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ON LACUNARY WAVELET SERIES

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We prove that the Hölder singularities of random lacunary wavelet series are chirps located on random fractal sets. We determine the Hausdorff dimensions of these singularities, and the a.e. modulus of continuity of the series. Lacunary wavelet series thus turn out to be a new example of multifractal functions.

1. Introduction and statement of results. The success of wavelet techniques in many fields of applications is largely due to the following remarkable property: many signals, images, or mathematical functions can be accurately represented in a wavelet basis using very few nonvanishing coefficients. This is the case for piecewise smooth functions, of a large class of images (see [6] or [7]), and of solutions of nonlinear hyperbolic equations (see [4]); the starting point of the denoising algorithm *wavelet* shrinkage of [6] and [7] is based on the remark that, since images composed of piecewise smooth parts have few nonzero wavelet coefficients, they can be efficiently denoised by setting to zero all small wavelet coefficients; this amounts to approximating the noisy image by a lacunary wavelet series. Similarly, recent techniques related to nonlinear approximation have been developed to give a functional framework fit to study functions which have a few numerically nonvanishing wavelet coefficients (see [3]).

Though many signals or functions have thus been shown to be accurately approximated by lacunary wavelet series, their properties have never been investigated. Our purpose is to study a probabilistic model of such series. We will see that, though this model is extremely simple, the corresponding random functions have a subtle local behavior: they exhibit a whole range of very oscillatory Hölder singularities, called *chirps* located on random fractal sets. Note that in different contexts wavelet methods have already been used to study stochastic processes; for instance, see [11] for processes with stationary increments and [2] for some Gaussian processes.

Let us now describe the model we will study. We use $2^d - 1$ wavelets $\psi^{(i)}$ in the Schwartz class and such that the set of functions $2^{dj/2}\psi^{(i)}(2^j x - k)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^d$ form an orthonormal basis of $L^2(\mathbb{R}^d)$ (see [16]); these conditions imply that all moments of the wavelets $\psi^{(i)}$ vanish. Since we are interested in local properties of wavelet expansions, it is more convenient to work with periodic wavelets which are obtained by a periodization of the above basis (see [17]);

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we thus obtain the functions

$$\psi_{0,0}(x)=1,$$

(1) $$\psi^{(i)}_{j,k}(x)=\sum_{l\in\mathbf{Z}^d}\psi^{(i)}(2^j(x-l)-k),\quad j\in\mathbb{N},\quad k\in\{0,\dots,2^j-1\}^d$$

for $x\in T(=\mathbb{R}^d/\mathbf{Z}^d)$. The $2^{dj/2}\psi^{(i)}_{j,k}$ form an orthonormal basis of $L^2(T)$. Note that we have chosen an L^∞ normalization for the $\psi^{(i)}_{j,k}$ which is more convenient for studying Hölder regularity; thus the wavelet coefficients of a function f are given by

$$C^{(i)}_{j,k}=2^{dj}\int_T f(t)\psi^{(i)}_{j,k}(t)\,dt.$$

The index (i) plays no specific role in the proofs, so we will forget it from now on and write $C_{j,k}$ for the wavelet coefficients and $\psi_{j,k}$ for the wavelets.

The random process F that we will study depends on two parameters $\eta\in(0,d)$ and $\alpha>0$ and is defined by its wavelet coefficients as follows: for each $j\geq 0$ we pick at random and independently $[2^{\eta j}]$ locations $k\in\{0,\dots,2^j-1\}^d$, and the corresponding wavelet coefficients $C_{j,k}$ take the value $2^{-\alpha j}$; we set to 0 all other wavelet coefficients. The locations of the nonvanishing coefficients are chosen independently for each value of j.

The parameter η characterizes the lacunarity of the wavelet series; the parameter α is related to its uniform Hölder regularity; indeed, a standard result of [17] shows that if $\alpha\notin\mathbb{N}$, $F\in C^\alpha(\mathbb{R})$, and this uniform regularity is best possible (if α is an integer, the Hölder spaces have to be replaced by either the Zygmund class if $\alpha=1$ or iterated primitives of functions of the Zygmund class if $\alpha=1$). Nonetheless F has a much larger pointwise Hölder regularity at "most" locations; this pointwise regularity is estimated with the help of the Hölder exponent, which is defined as follows. Recall that if h is a positive real number and $x_0\in\mathbb{R}^d$, a function f is $C^h(x_0)$ if there exists a polynomial P of degree less than h and a constant C such that, if $|x-x_0|$ is small enough,

(2) $$|f(x)-P(x-x_0)|\leq C|x-x_0|^h.$$

The Hölder exponent of f at x_0 [denoted by $h(x_0)$] is the supremum of all the values of h such that (2) holds. One of our purposes is to study the random sets of points where F has a given Hölder exponent. Actually, we are interested in a more complete local study than only the determination of Hölder exponents. Indeed, a given Hölder exponent α at x_0 allows for many different behaviors near x_0: for instance, cusplike singularities, such as $|x-x_0|^\alpha$ or very oscillatory behaviors, such as

(3) $$g_{\alpha,\beta}(x)=|x-x_0|^\alpha\sin\left(\frac{1}{|x-x_0|^\beta}\right)$$

for $\beta>0$. The functions $g_{\alpha,\beta}$ are the most simple examples of chirps at x_0. A general definition can be derived from this one-dimensional example as follows.

If $f \in L^\infty(\mathbb{R})$, denote by $f^{(-n)}$ an iterated primitive of f of order n. A consequence of the oscillations of (3) near x_0 is that $g^{(-n)}_{\alpha,\beta}$ is $C^{\alpha+n(\beta+1)}(x_0)$ (the increase of regularity at x_0 is not 1 at each integration, as would be expected for an arbitrary function, but $\beta+1$). This remark motivated the following definition of [18].

DEFINITION 1. Let $h \geq 0$ and $\beta > 0$. A function $f \in L^\infty(\mathbb{R}^d)$ is a chirp of type (h, β) at x_0 if, for every $n \geq 0$, f can be written as a finite sum

$$f = \sum_{|\alpha|\leq n} \partial^\alpha g_\alpha,$$

where $g_\alpha \in C^{\beta n+|\alpha|+h}(x_0)$.

The characterization of chirps given by Proposition 1 shows that this definition recaptures the oscillatory behavior of $g_{\alpha,\beta}$. Before stating this proposition, we need to recall the following definition.

DEFINITION 2. A bounded function f defined outside a ball centered at the origin is infinitely oscillating if $\forall n \in \mathbb{N}$, there exists a finite number of bounded functions g_α such that

$$f = \sum_{|\alpha|=n} \partial^\alpha g_\alpha.$$

For instance, in dimension $d = 1$, the sine function is infinitely oscillating. The following proposition is proved in [18].

PROPOSITION 1. *A function $f \in L^\infty(\mathbb{R}^d)$ is a chirp of type (h, β) at x_0 if and only if there exists a function $r(x)$ which is C^∞ in a neighborhood of x_0 and $\varepsilon > 0$ such that if $|x - x_0| \leq \varepsilon$,*

$$f(x) = |x-x_0|^h g\left(\frac{(x-x_0)}{|x-x_0|^{\beta+1}}\right) + r(x-x_0),$$

where the function g is defined outside a neighborhood of the origin and is infinitely oscillating and r is C^∞. Furthermore, the following wavelet characterization holds: a function f is a chirp of type (h, β) at x_0 if and only if the following conditions are fulfilled:

(i) *f is $C^h(x_0)$.*
(ii) *In the domain $|x_0 - k2^{-j}|^{1+\beta} \leq 2^{-j} \leq |x_0 - k2^{-j}|$, $\forall N \in \mathbb{N}$,*

$$|C_{j,k}| \leq C_N\left(2^j|x_0 - k2^{-j}|^{1+\beta}\right)^N |x_0 - k2^{-j}|^h. \tag{4}$$

(iii) *In the domain $|x_0 - k2^{-j}| \leq 2^{-j}$, $\forall N \in \mathbb{N}$,*

$$|C_{j,k}| \leq C_N 2^{-Nj}. \tag{5}$$

We will define chirp exponents which extend the notion of Hölder exponents. In order to state a definition, we need the following property that will be proved in Section 3.

PROPOSITION 2. *If $f \in L^\infty$ is a chirp of type (h, β) at x_0 with $\beta > 0$ and if $f \in C^{h'}(x_0)$ for $h' > h$, then $\forall h'' < h'$, f is a chirp of type (h'', β) at x_0.*

This proposition implies that the interior of the set of ordered pairs (h, β) such that f is a chirp of type (h, β) at x_0 is a rectangle $(0, h_0) \times (0, \beta_0)$. We can thus use h_0 and β_0 to define chirp exponents.

DEFINITION 3. The chirp exponents of a function f at x_0 are:

(i) $h(x_0) = \sup\{h\colon \text{such that } f \text{ is } C^h(x_0)\}$.
(ii) $\beta(x_0) = \sup\{\beta\colon \exists h \text{ such that } f \text{ is a chirp of type } (h, \beta) \text{ at } x_0\}$.

The exponent $h(x_0)$ is the Hölder exponent at $x_0$which was defined above, and we will call $\beta(x_0)$ the *oscillation exponent* at x_0. When the Hölder exponent of a function is widely changing from point to point, one is interested in obtaining some information about the geometry of the locations of the Hölder singularities of the function. This motivated the definition of the spectrum of singularities $d(h)$. This function associates to each positive h the Hausdorff dimension of the set A_h of the points x where the Hölder exponent is h (see [1] and [9] where this notion is studied in the context of fully developed turbulence, where it was first introduced, and [13] for some mathematical results). If we characterize singularities by their Hölder and oscillation exponent, we are naturally led to define the chirp spectrum as follows.

DEFINITION 4. The chirp spectrum $d(h, \beta)$ of a function f is the Hausdorff dimension of the set of points where f has the chirp exponents (h, β).

We denote by $\dim_H(E)$ the Hausdorff dimension of the set E, and we use the usual convention $\dim_H(\varnothing) = -\infty$. We will determine the chirp spectrum of lacunary wavelet series, and we will also determine their almost everywhere regularity; more general moduli of continuity than those supplied by the Hölder conditions will be needed.

DEFINITION 5. A modulus of continuity is any continuous increasing function ω defined on $[0, 1]$ such that

$$\omega(0) = 0, \qquad (6)$$
$$\exists C > 0 \quad \text{such that } \omega(2x) \leq C\omega(x).$$

Let ω be a function satisfying (6); ω is a modulus of continuity at x_0 of f if there exists a polynomial P and a constant C such that, if $|x - x_0|$ is small enough,

$$|f(x) - P(x - x_0)| \leq C\omega(|x - x_0|).$$

We can now state the main results of this paper.

THEOREM 1. *The chirp spectrum of almost every sample path of the lacunary wavelet series F is supported by the segment $h = \alpha(\beta+1)$, $h \in [\alpha, d\alpha/\eta]$; on this segment,*

$$d(h, \beta) = \eta(\beta + 1).$$

A function ω satisfying (6) *is a.s. almost everywhere a modulus of continuity of F if and only if*

$$\int_0^1 \omega^{-\eta/\alpha}(x) x^{d-1}\, dx < \infty. \tag{7}$$

REMARKS.

1. Since for each h there is at most one β such that $d(h, \beta) \neq -\infty$, the spectrum of singularities of F is $d(h) = h\eta/\alpha$ for $h \in [\alpha, d\alpha/\eta]$.
2. The same argument applied to primitives of F shows that the chirp exponents of F are almost surely almost everywhere.

$$(h, \beta) = \left(\frac{d\alpha}{\eta}, \frac{1}{\eta} - 1\right).$$

A strong local oscillatory behavior such as in (3) is very remarkable, and it was commonly believed that it could only be found at isolated points of a function; Y. Meyer disproved this opinion by showing that the Riemann function $\sum n^{-2} \sin(\pi n^2 x)$ has a dense set of chirps of type (3/2,1); see [14]. Lacunary wavelet series are more remarkable under this respect, since they exhibit chirps almost everywhere (a function cannot have chirps everywhere; see [10]).

3. The assertion expressed in the theorem is stronger than stating that, for each $h, d(h, \beta)$ has almost surely a given value, which would not be sufficient to determine the spectrum of singularities of almost every sample path.
4. The parameters α and η can also be related to the Besov regularity of the sample paths. Indeed, using the normalization we chose for wavelet coefficients, the following characterization holds (see [17]). A function f belongs to $B_p^{s,\infty}$ if and only if its wavelet coefficients satisfy

$$\exists\, C\, \forall j \sum_k |C_{j,k}|^p 2^{j(sp-d)} \leq C.$$

It follows that sample paths of lacunary wavelet series belong to $B_p^{s,\infty}$ if and only if

$$\eta - d \leq (\alpha - s)p.$$

Almost everywhere moduli of continuity will be studied in Section 2 and the chirp spectrum will be determined in Section 3.

2. Almost everywhere regularity. In order to determine the almost everywhere regularity of F, we will need the following result of [14].

PROPOSITION 3. *Let ω be a modulus of continuity at x_0 of a function $f \in L^\infty(\mathbb{R}^d)$. There exists C' such that the wavelet coefficients of f satisfy for all j and k such that j is large enough and $|k2^{-j} - x_0|$ small enough,*

$$|C_{j,k}| \le C'\Big(\omega(2^{-j}) + \omega(|k2^{-j} - x_0|)\Big). \tag{8}$$

We now prove the last point of Theorem 1. Suppose that ω is an almost everywhere modulus of continuity of F; (8) will hold for almost every x_0, but the constant C' may depend on x_0. However, we can replace ω by a larger modulus of continuity ω' such that $\omega(x)/\omega'(x) \to 0$ when $x \to 0$ and without altering the integrability condition (7); thus we can pick $C' = 1/3$ almost everywhere when we replace ω by ω' in (8). [Of course there is no uniformity: the j's and k's such that (8) holds depend on x_0.] Since ω' is continuous and increasing, we can define S_j by

$$\omega'(2^{-\eta j/d} S_j) = 2^{-\alpha j} \quad \text{if } j \ge 0. \tag{9}$$

Suppose first that $\sum S_j^d = +\infty$. We can pick a subsequence j_n such that $\sum_{n=1}^\infty S_{j_n}^d = +\infty$ $S_{j_n}^d \ge 1/j_n^2$. For each j let F_j denote the set of points $k2^{-j}$ such that $C_{j,k}$ is not vanishing. If $c > 0$ and $k2^{-j} \in F_j$, let

$$B_{j,k}^c = B(k2^{-j}, c2^{-\eta j/d} S_j)$$

(where $B(x, r)$ denotes the closed ball with center x and radius r). The centers of the balls $B_{j,k}^c$ are not exactly equidistributed on T; nonetheless, instead of choosing the subsets F_j, we can equivalently pick an infinite sequence t_n of points on T, such that the t_n are independent and equidistributed and then define, if $n_j = \sum_{l>j} [2^{\eta l}]$ and $L_j = \{n_j, \dots, n_{j+1} - 1\}$,

$$F_j = \{[t_n 2^j] 2^{-j}\} \quad \text{for } n \in L_j. \tag{10}$$

The ball $\bar{B}_{j_n,k}^c$ centered at t_n of radius $(c/2)2^{-\eta j_n/d} S_{j_n}$ is included in $B_{j_n,k}^c$ because the distance between the centers of $B_{j_n,k}^c$ and $\bar{B}_{j_n,k}^c$ is bounded by

$$\sqrt{d}\, 2^{-j_n} \le \frac{c}{2} 2^{-\eta j_n/d} S_{j_n}.$$

Thus, using the Borel–Cantelli lemma, $\forall\, c > 0$ almost every point of T belongs to $\limsup \bar{B}_{j_n,k}^c$ hence to $\limsup B_{j_n,k}^c$. Since the $B_{j_n,k}^c$ are increasing with c, almost every point of T belongs to

$$\bigcap_{c>0} \limsup B_{j_n,k}^c.$$

Let x_0 be a point of this set. There exists a subsequence j'_n of j_n, and there exist points $\lambda'_n = k2^{-j'_n}$ with $k \in F_{j'_n}$ and a sequence $c_n \to 0$ such that

$$\forall\, n,\ |\lambda'_n - x_0| \le c_n 2^{-\eta j'_n/d} S_{j'_n}.$$

Since (8) holds with $C' = 1/3$,

$$2^{-\alpha j'_n} \leq \tfrac{1}{3}\Big(\omega'(2^{-j'_n}) + \omega'(|\lambda'_n - x_0|)\Big) \leq \tfrac{1}{3}\Big(\omega'(2^{-j'_n}) + \omega'(c_n 2^{-\eta j'_n/d} S_{j'_n})\Big);$$

hence for each n, either $2^{-\alpha j'_n} < \omega'(c_n 2^{-\eta j'_n/d} S_{j'_n})$ or $2^{-\alpha j'_n} < \omega'(2^{-j'_n})$. The first inequality cannot hold for an infinite number of values of n for the following reason: since $c_n \to 0$ and ω' is increasing, it follows that

$$\omega'(c_n 2^{-\eta j'_n/d} S_{j'_n}) < \omega'(2^{-\eta j'_n/d} S_{j'_n}) = 2^{-\alpha j'_n}.$$

If the second inequality held for an infinite number of values of n, then

$$\omega'(2^{-\eta j'_n/d} S_{j'_n}) = 2^{-\alpha j'_n} \leq \omega'(2^{-j'_n})$$

holds, thus

$$2^{-\eta j'_n/d} S_{j'_n} \leq 2^{-j'_n}$$

(because ω' is increasing) so that $S_{j'_n} \leq 2^{(\eta/d-1)j'_n}$, which is contradictory with $S_{j_n} \geq (1/j_n)^{2/d}$ for n large enough, since $\eta < d$. Thus we have proved that if $\sum S_j^d = +\infty$, ω is not an a.e. modulus of continuity.

Let us now show that $\sum S_j^d$ diverges if and only if $\int_0^1 (\omega(x)^{-\eta/\alpha} x^{d-1})\, dx$ diverges. We split the interval of integration $[0, 1]$ into the sequence of intervals of ends $2^{-\eta j/d} S_j$. Since ω is increasing, using (9),

$$\sum_{j=1}^{\infty} \left(\frac{2^{-\alpha j}}{3C'}\right)^{-\eta/\alpha} \int_{2^{-\eta j/d} S_j}^{2^{-\eta(j-1)/d} S_{j-1}} x^{d-1}\, dx \leq \int_0^1 (\omega(x)^{-\eta/\alpha} x^{d-1})\, dx$$
$$\leq \sum_{j=1}^{\infty} \left(\frac{2^{-\alpha(j-1)}}{3C'}\right)^{-\eta/\alpha} \int_{2^{-\eta j/d} S_j}^{2^{-\eta(j-1)/d} S_{j-1}} x^{d-1}\, dx;$$

the first and last terms are equivalent to

$$\sum_{j=1}^{\infty} (2^{-\alpha j})^{-\eta/\alpha} \left(\left(2^{-\eta(j-1)/d} S_{j-1}\right)^d - \left(2^{-\eta j/d} S_j\right)^d \right),$$

which is equivalent to $\sum S_j^d$. It follows that, with probability 1,

$$\text{(11)} \qquad \int_0^1 \omega^{-\eta/\alpha}(x) x^{d-1}\, dx = \infty \Rightarrow \text{a.e. } \omega \text{ is not a modulus of continuity.}$$

In order to prove the converse implication, let ω be a continuous increasing function satisfying (8) and let S_j be defined by (9). Suppose now that $\sum S_j^d < +\infty$.

First, note that, because of (6), (7) is equivalent to $\sum 2^{-dj} \omega(2^{-j})^{-\eta/\alpha} < \infty$, which implies $\omega(2^{-j}) \geq 2^{-\alpha d j/\eta}$ for j large enough. Because of (6), it implies that, if $|x - x_0|$ is small enough,

$$\text{(12)} \qquad \omega(|x - x_0|) \geq |x - x_0|^{-\alpha d/\eta}.$$

If $k2^{-j} \in F_j$, let

$$B_{j,k} = B(k2^{-j}, 2 \cdot 2^{-\eta j/d} S_j).$$

Using again the Borel–Cantelli lemma, for almost every point x_0 of T, $x_0 \notin \limsup B_{j,k}$. Let x_0 be such a point; the wavelet coefficients of F satisfy

$$(13) \qquad \exists J \, \forall j \geq J, \qquad C_{j,k} = 0 \quad \text{if } |x_0 - k2^{-j}| \leq 2 \cdot 2^{-\eta j/d} S_j.$$

Recall that $F(x) = \sum C_{j,k} \psi_{j,k}(x)$. The function $\sum_{j \leq J} C_{j,k} \psi_{j,k}(x)$ belongs to $C^\infty(\mathbb{R}^d)$; therefore, since ω satisfies (12), we can forget about this part. Consider now the remaining term

$$\sum_{j > J, \, |x_0 - k2^{-j}| > 2 \cdot 2^{-\eta j} S_j} C_{j,k} \psi_{j,k}(x).$$

For x given, let j_1 be the first integer satisfying

$$|x_0 - x| \geq 2^{-\eta j_1/d} S_{j_1}.$$

Let $\Delta_j f(x) = \sum_k C_{j,k} \psi_{j,k}(x)$. Using the fast decay of ψ and its partial derivatives, it follows that $\forall j \geq J, \forall N$ and for every multiindex β,

$$\left| \partial^\beta \Delta_j f(x_0) \right| \leq 2^{(|\beta| - \alpha) j} \frac{C_{N,\beta}}{(2^j 2^{-\eta j/d} S_j)^N},$$

and, uniformly on the segment with ends x_0 and x; $\forall j \in \{J, \ldots, j_1\}, \forall N$ and for every multiindex β,

$$(14) \qquad \left| \partial^\beta \Delta_j(f)(u) \right| \leq 2^{(|\beta| - \alpha) j} \frac{C_{N,\beta}}{(2^j 2^{-\eta j/d} S_j)^N}.$$

We denote by P_j the Taylor polynomial of $\Delta_j(f)$ of degree $[\alpha/\eta]$ at x_0.

Now, let us make the further assumption

$$(15) \qquad \exists A \text{ such that } S_j \geq \frac{1}{j^A} \text{ for } j \text{ large enough.}$$

Since $\eta < d$, (14) implies that, uniformly on the segment of ends x_0 and $x, \forall j \in \{J, \ldots, j_1\}$, $\forall N$ and for every multiindex β,

$$(16) \qquad \left| \partial^\beta \Delta_j f(u) \right| \leq \frac{C'_{N,\beta}}{2^{jN}},$$

and $\forall j \geq J, \forall N$ and for every multiindex β,

$$(17) \qquad \left| \partial^\beta \Delta_j f(x_0) \right| \leq \frac{C'_{N,\beta}}{2^{jN}}.$$

It follows that $\sum_{j=1}^\infty P_j(x - x_0)$ is convergent and

$$\left| \sum_{j=J}^{j_1} (\Delta_j f(x) - P_j(x - x_0)) + \sum_{j=j_1+1}^\infty \Delta_j f(x) - \sum_{j=j_1+1}^\infty P_j(x - x_0) \right| \leq C\omega(|x - x_0|)$$

[we apply Taylor's formula and (16) to each term of the first sum, we bound each term of the second sum by $C2^{-\alpha j}$ and we use (17) to bound each term of the last sum], hence the converse part of (11) provided that (15) holds. Now (15) means that there exists A such that the modulus of continuity ω that we consider is smaller than $|x-x_0|^{\alpha d/\eta}|\log(|x-x_0|)|^A$. If ω does not satisfy this property, let

$$\omega'(|x-x_0|) = \inf\Big(\omega(|x-x_0|), |x-x_0|^{\alpha d/\eta}|\log(|x-x_0|)|^{2\alpha/\eta}\Big).$$

This is also a modulus of continuity which satisfies (15), so that Theorem 1 holds for ω', hence a fortiori also for ω, which is larger.

3. Determination of the chirp spectrum. We first prove the property of chirps exponents stated in Proposition 2. It requires the following result of [12] which relates pointwise regularity to decay conditions of the wavelet coefficients.

PROPOSITION 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function. if f is $C^h(x_0)$, for all j and k such that $j\geq 0$ and $|x_0-k2^{-j}|\leq 1$,*

$$|C_{j,k}| \leq C2^{-hj}(1+|2^j x_0-k|)^h. \tag{18}$$

Conversely, suppose that there exists $h'<h$ such that for all j and k such that $j\geq 0$ and $|2^j x_0-k|\leq 1$,

$$|C_{j,k}| \leq C2^{-hj}(1+|2^j x_0-k|)^{h'}. \tag{19}$$

Then f is $C^h(x_0)$.

PROOF OF PROPOSITION 2. Since f has a chirp of exponents (h,β) at x_0, the fast decay condition (5) holds, and since f is $C^{h'}(x_0)$, using the criterium given by (4), we only have to check that $\forall N$,

$$|C_{j,k}| \leq C_N\Big(2^j|x_0-k2^{-j}|\Big)^N|x_0-k2^{-j}|^{h''} \quad \text{for}$$
$$|x_0-k2^{-j}|^{1+\beta} \leq 2^{-j} \leq |x_0-k2^{-j}|.$$

Since f has a chirp of exponents (h,β) at x_0, (4) implies that $\forall N$,

$$|C_{j,k}| \leq C_N\Big(2^j|x_0-k2^{-j}|\Big)^N|x_0-k2^{-j}|^{h} \quad \text{for}$$
$$|x_0-k2^{-j}|^{1+\beta} \leq 2^{-j} \leq |x_0-k2^{-j}|$$

and since f is $C^{h'}(x_0)$, $|C_{j,k}|\leq C|x_0-k2^{-j}|^{h'}$ in the same domain. The result follows by taking a weighted geometric average of these two estimates.

In order to prove the first part of Theorem 1, we reduce it to a problem about random balls on the torus T.

Let $\delta \in [0,1]$. Denote by $B^{\delta}_{j,k}$ the ball centered at $k2^{-j}(k \in F_j)$ and of radius $2^{-\delta j}$. Let

$$E_\delta = \limsup_{j\to\infty} \bigcup_k B^{\delta}_{j,k}, \qquad G_\delta = \bigcap_{\delta'<\delta} E_{\delta'} - \bigcup_{\delta'>\delta} E_{\delta'} \quad \text{if } \delta < 1 \text{ and } G_1 = \bigcap_{\delta'<1} E_{\delta'}.$$

Note that the E_δ are decreasing (in δ).

LEMMA 1. *If $\delta < \eta/d$, $E_\delta = T$ a.s.*

The proof of Lemma 1 uses the following proposition of [15] concerning random coverings of the torus T. Let v_n be a decreasing sequence of positive numbers smaller than 1. We distribute at random and independently balls B_n of volume v_n on T. Let

$$E = \limsup B_n.$$

PROPOSITION 5. *If $\limsup_{n\to\infty}(\sum_{j=1}^n v_j - d \log n) = +\infty$, $E = T$ a.s.*

PROOF OF LEMMA 1. Using the same argument as at the beginning of Section 2, we can reduce the computation of $\limsup_{k2^{-j}\in F_j} B(k2^{-j}, 2\cdot 2^{-\delta j})$ to the case of balls of radius two times smaller, but of centers i.i.d. on T with the Lebesgue measure. We apply Proposition 5 to these smaller balls, and if v_n denotes their volumes, $\exists\, C, C' > 0$ such that

$$\sum_{n_j}^{n_{j+1}} v_n - d \log n_{j+1} \geq C[2^{nj}]2^{-d\delta j} - C' j,$$

which tends to ∞ if $\eta > d\delta$, hence Lemma 1. Thus every point of the torus belongs to one of the $(G_\delta)_{\delta\in[\eta/d,\,1]}$. The following proposition shows that the points that belong to a given G_δ have the same regularity and oscillation.

PROPOSITION 6. *If $x \in G_\delta$, the chirp exponents of F at x are $(h, \beta) = (\alpha/\delta, 1/\delta - 1)$.*

PROOF OF PROPOSITION 6. If $x \in E_\delta$, there exists an infinite number of points $k2^{-j}$ $(k \in F_j)$ such that $|x - k2^{-j}| \leq 2^{-\delta j}$, so that, using Propositon 4, the Hölder exponent of F at x is at most α/δ. Conversely, if $x \notin E_\delta$, inside a domain $|x - k2^{-j}| \leq 2^{-\delta j}$ the wavelet coefficients of F vanish for j large enough. Thus, if $C_{j,k}$ is a nonvanishing wavelet coefficient,

$$|C_{j,k}| = 2^{-\alpha j} = (2^{-\delta j})^{\alpha/\delta} \leq |x - k2^{-j}|^{\alpha/\delta}.$$

Using again Proposition 4, $F \in C^{\alpha/\delta}(x)$.

Since the wavelet coefficients of F vanish in the domain $|x_0 - k2^{-j}|^{1/\delta} \leq 2^{-j}$, the oscillation exponent is at least β. Since there are coefficients of size $2^{-\alpha j}$ just outside this domain, the oscillation exponent cannot be larger; hence Proposition 6 holds.

The determination of the spectrum of F is thus reduced to the computation of the Hausdorff dimensions of the sets G_δ. Note that E_δ is included in

$$\bigcup_{j\geq J,\, k\in F_j} B(k2^{-j}, 2^{-\delta j})$$

so that, using these balls as a covering, we obtain $\dim E_\delta \leq \eta/\delta$, hence $\dim G_\delta \leq \eta/\delta$. The lower bound for the Hausdorff dimension of G_δ is a consequence of a general technique that we will develop in the next section in order to obtain lower bounds for the Hausdorff dimension of a fairly general class of fractal sets.

The main result proved in Section 4 is the following. Let λ_n be a sequence of points in $T = [0,1]^d$ and $\varepsilon_n > 0$. We consider the sets

$$E_a = \limsup_{N\to\infty} \bigcup_{n\geq N} B(\lambda_n, \varepsilon_n^a)$$

[E_a is the set of points that belong to an infinite number of balls $B(\lambda_n, \varepsilon_n^a)$]. The function $a \to \dim_H(E_a)$ is decreasing. Furthermore, if

$$A = \sup\{\alpha\colon \textstyle\sum \varepsilon_n^\alpha = \infty\} = \inf\{\alpha\colon \textstyle\sum \varepsilon_n^\alpha < \infty\},$$

using the covering by the balls $B(\lambda_n, \varepsilon_n^a)$, one easily obtains $\dim_H(E_a) \leq A/a$. This upper bound often turns out to be sharp in situations where the λ_n are "equidistributed" in some sense. However, this type of information is often hard to obtain or to handle; sometimes a different kind of information is easily available. For an a small enough, we may know that almost every point of T belongs to E_a (it is the case in problems related to diophantine or dyadic approximation, or if the λ_n are independent equidistributed random variables). We will prove that this sole information yields a lower bound on $\dim_H(E_b)$ for $b > a$. In practice, a more precise result is often required: one needs to obtain a positive Hausdorff measure for E_a. Let us now recall the definition of a Hausdorff measure.

Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function satisfying $h(0) = 0$, and let A be a bounded subset of $\mathbb{R}^d$. If $|B|$ denotes the diameter of the ball B, let

$$\mathscr{H}_\varepsilon^h(A) = \inf_{\mathscr{U}} \Bigg(\sum_{(u_i)_{i\in N}\in\mathscr{U}} h(|u_i|) \Bigg)$$

where the infimum is taken on all coverings $\mathscr{U}$ by families of balls $(u_i)_{i\in\mathbb{N}}$ of radius at most ε. The $\mathscr{H}^h$-measure of A can be defined as

$$\mathscr{H}^h(A) = \lim_{\varepsilon\to 0} \mathscr{H}_\varepsilon^h(A).$$

We will prove the following theorem in the next section.

THEOREM 2. *Let $h_c(x) = (\log x)^2 |x|^c$. If almost every x belongs to E_a,*

$$\forall\, b > a, \qquad \mathscr{H}^{h_{da/b}}(E_b) > 0.$$

(*In particular, the Hausdorff dimension of E_b is larger than da/b.*)

Here again, using the same argument as at the beginning of Section 2, we reduce the problem to the case where the points of F_j are independent and equidistributed on T; hence, by the Borel–Cantelli lemma, the hypothesis of Theorem 2 is satisfied for $a = \eta/d$, hence Theorem 2 implies that the $h_{\eta/\delta}$ measure of E_δ is positive. Since the $h_{\eta/\delta}$ measure of $E_{\delta'}$ vanishes if $\delta' > \delta$, it follows that the Hausdorff dimension of G_δ is η/δ, hence the formula for the spectrum of chirps in Theorem 1.

4. A priori lower bounds of the dimension of some fractals. The idea of the proof of Theorem 2 is to construct a generalized Cantor set K included in E_b and simultaneously a probability measure μ supported by this set, with specific scaling properties. The "mass distribution principle" will allow us to deduce from these scaling properties a lower bound for the $\mathscr{H}^{h_{da/b}}$ Hausdorff measure of E_b. The generalized Cantor set and the measure will be constructed using an iterative procedure.

After perhaps reordering the sequence $(\lambda_n, \varepsilon_n)$, we can suppose that ε_n is nonincreasing. Let $b > a$ fixed. We introduce the notations

$$B_n = B(\lambda_n, \varepsilon_n^a)$$

and

$$\tilde{B}_n = B(\lambda_n, \varepsilon_n^b).$$

[More generally, if B is the ball $B(\lambda, e)$, $\tilde{B}$ will denote the ball $B(\lambda, e^{b/a})$.]

We now construct the first generation of the balls of the Cantor set K. First we will select a finite subsequence $B_{\phi(n)}$ of B_n as follows. Denote by $5B_n$ the ball of same center as B_n and of diameter $5|B_n|$. We first choose $\phi(1) = 1$ (i.e., we select B_1); $\phi(2)$ is the first index such that $B_{\phi(2)}$ is not included in $5B_{\phi(1)}$; $\phi(3)$ is the first index such that $B_{\phi(3)}$ is not included in $5B_{\phi(1)} \cup 5B_{\phi(2),\dots}$. We stop this extraction at the first index N such that

$$\operatorname{mes}\left(\bigcup_{i=1}^{N} 5B_{\phi(i)}\right) \geq \tfrac{1}{2} \tag{20}$$

[where $\operatorname{mes}(A)$ denotes the d-dimensional Lebesgue measure of A]. The index N exists because each ball B_n which has not been selected among the $B_{\phi(i)}$ is included in one of the $5B_{\phi(i)}$ previously selected (because ε_n is decreasing), so that

$$\bigcup_{i=1}^{\phi(N)} 5B_i \subset \bigcup_{i=1}^{N} 5B_{\phi(i)}. \tag{21}$$

Since almost every x belongs to E_a, $\operatorname{mes}(\bigcup_{i=1}^{n} B_i) \to 1$, and (20) follows if N is large enough.

By construction, the balls $B_{\phi(i)}$ thus selected are disjoint, and (20) implies that

$$\operatorname{mes}\left(\bigcup_{i=1}^{N} B_{\phi(i)}\right) \geq \frac{1}{2 \cdot 5^d}. \tag{22}$$

The N balls $\tilde{B}_{\phi(i)}$ are the first generation balls of our generalized Cantor set. The measure μ will be supported by the union of these balls, and we take

$$\forall\, i \;\; \mu(\tilde{B}_{\phi(i)}) = \frac{\operatorname{mes}(B_{\phi(i)})}{\sum_{j=1}^{N} \operatorname{mes}(B_{\phi(j)})};$$

(22) implies that

$$\mu(\tilde{B}_{\phi(i)}) \le 2 \cdot 5^d |\tilde{B}_{\phi(i)}|^{da/b}. \tag{23}$$

We will not construct the second generation balls by subdivising each $\tilde{B}_{\phi(i)}$. Let n be such that

$$\frac{1}{\varepsilon_n} \ge \exp\left(\frac{1}{\varepsilon_{\phi(N)}}\right). \tag{24}$$

Let us consider one of the balls $\tilde{B}_{\phi(i)}$; since $\bigcup_{j\ge n} B_j$ covers almost every point of $\tilde{B}_{\phi(i)}$, we can as above select a finite number of balls $B_{\phi(i,\,1)}, \dots, B_{\phi(i,\,N(i))}$ from the sequence $(B_j)_{j\ge n}$ such that

$$\operatorname{mes}\left(\bigcup_{i=1}^{N(i)} 5B_{\phi(i,\,j)}\right) \ge \tfrac{1}{2}\operatorname{mes}(\tilde{B}_{\phi(i)}).$$

The $B_{\phi(i,\,j)}$ are disjoint, so that

$$\operatorname{mes}\left(\bigcup_{i=1}^{N(i)} B_{\phi(i,\,j)}\right) \ge \frac{1}{2\cdot 5^d}\operatorname{mes}(\tilde{B}_{\phi(i)}). \tag{25}$$

The balls $\tilde{B}_{\phi(i,\,j)}$ are the second generation balls in the construction of K, and we take

$$\mu(\tilde{B}_{\phi(i,\,j)}) = \mu(\tilde{B}_{\phi(i)})\frac{\operatorname{mes}(B_{\phi(i,\,j)})}{\sum_{j=1}^{N(i)} \operatorname{mes}(B_{\phi(i,\,j)})}. \tag{26}$$

Thus

$$\mu(\tilde{B}_{\phi(i,\,j)}) \le 2 \cdot 5^d |\tilde{B}_{\phi(i,\,j)}|^{da/b}\frac{\mu(\tilde{B}_{\phi(i)})}{\operatorname{mes}(\tilde{B}_{\phi(i)})}. \tag{27}$$

This construction is iterated, and we thus obtain a generalized Cantor set K and a probability measure μ supported by K.

The balls thus constructed at each generation are called the *fundamental balls* of the generalized Cantor set. Note that the fundamental balls constructed are indexed by a tree, and the diameters of the balls at a given depth of the tree need not be of the same order of magnitude. If B is a fundamental ball, we will denote by $\hat{B}$ the “father” of B, that is, the fundamental ball from which B was directly obtained.

The diameters of the fundamental balls have been chosen such that, if B is any fundamental ball of the nth generation,

$$\frac{1}{|B|} \geq \exp\left(\sup\left(\frac{1}{|J|}\right)\right), \tag{28}$$

where the supremum is taken on all fundamental balls J of the previous generation.

We will now check that, if B is an arbitrary open ball,

$$\mu(B) \leq C|B|^{da/b}(\log|B|)^2; \tag{29}$$

following Principle 4.2 of [8], the Hausdorff measure of E_b constructed with the dimension function $h_{da/b}$ will then be positive.

We first check that (29) holds for the fundamental balls, by induction on the generation of the ball; (23) asserts that it is true for the first generation. Suppose now that B is any ball of the nth generation. The analogue of (27) at the nth generation states that

$$\mu(B) \leq 2 \cdot 5^d |B|^{da/b} \frac{\mu(\hat{B})}{\operatorname{mes}(\hat{B})},$$

which, using the induction hypothesis, is bounded by

$$2 \cdot 5^d |B|^{da/b} |\hat{B}|^{(da/b)-b} (\log|\hat{B}|)^2,$$

which, because of (28), is bounded by $2 \cdot 5^d |B|^{da/b} |\log|B|| \log(\log(|B|))^2$. Thus (29) holds for the balls of generation n.

Let now D be an arbitrary open ball. If D does not intersect the Cantor set, $\mu(D) = 0$. Otherwise, let B be the fundamental ball of smallest generation which intersects D and such that two children of B at least intersect D. Clearly, there exists exactly one such ball. Denote by $\tilde{B}_1, \dots, \tilde{B}_p$ the children of B that intersect D. If B is small enough, $|\tilde{B}_i| \ll |B_i|$ (because $|\tilde{B}_i| = |B_i|^{b/a}$), so that

$$\forall\, i = 1, \dots, p, \ \operatorname{mes}, \ (B_1 \cap D) \geq C \operatorname{mes}(B_i),$$

where the constant C depends only on the dimension d; this estimate holds because $|D|$ is larger than $|B_i|$ and D contains points of $\tilde{B}_i$ which are close to the center of B_i;

$$\mu(D) \leq \sum_{i=1}^{p} \mu(\tilde{B}_i),$$

which, by (26) and (25) applied at the corresponding generation is bounded by

$$\mu(B) \frac{2 \cdot 5^d \sum_{i=1}^{p} \operatorname{mes}(B_i)}{\operatorname{mes}(B)}.$$

If $|D| \geq |B|$, (29) holds because it holds for B, and $\mu(D) \leq \mu(B)$; Otherwise, since $\text{mes}(B_i) \leq C\,\text{mes}(B_i \cap D)$ and $B_i \cap B_j = \varnothing$, it follows that

$$\begin{aligned}\mu(D) &\leq C\mu(B)\frac{\text{mes}(D)}{\text{mes}(B)}\\ &\leq C|B|^{da/b}(\log(|B|))^2\frac{|D|^d}{|B|^d}\\ &\leq C|D|^{da/b}(\log(|D|))^2\end{aligned}$$

because $a < b$ so that the function $r \to r^{d(a/b-1)}(\log r)^2$ is decreasing near the origin.

5. Concluding remarks.

5.1. *Failure of the multifractal formalism.* Theorem 1 shows that, when $\eta \neq d$, Lacunary wavelet series are a.e. multifractal functions (i.e., their spectrum of singularities is supported by an interval of nonempty interior). This notion was introduced in [9] in the context of fully developed turbulence. Frisch and Parisi proposed a formula, known as the "multifractal formalism" in order to compute the spectrum of singularities of a function f; let us describe an equivalent wavelet formulation, proposed in [1], which is more adapted to numerical computations. Let

$$\zeta(q) = \liminf_{j\to\infty} \frac{\log(\sum_k |C_{j,k}|^q)}{-j\log 2}.$$

The multifractal formalism asserts that the spectrum of singularities can be recovered from $\zeta(q)$ by

$$d(h) = \inf_q (qh - \zeta(q)). \tag{30}$$

This formula is valid for many multifractal functions (see [13] for a mathematical discussion). Nonetheless, let us check that the multifractal formalism fails for the sample paths of lacunary wavelet series when $\eta \neq d$; here $\zeta(q) = \alpha q - \eta$ and (30) would yield

$$d(h) = \begin{cases} \eta, & \text{if } h = \alpha,\\ -\infty, & \text{else,}\end{cases}$$

which is the right spectrum only in the nonlacunary case $\eta = d$. The results of this paper thus show the need for a more general multifractal formalism when dealing with functions which have sparse wavelet expansions (which, as was pointed out in the introduction, is a very common situation in signal analysis).

5.2. *Almost everywhere directional regularity.* We proved that the a.e. Hölder exponent of F is $d\alpha/\eta$. Note that, in dimension larger than 1, the definition of pointwise Hölder regularity given by (2) involves a uniform bound in all directions. This does not prevent the function F being smoother when one considers its traces on fixed directions. Let us introduce some definitions concerning directional regularity. If u is a unit vector of $\mathbb{R}^d$, f is $C^\alpha(x_0, u)$ if the one-variable function $t \to f(x_0 + tu)$ is $C^\alpha(0)$. The following result holds.

PROPOSITION 7. *For a.e. x_0 and a.e. direction u:*

(i) *If $\eta \le d-1$, F is $C^\infty(x_0, u)$.*
(ii) *If $\eta > d-1$, F is $C^{\alpha/(\eta-d+1)}(x_0, u)$.*

Let us just sketch the proof of this result since it follows the line of the other wavelet regularity proofs of this paper.

We consider a fixed point x_0 and a given direction u. We showed that, with probability arbitrarily close to 1,

$$\exists\, A,\ J,\ \forall\, j \ge J, \qquad C_{j,k} = 0 \quad \text{if } |x_0 - k2^{-j}| \le \frac{1}{j^A} 2^{-nj/d}.$$

Denote by D the straight line $x_0 + \mathbb{R}u$. Among the nonvanishing wavelet coefficients of F, we separate two cases.

Case 1. $\mathrm{dist}(k2^{-j}, D) \ge j2^{-j}$. Because of the fast decay of the wavelets, the contribution of these coefficients to the directional regularity is C^∞ (if the wavelets were compactly supported, these wavelets would actually bring no contribution at all).

Case 2. $\mathrm{dist}(k2^{-j}, D) < j2^{-j}$. There are at most $Cj^A 2^{(1-d)j} 2^{\eta j}$ such coefficients (for an A large enough) which are equidistributed in this domain. The distance of the closest one to x_0 is therefore at least $(C/j^{A'}) 2^{(-\eta+d-1)j}$ (for an A' large enough). The Hölder exponent of F at x_0 in the direction u is therefore at least $\alpha/(\eta-d+1)$. It cannot be larger since there also exists a nonvanishing wavelet coefficient, whose distance to x_0 is bounded by $Cj^{A''} 2^{(-\eta+d-1)j}$ (for an A'' large enough).

Note that there clearly exists a dense set of directions for which the directional Hölder regularity is α/η, so that the directional regularity at a.e. point x_0 changes strongly from direction to direction. This study might probably be pushed further to prove a multifractal directional regularity at a generic point x_0.

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