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LIMIT THEORY FOR RANDOM SEQUENTIAL PACKING AND DEPOSITION

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Consider sequential packing of unit balls in a large cube, as in the Rényi car-parking model, but in any dimension and with finite input. We prove a law of large numbers and central limit theorem for the number of packed balls in the thermodynamic limit. We prove analogous results for numerous related applied models, including cooperative sequential adsorption, ballistic deposition, and spatial birth-growth models.

The proofs are based on a general law of large numbers and central limit theorem for "stabilizing" functionals of marked point processes of independent uniform points in a large cube, which are of independent interest. "Stabilization" means, loosely, that local modifications have only local effects.

1. Introduction. Consider the following prototype random packing model. Unit volume open balls $B_{1,n}, B_{2,n}, \dots,$ arrive sequentially and uniformly at random in the d-dimensional cube Q_n having volume n and centered at the origin. Let the first ball $B_{1,n}$ be *packed*, and recursively for $i = 2, 3, \dots,$ let the ith ball $B_{i,n}$ be packed iff $B_{i,n}$ does not overlap any ball in $B_{1,n}, \dots, B_{i-1,n}$ which has already been packed. If not packed, the ith ball is discarded. Given a positive integer k, let $N_{n,d}(k) := N(\{B_{1,n}, \dots, B_{k,n}\})$ be the number of balls packed out of the first k arrivals. One studies the (random) packing number $N_{n,d}(k)$, or the (necessarily finite) limit $N_{n,d}(\infty) := \lim_{k\to\infty} N_{n,d}(k)$.

Packing models of this type arise in diverse disciplines, including the study of microscopic physical, chemical, and biological processes and macroscopic ecological and sociological systems. In statistical mechanics and biology, they are fundamental to the description of the irreversible deposition of colloidal particles or proteins onto a substrate (i.e., a surface). In this context, the prototype packing model described above is known as the Random Sequential Adsorption (RSA) model for hard spheres on a continuum surface; we will call this the "basic RSA model."

There is a vast literature involving versions of the basic RSA model on continuum and lattice substrates. The abundance of experimental results contrasts sharply with the paucity of rigorous theoretical results, particularly in more than

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one dimension; see Evans [11] for an extensive survey on lattice RSA results and Senger, Voegel and Schaaf [33] for a survey on continuum RSA results. Other surveys include Bartelt and Privman [3], Adamczyk, Siwek, Zembala and Belouschek [1] and Talbot, Tarjus, Van Tassel and Viot [37]. A recent voluminous special issue of Colloids and Surfaces [30] is dedicated to RSA research and contains further surveys.

In addition to their fundamental role in adsorption modeling, sequential packing models arise in the study of polymer reactions (Flory [12]; González, Hemmer and Høye [13]). Packing also arises in the study of hard core interactions in physical and materials science, and spatial growth models in crsytallography and biology (Evans [11], Section III). A classical result of Rényi [32] on car parking is effectively concerned with $N_{n,1}(\infty)$. In modeling communication protocols (Coffman, Flatto, Jelenković and Poonen [9]), RSA is called *on-line packing*.

In the sequel we shall refer to $N_{n,d}(k)$ and $N_{n,d}(\infty)$ as a *fixed input packing number* and *infinite input packing number*, respectively. Also of interest is the *Poisson input packing number* $N_{n,d}(Po(\lambda)) := N(\{B_{1,n}, \dots, B_{Po(\lambda),n}\})$, where $Po(\lambda)$ denotes an independent Poisson random variable with parameter λ. This is quite natural if one imagines the evolving configuration of packed balls as a continuous-time Markov chain running for a fixed amount of time.

In this paper we restrict attention to the fixed input and Poisson input packing numbers, and take a thermodynamic limit with the (expected) number of incoming balls proportionate to n, with constant of proportionality denoted τ. Our main purpose is to prove a law of large numbers (LLN) and a central limit theorem (CLT), for the basic RSA model (Theorems 1.1 and 1.2) as well as for diverse related packing and deposition models. These include models with random shapes/types, spatial birth-growth models, cooperative sequential adsorption, and multilayer ballistic deposition. This section treats the limit theory of the basic RSA model, whereas Section 2 treats the diverse related models.

Our CLTs and LLNs for all these models are proved via a general CLT (Theorem 3.1 below) and a general LLN (Theorem 3.2 below) for functionals of "marked" binomial and Poisson point processes. Theorem 3.1 extends a CLT for unmarked processes in Penrose and Yukich [26], which was itself inspired in part by methods of Kesten and Lee [17], Lee [19]. Theorem 3.2 is new, even for unmarked processes. The general CLT and LLN are of independent interest and potentially useful in other contexts, for example that of Boolean models over binomial point processes; see Section 3 for further discussion.

Our LLNs are quite "strong;" they hold not only with *convergence of means* (c.m.) but also with *complete convergence* (c.c.). Given a sequence of random variables X_n (possibly on distinct probability spaces) and a constant x, we shall say $X_n \to x$ c.m. if $\lim_{n\to\infty} EX_n = x$, and $X_n \to x$ c.c. if $\sum_n P[|X_n - x| > \varepsilon] < \infty$ for all $\varepsilon > 0$. Complete convergence implies almost sure convergence of X_n' to x for any sequence of coupled variables X_n' for which X_n' and X_n are identically distributed for all n. Write c.m.c.c. if both types of convergence hold.

THEOREM 1.1 (LLN for basic RSA). *For all $\tau \in (0,\infty)$ and all $d \geq 1$ there is a constant $\alpha(d,\tau)$ such that*

$$\frac{N_{n,d}([\tau n])}{n} \to \alpha(d,\tau) \quad \text{c.m.c.c.} \tag{1.1}$$

and

$$\frac{N_{n,d}(Po(\tau n))}{n} \to \alpha(d,\tau) \quad \text{c.m.c.c.} \tag{1.2}$$

Turning to the CLT, we write $\xrightarrow{\mathcal{D}}$ for convergence in distribution as $n \to \infty$ and $\mathcal{N}(0,\sigma^2)$ for a random variable having normal distribution with mean zero and variance σ^2. Our result gives convergence of variances as well as convergence to the normal distribution.

THEOREM 1.2 (CLT for basic RSA). *For all $\tau \in (0,\infty)$ and all $d \geq 1$ there exist constants $0 < \eta_{d,\tau} \leq \sigma_{d,\tau} < \infty$ such that*

$$\frac{N_{n,d}(Po(\tau n)) - EN_{n,d}(Po(\tau n))}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2_{d,\tau}) \tag{1.3}$$

and

$$n^{-1}\operatorname{Var} N_{n,d}\big(Po(\tau n)\big) \to \sigma^2_{d,\tau}, \tag{1.4}$$

while

$$\frac{N_{n,d}([\tau n]) - EN_{n,d}([\tau n])}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\eta^2_{d,\tau}) \tag{1.5}$$

and

$$n^{-1}\operatorname{Var} N_{n,d}([\tau n]) \to \eta^2_{d,\tau}. \tag{1.6}$$

REMARKS. (i) Previous rigorous mathematical work on basic RSA (and variants) has been restricted to dimension $d = 1$. See Ney [22], Dvoretzky and Robbins [10], Mannion [20], Coffman, Flatto, Jelenković and Poonen [9], Coffman, Flatto and Jelenković [8], Itoh and Shepp [15]. In particular, the c.m. part of (1.1) and the Poisson input CLT results (1.3) and (1.4) were already known for $d = 1$, with explicit formulae for the limiting constants $\alpha(1,\tau)$ and $\sigma_{1,\tau}$ (see Coffman, Flatto, Jelenković and Poonen [9], equation (2) and Theorems 13, 14). Theorems 1.1 and 1.2 extend these results to higher dimensions, albeit without giving explicit formulae for the limiting constants. Even for $d = 1$ the fixed input CLTs (1.5) and (1.6) of Theorem 1.2 are apparently new, and answer a question of Coffman, Flatto, Jelenković and Poonen [9]. Note that Poissonization of the input contributes extra randomness which shows up in the limiting variance.

(ii) An alternative appproach to the LLN is given in simultaneous work of Penrose [24], which is mainly concerned with infinite input but also applicable to Poisson input. [24] does not address fixed input, complete convergence, or CLTs.

(iii) The physical sciences literature contains many simulation and analytical studies of RSA in dimensions $d = 2, 3$. See the survey articles mentioned previously for discussion of these, and also Pomeau [29], Swendsen [36], Caser and Hilhorst [4] for analytical results. A principal object of interest in these studies is the limit $\alpha(d, \tau)$ (known as the *coverage function*) and/or the analogously defined infinite-input limit $\alpha(d, \infty)$ (the *jamming coverage*). Beginning with the oft-quoted paper of Widom [40], these limits have long been assumed to exist, at least in the c.m. sense, without proof. Theorem 1.1, and the LLNs for infinite input in [24], establish rigorously the existence of these limits. Theorem 1.2 provides rigorous information regarding the asymptotics of the fluctuations of the packing number about the limit $\alpha(d, \tau)$.

(iv) At a technical level, sequential packing presents an interesting challenge because it yields dependently thinned point processes having finite range interactions but with long range dependence, apparently lacking any obvious subadditivity structure. This makes it harder to prove LLNs or CLTs by standard techniques.

(v) The unifying thread which ties together the limit theory for functionals in this paper involves the notion of "add one cost," namely the effect on the functional of inserting a single point into a (marked) binomial or Poisson point process. In fact, the hypotheses for both the general CLT and the general LLN are framed in terms of "add one cost." The add one cost for packing functionals is estimated using methods from first passage percolation and, in particular, bounds for the growth rate of a spatial epidemic.

(vi) The proof of Theorems 1.1 and 1.2 shows that the cubes Q_n, $n \geq 1$, can be replaced by more general regions. We refer to Section 3 for the exact conditions on the regions Q_n, $n \geq 1$.

(vii) In the *off-line packing* problem, all balls are collectively and simultaneously available for packing, that is, there are no sequential arrivals. Off-line packing is not treated in this paper, although in $d = 1$ it is amenable to methods related to those used here.

NOTATION. For $x \in \mathbb{R}^d$, $r \geq 0$, and $A \subset \mathbb{R}^d$, let $x + A$ denote the translated set $\{x + y : y \in A\}$, let $r \cdot A := \{ry : y \in A\}$ let $|x|$ denote the Euclidean norm of x, and let $|A|$ denote the volume of A. Let $B_r(x)$ denote the ball $\{y \in \mathbb{R}^d : |y - x| \leq r\}$. Let $\operatorname{diam}(A) := \sup\{|x - y| : x, y \in A\}$, let ∂A denote the intersection of the closure of A with that of its complement, and let $\partial_r(A) := \bigcup_{x \in \partial A} B_r(x)$, the r-neighborhood of the boundary of A. Let $\mathbf{0}$ be the origin in $\mathbb{R}^d$. Let C denote a positive finite constant whose value is unimportant and which may change from line to line.

2. Related models. There are a multitude of variants of the basic RSA packing model. Our methods yield LLNs and CLTs for many of these variants. We limit the discussion here to four widely used such models: (i) models with unit volume balls replaced by particles of random size/shape/charge, (ii) time dependent models, (iii) cooperative sequential adsorption models, and (iv) ballistic deposition models.

2.1. *Random shapes and types.* In the basic RSA model, each incoming particle is required to be a unit volume ball. One can relax this by allowing random shapes such as balls of random radius or ellipses of random orientation. Many simulation studies have considered RSA models of this kind. See for example Sections 6.2 and 7 of [3]. More generally, one may consider particles having *random type*, not necessarily representing size or shape. For example, Itoh and Shepp [15] considered a one-dimensional packing model where there are two types, which they call "spins;" particles of the same type exclude each other up to range a and particles of different type exclude each other up to range 1.

We thus consider a generalized RSA model, where each particle has a random type, representing its shape, or "spin," or any other property of the particle. Let $(\mathbb{F}, \mathcal{F})$ be an arbitrary measurable space (the space of possible types), and let P_1 be a probability measure on $(\mathbb{F}, \mathcal{F})$, representing the distribution of types of random inputs. Suppose

$$\Phi\colon \mathbb{R}^d \times \mathbb{F} \times \mathbb{F} \to \{0, 1\}$$

is a measurable function (with respect to product measure), the so-called "exclusion function," where $\Phi(x, \Xi', \Xi)$ takes the value 0 if a particle of type Ξ' excludes any subsequently arriving particle of type Ξ with relative displacement x. For example, in the case of spatial exclusion by particles with random shape, one could let $\mathbb{F}$ be a space of subsets of $\mathbb{R}^d$ and set $\Phi(x, \Xi', \Xi)$ to be 1 if and only if $\Xi' \cap (x + \Xi) = \emptyset$. Assume the exclusion function has finite range, in a sense to be made precise later on.

Let the ith incoming particle have location $X_{i,n}$ and type Ξ_i, where $X_{1,n}$, $X_{2,n}, \ldots$ are independent and uniformly distributed over the cube Q_n (as in the basic RSA model), and $(\Xi_1, \Xi_2, \ldots)$ is a sequence of i.i.d. random elements of $\mathbb{F}$ with common probability distribution P_1, independent of $(X_{1,n}, X_{2,n}, \ldots)$. For example, if the particles are random shape sets in $\mathbb{R}^d$, then the types Ξ_i are i.i.d. in some space of (uniformly bounded) closed sets (see [35, 21] for background on random closed sets) and the region of space actually occupied by the ith particle is $X_{i,n} + \Xi_i$.

Recursively, let the ith particle be accepted (i.e., packed) if $\Phi(X_{i,n} - X_{j,n}, \Xi_j, \Xi_i) = 1$ for all $j < i$ such that j was accepted; otherwise the ith particle is rejected.

Of interest are various counts concerned with accepted particles. These include the total number of particles, out of the first k to arrive, to be accepted, or the total volume of accepted sets, or the number of accepted sets of a particular kind, for example, sets having volume bounded by some parameter α. The latter has been studied for $d = 1$ by Ney [22] and Mannion [20]. Studying the total volume of accepted sets is equivalent to studying the wasted (vacant) space, for which Coffman, Flatto and Jelenković [8] have obtained rather precise asymptotics in dimension $d = 1$ only. Theorem 2.1 below will provide LLNs and CLTs for a large class of counts associated with accepted particles.

2.2. *Time-dependent models.* We describe two models where the acceptance probability for an incoming particle depends not only on the locations and types of nearby accepted particles, but also on the time since arrival for these particles.

2.2.1. *Models with desorption.* Consider a generalization of the basic packing model, with arrivals at the points of a homogeneous space–time Poisson process of unit intensity on $Q_n \times [0, \infty)$, in which a packed ball remains in place for a random period of time (possibly infinity) at the end of which it is removed (i.e., desorbs). This is a dynamic model, which among other things, describes the reversible deposition of particles on substrates, and is discussed in the surveys of Senger, Voegel and Schaaf (page 267 of [33]) and Talbot, Tarjus, Van Tassel and Viot (Section 7 of [37]). Numerous experimental results are described in [37].

Assume that the times before desorption for particles are i.i.d. with some arbitrary distribution on $[0, \infty]$, and independent of the locations of the balls. Let $N_n(Po(n\tau))$ be the number of adsorbed particles in the set Q_n at time τ. The scientific literature [33] implicitly assumes the existence of a coverage function $\lim_{n\to\infty} EN_n(Po(n\tau))/n$ representing the limiting mean coverage in the thermodynamic limit. Theorem 2.1 below puts this on rigorous footing.

Desorption models can be put into the context of the "random type" model described in Section 2.1, if we allow the function Φ, representing acceptance probability, to depend not only on the location and type of nearby particles but also on the time since arrival. For the basic RSA model with desorption, let the space of types $\mathbb{F}$ be the interval $[0, \infty]$ and let the type Ξ of an incoming particle represent its time to desorption. Then take the domain of the exclusion function Φ to be $\mathbb{R}^d \times \mathbb{F} \times \mathbb{F} \times [0, \infty]$, and let $\Phi(x, \Xi', \Xi, t)$ take the value 0 if unit volume balls centered at 0 and x overlap each other *and* $\Xi' \geq t$. Then, recursively, let an object arriving at X with type Ξ at time T be accepted (i.e., packed) if $\Phi(X - X', \Xi', \Xi, T - T') = 1$ for all previously packed items (X', Ξ', T').

The contribution of a particle to the count of adsorbed points $N_n(Po(n\tau))$ is determined not only by whether it is accepted or not, but also by whether, if accepted, it desorbs before time τ. This contribution is therefore a function of the particle's type and time of arrival, and Theorem 2.1 below is framed to take this into account.

2.2.2. *Spatial birth-growth models.* Consider the following generalized version of a classical birth-growth model on $\mathbb{R}^d$: cells are formed at random locations $X_i \in \mathbb{R}^d$ at times T_i, $i = 1, 2, \dots$, according to a unit intensity homogeneous spatial–temporal Poisson point process $\Psi := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \infty)\}$. When a new cell is formed, its center X_i is called its "seed." Initially (at time T_i) the new cell takes the form of a ball of (possibly random) radius $\rho_i \geq 0$ centered at X_i (or just the point X_i if $\rho_i = 0$) and then grows with a constant speed v, radially in all directions compatible with nonoverlap with other cells: wherever a cell touches another, it stops growing in that direction. New cells form only in the *uncovered* space in $\mathbb{R}^d$. That is, if a new seed appears at X_i such that the ball of radius ρ_i centered at X_i overlaps any of the existing cells, then it is discarded. Ultimately, the cells tessellate $\mathbb{R}^d$. Assume that ρ_i, $i = 1, 2, 3, \dots$, are i.i.d. and independent of Ψ, and that there is a constant $r_2 > 0$ such that $P[\rho_i \leq r_2] = 1$.

In the special case where the growth rate v is zero a.s., this model reduces to the packing model with balls of random radius as already described. In the alternative special case where all the initial radii ρ_i of cells are zero a.s., our model is referred to as the *Johnson–Mehl model* in Stoyan, Kendall and Mecke [35], where further references can be found. It was originally studied to model crystal growth (Kolmogorov [18]). A slightly different spatial birth-growth model due to Pielou [28] is discussed in Section 2.4.1 below.

Let $N^*(Q_n, \tau)$ denote the number of accepted seeds inside the window Q_n by time τ. Quine and Robinson [31] (for $d = 1$) and later Chiu and Quine [6] (for general d) established the asymptotic normality of $N^*(Q_n, \tau)$ in the case of the Johnson–Mehl model. They allow the intensity of Ψ to be nonhomogeneous in time. We simplify here by assuming time-homogeneity but in fact our methods can be adapted to allow for nonhomogeneity in time, at the cost of some extra notation.

We consider a modification of this model in which all seeds outside $Q_n \times [0, \infty)$ are automatically rejected, while the rules for seeds inside $Q_n \times [0, \infty)$ are as described above. Let $N(Q_n, \tau)$ denote the number of accepted seeds up to time τ for this modified model. Unlike $N^*(Q_n, \tau)$, this is not the restriction to Q_n of a stationary point process, but incorporates edge effects and is arguably more realistic.

Again the model can be described in terms of packing of objects of random type, with account taken of time since arrival. In this case the type of an object is $\Xi_i := \rho_i$ and the exclusion function $\Phi(x, \Xi_j, \Xi_i, t)$ is 1 if $|x| > \rho_j + vt + \rho_i$, and zero otherwise. Since we are interested only in the evolution of the process up to a finite time τ, there is a finite range of interaction between two particles: no particle excludes any other particle appearing at a distance more than $2r_2 + v\tau$ from it.

As a special case of Theorem 2.1 below, one obtains a LLN and a CLT for the functional $N(Q_n, \tau)$, as well as convergence of its variance. Theorem 2.1 also shows that the asymptotic normality of $N(Q_n, \tau)$ continues to hold if the Poisson input is replaced by fixed input.

2.3. *Cooperative sequential adsorption.* Cooperative sequential adsorption (CSA) is a generalization of RSA, in which the probability that an arriving particle centered at $x \in \mathbb{R}^d$ is accepted depends on the local configuration of previously accepted particles near x. CSA models are ubiquitous in the scientific literature; we refer to Evans [11] for a survey (mainly concerned with the lattice models), and to Adamczyk, Siwek, Zembala and Belouschek [1] and Senger, Voegel and Schaaf [33] for discussions of "soft sphere" models which are a special case of continuum CSA.

We restrict attention to CSA with a finite range $r_3 > 0$ of interactions. While many models have infinite range [11], we note that the basic RSA model is a special case of the finite range CSA model. We assume the acceptance probability takes the form of a product of probabilities associated with each of the existing nearby particles. This simplifies the notation and is in keeping with most CSA models seen in the literature, although our methods are also applicable to acceptance probabilities with finite range that do not take the form of a product.

It is easy to imagine combining the idea of CSA as just described with those of random type and time-dependence described in Sections 2.1 and 2.2, and therefore we now give results for a *generalized CSA model* allowing for all these effects. The assumption that the acceptance probability has the form of a product means that in effect, we simply extend the range of the function Φ described in earlier sections from $\{0, 1\}$ to the interval $[0, 1]$, and take products to give the acceptance probability.

Formally, the generalized CSA model goes as follows. The dimension $d \geq 1$ is fixed but arbitrary. Let $(\mathbb{F}, \mathcal{F}, P_1)$ be a probability space with $\mathbb{F}$ representing the space of possible types and P_1 representing the distribution of types of incoming particles. Define a product measurable function

$$\Phi\colon \mathbb{R}^d \times \mathbb{F} \times \mathbb{F} \times [0, \infty] \to [0, 1]. \tag{2.1}$$

Assume finite range interactions, that is, assume there is a constant $r_3 > 0$ such that for $(P_1 \times P_1)$-almost all (Ξ, Ξ'),

$$\Phi(x, \Xi', \Xi, t) = 1 \qquad \text{if } |x| > r_3. \tag{2.2}$$

Given $\tau > 0$, and given the window $Q_n \subset \mathbb{R}^d$, we suppose that each incoming particle is represented by a triple (X, Ξ, T) of independent variables, with X distributed uniformly at random over Q_n and representing the particle's location, with Ξ taking its value in $\mathbb{F}$ with the distribution P_1 and representing the type of the particle, and with T distributed uniformly over $[0, \tau]$ and representing its time of arrival. The total number of incoming particles is Poisson with parameter τn for the Poisson input model, and is $[\tau n]$ for the fixed input model.

The decision on whether to accept a particle is made sequentially in the order of arrival. If a particle is represented by the triple (X, Ξ, T), let it be accepted with probability

$$\prod \Phi(X - X', \Xi', \Xi, T - T'),$$

with the product taken over all accepted particles (X', Ξ', T') for which $T' < T$ (and the product over the empty set is taken to be 1).

Let h: $\mathbb{F} \times [0, \infty] \to [0, \infty)$ be a bounded measurable test function, and define $G_n([\tau n])$, respectively $G_n(Po(\tau n))$, to be the sum

$$\sum h(\Xi, T), \tag{2.3}$$

with the sum taken over accepted particles in the fixed input generalized CSA model, respectively the Poisson input generalized CSA model, as just described. To avoid uninteresting degenerate cases, assume also that

$$\int_0^\tau \int_{\mathbb{F}} \int_{\mathbb{F}} \int_{\mathbb{R}^d} \big(1 - \Phi(x, \Xi', \Xi, t)\big)\, dx\, dP_1(\Xi)\, dP_1(\Xi')\, dt > 0; \tag{2.4}$$

$$\int_0^\tau \int_{\mathbb{F}} h(\Xi, t)\, dP_1(\Xi)\, dt > 0. \tag{2.5}$$

Then the following limit result holds. It clearly generalizes Theorems 1.1 and 1.2.

THEOREM 2.1 (LLN and CLT for generalized CSA model). (a) *There exists a constant* $G := G(d, \mathbb{F}, \mathcal{F}, P_1, \Phi, h, \tau)$ *such that*

$$\frac{G_n([\tau n])}{n} \to G \quad \text{c.m.c.c.}; \qquad \frac{G_n(Po(\tau n))}{n} \to G \quad \text{c.m.c.c.}$$

(b) *There exist constants* $0 < \eta \le \sigma < \infty$ (*dependent on* $d, \mathbb{F}, \mathcal{F}, P_1, \Phi, h, \tau$) *such that*

$$\frac{G_n(Po(\tau n)) - EG_n(Po(\tau n))}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \tag{2.6}$$

and $n^{-1}\,\mathrm{Var}(G_n(Po(\tau n))) \to \sigma^2$, *while*

$$\frac{G_n([\tau n]) - EG_n([\tau n])}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \eta^2) \tag{2.7}$$

and $n^{-1}\,\mathrm{Var}(G_n([\tau n])) \to \eta^2$.

2.4. *Ballistic deposition.* Ballistic deposition (BD) models are extensions of the basic RSA model, representing deposition of particles in the presence of a gravitational field. Each incoming particle occupies a region of $(d+1)$-dimensional space (typically, a Euclidean ball); we assume $d = 2$ or $d = 1$, and the $(d+1)$st coordinate represents "height." An incoming particle falls perpendicularly from above towards a substrate taking the form of a d-dimensional surface embedded in $(d+1)$-space. We represent the substrate by the surface $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$, which we identify with the lower-dimensional space $\mathbb{R}^d$.

The downward motion of an incoming particle is vertical until it hits the substrate or one of the particles already adsorbed. At this point there may be some lateral and/or vertical motion (displacement) of the incoming particle before it either is accepted (adsorbed) and comes to rest, or is rejected.

The lateral/vertical displacement of the incoming particle, and also the decision on whether to accept or reject it, depends on the positions of already accepted particles according to some mechanism which could be deterministic (as for RSA) or stochastic (as for CSA). For simplicity, we restrict attention here to deterministic mechanisms, and also assume the incoming particles are open Euclidean balls of identical radius (as for the basic RSA model); however, both these assumptions can be relaxed using ideas discussed in Sections 2.1 and 2.3.

In keeping with the rest of this article, we restrict attention to continuum models, although lattice models of BD are also of interest; see [11]. We assume that the location of each incoming particle, that is, the position on the surface $\mathbb{R}^d$ at which it would land if unhindered by existing particles, is uniformly distributed over the window Q_n (i.e., the d-cube of volume n). We describe two of the many BD models which have been proposed: (i) monolayer BD with a rolling mechanism and (ii) multi-layer BD.

2.4.1. *Monolayer ballistic deposition with a rolling mechanism.* "Monolayer" refers here to the fact that all accepted particles lie on the surface of the substrate, as in the basic RSA model. The model is as follows.

Particles fall sequentially from above, vertically towards the adsorption surface $\mathbb{R}^d$ (strictly speaking, $\mathbb{R}^d \times \{0\}$). If a particle reaches the surface $\mathbb{R}^d$, it is irreversibly fixed on it. Otherwise, if the particle contacts a previously deposited particle, it then rolls, following the path of steepest descent until it reaches a stable position. The rolling process does not displace already deposited particles, that, is there is no updating of existing particles. If the particle reaches the adsorption surface, it is fixed there; otherwise it is removed from the system and the next sequenced particle is considered. For $d = 1$, the model dates back to Solomon ([34], page 129). For $d = 2$, Senger, Voegel and Schaaf [33] describe the many recent experimental results; Choi, Talbot, Tarjus and Viot [7] give both experimental and analytical results. Current work of Penrose [25] gives rigorous results on the infinite input version of this model.

All accepted particles lie on the substrate, and so can be represented by points in $\mathbb{R}^d$. The position of an accepted particle is a translate (or *displacement*) of the original location in $\mathbb{R}^d$ above which it originally comes in. The displacement and the decision on whether to accept the particle are both determined by the original location of the arrival of the particle, and the positions (after displacement) of previously accepted particles. We claim that there is a uniform bound on the size of the possible displacement. This is clear when $d = 1$, and a proof for $d = 2$ is given in [25]; see also [7]. There are also numerous other monolayer BD models satisfying this condition of uniformly bounded displacements. Our methods yield a LLN and CLT for the total number of accepted particles in a fixed input or Poisson input setting; see Section 2.4.3.

Pielou [28] proposes another related model in the study of biological populations in a region of $\mathbb{R}^2$. This resembles the spatial growth model discussed in Section 2.2.2 above, but with *infinite* growth rate of cells, up to a maximum cell size. Whenever a new seed appears, its initial cell radius is some specified minimum, and if that entails overlap with an existing cell the new seed is discarded as in the grain-growth model described earlier. If accepted, the new cell instantly grows up to a disk of the specified maximum radius or the smallest radius at which it touches an existing cell, whichever is the smaller, and thereafter the cell remains unchanged. This resembles the BD model just described, except that now it is the size of the particle (cell) rather than the location of its center that adjusts itself in a manner determined by the nearby existing (adjusted) cells, before the arrival of the next particle. One might be interested either in the total number of cells or their total area, and can obtain a LLN and CLT for these in much the same manner as for the BD model just described.

2.4.2. *Multilayer ballistic deposition.* In multilayer BD, a particle may attach itself to previously adsorbed particles instead of to the substrate. In the simplest form of continuum multilayer BD, each particle falls vertically towards the substrate Q_n (as described above) and as soon as it encounters either the substrate or another particle, it sticks (and remains in that place forever). Therefore each particle is accepted, and has a vertical displacement (or *height*) relative to the position it would occupy if it were to fall to the substrate unhindered by other particles. There have been many empirical studies of this multilayer BD model (first proposed by Vold [39]) and its variants; see, for example, Jullien and Meakin [16], Vicsek [38], Talbot, Tarjus, Van Tassel and Viot [37]. However there is even less rigorous limit theory for BD models than for RSA models.

Variants of the BD model include those with displacement of incoming particles by a rolling mechanism, and those where there is a possibility of rejection of an incoming particle. We restrict attention to cases involving a uniform bound on the possible lateral displacement induced by the rolling mechanism. Reference [16] provides a variety of models which incorporate rolling subject to this restriction and ([37], pages 321–322) describes a family of models incorporating both (uniformly bounded) rolling and possible rejection.

There are a number of quantities of interest, including the number of accepted particles (in cases where particles are not all accepted), the *total height* of accepted particles, and the *total volume* of the agglomoration of particles, including "empty space" trapped below overhanging particles. More precisely, let a part of a particle's surface be denoted "exposed" if it does not have any other parts of particles lying above it. The contribution of a particle to the total volume is the volume of the region (if any) lying below its exposed surface (and above the substrate).

2.4.3. *Generalized ballistic deposition.* We describe a model incorporating both types of BD described above as special cases. Particles arrive sequentially at locations $X_{1,n}, X_{2,n}, \dots$, which are independent and uniformly distributed over the region Q_n; as mentioned above, $X_{i,n}$ represents the position in $\mathbb{R}^d$ at which the ith particle would land if it fell unhindered to the substrate. The ith particle is either accepted or rejected, and if accepted, receives a $(d+1)$-dimensional displacement $\xi_i := (\xi_i^{\rightarrow}, \xi_i^{\uparrow})$, with $\xi_i^{\rightarrow} \in \mathbb{R}^d$ representing lateral displacement, and $\xi_i^{\uparrow} \in [0, \infty)$, representing vertical displacement ("height").

Assume that there is a constant r_4 providing a uniform bound both on the range of interaction and on the lateral displacement distance. Specify a measurable function $S \mapsto \Psi(S)$, taking values in $\{0, 1\} \times B_{r_4}(\mathbf{0}) \times [0, \infty)$ and defined for all finite subsets S of $B_{r_4}(\mathbf{0}) \times [0, \infty)$. We use the notation

$$\Psi(S) := \big(\Psi_0(S), \Psi_{\rightarrow}(S), \Psi_{\uparrow}(S)\big)$$

with $\Psi_0(S) \in \{0, 1\}$, $\Psi_{\rightarrow}(S) \in B_{r_4}(\mathbf{0})$, and $\Psi_{\uparrow}(S) \in [0, \infty)$. Assume that if S is the empty set, then $\Psi(S) = (1, \mathbf{0}, 0)$. As a further condition on the function Ψ, to reflect the idea that height builds up by the stacking of particles, assume that there is a constant r_5 such that for all S,

$$\psi_{\uparrow}(S) \le \max\{y_{\uparrow} : y = (y_{\rightarrow}, y_{\uparrow}) \in S\} + r_5. \tag{2.8}$$

Suppose, inductively, that it has been determined which of particles $1, 2, \dots, i-1$ are to be accepted and what their displacements are. Let $J(i)$ be the set (possibly empty) of $j < i$ for which X_j is accepted and $|X_j + \xi_j^{\rightarrow} - X_i| \le r_4$, and let

$$S_i := \big\{(X_j + \xi_j^{\rightarrow} - X_i, \xi_j^{\uparrow}) : j \in J(i)\big\}.$$

Then if particle i arrives at location X_i, it is accepted if and only if $\Psi_0(S_i) = 1$, and if accepted its displacement is $\xi_i := (\Psi_{\rightarrow}(S_i), \Psi_{\uparrow}(S_i))$.

Again we consider the thermodynamic limit. Other limits are also of interest, but the thermodynamic limit represents a fixed expected number (namely τ) of incoming particles per unit area as in Talbot, Tarjus, Van Tassel and Viot ([37], page 321), where attention is focused on the first few layers. If $N_n(k)$ denotes the number of items accepted in the BD model just described (which includes the basic RSA model as a special case), then for any $\tau \in (0, \infty)$ the variables $N_n([\tau n])$ and $N_n(Po(\tau n))$ satisfy a LLN and CLT as $n \to \infty$. The proof is virtually the same as for Theorems 1.1 and 1.2, using the fact that an incoming particle is affected only by the positions and displacements of previously accepted particles whose original location (before displacement) was within distance $2r_4$ of that of the new particle.

Next we consider the total height of accepted particles. Let $D_n(k)$ denote the total height of accepted particles, out of the first k to arrive. That is, $D_n(k) := \sum \xi_j^{\uparrow}$, the sum being over all accepted particles with index $j \le k$. To avoid degeneracy, assume the parameters of the model are such that there exists an integer $k > 0$ such that with positive probability, particle k is accepted and $\xi_k^{\uparrow} > 0$.

THEOREM 2.2 (LLN and CLT for total height in generalized BD). (a) *For all* $\tau \in (0, \infty)$, *there is a constant* $D := D(d, \Psi, \tau)$ *such that*

$$\frac{D_n([\tau n])}{n} \to D \quad \text{c.m.c.c.}; \qquad \frac{D_n(Po(\tau n))}{n} \to D \quad \text{c.m.c.c.} \tag{2.9}$$

(b) *For all* $\tau \in (0, \infty)$ *there exist constants* $0 < \eta_\tau \le \sigma_\tau < \infty$ (*also dependent on* d *and* Ψ) *such that*

$$\frac{D_n(Po(\tau n)) - E D_n(Po(\tau n))}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\tau^2) \tag{2.10}$$

and $n^{-1} \operatorname{Var}(D_n(Po(\tau n))) \to \sigma_\tau^2$, *while*

$$\frac{D_n([\tau n]) - E D_n([\tau n])}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \eta_\tau^2) \tag{2.11}$$

and $n^{-1} \operatorname{Var}(D_n([\tau n])) \to \eta_\tau^2$.

A LLN and CLT for the total volume of the generalized BD process can be derived by very similar means to those we shall use in the case of the total height. In [27], we establish a thermodynamic limit and Gaussian fluctuations for the height and surface width of the random interface formed in the generalized BD model and we explicitly describe the limiting constant D.

3. A general CLT and LLN for marked point processes. To develop the limit theory for packing and deposition numbers, we shall formulate in this section a general CLT and a general LLN for stabilizing functionals H of marked spatial point processes. The CLT is an extension of a general CLT developed by Penrose and Yukich [26] in the context of unmarked point processes. The general LLN is new.

Let $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, P_{\mathcal{K}})$ be a probability space. We assume marks take values in $\mathcal{K}$ and are i.i.d. with distribution $P_{\mathcal{K}}$. Let H be a real-valued functional defined for all finite (marked) point sets $\mathcal{X} \subset \mathbb{R}^d$. Thus H is defined on sets of the form $\{(x_i, M_{x_i})\}_{i=1}^n \subset \mathbb{R}^d \times \mathcal{K}$, where for all $x \in \mathbb{R}^d$ the mark is denoted by M_x, $M_x \in \mathcal{K}$. If $\mathcal{X} := \{x_i\}_{i=1}^n$, then we will abbreviate notation by suppressing mention of the marks and simply write $H(\mathcal{X})$ for $H(\{(x_i, M_{x_i})\}_{i=1}^n)$.

Let S_y be the shift operator on $\mathbb{R}^d \times \mathcal{K}$ which sends (x, M_x) to $(x + y, M_x)$. Then H is *translation-invariant* if for any $\mathcal{X}$ and any $y \in \mathbb{R}^d$ we have $H(\mathcal{X}) = H(S_y(\mathcal{X}))$.

Given a translation invariant H, define the "add one cost," by which we mean the increment in H caused by inserting a marked point at the origin into a finite marked point set $\mathcal{X} \subset \mathbb{R}^d$, formally defined by

$$\Delta(\mathcal{X}) := H(\mathcal{X} \cup \{\mathbf{0}\}) - H(\mathcal{X}), \tag{3.1}$$

where it is always assumed that the inserted point at the origin has a $\mathcal{K}$-valued mark with distribution $P_{\mathcal{K}}$.

For "typical" marked point sets $\mathcal{X}$, it is conceivable that the add one cost is not affected by changes in $\mathcal{X}$ which are far from the origin. We formalize this notion of stability as follows.

Let $\mathcal{P} := \mathcal{P}_\tau$ be a homogeneous Poisson process of intensity τ on $\mathbb{R}^d$ with each point carrying a $\mathcal{K}$-valued mark with distribution $P_{\mathcal{K}}$. Formally, $\mathcal{P}$ is a Poisson process on the product of $\mathbb{R}^d \times \mathcal{K}$ with mean measure τ times the product of Lebesgue measure on $\mathbb{R}^d$ with the measure $P_{\mathcal{K}}$ on $\mathcal{K}$. Each point of $\mathcal{P}$ is a pair (X, M) with $X \in \mathbb{R}^d$, $M \in \mathcal{K}$, but we view it as a point X in $\mathbb{R}^d$ carrying a mark $M := M_X$.

The functional H is *strongly stabilizing at intensity* τ if there exist a.s. finite random variables $R := R(\tau)$ (a *radius of stabilization* of H) and $\Delta_\tau(\infty)$ such that with probability 1, $\Delta((\mathcal{P} \cap B_R(\mathbf{0})) \cup \mathcal{A}) = \Delta_\tau(\infty)$ for all finite marked $\mathcal{A} \subset (\mathbb{R}^d \setminus B_R(\mathbf{0}))$. Thus, R is a radius of stabilization if the add one cost is unaffected by changes in the configuration outside the ball $B_R(\mathbf{0})$.

Throughout the sequel, $\tau > 0$ is a constant and $(Q_n)_{n\geq 1}$ denotes a sequence of bounded Borel subsets ("regions" or "windows") of $\mathbb{R}^d$, satisfying the following conditions. First, $|Q_n| = n$ for all n; second, $\bigcup_{n\geq 1}\bigcap_{m\geq n} Q_m = \mathbb{R}^d$; third, $\lim_{n\to\infty}(|\partial_r Q_n|/n) = 0$ for all $r > 0$ (the *vanishing relative boundary* condition), and fourth, $\limsup_{n\to\infty}(n^{-1}\,\mathrm{diam}(Q_n)) < \infty$ (the *linear boundedness* condition on Q_n). Subject to these conditions, the choice of $(Q_n)_{n\geq 1}$ is arbitrary. Note that in particular the cubes defined at the outset of this paper satisfy these conditions, as do sets Q_n of the form $n^{1/d} \cdot C$, where C is a bounded convex set of unit volume containing the origin. Let $\mathcal{B}$ denote the collection of all regions obtained by translating any of the sets Q_n; that is, $\mathcal{B} := \{x + Q_n : x \in \mathbb{R}^d,\ n \geq 1\}$.

Suppose $Q \in \mathcal{B}$. Let $U_{1,Q}, U_{2,Q}, \dots$ be independent identically distributed uniform variables on Q, and let $M_1, M_2, \dots$ be independent $\mathcal{K}$-valued variables with common distribution $P_{\mathcal{K}}$, independent of $(U_{1,Q}, U_{2,Q}, \dots)$. Let $\mathcal{U}_{m,Q}$ be a marked point process consisting of m independent uniform variables on Q. Thus

$$\mathcal{U}_{m,Q} := \{(U_{1,Q}, M_1), \dots, (U_{m,Q}, M_m)\}$$

which is a point process in $Q \times \mathcal{K}$ but which we view as a marked binomial point process on Q with each point carrying a $\mathcal{K}$-valued mark with distribution $P_{\mathcal{K}}$. Let $\mathcal{P}_{\tau,n}$ be a homogeneous marked Poisson process on Q_n of intensity τ (e.g., the restriction of $\mathcal{P}_\tau$ to Q_n). Our general results refer to the point processes $\mathcal{P}_{\tau,n}$ and $\mathcal{U}_{[n\tau],Q_n}$.

Our general CLT also requires a moment condition for the add one cost of inserting a marked point at the origin into the marked binomial process $\mathcal{U}_{m,Q}$. Given $p \geq 1$, the functional H satisfies the *bounded pth moments condition* on $\mathcal{B}$ at intensity τ if

$$\sup_{Q\in\mathcal{B}\,:\,\mathbf{0}\in Q}\ \sup_{m\in[\tau|Q|/2,3\tau|Q|/2]} \{E[\Delta(\mathcal{U}_{m,Q})^p]\} < \infty. \tag{3.2}$$

The functional H is *polynomially bounded* if there exist constants C and β_1 such that for all finite marked point sets $\mathcal{X} \subset \mathbb{R}^d$

$$|H(\mathcal{X})| \le C(\operatorname{diam} \mathcal{X} + \operatorname{card} \mathcal{X})^{\beta_1}, \tag{3.3}$$

in which case we refer to β_1 as an *index* of polynomial boundedness.

We extend the definition of $\mathcal{N}(0, \sigma^2)$ to the case $\sigma = 0$ by taking $\mathcal{N}(0, 0)$ to be a degenerate random variable taking the value 0 with probability 1. Our general CLT goes as follows. It is proved by straightforward modifications of the proof of Theorem 2.1 in Penrose and Yukich [26], which we leave to the reader.

THEOREM 3.1 (General CLT for marked point processes). *Let $d \ge 1$ and suppose H is a translation invariant functional defined on finite marked point sets in $\mathbb{R}^d$. Let $\tau > 0$, and assume that H is strongly stabilizing at intensity τ, satisfies the bounded fourth moments condition on $\mathcal{B}$ at intensity τ, and is polynomially bounded. Then there exist constants σ_τ^2 and $\eta_\tau^2 := \sigma_\tau^2 - (E\Delta_\tau(\infty))^2$ such that as $n \to \infty$, $n^{-1}\operatorname{Var}(H(\mathcal{P}_{\tau,n})) \to \sigma_\tau^2$ and*

$$\frac{H(\mathcal{P}_{\tau,n}) - EH(\mathcal{P}_{\tau,n})}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\tau^2), \tag{3.4}$$

while $n^{-1}\operatorname{Var}(H(\mathcal{U}_{[\tau n], Q_n})) \to \eta_\tau^2$ and

$$\frac{H(\mathcal{U}_{[\tau n], Q_n}) - EH(\mathcal{U}_{[\tau n], Q_n})}{n^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \eta_\tau^2). \tag{3.5}$$

Also, σ_τ^2 and η_τ^2 are independent of the choice of (Q_n). If the distribution of $\Delta_\tau(\infty)$ is nondegenerate, then $\eta_\tau^2 > 0$, and hence also $\sigma_\tau^2 > 0$.

We anticipate that there will be other applications of this CLT besides those to packing, for example, to Boolean models over binomial point sets, adding to known CLTs for Boolean models over Poisson point processes found, for example, in Heinrich and Molchanov [14], Penrose [23]. Numerous functionals of Boolean models are of interest in statistical estimation; see Molchanov [21].

Next, we give a general LLN. We consider functionals H of finite marked point processes, as described above, but now require some extra structure on H, as follows.

Suppose $\xi(\mathcal{X}, x)$ is a measurable real-valued function defined for all pairs $(\mathcal{X}, x)$, where $\mathcal{X}$ is a finite marked subset of $\mathbb{R}^d$ and x is an element of $\mathcal{X}$. Recalling the definition above of the shift operator S_y on $\mathbb{R}^d \times \mathcal{K}$, we say that ξ is *translation-invariant* if for any finite marked set $\mathcal{X} \subset \mathbb{R}^d$ and any $x \in \mathcal{X}$ and $y \in \mathbb{R}^d$, we have $\xi(\mathcal{X}, x) = \xi(S_y(\mathcal{X}), S_y(x))$. If ξ is translation-invariant, it clearly induces a translation-invariant functional H by

$$H(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(\mathcal{X}, x). \tag{3.6}$$

Our LLN applies only to functionals H generated in this way. In practice, this includes many functionals of interest, including all packing and deposition functionals considered in this paper, and also all the functionals of unmarked point processes considered as examples in Penrose and Yukich [26]. We define stabilization and moments conditions for ξ (rather than H) as follows.

The functional ξ is *strongly stabilizing at intensity* τ if there exist a.s. finite random variables $R' := R'(\tau)$ (a *radius of stabilization for* ξ with respect to $\mathcal{P}_\tau$) and $\xi_\infty := \xi_\infty(\tau)$ (the *limit* of ξ) such that with probability 1, $\xi((\mathcal{P}_\tau \cap B_{R'}(\mathbf{0})) \cup \{\mathbf{0}\} \cup \mathcal{A}, \mathbf{0}) = \xi_\infty$ for all finite marked $\mathcal{A} \subset (\mathbb{R}^d \setminus B_{R'}(\mathbf{0}))$. Thus, R' is a radius of stabilization for ξ if the contribution of the origin to the induced functional H is unaffected by changes in the configuration outside the ball $B_{R'}(\mathbf{0})$.

Given $p \geq 1$, the functional ξ satisfies the *bounded pth moments condition* on $\mathcal{B}$ at intensity τ if

$$\sup_{Q \in \mathcal{B} : \mathbf{0} \in Q} \ \sup_{m \in [\tau|Q|/2, 3\tau|Q|/2]} \{E[\xi(\mathcal{U}_{m,Q} \cup \{\mathbf{0}\}, \mathbf{0})^p]\} < \infty. \tag{3.7}$$

In practice, the types of calculations needed to check stabilization and moments conditions for ξ are similar to those needed to check stabilization and moments for the induced H, although we are not aware of any precise correspondence.

The general LLN goes as follows. The proof is deferred to Section 6.

THEOREM 3.2 (General LLN for marked point processes). *Suppose ξ is a translation invariant functional defined on pairs $(\mathcal{X}, x)$ consisting of a marked point set $\mathcal{X}$ in $\mathbb{R}^d$, and an element x of $\mathcal{X}$. Let H be the functional induced by ξ using* (3.6), *and assume H is polynomially bounded with index β_1.*

Let $\tau > 0$, and suppose that there exists $p > 2(1 + \beta_1)$, such that H satisfies the bounded pth moments condition on $\mathcal{B}$ at intensity τ. Suppose also that ξ is strongly stabilizing with respect to $\mathcal{P}_\tau$, with limit ξ_∞, and ξ satisfies the bounded qth moments condition on $\mathcal{B}$ at intensity τ, for some $q > 1$. Then as $n \to \infty$,

$$n^{-1} H(\mathcal{U}_{[\tau n], Q_n}) \to \tau E\xi_\infty \quad \text{c.m.c.c.} \tag{3.8}$$

and

$$n^{-1} H(\mathcal{P}_{\tau,n}) \to \tau E\xi_\infty \quad \text{c.m.c.c.} \tag{3.9}$$

Theorem 3.2 will have direct application in this paper to packing and deposition functionals. As with the general CLT, we anticipate that there may be other applications. In particular, by modifying arguments in [26], Theorem 3.2 yields a LLN for the functionals of (unmarked) point processes considered as examples in [26]; however for many of those functionals there are other ways to obtain a LLN, although not always with an explicit identification of the limit.

Theorem 3.2, together with stabilization results of Lee ([19], Proposition 1), should also yield a c.m.c.c. LLN for sums of the form $\sum_{e_i} h(e_i)$, where e_i,

$1 \le i \le n-1$, denote the lengths of the edges in the minimal spanning tree on n i.i.d. uniform points in Q_n and where h has at most polynomial growth and is not necessarily monotone. This would generalize Aldous and Steele [2] who treat the case $h(x) = x^d$.

To conclude this section, we demonstrate how to apply the above marked point process setup to the basic RSA model. Let the mark space be $\mathcal{K} := [0, \tau]$ and let $P_{\mathcal{K}}$ be the uniform distribution on $[0, \tau]$. We need to define a "packing" functional $H := H_{\mathrm{RSA}}$ which is appropriate for this model. This is done by using the marks to determine arrival times of balls, as follows.

Given a marked set $\mathcal{X} := \{x_1, \dots, x_k\}$, we order $\mathcal{X}$ according to the natural ordering on the marks $\{M_{x_1}, \dots, M_{x_k}\}$ (if $M_x = M_y$ for some $x, y \in \mathcal{X}$, then the order of x and y is chosen using the lexicographic order on $\mathbb{R}^d$). The marks represent the arrival times of balls. Observe that the order is preserved under insertion and deletion of points.

Sequentially for $i = 1, 2, \dots, k$, let B_i be the unit volume ball centered at x_i, and let B_i be packed if it does not overlap any ball B_j, $j < i$, that was itself packed. Let $H_{\mathrm{RSA}}(\mathcal{X})$ denote the number of balls $B_1, \dots, B_k$ which are packed.

It is clear that H_{RSA} defined in this way is a translation-invariant functional defined for all finite marked point sets in $\mathbb{R}^d$. Moreover, $H_{\mathrm{RSA}}(\mathcal{U}_{[\tau n], Q_n})$ and $H_{\mathrm{RSA}}(\mathcal{P}_{\tau,n})$ are realizations of the packing numbers $N_{n,d}([\tau n])$ and $N_{n,d}(Po(\tau n))$, respectively, appearing in Theorems 1.1 and 1.2. Therefore to prove those theorems, it will suffice to show that the packing functional H_{RSA} satisfies the conditions of Theorems 3.1 and 3.2.

4. Percolation estimates. In this section we describe an oriented graph on random points in $\mathbb{R}^{d+1}$ which is a continuum version of oriented percolation on $\mathbb{Z}^d$. We prove two lemmas bounding the size of a "cluster at the origin" for this graph; this will yield stabilization and moment conditions for packing and deposition functionals.

The oriented graph is defined as follows. In addition to the parameter τ representing mean point density, it has a parameter $r > 0$, representing range of interaction. Given a marked point set $\mathcal{X} \subset \mathbb{R}^d$, with marks $M_x (x \in \mathcal{X})$ taking values in $[0, \tau]$, make $\mathcal{X}$ into the vertex set of an oriented graph by including an edge from x to y whenever (i) $|x - y| \le r$ and (ii) $M_x \le M_y$. Given $x \in \mathcal{X}$, let $A_{\mathrm{out}}(x, \mathcal{X}, r)$ be the set of points in $\mathcal{X}$ that can be reached from x by a directed path in this graph (along with x itself). Let $A_{\mathrm{in}}(x, \mathcal{X}, r)$ be the set of points in $\mathcal{X}$ from which the point x can be reached by a directed path in this graph (along with x itself). Finally, let

$$A_{\mathrm{out,in}}(x, \mathcal{X}, r) := \bigcup_{y \in A_{\mathrm{out}}(x, \mathcal{X}, r)} A_{\mathrm{in}}(y, \mathcal{X}, r).$$

In percolation terminology, $A_{\mathrm{out,in}}(x, \mathcal{X}, r)$ is a sort of "cluster" associated with the point x. The following lemma demonstrates its relevance to packing.

LEMMA 4.1. *Let r_1 be the diameter of a unit volume ball. Suppose $\mathcal{X}$ and $\mathcal{X}'$ are finite marked point sets in $\mathbb{R}^d$, such that $\mathbf{0} \in \mathcal{X} \cap \mathcal{X}'$ and $A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{X}, r_1) = A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{X}', r_1)$, with the equality referring to the mark values as well as locations. Then*

$$H_{\mathrm{RSA}}(\mathcal{X}) - H_{\mathrm{RSA}}(\mathcal{X} \setminus \{\mathbf{0}\}) = H_{\mathrm{RSA}}(\mathcal{X}') - H_{\mathrm{RSA}}(\mathcal{X}' \setminus \{\mathbf{0}\}). \tag{4.1}$$

PROOF. Recall that in the definition of the packing functional H_{RSA}, $\mathcal{X}$ represents the set of centers of balls in the basic RSA model, and the marks represent their arrival times. Some thought (see [24] for more details) shows that (i) the set of points whose packing status is affected by a deletion at the origin is a subset of $A_{\mathrm{out}}(\mathbf{0}, \mathcal{X}, r_1)$, and (ii) the packing status of a point $x \in \mathcal{X}$ is determined by the graph structure of $A_{\mathrm{in}}(x, \mathcal{X}, r_1)$. Together, these observations imply the equality (4.1) of add one costs. $\square$

Lemma 4.1 demonstrates the potential relevance to packing of the following lemmas concerned with the percolation clusters, which will be formally applied to packing functionals in the next section.

LEMMA 4.2. *Let $\tau,\ r > 0$ and let $\mathcal{P}_\tau$ be a Poisson point process with intensity τ on $\mathbb{R}^d$ with points carrying uniform marks on $[0, \tau]$. Then $\mathrm{diam}(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r))$ is a.s. finite and its distribution has an exponentially decaying tail, that is,*

$$\limsup_{t\to\infty} t^{-1} \log P\big[\mathrm{diam}\big(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r)\big) > t\big] < 0.$$

PROOF. Since M_x is uniformly distributed on $[0, \tau]$, the points $(x, M_x)_{x\in\mathcal{P}_\tau}$ form a rate one Poisson point process on $\mathbb{R}^d \times [0, \tau]$. It is helpful to assume this Poisson process is the restriction to $\mathbb{R}^d \times [0, \tau]$ of a unit intensity homogeneous Poisson process $\mathcal{P}$ on the whole of $\mathbb{R}^d \times [0, \infty)$. Extending the previously defined graph structure on marked point sets in the natural way, we assume there is an oriented edge from point (X, T) to (Y, U) whenever (X, T) and (Y, U) are points of $\mathcal{P}$ satisfying $T \le U$ and $|X - Y| \le r$.

The proof uses discretization. Let $K > 2r$. Divide $\mathbb{R}^d$ into cubes of side K, centered at the points of the lattice $K\mathbb{Z}^d$. We thus identify each cube with the corresponding point of $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ let $T(x, y)$ be the smallest $t > 0$ such that there exist Poisson points (X, T_X) and (Y, T_Y) with X in cube x, Y in cube y and $T_Y \le t$, with an oriented graph path from (X, T_X) to (Y, T_Y).

It is useful to think of the induced graph on $\mathcal{P}_\tau \cup \{\mathbf{0}\}$ as representing the spread of an epidemic in which new points born in the r-neighborhood of existing infected points are themselves instantly (and permanently) infected. Then $T(x, y)$ is the time it takes the infection to reach cube y assuming that all points born in cube x are infected.

Let $\|\cdot\|$ be the l_∞ (maximum-component) norm on $\mathbb{R}^d$. For x and y in $\mathbb{Z}^d$, a *path* γ from x to y (written $\gamma\colon x \rightsquigarrow y$) is a sequence $x_0 := x, x_1, x_2, \dots, x_n := y$ of distinct elements of $\mathbb{Z}^d$, with $\|x_i - x_{i-1}\| = 1$ for each $i \in \{1, 2, \dots, n\}$. For such a path we write $|\gamma| = n$. Note that there are 3^d distinct vectors with entries 0, 1, or -1 and thus $3^d - 1$ l_∞-neighbors of a given lattice point. Hence, the number of paths of length n from any given starting point $x_0 = x$ is at most 3^{dn}.

Given a path $\gamma := (x_0, x_1, \dots, x_n)$, define S_γ as follows (it is a lower bound for the time it would take for the infection to pass from cube x to cube y along the path γ, if only infections along that path were allowed). Set $S_{\gamma,0} := 0$ and inductively for $i = 1, \dots, n$ define

$$S_{\gamma,i} := S_{\gamma,i-1} + W_{\gamma,i}$$

where $W_{\gamma,i}$ is the time from $S_{\gamma,i-1}$ to the next Poisson arrival time in the cube associated with x_i, and finally set $S_\gamma = S_{\gamma,n}$. Then $S_\gamma := \sum_{i=1}^n W_{\gamma,i}$ where the $W_{\gamma,i}$, $1 \le i \le n$, are independent exponential variables with parameter K^d.

We claim that there exists a finite constant C such that for all $x, y \in \mathbb{Z}^d$,

$$P[T(x,y) \le \tau] \le C 3^{-\|x-y\|}. \tag{4.2}$$

To see this, observe that if $T(x,y) \le \tau$ then $S_\gamma \le \tau$ for some path $\gamma\colon x \rightsquigarrow y$, and so by Boole's inequality we have

$$P[T(x,y) \le \tau] \le \sum_{\gamma\colon x \rightsquigarrow y} P[S_\gamma \le \tau].$$

For $\theta > 0$, setting W to be an exponential variable with parameter K^d we have

$$P[S_\gamma \le \tau] \le \frac{E[e^{-\theta S_\gamma}]}{e^{-\theta\tau}} = e^{\theta\tau}\big(E[e^{-\theta W}]\big)^{|\gamma|}.$$

Let $\alpha := 3^{-(d+1)}$. Take $\theta > 0$ with $E[e^{-\theta W}] \le \alpha$. Then $P[S_\gamma \le \tau] \le e^{\theta\tau}\alpha^{|\gamma|}$, and hence, since the number of paths of length n starting from x is at most 3^{dn},

$$\begin{aligned} P[T(x,y) \le \tau] &\le \sum_{n=\|x-y\|}^{\infty} \sum_{\gamma\colon x \rightsquigarrow y, |\gamma|=n} P[S_\gamma \le \tau] \\ &\le \sum_{n=\|x-y\|}^{\infty} e^{\theta\tau} 3^{dn}\alpha^n = \frac{e^{\theta\tau}(3^d\alpha)^{\|x-y\|}}{1 - 3^d\alpha}. \end{aligned}$$

This gives us (4.2).

Set $R^* := \operatorname{diam}(A_{\mathrm{out}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r))$. We now show that the distribution of R^* has an exponentially decaying tail. Given $\lambda > 0$, let $\{x_{\lambda,i}\}_{i=1}^{\nu(\lambda)}$ be the set of cubes having nonempty intersection with the boundary of $B_\lambda(\mathbf{0})$, and observe that

$\nu(\lambda) = O(\lambda^{d-1})$. If $R^* \geq 2\lambda$ then at least one of these cubes is infected by time τ, so by Boole's inequality,

$$P[R^* \geq 2\lambda] \leq \sum_{i=1}^{\nu(\lambda)} P[T(\mathbf{0}, x_{\lambda,i}) \leq \tau].$$

By (4.2) we obtain the desired exponential decay in λ.

Next, let R be the radius of the smallest ball centered at the origin containing $A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r)$. If $R^* \leq \lambda$ but $R \geq 2\lambda$, then there must be an oriented path in the oriented graph induced by the marked point set $\mathcal{P}_\tau$, that starts outside the ball $B_{2\lambda}(\mathbf{0})$ but ends inside the ball $B_\lambda(\mathbf{0})$. This implies that $T(x_{2\lambda,i}, x_{\lambda,j}) \leq \tau$ for some $i \leq \nu(2\lambda)$, $j \leq \nu(\lambda)$. Therefore by Boole's inequality

$$P[R^* \leq \lambda, R > 2\lambda] \leq \sum_{i=1}^{\nu(2\lambda)} \sum_{j=1}^{\nu(\lambda)} P[T(x_{2\lambda,i}, x_{\lambda,j}) \leq \tau].$$

Since the number of terms in this sum is $O(\lambda^{2(d-1)})$, the above expression decays exponentially in λ by (4.2). Combined with the previous claim, this demonstrates exponential decay for the tail of the distribution of R, and this completes the proof. $\square$

The next lemma will help show that packing functionals satisfy the bounded pth moments condition. For $Q \in \mathcal{B}$, $m \in \mathbb{N}$, and $r > 0$ define $R_{m,Q,r}$ by

$$R_{m,Q,r} := \mathrm{diam}\big(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{U}_{m,Q} \cup \{\mathbf{0}\}, r)\big). \tag{4.3}$$

LEMMA 4.3. *Let $\tau > 0$, $r > 0$. Then there is a constant $\delta := \delta(\tau, r) > 0$ such that*

$$\sup_{Q \in \mathcal{B} : \mathbf{0} \in Q} \ \sup_{m \in [\tau|Q|/2, 3\tau|Q|/2]} E \exp(\delta R_{m,Q,r}) < \infty.$$

PROOF. We will embed the binomial process $\mathcal{U}_{m,Q}$ on Q into a higher intensity Poisson process $\mathcal{P}_{2\tau}$ on $\mathbb{R}^d$, and then use the epidemic arguments from the proof of Lemma 4.2.

By standard large deviations arguments, there are constants $C > 0, \alpha > 0$ such that for all integers $n > 0$

$$P[Po(2\tau n) \leq 3\tau n/2] \leq C \exp(-\alpha n),$$

and therefore for all n, all $Q \in \mathcal{B}$ with $|Q| = n$, and all $m \leq 3\tau|Q|/2$, we can find a coupling of the point processes $\mathcal{P}_{2\tau}$ and $\mathcal{U}_{m,Q}$ such that if $E_{m,Q}$ denotes the event that $\mathcal{U}_{m,Q} \subset \mathcal{P}_{2\tau}$, we have that

$$P[E_{m,Q}^c] \leq C \exp(-\alpha n). \tag{4.4}$$

Let $R_{2\tau,r}$ be defined by

$$R_{2\tau} := \mathrm{diam}\big(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_{2\tau} \cup \{\mathbf{0}\}, r)\big).$$

Clearly for any point sets $\mathcal{X} \subseteq \mathcal{X}'$ and any $x \in \mathcal{X}$ we have $A_{\mathrm{out,in}}(x, \mathcal{X}, r) \subseteq A_{\mathrm{out,in}}(x, \mathcal{X}', r)$. Therefore on the event $E_{m,Q}$ we have $R_{m,Q,r} \le R_{2\tau,r}$.

By linear boundedness of the sequence $(Q_n)_{n\ge1}$, there is a constant $\beta > 0$ such that for any $Q \in \mathcal{B}$ with $|Q| = n$, and any $m \in [\tau n/2, 3\tau n/2]$ we have $R_{m,Q,r} \le \mathrm{diam}(Q_n) \le \beta n$, and hence

$$\begin{aligned} E\exp(\delta R_{m,Q,r}) &= E\big[\exp(\delta R_{m,Q,r})(\mathbb{1}_{E_{m,Q}} + \mathbb{1}_{E^c_{m,Q}})\big] \\ &\le E\exp(\delta R_{2\tau,r}) + P[E^c_{m,Q}]\exp(\delta\beta n). \end{aligned}$$

Provided δ is small enough, the last two terms are finite uniformly over n since $R_{2\tau,r}$ has a finite exponential moment (by Lemma 4.2) and since $P[E^c_{m,Q}]$ decays exponentially by (4.4). $\square$

5. Proof of limit theorems for packing. Equipped with the general limit theorems of Section 3, and the percolation estimates from Section 4, we now prove our main results for the RSA, CSA and BD models.

5.1. *Proof of Theorems* 1.1 *and* 1.2. We prove Theorem 1.2 by verifying that the packing functional H_{RSA} defined in Section 3 satisfies the conditions of Theorem 3.1. We check the conditions as follows. Let r_1 be the diameter of a unit volume ball.

(i) *Polynomially bounded.* This is trivial since $H_{\mathrm{RSA}}(\mathcal{X}) \le \mathrm{card}(\mathcal{X})$.

(ii) *Strong stabilization.* Set

$$R := \mathrm{diam}\big(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r_1)\big) + r_1.$$

Then R is almost surely finite by Lemma 4.2. Also, the set $A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P}_\tau \cup \{\mathbf{0}\}, r_1)$ is insensitive to changes to the configuration of $\mathcal{P}_\tau$ outside $B_R(\mathbf{0})$, and by Lemma 4.1, so is the add one cost of inserting a point at the origin to this Poisson process. In other words, R is a radius of stabilization for H_{RSA}.

(iii) *Bounded fourth moments condition.* Given $Q \in \mathcal{B}$ with $\mathbf{0} \in Q$, and given $m \in [\tau|Q|/2, 3\tau|Q|/2]$, recall from (4.3) that $R_{m,Q,r}$ denotes the diameter of the cluster $A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{U}_{m,Q} \cup \{\mathbf{0}\}, r)$. As in (ii) above, the set of points of $\mathcal{U}_{m,Q}$ affected by an insertion at the origin is a subset of the ball of radius R_{m,Q,r_1} centered at the origin. Since the number of centers of disjoint unit volume balls that can be packed in a set S is bounded by the volume of the $(r_1/2)$-neighborhood of S, there is a constant C such that

$$\Delta(\mathcal{U}_{m,Q})^4 \le C(R_{m,Q,r_1} + r_1)^{4d}.$$

Lemma 4.3 shows that the expectation of the last expression is bounded uniformly over in $Q \in \mathcal{B}$ with $\mathbf{0} \in Q$ and $m \in [\tau|Q|/2, 3\tau|Q|/2]$.

(iv) *Nondegeneracy of* $\Delta(\infty)$. Let E_1 be the event that the ball centered at the origin is rejected, that is, the event that there is a unit volume ball which intersects $B_{r_1/2}(\mathbf{0})$ and which is packed before the origin is inserted. Then $P(E_1) > 0$ and on E_1 we have

$$H_{\mathrm{RSA}}(\{\mathbf{0}\} \cup \mathcal{P}_\tau \cap Q) - H_{\mathrm{RSA}}(\mathcal{P}_\tau \cap Q) = 0$$

for any large cube Q. On the other hand, if E_2 is the event that none of the balls centered at points of $\mathcal{P}_\tau$ is centered within a distance r_1 of the origin, then $P[E_2] > 0$ and on E_2 we have

$$H_{\mathrm{RSA}}(\{\mathbf{0}\} \cup \mathcal{P}_\tau \cap Q) - H_{\mathrm{RSA}}(\mathcal{P}_\tau \cap Q) = 1$$

for any cube Q. Thus $\Delta(\infty)$ is nondegenerate.

The packing functional H_{RSA} thus satisfies all of the conditions of Theorem 3.1, proving Theorem 1.2.

Next we prove Theorem 1.1 by verifying that the packing functional H_{RSA} satisfies the conditions of Theorem 3.2. Note first that $H_{\mathrm{RSA}}(\mathcal{X}) = \sum_{x \in \mathcal{X}} \xi(\mathcal{X}, x)$, where $\xi(\mathcal{X}, x)$ takes the value 1 if the ball arriving at x is packed, and zero if not. Clearly ξ is translation-invariant.

In addition to those conditions already verified in the proof of Theorem 1.2, we need to check that ξ is strongly stabilizing and satisfies the bounded second moments condition on $\mathcal{B}$ at intensity τ. Strong stabilization of ξ follows from Lemma 4.2, while the second moments condition is immediate from the fact that the range $\{0, 1\}$ of possible values for $\xi(\mathcal{X}, x)$ is bounded. Finally, since the index of polynomial boundedness of H_{RSA} is 1, we need to check that H_{RSA} satisfies the bounded pth moments condition on $\mathcal{B}$ for some $p > 4$. However, the proof of the pth moments condition already given for $p = 4$ in proving Theorem 1.2 clearly carries through to any $p > 4$.

The packing functional H thus satisfies all of the conditions of Theorem 3.2, proving Theorem 1.1.

5.2. *Proof of Theorem* 2.1. Recall the notation of the generalized CSA model of Section 2.3. To apply the general results on marked point processes from Section 3 to CSA, we need a mark space which is richer than that for basic RSA. We take $\mathcal{K} := \mathbb{F} \times [0, \tau] \times [0, 1]$ as the mark space, where the distribution $P_{\mathcal{K}}$ of marks is given by the product of the measure P_1 on the space of types $\mathbb{F}$ (see Section 2.3), the uniform probability distribution on $[0, \tau]$, and the uniform probability distribution on $[0, 1]$.

Define the generalized CSA functional H_{CSA} on finite marked subsets of $\mathbb{R}^d$ as follows. Given a finite marked set $\mathcal{X} \subset \mathbb{R}^d$, each point x of $\mathcal{X}$ carries a mark $M_x := (\Xi_x, T_x, V_x)$, with $x \in \mathbb{R}^d$ representing a location of an incoming particle, $\Xi_x \in \mathbb{F}$ its type, $T_x \in [0, \tau]$ its time of arrival, and $V_x \in [0, 1]$ its "acceptability."

Enumerate the elements of $\mathcal{X}$ as $\{x_1, \dots, x_k\}$, choosing the order according to the natural ordering on the marks $\{T_{x_1}, \dots, T_{x_k}\}$ (if $T_x = T_y$ for some $x, y \in \mathcal{X}$, then the order of x and y is chosen using the lexicographic order on $\mathbb{R}^d$). For simplicity write (Ξ_i, T_i, V_i) for $(\Xi_{x_i}, T_{x_i}, V_{x_i})$.

Sequentially for $i = 1, 2, \dots, k$, let the ith incoming particle x_i be accepted if and only if

$$V_i \leq \prod \Phi(x_i - x_j, \Xi_j, \Xi_i, T_i - T_j)$$

where the product is over all accepted particles x_j, $j < i$ (and the product over the empty set is taken to be 1). Let $H_{\mathrm{CSA}}(\mathcal{X}) := \sum h(\Xi_i, T_i)$, with the sum taken over all $i \in \{1, 2, \dots, k\}$ for which x_i is accepted, and with h being the test function appearing in the definition (2.3) of the CSA process. Then H_{CSA} is our CSA functional.

Defined in this way, H_{CSA} is clearly a translation-invariant functional of finite marked point sets in $\mathbb{R}^d$. Moreover, by including an "acceptability" component V in the mark at each point, we have defined the deterministic functional H_{CSA} of marked points, which, on points with random marks, mimics the randomized acceptance mechanism in the description of generalized CSA in Section 2.3. In particular, $H_{\mathrm{CSA}}(\mathcal{U}_{[\tau n], Q_n})$ and $H_{\mathrm{CSA}}(\mathcal{P}_{\tau, n})$ are realizations of the variables $G_n([\tau n])$ and $G_n(Po(\tau n))$, respectively, appearing in Theorem 2.1. Therefore to prove this theorem, it suffices to show that H_{CSA} satisfies the conditions of Theorems 3.1 and 3.2.

Recall from (2.2) that r_3 is the range of interaction in the definition of generalized CSA. Given a marked point set $\mathcal{X} \subset \mathbb{R}^d$ with marks (Ξ_x, T_x, V_x), $x \in \mathcal{X}$, define an oriented graph on vertex set $\mathcal{X}$ by including an edge from x to y whenever $|x - y| \leq r_3$ and $T_x \leq T_y$. In other words, the graph is constructed just as in Section 4 by taking only the second ("time of arrival") component of the mark, and ignoring the other components. Using this graph, let $A_{\mathrm{out,in}}(x, \mathcal{X}, r_3)$ be defined just as in Section 4.

A similar argument to Lemma 4.1 shows that if $\mathcal{X}$ and $\mathcal{X}'$ are finite marked point sets in $\mathbb{R}^d$, such that $\mathbf{0} \in \mathcal{X} \cap \mathcal{X}'$ and $A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{X}, r_3) = A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{X}', r_3)$, with the equality referring to the marks as well as locations, then

$$H_{\mathrm{CSA}}(\mathcal{X}) - H_{\mathrm{CSA}}(\mathcal{X} \setminus \{\mathbf{0}\}) = H_{\mathrm{CSA}}(\mathcal{X}') - H_{\mathrm{CSA}}(\mathcal{X}' \setminus \{\mathbf{0}\}). \tag{5.1}$$

PROOF OF THEOREM 2.1(b). It suffices to show that H_{CSA} satisfies the conditions of Theorem 3.1. Since the test function h is assumed to be bounded, we have $H_{\mathrm{CSA}}(\mathcal{X}) \leq C \operatorname{card}(\mathcal{X})$ so that H_{CSA} is polynomially bounded with index 1.

If we set $R := \operatorname{diam}(A_{\mathrm{out,in}}(\mathbf{0}, \mathcal{P} \cup \{\mathbf{0}\}, r_3)) + r_3$, then R is almost surely finite by Lemma 4.2 and is a radius of stabilization for H_{CSA} by (5.1).

For the bounded pth moments condition, let Q be an arbitrary translate of Q_n, and let $m \in [n/2, 3n/2]$. Writing $R_{m,Q}$ for R_{m,Q,r_3} as defined at (4.3), note that by (5.1) again and the boundedness of h,

$$\begin{aligned}\Delta(\mathcal{U}_{m,Q})^p &\le C\,\mathrm{card}\big(\mathcal{U}_{m,Q}\cap B_{R_{m,Q}}(\mathbf{0})\big)^p\\ &= C\sum_{j=0}^{\infty}\mathrm{card}\big(\mathcal{U}_{m,Q}\cap B_{R_{m,Q}}(\mathbf{0})\big)^p\mathbb{1}_{\{j\le R_{m,Q}<j+1\}}\\ &\le C\sum_{j=0}^{\infty}\mathrm{card}\big(\mathcal{U}_{m,Q}\cap B_{j+1}(\mathbf{0})\big)^p\mathbb{1}_{\{R_{m,Q}\ge j\}}.\end{aligned}$$

Taking expectations, using Cauchy–Schwarz on the summands and the binomial moment bound

$$E[Bi(n,q)^{2p}]\le C(nq)^{2p},$$

together with the uniform exponential decay of the tails of $R_{m,Q}$ given in Lemma 4.3, we deduce that

$$\sup_{Q\in\mathcal{B}:\,\mathbf{0}\in Q}\ \sup_{m\in[\tau|Q|/2,3\tau|Q|/2]} E\big[\big(\Delta_{m,Q}(\mathbf{0})\big)^p\big]<\infty.$$

Thus H_{CSA} satisfies the bounded pth moment condition for any $p\ge 1$, in particular $p=4$.

The proof of nondegeneracy essentially follows the proof of nondegeneracy for the basic RSA model. One uses r_3 (the range of interactions) instead of r_1 in the argument, and appeals to (2.4) to be assured that $P[E_1]>0$, and to (2.5) to be assured that $\Delta(\infty)>0$ with positive probability on event E_2. $\square$

PROOF OF THEOREM 2.1(a). We apply Theorem 3.2 to H_{CSA}. By the definition H_{CSA}, it is clear that $H_{\mathrm{CSA}}(\mathcal{X})$ takes the desired form of a sum $\sum_{x\in\mathcal{X}}\xi(x,\mathcal{X})$, where ξ takes the value $h(\Xi_x,T_x)$ if x is accepted and zero if not. Since h is assumed uniformly bounded, so is ξ and therefore ξ satisfies the bounded second moments condition. Also, by Lemma 4.2 and a similar argument to the one leading up to (5.1), ξ is strongly stabilizing. Therefore, Theorem 3.2 applies. $\square$

5.3. *Proof of Theorem* 2.2. Given a marked point set $\mathcal{X}\subset\mathbb{R}^d$, with distinct marks taking values in $[0,\tau]$, let $H_{\mathrm{BD}}(\mathcal{X})$ be the total height of accepted particles if the points of $\mathcal{X}$ are the locations of incoming particles for the generalized BD process, with arrival times given by the marks. Note that $H_{\mathrm{BD}}(\mathcal{X})$ takes the form of a sum $\sum\xi(x,\mathcal{X})$ with $\xi(x,\mathcal{X})$ here taking the value equal to the height of x (if x is accepted) and taking the value 0 if x is rejected.

Recall that the generalized BD model assumes that the range of interaction and the lateral displacement are both uniformly bounded by some constant r_4. What happens to an incoming particle, given the location at which it arrives, is clearly determined by the location, acceptance status and displacement of particles previously arriving within a distance $2r_4$ of the new particle's location. Therefore, a particle at x affects only points in $A_{\text{out}}(x, \mathcal{X}, 2r_4)$, as described in Section 4, and the amount of the effect on such points is determined entirely by the set $A_{\text{out,in}}(x, \mathcal{X}, 2r_4)$ (cf. Lemma 4.1). In particular, Lemma 4.2 shows that the deposition functional H_{BD} is strongly stabilizing.

To check the moments condition, observe first that for any point $x \in \mathcal{X}$, the height of the particle x is at most r_5 times the number of points in $A_{\text{in}}(x, \mathcal{X}, 2r_4)$. This fact implies, firstly, that the functional H is polynomially bounded with index $\beta_1 = 2$, and secondly, that the total change caused by an insertion of x into $\mathcal{X}$ is bounded by a constant times the square of the number of points in $A_{\text{out,in}}(x, \mathcal{X}, 2r_4)$. Therefore, now writing $R_{m,Q}$ for $R_{m,Q,2r_4}$, we have

$$\begin{aligned}
\Delta(\mathcal{U}_{m,Q})^p &\le C \sum_{j=0}^{\infty} \operatorname{card}\big(\mathcal{U}_{m,Q} \cap B_{R_{m,Q}}(\mathbf{0})\big)^{2p} \mathbb{1}_{\{j \le R_{m,Q} < j+1\}} \\
&\le C \sum_{j=0}^{\infty} \operatorname{card}\big(\mathcal{U}_{m,Q} \cap B_{j+1}(\mathbf{0})\big)^{2p} \mathbb{1}_{\{R_{m,Q} \ge j\}}.
\end{aligned}$$

Using Cauchy–Schwarz on the summands, the binomial moment bound

$$E[Bi(n,q)^{4p}] \le C(nq)^{4p},$$

together with the uniform exponential decay of the tails of $R_{m,Q}$ (Theorem 4.3), we can show the pth moment of $\Delta(\mathcal{U}_{m,Q})$ is bounded uniformly over $Q \in \mathcal{B}$ and $m \in [|Q|\tau/2, 3|Q|\tau/2]$.

This gives us the bounded pth moments condition for H_{BD}, and also the bounded second moments condition on ξ.

The proof of nondegeneracy is similar to those given earlier. There is a positive probability that no particle arrives within distance $2r_4$ of the origin, in which case the inserted particle there does not change the total height. By the nondegeneracy condition given just before the statement of the theorem, there is also a positive probability that the inserted particle has a positive height and has no subsequent particles arriving subsequently within a distance $2r_4$ of it, in which case the inserted particle adds to the total height.

Therefore, the functional H_{BD} satisfies all the conditions of Theorems 3.1 and 3.2, giving us the result.

6. Proof of the general LLN. In this section we prove Theorem 3.2, the general LLN for functionals of the form $H(\mathcal{X}) = \sum_{x \in \mathcal{X}} \xi(\mathcal{X}, x)$. The proof uses the following Coupling Lemma.

LEMMA 6.1. *If ξ is strongly stabilizing at intensity τ with limit ξ_∞, then there exist coupled random variables ξ'_n, $n \ge 1$, and ξ'_∞, such that ξ'_n has the same distribution as $\xi(\mathcal{U}_{[n\tau],Q_n}, U_{1,Q_n})$ for each n, and ξ'_∞ has the same distribution as ξ_∞, and $P[\xi'_n \ne \xi'_\infty] \to 0$ as $n \to \infty$.*

PROOF. The proof is similar to that of Lemma 4.2 of [26]. On a suitable probability space, let $\mathcal{P}'_\tau$ be a marked homogeneous Poisson process of rate τ on $\mathbb{R}^d$, and for each n let $(V_{j,n},\ j \geq 0)$ be a sequence of (marked) independent variables uniformly distributed over Q_n and independent of $\mathcal{P}'_\tau$. Let R' be a radius of stabilization for the function ξ with respect to $\mathcal{P}'_\tau$, and let ξ'_∞ be the corresponding limit of ξ. By definition, ξ'_∞ has the same distribution as ξ_∞.

Recall from Section 3 the definition of the shift map S_y. The translated point process $S_{V_{0,n}}(\mathcal{P}'_\tau)$ is also a homogeneous Poisson process of rate τ on $\mathbb{R}^d$. Let $M_n - 1$ be the number of points of this translated point process lying in Q_n. Let the points of $S_{V_{0,n}}(\mathcal{P}'_\tau)$ lying in Q_n, taken in an order chosen uniformly at random from the $(M_n-1)!$ possibilities, be denoted $U'_{2,n}, U'_{3,n}, \dots, U'_{M_n,n}$. Set $U'_{1,n} = V_{0,n}$ and for $j \geq 1$ set $U'_{M_n+j,n} = V_{j,n}$. Then the points $U'_{2,n}, U'_{3,n}, \dots$ are independent and uniformly distributed over Q_n.

Define the marked point process $\mathcal{U}'_{[\tau n],Q_n} := \{U'_{1,n}, \dots, U'_{[\tau n],n}\}$ and set $\xi'_n := \xi(\mathcal{U}'_{[\tau n],Q_n}, U'_{1,n})$. By the conclusion of the preceding paragraph, ξ'_n has the same distribution as $\xi(\mathcal{U}_{[\tau n],Q_n}, U_{1,Q_n})$.

Let $\varepsilon > 0$ and choose K such that $P[R' > K] \leq \varepsilon$. Observe that the point process $\mathcal{U}'_{[\tau n],n}$ is obtained by starting with the points in Q_n of the translated Poisson process $S_{V_{0,n}}(\mathcal{P}'_\tau)$, adding or removing $|M_n - [\tau n]|$ points as appropriate to modify the number of points in Q_n to $[\tau n] - 1$, and finally inserting an extra point at $V_{0,n}$. The number of added/removed points is $O(n^{1/2})$ in probability, and since $|Q_n| = n$, an estimate using Boole's inequality shows that the probability that any point (other than the special point $V_{0,n}$) is added or removed in the ball $B_K(V_{0,n})$ tends to zero.

By the vanishing relative boundary condition on Q_n (see Section 3), the probability that the ball $B_K(V_{0,n})$ is contained in Q_n tends to 1. If this occurs, and if also no point is added or removed in this ball, then $\mathcal{U}'_{[\tau n],Q_n} \cap B_K(V_{0,n})$ is a translate of the point process $\{\mathbf{0}\} \cup (\mathcal{P}'_\tau \cap B_K(\mathbf{0}))$. If also $R' \leq K$, then $\xi'_n = \xi'_\infty$, and therefore $P[\xi'_n \neq \xi'_\infty] \leq 2\varepsilon$ for large enough n. The result follows. $\square$

PROOF OF THEOREM 3.2. The proof has three steps: the first step uses Lemma 6.1 to show convergence of the mean $EH(\mathcal{U}_{[\tau n],n})/n$, the second step uses a variant of Azuma's inequality to strengthen this to complete convergence, and the third step deduces (3.9) from (3.8).

Step 1. By definition (3.6) of the functional H and exchangeability,

$$[\tau n]^{-1} EH(\mathcal{U}_{[\tau n],Q_n}) = E\xi(\mathcal{U}_{[\tau n],Q_n}, U_{1,Q_n}) = E[\xi'_n], \tag{6.1}$$

with ξ'_n and ξ'_∞ given by Lemma 6.1. Since ξ is assumed to satisfy the bounded qth moments condition at intensity τ for some $q > 1$, the qth moment of $\xi(\mathcal{U}_{[n\tau],Q_n}, U_{1,Q_n})$, and therefore that of ξ'_n, is uniformly bounded. So ξ'_n are uniformly integrable random variables which converge in probability to ξ'_∞, and therefore

$E[\xi_n'] \to E[\xi_\infty'] = E[\xi_\infty]$. Therefore, the expression (6.1) tends to $E[\xi_\infty]$, completing Step 1.

Step 2. Now we show complete convergence of $n^{-1}H(\mathcal{U}_{[\tau n],Q_n})$. Write for simplicity U_i instead of U_{i,Q_n}, $1 \le i \le [\tau n]$. By the definition of complete convergence it suffices to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\big[|H(U_1,\dots,U_{[\tau n]}) - EH(U_1,\dots,U_{[\tau n]})| > \varepsilon n\big] < \infty. \tag{6.2}$$

Let $\mathcal{F}_i$ denote the σ-field generated by the random variables $U_1,\dots,U_i$ and let $\mathcal{F}_0$ be the trivial σ-field. Consider the martingale difference representation

$$H(U_1,\dots,U_{[\tau n]}) - EH(U_1,\dots,U_{[\tau n]}) = \sum_{i=1}^{[\tau n]} d_i,$$

with $d_i := E(H(U_1,\dots,U_{[\tau n]}) \mid \mathcal{F}_i) - E(H(U_1,\dots,U_{[\tau n]}) \mid \mathcal{F}_{i-1})$. Notice that

$$d_i = E[H(U_1,\dots,U_i,\dots,U_{[\tau n]}) - H(U_1,\dots,U_i',\dots,U_{[\tau n]}) \mid \mathcal{F}_i]$$

where U_i' is an independent copy of U_i. We claim that the bounded pth moments assumption on H implies that the variables d_i have uniformly bounded pth moments. To see this, observe that

$$\begin{aligned} &E\big|H(\mathcal{U}_{[\tau n],Q_n}) - H(\mathcal{U}_{[\tau n]-1,Q_n})\big|^p \\ &\qquad = \int_{Q_n} E\big|H(\mathcal{U}_{[\tau n]-1,Q_n} \cup \{x\}) - H(\mathcal{U}_{[\tau n]-1,Q_n})\big|^p \frac{dx}{n} \end{aligned}$$

which is bounded uniformly in n by the bounded pth moments condition on H. Using this bound twice, and the triangle inequality, one obtains for all $1 \le i \le [\tau n]$,

$$E\big|H(U_1,\dots,U_i,\dots,U_{[\tau n]}) - H(U_1,\dots,U_i',\dots,U_{[\tau n]})\big|^p \le C,$$

and so by the conditional Jensen inequality,

$$E|d_i|^p \le EE\big(\big|H(U_1,\dots,U_{[\tau n]}) - H(U_1,\dots,U_i',\dots,U_{[\tau n]})\big|^p \mid \mathcal{F}_i\big) \le C. \tag{6.3}$$

Choose γ to satisfy $\gamma < 1/2$ and $p\gamma > \beta_1 + 1$. By the condition $p > 2(\beta_1 + 1)$, such γ exists.

To show (6.2) we use the following modification of Azuma's inequality (see, e.g., Lemma 1 of Chalker, Godbole, Hitczenko, Radcliff and Ruehr [5]). For any martingale difference sequence d_i, $i \ge 1$, and for all sequences w_i, $i \ge 1$, of positive numbers we have for all $t > 0$ that

$$P\left[\left|\sum_{i=1}^{[\tau n]} d_i\right| > t\right] \le 2\exp\left(\frac{-t^2}{32\sum_{i=1}^{[\tau n]} w_i^2}\right) + \big(1 + 2t^{-1}\sup_i \|d_i\|_\infty\big)\sum_{i=1}^{[\tau n]} P[|d_i| > w_i].$$

Letting $w_i := n^\gamma$, $t := \varepsilon n$, using (6.3), Markov's inequality, and noting that $\sup_i \|d_i\|_\infty \le Cn^{\beta_1}$ by polynomial boundedness, we obtain

$$P\left[\left|\sum_{i=1}^{[\tau n]} d_i\right| > \varepsilon n\right] \le 2\exp\left(\frac{-n^2}{Cn^{1+2\gamma}}\right) + (1 + Cn^{\beta_1 - 1})\frac{n}{n^{p\gamma}},$$

which is summable in n by the choice of γ. This completes the proof of Step 2, and therefore of (3.8).

Step 3. We give only a sketch. Let M_n be the number of points of $\mathcal{P}_{\tau,n}$, a Poisson variable with mean $n\tau$. Let $\lambda_\tau := \tau E\xi_\infty$. Then by conditioning on the value of M_n, we obtain

$$\begin{aligned}\left|E[H(\mathcal{P}_{\tau,n}) - n\lambda_\tau]\right| \le{}& \max_{m\,:\,|m-\tau n|\le n^{3/4}} \left|E[H(\mathcal{U}_{m,Q_n}) - n\lambda_\tau]\right| \\ &+ \left|E[(H(\mathcal{P}_{\tau,n}) - n\lambda_\tau)\mathbb{1}_{\{|M_n - n\tau| > n^{3/4}\}}]\right|\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}P\left[|H(\mathcal{P}_{\tau,n}) - n\lambda_\tau| > \varepsilon n\right] \le{}& \max_{m\,:\,|m-\tau n|\le n^{3/4}} P\left[|H(\mathcal{U}_{m,Q_n}) - n\lambda_\tau| > \varepsilon n\right] \\ &+ P\left[|M_n - \tau n| > n^{3/4}\right].\end{aligned} \tag{6.5}$$

By repeating Steps 1 and 2 with the point process $\mathcal{U}_{[\tau n],Q_n}$ replaced by $\mathcal{U}_{m_n,Q_n}$ for an arbitrary sequence (m_n) satisfying $|m_n - n| \le n^{3/4}$ for all n, one finds that the first term in the right hand side of (6.4) tends to zero while the first term in the right hand side of (6.5) is summable in n.

By a standard argument applying Markov's inequality to the moment generating function, $P[|M_n - \tau n| > n^{3/4}]$ decays exponentially in a power of n. Therefore the second term of (6.5) is summable. Also, by Cauchy–Schwarz and polynomial boundedness, the second term of (6.4) tends to zero. Therefore (6.4) tends to zero and (6.5) is summable; together, these imply that $n^{-1}H(\mathcal{P}_{\tau,n}) \to \lambda_\tau$ c.m.c.c., completing Step 3. $\square$

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