infty(\mathcal{P}_\tau)] = \frac{1}{2} E \sum_{e \in \mathcal{E}(\mathbf{0}; G(\mathcal{P}_{1,0}))} \phi(\tau^{-1/d} |e|).$$

Then (2.15) follows from Theorem 2.1. \square

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REFERENCES

- [1] ALDOUS, D. and STEELE, J. M. (1992). Asymptotics for Euclidean minimal spanning trees on random points. *Probab. Theory Related Fields* **92** 247–258.
- [2] Alexander, K. S. (1995). Simultaneous uniqueness of infinite clusters in stationary random labeled graphs. *Comm. Math. Phys.* **168** 39–55.
- [3] BEZUIDENHOUT, C., GRIMMETT, G. and LÖFFLER, A. (1998). Percolation and minimal spanning trees. *J. Statist. Phys.* **92** 1–34.
- [4] BORG, I. and GROENEN, P. (1997). *Modern Multidimensional Scaling: Theory and Applications*. Springer, New York.
- [5] BRITO, M., QUIROZ, A. and YUKICH, J. E. (2002). Graph theoretic procedures for dimension identification. *J. Multivariate Anal.* **81** 67–84.
- [6] DEVROYE, L. (1988). The expected size of some graphs in computational geometry. *Comput. Math. Appl.* **15** 53–64.
- [7] EPPSTEIN, D., PATERSON, M. S. and YAO, F. F. (1997). On nearest-neighbor graphs. *Discrete Comput. Geom.* **17** 263–282.
- [8] FÜREDI, Z. (1997). The expected size of a random sphere of influence graph. In *Intuitive Geometry* (I. Bárány and K. Böröczky, eds.) 319–326. János Bolyai Math. Society, Budapest, Hungary.
- [9] GRIMMETT, G. (1989, 1999). *Percolation*. Springer, New York.
- [10] HALL, P. (1988). *Introduction to the Theory of Coverage Processes*. Wiley, New York.
- [11] HAYEN, A. and QUINE, M. P. (2000). The proportion of triangles in a Poisson–Voronoi tessellation of the plane. *Adv. in Appl. Probab.* **32** 67–74.
- [12] HENZE, N. (1987). On the fraction of random points with specified nearest neighbor interrelations and degree of attraction. *Adv. in Appl. Probab.* **19** 873–895.
- [13] HITCZENKO, P., JANSON, S. and YUKICH, J. E. (1999). On the variance of the random sphere of influence graph. *Random Structures Algorithms* **14** 139–152.
- [14] JAROMCZYK, J. W. and TOUSSAINT, G. T. (1992). Relative neighborhood graphs and their relatives. *Proc. IEEE* **80** 1502–1517.
- [15] JIMENEZ, R. and YUKICH, J. E. (2002). Strong laws for Euclidean graphs with general edge weights. *Statist. Probab. Lett.* **56** 251–259.
- [16] KESTEN, H. and LEE, S. (1996). The central limit theorem for weighted minimal spanning trees on random points. *Ann. Appl. Probab.* **6** 495–527.
- [17] MATHERON, G. (1975). *Random Sets and Integral Geometry*. Wiley, New York.
- [18] MCGIVNEY, K. (1997). Probabilistic limit theorems for combinatorial optimization problems. Ph.D. dissertation, Lehigh Univ.
- [19] MCGIVNEY, K. and YUKICH, J. E. (1999). Asymptotics for Voronoi tessellations on random samples. *Stochastic Process. Appl.* **83** 273–288.
- [20] MOLCHANOV, I. (1997). *Statistics of the Boolean Model for Practitioners and Mathematicians*. Wiley, New York.
- [21] PENROSE, M. D. (1996). The random minimal spanning tree in high dimensions. *Ann. Probab.* **24** 1903–1925.
- [22] PENROSE, M. D. (2001). Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.* **218** 153–176.
- [23] PENROSE, M. D. and YUKICH, J. E. (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11** 1005–1041.

- [24] PENROSE, M. D. and YUKICH, J. E. (2001). Limit theory for random sequential packing and deposition. *Ann. Appl. Probab.* **12** 272–301.
- [25] RUDIN, W. (1987). *Real and Complex Analysis*, 3rd ed. McGraw-Hill, New York.
- [26] SMITH, W. D. (1989). Studies in computational geometry motivated by mesh generation. Ph.D. dissertation, Princeton Univ.
- [27] STEELE, J. M. (1997). *Probability Theory and Combinatorial Optimization*. SIAM, Philadelphia.
- [28] STEELE, J. M., SHEPP, L. and EDDY, W. (1987). On the number of leaves of a Euclidean minimal spanning tree. *J. Appl. Probab.* **24** 809–826.
- [29] YUKICH, J. E. (1998). *Probability Theory of Classical Euclidean Optimization Problems. Lecture Notes in Math.* **1675**. Springer, Berlin.
- [30] YUKICH, J. E. (1999). Asymptotics for weighted minimal spanning trees on random points. *Stochastic Process. Appl.* **85** 123–138.

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