*The Annals of Applied Probability*
2002, Vol. 12, No. 4, 1149–1173

# GAUSSIAN APPROXIMATION THEOREMS FOR URN MODELS AND THEIR APPLICATIONS

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We consider weak and strong Gaussian approximations for a two-color generalized Friedman's urn model with homogeneous and nonhomogeneous generating matrices. In particular, the functional central limit theorems and the laws of iterated logarithm are obtained. As an application, we obtain the asymptotic properties for the randomized-play-the-winner rule. Based on the Gaussian approximations, we also get some variance estimators for the urn model.

**1. Introduction.** Adaptive designs in clinical trials have received considerable attention in the literature. The goal of adaptive designs is to pursue higher survival rates in a long run of clinical trials while not significantly affecting the accuracy of the statistical inferences on all treatments involved in the trials. In these designs, more patients are sequentially to be assigned to better treatments, based on outcomes of previous treatments in clinical trials. A very important class of adaptive designs is based on the generalized Friedman's urn (GFU) model [also called the generalized Pólya urn (GPU) in the literature] which has been used in clinical trials, bioassay and psychophysics. For more detailed references, the reader is referred to Flournoy and Rosenberger (1995), Rosenberger (1996), Rosenberger and Grill (1997). Athreya and Karlin (1968) first considered the asymptotic properties of the GFU model with homogeneous generating matrix. Smythe (1996) defined the extended Pólya urn model (EPU) (a special class of GFU) and considered its asymptotic normality. In applications, it is quite often that the generating matrices are not homogeneous. Examples can be found in Coad (1991) and Hu and Rosenberger (2000) as well as Bai, Hu and Shen (2002). For the nonhomogeneous case, Bai and Hu (1999) establish strong consistency and asymptotic normality of the GFU model. Statistical inference about adaptive designs is considered in Wei, Smythe, Lin and Park (1990), Rosenberger and Sriram (1997) for the homogeneous case and Hu, Rosenberger and Zidek (2000) for the nonhomogeneous case.

In this paper, we consider a two-color GFU model with $W_0$ white and $\overline{W}_0$ black balls with $T_0 = W_0 + \overline{W}_0$. Balls are drawn at random in succession, their

Received November 2000; revised January 2002.

[1]Supported in part by National University of Singapore, Grant R-155-000-015-112.

[2]Supported in part by National Natural Science Foundation of China Grant 10071072 and a grant from Zhejiang Province, Natural Science Foundation.

*AMS 2000 subject classifications.* Primary 62G10; secondary 60F15, 62E10.

*Key words and phrases.* Gaussian approximation, the law of iterated logarithm, functional central limit theorems, urn model, nonhomogeneous generating matrix, randomized play-the-winner rule.

color noticed and then replaced in the urn, together with new black and white balls. Replacements are controlled by a sequence of *rule* matrices $\mathbf{R}_i = \left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$ as follows: at stage $i$, if a white ball is drawn, it is returned to the urn with $A_i$ white and $B_i$ black balls. Otherwise, when a black is drawn, it is returned with $C_i$ white and $D_i$ black balls. Negative entries in $\mathbf{R}_i$ are allowed and correspond to removals. After $n$ splits and generations, the numbers of white and black balls in the urn are denoted by $W_n$ and $\overline{W}_n$, respectively, and $T_n = W_n + \overline{W}_n$ is the total number of balls.

In a two-arm clinical trial, the white and black balls represent treatments 1 and 2, respectively. If a white ball is drawn at the $i$th stage, then the treatment 1 is assigned to the $i$th patient. The rule $\mathbf{R}_i$ is usually a function of $\xi(i)$, a random variable associated with the $i$th stage of the clinical trial, which may include measurements on the $i$th patient and the outcome of the treatment at the $i$th stage. The sequence of the expectations of the rules

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{E}A_i & \mathbf{E}B_i \\ \mathbf{E}C_i & \mathbf{E}D_i \end{bmatrix} =: \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

are called *generating matrices*. The GFU model is called *homogeneous* if $\mathbf{H}_i = \mathbf{H}$ for all $i$.

When $\mathbf{R}_i = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a deterministic matrix for all $i$, Gouet (1993) established the weak invariance principle for the urn process $\{W_n\}$. This leads us to show that the urn process $\{W_n\}$ can be weakly and strongly approximated by a Gaussian process for both the homogeneous and nonhomogeneous cases. As an application, we establish the weak invariance principle and the law of the iterated logarithm for $\{W_n\}$. The technique used here is the Gaussian approximation of a process, which is different from Gouet (1993) as well as others. Some results of Bai and Hu (1999, 2000), if reduced to the two-arm case, can also be obtained as special cases of the results in the present paper.

The paper is organized as follows. In Section 2, we first describe the model and some important assumptions. Then some main theorems are presented. The proofs are given in Section 3. In Section 4, we apply the results to the randomized play-the-winner rule [Wei (1979)] to get its asymptotic properties. The asymptotic results in Section 2 depend on an unknown variance. Based on $W_n$, we obtain two variance estimators of the GFU model by using the Gaussian approximation.

## 2. Main results.

2.1. *Notation and assumptions.* Suppose that there is a sequence of increasing $\sigma$-fields $\{\mathcal{F}_n\}$ and that $W_n$, $A_n$ and $C_n$ are three sequences of random variables which are adapted to $\{\mathcal{F}_n\}$ and satisfy the following model:

$$W_n = W_{n-1} + I_n A_n + (1 - I_n) C_n, \tag{2.1}$$

where $(A_n, C_n)$ is the adding rule at the stage $n$ and $I_n$ is the result of the $n$th draw with $I_n = 1$ or 0 according to whether a white ball or a black is drawn. We assume that for each $n$, $(A_n, C_n)$ is conditionally independent of $I_n$ when given $\mathcal{F}_{n-1}$ and $\mathbf{P}(I_n = 1|\mathcal{F}_{n-1}) = W_{n-1}/T_{n-1}$, where $T_n = W_n + \overline{W}_n$ is the total number of all balls in the urn at stage $n$. Write

$$\mathbf{E}(A_n|\mathcal{F}_{n-1}) = a_n, \qquad \mathbf{E}(C_n|\mathcal{F}_{n-1}) = c_n,$$

where $a_n$ and $c_n$ are assumed to be nonrandom. The model is called homogeneous if $a_i = a$ and $c_i = c$ for all $i$.

We need the following assumptions.

ASSUMPTION 2.1. *$T_n = ns + \beta$, where $\beta > 0$ is the number of the balls in the initial urn and $s$ is the number of balls added to the urn at each stage. Without loss of generality, we assume $\beta = 1$ and $s = 1$.*

In some cases, the number of balls added to the urn at each stage is random. Thus, $T_n$ may be a random variable and Assumption 2.1 may not be satisfied. In such cases, we shall assume that $T_n$ is not far away from $ns + \beta$. And thus in those cases, we shall make an assumption on the distance of $T_n$ from $ns + \beta$ instead of Assumption 2.1. For example, we may assume that $T = ns + \beta + o(\sqrt{n})$ in $L_2$ when we consider the $L_2$-approximations.

ASSUMPTION 2.2. *$a_n \to a$ and $c_n \to c$ as $n \to \infty$. Denote $\rho_n = a_n - c_n$, $\rho = a - c$ and $\mu = c/(1-\rho)$. Assume $\rho \le 1/2$.*

ASSUMPTION 2.3. *For some $C > 0$ and $0 < \varepsilon \le 1$, the rule $(A_n, C_n)$ satisfies*

$$\mathbf{E}|A_n|^{2+\varepsilon} \le C < \infty, \qquad \mathbf{E}|C_n|^{2+\varepsilon} < \infty \qquad \textit{for all } n$$

*and also*

$$\mathrm{Var}(A_n|\mathcal{F}_{n-1}) \to V_a \quad a.s., \qquad \mathrm{Var}(C_n|\mathcal{F}_{n-1}) \to V_c \quad a.s.,$$

*where $V_a$ and $V_c$ are nonrandom nonnegative numbers.*

ASSUMPTION 2.4. *$|a_n - a| + |c_n - c| = o((\log\log n)^{-1})$ and $|\mathrm{Var}(A_n|\mathcal{F}_{n-1}) - V_a| + |\mathrm{Var}(C_n|\mathcal{F}_{n-1}) - V_c| = o((\log\log n)^{-1})$ a.s.*

ASSUMPTION 2.5. *For some $0 < \varepsilon \le 1$, $|a_n - a| + |c_n - c| = o((\log n)^{-1-\varepsilon})$ and $|\mathrm{Var}(A_n|\mathcal{F}_{n-1}) - V_a| + |\mathrm{Var}(C_n|\mathcal{F}_{n-1}) - V_c| = o((\log n)^{-1-\varepsilon})$ a.s.*

ASSUMPTION 2.6. *$|a_n - a| + |c_n - c| = O(n^{-1/2})$, $|\mathrm{Var}(A_n|\mathcal{F}_{n-1}) - V_a| + |\mathrm{Var}(C_n|\ \mathcal{F}_{n-1}) - V_c| = O(n^{-1/2})$ a.s. and*

$$\mathbf{E}|A_n|^4 \le C < \infty, \quad \mathbf{E}|C_n|^4 \le C < \infty \qquad \textit{for all } n.$$

2.2. *Main results.* Denote

$$\sigma_M^2 = \mu V_a + (1-\mu)V_c + \rho^2\mu(1-\mu), \qquad \sigma = \sigma_M/\sqrt{1-2\rho}, \tag{2.2}$$

$e_0 = 1$ and

$$e_n = \sum_{k=0}^{n-1} \rho_{k+1}\frac{e_k}{k+1} + \sum_{k=1}^{n} c_k, \tag{2.3}$$

for all $n \le 1$.

The following are the first two approximations related to the law of the iterated logarithm and the invariance principle.

THEOREM 2.1. *If* $\rho < 1/2$ *and* $T_n = n + 1 + o((n\log\log n)^{1/2})$ *a.s., then under Assumptions* 2.2, 2.3, *there exists a probability space on which the sequence* $\{W_n\}$ *and a standard Brownian motion* $W(\cdot)$ *are so defined that*

$$W_n - e_n - G_n = o\big((n\log\log n)^{1/2}\big) \quad a.s. \tag{2.4}$$

*Also, if we further assume* $T_n = n + 1 + o((n\log\log n)^{1/2})$ *in* $L_1$, *then*

$$W_n - \mathbf{E}W_n - G_n = o\big((n\log\log n)^{1/2}\big) \quad a.s., \tag{2.5}$$

*where*

$$G_t = t^\rho \int_0^t \frac{dW(s\sigma_M^2)}{s^\rho}, \qquad t \ge 0 \tag{2.6}$$

*and*

$$\{G_t;\ t\ge 0\} \overset{\mathcal{D}}{=} \{\sigma t^\rho W(t^{1-2\rho});\ t \ge 0\}. \tag{2.7}$$

*In addition, if*

$$\sum_{k=1}^{n} \{(a_k - a)\mu + (c_k - c)(1-\mu)\} = o(\sqrt{n}), \tag{2.8}$$

*then*

$$W_n - n\mu - G_n = o\big((n\log\log n)^{1/2}\big) \quad a.s. \tag{2.9}$$

THEOREM 2.2. *Under Assumptions* 2.2 *and* 2.3, *if* $\rho < 1/2$ *and* $T_n = n + 1 + o(\sqrt{n})$ *in* $L_2$, *then*

$$\max_{k\le n} |W_k - e_k - G_k| = o(\sqrt{n}) \qquad in\ L_2 \tag{2.10}$$

*and*

$$\max_{k\le n} |W_k - \mathbf{E}W_k - G_k| = o(\sqrt{n}) \qquad in\ L_2. \tag{2.11}$$

*Furthermore, if condition* (2.8) *is also satisfied, then*

$$\max_{k\le n} |W_k - k\mu - G_k| = o(\sqrt{n}) \qquad in\ L_2. \tag{2.12}$$

From Theorems 2.1 and 2.2, it is easily seen that

COROLLARY 2.1. *Assume* $\rho < 1/2$, *and* $T_n = n + 1 + o(\sqrt{n})$ *in* $L_2$, *then under Assumptions* 2.2, 2.3,

$$n^{1/2}(W_{[nt]} - \mathbf{E}W_{[nt]}) \Longrightarrow \sigma t^{\rho} W(t^{1-2\rho}); \tag{2.13}$$

*if* $T_n = n + 1 + o((n \log\log n)^{1/2})$ *a.s. and in* $L_1$, *then under Assumptions* 2.2, 2.3,

$$\limsup_{n\to\infty} \frac{W_n - \mathbf{E}W_n}{\sqrt{2n \log\log n}} = \sigma \quad a.s. \tag{2.14}$$

*Furthermore*, *if condition* (2.8) *is also satisfied*, *then* $\mathbf{E}W_n$ *can be placed by* $n\mu$.

REMARK. (2.13) was first established by Gouet (1993) in the case of $A_n = a$ and $C_n = c$ for all $n$. Result (2.14) is new. For the random and non-homogeneous Pólya's urn, Bai and Hu (1999) showed that

$$n^{-1/2}(W_n - \mathbf{E}W_n) \xrightarrow{\mathcal{D}} N(0, \sigma) \tag{2.15}$$

under the condition

$$\sum_{k=1}^{\infty} \frac{|a_k - a| + |c_k - c|}{k} < \infty. \tag{2.16}$$

Also, the result of Bai and Hu (2000) implies that

$$n^{-1/2}(W_n - n\mu) \xrightarrow{\mathcal{D}} N(0, \sigma),$$

but the following condition is needed:

$$\sum_{k=1}^{\infty} \frac{|a_k - a| + |c_k - c|}{\sqrt{k}} < \infty. \tag{2.17}$$

Obviously, condition (2.17) is stronger than (2.8). But, Bai and Hu (1999, 2000) studied the multicolor urn models.

Assumptions 2.2 and 2.3 used in Theorems 2.1 and 2.2 are very weak and standard, but the rates of the approximations obtained are slow. The next three theorems give faster rates for strong approximations.

THEOREM 2.3. *If* $\rho < 1/2$ $T_n = n + 1 + o(\sqrt{n})$ *a.s.*, *then under Assumptions* 2.2, 2.3 *and* 2.4,

$$W_n - e_n - G_n = o(\sqrt{n}) \quad a.s. \tag{2.18}$$

*And if also* $T_n = n + 1 + o(\sqrt{n})$ *in* $L_1$, *then*

$$W_n - \mathbf{E}W_n - G_n = o(\sqrt{n}) \quad a.s. \tag{2.19}$$

*Furthermore*, *if* (2.8) *holds*, *then*

$$W_n - n\mu - G_n = o(\sqrt{n}) \quad a.s. \tag{2.20}$$

THEOREM 2.4. *If* $\rho < 1/2$ *and* $T_n = n + 1 + o(n^{1/2}(\log n)^{-1/2-\varepsilon})$ *a.s., then under Assumptions* 2.2, 2.3 *and* 2.5,

$$W_n - e_n - G_n = o\big(n^{1/2}(\log n)^{-1/2-\varepsilon/3}\big) \quad a.s.$$

*and if also* $T_n = n + 1 + o(n^{1/2}(\log n)^{-1/2-\varepsilon})$ *in* $L_1$, *then*

$$W_n - \mathbf{E}W_n - G_n = o\big(n^{1/2}(\log n)^{-1/2-\varepsilon/3}\big) \quad a.s.$$

THEOREM 2.5. *If* $\rho < 1/2$, *then under Assumptions* 2.1, 2.2 *and* 2.6 *we have*

$$W_n - e_n - G_n = o(n^{1/2-\delta}) \qquad a.s.\ \forall\, 0 < \delta < (1/2 - \rho) \wedge (1/4)$$

*and*

$$W_n - \mathbf{E}W_n - G_n = o(n^{1/2-\delta}) \qquad a.s.\ \forall\, 0 < \delta < (1/2 - \rho) \wedge (1/4),$$

*where* $a \wedge b = \min(a, b)$.

It is known that the best convergence rate of Skorokhod embedding is $O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4})$. Theorem 2.5 gives an approximation close to this rate. In the remainder of this section, we give a strong approximation in the case of $\rho = 1/2$.

THEOREM 2.6. *Suppose* $\rho = 1/2$ *and* $T_n = n + 1 + o(n^{1/2}(\log n)^{1/2-\varepsilon})$ *a.s. Then under Assumptions* 2.2, 2.3, 2.5 *and* (2.16) *there exists a* $\delta > 0$ *such that*

$$W_n - e_n - \widehat{G}_n = o\big(n^{1/2}(\log n)^{1/2-\delta}\big) \quad a.s. \tag{2.21}$$

*Also if* $T_n = n + 1 + o(n^{1/2}(\log n)^{1/2-\varepsilon})$ *in* $L_1$, *then*

$$W_n - \mathbf{E}W_n - \widehat{G}_n = o\big(n^{1/2}(\log n)^{1/2-\delta}\big) \quad a.s.,$$

*where*

$$\widehat{G}_t = t^{1/2}\int_1^t \frac{dW(s\sigma_M^2)}{s^{1/2}}, \qquad t \ge 0 \tag{2.22}$$

*and*

$$\{\widehat{G}_t; t > 0\} \stackrel{\mathcal{D}}{=} \{\sigma_M t^{1/2} W(\log t); t > 0\}. \tag{2.23}$$

*Furthermore*, *if condition* (2.17) *is satisfied*, *then*

$$W_n - n\mu - \widehat{G}_n = o\big(n^{1/2}(\log n)^{1/2-\delta}\big) \quad a.s.$$

The following corollary comes from Theorem 2.6 immediately.

COROLLARY 2.2. *Under the conditions in Theorem* 2.6, *we have*

$$(n^t \log n)^{1/2}\big(W_{[n^t]} - \mathbf{E}W_{[n^t]}\big) \Longrightarrow \sigma_M W(t),$$

*and*

$$\limsup_{n\to\infty} \frac{W_n - \mathbf{E}W_n}{\sqrt{2n(\log n)(\log\log\log n)}} = \sigma_M \quad a.s.$$

*Furthermore*, *if condition* (2.17) *is satisfied*, *then* $\mathbf{E}W_{[n^t]}$ *and* $\mathbf{E}W_n$ *can be replaced by* $n^t\mu$ *and* $n\mu$, *respectively*.

**3. Proofs.** Recalling (2.1), write

$$\begin{aligned}
W_n &= W_0 + \sum_{k=1}^n (A_k - C_k)I_k + \sum_{k=1}^n C_k \\
&= W_0 + \sum_{k=1}^n \big\{(A_k - C_k)I_k - \mathbf{E}[(A_k - C_k)I_k|\mathcal{F}_{k-1}] + (C_k - c_k)\big\} \\
&\qquad + \sum_{k=0}^{n-1} \rho_{k+1}\frac{W_k}{T_k} + \sum_{k=1}^n c_k \\
&= W_0 + M_n + \sum_{k=0}^{n-1} \rho_{k+1}\frac{W_k}{k+1} + \sum_{k=0}^{n-1} \rho_{k+1}\frac{W_k}{T_k}\left(\frac{k+1-T_k}{k+1}\right) + \sum_{k=1}^n c_k,
\end{aligned} \tag{3.1}$$

where

$$M_n := \sum_{k=1}^n \Delta M_k = \sum_{k=1}^n \big\{(A_k - C_k)I_k - \mathbf{E}[(A_k - C_k)I_k|\mathcal{F}_{k-1}] + (C_k - c_k)\big\}$$

is a martingale with

$$\begin{aligned}
&\mathbf{E}[(\Delta M_n)^2|\mathcal{F}_{n-1}] \\
&\quad = \mathbf{E}\big[\big((A_n - C_n)I_n + C_n - c_n\big)^2|\mathcal{F}_{n-1}\big] - \left((a_n - c_n)\frac{W_{n-1}}{T_{n-1}}\right)^2 \\
&\quad = \mathbf{E}\big[(A_n - C_n)^2 I_n + 2(A_n - C_n)(C_n - c_n)I_n + (C_n - c_n)^2|\mathcal{F}_{n-1}\big] \\
&\qquad - \left((a_n - c_n)\frac{W_{n-1}}{T_{n-1}}\right)^2 \\
&\quad = \frac{W_{n-1}}{T_{n-1}}\mathbf{E}\big[(A_n - C_n)^2 + 2(A_n - C_n)(C_n - c_n)|\mathcal{F}_{n-1}\big] + \mathrm{Var}(C_n|\mathcal{F}_{n-1}) \\
&\qquad - \left((a_n - c_n)\frac{W_{n-1}}{T_{n-1}}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{W_{n-1}}{T_{n-1}} \operatorname{Var}(A_n|\mathcal{F}_{n-1}) + \left(1 - \frac{W_{n-1}}{T_{n-1}}\right) \operatorname{Var}(C_n|\mathcal{F}_{n-1}) \\
&\quad + \rho_n^2 \frac{W_{n-1}}{T_{n-1}} \left(1 - \frac{W_{n-1}}{T_{n-1}}\right) \qquad (3.2) \\
&= \mu \operatorname{Var}(A_n|\mathcal{F}_{n-1}) + (1-\mu) \operatorname{Var}(C_n|\mathcal{F}_{n-1}) + \rho_n^2 \mu(1-\mu) \\
&\quad + O\left(\frac{W_{n-1}}{T_{n-1}} - \mu\right) \\
&= \mu V_a + V_c(1-\mu) + \rho^2 \mu(1-\mu) + o(1) + O\left(\frac{W_{n-1}}{T_{n-1}} - \mu\right) \\
&= \sigma_M^2 + o(1) + O\left(\frac{W_{n-1}}{T_{n-1}} - \mu\right) \quad a.s.
\end{aligned}
$$

under Assumptions 2.2 and 2.3.

By the Skorokhod embedding theorem [cf. Hall and Heyde (1980)], there exists an $\mathcal{F}_n$-adapted sequence of nonnegative random variables $\{\tau_n\}$ and a standard Brownian motion $W$, such that

$$
\mathbf{E}[\tau_n|\mathcal{F}_{n-1}] = \mathbf{E}[(\Delta M_n)^2|\mathcal{F}_{n-1}], \qquad \mathbf{E}|\tau_n|^{1+\varepsilon/2} \le C\mathbf{E}|\Delta M_n|^{2+\varepsilon} \tag{3.3}
$$

and

$$
\left\{W\left(\sum_{i=1}^n \tau_i\right);\ n = 1, 2, \dots\right\} \stackrel{\mathcal{D}}{=} \{M_n;\ n = 1, 2, \dots\}.
$$

Without loss of generality, we write

$$
M_n = W\left(\sum_{i=1}^n \tau_i\right), \qquad n = 1, 2, \dots. \tag{3.4}
$$

On the other hand, from (2.3) and (3.1), it follows that

$$
W_n - e_n = W_0 + M_n + \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k - e_k}{k+1} + \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k}{T_k} \left(\frac{k+1-T_k}{k+1}\right). \tag{3.5}
$$

If Assumption 2.1 is satisfied, that is, $T_k = k+1$, then (3.5) becomes

$$
W_n - e_n = W_0 + M_n + \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k - e_k}{k+1}. \tag{3.6}
$$

So it is natural that $W_n$ may be approximated by a Gaussian process, and what we need to show is how $W_n - e_n$ can be approximated by a related Gaussian process when $M_n$ can.

Before proving the theorems, we need some lemmas first. The first two are on the convergence rates of a real sequence of type (3.6).

LEMMA 3.1. *Let $\rho_n$ and $p_n$ be two sequences of real numbers. Define $\{q_n\}$ by*

$$q_1 = p_1 \quad \text{and} \quad q_n = p_n + \sum_{k=1}^{n-1} \rho_k \frac{q_k}{k}.$$

*Then*

$$q_n = \sum_{k=1}^{n} p_k r_{n,k}, \tag{3.7}$$

*where $r_{n,n} = 1$ and*

$$r_{n,k} = \frac{\rho_k}{k} \prod_{i=k+1}^{n-1} \left(1 + \frac{\rho_i}{i}\right), \qquad k = 1, 2, \dots, n-1, \ n = 1, 2, \dots.$$

*Here we define $\prod_{i=k+1}^{k}(\cdot) = 1$. Furthermore, if $\rho_k \to \rho$, then for $\forall\, \varepsilon > 0$, there is a constant $C > 0$ such that*

$$|r_{n,k}| \le C k^{-1} (n/k)^{\rho+\varepsilon}, \qquad k = 1, 2, \dots, n, \ n = 1, 2, \dots.$$

*And if*

$$\sum_{k=1}^{\infty} |\rho_k - \rho|/k < \infty, \tag{3.8}$$

*then*

$$|r_{n,k}| \le C k^{-1} (n/k)^{\rho}, \qquad k = 1, 2, \dots, n, \ n = 1, 2, \dots.$$

PROOF. When $n = 1$, we have $q_1 = p_1 = r_{1,1} p_1$. Thus (3.7) is true for $n = 1$. By induction, we have

$$q_n = p_n + \sum_{k=1}^{n-1} \frac{\rho_k}{k} \sum_{j=1}^{k} p_j r_{kj} = p_n r_{n,n} + \sum_{j=1}^{n-1} p_j \sum_{k=j}^{n-1} \frac{\rho_k}{k} r_{k,j} = \sum_{j=1}^{n} p_j r_{n,j},$$

where the last step follows from

$$\sum_{k=j}^{n-1} \frac{\rho_k}{k} r_{k,j} = \frac{\rho_j}{j} \left(1 + \sum_{k=j+1}^{n-1} \frac{\rho_k}{k} \prod_{i=j+1}^{k-1} \left(1 + \frac{\rho_i}{i}\right)\right) = r_{n,j}.$$

The first part of the conclusion is proved. The second part is obvious since

$$\log \prod_{i=k}^{n-1} \left(1 + \frac{\rho_i}{i}\right) = \sum_{i=k}^{n-1} \log\left(1 + \frac{\rho_i}{i}\right) = \sum_{i=k}^{n-1} \frac{\rho_i}{i} + O(1)$$
$$= \sum_{i=k}^{n-1} \frac{\rho}{i} + \sum_{i=k}^{n-1} \frac{\rho_i - \rho}{i} + O(1). \qquad \square$$

LEMMA 3.2. *Let $p_n$, $\rho_n$ and $q_n$ be defined as in Lemma 3.1. If $\rho_n \to \rho$ and $p_n = o(n^{\rho+\delta}\delta_n)$ [or $O(n^{\rho+\delta}\delta_n)$] where $\delta > 0$ and $\{\delta_n\}$ is a nondecreasing sequence of positive numbers, then*

$$q_n = o(n^{\rho+\delta}\delta_n) \qquad \big[\textit{corresp.}\ q_n = O(n^{\rho+\delta}\delta_n)\big].$$

*If (3.8) holds and $p_n = o(n^\rho \delta_n)$ [corresp. $= O(n^\rho \delta_n)$] where $\delta_n$ is a sequence of positive numbers, then*

$$q_n = o\left(n^\rho \sum_{k=1}^n \delta_k/k\right) \qquad \left(\textit{corresp.}\ q_n = O\left(n^\rho \sum_{k=1}^n \delta_k/k\right)\right).$$

By Lemma 3.1, the proof is easy.

The definition of $e_n$ seems complicated. But, the following two lemmas tell us that it can be replaced by $\mathbf{E}W_n$ in most cases, or by $n\mu$ in some cases.

LEMMA 3.3. (a) *Suppose that Assumptions 2.1 and 2.2 are satisfied. If $\rho < 1/2$, then*

$$\mathbf{E}W_n - e_n = o(n^{1/2-\delta}) \qquad \forall\, 0 \le \delta < (1/2-\rho) \wedge 1/2.$$

*If $\rho = 1/2$ and (3.8) holds, then*

$$\mathbf{E}W_n - e_n = o(n^{1/2}).$$

(b) *Suppose $\rho < 1/2$, Assumption 2.2 is true and $T_n = n+1+o((n\log\log n)^{1/2})$ in $L_1$. Then*

$$\mathbf{E}W_n - e_n = o\big((n\log\log n)^{1/2}\big).$$

(c) *Suppose $\rho < 1/2$, Assumption 2.2 and $T_n = n+1+o(\sqrt{n})$ in $L_1$. Then*

$$\mathbf{E}W_n - e_n = o(\sqrt{n}).$$

(d) *Suppose Assumption 2.2 and $T_n = n+1+o(n^{1/2}(\log n)^{-1/2-\varepsilon})$ in $L_1$ for some $\varepsilon > 0$. If $\rho < 1/2$, then*

$$\mathbf{E}W_n - e_n = o\big(n^{1/2}(\log n)^{-1/2-\varepsilon}\big).$$

*If $\rho = 1/2$ and (3.8) holds, then*

$$\mathbf{E}W_n - e_n = o\big(n^{1/2}(\log n)^{1/2-\varepsilon}\big).$$

PROOF. We give the proof of (a) only. By (3.5),

$$\mathbf{E}W_n - e_n = \sum_{k=0}^{n-1} \rho_{k+1}\frac{\mathbf{E}W_k - e_k}{k+1} + O(1) = \sum_{k=0}^{n-1} \rho_{k+1}\frac{\mathbf{E}W_k - e_k}{k+1} + o(n^{1/2-\delta}).$$

By Lemma 3.2, it follows that if $\rho < 1/2$, then

$$|\mathbf{E}W_n - e_n| = o(n^{1/2-\delta})$$

since $\varepsilon =: 1/2 - \delta - \rho > 0$. If $\rho = 1/2$ and (3.8) holds, then

$$|\mathbf{E}W_n - e_n| = o\left(n^{1/2} \sum_{k=1}^{n} k^{-1-1/2}\right) = o(n^{1/2}). \qquad \square$$

LEMMA 3.4. *Under Assumption* 2.2, *we have*

$$\frac{e_n}{n} \to \mu.$$

*Furthermore*, *if* (2.8) *holds and* $\rho < 1/2$, *then*

$$e_n - n\mu = o(\sqrt{n})$$

*and if* $\rho = 1/2$ *and condition* (2.17) *is satisfied*, *then*

$$e_n - n\mu = O(\sqrt{n}).$$

PROOF. By (2.3),

$$e_n - n\mu = \sum_{k=0}^{n-1} \rho_{k+1} \frac{e_k - (k+1)\mu}{k+1} + \sum_{k=1}^{n} \{(a_k - a)\mu + (c_k - c)(1-\mu)\}. \tag{3.9}$$

The first two conclusions follow from Lemma 3.2 easily by taking $p_n = o(n^{\rho+1-\rho})$ and $p_n = o(n^{\rho+1/2-\rho})$, respectively. Now, assume $\rho = 1/2$ and (2.17). Take $b_n = n^{1/2}\delta_n$, where

$$\delta_n = \frac{\sum_{k=1}^{n} \{(a_k - a)\mu + (c_k - c)(1-\mu)\}}{\sqrt{n}}.$$

Then, by the second part of Lemma 3.2,

$$\begin{aligned} e_n - n\mu &= O\left(n^{1/2} \sum_{k=1}^{n} \delta_k / k\right) \\ &= O\left(n^{1/2} \sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \{(a_i - a)\mu + (c_i - c)(1-\mu)\}}{k^{3/2}}\right) \\ &= O\left(n^{1/2} \sum_{i=1}^{n} (|a_i - a| + |c_i - c|) \sum_{k=i}^{n} k^{-3/2}\right) \\ &= O\left(n^{1/2} \sum_{i=1}^{n} \frac{|a_i - a| + |c_i - c|}{\sqrt{i}}\right) = O(\sqrt{n}). \qquad \square \end{aligned}$$

Define

$$\overline{G}_0 = 0, \qquad \overline{G}_n = W(n\sigma_M^2) + \rho \sum_{k=1}^{n-1} \frac{\overline{G}_k}{k}, \tag{3.10}$$

where $\sum_{k=1}^{0}(\cdot) = 0$. The next two lemmas tell us how $\overline{G}_n$ is close to $G_n$ or $\widehat{G}_n$, where $G_n$ and $\widehat{G}_n$ are defined in (2.6) and (2.22), respectively.

LEMMA 3.5. *If $\rho < 1/2$, we have for all $0 \le \delta < 1/2 - \rho$,*

$$\overline{G}_n - G_n = o(n^{1/2-\delta}) \quad a.s. \tag{3.11}$$

*and*

$$\Big\| \max_{k \le n} |\overline{G}_k - G_k| \Big\|_2 = o(n^{1/2-\delta}). \tag{3.12}$$

PROOF. By the Taylor expansion,

$$\begin{aligned} G_n - G_{n-1} &= n^\rho \int_{n-1}^{n} \frac{dW(s\sigma_M^2)}{s^\rho} + \left(1 + \frac{1}{n-1}\right)^\rho G_{n-1} - G_{n-1} \\ &= n^\rho \int_{n-1}^{n} \frac{dW(s\sigma_M^2)}{s^\rho} + \rho \frac{G_{n-1}}{n-1} + \frac{\rho(\rho-1)}{2(n-1)^2}(1+\xi_{n-1})^{\rho-2} G_{n-1}, \end{aligned}$$

where $\xi_{n-1} \in [0, 1]$ is a real number. It follows that

$$G_n = \rho \sum_{k=1}^{n-1} \frac{G_k}{k} + \sum_{k=1}^{n} k^\rho \int_{k-1}^{k} \frac{dW(s\sigma_M^2)}{s^\rho} + \frac{\rho(\rho-1)}{2} \sum_{k=1}^{n-1} \frac{(1+\xi_k)^{\rho-2}}{k^2} G_k.$$

Then,

$$\begin{aligned} G_n - \overline{G}_n &= \rho \sum_{k=1}^{n-1} \frac{G_k - \overline{G}_k}{k} + \sum_{k=1}^{n} k^\rho \int_{k-1}^{k} \left(\frac{1}{s^\rho} - \frac{1}{k^\rho}\right) dW(s\sigma_M^2) \\ &\quad + \frac{\rho(\rho-1)}{2} \sum_{k=1}^{n-1} \frac{(1+\xi_k)^{\rho-2}}{k^2} G_k \qquad (3.13) \\ &= \rho \sum_{k=1}^{n-1} \frac{G_k - \overline{G}_k}{k} + \sum_{k=1}^{n} Z_k + \frac{\rho(\rho-1)}{2} \sum_{k=1}^{n-1} \frac{(1+\xi_k)^{\rho-2}}{k^2} G_k, \end{aligned}$$

where $\{Z_k;\ k = 1, 2, \ldots\}$ is a sequence of independent normal variables with $\mathbf{E}Z_k = 0$ and

$$\mathbf{E}Z_k^2 = \sigma_M^2 k^{2\rho} \int_{k-1}^{k} \left(\frac{1}{s^\rho} - \frac{1}{k^\rho}\right)^2 ds \le C k^{2\rho} \frac{1}{k^{2\rho+2}} \le C k^{-2}.$$

It follows that $\sum_{k=1}^n Z_k = O(1)$ in $L_2$, and $\sum_{k=1}^n Z_k = O(1)$ a.s. by the three-series theorem. Also

$$\left|\frac{\rho(\rho-1)}{2}\sum_{k=1}^{n-1}\frac{(1+\xi_k)^{\rho-2}}{k^2}G_k\right| \le \frac{|\rho(\rho-1)|}{2}\sum_{k=1}^{n-1}\frac{|G_k|}{k^2} < \infty \quad \text{a.s. and in } L_2.$$

So,

$$\begin{aligned} G_n - \overline{G}_n &= \rho\sum_{k=1}^{n-1}\frac{G_k-\overline{G}_k}{k} + O(1) \\ &= \rho\sum_{k=1}^{n-1}\frac{G_k-\overline{G}_k}{k} + o(n^{1/2-\delta}) \quad \text{a.s. and in } L_2. \end{aligned} \tag{3.14}$$

Hence, from Lemma 3.2 it follows that

$$G_n - \overline{G}_n = o(n^{1/2-\delta}) \qquad \text{a.s. and in } L_2\ \forall\, 0 \le \delta < 1/2-\rho.$$

The assertion (3.11) is proved. Finally,

$$\max_{m\le n}|G_m - \overline{G}_m| \le |\rho|\sum_{k=1}^{n-1}\frac{|G_k-\overline{G}_k|}{k} + \max_{m\le n}\left|\sum_{k=1}^{m} Z_k\right| + \frac{|\rho(\rho-1)|}{2}\sum_{k=1}^{n-1}\frac{|G_k|}{k^2}.$$

It follows that

$$\begin{aligned} \left\|\max_{m\le n}|G_m-\overline{G}_m|\right\|_2 &\le |\rho|\sum_{k=1}^{n-1}\frac{\|G_k-\overline{G}_k\|_2}{k} + \left\|\max_{m\le n}|\sum_{k=1}^{m}Z_k\right\|_2 \\ &\quad + \frac{|\rho(\rho-1)|}{2}\sum_{k=1}^{n-1}\frac{\|G_k\|_2}{k^2} \\ &\le |\rho|\sum_{k=1}^{n-1} o(k^{1/2-\delta-1}) + O(1) + \sum_{k=1}^{n-1} O(k^{1/2-2}) = o(n^{1/2-\delta}). \end{aligned}$$

The conclusion (3.12) follows. □

LEMMA 3.6. *If $\rho = 1/2$, we have*

$$\overline{G}_n - \widehat{G}_n = o(n^{1/2}) \quad a.s.$$

PROOF. Similarly to (3.13),

$$\widehat{G}_n - \overline{G}_n = \rho\sum_{k=1}^{n-1}\frac{\widehat{G}_k-\overline{G}_k}{k} + W(\sigma_M^2) + \sum_{k=2}^{n} Z_k + \frac{\rho(\rho-1)}{2}\sum_{k=1}^{n-1}\frac{(1+\xi_k)^{\rho-2}}{k^2}\widehat{G}_k.$$

So, just as in (3.14), we have

$$\widehat{G}_n - \overline{G}_n = \rho \sum_{k=1}^{n-1} \frac{\widehat{G}_k - \overline{G}_k}{k} + o(n^{1/2-\delta}) \qquad \text{a.s. } \forall\, 0 < \delta < 1/2.$$

Applying the second part of Lemma 3.2, we conclude that

$$\widehat{G}_n - \overline{G}_n = o\left(n^{1/2} \sum_{k=1}^{n} k^{-1-\delta}\right) = o(\sqrt{n}) \quad \text{a.s.} \qquad \square$$

Now we are in position to prove the main theorems.

PROOF OF THEOREM 2.1. We first show the two processes are equal in law. Since $\mathbf{E}G_t = 0$ and for $t \geq s$,

$$\begin{aligned}\mathbf{E}G_s G_t &= t^\rho s^\rho \mathbf{E}\left(\int_0^s \frac{dW(x\sigma_M^2)}{x^\rho}\right)^2 \\ &= t^\rho s^\rho \int_0^s \frac{\sigma_M^2}{x^{2\rho}}\, dx = \sigma^2 t^\rho s^\rho s^{1-2\rho} = \mathbf{E}\big(\sigma t^\rho W(t^{1-2\rho})\big)\big(\sigma s^\rho W(s^{1-2\rho})\big).\end{aligned}$$

This shows that the two Gaussian processes have the same mean and covariance functions, which implies (2.7).

Note that (2.5) follows from (2.4) and Lemma 3.3(b) whereas (2.9) follows from (2.4) and Lemma 3.4. To complete the proof of Theorem 2.1, it suffices to prove (2.4). To this end, we shall first show how $M_n$ can be approximated by $W(n\sigma_M^2)$. Let $\tau_n$ be defined as in (3.3) and (3.4) through the Skorohod embedding theorem. Note that

$$\begin{aligned}&\mathbf{E}|\Delta M_n|^{2+\varepsilon} \\ &\quad = \mathbf{E}\big|(A_n - C_n)I_n - \mathbf{E}[(A_n - C_n)I_n|\mathcal{F}_{n-1}] + (C_n - \mathbf{E}[C_n|\mathcal{F}_{n-1}])\big|^{2+\varepsilon} \\ &\quad \leq C(\mathbf{E}|A_n|^{2+\varepsilon} + \mathbf{E}|C_n|^{2+\varepsilon}) < C < \infty,\end{aligned}$$

where $C$ is a generic notation for positive constants; that is, it may take different values at different appearances.

It then follows that $\mathbf{E}|\tau_n|^{1+\varepsilon/2} < C < \infty$. Hence,

$$\sum_{n=1}^{\infty} \mathbf{E}\left|\frac{\tau_n}{n^{1-\varepsilon/3}}\right|^{1+\varepsilon/2} < \infty.$$

So, by the law of large numbers for martingales [cf. Theorem 20.11 of Davidson (1994)],

$$\sum_{k=1}^{n} \tau_k - \sum_{k=1}^{n} \mathbf{E}[(\Delta M_k)^2|\mathcal{F}_{k-1}] = \sum_{k=1}^{n} (\tau_k - \mathbf{E}[\tau_k|\mathcal{F}_{k-1}]) = o(n^{1-\varepsilon/3}) \quad \text{a.s.} \tag{3.15}$$

Obviously, by (3.2),

$$\mathbf{E}[(\Delta M_n)^2|\mathcal{F}_{n-1}] = O(1) \quad \text{a.s.}$$

Thus,

$$\sum_{k=1}^{n} \tau_k = O(n) \quad \text{a.s.}$$

Then by (3.4) and the law of iterated logarithm of a Brownian motion,

$$M_n = O\big((n \log\log n)^{1/2}\big) \quad \text{a.s.}$$

which, together with (3.5) and Lemma 3.2, implies

$$W_n - e_n = O\big((n \log\log n)^{1/2}\big) \quad \text{a.s.} \tag{3.16}$$

By (3.2) and (3.16) and Lemma 3.4, it follows that

$$\begin{aligned}\mathbf{E}[(\Delta M_n)^2|\mathcal{F}_{n-1}] &= \sigma_M^2 + o(1) + O\left(\frac{W_{n-1}}{T_{n-1}} - \frac{e_{n-1}}{T_{n-1}}\right) + O\left(\frac{e_{n-1}}{T_{n-1}} - \mu\right) \\ &= \sigma_M^2 + o(1) \quad \text{a.s.}\end{aligned}$$

So,

$$\sum_{k=1}^{n} \tau_k = n\sigma_M^2 + o(n) \quad \text{a.s.} \tag{3.17}$$

Thus by the properties of a Brownian motion [cf. Theorem 1.2.1 of Csörgő and Révész (1981)], we get the following approximation of $M_n$:

$$M_n = W\left(\sum_{k=1}^{n} \tau_k\right) = W(n\sigma_M^2) + o\big((n \log\log n)^{1/2}\big) \quad \text{a.s.} \tag{3.18}$$

Recalling the definition of $\overline{G}_n$ in (3.10) and noticing (3.11), the proof of (2.4) reduces to showing that

$$W_n - e_n - \overline{G}_n = o\big((n \log\log n)^{1/2}\big) \quad \text{a.s.} \tag{3.19}$$

Note that

$$\begin{aligned}\overline{G}_n &= W(n\sigma_M^2) + \rho \sum_{k=1}^{n-1} \frac{\overline{G}_k}{k} \\ &= W(n\sigma_M^2) + \rho \sum_{k=1}^{n-1} \frac{\overline{G}_k}{k+1} + \rho \sum_{k=1}^{n-1} \frac{\overline{G}_k}{k(k+1)} \\ &= W(n\sigma_M^2) + \sum_{k=0}^{n-1} \rho_{k+1} \frac{\overline{G}_k}{k+1} \\ &\quad + \rho \sum_{k=1}^{n-1} \frac{\overline{G}_k}{k(k+1)} + \sum_{k=0}^{n-1} (\rho - \rho_{k+1}) \frac{\overline{G}_k}{k+1}.\end{aligned} \tag{3.20}$$

Note that by (3.11) and (2.7),

$$\overline{G}_n = O\big((n\log\log n)^{1/2}\big) \quad \text{a.s.}$$

It follows that

$$\begin{aligned}\overline{G}_n &= W(n\sigma_M^2) + \sum_{k=0}^{n-1}\rho_{k+1}\frac{\overline{G}_k}{k+1}\\ &\quad + \rho\sum_{k=1}^{n-1}\frac{o(1)}{k+1} + \sum_{k=0}^{n-1}(\rho-\rho_{k+1})\frac{O((k\log\log k)^{1/2})}{k+1}. \qquad (3.21)\\ &= W(n\sigma_M^2) + \sum_{k=0}^{n-1}\rho_{k+1}\frac{\overline{G}_k}{k+1} + o\big((n\log\log n)^{1/2}\big) \quad \text{a.s.}\end{aligned}$$

By (3.5), (3.18), (3.21) and

$$\sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k}{T_k}\bigg(\frac{k+1-T_k}{k+1}\bigg) = o\Bigg(\sum_{k=0}^{n-1}\frac{(k\log\log k)^{1/2}}{k+1}\Bigg) = o\big((n\log\log n)^{1/2}\big) \quad \text{a.s.}$$

we conclude that

$$W_n - e_n - \overline{G}_n = \sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k - e_k - \overline{G}_k}{k+1} + o\big((n\log\log n)^{1/2}\big) \quad \text{a.s.}$$

Hence by Lemma 3.2, we have proved (3.19). □

PROOF OF THEOREM 2.2. Noticing that (2.11) and (2.12) are consequences of (2.10) and application of Lemmas 3.3 and 3.4, we need only to show (2.10). Define $\nu_n = \sum_{k=1}^n \tau_k - n\sigma_M^2$. Then by (3.17),

$$\nu_n = o(n) \quad \text{a.s.} \qquad (3.22)$$

First, we show that

$$\max_{k\le n}|M_k - W(k\sigma_M^2)| = o\big(\sqrt{n}\big) \qquad \text{in } L_2. \qquad (3.23)$$

Note that (3.22) implies that $\max_{k\le n}|\nu_k|/n \to 0$ in probability, and then

$$\begin{aligned}&\mathbf{E}\max_{k\le n}|M_k - W(k\sigma_M^2)|^2\\ &\quad = \mathbf{E}\max_{k\le n}|M_k - W(k\sigma_M^2)|^2 I\Big\{\max_{k\le n}|\nu_k| \le \varepsilon n\Big\}\\ &\qquad + \mathbf{E}\max_{k\le n}|M_k - W(k\sigma_M^2)|^2 I\Big\{\max_{k\le n}|\nu_k| > \varepsilon n\Big\}\\ &\quad \le \mathbf{E}\sup_{0\le t\le n(1+\sigma_M^2)}\ \sup_{0\le s\le \varepsilon n}|W(t+s) - W(t)|^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mathbf{E}\max_{k\le n}|M_k|^2 I\Big\{\max_{k\le n}|\nu_k| > \varepsilon n\Big\}\\
&\quad + 2\mathbf{E}\max_{k\le n}|W(k\sigma_M^2)|^2 I\Big\{\max_{k\le n}|\nu_k| > \varepsilon n\Big\}\\
&\le n\mathbf{E}\sup_{0\le t\le 1+\sigma_M^2}\sup_{0\le s\le\varepsilon}|W(t+s)-W(t)|^2\\
&\quad + 2\bigg(\Big\|\max_{k\le n}|M_k|\Big\|_{2+\varepsilon}^2 + \Big\|\max_{k\le n}|W(k\sigma_M^2)|\Big\|_{2+\varepsilon}^2\bigg)\\
&\qquad \times \bigg(\mathbf{P}\Big(\max_{k\le n}|\nu_k| > \varepsilon n\Big)\bigg)^{(2+\varepsilon)/\varepsilon}\\
&\le n\mathbf{E}\sup_{0\le t\le 1+\sigma_M^2}\sup_{0\le s\le\varepsilon}|W(t+s)-W(t)|^2 + Cn\bigg(\mathbf{P}\Big(\max_{k\le n}|\nu_k| > \varepsilon n\Big)\bigg)^{(2+\varepsilon)/\varepsilon}\\
&= o(n) \qquad \text{as } n\to\infty \text{ and then } \varepsilon\to 0.
\end{aligned}
$$

The assertion (3.23) is proved. Now, let $\overline{G}_n$ be defined through (3.10). By Lemma 3.5, to prove (2.10), it is enough to show that

$$
\max_{k\le n}|W_k - e_k - \overline{G}_k| = o(\sqrt{n}) \qquad \text{in } L_2. \tag{3.24}
$$

By (3.5) and (3.20), we have

$$
\begin{aligned}
W_n - e_n - \overline{G}_n &= W_0 + M_n - W(n\sigma_M^2) + \sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k - e_k - \overline{G}_k}{k+1}\\
&\quad + \rho\sum_{k=1}^{n-1}\frac{\overline{G}_k}{k(k+1)} + \sum_{k=0}^{n-1}(\rho-\rho_{k+1})\frac{\overline{G}_k}{k+1}\\
&\quad + \sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k}{T_k}\bigg(\frac{k+1-T_k}{k+1}\bigg).
\end{aligned} \tag{3.25}
$$

By (3.12) and (2.7), we know that $\|\overline{G}_n\|_2 = O(\sqrt{n})$. It follows that

$$
\begin{aligned}
&\bigg\|\rho\sum_{k=1}^{n-1}\frac{\overline{G}_k}{k(k+1)} + \sum_{k=0}^{n-1}(\rho-\rho_{k+1})\frac{\overline{G}_k}{k+1}\bigg\|_2\\
&\qquad \le |\rho|\sum_{k=1}^{n-1}\frac{\|\overline{G}_k\|_2}{k(k+1)} + \sum_{k=0}^{n-1}|\rho-\rho_{k+1}|\frac{\|\overline{G}_k\|_2}{k+1}\\
&\qquad \le |\rho|\sum_{k=1}^{n-1}\frac{O(\sqrt{k})}{k(k+1)} + \sum_{k=0}^{n-1}|\rho-\rho_{k+1}|\frac{O(\sqrt{k})}{k+1} = o(\sqrt{n}),
\end{aligned}
$$

which, together with (3.23) and

$$\left\| \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k}{T_k} \left( \frac{k+1-T_k}{k+1} \right) \right\|_2 \leq \sum_{k=0}^{n-1} |\rho_{k+1}| \frac{\|k+1-T_k\|_2}{k+1} = o(\sqrt{n}),$$

implies

$$W_n - e_n - \overline{G}_n = \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k - e_k - \overline{G}_k}{k+1} + o(\sqrt{n}) \qquad \text{in } L_2.$$

Thus by Lemma 3.2,

$$W_n - e_n - \overline{G}_n = o(\sqrt{n}) \qquad \text{in } L_2. \tag{3.26}$$

Finally, by (3.25) we have

$$\begin{aligned}\max_{k \leq n} |W_k - e_k - \overline{G}_k| \leq |W_0| + \max_{k \leq n} |M_k - W(k\sigma_M^2)| \\ + \sum_{k=0}^{n-1} |\rho_{k+1}| \frac{|W_k - e_k - \overline{G}_k|}{k+1} + |\rho| \sum_{k=1}^{n-1} \frac{|\overline{G}_k|}{k(k+1)} \\ + \sum_{k=0}^{n-1} |\rho - \rho_{k+1}| \frac{|\overline{G}_k|}{k+1} + \sum_{k=0}^{n-1} |\rho_{k+1}| \frac{|k+1-T_k|}{k+1}.\end{aligned}$$

Thus, by (3.12), (3.23) and (3.26), it follows that

$$\begin{aligned}\left\| \max_{k \leq n} |W_k - e_k - \overline{G}_k| \right\|_2 = o(\sqrt{n}) + \sum_{k=0}^{n-1} |\rho_{k+1}| \frac{o(\sqrt{k})}{k+1} + |\rho| \sum_{k=1}^{n-1} \frac{O(\sqrt{k})}{k(k+1)} \\ + \sum_{k=0}^{n-1} |\rho - \rho_{k+1}| \frac{O(\sqrt{k})}{k+1} + \sum_{k=0}^{n-1} |\rho_{k+1}| \frac{o(\sqrt{k})}{k+1} = o(\sqrt{n}).\end{aligned}$$

The assertion (3.24) is proved. □

PROOF OF THEOREM 2.3. It is enough to show (2.19). First we show that

$$M_n - W(n\sigma_M^2) = o(\sqrt{n}) \quad \text{a.s.} \tag{3.27}$$

By Assumption 2.4,

$$\sum_{k=1}^{n} \{(a_k - a)\mu + (c_k - c)(1 - \mu)\} = o(n(\log\log n)^{-1}).$$

It follows by Lemma 3.2 and (3.9) that

$$\frac{e_n}{n} - \mu = o((\log\log n)^{-1}).$$

And then by (3.2) and (3.16),

$$\begin{aligned}\mathbf{E}[(\Delta M_n)^2|\mathcal{F}_{n-1}] &= \mu\mathrm{Var}(A_n|\mathcal{F}_{n-1}) + (1-\mu)\mathrm{Var}(C_n|\mathcal{F}_{n-1}) + \rho_n^2\mu(1-\mu) \\ &\quad + O\left(\frac{W_{n-1}}{T_{n-1}} - \frac{e_{n-1}}{n}\right) + O\left(\frac{e_{n-1}}{n} - \mu\right) \\ &= \sigma_M^2 + o\big((\log\log n)^{-1}\big) \quad \text{a.s.},\end{aligned}$$

which, together with (3.15), implies

$$\sum_{k=1}^{n} \tau_k = n\sigma_M^2 + o\big(n(\log\log n)^{-1}\big) \quad \text{a.s.}$$

Then by Theorem 1.2.1 of Csörgő and Révész (1981) again,

$$\begin{aligned}M_n = W\left(\sum_{k=1}^{n}\tau_k\right) &= W(n\sigma_M^2) + o\big(\big(n(\log\log n)^{-1}\big)^{1/2}(\log\log n)^{1/2}\big) \\ &= W(n\sigma_M^2) + o(\sqrt{n}) \quad \text{a.s.},\end{aligned}$$

from which (3.27) follows. Next, by (3.20),

$$\begin{aligned}\overline{G}_n &= W(n\sigma_M^2) + \sum_{k=0}^{n-1}\rho_{k+1}\frac{\overline{G}_k}{k+1} \\ &\quad + \rho\sum_{k=1}^{n-1}\frac{o(1)}{k+1} + \sum_{k=0}^{n-1} o\big((\log\log k)^{-1}\big)\frac{O((k\log\log k)^{1/2})}{k+1} \qquad (3.28) \\ &= W(n\sigma_M^2) + \sum_{k=0}^{n-1}\rho_{k+1}\frac{\overline{G}_k}{k+1} + o(\sqrt{n}) \quad \text{a.s.}\end{aligned}$$

Hence by (3.5), (3.27), (3.28) and

$$\sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k}{T_k}\left(\frac{k+1-T_k}{k+1}\right) = o\left(\sum_{k=0}^{n-1}\frac{\sqrt{k}}{k+1}\right) = o(\sqrt{n}) \quad \text{a.s.},$$

we conclude that

$$W_n - e_n - \overline{G}_n = \sum_{k=0}^{n-1}\rho_{k+1}\frac{W_k - e_k - \overline{G}_k}{k+1} + o(\sqrt{n}) \quad \text{a.s.}$$

By Lemma 3.2, it follows that

$$W_n - e_n - \overline{G}_n = o(\sqrt{n}) \quad \text{a.s.}$$

The rest of the proof is similar to that of Theorem 2.1. □

The proofs of Theorems 2.4 and 2.5 are similar to that of Theorem 2.3, and the details are omitted.

PROOF OF THEOREM 2.6. Assertion (2.23) can be easily verified by showing that the two processes have identical covariance functions. Also, by Lemmas 3.3 and 3.4, to prove the theorem, it is enough to show (2.21). Following the lines of the proof of Theorem 2.3, one can show that

$$M_n = W(n\sigma_M^2) + o\big(n^{1/2}(\log n)^{-1/2-\varepsilon/3}\big) \quad \text{a.s.}$$

Also, similar to (3.28),

$$\begin{aligned}\overline{G}_n &= W(n\sigma_M^2) + \sum_{k=0}^{n-1} \rho_{k+1} \frac{\overline{G}_k}{k+1} \\ &\quad + \rho \sum_{k=1}^{n-1} \frac{o(1)}{k+1} + \sum_{k=0}^{n-1} o\big((\log k)^{-1-\varepsilon}\big) \frac{O((k \log\log k)^{1/2}}{k+1} \\ &= W(n\sigma_M^2) + \sum_{k=0}^{n-1} \rho_{k+1} \frac{\overline{G}_k}{k+1} + o\big(n^{1/2}(\log n)^{-1-\varepsilon/2}\big) \quad \text{a.s.}\end{aligned}$$

Hence

$$W_n - e_n - \overline{G}_n = \sum_{k=0}^{n-1} \rho_{k+1} \frac{W_k - e_k - \overline{G}_k}{k+1} + o\big(n^{1/2}(\log n)^{-1/2-\varepsilon/3}\big) \quad \text{a.s.}$$

By the second part of Lemma 3.2, it follows that

$$W_n - e_n - \overline{G}_n = o(1)n^{1/2} \sum_{k=1}^{n} k^{-1}(\log k)^{-1/2-\varepsilon/3} = o\big(n^{1/2}(\log n)^{1/2-\varepsilon/3}\big) \quad \text{a.s.}$$

Finally, by Lemma 3.6,

$$\overline{G}_n - \widehat{G}_n = o(\sqrt{n}) \quad \text{a.s.}$$

The proof is complete. □

## 4. Some applications.

4.1. *Asymptotic properties of the randomized-play-the-winner rule.* The randomized-play-the-winner (RPW) rule was introduced by Wei and Durham (1978) and it can be formulated as a GFU model [Wei (1979)] as follows: Assume there are two treatments (say, T1 and T2), with dichotomous response (success and failure). For the $i$th patient, if a white ball is drawn, the patient is assigned to the treatment T1, and otherwise, the patient is assigned to the treatment T2. The ball is then replaced in the urn and the patient response is observed. A success on treatment T1 or a failure on treatment T2 generates a white ball to the urn; a success on treatment T2 or a failure on treatment T1 generates a black ball to the urn.

Let $p_1 = \mathbf{P}(success|\text{T1})$, $p_2 = \mathbf{P}(success|\text{T2})$, $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. It is easy to see that

$$\mathbf{R} = \begin{bmatrix} I(success|\text{T1}) & 1 - I(success|\text{T1}) \\ 1 - I(success|\text{T2}) & I(success|\text{T2}) \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} p_1 & q_1 \\ q_2 & p_2 \end{bmatrix},$$

where $I$ is an indicator function. From the results of Section 2, we have the following corollaries.

COROLLARY 4.1. *If* $q_1 + q_2 > 1/2$, *then*:

(i)

$$n^{-1/2}\left(W_n - \frac{q_2 n}{q_1 + q_2}\right) \to N(0, \sigma^2) \quad \textit{in distribution}$$

*and further*, *we have*

(ii)

$$\limsup_{n\to\infty} \frac{W_n - q_2 n/(q_1 + q_2)}{\sqrt{2n \log\log n}} = \sigma \quad a.s.,$$

*where* $\sigma^2 = q_1 q_2/[(2(q_1 + q_2) - 1)(q_1 + q_2)^2]$.

It is easy to see that $T_n = n + \beta$ and Assumptions 2.2 and 2.3 hold. From Corollary 2.1, we can obtain both (i) and (ii). The result (i) has been studied in Smythe and Rosenberger (1995) for the homogeneous case and Bai and Hu (1999) for the nonhomogeneous generating matrices. The result (ii) is new. When $q_1 + q_2 = 1/2$, the following similar results are true.

COROLLARY 4.2. *If* $q_1 + q_2 = 1/2$, *then*:

(i)

$$(n \log n)^{-1/2}\left(W_n - \frac{q_2 n}{q_1 + q_2}\right) \to N(0, \sigma_W^2) \quad \textit{in distribution}$$

*and further*, *we have*

(ii)

$$\limsup_{n\to\infty} \frac{W_n - q_2 n/(q_1 + q_2)}{\sqrt{2n(\log n)(\log\log\log n)}} = \sigma_W \quad a.s.,$$

*where* $\sigma_W^2 = q_1 q_2/(q_1 + q_2)^2$.

4.2. *Variance estimation.* From Corollary 3.1, we know that under Assumptions 2.1, 2.2, 2.3 and condition (2.8),

$$\frac{W_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{\mathcal{D}} N(0,1), \tag{4.1}$$

where $\sigma$ is defined as in (2.2). The result (4.1) gives us the limit distribution of $W_n/n$ which is an estimator of $\mu$. But (4.1) is difficult to apply since the value of $\sigma$ is unknown. So it is important to find a consistent estimate of $\sigma$ from the sample $\{W_n\}$.

Inspired by Shao (1994), we define two estimators as follows:

$$\widehat{\sigma}_{1,n} = \frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\left|\frac{W_i}{i} - \frac{W_n}{n}\right| \quad \text{and} \quad \widehat{\sigma}^2_{2,n} = \frac{1}{\log n}\sum_{i=1}^{n}\left(\frac{W_i}{i} - \frac{W_n}{n}\right)^2. \tag{4.2}$$

The following two theorems establish the weak and strong consistency of the estimators, respectively.

THEOREM 4.1. *Suppose $\rho < 1/2$. Under Assumptions* 2.2, 2.3, (2.8) *and that $T_n = n + 1 + o(\sqrt{n})$ in $L_2$, we have*

$$\widehat{\sigma}_{1,n} \to \sqrt{\frac{2}{\pi}}\sigma \quad \textit{and} \quad \widehat{\sigma}^2_{2,n} \to \sigma^2 \quad \textit{in } L_2. \tag{4.3}$$

THEOREM 4.2. *Suppose $\rho < 1/2$. Under Assumptions* 2.2, 2.3, 2.4, (2.8) *and that $T_n = n + 1 + o(\sqrt{n})$ a.s.*,

$$\widehat{\sigma}_{1,n} \to \sqrt{\frac{2}{\pi}}\sigma \quad \textit{and} \quad \widehat{\sigma}^2_{2,n} \to \sigma^2 \quad a.s. \tag{4.4}$$

The proofs of Theorems 4.1 and 4.2 are based on the Gaussian approximations and the following lemma.

LEMMA 4.1. *Suppose $\rho < 1/2$. Let $\{G_t;\ t \ge 0\}$ be as in* (2.6), *and let*

$$V_{1,n} = \frac{1}{\log n}\sum_{i=1}^{n}\frac{|G_i|}{i^{3/2}} \quad \textit{and} \quad V^2_{2,n} = \frac{1}{\log n}\sum_{i=1}^{n}\frac{G_i^2}{i^2}. \tag{4.5}$$

*Then*

$$V_{1,n} \to \sqrt{\frac{2}{\pi}}\sigma, \quad V^2_{2,n} \to \sigma^2 \quad a.s. \textit{ as well as in } L_2. \tag{4.6}$$

PROOF. Obviously,

$$\mathbf{E}V_{1,n} \to \sqrt{\frac{2}{\pi}}\sigma \quad \text{and} \quad \mathbf{E}V^2_{2,n} \to \sigma^2. \tag{4.7}$$

Also, by (2.6), $\mathrm{Cov}(G_i/\sqrt{i}, G_j/\sqrt{j}) = \sigma^2 (i/j)^{1/2-\rho}$ for all $i \le j$. It follows that

$$\mathrm{Cov}\left(\frac{|G_i|}{\sqrt{i}}, \frac{|G_j|}{\sqrt{j}}\right) \le \sigma^2 (i/j)^{1/2-\rho}$$

and

$$\mathrm{Cov}\left(\frac{G_i^2}{i}, \frac{G_j^2}{j}\right) = 2\sigma^4 (i/j)^{1-2\rho}.$$

Then

$$\begin{aligned}\mathrm{Var}(V_{1,n}) &= \frac{1}{(\log n)^2}\left\{\sum_{i=1}^{n} \frac{1}{i^2}\mathrm{Var}\left(\frac{|G_i|}{\sqrt{i}}\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{ij}\mathrm{Cov}\left(\frac{|G_i|}{\sqrt{i}}, \frac{|G_j|}{\sqrt{j}}\right)\right\} \\ &\le C\frac{1}{(\log n)^2}\sum_{i=1}^{n-1}\sum_{j=i}^{n}\frac{1}{ij}(i/j)^{1/2-\rho} \le \frac{C}{\log n}\end{aligned} \tag{4.8}$$

and

$$\mathrm{Var}(V_{1,n}^2) \le C\frac{1}{(\log n)^2}\sum_{i=1}^{n-1}\sum_{j=i}^{n}\frac{1}{ij}(i/j)^{1-2\rho} \le \frac{C}{\log n}. \tag{4.9}$$

The estimates (4.7)–(4.9) directly imply the $L_2$ convergence part of (4.6). By some standard calculation, the three estimates also imply the a.s. convergence of (4.6) [cf. Shao (1994)]. □

Now we start to prove the main theorems for the consistency of the variance estimators.

PROOF OF THEOREM 4.1. Let $V_{1,n}$ and $V_{2,n}$ be defined as in (4.5). Since

$$\begin{aligned}\left|\left|\frac{W_i}{i} - \frac{W_n}{n}\right| - \left|\frac{G_i}{i}\right|\right| &= \left|\left|\frac{W_i - i\mu}{i} - \frac{W_n - n\mu}{n}\right| - \left|\frac{G_i}{i}\right|\right| \\ &\le \left|\frac{W_i - i\mu}{i} - \frac{W_n - n\mu}{n} - \frac{G_i}{i}\right| \\ &\le \left|\frac{W_i - i\mu - G_i}{i}\right| + \left|\frac{W_n - n\mu}{n}\right|,\end{aligned}$$

we have

$$\|\widehat{\sigma}_{1,n} - V_{1,n}\|_2 \le \frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\left\|\frac{W_i - i\mu - G_i}{i}\right\|_2 + \frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{\sqrt{i}}\left\|\frac{W_n - n\mu}{n}\right\|_2.$$

Also, if we define $\|\cdot\|$ to be the Euclidean norm in $\mathcal{R}^n$, and write $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$, where $x_i = \frac{W_i - i\mu}{i}$, $y_i = \frac{W_n - n\mu}{n}$, $z_i = \frac{G_i}{i}$, $i = 1, \dots, n$, then

$$|\widehat{\sigma}_{2,n} - V_{2,n}| = \frac{1}{(\log n)^{1/2}} \big| \|\mathbf{x} - \mathbf{y}\| - \|\mathbf{z}\| \big| \le \frac{1}{(\log n)^{1/2}} \|\mathbf{x} - \mathbf{y} - \mathbf{z}\|$$

$$\le \frac{\|\mathbf{x} - \mathbf{z}\|}{(\log n)^{1/2}} + \frac{\|\mathbf{y}\|}{(\log n)^{1/2}}.$$

So,

$$\begin{aligned}\|\widehat{\sigma}_{2,n} - V_{2,n}\|_2 &\le \frac{1}{(\log n)^{1/2}} \big\| \|\mathbf{x} - \mathbf{z}\| \big\|_2 + \frac{1}{(\log n)^{1/2}} \big\| \|\mathbf{y}\| \big\|_2 \\ &= \frac{1}{(\log n)^{1/2}} \left( \sum_{i=1}^n \mathbf{E} \left( \frac{W_i - i\mu - G_i}{i} \right)^2 \right)^{1/2} \\ &\quad + \frac{1}{(\log n)^{1/2}} \left( \sum_{i=1}^n \mathbf{E} \left( \frac{W_n - n\mu}{n} \right)^2 \right)^{1/2}.\end{aligned}$$

From Theorem 2.2 it follows that

$$\|\widehat{\sigma}_{1,n} - V_{1,n}\|_2 = \frac{1}{\log n} \sum_{i=1}^n \frac{1}{i} o(1) + \frac{1}{\log n} \sum_{i=1}^n \frac{1}{\sqrt{i}} O\big(1/\sqrt{n}\big) = o(1)$$

and

$$\|\widehat{\sigma}_{2,n} - V_{2,n}\|_2 = \frac{1}{(\log n)^{1/2}} \left( \sum_{i=1}^n o\left(\frac{1}{i}\right) \right)^{1/2} + \frac{1}{(\log n)^{1/2}} \left( \sum_{i=1}^n O\left(\frac{1}{n}\right) \right)^{1/2} = o(1).$$

Then, by Lemma 4.1 we have proved the theorem. □

By applying Theorem 2.3 instead of Theorem 2.2, the proof of Theorem 4.2 is similar to that of Theorem 4.1.

**Acknowledgments.** Special thanks go to an anonymous referee for constructive comments which led to a much improved version of the paper. Professor Hu is also affiliated with Department of Statistics and Applied Probability, National University of Singapore.

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