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EPIDEMICS WITH RECOVERY IN $D = 2$

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We consider a modification of the spatial epidemic with removal that has regrowth of susceptibles. We show that if the original epidemic is supercritical, then the modified process has a nontrivial stationary distribution.

1. Introduction. In this paper we will study a process that has been used to model the spread of epidemics and forest fires (see [4], [13], [14] and references therein). In the model, each site $z \in Z^2$ can be one of three states 0, 1 or 2. In the epidemic interpretation, 0 = healthy, 1 = infected, 2 = removed = immune or dead. In the forest fire interpretation, 0 = alive, 1 = on fire and 2 = burnt. The main result of the paper makes more sense for epidemics but we will use the forest fire interpretation when we give the proofs and describe the dynamics:

1. A burning tree sends out sparks at rate α.
2. A spark emitted from x flies to one of the four nearest neighbors $x + (1, 0)$, $x + (0, 1)$, $x + (-1, 0)$ or $x + (0, -1)$ chosen at random with equal probabilities.
3. If the spark hits a live tree, the tree catches fire immediately and begins to emit sparks. It burns for an exponentially distributed amount of time with mean 1, then burns out.
4. Finally, burnt sites return to life at rate β.

At first glance, the spontaneous reappearance of trees may not seem reasonable. However, there are some situations (e.g., pine forests) where there are seeds or seedlings everywhere that do not grow in the presence of adult trees but have a chance to grow after a fire destroys the adult trees. The spontaneous regrowth is quite natural for epidemics of nonfatal diseases like measles that upon recovery confers lifetime immunity. New susceptibles are born and immune individuals die. We combine the two transitions into the one in step 4 to keep a constant population size.

The case $\beta = 0$ is called the spatial epidemic with removal (see [4] for the historical background and the facts we quote below). In this case it is known that there is a critical value $\alpha_c(0) \in (0, \infty)$ so that if $\alpha > \alpha_c(0)$ and we start with one burning tree in an otherwise virgin forest, then the fire has positive

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probability of not going out. Here the 0 refers to the value of β. Cox and Durrett [4] showed that when the forest fire does not go out, it expands linearly and has an asymptotic shape. We will use some of their methods to show:

THEOREM. *If* $\alpha > \alpha_c(0)$ *and* $\beta > 0$, *there exists a nontrivial stationary distribution, that is, one that assigns no mass to the "all healthy" state.*

The last result is not surprising in view of the behavior of the contact process (the extreme case $\beta = \infty$ in which the transition from 2 to 0 is instantaneous). In that case, there is a critical value $\alpha_c(\infty)$ so that if $\alpha > \alpha_c(\infty)$, then (a) the fire has positive probability of not going out starting from one burning tree in an otherwise virgin forest and (b) there is a stationary distribution concentrated on the set of configurations with infinitely many burning trees, which is the limit in distribution whenever the fire does not go out. We conjecture that when $\beta > 0$, there is a critical value $\alpha_c(\beta)$ so that conclusions (a) and (b) hold if (and only if) $\alpha > \alpha_c(\beta)$ but we do not know how to show this.

Before turning to the details of the proof of our theorem, we would like to discuss three modifications of the model for which we conjecture that the theorem holds.

1. Suppose we increase the possible states to $\{0, 1, \dots, N-1\}$ and declare that for $1 \le j \le N-1$, transitions $j \to j+1 \pmod N$ occur at rate 1 (i.e., $N-1 \to 0$ occurs at rate 1). This is a continuous-time version of the Greenberg–Hastings model (see [9]). A fairly straightforward generalization of the proof below shows that if $\alpha > \alpha_c(0)$, then this model has a stationary distribution for any N.
2. While reasonable for epidemics, the spontaneous reappearance of trees is not natural for forests. In this case, it would be more natural to suppose that new trees appear at a rate β times the number of occupied neighbors. Our theorem undoubtedly is true for this variation of the model but we have not been able to generalize the proof. The main technical problem is that we must rely on unburned trees to regenerate the forest and hence we must show that there is always a fairly dense set of survivors.
3. Perhaps the most embarrassing of our assumptions is that we allow the fire to spread only to nearest neighbors. This is needed for the proof, which relies on ancient (circa 1980) results in percolation theory. The recent revolution in percolation technology (see [2], [10]) makes it clear that the results of [4] and this paper remain valid for forest fires in which sparks go from x to y at rate $\alpha f(y-x)$ if we assume f is symmetric ($f(z) = f(-z)$) and has finite range [$f(z) \neq 0$ for only finitely many z]. Zhang Yu (see [2]) has taken the first step in this direction by generalizing the results of [4].

SKETCH OF THE PROOF. The rest of the paper is devoted to the proof of our theorem. The proof is somewhat complicated, so in what remains of the Introduction we will describe the ideas on which the proof is based. To be strictly shorter than the proof itself, our heuristic discussions will have to ignore some technicalities. If the reader finds the discussion confusing, he can skip to the beginning of Section 2 and plunge headfirst into the details. The only part of the Introduction that we rely on later is the claim that it suffices to prove Lemma 1.1.

We will prove our theorem using the technique explained in [7]. Let $B = (-L, L)^2$ and $B_m = mLe_1 + B$, where $e_1 = (1, 0)$ is the first unit vector. Let $\mathscr{L} = \{(m, n) \in Z^2\colon\ m + n \text{ is even}\}$. We say that $(m, n) \in \mathscr{L}$ is *occupied* if there are more than $M = L^{1/2}$ burning trees in B_m at some time $t \in [n\Gamma L, (n+1)\Gamma L]$ and we have at least one burning tree in B_m at all times $t \in [(n+1)\Gamma L, (n+2)\Gamma L]$. For convenience we will suppose that L is chosen so that $M = L^{1/2}$ is an integer. We will show:

1.1 LEMMA. *Γ and L can be chosen so that the set of occupied sites dominates the set of wet sites in a one-dependent oriented percolation process on $\mathscr{L}$ with parameter $p = 1 - 6^{-36}$.*

The reader should note that the space-time blocks $B_m \times [n\Gamma, (n+2)\Gamma]$ in which the process has the desired properties are called occupied while points in the percolation process are called wet and dry. (A.1) in the Appendix implies that if the percolation process starts with all sites wet at time 0, then for all even n,

$$(1.2) \qquad P(\text{some site } (2k, n) \text{ with } |k| \le K \text{ is wet}) \ge 1 - \varepsilon_K$$

and $\varepsilon_K \to 0$ as $K \to \infty$. With Lemma 1.1 and (1.2) in hand, standard arguments take over to give our theorem. We start the process from an initial configuration that has more than M burning trees in each box $2mLe_1 + (-L, L)^2$, $m \in Z$, take the Cesaro average of the distributions from times 0 to T and extract a convergent subsequence. Because our process has the Feller property, the limit μ is a stationary distribution (see part (d) of Proposition 1.8 in [12]). Lemma 1.1 and (1.2) imply μ concentrates on configurations with at least one burning site. Being a stationary distribution, μ must concentrate on configurations with infinitely many burning sites (or there would be positive probability of having no burning trees).

It suffices then to prove Lemma 1.1. Most of the work goes into showing that if $\Gamma < \infty$ and we start with at least $M = L^{1/2}$ burning trees in B, then for large L, with high probability, we can *keep the fire burning*, that is, have at least one burning tree in B at all times $t \in [0, 2\Gamma L]$. (Here and in what follows, *italics* indicates that we are giving a technical meaning to a phrase.) Once we demonstrate that we can keep the fire burning, we will have enough control over the fire to show that if Γ is large, the fire will spread into the neighboring boxes B_1 and B_{-1} by time ΓL and produce at least M burning trees there by time $2\Gamma L$.

To keep the fire burning, we will (a) do nothing as long as the number of burning trees is larger than M and (b) show that if L is large and there are exactly $M = L^{1/2}$ trees, then, no matter how they are arranged, the number of burning trees in B will, with high probability, reach $\delta L^{0.7}$ before hitting 0. Here and in what follows, δ is a small positive number, whose value is unimportant and will change from line to line. The first step in proving (b) is a covering lemma essentially due to Besicovitch [3]. (Here and for the rest of the introduction, we use the numbering in the body of the text.)

2.2 LEMMA. *If there are at most M burning trees, then we can cover an area of size at least $L^2/11$ in B by disjoint squares with three properties*: (i) *they contain no burning trees in their interiors*, (ii) *they have a burning tree in at least one corner and* (iii) *their sides are longer than $L^{0.7}$*.

BURNING A MEADOW. The squares satisfying (i)–(iii) give us opportunities for large fires and we call them *meadows*. Our first goal is to show:

(∗) There are constants ρ, u, λ and Λ so that if $[0, K]^2$ is a meadow at time 0, then with probability at least ρ, we will have a *nice fire* in the meadow, that is, a fraction ρ of the trees in $[0, K]^2$ will burn and will stay on fire for at least u units of time during $[\lambda K, \Lambda K]$.

To explain why such fires are nice, note that the number of tree-hours of burning is at least $u\rho K^2$, so at some time we must have $(u\rho/\Lambda)K$ burning trees and by (iii) in Lemma 2.2, this is at least $(u\rho/\Lambda)L^{0.7}$. The proof of (∗) occurs in two steps. First, we pick S large and wait S units of time so that even if the meadow were completely burnt, then there will be enough trees for a strong fire. There are two technical problems here: (i) We have to consider a bond-site version of the forest fire model in [4] in which sites are present with probability $1 - \exp(-\beta S)$ and absent with probability $\exp(-\beta S)$ and show that if the fire is supercritical with all trees present, that is, $\alpha > \alpha_c(0)$, then it is for large S and the techniques of [4] can be used; (ii) we have to show that not too much of the meadow burns by time S. Here our nightmare is that all trees on the boundary of the meadow are on fire. To cope with this scenario, suppose the meadow is $[0, K] \times [0, K]$ and let $\mathscr{M}$ be the five-sided region formed by connecting the following points in the order indicated (J, J), $(3J, J)$, $(3K/4, K/4)$, $(K/4, 3K/4)$, $(J, 3J)$, (J, J) (see Figure 1 for a picture). A straightforward large deviations estimate for first passage percolation gives:

3.6 LEMMA. *If $J \geq J_1$ and $K \geq 8J$, the probability that the fire reaches $\mathscr{M}$ by time S is smaller than $\frac{1}{2}$, independent of the number of burning trees on the boundary or the initial state of the meadow* (*assuming it contains no burning trees*).

Once the S units of waiting time elapses, we are ready to burn $\mathscr{M}$. In this stage, we ignore the further regrowth of trees so extra burning trees only help us and we can use the results of [4] (as generalized above). Let $\mathscr{N}$ be the trapezoid with vertices at $(3K/8, K/8)$, $(3K/4, K/4)$, $(K/4, 3K/4)$, $(K/8, 3K/8)$ (again see Figure 1.) We restrict our attention to $\mathscr{N}$ to have the following consequence of the proof of Lemma 3.6.

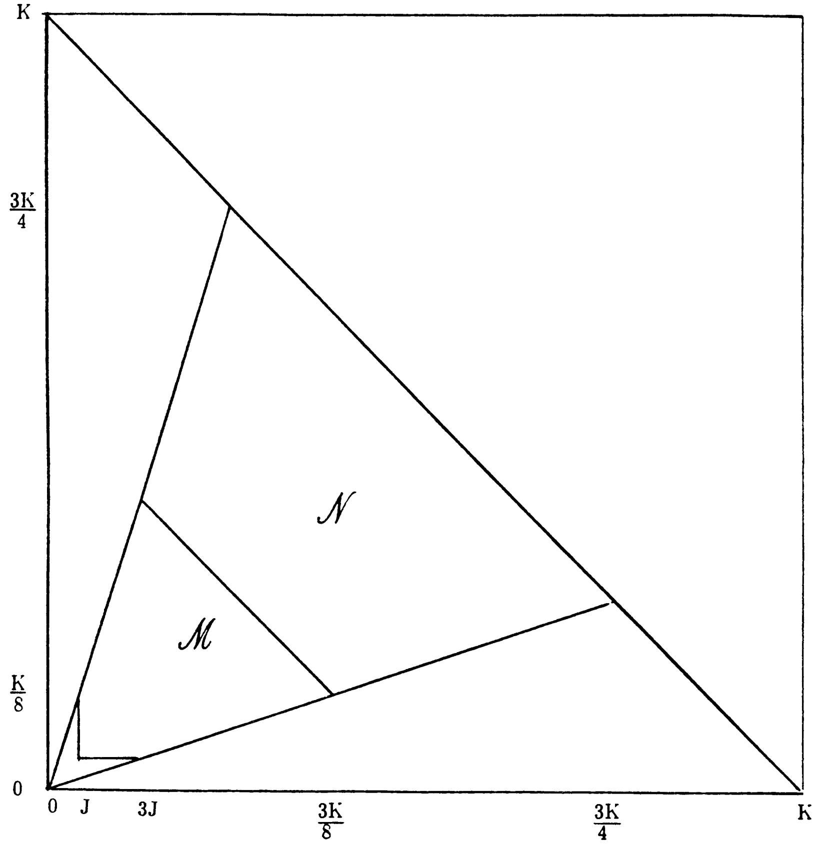

FIG. 1.

3.8 LEMMA. *There are constants δ, $\lambda > 0$ so that with probability at least $1 - e^{-\delta K}$, the fire does not reach $\mathscr{N}$ before time λK.*

The final step in the proof of $(*)$ is to use a block construction from [4] to show there is a constant $\rho > 0$ so that for $J \geq J_1$ and $K \geq 8J$, the probability that the fire burns at least a fraction ρ of $\mathscr{N}$ is at least ρ. In our construction, we only use sites that burn for at least u units of time and bonds with passage times less than or equal to U so all the burning gets accomplished by time ΛK.

KEEPING THE FIRE BURNING. To keep the fire burning in B for $2\Gamma L$ units of time (i.e., to have at least one burning tree at all times), we have to deal with two extreme cases: (i) There are a large number of meadows spread evenly throughout the square $(-L, L)^2$ so the largest meadow is small and hence a nice fire does not last very long; (ii) the number of meadows is small so *failure* (i.e., no meadow has a successful burn) is likely. If we define a large number of meadows to be at least $A \log L$ and A is large, then case (i) is not a problem because each nice fire uses up $\lambda L^{0.7}$ units of time and the probability of a *success* (i.e., a nice fire in some meadow) is so close to 1 that if we always found ourselves in case (i), then with high probability, $2\Gamma L$ units of time would elapse before the first failure.

Case (ii) is more complicated. Ignoring a technicality that we will confront in the proof, the first step is to show:

4.1 LEMMA. *For any $F < \infty$, if L is large, then with high probability, we can tolerate F failures in a row without having the number of burning trees drop below $L^{1/4}/3$.*

To prove Lemma 4.1, we show that failures almost never last longer than $C \log\log L$, so for large L then with high probability, at least $L^{-1/4F}$ of the trees that were burning originally stay burning after one failure. This estimate uses ideas from percolation theory and relies on the fact that the number of meadows is $< A \log L$.

When there are k meadows, Lemma 2.2 implies that the largest meadow has length greater than or equal to $L/(11k)^{1/2}$ and by definition, a nice fire in this meadow uses $\delta k^{-1/2} L$ units of time. We call a nice fire in the largest meadow a *big success*. Now, if the number of meadows were always equal to k, the expected amount of time used up by big successes before we have a total of F failures (and hence before we experience F failures in a row) is at least

$$F \cdot \delta k^{-1/2} L \cdot \rho(1-\rho)^{-k}. \tag{1.3}$$

To see this, recall that if we have disjoint events A and B with probabilities p_A and p_B, then the expected number of times that A occurs before B occurs is p_A/p_B.

If F is large, the quantity in (1.3) is much larger than $2\Gamma L$ for all values of k. The arguments above only show that the expected value has the desired behavior when the number of meadows is constant, but a second moment computation [see Lemma 4.3] shows that even if the number of meadows varies, the elapsed time is larger than $2\Gamma L$ with high probability.

INVADING THE NEIGHBORS. At this point we have shown (modulo a few details) that if there are at least $M = L^{1/2}$ burning trees in $B_0 = (-L, L)^2$ at some time in $[0, \Gamma L]$, then with high probability we can keep the fire burning until time $2\Gamma L$. Our last chore is to show that we can get M burning trees in $B_1 = (0, 2L) \times (-L, L)$ at some time in $[\Gamma L, 2\Gamma L]$. If we have a burning tree in B_1, then the construction used to burn a meadow gives a probability greater than or equal to ρ of producing M burning trees in B_1. If we have no burning trees in B_1, then the square is itself a large meadow, and a simple modification of the construction gives us a positive probability to bring a large enough fire from the rightmost burning tree in B_0.

The paper is organized as follows: In Section 2, we prove the covering lemma is 2.2. Sections 3–5 give the details for the paragraphs labelled burning a meadow, keeping the fire burning and invading the neighbors. Finally, an Appendix contains the facts concerning oriented percolation we use and indicates how they can be obtained from results in [5].

2. A covering lemma. The first step in proving Lemma 1.1 is to show that if there are only $M = L^{1/2}$ burning trees in $(-L, L)^2$, then we can cover a region of area at least $L^2/11$ by reasonably large disjoint squares that contain no burning trees and have a burning tree in at least one corner. From a covering lemma in [15] (page 137):

2.1 LEMMA. *If we have a finite collection of squares, then we can find a subcollection of disjoint squares that covers at least $\frac{1}{9}$ of the area.*

SKETCH OF PROOF OF LEMMA 2.1. This lemma is due to Besicovitch [3] and is easy to prove. We start by choosing the biggest square and discarding all squares that have a nonempty intersection with the biggest square. Among the squares that are left, we choose the biggest one and discard all squares that have a nonempty intersection with it. Continuing in this way gives us a subcollection that has the following property: If we make all the squares in the subcollection three times as large, their union contains all the squares (otherwise we get a contradiction to the way things are chosen) and hence the subcollection covers at least $\frac{1}{9}$ of the area.

2.2 LEMMA. *If there are at most M burning trees, then we can cover an area of size at least $L^2/11$ in B by disjoint squares with three properties*: (i) *they contain no burning trees in their interiors*, (ii) *they have a burning tree in at least one corner and* (iii) *their sides are longer than $L^{0.7}$*.

PROOF. We apply Lemma 2.1 to the collection that consists of all the squares contained in B that have corners at points of the integer lattice, that do not contain any burning trees in their interior and that have at least one burning tree on their boundary. It is easy to see that each nonburning tree in $(-L, L)^2$ is in the interior of some square in the original collection. This means that the original collection covers an area of at least $(2L - 1)^2 - L^{1/2}$, so for large L, the subcollection covers an area larger than $(2L)^2/10$.

The squares we have chosen have a burning tree on the boundary. If there is a burning tree at a corner, we are happy. If not, then making a cut at a location of a burning tree perpendicular to the side, picking the larger of the two pieces and then making a second cut to make the region square again gives a collection of squares that covers an area larger than $L^2/10$. To fulfill property (iii), we discard all squares with sides shorter than $L^{0.7}$. Since we have at most $L^{1/2}$ burning trees and each burning tree can be on the boundary of at most four squares, the total area of the squares that are smaller than $(L^{0.7})^2$ is at most $4L^{1.9}$. By choosing L large, we can guarantee that the remaining squares cover an area of at least $L^2/11$.

3. Burning a meadow. The squares satisfying (i)–(iii) in Lemma 2.2 give us opportunities for large fires and we call them *meadows*. The goal of this section is to prove $(*)$. To explain why such fires are nice, note that the number of tree-hours of burning is at least $u\rho K^2$, so at some time we must have $(u\rho/\Lambda)K$ burning trees and by (iii) in Lemma 2.2, this is at least $(u\rho/\Lambda)L^{0.7}$. The proof of $(*)$ is divided into three phases, which for ease of later reference we label a, b and c.

(a) We begin the proof of $(*)$ by introducing some notation, constructing our process, and recalling its connection with percolation when $\beta = 0$. For $x, y \in Z^2$ with $\|x - y\| = 1$, let $\{U_n^{x,y},\ n \geq 1\}$, $\{V_n^x,\ n \geq 1\}$ and $\{W_n^x,\ n \geq 1\}$ be the arrival times of independent Poisson processes with rates $\alpha/4$, 1 and β, respectively. As the reader can probably guess from the rates: At time $U_n^{x,y}$, a spark jumps from x to y if x is on fire; at time V_n^x the tree at x goes out if it was burning; at time W_n^x a new tree appears at x if the site was burnt. A result of Harris [11] implies that we can use this graphical representation to construct the process starting from any initial configuration. One of the reasons for using this construction is the following fact, which we ask the reader to check as he goes along: Given the initial state of the meadow, occurrence of a nice fire is determined by events in the Poisson processes of the points in the meadow, so for disjoint meadows, these events are independent.

Suppose now that $\beta = 0$ and we start with a burning tree at 0 in an otherwise virgin forest. Let t_x be the first time the tree at x catches fire. If $t_x < \infty$, let T_x [resp. $e(x, y)$] be the time lag until the first arrival in V_n^x (resp., $U_n^{x,y}$) after time t_x. If $t_x = \infty$, generate T_x and $e(x, y)$ by using independent r.v.'s with the appropriate distributions. It is easy to see that $\{T_x,\ x \in Z^2\}$ and $\{e(x, y)\colon\ x, y \in Z^2,\ \|x - y\| = 1\}$ are independent r.v.'s with $P(T_x > t) = e^{-t}$ and $P(e(x, y) > t) = e^{-\alpha t/4}$. T_x is the amount of time the tree at x will burn if it ever catches fire. $e(x, y)$ is the time lag from when the tree at x catches fire until it first tries to send a spark to y. Let

$$\tau(x, y) = \begin{cases} e(x, y), & \text{if } T_x > e(x, y), \\ \infty, & \text{if } T_x \leq e(x, y), \end{cases}$$

and say the oriented bond (x, y) is *open* if $\tau(x, y) < \infty$. Let C_0 be the set of points that can be reached from 0 by a path of open bonds. In [4] it was shown that C_0 coincides with the set of sites that will ever burn when we start with a tree on fire at 0 and all other sites occupied by trees and that the passage time to x in the corresponding first passage percolation process is t_x.

To treat the system with $\beta > 0$, we will consider a bond-site percolation problem in which the sites are open with probability $1 - \exp(-\beta S)$. This corresponds to a fire in a forest that was initially all burnt but has regrown for S units of time. To get control over the times at which the burning occurs, we will further modify the percolation process so that we only use x with $T_x > u$ and bonds (x, y) with $e(x, y) \leq U$. In the modified system, a path is said to be open if all its bonds and sites are. Let $\overline{C}_0$ be the set of points that can be reached by an open path and let $\overline{R}(2m, m)$ be the probability that there is an open path crossing $(0, 2m) \times (0, m)$ from left to right. It is easy to use results in Section 2 of [4] to show:

3.1 LEMMA. *If $\alpha > \alpha_c(0)$, then we can pick H, u, U and S so that $P(\overline{R}(2H, H)) > 0.98$ and hence $P(|\overline{C}_0| = \infty) > 0$.*

PROOF. Let $R(K, L)$ [resp., $\overline{R}(K, L)$] be the probability that there is a left to right crossing of $(0, K) \times (0, L)$ in the bond percolation (resp., modified bond-site percolation) model defined above. [4] has shown

(3.2) If $\alpha > \alpha_c(0)$, then $R(2m, m) \to 1$ as $m \to \infty$.

(3.3) If $R(2m, m) = 1 - \lambda/49$ for some $\lambda < 1$, then

$$R(2^k m, 2^{k-1} m) \geq 1 - \tfrac{1}{49} \exp(2^{k-1} \log \lambda).$$

The last result is proved by showing

$$R(4m, m) \geq 1 - 7\{1 - R(2m, m)\} \tag{3.4}$$

(see Figure 3 of [4]) and then observing that crossings of disjoint rectangles are independent so

$$R(4m, 2m) \geq 1 - \{1 - R(4m, m)\}^2, \tag{3.5}$$

(3.4) and (3.5) generalize immediately to the modified bond-site model. Once they hold, (3.3) follows by simple algebra (see [4], page 182) so that the result holds for the modified bond-site model as well.

(3.3) shows that if we can find S and H so that $\overline{R}(2H, H) \geq 0.98$, then $\overline{R}(2^k H, 2^{k-1} H) \to 1$ exponentially fast and the conclusion $P(|\overline{C}_0| = \infty) > 0$ follows from the construction drawn in Figure 4 of [4]. To complete the proof of Lemma 3.1, we observe that (3.2) implies we can pick H so that $R(2H, H) \geq 0.99$, so if we pick S, u and U so that

$$(2L + 1)^2 (e^{-\beta S} + (1 - e^{-u}) + 4e^{-\alpha U/4}) < 0.01,$$

it follows that $\overline{R}(2H, H) \geq 0.98$ and we have proved Lemma 3.1. □

(b) The estimate in Lemma 3.1 will give us good control over a fire in a forest that has regrown for S units of time. The next step is to show that the fire will not burn too much of the meadow during the S units of time that we wait for the forest to regrow. Suppose the meadow is $[0, K] \times [0, K]$ with a burning tree at the origin. Let $\mathscr{M}$ be the five-sided region formed by connecting the following points in the order indicated (J, J), $(3J, J)$, $(3K/4, K/4)$, $(K/4, 3K/4)$, $(J, 3J)$, (J, J) (see Figure 1 in the Introduction for a picture).

3.6 LEMMA. *If $J \geq J_1$ and $K \geq 8J$, the probability that the fire reaches $\mathscr{M}$ by time S is smaller than $\frac{1}{2}$, independent of the number of burning trees on the boundary or the initial state of the meadow (assuming it contains no burning trees).*

PROOF. It suffices to consider the situation in which all the trees on the boundary are on fire, there is a virgin forest in the interior and the model is modified so that trees stay burning forever. The key to the proofs is a large deviations estimate for the associated first passage percolation process. To formulate the estimate, we need several definitions. Let $e(x,y)$ be i.i.d. with $P(e(x,y) > t) = e^{-\alpha t/4}$. A sequence $x_0 = x, x_1, \dots, x_k = y$ is said to be a *path* from x to y if for $1 \le j \le k$, x_j is a neighbor of x_{j-1}. A path is said to be *self-avoiding* if it does not visit the same site twice. The *travel time* of a path is defined to be $e(x_0, x_1) + \cdots + e(x_{k-1}, x_k)$. The *passage time* from x to y is defined to be the infimum of the travel times over all paths from x to y. In computing the infimum, we can obviously restrict our attention to self-avoiding paths. Finally, let $F_{\delta,k} =$ {there is a self-avoiding path of length k with travel time $\le \delta k$}.

3.7 LEMMA. *If $\delta > 0$ is small, $P(F_{\delta,k}) \le (\frac{3}{4})^{k-1}$.*

PROOF. Let $X_1, \dots, X_k$ be i.i.d. with $P(X_j > t) = e^{-\alpha t/4}$ and let $S_k = X_1 + \cdots + X_k$.

$$e^{-\alpha\delta k} P(S_k \le \delta k) \le E e^{-\alpha S_k} = \left(\tfrac{1}{5}\right)^k.$$

Pick δ so that $e^{\alpha\delta}/5 < \frac{1}{4}$ and hence $P(S_k \le \delta k) \le (\frac{1}{4})^k$. The number of self-avoiding paths of length k is at most $4 \cdot 3^{k-1}$, so $P(F_{\delta,k}) \le (\frac{3}{4})^{k-1}$.

To prove Lemma 3.6 now, pick J_0 so that $J_0\delta \ge S$. By considering the points on the boundary of $\mathscr{M}$ in turn, it is easy to see that for $J \ge J_0$, the probability that the fire reaches $\mathscr{M}$ by time S is at most

$$(2J+1)\left(\tfrac{3}{4}\right)^{J-1} + 6 \sum_{j=J+1}^{K/4} \left(\tfrac{3}{4}\right)^{j-1} + (K/2)\left(\tfrac{3}{4}\right)^{K/4-1}.$$

The 6 comes from the fact that the segment $(3J, J), (3K/4, K/4)$ has slope $\frac{1}{3}$, so there are three boundary points at each height between $J+1$ and $K/4$. If we pick $J_1 > J_0$, then the last quantity is at most $\frac{1}{2}$ whenever $J \ge J_1$ and $K \ge 8J$ and the proof of Lemma 3.6 is complete. □

Lemma 3.6 implies that with probability at least $\frac{1}{2}$, the fire does not reach $\mathscr{M}$ by time S. The last step in preparing to burn the meadow is to make a path for the fire to enter the meadow. Consider the path $(1,0)$, $(1,1)$, $(2,1)$, $(2,2), \dots, (2J, 2J)$ and let E_0 be the event that:

(i) no site in this path is attacked by a spark by time S,
(ii) all sites in the path in the interior of the square are regrown by time S,
(iii) the fire at $(0,0)$ does not burn out by time S,
(iv) if there is a fire at $(1,0)$, it does not go out by time S,
(v) if there is no fire at $(1,0)$, the site is regrown by time S.

$P(E_0) = \delta(J) > 0$. Let E_1 denote the event that the fire does not reach $\mathscr{M}$ by time S. E_1 is a decreasing event concerning the independent variables $e(x,y)$, so it is independent of (ii)–(v) and Harris' inequality (see [6], page 129) implies it is positively correlated with (i). The last two observations imply $P(E_0 \cap E_1) \geq \delta(J)/2 > 0$.

On $E_0 \cap E_1$, we have a burning tree at the origin that is connected by a fuse to the region $\mathscr{M}$ which has regrown undisturbed for S units of time. Having reached this stage, we ignore the further regrowth of trees so that the set of trees that will burn is larger than the cluster containing 0 in the obvious associated bond-site percolation model. Sites in $\mathscr{M}$ that regrew before time S are open; and we define T_x and $e(x, y)$ for sites $x \in \mathscr{M}$ by looking at the Poisson processes after the fire first reaches x. To have a lower bound on the time the fire starts, we will concentrate on burning $\mathscr{N}$, the trapezoid with vertices at $(3K/8, K/8)$, $(3K/4, K/4)$, $(K/4, 3K/4)$, $(K/8, 3K/8)$ (again see Figure 1). While the proof of Lemma 3.6 is fresh in the reader's mind, we will prove:

3.8 Lemma. *There are constants $\delta, \lambda > 0$ so that with probability at least $1 - e^{-\delta K}$, the fire does not reach $\mathscr{N}$ before time λK.*

Proof. Projecting $\mathscr{N}$ onto the x-axis shows that it has at most $2K$ boundary points. All boundary points are at least $K/8$ units from the boundary, so Lemma 3.7 implies that the probability in question is bounded by $2K(\frac{3}{4})^{K/8-1}$.

(c) We have come now to the key to the proof of $(*)$: Using sponge crossing constructions that generalize the ones in Section 2 of [4], we will show that the fire in $\mathscr{M}$ dominates oriented percolation on $\mathscr{L} = \{(m, n) \in Z^2 \colon m + n \text{ is even}\}$ with parameter p close enough to 1. Let $\phi(m, n) = (2.5n + 0.5m + 1, 2.5n - 0.5m + 1)J$ map $\mathscr{L}$ into R^2. We say that $(0, 0)$ is *open* if (a) there are bottom to top crossings of

$$(J, 2J) \times (J, 3J), \qquad (2J, 3J) \times (J, 4J), \qquad (3J, 4J) \times (2J, 5J)$$

and (b) there are left to right crossings of

$$(J, 3J) \times (J, 2J), \qquad (J, 4J) \times (2J, 3J), \qquad (2J, 5J) \times (3J, 4J)$$

(see Figure 2). We call the crossings in (a), V_1, V_2 and V_3, and those in (b), H_1, H_2 and H_3. We say that $(m, n) \in \mathscr{L}$ is *open* if V_1, V_2, V_3, H_1, H_2 and H_3 occur in the system translated by $-(2.5n + 0.5m,\ 2.5n - 0.5m)J$. Let $Q(m, n) = \phi(m, n) + [0, J]^2$. Note that $\phi(0, 0) = (1, 1)$ and $\phi(1, 1) = (4, 3)$ so the line through these points has slope $\frac{2}{3}$. The rectangles in the definition of "$(0, 0)$ is open" fit in the meadow $\mathscr{M}$, so this is true for their translates by $\phi(m, n)$ with $(m, n) \in \mathscr{L}$ as long as $\phi(m, n) \cdot (1, 1) \leq K - 7J$.

The events just described are designed so that if $(0, 0)$ and $(-1, 1)$ are open, then there will be paths from $Q(0, 0)$ to $Q(-2, 2)$ and to $Q(0, 2)$. [To reach $Q(0, 2)$ from $Q(-1, 1)$, we use the translates of H_1, V_2, H_3; to reach $Q(-2, 2)$ we use translates of H_1, V_2, H_2, V_3. We start each sequence with a translate of H_1, since that path must intersect V_3.] With the last comparison in mind, we

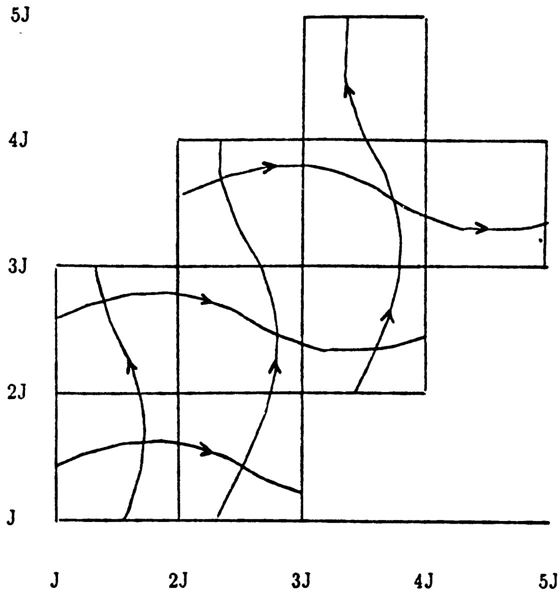

FIG. 2.

say that y can be reached from x (and write $x \to y$) if there is a sequence of open sites $x_0 = x, x_1, \dots, x_k = y$ so that for $1 \le j \le k$, we have $x_j - x_{j-1} \in \{(1,1),(-1,1)\}$. Let $\mathscr{C}_0 = \{x\colon (0,0) \to x\}$ and $\Omega_\infty = \{|\mathscr{C}_0| = \infty\}$. The events $G_{m,n} = \{(m,n) \text{ is open}\}$ are *one-dependent*, that is, if $(m_1, n_1), \dots, (m_k, n_k)$ have $(|m_i - m_j| + |n_i - n_j|)/2 > 1$ whenever $i \neq j$, then $G_{m_1,n_1}, \dots, G_{m_k,n_k}$ are independent, so it follows from (A.2) and (A.10) that:

3.9 LEMMA. *If $P(G_{m,n}) > 1 - 6^{-36}$, then $P(\Omega_\infty) \ge \frac{1}{2}$ and furthermore on Ω_∞, we have*

$$\liminf_{n \to \infty} \frac{1}{N} \left| \mathscr{C}_0 \cap \{(m,n)\colon m \le N\} \right| \ge \frac{1}{6}.$$

Combining the choices of S and H with (3.2) and (3.3) shows that if we let $J = 2^k H$ and pick k large enough, then $J \ge J_1$ and $\overline{R}(4J, J) \ge 1 - 6^{-37}$, so $P(G_{m,n}) > 1 - 6^{-36}$.

The last observation, as we will now explain, completes the proof of $(*)$. It is clear that the events we desire have positive probability and burn a positive fraction of $\mathscr{N}$. We only use trees that burn for u units of time so that condition is fulfilled. Lemma 3.8 implies that with high probability the fire does not reach $\mathscr{N}$ by time λK. To check that burning occurs by time ΛK, note that since we only use bonds with passage time at most U and each crossing of $(0, 3J) \times (0, J)$ contains at most $6J^2$ bonds, there is an absolute upper bound on the time that elapses in each crossing used in the construction.

4. Keeping the fire burning. The goal of this section is to show that if $\Gamma < \infty$ and we start with at least $M = L^{1/2}$ burning trees in $B = (-L, L)^2$, then for large L, we can with high probability *keep the fire burning*, that is, have at least one burning tree in B at all times $t \in [0, 2\Gamma L]$. To keep the fire burning, we will do nothing as long as the number of burning trees is greater than M. When the number of burning trees is less than or equal to M, Lemma 2.2 implies that a positive fraction of B can be covered by meadows and $(*)$ in Section 3 implies that there is a probability of at least ρ of having a nice fire in each one. As explained at the beginning of Section 3, if we get at least one nice fire, we will end up with $\delta L^{0.7}$ burning trees, which is larger than M if L is large.

When we try to find a nice fire, there are two things to worry about: (i) The remaining burning trees are spread evenly throughout the square $(-L, L)^2$, so the largest meadow and hence the time that elapses when it burns is small; (ii) all the burning trees are in one clump so the number of meadows is small and it is likely there will be no nice fire. To cope with these conflicting fears, we will consider two cases depending on the number of meadows. Before turning to the details, we would like to observe that by separating our trials into two piles, it suffices to prove that we can keep the fire burning when we assume the system is always in Case 1 or Case 2.

CASE 1. Suppose that the number of meadows is greater than or equal to $A \log L$, where $A = -1/\log(1 - \rho)$. The probability of *failure* (i.e., no meadow has a nice fire) is at most $(1 - \rho)^{A \log L} = L^{-1}$, so the probability of a failure in $L^{0.5}$ trials is small. Since each meadow has length $K \geq L^{0.7}$, each nice fire takes at least $\lambda L^{0.7}$ units of time and with high probability, we can keep the fire burning for at least $\lambda L^{1.2}$ units of time. For any $\Gamma < \infty$, $\lambda L^{1.2} > \Gamma L$ for large L and the proof for this case is complete.

CASE 2. The number of meadows is $< A \log L$. Unfortunately, this case is much more complicated. The good news is that Lemma 2.2 implies that the biggest meadow has length $\geq L/(11A \log L)^{1/2}$ and a nice fire in the biggest meadow uses up at least $\gamma L(\log L)^{-1/2}$ units of time where $\gamma = \lambda/(11A)^{1/2}$. We call a nice fire in the biggest meadow a *big success* and use these events to use up $2\Gamma L$ units of time.

Now for the bad news. We can expect to have several failures (i.e., no nice fire in any meadow) before the big successes use up $2\Gamma L$ units of time, so we have to make sure there are a reasonable number of burning trees after a failure. To do this, it is useful to introduce a notion weaker than a nice fire so that the duration of a bad outcome can be estimated using percolation theory. We say that a *successful burn* occurs if the events E_0 and E_1 in the regrowth phase of the construction (i.e., part (b) of Section 3) occur and there is percolation on $\mathscr{L}$ in part (c) of Section 3. The events in the regrowth phase are known after S units of time so we only have to estimate the duration of an unsuccessful attempt at percolation. To do this, we begin by observing that (a) after $(4\Gamma/\rho\gamma)(\log L)^{1/2}$ trials it is likely that the big successes have used up $2\Gamma L$ units of time and (b) there are at most $A \log L$ meadows in each trial, so we are concerned with at most $C(\log L)^{3/2}$ percolation trials. We say that a *bad outcome* occurs if there is an unsuccessful burn in each meadow. Using a standard exponential estimate from percolation leads to:

4.1 LEMMA. *For any $F < \infty$, if L is large, then with high probability, the number of burning trees will not drop below $L^{1/4}/3$ before we see* (i) *F bad outcomes in a row or* (ii) $(4\Gamma/\rho\gamma)(\log L)^{1/2}$ *trials.*

PROOF. (A.2) implies that if sites are open with probability $p > 1 - 6^{-36}$, then the probability that $\mathscr{C}_0$ reaches $\{(m,n)\colon\ -n \le m \le n\}$ but does not percolate is at most 2^{-2n-1}. Since there is an upper bound on the passage times of the bonds used in the construction, this implies that the probability an unsuccessful burn lasts for t units of time is at most $Ce^{-\delta t}$. This means that if L is large, the longest failure in our at most $C(\log L)^{3/2}$ attempts will, for large L, take less than $(2/\delta)\log\log L$ units of time with high probability. The probability a burning tree does not go out in $(2/\delta)\log\log L$ units of time is thus at least $L^{-1/4F}$ for large L. Since we start with $M = L^{1/2}$ trees, the last estimate implies that even after F consecutive unsuccessful burns, the expected number of trees that remain is at least $L^{1/4}$. To see that the number of survivors is larger than $L^{1/4}/3$ with high probability, let $X_1, \dots, X_M \in \{0, 1\}$ be i.i.d. with $P(X_i = 1) = L^{-1/4}$ and let $W = X_1 + \cdots + X_M$. Now

$$Ee^{-W} = \left\{1 + L^{-1/4}(e^{-1} - 1)\right\}^M \le \exp\left(-L^{1/4}/2\right),$$

since $(1 + x) \le e^x$ and $e > 2$. So

$$P\left(W \le L^{1/4}/3\right) \le e^{L^{1/4}/3} Ee^{-W} \le e^{-L^{1/4}/6},$$

which completes the proof of Lemma 4.1. □

In the proof of the last result, we have emphasized the distinction between an unsuccessful burn and a nice fire. The next result implies that we can ignore the difference.

4.2 LEMMA. *In a meadow of side K, P (nice fire|successful burn) $\ge$* $1 - CK^{-2}$.

REMARK. Since we will inspect at most $C(\log L)^{3/2}$ meadows in the first $(4\Gamma/\rho\gamma)(\log L)^{1/2}$ trials and all the meadows have $K \ge L^{0.7}$, it follows that with high probability all successful burns will be nice fires.

PROOF. δK generations of the percolation process fit inside of $\mathscr{M}$. Using (A.9) and summing from $n = \delta K/2$ to δK shows that when percolation occurs, the probability of burning fewer than ρK^2 trees in $\mathscr{N}$ is at most CK^{-2}. Lemma 3.8 implies that the probability that the fire reaches $\mathscr{N}$ by time λK is at most $\exp(-\delta K)$. Combining the last two estimates and recalling that we only use trees that burn for u units of time and there is an absolute upper bound on each crossing used in the construction proves Lemma 4.2.

The next step is to show that if F is large, then the total time used up by big successes will, with high probability, reach $2\Gamma L$ before we experience a total of F failures, which in turn occurs before we have F consecutive bad outcomes. The key to this is the following.

4.3 LEMMA. *Let $K_m \geq 1$ and X_m be sequences adapted to an increasing sequence of σ-fields $\mathscr{F}_m$ and suppose*

$$P\big(X_m = \theta K_m^{-1/2} | \mathscr{F}_{m-1}\big) = \varepsilon_m,$$
$$P\big(X_m = 0 | \mathscr{F}_{m-1}\big) = 1 - \varepsilon_m,$$

where $\varepsilon_m = \rho a K_m^{1/2}(1-\rho)^{K_m}$ and a is chosen so that $ax^{1/2}(1-\rho)^x \leq 1$ for all $x \geq 0$. Let $S_n = X_1 + \cdots + X_n$ and $N = \inf\{n: \sum_{m=1}^{n}(1-\rho)^{K_m} \geq B^{1/2}\}$. Then

$$P\big(S_N \leq 2^{-1}\theta\rho a B^{1/2}\big) \leq 4(B^{1/2}+1)/(\rho a B).$$

REMARK. Here K_m is the number of meadows on the mth attempt and $X_m L$ is a lower bound for the duration of a nice fire in the largest meadow. To make the computation simple, we have chosen to lower the success probability by a factor of $ak^{1/2}(1-\rho)^k$, which is less than or equal to 1 by the choice of a.

PROOF. Let $\mu_m = E(X_m | \mathscr{F}_{m-1}) = \theta\rho a(1-\rho)^{K_m}$, $Y_m = X_m - \mu_m$ and $T_n = Y_1 + \cdots + Y_n$. T_n is a martingale w.r.t. $\mathscr{F}_n$. Let

$$\sigma_m^2 = \operatorname{var}\big(X_m | \mathscr{F}_{m-1}\big) \leq E\big(X_m^2 | \mathscr{F}_{m-1}\big) \leq \theta^2\rho a(1-\rho)^{K_m}$$

(since $K_m \geq 1$) and let $v_n = \sigma_1^2 + \cdots + \sigma_n^2$. $T_n^2 - v_n$ is a martingale w.r.t. $\mathscr{F}_n$. Using the optional stopping theorem at time $N \wedge n$ gives $ET_{N\wedge n}^2 = Ev_{N\wedge n}$. Letting $n \to \infty$, using Fatou's lemma on the left and the monotone convergence theorem on the right gives $ET_N^2 \leq Ev_N$. From the definition of N, it follows that

$$B^{1/2} \leq \sum_{m=1}^{N} (1-\rho)^{K_m} \leq B^{1/2} + 1.$$

This implies $\mu_1 + \cdots + \mu_N \geq \theta\rho a B^{1/2}$ and $v_N \leq \theta^2\rho a(B^{1/2}+1)$. Combining the last two observations and recalling $ET_N^2 \leq Ev_N$ gives

$$P\big(S_N \leq 2^{-1}\theta\rho a B^{1/2}\big) \leq ET_N^2/\big(2^{-1}\theta\rho a B^{1/2}\big)^2 \leq 4(B^{1/2}+1)/(\rho a B),$$

completing the proof of Lemma 4.3. □

If we take F large, then Lemma 4.3 implies $P(S_N \leq 2\Gamma) \leq \varepsilon$. To see that with high probability we do not have F failures by time N, let $F_m =$ "a failure occurs on the mth attempt." Now $P(F_m|\mathscr{F}_{m-1}) \le (1-\rho)^{K_m}$, so taking expected values shows that the expected number of failures by the Nth attempt is at most $F^{1/2}+1$ and Chebyshev's inequality implies that, for large F, the probability of F failures by time N is small.

5. Invading the neighbors. At this point, we have shown that if there are at least $M = L^{1/2}$ burning trees in $B_0 = (-L, L)^2$ at some time in $[0, \Gamma L]$, then with high probability we can keep the fire burning until time $2\Gamma L$. Our last chore is to show that we can get M burning trees in $B_1 = (0, 2L) \times (-L, L)$ at some time in $[\Gamma L, 2\Gamma L]$. If we have a burning tree in B_1, then the construction used in Section 3 gives a probability greater than or equal to ρ of producing M burning trees in B_1. If we have no burning trees in B_1, then the rightmost burning tree in B_0 is at a point (x, y) with $x \le 0$. Choosing the square $(x,y) + [0, L]^2$ when $y \le 0$ and $(x, y) + [0, L] \times [0, -L]$ when $y > 0$ gives a large meadow with no burning trees. The construction used in Section 3 gives us a positive probability of burning a positive fraction of this square and when this occurs the rightmost burning tree has x-coordinate $\bar{x} \ge x + 0.51L$ with high probability [see (A.3)–(A.5)]. Thus at most two successes will create a burning tree in B_1, which gives us probability ρ of getting M burning trees. Each cycle of the procedure above has probability at least ρ^3 of creating M burning trees in B_1 and uses up less than $4\Lambda L$ units of time [by $(*)$ since in the first two steps we have $K = L$ and in the third we have $K \le 2L$]. From this it is clear that if Γ is a large multiple of $4\Lambda/\rho^3$, we will get M burning trees in B_1 with high probability. The proof of Lemma 1.1 and hence of our theorem is complete.

APPENDIX

One-dependent oriented percolation. Let $W_n^A = \{m\colon (k,0) \to (m,n)$ for some $k \in A\}$ and $\tau^A = \inf\{n\colon W_n^A = \varnothing\}$. We will write W_n^0 and τ^0 when $A = \{0\}$. The first thing to be proved about oriented percolation is that if p is close to 1, then $P(\tau^0 = \infty) > 0$. This is done in Section 10 of [5] by a contour argument which has the following consequences. Here and throughout this section we assume $p \ge 1 - 6^{-36}$.

$$P(\tau^{[-N,N]} < \infty) \le 2^{-2N-1} \quad \text{for even } N \ge 0, \tag{A.1}$$

$$P(N \le \tau^0 < \infty) \le 2^{-2N-1} \quad \text{for } N \ge 0. \tag{A.2}$$

To get the second result, we observe that if the process survives up to time N and then dies out, the contour must have length at least $2N + 2$.

Let $r_n^A = \sup W_n^A$, $l_n^A = \inf W_n^A$. We will write r_n^- and so on when $A = \{0, -2, -4, \dots\}$. Results from Section 3 in [5] imply that on $\{W_n^0 \ne \varnothing\}$,

$$W_n^0 = W_n^{2Z} \cap \left[l_n^0, r_n^0\right], \tag{A.3}$$

$$W_n^0 = W_n^- \cap \left[l_n^0, \infty\right) \quad \text{and hence } r_n^0 = r_n^-. \tag{A.4}$$

Our next result show that $r_n^0 - l_n^0$ grows linearly on $\{\tau^0 = \infty\}$. Noticing that $6^{2/3} > 3.3$ and then using the bound from [5] (at the bottom of page 1030) gives

$$P(r_n^- \le n/3) \le 2\left(\tfrac{10}{11}\right)^n. \tag{A.5}$$

To show that $|W_n^0|$ grows linearly when $\tau_0 = \infty$, we introduce the dual percolation process. We say that y can be reached from x by a dual path (and write $x \overset{*}{\to} y$) if there is a sequence of open sites $x_0 = x, x_1, \dots, x_k = y$ so that for $1 \le j \le k$, we have $x_{j-1} - x_j \in \{(1,1), (-1,1)\}$. Let $\hat{W}_s^{(m,n)} = \{k\colon (m,n) \overset{*}{\to} (k, n-s)\}$. It follows immediately from the definition that

$$\{m \in W_n^{2Z}\} = \{\hat{W}_n^{(m,n)} \ne \varnothing\}. \tag{A.6}$$

The last observation and (A.1) imply

$$P\left(W_n^{2Z} \cap [-N, N] = \varnothing\right) \le 2^{-2N-1} \tag{A.7}$$

which proves the claim we made in (1.2).

Our last goal is to prove the second claim in (3.6). The first step is to observe that using (A.2) for the dual process implies that if $a = 2/\log 2$, then

$$P\left(\hat{W}_{a \log n}^{(m,n)} \ne \varnothing, \hat{W}_n^{(m,n)} = \varnothing\right) \le 2^{-1} n^{-4}. \tag{A.8}$$

The variables $\hat{W}_{a \log n}^{(m,n)}$ are independent when $|m_1 - m_2| > 2a \log n$. Let

$$X_m = 1_{(\hat{W}_{a \log n}^{(m,n)} \ne \varnothing)} - P\left(\hat{W}_{a \log n}^{(m,n)} \ne \varnothing\right).$$

Dividing the integers into groups $I_{n,i} = \{k(2a \log n) + i, k \in Z\}$, computing the eighth moment of the sum of X_m over $[-n/3, n/3] \cap I_{n,i}$, using Chebyshev's inequality and then combining the estimates shows

$$P\left(\sum_{m=-n/3}^{n/3} X_m \le n/6 \right) \le (C \log n)(n/\log n)^{-4}.$$

Combining the last result with (A.8) and (A.6) gives

$$P\left(\left|W_n^{2Z} \cap [-n/3, n/3]\right| \le n/6\right) \le Cn^{-3},$$

then using (A.3) and (A.5) gives

$$P\left(0 < |W_n^0| \le n/6\right) \le Cn^{-3}. \tag{A.9}$$

Using the Borel–Cantelli lemma now gives

$$\liminf_{n \to \infty} |W_n^0|/n \ge \tfrac{1}{6} \quad \text{a.s. on } \tau^0 = \infty. \tag{A.10}$$

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