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IMPLICIT RENEWAL THEORY AND TAILS OF SOLUTIONS OF RANDOM EQUATIONS[1]

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For the solutions of certain random equations, or equivalently the stationary solutions of certain random recurrences, the distribution tails are evaluated by renewal-theoretic methods. Six such equations, including one arising in queueing theory, are studied in detail. Implications in extreme-value theory are discussed by way of an illustration from economics.

1. Introduction. Standard renewal theory can be seen as the study of the asymptotics of a function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies a renewal equation

$$f - f * \mu = g$$

in which μ is a known probability measure and g a known function. By "implicit renewal theory" is meant a variant in which g is not known and indeed is an integral involving f itself. Renewal-theoretic methods can still give conclusions about f in those circumstances. In what follows we establish a specific implicit renewal theorem in order to study certain random equations, including the following examples which will be discussed in detail:

$$R =_L Q + MR, \qquad R \text{ independent of } (M, Q) \tag{1.1}$$

($=_L$ denoting equality of probability laws);

$$R =_L \max(Q, MR), \qquad R \text{ independent of } (M, Q); \tag{1.2}$$

$$R =_L Q + M\max(L, R), \qquad R \text{ independent of } (L, M, Q); \tag{1.3}$$

$$R =_L Q \curlyvee MR, \qquad R \text{ independent of } (M, Q), \tag{1.4}$$

where

$$a \curlyvee b := \begin{cases} a, & \text{if } |a| > |b|, \\ b, & \text{otherwise;} \end{cases} \tag{1.5}$$

$$R =_L [Q + MR], \qquad R \text{ independent of } (M, Q) \tag{1.6}$$

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([ ] being integer part);

(1.7) $R =_L \sqrt{MR^2 + NR + Q}$, R independent of (M, N, Q).

The first few of these equations are important for applications, while the latter ones are included mainly to demonstrate the breadth of the method. Their pattern is that $\Psi(\cdot)$ is a given random real-valued function on $\mathbb{R}$ and R is a random variable independent of $\Psi(\cdot)$ such that $\Psi(R)$ has the same probability law as R. The random function $\Psi(\cdot)$ is such that $\Psi(t)$, for $|t|$ large, is approximately Mt where M is a random variable on which we shall impose moment conditions. We shall prove that then the tails of the law of R are asymptotic to a power, and under extra conditions evaluate the rate of approach.

The plan of the paper is as follows. Implicit renewal theory is set out in Section 2 and the rate results in Section 3. They are specialized to the above random equations in Sections 4–8, proofs are in Section 9 and the paper ends with Section 10, a brief discussion of the implications for extreme-value theory, with an illustration from economics.

We close this section with a list, to be referred to as needed, of notation and conventions that will apply throughout.

$x^+ := x \vee 0$, $x^- := (-x) \vee 0$, $x \in \mathbb{R}$.

$a \vee b + c := (a \vee b) + c$, $a \vee bc := a \vee (bc)$, etc. Thus $\vee$ binds more tightly than $+$, $-$ but less tightly than $\cdot$, $/$.

$0^a \log 0 := 0$ for $a \geq 0$.

An *arithmetic* probability law on $\mathbb{R}$ is one that is centred lattice: concentrated on $\{n\lambda\colon n \in \mathbb{Z}\}$ for some $\lambda > 0$.

A *spread-out* law is one for which some convolution power has an absolutely continuous component.

$$\|X\|_p := \begin{cases} E|X|^p, & \text{if } 0 < p \leq 1, \\ (E|X|^p)^{1/p}, & \text{if } 1 \leq p < \infty. \end{cases}$$

$*$ denotes one of three sorts of convolution; the arguments determine which sort. For suitable real functions f, g on $\mathbb{R}$,

$$f * g(t) := \int_{\mathbb{R}} f(t-u) g(u)\, du, \qquad t \in \mathbb{R};$$

for measures μ, ν on $\mathscr{B}$, the Borel σ-algebra in $\mathbb{R}$,

$$\mu * \nu(B) := \iint_{\{(x,y)\colon x+y \in B\}} d(\mu \times \nu), \qquad B \in \mathscr{B};$$

for a function f and measure μ,

$$f * \mu(t) := \int_{\mathbb{R}} f(t-u) \mu(du), \qquad t \in \mathbb{R},$$

all integrals assumed absolutely convergent.

$\mu^{(n)}$ denotes nth convolution power of μ; $\mu^{(0)}$ is unit mass at 0, also denoted δ_0.

$$\check{f}(t) := \int_{-\infty}^{t} e^{-(t-u)} f(u)\, du, \qquad t \in \mathbb{R},$$

$$\tilde{\mu}(\theta) := \int_{\mathbb{R}} e^{\theta t}\mu(dt), \qquad \hat{\mu}(\theta) := \int_{\mathbb{R}} e^{i\theta t}\mu(dt), \qquad \hat{f}(\theta) := \int_{\mathbb{R}} e^{i\theta t} f(t)\, dt, \qquad \theta \in \mathbb{C}.$$

2. Implicit renewal theory. Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $\Psi\colon \mathbb{R} \times \Omega \to \mathbb{R}$ be $\mathscr{B} \times \mathscr{A}$-measurable. So Ψ can be considered a random element of $\mathscr{M}(\mathbb{R}, \mathbb{R})$, the space of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$, and each $\Psi(t) := \Psi(t, \cdot\,)$ is a r.v. Suppose R is a r.v. independent of Ψ and such that the composite r.v. $\Psi \circ R$, that is, the map $\Omega \to \mathbb{R}$, $\omega \mapsto \Psi(R(\omega), \omega)$, has the same probability law as R. So R is to satisfy the random equation

$$R =_L \Psi \circ R, \qquad \Psi \text{ and } R \text{ independent}. \tag{2.1}$$

The classical random equation $S =_L (X + S)^+$ of random walk and queueing theory leads to exponential decay of the tail of its solution under suitable circumstances, and this model is included in our framework by making an exponential transformation of (1.2) (see Section 5). The only other equation for which power-law tail decay has previously been proved appears to be the random difference equation of Kesten (1973), of which the one-dimensional case is (1.1). We discuss it in Section 4.

Questions of existence and uniqueness of R need not delay us in the general setting as they have been discussed elsewhere: see Borovkov [(1984, Section 4.6], Brandt, Franken and Lisek (1984), Letac (1986) and references therein. For our purposes a result of Letac suffices. On a suitably enlarged probability space we may suppose independent copies $\Psi_1, \Psi_2, \dots$ of Ψ to exist, and may define

$$Z_n(t) := \Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_n(t) = \Psi_1(\Psi_2(\cdots \Psi_n(t) \cdots)), \qquad n = 1, 2, \dots.$$

THEOREM 2.1 (Letac's principle). *Assume Ψ has continuous paths, i.e., for each $\omega \in \Omega$ the map $t \mapsto \Psi(t, \omega)$ is continuous. Suppose $Z := \lim_{n\to\infty} Z_n(t)$ exists a.s. and does not depend on t. Then the law of Z is a law for R satisfying (2.1) and is the unique such law. Also the sequence $W_n(t) := \Psi_n \circ \Psi_{n-1} \circ \cdots \circ \Psi_1(t)$, where $n = 1, 2, \dots,$ has this law as its limit law, whatever the initial t.*

[The formulation of Letac (1986) is in terms of environmental variables: (2.1) is replaced by

$$R =_L f(R, Y), \qquad Y \text{ and } R \text{ independent}, \tag{2.2}$$

where Y is an F-valued r.v. for some measurable space $(F, \mathscr{F})$ and $f\colon \mathbb{R} \times F \to \mathbb{R}$ is a fixed product-measurable function. Our formulation can be put in that way, for one can let F be $\mathscr{M}(\mathbb{R}, \mathbb{R})$ and define $f(t, g) := g(t)$ for $g \in F$, $t \in \mathbb{R}$; thus (2.2) becomes (2.1) on identifying Ψ with Y.]

The last statement of the above result identifies R as having the limit stationary law for the random recurrence relation $W_n = \Psi_n(W_{n-1})$, $W_0 := t$. The sorts of recurrence relations to be considered are those in which the action of Ψ on its argument t, for $|t|$ large, is close to multiplication by a r.v. M that satisfies a "Cramér condition" $E|M|^\kappa = 1$, for some $\kappa > 0$. If the range-space is $\mathbb{R}^+$ we may set $V_n := \log W_n$, $\Phi_n(t) := \log \Psi_n(e^t)$. Then the recurrence relation becomes $V_n = \Phi_n(V_{n-1})$ and we have for large t that $\Phi(t) \doteq L + t$ where $Ee^{\kappa L} = 1$. This formulation is essentially that of the Markov additive processes of Aldous (1989). The content of Aldous (1989), Sections C11 and C33, thus provides a heuristic for some cases of our results.

Conditions and preliminary properties for M are as follows.

LEMMA 2.2. *Let M be a r.v. such that, for some $\kappa > 0$,*

(2.3) $$E|M|^\kappa = 1,$$

(2.4) $$E|M|^\kappa \log^+|M| < \infty$$

and

(2.5) *the conditional law of* $\log|M|$, *given* $M \neq 0$, *is nonarithmetic.*

Then

(2.6) $$-\infty \leq E\log|M| < 0$$

and

(2.7) $$m := E|M|^\kappa \log|M| \in (0,\infty).$$

The following *implicit renewal theorem* says that, if the tails of R satisfy a certain integrability condition involving M, then they are asymptotic to a power.

THEOREM 2.3. *Let M satisfy the conditions of Lemma 2.2 and let R be independent of M.*

CASE 1. *Suppose $M \geq 0$ a.s. If*

(2.8) $$\int_0^\infty |P(R > t) - P(MR > t)| t^{\kappa-1}\, dt < \infty$$

or, respectively,

(2.9) $$\int_0^\infty |P(R < -t) - P(MR < -t)| t^{\kappa-1}\, dt < \infty,$$

then

(2.10) $$P(R > t) \sim C_+ t^{-\kappa}, \qquad t \to \infty,$$

respectively

(2.11) $$P(R < -t) \sim C_- t^{-\kappa}, \qquad t \to \infty,$$

where

$$C_+ = \frac{1}{m}\int_0^\infty (P(R > t) - P(MR > t))t^{\kappa-1}\,dt, \tag{2.12}$$

$$C_- = \frac{1}{m}\int_0^\infty (P(R < -t) - P(MR < -t))t^{\kappa-1}\,dt. \tag{2.13}$$

CASE 2. *Suppose* $P(M < 0) > 0$. *If both* (2.8) *and* (2.9) *are satisfied, then both* (2.10) *and* (2.11) *hold, with*

$$C_+ = C_- = \frac{1}{2m}\int_0^\infty (P(|R| > t) - P(|MR| > t))t^{\kappa-1}\,dt. \tag{2.14}$$

Note that in Case 1 when both conditions are satisfied, and in Case 2,

$$C := C_+ + C_- = \frac{1}{m}\int_0^\infty (P(|R| > t) - P(|MR| > t))t^{\kappa-1}\,dt. \tag{2.15}$$

The theorem has content only when R has infinite absolute moment of order κ. For, if $E|R|^\kappa < \infty$, (2.15) yields $C = (\kappa m)^{-1}(E|R|^\kappa - E|MR|^\kappa)$, which, by the independence of M and R and (2.3), is zero. (2.10) is then to be interpreted as $P(|R| > t) = o(t^{-\kappa})$, which we have in any case from $E|R|^\kappa < \infty$.

The last remark makes it clear that instances where $C_+ + C_- = 0$ are those to be avoided. Choice of the *correct* M, and hence κ, is crucial.

For use in random equations it will be convenient to specialize Theorem 2.3 to the following form.

COROLLARY 2.4. *Suppose R satisfies* (2.1) *and that M is defined on* $(\Omega, \mathscr{A}, P)$, *satisfies the conditions of Lemma* 2.2 *and is such that R is independent of* (Ψ, M). *Then in Theorem* 2.3 *conditions* (2.8) *and* (2.9) *may be replaced, respectively, by*

$$E|(\Psi(R)^+)^\kappa - ((MR)^+)^\kappa| < \infty, \tag{2.16}$$

$$E|(\Psi(R)^-)^\kappa - ((MR)^-)^\kappa| < \infty \tag{2.17}$$

and the formulae (2.12), (2.13) *and* (2.14), *respectively, by*

$$C_+ = \frac{1}{\kappa m}E\big((\Psi(R)^+)^\kappa - ((MR)^+)^\kappa\big), \tag{2.18}$$

$$C_- = \frac{1}{\kappa m}E\big((\Psi(R)^-)^\kappa - ((MR)^-)^\kappa\big), \tag{2.19}$$

$$C_+ = C_- = \frac{1}{2\kappa m}E(|\Psi(R)|^\kappa - |MR|^\kappa). \tag{2.20}$$

[Although $\Psi(R) =_L R$ you cannot replace $\Psi(R)$ by R in these expressions, for the pair $(\Psi(R), MR)$ does not have the same law as (R, MR).]

Only for one particular Ψ does it appear that these formulae can yield explicit expressions for C_+ or C_-, not involving the unknown R: see Corollary 4.4. But they will give bounds on C_+ and C_- for most Ψ. We calculate such bounds for one case in Corollary 4.3 and leave others to the reader.

Our method for the core result above, Theorem 2.3, has benefitted from ideas in Grincevičius (1975). The results of the latter are partly rediscoveries of the content of Kesten (1973), but Grincevičius's approach is distinct. We have taken care to avoid any version of one step in his approach that seems to be incomplete: The need for an extended form of Choquet–Deny lemma to obtain a smoothed version of $t^{\kappa}P(R > t)$, not known to be bounded, from a renewal equation in which it appears.

The basis for the renewal theory in the present section is ultimately the (two-sided) key renewal theorem. For various reason we have been unable to use other authors' renewal-theoretic superstructures built similarly on that base. Thus in part of the proof of Theorem 2.3 we could have employed results about renewal theory for Markov chains from Kesten (1974), but had to avoid doing so because a later proof (of Theorem 3.2) would then have fallen foul of the lack of a Stone-type decomposition of renewal measures in the Markov-chain case. The explicit proof of Theorem 2.3 is needed to build on. However, we have followed Kesten (1974) for a number of steps, and have kept mostly to its notation so that the dependence is visible.

3. Rates of approach. This section contains the rate results for implicit renewal theory, in which we quantify the approach of $t^{\kappa}P(R > t)$ to C_+. The tools employed are an explicit-rate version of Stone's (1966) decomposition of an exponentially decaying renewal measure, and Tauberian remainder theory. We give the Stone decomposition first, as its conditions will be needed in the substantive theorems.

THEOREM 3.1. *Let η be a probability law on $\mathbb{R}$ with finite second moment and positive first moment m, such that $\tilde{\eta}(\beta) < \infty$ for some $\beta > 0$. Suppose that η is spread out, so for some n_0 we have*

$$\eta^{(n_0)} = (1-\delta)\phi_0 + \delta\phi_1, \tag{3.1}$$

where $\delta \in [0,1)$ is constant and ϕ_0, ϕ_1 are probability measures with ϕ_0 absolutely continuous. Suppose that β has been taken so small that $\delta\tilde{\phi}_1(\beta) < 1$. Suppose also that $\hat{\eta}(\theta) \neq 1$ on $\Im\theta = -\beta$. Then the renewal measure $\nu := \sum_0^\infty \eta^{(n)}$ may be written $\nu = \nu_0 + \nu_1$, where ν_1 is a finite measure such that $\tilde{\nu}_1(\beta) < \infty$, and ν_0 is absolutely continuous with a continuous bounded density $p(\cdot)$ such that

$$p(t) = \frac{1}{m} - \frac{1}{2\pi}\int_{\mathscr{C}} e^{-i\theta t}\frac{d\theta}{1-\hat{\eta}(\theta)} + o(e^{-\beta t}), \qquad t \to \infty. \tag{3.2}$$

(Here $\mathscr{C}$ is a simple closed contour in the domain $D := \{\theta\colon -\beta < \Im\theta < 0\}$, enclosing all the zeroes of $1 - \hat{\eta}$ in D.)

Regarding (3.2), note that $\tilde{\eta}(\beta) < \infty$ implies $\hat{\eta}$ is holomorphic in D and continuous in its closure $\overline{D}$. By the Riemann–Lebesgue lemma, $\hat{\phi}_0(\theta) \to 0$ as $|\Re\theta| \to \infty$, uniformly in $-\beta \le \Im\theta \le 0$. Since $|\delta\hat{\phi}_1(\theta)| \le \delta\tilde{\phi}_1(\beta) < 1$, the values of θ where $1 - \hat{\eta}(\theta) = 0$ are restricted to some bounded subset of $\overline{D}$. On the real axis, $1 - \hat{\eta}(\theta) = 0$ only at $\theta = 0$, and since $\hat{\eta}(\theta) = 1 + im\theta + o(\theta)$ as $\theta \to 0$, the origin is an isolated zero of $\hat{\eta}$. All this together with our assumption that $1 - \hat{\eta}(\theta) \ne 0$ on $\Im\theta = -\beta$ implies that there can be at most finitely many zeroes of $1 - \hat{\eta}$ in D. So we may find a suitable contour $\mathscr{C}$ enclosing them. If there are no such zeros in D the value of the contour integral in (3.2) is 0. Otherwise, let the zeroes be $\theta_1, \dots, \theta_{n_1}$, say, with respective multiplicities $k_1, \dots, k_{n_1}$. Then (3.2) may be written more explicitly as

$$p(t) = \frac{1}{m} + \sum_{l=1}^{n_1} \sum_{j=1}^{k_l} \frac{t^{j-1}}{(j-1)!} \Re\left(e^{-i\theta_l t}(-i)^j a_{l,j}\right) + o(e^{-\beta t}),$$

where $a_{l,j}$ is the residue of $(\theta - \theta_l)^{j-1}/(1 - \hat{\eta}(\theta))$ at θ_l.

It is also worth remarking that the condition that $\hat{\eta}(\theta) \ne 1$ on the line $\Im\theta = -\beta$ is merely a technicality. We could allow zeroes on that line by imposing analyticity and other conditions there. Alternatively, if $\tilde{\eta}(\beta) < \infty$, then for suitable small $\varepsilon > 0$ it will be the case that $\hat{\eta}(\theta) \ne 1$ on $\Im\theta = -(\beta - \varepsilon)$, and so the theorem will hold with the error term worsened to $o(e^{-(\beta-\varepsilon)t})$.

Our first rate result is for the $M \ge 0$ case, as follows. [Modulus signs appear in (3.3) and (3.4) for later re-use of these equations.]

THEOREM 3.2. *Let $M \ge 0$ be independent of R and satisfy* (2.3). *Suppose that*

$$E|M|^{\kappa+\beta} < \infty \tag{3.3}$$

for some $\beta > 0$ and that

(3.4) *the conditional law of* $\log|M|$, *given $M \ne 0$, is spread out.*

Then the probability measure $\eta(dx) := e^{\kappa x}P(\log M \in dx)$ is spread out. Suppose it satisfies (3.1) *and the subsequent conditions in Theorem* 3.1. *Let $\mathscr{C}$ be as in Theorem* 3.1 *and set*

$$g_1(t) := e^{\kappa t}\big(P(R > e^t) - P(MR > e^t)\big), \tag{3.5}$$

$$g_{-1}(t) := e^{\kappa t}\big(P(R < -e^t) - P(MR < -e^t)\big). \tag{3.6}$$

(i) *If*

$$\int_0^\infty |P(R > -t) - P(MR > t)| t^{\kappa+\beta-1}\,dt < \infty, \tag{3.7}$$

then

$$t^\kappa P(R > t) = C_+ - \frac{1}{2\pi}\Re\int_{\mathscr{C}} e^{-i\theta t}\frac{\hat{g}_1(\theta)}{1 - \hat{\eta}(\theta)}\,d\theta + O(t^{-\beta/2}), \qquad t \to \infty. \tag{3.8}$$

(ii) *If*

$$\int_0^\infty |P(R < -t) - P(MR < -t)| t^{\kappa+\beta-1}\, dt < \infty, \tag{3.9}$$

then

$$t^\kappa P(R < -t) = C_- - \frac{1}{2\pi} \Re \int_{\mathscr{C}} e^{-i\theta t} \frac{\hat{g}_{-1}(\theta)}{1 - \hat{\eta}(\theta)}\, d\theta + O(t^{-\beta/2}), \qquad t \to \infty. \tag{3.10}$$

(iii) *If R satisfies* (2.1) *and is independent of* (M, Ψ), *then the assumptions* (3.7) *and* (3.9) *may be replaced, respectively, by*

$$E\left|\left(\Psi(R)^+\right)^{\kappa+\beta} - \left((MR)^+\right)^{\kappa+\beta}\right| < \infty, \tag{3.11}$$

$$E\left|\left(\Psi(R)^-\right)^{\kappa+\beta} - \left((MR)^-\right)^{\kappa+\beta}\right| < \infty. \tag{3.12}$$

In the above we have left the contour $\mathscr{C}$ enclosing all the zeroes of $1 - \hat{\eta}$ in $D = \{\theta\colon 0 < -\Im\theta < \beta\}$, but it needs only to lie in D and enclose all the zeroes in $\{\theta\colon 0 < -\Im\theta \le \beta/2\}$, as the contributions of any others are covered by the $O(t^{-\beta/2})$ remainder term. The latter term is probably not sharp, however, and probably should be $O(t^{-\beta})$, but it seems to be the best that a method employing Tauberian remainder theory can do. An example in Lyttkens [(1956), Section 12] shows that the factor $\frac{1}{2}$ is sharp in the Tauberian remainder theorem we use.

Tauberian remainder theory is needed because we obtain (3.8) and (3.10) in a smoothed form and have to unsmooth. The Beurling–Ganelius Tauberian remainder theorem is employed. As it comes from the rather inaccessible Ganelius (1962), we restate it as Theorem 9.6. [It does not unfortunately appear in Ganelius (1971), the most readable introduction to the area.]

A more explicit form of (3.8) is

$$t^\kappa P(R > t) = C_+ + \sum_{l=1}^{n_1} \sum_{j=1}^{k_1} \frac{t^{j-1}}{(j-1)!} \Re\left(r_{l,j}(-i)^j e^{-i\theta_l t}\right) + O(t^{-\beta/2}), \tag{3.13}$$

where the θ_l are the zeroes of $1 - \hat{\eta}(\theta)$ in $0 < -\Im\theta \le \beta/2$, with respective multiplicities k_l, and $r_{l,j}$ is the residue of $(\theta - \theta_l)^{j-1}\hat{g}_1(\theta)/(1 - \hat{\eta}(\theta))$ at θ_l. The constants $r_{l,j}$ thus depend, through g_1, on the unknown probability law of R; however the *form* of the right-hand side of (3.13) does not.

We remark that if $|M|$ satisfies the conditions on M in the above theorem, then the conclusion

$$t^\kappa P(|R| > t) = C_+ + C_- - \frac{1}{2\pi} \Re \int_{\mathscr{C}} e^{-i\theta t} \frac{\hat{g}(\theta)}{1 - \hat{\eta}(\theta)}\, d\theta + O(t^{-\beta/2}), \qquad t \to \infty,$$

where $g(t) := e^{\kappa t}(P(|R| > e^t) - P(|MR| > e^t))$, may be established with the conditions (3.7) and (3.9) replaced by

$$\int_0^\infty |P(|R| > t) - P(|MR| > t)| t^{\kappa+\beta-1}\, dt < \infty. \tag{3.14}$$

For this follows by simply applying the result to $|M|, |R|$. The expectation formula that can here be used in place of (3.14) is $E||\Psi(R)|^{\kappa+\beta} - |MR|^{\kappa+\beta}| < \infty$.

Our second rate result, for M that can take negative values, involves a probability law η now defined as follows. Let $M_1, M_2, \dots$ be independent, all with the law of M, and let $N_1^{(+)} := \inf\{n \geq 1: M_1 \cdots M_n > 0\}$. Then

$$\eta(B) := E\left(|M_1 \cdots M_{N_1^{(+)}}|^\kappa \mathbf{1}\left\{\sum_{j=1}^{N_1^{(+)}} \log|M_j| \in B\right\}\right), \qquad B \in \mathscr{B}, \tag{3.15}$$

where the integrand is to be interpreted as 0 on the event that $M_1 \cdots M_n \leq 0$ for all $n \geq 1$.

THEOREM 3.3. *Suppose $P(M < 0) > 0$. Let M be independent of R and satisfy* (2.3), (3.4) *and* (3.3) *for some $\beta > 0$, and suppose β is so small that*

$$EM^{\kappa+\beta}\mathbf{1}_{M>0} < 1. \tag{3.16}$$

Then η, defined in (3.15), *is spread out. Suppose it satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *Let $\mathscr{C}$ be as in Theorem* 3.1, *define g_j for $j = 1, -1$ as in Theorem* 3.2 *and define measures μ_+, μ_- on $\mathbb{R}$ by*

$$\mu_+(B) := EM^\kappa \mathbf{1}_{M>0}\mathbf{1}_{\log M \in B},$$

$$\mu_-(B) := E|M|^\kappa \mathbf{1}_{M<0}\mathbf{1}_{\log|M| \in B}, \qquad B \in \mathscr{B}.$$

(i) *If* (3.7) *and* (3.9) *hold, then*

$$\begin{aligned} t^\kappa P(R > t) = C_+ - \frac{1}{2\pi}\Re\int_{\mathscr{C}} e^{-i\theta t}\bigg(&\frac{\hat{g}_1(\theta) + \hat{g}_{-1}(\theta)}{2(1 - \hat{\mu}_+(\theta) - \hat{\mu}_-(\theta))} \\ &+ \frac{\hat{g}_1(\theta) - \hat{g}_{-1}(\theta)}{2(1 - \hat{\mu}_+(\theta) + \hat{\mu}_-(\theta))}\bigg)\, d\theta \\ &+ O(t^{-\beta/2}), \qquad t \to \infty, \end{aligned} \tag{3.17}$$

and the same formula holds for $t^\kappa P(R < -t)$. (Recall that $C_+ = C_-$ in this case.)

(ii) *If R satisfies* (2.1) *and is independent of* (M, Ψ), *then* (3.7) *and* (3.9) *may be replaced by* (3.11) *and* (3.12).

An expansion of the conclusion along the lines of (3.13) can also be given. However, the zeroes of $1 - \hat{\eta}$ are likely to be hard to locate exactly, so the following corollary sums up the main import of the result.

COROLLARY 3.4. *Suppose $P(M<0)>0$. Let M be independent of R and satisfy* (2.3), (3.4) *and* (3.3) *for some $\beta>0$. If* (3.7) *and* (3.9) *hold, then, for some $\gamma>0$,*

$$t^\kappa P(R>t)=C_+ + O(t^{-\gamma}),$$

$$t^\kappa P(R<-t)=C_- + O(t^{-\gamma}),\qquad t\to\infty,$$

4. Random difference equations. In the present section $\Psi(\cdot)$ is given by

$$\Psi(t):=Q+Mt,\qquad t\in\mathbb{R}, \tag{4.1}$$

where Q and M are r.v.s. So $R=_L\sum_{k=1}^\infty Q_k\Pi_{k-1}$, where (Q_k,M_k) for $k=1,\dots$ are independent with the same law as (Q,M), and $\Pi_n:=M_1\cdots M_n$.

We obtain a new proof of Kesten [(1973), Theorem 5] and some extensions of the result. Admittedly this concerns only the "not so hard to prove" one-dimensional case of Kesten's theorems [(1973), page 208], but the proof of Kesten's Theorem 4, which is needed for Theorem 5, is formidable enough even in that case (occupying pages 236–244) that the present simpler alternative is worth having. The placing of (4.1) as one case of a general picture is also valuable, but for the random difference equation the real strength of the present approach lies in the extensions it allows. Thus bounds on C_+ and C_- in (2.10) and (2.11) will be obtained in all cases, and explicit values in some cases. A qualitative difference emerges between the case when $M\geq 0$ a.s., when C_+ and C_- will in general differ, and otherwise, when they coincide. The rates of approach of the tails to their asymptotes are also found.

The method presented here should extend to higher-dimensional cases, and further work is intended in that direction, where recent results of LePage (1983) are relevant. However the explicit extensions to Kesten's results that our method allows in the one-dimensional case make it appropriate to concentrate on that case first.

THEOREM 4.1. *Let Q and M be r.v.s on a common probability space and suppose that M satisfies the conditions of Lemma* 2.2 *and that*

$$E|Q|^\kappa<\infty. \tag{4.2}$$

Then there is a unique law for R satisfying (1.1). *For this law both* (2.10) *and* (2.11) *hold. If $M\geq 0$ a.s. then*

$$C_+=\frac{E\big(((Q+MR)^+)^\kappa-((MR)^+)^\kappa\big)}{\kappa m},$$

$$C_-=\frac{E\big(((Q+MR)^-)^\kappa-((MR)^-)^\kappa\big)}{\kappa m}, \tag{4.3}$$

while otherwise

$$C_+ = C_- = \frac{1}{2\kappa m} E(|Q + MR|^\kappa - |MR|^\kappa). \tag{4.4}$$

Finally, $C_+ + C_- > 0$ *if and only if*

$$\text{for each fixed } c \in \mathbb{R}, \qquad P(Q = (1-M)c) < 1. \tag{4.5}$$

Apart from the formulae for C_+ and C_-, this is Kesten [(1973), Theorem 5]. Existence and uniqueness of the probability law solving (1.1) is fully discussed in Vervaat (1979) [see also Grincevičius (1981)] and follows under the given conditions from Letac's principle. Our proof of the remaining assertions is in Section 9. For the last assertion we use Grincevičius's extension of Lévy's symmetrization inequality, in which med denotes median.

PROPOSITION 4.2 [Grincevičius (1980)]. *Suppose* (Q_n, M_n), *for* $n = 1, 2, \ldots,$ *are independent and each has the law of* (Q, M). *Let*

$$\Pi_j := \prod_1^j M_k, \qquad R_n := \sum_{k=1}^n \Pi_{k-1} Q_k,$$

$$\Pi_{j,n} := \prod_{j+1}^n M_k, \qquad R_{j,n} := \sum_{k=j+1}^n \Pi_{j,k-1} Q_k$$

(*so that* $R_n = R_j + \Pi_j R_{j,n}$). *Then*

$$P\Big(\max_{j=1,\ldots,n} \big(R_j + \Pi_j \operatorname{med}(R_{j,n} + \Pi_{j,n} y)\big) > x\Big)$$
$$\le 2P(R_n + \Pi_n y > x), \qquad x, y \in \mathbb{R}.$$

[The assumption $P(M = 0) = 0$, needed for other purposes in Grincevičius (1981), is superfluous and has been removed.]

The following consequences of Proposition 4.2 are interesting in their own right and include the form that we will use later. Under the conditions of Letac's principle, in particular under the conditions of Theorem 4.1, one may let $n \to \infty$ in the inequality. Then $R_n \to \tilde{R}$ a.s. and $\tilde{R}$ has the law of R. For each fixed j, $R_{j,n} \to \sum_{k=j+1}^\infty \Pi_{j,k-1} Q_k$ a.s., and the limit also has the law of R. Thus the $y = 0$ case gives

$$P\Big(\sup_{j \in \mathbb{N}} (R_j + \Pi_j \operatorname{med} R) > x\Big) \le 2P(R > x), \qquad x \ge 0.$$

Combine this with the corresponding version with the signs of R, R_j reversed to get

$$P\Big(\sup_{j \in \mathbb{N}} |R_j + \Pi_j \operatorname{med} R| > x\Big) \le 2P(|R| > x), \qquad x \ge 0. \tag{4.6}$$

Now we investigate consequences of Theorem 4.1, in particular of the formulae for C_+, C_- there.

COROLLARY 4.3. *When* $0 < \kappa \le 1$,

$$C_+ + C_- \le \frac{1}{\kappa m} E|Q|^\kappa,$$

while when $\kappa \ge 1$,

$$C_+ + C_- \le \frac{2^{\kappa-1}}{m}\big(E|Q|^\kappa + E|Q||M|^{\kappa-1}E|R|^{\kappa-1}\big). \tag{4.7}$$

The latter bound is finite because $E|Q||M|^{\kappa-1} < \infty$ by the assumptions of Theorem 4.1 and Hölder's inequality, while $E|R|^{\kappa-1} < \infty$ by the conclusions of Theorem 4.1.

When $P(M < 0) > 0$ the constants C_+ and C_-, being equal, are individually bounded by half the above amounts.

To make the bound (4.7) explicit, employ (1.1) to bound $E|R|^{\kappa-1}$. Thus, since $\|\ \|_p$ as defined in Section 1 satisfies the triangle inequality, we are led to

$$\|R\|_{\kappa-1} \le \|Q\|_{\kappa-1}/(1 - \|M\|_{\kappa-1}) < \infty, \tag{4.8}$$

whence a bound in (4.7) involving only Q and M.

The next two results detail all cases when the constants in Theorem 4.1 can be made explicit, that is, not involving the unknown probability law of R.

COROLLARY 4.4. *In Theorem* 4.1 *assume that* $M \ge 0$, $Q \ge 0$ *a.s., so that* $R \ge 0$ *a.s. and* $C_- = 0$. *If* κ *is an integer, then*

$$C_+ = \frac{1}{\kappa m}\sum_{j=0}^{\kappa-1}\binom{\kappa}{j} EM^jQ^{\kappa-j}ER^j, \tag{4.9}$$

where the moments ER^j *may be found iteratively* [*Vervaat* (1979), *Theorem* 5.1] *from the equations*

$$ER^j = \sum_{k=0}^{j}\binom{j}{k} EM^kQ^{j-k}ER^k, \qquad j = 1, 2, \dots, \kappa - 1. \tag{4.10}$$

In particular, when $\kappa = 1$, *i.e.*, $EM = 1$, $C_+ = EQ/EM \log M$, *while when* $\kappa = 2$, *i.e.*, $EM^2 = 1$,

$$C_+ = \frac{1}{m}\left(\frac{1}{2}EQ^2 + \frac{EQEQM}{1 - EM}\right). \tag{4.11}$$

COROLLARY 4.5. *In Theorem* 4.1 *suppose* κ *is an even integer. Then* $C_+ + C_-$ *equals the right-hand side of* (4.9). *If, additionally,* $P(M < 0) > 0$, *then both of* C_+ *and* C_- *are equal to half that quantity. The* ER^j *are again available from* (4.10).

In particular, when $\kappa = 2$, *i.e.*, $EM^2 = 1$, $C_+ + C_-$ *equals the right-hand side of* (4.11).

There seems to be just one class of random difference equations to which our results apply and whose stationary laws are known explicitly. We give it here because it provides a useful check on our formula for C_+. The $k = 1, 2$ cases of the next result are due to Chamayou and Letac (1989). By the notation $X \sim \beta_{a,b}$, where $a, b > 0$, is meant that X has probability density $x^{a-1}(1+x)^{-a-b}\mathbf{1}_{x>0}/B(a,b)$.

PROPOSITION 4.6. *Fix* $k \in \mathbb{N}$, *and positive numbers* $a_1, \dots, a_k, b$. *Put* $a_{k+1} := a_1$. *Let* $R, Y_1, \dots, Y_k$ *be independent with* $R \sim \beta_{a_1, b}$ *and* $Y_j \sim \beta_{a_{j+1}, a_j + b}$ *for* $j = 1, \dots, k$. *Set* $M := Y_k \cdots Y_1$ *and*

$$Q := Y_k + Y_k Y_{k-1} + Y_k Y_{k-1} Y_{k-2} + \cdots + Y_k Y_{k-1} \cdots Y_1.$$

Then $Q + MR =_L R$.

For this model we have $P(R > r) \sim b^{-1} r^{-b}/B(a, b)$, so $\kappa = b$ and the claim made by (4.3) is that, whatever k,

$$\frac{1}{bB(a_1, b)} = \frac{E\big((Q + MR)^b - (MR)^b\big)}{bm}.$$

One may verify this directly. It is left as an exercise.

Consider now the rates of approach of the tails of R to their asymptotes.

THEOREM 4.7. *Suppose that* R *solves* (1.1), *where* M *satisfies* (2.3) *for some* $\kappa > 0$ *and, for some* $\beta \in (0, 1)$, (3.3) *and* (3.4) *hold and* $E|Q|^{\kappa+\beta} < \infty$. *Define* $\eta(dx) := e^{\kappa x} P(\log M \in dx)$ *if* $M \geq 0$ *a.s., and otherwise define* η *by* (3.15). *Suppose it satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *If* $P(M < 0) > 0$, *suppose* (3.16) *also. Then if* $M \geq 0$ *a.s., both* (3.8) *and* (3.10) *hold, while otherwise* (3.17) *holds and* $t^\kappa P(R < -t)$ *satisfies the same formula.*

Applications of random difference equations are listed in Kesten (1973). We discuss one new application in Section 10. There is also a statistical literature on random difference equations: see Nicholls and Quinn (1982), Pötzelberger (1990) and references therein.

5. A random extremal equation.

Here $\Psi(\cdot)$ is given by

$$\Psi(t) := \max(Q, Mt), \qquad t \in \mathbb{R},$$

where $M \geq 0$ a.s. In the notation of Theorem 2.1, iteration of Ψ gives $Z_n(t) = \max(t\Pi_n, \bigvee_{k=1}^n Q_k \Pi_{k-1})$, where $\Pi_k := M_1 \cdots M_k$, and (M_k, Q_k) are independent, each with the law of (M, Q). (Without the insistence on $M \geq 0$, no straightforward iteration of Ψ is valid.) An easy application of Letac's principle gives the following:

PROPOSITION 5.1. *If* $M \geq 0$ *a.s.,* $-\infty \leq E \log M < 0$ *and* $E \log(1 \vee Q) < \infty$, *then* $\tilde{R} := \bigvee_{k=1}^\infty Q_k \Pi_{k-1}$ *is a.s. finite and its law is the unique law such that* (1.2) *holds.*

If $Q = 1$ a.s., then on writing $S := \log R$, $X := \log M$ we find $S =_L \bigvee_{k=0}^{\infty}(X_1 + \cdots + X_k)$ so that we have the classical setup of the overall maximum of a downward-drifting random walk, the stationary waiting time of a G/G/1 queue and so on. Alternatively, if $M = e^{B-A}$ and $Q = e^B$, then $S = \log R$ satisfies $S =_L B + (S - A)^+$, which is the recurrence relation for the stationary *queueing time* (waiting time plus service time) in a G/G/1 queue. In either case our first theorem below then gives exponential decay of the upper tail of S, retrieving the well-known estimates found in Feller [(1971), Section XII.5]. We follow it with the rate result.

THEOREM 5.2. *Suppose that $M \ge 0$ a.s., that M satisfies the conditions of Lemma* 2.2 *and that $E(Q^+)^\kappa < \infty$. Then there is a unique law for R satisfying* (1.2), *and $P(R > t) \sim C_+ t^{-\kappa}$ as $t \to \infty$, where $C_+ = (\kappa m)^{-1}E((Q^+ \vee MR^+)^\kappa - (MR^+)^\kappa)$. Also $C_+ > 0$ if and only if $P(Q > 0) > 0$.*

THEOREM 5.3. *Suppose that R solves* (1.2), *where $M \ge 0$ satisfies* (2.3) *for some $\kappa > 0$ and, for some $\beta > 0$,* (3.3) *and* (3.4) *hold and $E(Q^+)^{\kappa+\beta} < \infty$. Suppose $\eta(dx) := e^{\kappa x}P(\log M \in dx)$ satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *Then both* (3.8) *and* (3.10) *hold.*

Specializing this to the stationary waiting time S of a G/G/1 queue as above, we obtain

$$e^{\kappa t}P(S > t) = C_+ - I(e^t) + O(t^{-\beta/2}), \qquad t \to \infty,$$

where $I(t)$ is the term following C_+ in (3.8). This extends the known asymptotic behaviour of S, where one has [Borovkov (1976), Section 22.3] that $e^{\kappa t}P(S > t) = C_+ + O(e^{-\gamma t})$ for some unidentified γ.

6. A model due to Letac. Model E of Letac (1986) is

$$\Psi(t) := Q + M\max(L, t), \qquad t \in \mathbb{R}, \tag{6.1}$$

where $M \ge 0$ a.s. From Letac (1986) we obtain, with a slight correction, the following formula for iteration of Ψ, in the notation of Theorem 2.1:

$$Z_n(t) = \max\left(\sum_{k=1}^{n} Q_k\Pi_{k-1} + t\Pi_n, \left(\sum_{k=1}^{m} Q_k\Pi_{k-1} + L_m\Pi_m\right)_{m=1}^{n}\right), \tag{6.2}$$

where $\Pi_k := M_1 \cdots M_k$, and (M_k, Q_k, L_k) are independent, each with the law of (M, Q, L). [The restriction $M \ge 0$ a.s. is not given in Letac (1986), but it is necessary for validity of (6.2).]

PROPOSITION 6.1. *If $M \ge 0$ a.s., $-\infty \le E\log M < 0$, $E\log(1 \vee Q) < \infty$ and $E\log(1 \vee L) < \infty$, then $\tilde{R} := \sup(\sum_{k=1}^{\infty} Q_k\Pi_{k-1}, (\sum_{k=1}^{m} Q_k\Pi_{k-1} + L_m\Pi_m)_{m=1}^{\infty})$ is a.s. finite and its law is the unique law such that* (1.3) *holds.*

With assumptions that include these we obtain a power-law tail for R.

THEOREM 6.2. *Suppose that* $M \geq 0$ *a.s., that* M *satisfies the conditions of Lemma* 2.2 *and that*

$$(6.3) \qquad E(ML^+)^\kappa < \infty, \qquad E|Q|^\kappa < \infty.$$

Then there is a unique law for R *satisfying* (1.3), *and* $P(R > t) \sim C_+ t^{-\kappa}$ *as* $t \to \infty$, *where*

$$C_+ = \frac{1}{\kappa m} E\Big(\big((Q + M\max(L, R))^+\big)^\kappa - \big((MR)^+\big)^\kappa\Big).$$

If, further, there is a constant c *such that* $Q - c(1 - M) \geq 0$ *a.s. and*

$$P(Q - c(1 - M) > 0) + P(M(L - c) > 0) > 0,$$

then $C_+ > 0$.

A necessary and sufficient condition for positivity of C_+ is lacking here. What is mainly needed is a criterion in Theorem 4.1 for the individual coefficients C_+, C_- to be positive. A sufficient condition is that $Q - c(1 - M)$ is symmetric for some c, and not a.s. zero, but this is far stronger than necessary. Clearly, this is also sufficient for C_+ to be positive in Theorem 6.2.

We turn to the rate result.

THEOREM 6.3. *Suppose* R *solves* (1.3), *where* $M \geq 0$ *satisfies* (2.3) *for some* $\kappa > 0$ *and, for some* $\beta \in (0, 1)$, (3.3) *and* (3.4) *hold and* $E|Q|^{\kappa+\beta} < \infty$, $E|ML^+|^{\kappa+\beta} < \infty$. *Suppose* $\eta(dx) := e^{\kappa x}P(\log M \in dx)$ *satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *Then both* (3.8) *and* (3.10) *hold.*

Two special cases of the model are worth remarking. First, if $Q = 0$ a.s., then Ψ becomes $\Psi(t) := ML \vee Mt$, a special case of the setup of Section 5, with $Q := ML$.

More interestingly, if $Q = 0$ a.s. and $L > 0$ a.s., then (1.3) becomes, in terms of $S := \log R$, $B := \log M$ and $A := \log L$,

$$S =_L B + A \vee S.$$

This model is discussed and applied in Helland and Nilsen (1976) and further in Letac (1986). Thus S will have the law of $\bigvee_{k=1}^\infty (A_k + B_1 + \cdots + B_k)$, and the results above give conditions for exponentially decaying upper tail, and exponential rate of approach thereto.

7. A largest-modulus equation. A two-sided variant of the model of Section 5 is obtainable in terms of the largest-modulus operator $\veebar$ defined in (1.5). Set $\Psi(t) := Q \veebar Mt$ for $t \in \mathbb{R}$, where we use the same conventions for $\veebar$ as for $\vee$: thus $a \veebar bc := a \veebar (bc)$. As usual, let $(M, Q), (M_1, Q_1), (M_2, Q_2), \ldots$ be i.i.d. and set $\Pi_n := M_1 \cdots M_n$. Application of Letac's principle gives that if $-\infty \leq E\log|M| < 0$ and $E\log^+|Q| < \infty$, then $\tilde{R} := \veebar_{k=1}^\infty Q_k \Pi_{k-1}$ is a.s. finite and its law is the unique law such that (1.4) holds.

PROPOSITION 7.1. *Suppose that M satisfies the conditions of Lemma* 2.2 *and that $E|Q|^\kappa < \infty$. Then there is a unique law for R satisfying* (1.4), *and for this law both* (2.10) *and* (2.11) *hold. If $M \geq 0$ a.s., then*

$$C_+ = \frac{E\big(((Q \vee MR)^+)^\kappa - ((MR)^+)^\kappa\big)}{\kappa m},$$

$$C_- = \frac{E\big(((Q \vee MR)^-)^\kappa - ((MR)^-)^\kappa\big)}{\kappa m},$$

while otherwise

$$C_+ = C_- = \frac{1}{2\kappa m} E\big((|Q|^\kappa - |MR|^\kappa)^+\big).$$

Further, $C_+ + C_- > 0$ if and only if $P(Q \neq 0) > 0$.

For a rate of approach we have the following result. Its proof is on the same lines as those of Theorems 4.7 and 5.3, and is omitted.

PROPOSITION 7.2. *Suppose R solves* (1.4), *where M satisfies* (2.3) *for some $\kappa > 0$ and, for some $\beta > 0$,* (3.3) *and* (3.4) *hold and $E|Q|^{\kappa+\beta} < \infty$. Define $\eta(dx) := e^{\kappa x} P(\log M \in dx)$ if $M \geq 0$ a.s., and otherwise define η by* (3.15). *Suppose it satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *If $P(M < 0) > 0$, suppose* (3.16) *also. Then if $M \geq 0$ a.s., both* (3.8) *and* (3.10) *hold, while otherwise* (3.17) *holds and $t^\kappa P(R < -t)$ satisfies the same formula.*

8. Two more equations. The method extends to variants of Section 4's random difference equation such as (1.6). If one takes $\Psi(t) := [Q + Mt]$ where the domain for t is $\mathbb{Z}$ rather than $\mathbb{R}$, then the continuity requirement in Letac's principle (Theorem 2.1) is trivially satisfied and the conditions of the principle are easily seen to hold if $-\infty \leq E \log|M| < 0$ and $E \log^+|Q| < \infty$.

Apart from its final statement, Theorem 4.1 remains true for this model, $[Q + MR]$ everywhere replacing $Q + MR$, with a virtually unchanged proof. There remains the last statement, that is, the condition that ensures $C_+ + C_- > 0$. It seems to be less than obvious in the present case, and indeed the only sufficient condition we have is that $M \geq 0$ a.s. and $Q \geq 1$ a.s. For then $R \geq 1$ a.s., and, in the formula

$$C_+ = \frac{1}{\kappa m} E\big([Q + MR]^\kappa - (MR)^\kappa\big),$$

we have $[Q + MR] > MR$, so $C_+ > 0$.

The difficulty in finding a necessary and sufficient condition here, or even a good sufficient condition, is again bound up with the lack of such a condition for the *individual* constants C_+, C_- in Theorem 4.1 to be positive.

The rate result, Theorem 4.7, also carries over verbatim to (1.6), with an essentially unchanged proof.

The other model for this section is (1.7), as an instance of a random polynomial equation. Here M, N, Q and so also R, are a.s. nonnegative. One may equivalently study $S := R^2$, satisfying

$$S =_L MS + N\sqrt{S} + Q, \qquad S \text{ independent of } (Q, N, M). \tag{8.1}$$

So the role of Ψ is now taken by $\Xi(t) := Mt + N\sqrt{t} + Q$. First we employ Letac's principle.

PROPOSITION 8.1. *Suppose $M \geq 0$, $N \geq 0$, $Q > 0$ a.s. and that (Q, N, M), (Q', N', M'), $((Q_n, N_n, M_n))_{n \geq 1}$ are i.i.d. Set $\Xi_n(t) := M_n t + N_n\sqrt{t} + Q_n$ for $t \geq 0$. If*

$$E \log^+ N < \infty \tag{8.2}$$

and

$$-\infty \leq E \log\left(M + \frac{N}{2\sqrt{Q'}} \right) < 0, \tag{8.3}$$

then $\tilde{S} := \lim_{n \to \infty} \Xi_1 \circ \Xi_2 \circ \cdots \circ \Xi_n(t)$ exists and is finite, almost surely, and its law is the unique law on $[0, \infty)$ satisfying (8.1).

Then S has power-law tail under suitable hypotheses and a rate of approach is available under stronger hypotheses.

THEOREM 8.2. *Suppose that M satisfies the conditions of Lemma* 2.2, *that the conditions of Proposition* 8.1 *hold, with* (8.2) *strengthened to $EN^\kappa < \infty$, and that $EQ^\kappa < \infty$. Then $P(S > t) \sim C_+ t^{-\kappa}$ as $t \to \infty$, where*

$$C_+ = \frac{1}{\kappa m} E\Big(\big(MS + N\sqrt{S} + Q\big)^\kappa - (MS)^\kappa \Big). \tag{8.4}$$

Also $C_+ > 0$ if and only if $P(Q > 0) + P(N > 0) > 0$.

THEOREM 8.3. *Suppose S solves* (8.1), *where $M \geq 0$, $N \geq 0$, $Q > 0$ a.s. and that* (8.3) *holds. Suppose that M satisfies* (2.3) *for some $\kappa > 0$ and that there exists $\beta > 0$, with $\beta < \min(1, \kappa)$, such that* (3.3) *and* (3.4) *hold and $EN^{\kappa+\beta} < \infty$, $EQ^{\kappa+\beta} < \infty$. Suppose $\eta(dx) := e^{\kappa x} P(\log M \in dx)$ satisfies* (3.1) *and the subsequent conditions in Theorem* 3.1. *Then, with S replacing R in both* (3.5) *and* (3.8), *formula* (3.8) *holds.*

9. Proofs.

PROOF OF LEMMA 2.2. Write $Y := \log|M| \in [-\infty, \infty)$. With the interpretation given in Section 1 of $0^t \log 0$ one finds

$$E|M|^u \log^- |M| = E|Y| e^{uY} \mathbf{1}_{Y<0} < \infty, \qquad u > 0. \tag{9.1}$$

The Mellin–Stieltjes transform of $|M|$ or Laplace–Stieltjes transform of Y is $f(u) := E\mathbf{1}_{M\neq 0}|M|^u = E\mathbf{1}_{Y>-\infty}e^{uY}$. Under (2.3) it is finite and continuous in $[0,\kappa]$, and because of (9.1) has finite derivative $E|M|^u \log|M|$ in the interior of that interval. Similarly f has second derivative $E|M|^u \log^2|M|$ there. Now (2.3) implies that $P(M \neq 0) > 0$, so (2.5) makes sense and implies in particular that $P(|M| \in \{0,1\}) < 1$. In turn that implies $f'' > 0$ in $(0,\kappa)$ so f is strictly convex in $[0,\kappa]$. If $P(M = 0) = 0$, then $f(0) = 1 = f(\kappa)$, so by the convexity $0 > f'(0) = E\log|M|$. On the other hand, if $P(M = 0) > 0$, then $E\log|M| = -\infty$ because $E\log^+|M| < \infty$ as a consequence of (2.4). So (2.6) is established in both cases.

By (2.4), (9.1) and the convexity we have also (2.7). □

Lemmas about the $\check{}$ smoothing defined in Section 1 (which is just Cesàro mean, with exponential arguments) are next. The same smoothing is used in Grincevičius (1975).

LEMMA 9.1. *If $f \geq 0$, $f \in L^1(\mathbb{R})$ and $f(t+\varepsilon) \geq \theta(\varepsilon)f(t)$ for all $\varepsilon > 0$ and $t \in \mathbb{R}$, where $\theta(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, then f is directly Riemann-integrable (dRi).*

PROOF. Without loss of generality it can be taken that $\theta(\varepsilon)\uparrow 1$ as $\varepsilon \downarrow 0$. For $\epsilon > 0$,

$$\varepsilon \sum_{n\in\mathbb{Z}} \inf_{[n\varepsilon,(n+1)\varepsilon]} f \geq \varepsilon\theta(\varepsilon)\sum_n f(n\varepsilon) \geq \theta^2(\varepsilon)\sum_n \int_{(n-1)\varepsilon}^{n\varepsilon} f = \theta^2(\varepsilon)\int f,$$

and, similarly,

$$\varepsilon \sum_{n\in\mathbb{Z}} \sup_{[n\varepsilon,(n+1)\varepsilon]} f \leq \frac{1}{\theta^2(\varepsilon)}\int f.$$

Let $\varepsilon \downarrow 0$, then these upper and lower sums converge to $\int f$. □

LEMMA 9.2. *If $f \in L^1(\mathbb{R})$, then $\check{f}$ is dRi.*

PROOF. By considering f^+, f^- separately, without loss of generality f may be assumed nonnegative. Now, for $\delta > 0$,

$$\check{f}(t+\delta) \geq e^{-t-\delta}\int_{-\infty}^{t} e^u f(u)\,du = e^{-\delta}\check{f}(t),$$

so the result follows from Lemma 9.1. □

LEMMA 9.3. *If $\int_0^t u^\kappa P(R > u)\,du \sim C_+ t$ as $t \to \infty$, then $P(R > t) \sim C_+ t^{-\kappa}$ as $t \to \infty$.*

PROOF. One can use Bingham, Goldie and Teugels [(1987), Exercise 1.11.14] with $U(t) := -t^\kappa P(R > t)$, $\rho := 0$, $\sigma := \kappa$ and $l(\cdot) \equiv 1$, but for completeness we give a direct proof. Fix $b > 1$. Then

$$\frac{b^{\kappa+1}-1}{\kappa+1}t^{\kappa+1}P(R>t) \ge \int_t^{bt} u^\kappa P(R>u)\,du \sim C_+(b-1)t, \qquad t\to\infty,$$

so

$$\liminf_{t\to\infty} t^\kappa P(R>t) \ge C_+(\kappa+1)\frac{b-1}{b^{\kappa+1}-1}.$$

On letting $b\downarrow 1$ one sees that the lim inf is at least C_+. The proof that $\limsup t^\kappa P(R>t) \le C_+$ is similar, starting from $\int_{bt}^t u^\kappa P(R>u)\,du$ with $0 < b < 1$. □

For the proof of Theorem 2.3 we may set up $R, M, M', M_1, M_2, \dots,$ independent r.v.s on a common probability space, where M' and the M_n all have the law of M. Set

$$(9.2) \qquad Y_k := \log|M_k|, \qquad V_k := \log|\Pi_k| = \sum_1^k Y_j, \qquad k\in\mathbb{N},$$

$$(9.3) \quad r(t) := e^{\kappa t}P(R>e^t), \qquad \delta_n(t) := e^{\kappa t}P(\Pi_n R > e^t), \qquad t\in\mathbb{R}.$$

In the proofs of this theorem and of Corollary 2.4 we need establish only (2.10) and the appropriate formula for C_+, as the conclusions for the other tail follow on considering $-R$ in place of R.

Proof of Theorem 2.3.

Case 1. $M\ge 0$ a.s. Start with a telescoping sum: For $n\in\mathbb{N}$, $t\in\mathbb{R}$,

$$\begin{aligned} P(R>e^t) &= \sum_{k=1}^n \big(P(\Pi_{k-1}R>e^t) - P(\Pi_k R>e^t)\big) + P(\Pi_n R>e^t)\\ &= \sum_{k=1}^n \big(P(e^{V_{k-1}}R>e^t) - P(e^{V_{k-1}}MR>e^t)\big) + P(e^{V_n}R>e^t) \qquad (9.4)\\ &= \sum_{k=0}^{n-1}\int_{\mathbb{R}}\big(P(R>e^{t-u}) - P(MR>e^{t-u})\big)P(V_k\in du)\\ &\quad + P(e^{V_n}R>e^t). \end{aligned}$$

The interval of integration does not include the point $-\infty$ though $P(V_k=-\infty)$ is permitted to be positive. Set

$$(9.5) \qquad \nu_n(dt) := e^{\kappa t}\sum_{k=0}^n P(V_k\in dt).$$

Then in terms also of g_1, defined in (3.5), the above says

$$r(t) = g_1 * \nu_{n-1}(t) + \delta_n(t), \qquad t\in\mathbb{R},\ n\in\mathbb{N}.$$

Apply the smoothing operator $\check{}$ to this. As the operator is just Lebesgue convolution with the kernel $K(t) := e^{-t}\mathbf{1}_{t>0}$, it follows that $(g_1 * \nu_{n-1})^{\vee} = \check{g}_1 * \nu_{n-1}$, and we deduce that

$$\check{r}(t) = \check{g}_1 * \nu_{n-1}(t) + \check{\delta}_n(t), \qquad t \in \mathbb{R}, n \in \mathbb{N}. \tag{9.6}$$

Let $\eta(du) := e^{\kappa u}P(Y_1 \in du)$. This measure places *no* mass at $-\infty$, and by (2.3), (2.5) and (2.7) is a proper nonarithmetic probability law on $\mathbb{R}$ with mean $m \in (0, \infty)$. Further,

$$\nu(dt) := \sum_0^\infty e^{\kappa t}P(V_k \in dt) \tag{9.7}$$

is its renewal measure $\Sigma_0^\infty \eta^{(n)}$. Because $m \neq 0$ this renewal measure has the property that $|f| * \nu(t) < \infty$ for all t whenever f is dRi. By (2.8) and Lemma 9.2, $\check{g}_1$ is dRi, so $|\check{g}_1| * \nu(t) < \infty$ for all t. Thus $E\Sigma_0^\infty e^{\kappa V_k}|\check{g}_1(t - V_k)| < \infty$. By the Fubini–Tonelli theorem, $E\Sigma_0^\infty e^{\kappa V_k}\check{g}_1(t - V_k)$ exists and the expectation and sum in it may be interchanged, so it is the limit as $n \to \infty$ of $\Sigma_0^n E e^{\kappa V_k}\check{g}_1(t - V_k)$. That is, for each fixed t, $\check{g}_1 * \nu(t) = \lim_{n\to\infty} \check{g}_1 * \nu_n(t)$. So from (9.6) follows

$$\check{r} = \check{g}_1 * \nu \tag{9.8}$$

provided we have $\check{\delta}_n(t) \to 0$ as $n \to \infty$ for each fixed t. That follows by dominated convergence from $\delta_n(t) \to 0$, which in turn is immediate from the fact that the Y_k are i.i.d. with negative mean, so $V_n \to -\infty$ a.s.

Apply the key renewal theorem [for two-sided walks on $\mathbb{R}$, cf. Athreya, McDonald and Ney (1978), Theorem 4.2] to (9.8):

$$\check{r}(t) \to \frac{1}{m}\int_{\mathbb{R}} \check{g}_1, \qquad t \to \infty.$$

The conclusion (2.10) now follows from Lemma 9.3. For (2.12) note that $\int_{\mathbb{R}}\check{g}_1 = \int_{\mathbb{R}} F * g_1 = \int_{\mathbb{R}} F \int_{\mathbb{R}} g_1 = \int_{\mathbb{R}} g_1$.

CASE 2a. $P(M > 0) > 0$ and $P(M < 0) > 0$. The notation set up before Case 1 is retained. Set $X_n := \operatorname{sgn} \Pi_n$. Again starting from the first equality of (9.4), this time

$$\begin{aligned} P(R > e^t) = &\sum_{k=0}^{n-1} \left(P\left(X_k = 1, R > e^{t-V_k}\right) - P\left(X_k = 1, MR > e^{t-V_k}\right)\right) \\ &+ \sum_{k=0}^{n-1} \left(P\left(X_k = -1, R < -e^{t-V_k}\right)\right. \\ &\qquad \left. - P\left(X_k = -1, MR < -e^{t-V_k}\right)\right) \\ &+ P\left(\Pi_n R > e^t\right). \end{aligned}$$

In terms also of g_1 and g_{-1}, defined in (3.5) and (3.6), the above becomes

$$r(t) = \sum_{k=0}^{n-1} E e^{\kappa V_k} g_{X_k}(t - V_k) + \delta_n(t). \tag{9.9}$$

Here $\delta_n(t)$ is at most $e^{\kappa t}P(|R| > e^{t-V_n})$ which tends to 0 as $n \to \infty$ for each fixed t, for the same reasons as in the Case 1 proof.

We now give the M_n a new probability law, under which probability and expectation will be denoted $\tilde{P}, \tilde{E}$. Under $\tilde{P}$ the r.v.s $M, M_1, M_2, \dots$ are to remain i.i.d. with

$$\tilde{P}(M \in dy) := |y|^{\kappa} P(M \in dy), \qquad y \in \mathbb{R}.$$

The definitions of Π_n, Y_n, V_n and X_n in terms of the M_k remain in force. (9.9) becomes

$$r(t) = \sum_{k=0}^{n-1} \tilde{E} g_{X_k}(t - V_k) + \delta_n(t), \qquad t \in \mathbb{R}, n \in \mathbb{N}.$$

The sum on the right is $g_1 * \nu_{n-1,1}(t) + g_{-1} * \nu_{n-1,-1}(t)$ where $\nu_{n,x}(dt) := \sum_{k=0}^{n} \tilde{P}(X_k = x, V_k \in dt)$. Just as with (9.6) it follows that

$$\check{r}(t) = \sum_{k=0}^{n-1} \tilde{E} \check{g}_{X_k}(t - V_k) + \check{\delta}_n(t), \qquad t \in \mathbb{R}, n \in \mathbb{N}.$$

Now $\mathbf{X} := (X_n)_{n \ge 0}$ is a Markov chain on $\{-1, 1\}$ with $X_0 = 1$ and transition matrix $\begin{pmatrix} p & q \\ q & p \end{pmatrix}$ where

$$p := \tilde{P}(M > 0) = E\mathbf{1}_{M>0}|M|^{\kappa},$$

$$q := \tilde{P}(M < 0) = E\mathbf{1}_{M<0}|M|^{\kappa}.$$

So $p > 0$, $q > 0$ and $p + q = 1$. Let η_+, η_- be the conditional laws of $\log|M|$ under $\tilde{P}$ given $M > 0$ and $M < 0$, respectively,

$$\eta_+(dy) := \tilde{P}(M > 0, \log|M| \in dy)/p,$$

$$\eta_-(dy) := \tilde{P}(M < 0, \log|M| \in dy)/q.$$

Conditional on $\mathbf{X}$ the r.v.s $Y_1, Y_2, \dots$ are independent, with conditional laws

$$\tilde{P}(Y_n \in \cdot\,|\mathbf{X}) = \mathbf{1}_{X_n = X_{n-1}} \eta_+(\cdot) + \mathbf{1}_{X_n \ne X_{n-1}} \eta_-(\cdot).$$

Let $0 = N_0^{(+)} < N_1^{(+)} < N_2^{(+)} < \cdots$ be those n for which $X_n = 1$ and let $N_0^{(-)} < N_1^{(-)} < N_2^{(-)} < \cdots$ be those n for which $X_n = -1$. Set $I_n^{(+)} := \max\{i\colon N_i^{(+)} \le n - 1\}$, $I_n^{(-)} := \max\{i\colon N_i^{(-)} \le n - 1\}$. Then

$$\check{r}(t) = \tilde{E} \sum_{k=0}^{I_n^{(+)}} \check{g}_1\big(t - W_k^{(+)}\big) + \tilde{E} \sum_{k=0}^{I_n^{(-)}} \check{g}_{-1}\big(t - W_k^{(-)}\big) + \check{\delta}_n(t), \qquad t \in \mathbb{R}, n \in \mathbb{N}, \tag{9.10}$$

where $W_k^{(\pm)} := V_{N_{k(\pm)}}$. The first aim will be to let $n \to \infty$ here; the second, to apply the key renewal theorem to the result. Note that $\check{\delta}_n(t) \to 0$ as $n \to \infty$ for each t.

Let η be the law of $Y_1 + \cdots + Y_{N_1^{(+)}}$. Then one sees that $(W_k^{(+)})$ is random walk with step law η. That is, $W_k^{(+)} = \sum_1^k Z_j^{(+)}$ where the $Z_j^{(+)}$ are independent, each with law η. We show that

$$(9.11)\qquad \eta = p\eta_+ + \sum_{n=2}^{\infty} q^2 p^{n-2} \eta_-^{(2)} * \eta_+^{(n-2)}$$

and from this derive the properties needed for renewal theory:

$$(9.12)\qquad \int_{\mathbb{R}} y\eta(dy) = 2m;$$

$$(9.13)\qquad \eta \text{ is nonarithmetic.}$$

Now $\tilde{P}(N_1^{(+)} = 1) = p$, $\tilde{P}(N_1^{(+)} = n) = q^2 p^{n-2}$ for all $n \geq 2$. If $N_1^{(+)} = 1$, then Y_1 has law η_+, while if $n \geq 2$, then Y_1 and $Y_{N_1^{(+)}}$ have law η_+ and the Y_k in between, if any, law η_-. We conclude (9.11). For (9.12) write m_+, m_- for the means of η_+, η_-. Then

$$\begin{aligned}\int y\eta(dy) &= pm_+ + \sum_{n=2}^{\infty} q^2 p^{n-2}(2m_- + (n-2)m_+)\\ &= 2(pm_+ + qm_-).\end{aligned}$$

This equals $2m$ because

$$m_+ = \boldsymbol{E}\mathbf{1}_{M>0}|M|^{\kappa}\log|M|/p,\ m_- = \boldsymbol{E}\mathbf{1}_{M<0}|M|^{\kappa}\log|M|/q.$$

Since, under $\tilde{P}$, $\log|M|$ is nonarithmetic, we can find a subset B of its support such that the additive group generated by B is dense in $\mathbb{R}$. (By Kronecker's theorem, B needs to have but two elements.) Let B_+, B_- be the intersections of B with the supports of $\mathbf{1}_{M>0}\log|M|$ and $\mathbf{1}_{M<0}\log|M|$, respectively. Let B^* consist of the elements of B_+ together with *twice* each element of B_-. So B^* generates an additive group dense in $\mathbb{R}$. For any $b \in B^*$, and any $\varepsilon > 0$, we have if $b \in B_+$ that

$$\tilde{P}\big(|Z_1^{(+)} - b| < \varepsilon\big) \geq p\eta_+(b - \varepsilon, b + \varepsilon) > 0,$$

and if $\frac{1}{2}b \in B_-$, that

$$\tilde{P}\big(|Z_1^{(+)} - b| < \varepsilon\big) \geq q^2\big(\eta_-\big(\tfrac{1}{2}(b-\varepsilon), \tfrac{1}{2}(b+\varepsilon)\big)\big)^2 > 0,$$

so b is contained in the support of $Z_1^{(+)}$, that is, of η. This establishes (9.13).

Let $\nu := \sum_0^{\infty}\eta^{(n)}$ be the renewal measure generated by η. By the properties of η proved above, any dRi function is ν-integrable and the key renewal theorem gives its asymptotics. By Lemma 9.2, applied to (2.8), $\check{g}_1$ is dRi, so $|\check{g}_1| * \nu(t) < \infty$ for all t. That is, $\tilde{E}\sum_0^{\infty}|\check{g}_1(t - W_k^{(+)})| < \infty$, and this in turn allows us to use dominated convergence to show that for each fixed t,

$$(9.14)\qquad \tilde{E}\sum_{k=0}^{I_n^{(+)}} \check{g}_1\big(t - W_k^{(+)}\big) \to \tilde{E}\sum_{k=0}^{\infty} \check{g}_1\big(t - W_k^{(+)}\big), \qquad n \to \infty,$$

since $I_n^{(+)} \to \infty$ a.s.

For the other expectation in (9.10), first, $W_k^{(-)} = \sum_0^k Z_j^{(-)}$ where the $Z_j^{(-)}$ are independent, $Z_j^{(-)}$ for $j \geq 1$ have law η, but $Z_0^{(-)}$ has a different law η_0

$(= \sum_1^\infty qp^{n-1}\eta_- * \eta_+^{(n-1)})$. As with $\check{g}_1$, $\check{g}_{-1}$ is dRi, whence by dominated convergence the corresponding statement to (9.14) holds, with $W_k^{(-)}$, $\check{g}_{-1}$ and $I_n^{(-)}$ in the appropriate positions. (9.10) now yields

$$\begin{aligned}\check{r}(t) &= \tilde{E}\sum_0^\infty \check{g}_1\big(t - W_k^{(+)}\big) + \tilde{E}\sum_0^\infty \check{g}_{-1}\big(t - W_k^{(-)}\big) \\ &= \check{g}_1 * \nu(t) + \check{g}_{-1} * \eta_0 * \nu(t), \qquad t \in \mathbb{R}.\end{aligned} \tag{9.15}$$

Now the key renewal theorem for the whole line, applied to each term on the right, gives as $t \to \infty$ that

$$\begin{aligned}\check{r}(t) &\to \frac{1}{2m}\int_{\mathbb{R}} \check{g}_1 + \frac{1}{2m}\int_{\mathbb{R}} \check{g}_{-1} * \eta_0 \\ &= \frac{1}{2m}\int_{\mathbb{R}} (g_1 + g_{-1}),\end{aligned}$$

and this is the right-hand side of (2.14). The conclusion (2.10) follows from Lemma 9.3.

CASE 2b. $M \le 0$ a.s. We first show that

$$\int_0^\infty |P(R > t) - P(MM'R > t)| t^{\kappa - 1}\, dt < \infty. \tag{9.16}$$

This will follow from (2.8) and

$$\int_0^\infty |P(MR > t) - P(MM'R > t)| t^{\kappa - 1}\, dt < \infty$$

if we can show the latter. But its left-hand side is

$$\begin{aligned}&\int_0^\infty \int_{(-\infty, 0)} |P(uR > t) - P(uM'R > t)| P(M \in du) t^{\kappa-1}\, dt \\ &= \int_{(-\infty,0)} \int_0^\infty |P(R < -v) - P(M'R < -v)| (-uv)^{\kappa - 1}(-u)\, dv\, P(M \in du) \\ &= E|M|^\kappa \int_0^\infty |P(R < -v) - P(M'R < -v)| v^{\kappa-1}\, dv,\end{aligned}$$

which is finite by (2.3) and (2.9). So (9.16) is proved and we can apply Case 1 to it. Thus (2.10) holds with

$$C_+ = \frac{1}{m_2}\int_0^\infty (P(R > t) - P(MM'R > t)) t^{\kappa - 1}\, dt,$$

where $m_2 := E|MM'|^\kappa \log|MM'|$. The integral is the sum of

$$\int_0^\infty (P(R > t) - P(MR > t)) t^{\kappa-1}\, dt$$

and

$$\int_0^\infty (P(MR > t) - P(MM'R > t))t^{\kappa-1}\,dt,$$

and by the proof of (9.16), with modulus signs removed, this latter integral equals

$$\int_0^\infty (P(R < -t) - P(MR < -t))t^{\kappa-1}\,dt.$$

So C_+ is as claimed in (2.14), since obviously $m_2 = 2m$. □

LEMMA 9.4. *For any r.v.s X, Y on a common probability space,*

$$\int_0^\infty |P(X > t) - P(Y > t)| t^{\kappa-1}\,dt = \frac{1}{\kappa} E|(X^+)^\kappa - (Y^+)^\kappa|, \tag{9.17}$$

finite or infinite. When finite, we have further that

$$\int_0^\infty (P(X > t) - P(Y > t))t^{\kappa-1}\,dt = \frac{1}{\kappa} E\big((X^+)^\kappa - (Y^+)^\kappa\big). \tag{9.18}$$

PROOF. The values of $P(X > t)$ and $P(X^+ > t)$ for $t > 0$ coincide, so it suffices to establish the formulae for nonnegative X, Y. The left-hand side of (9.17) is the sum of $\int_0^\infty P(X \le t < Y)t^{\kappa-1}\,dt$ and $\int_0^\infty P(Y \le t < X)t^{\kappa-1}\,dt$. The first of these is

$$E\mathbf{1}_{X<Y}\int_X^Y t^{\kappa-1}\,dt = \frac{1}{\kappa}E\mathbf{1}_{X<Y}(Y^\kappa - X^\kappa)$$

and the second similarly is $(1/\kappa)E\mathbf{1}_{Y<X}(X^\kappa - Y^\kappa)$. Putting them together one obtains (9.17). The calculations for (9.18) are then straightforward. □

PROOF OF COROLLARY 2.4. The left-hand side of (2.8) equals

$$\int_0^\infty |P(\Psi(R) > t) - P(MR > t)| t^{\kappa-1}\,dt,$$

whose finiteness follows from (2.16) and Lemma 9.4. Similarly (2.9) follows from (2.17). So Theorem 2.3 applies, and its formulae for C_+ lead to (2.18) by way of (9.18). □

Theorem 3.2 relies on the following inversion formula for the renewal density of a suitable modified renewal process. It is in effect given in Stone (1966), but as no proof is to be found there it seems appropriate to provide one.

LEMMA 9.5. *Let χ and μ be probability measures on $\mathbb{R}$. Suppose χ is absolutely continuous with density $q \in C^2$. Suppose $\int_{\mathbb{R}} x^2\mu(dx) < \infty$, $m := \int_{\mathbb{R}} x\mu(dx) > 0$ and that μ has an absolutely continuous component. Then the measure $\sum_0^\infty \chi * \mu^{(n)}$ is absolutely continuous with continuous density $p(\cdot)$*

satisfying

$$p(x) - \frac{1}{m}\chi(-\infty, x] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-\hat{\mu}(\theta)} - \frac{1}{-im\theta}\right)d\theta, \qquad x \in \mathbb{R}.$$

PROOF. Since $\hat{\chi} \in L^1(\mathbb{R})$ it suffices to prove

$$(9.19)\qquad \begin{aligned} &p(x) - \frac{1}{m}\chi(-\infty, x] - q(x) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-\hat{\mu}(\theta)} - \frac{1-im\theta}{-im\theta}\right)d\theta, \qquad x \in \mathbb{R}. \end{aligned}$$

Now, for real θ,

$$\hat{\mu}(\theta) = 1 + im\theta + A(\theta)\theta^2,$$

where $A(\theta) = O(1)$ as $\theta \to 0$. Take θ_0 so small that $|\theta \mathfrak{J} A(\theta)| \le \frac{1}{2}m$ for $-\theta_0 \le \theta \le \theta_0$. Then for such θ and for $\frac{1}{2} \le r \le 1$ we have

$$\begin{aligned} &\left|\frac{1}{1-r\hat{\mu}(\theta)} - \frac{1-im\theta}{1-r-im\theta}\right| \\ &= \frac{r\theta^2|m^2 - (1-im\theta)A(\theta)|}{|1-r-irm\theta + rA(\theta)\theta^2||1-r-im\theta|} \\ &\le \frac{\theta^2|m^2 - (1-im\theta)A(\theta)|}{|rm\theta - r\theta^2\mathfrak{J}A(\theta)||m\theta|} \le \frac{4|m^2 - (1-im\theta)A(\theta)|}{m^2}, \end{aligned}$$

which is bounded over the given range of θ and r. On the other hand, for $|\theta| > \theta_0$ and $\frac{1}{2} \le r \le 1$,

$$\left|\frac{1}{1-r\hat{\mu}(\theta)} - \frac{1-im\theta}{1-r-im\theta}\right| \le \frac{1}{1-\sup_{|\theta|\ge\theta_0}|\hat{\mu}(\theta)|} + 1 + \frac{1}{m\theta_0},$$

which is finite because μ has an absolutely continuous component ("strongly nonlattice" would suffice at this point). We may now, since $\hat{\chi} \in L^1(\mathbb{R})$, apply dominated convergence to conclude that

$$(9.20)\qquad \begin{aligned} &\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-r\hat{\mu}(\theta)} - \frac{1-im\theta}{1-r-im\theta}\right)d\theta \\ &\to \int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-\hat{\mu}(\theta)} - \frac{1-im\theta}{-im\theta}\right)d\theta \end{aligned}$$

as $r \uparrow 1$. Let us write

$$p_r(x) := \int_{\mathbb{R}} q(x-y)\sum_{n=0}^{\infty} r^n\left(\mu^{(n)}(dy) - E_m^{(n)}(dy)\right), \qquad x \in \mathbb{R},$$

where $E_m(dy) := \mathbf{1}_{y>0} m e^{-my}\, dy$. Then p_r is continuous and

$$\int_{\mathbb{R}} e^{i\theta x} p_r(x)\, dx = \hat{\chi}(\theta)\left(\frac{1}{1 - r\hat{\mu}(\theta)} - \frac{1 - im\theta}{1 - r - im\theta}\right) \in L^1(\mathbb{R}).$$

So the left-hand side of (9.20) equals $2\pi p_r(x)$, for each x. As $r \uparrow 1$,

$$\sum_0^\infty r^n \int_{\mathbb{R}} q(x-y)\mu^{(n)}(dy) \uparrow \sum_0^\infty \int_{\mathbb{R}} q(x-y)\mu^{(n)}(dy) = p(x),$$

while

$$\begin{aligned}\sum_{n=0}^\infty \int_{\mathbb{R}} q(x-y) E_m^{(n)}(dy) &\uparrow \sum_{n=0}^\infty \int_{\mathbb{R}} q(x-y) E_m^{(n)}(dy) \\ &= \int_{\mathbb{R}} q(x-y)\left(\delta_0(dy) + \frac{1}{m}\mathbf{1}_{y>0}\, dy\right) \\ &= q(x) + \frac{1}{m}\chi(-\infty, x],\end{aligned}$$

so we find on subtracting that $p_r(x) \to p(x)$, which proves (9.19) as desired. □

PROOF OF THEOREM 3.1. (3.1) says that $\eta^{(n_0)}$ is the sum of an absolutely continuous measure η_0, with density $d(\cdot)$, say, and a measure η_1 that has $\tilde{\eta}_1(\beta) < 1$. We need to alter this decomposition to obtain one with additional properties. First we replace the density $d(\cdot)$ by $c \wedge d(\cdot)$ and the measure η_1 by $\eta_1(dx) + (d(x) - c)^+\, dx$, where c is constant. This converts $d(\cdot)$ to a bounded function and, if c is taken large enough, the property $\tilde{\eta}_1(\beta) < 1$ is retained. Thus we may assume $d(\cdot)$ is bounded. Then, for $m \ge 2$, $\eta^{(mn_0)} = \eta_{0,m} + \eta_{1,m}$ where $\eta_{0,m} := \sum_{j=2}^m \binom{m}{j}\eta_0^{(j)} * \eta_1^{(m-j)}$ and $\eta_{1,m} := m\eta_0 * \eta_1^{(m-1)} + \eta_1^{(m)}$. We may take m so large that

$$\tilde{\eta}_{1,m}(\beta) = (m\tilde{\eta}_0(\beta) + \tilde{\eta}_1(\beta))\tilde{\eta}_1^{m-1}(\beta) < 1.$$

Now $\eta_{0,m}$ is the convolution of $\eta_0^{(2)}$ with some other measure, and $\eta_0^{(2)}$ has density $d * d$ which, since d is bounded and integrable, is continuous [cf. Hewitt and Stromberg (1965), Theorem 21.33]. So $\eta_{0,m}$ has a continuous density. Finally, we may approximate this density from below, by a C^2 function f of compact support, so closely that $\int_{\mathbb{R}} e^{\beta x} f(x)\, dx$ differs from $\tilde{\eta}_{0,m}(\beta)$ by as little as desired. We gain a new decomposition $\eta^{(mn_0)} = \mu_0 + \mu_1$ where the measure μ_0 has density f and the measure μ_1 still has $\tilde{\mu}_1(\beta) < 1$. In the formulation (3.1) we may thus assume additionally that the density of ϕ_0 is C^2 with compact support.

Set $\nu_2 := \sum_0^\infty \eta^{(nn_0)}$. Then $\nu_2 = \delta_0 + \eta^{(n_0)} * \nu_2$ whence, on substituting (3.1),

$$(\delta_0 - \delta\phi_1) * \nu_2 = \delta_0 + (1-\delta)\phi_0 * \nu_2$$

and so

$$\nu_2 = \left(\sum_0^\infty \delta^n \phi_1^{(n)}\right) * (\delta_0 + (1-\delta)\phi_0 * \nu_2).$$

Set $\nu_1 := (\sum_{n=0}^{n_0-1}\eta^{(n)}) * (\sum_{n=0}^\infty \delta^n \phi_1^{(n)})$ and $\nu_0 := (1-\delta)\phi_0 * \nu_1 * \nu_2$. Now $\nu = (\sum_0^{n_0-1}\eta^{(n)}) * \nu_2$, so on substituting the above expression for ν_2 we find

$$\nu = \nu_1 * (\delta_0 + (1-\delta)\phi_0 * \nu_2) = \nu_1 + \nu_0.$$

This is the claimed decomposition. Observe that ν_1 has total mass $n_0/(1-\delta)$ and

$$\tilde{\nu}_1(\beta) = \sum_{k=0}^{n_0-1} \tilde{\eta}^k(\beta) \sum_{n=0}^\infty \delta^n \tilde{\phi}_1^n(\beta) < \infty.$$

By construction, ν_0 is absolutely continuous, with continuous density p, say, and by way of Blackwell's theorem applied to the renewal measure ν_2 it is clear that $p(x) \to 0$ as $x \to -\infty$. To find out about the behaviour of p at $+\infty$ we set $\chi := (1-\delta)n_0^{-1}\phi_0 * \nu_1$ and observe that $\chi * \sum_0^\infty \eta^{(nn_0)}$ has density $n_0^{-1}p$, so we may apply Lemma 9.5 with μ, m, p therein replaced by $\eta^{(n_0)}, n_0 m, n_0^{-1}p$, to conclude

$$\frac{1}{n_0}p(x) - \frac{\chi(-\infty, x]}{n_0 m} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-\hat{\eta}^{n_0}(\theta)} - \frac{1}{-in_0 m\theta}\right) d\theta.$$

From properties of ϕ_0 and ν_1 we find that $\chi(x,\infty) = o(e^{-\beta x})$ as $x \to \infty$, so

$$\begin{aligned} &p(x) - \frac{1}{m} \\ (9.21)\qquad &= \frac{n_0}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\left(\frac{1}{1-\hat{\eta}^{n_0}(\theta)} - \frac{1}{-in_0 m\theta}\right) d\theta + o(e^{-\beta x}).\end{aligned}$$

To find how the integral behaves as $x \to \infty$ it will be simplest to assume that $1 - \hat{\eta}(\theta)$ has just one zero in D, located at $\theta = \theta_0$ and of order k, say. The general case will follow immediately by adding in similar terms for the other zeroes. Let a_j be the residue of $(\theta - \theta_0)^{j-1}/(1-\hat{\eta}(\theta))$ at $\theta = \theta_0$. Then for $j = 1, \dots, k$ the residue of $n_0(\theta-\theta_0)^{j-1}/(1-\hat{\eta}^{n_0}(\theta))$ at $\theta = \theta_0$ is also a_j. So

$$w(\theta) := \frac{n_0}{1-\hat{\eta}^{n_0}(\theta)} - \frac{1}{-im\theta} - \sum_1^k \frac{a_j}{(\theta-\theta_0)^j}$$

is holomorphic in D, and bounded and continuous in $\overline{D}$. By Cauchy's theorem, as in Stone [(1966), page 275],

$$\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)w(\theta)\, d\theta = \int_{\mathbb{R}} e^{-x(\beta+i\theta)}\hat{\chi}(\theta - i\beta)w(\theta - i\beta)\, d\theta$$

and the right-hand side is $e^{-\beta x}$ times an integral that, by the Riemann–

Lebesgue lemma, tends to 0 as $x \to \pm\infty$. Returning to (9.21), we thus find that

$$p(x) - \frac{1}{m} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\hat{\chi}(\theta)\sum_1^k \frac{a_j}{(\theta-\theta_0)^j}\,d\theta + o(e^{-\beta x}), \quad x \to \infty. \tag{9.22}$$

Now

$$\hat{\chi}(\theta) = \frac{\sum_0^{n_0-1}\hat{\eta}^n(\theta)}{n_0}\left(1 - \frac{1-\hat{\eta}^{n_0}(\theta)}{1-\delta\hat{\phi}_1(\theta)}\right),$$

so $\hat{\chi}(\theta_0) = 1$ [note that $|\delta\hat{\phi}_1(\theta_0)| \le \delta\tilde{\phi}_1(\beta) < 1$] and indeed $\hat{\chi}(\theta) - 1$ has a kth order zero at θ_0. So the principal part of $\hat{\chi}(\theta)\sum_1^k a_j/(\theta-\theta_0)^j$ at θ_0 is just $\sum_1^k a_j/(\theta-\theta_0)^j$ and the rest is disposed of by a further application of Cauchy's theorem and the Riemann–Lebesgue lemma, as above. We conclude that (9.22) continues to hold when $\hat{\chi}(\theta)$ is deleted from the integrand. We may then write

$$p(x) - \frac{1}{m} = -\frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-ix(\theta-i\beta)}\sum_1^k \frac{a_j}{(\theta-i\beta-\theta_0)^j}\,d\theta - \int_{\mathbb{R}} e^{-ix\theta}\sum_1^k \frac{a_j}{(\theta-\theta_0)^j}\,d\theta\right) + o(e^{-\beta x}),$$

since the introduced term is $o(e^{-\beta x})$. By one more use of Cauchy's theorem we find

$$p(x) - \frac{1}{m} = -\frac{1}{2\pi}\int_{\mathscr{C}} e^{-ix\theta}\sum_1^k \frac{a_j}{(\theta-\theta_0)^j}\,d\theta + o(e^{-\beta x}), \qquad x \to \infty.$$

This integral is not altered if we extend the sum to run over $j = 1, 2, \dots,$ since to do so merely adds a holomorphic function to the integrand. The conclusion (3.2) now follows. □

Proof of Theorem 3.2. (i) Assume (3.7). By Theorem 2.3, (2.10) holds. (3.7) gives $e^{\beta t}g_1(t) \in L^1(\mathbb{R})$. That is not enough to force $e^{\beta t}g_1(t)$ to be $o(1)$ at $+\infty$, so we need a smoothing transform in the present proof also, despite the spreadout assumption. Thus pick $b > \beta$ and set

$$K(t) := be^{-bt}\mathbf{1}_{t>0}, \qquad t \in \mathbb{R}.$$

It is easy to check that $e^{\beta t}K * g_1(t) \in L^1(\mathbb{R})$. Further, one can prove that $e^{\beta t}K * g_1(t)$ is dRi by the method of proof of Lemma 9.2. So $e^{\beta t}K * g_1(t) \to 0$ as $t \to \infty$.

In place of (9.8) we may prove

$$K * r = K * g_1 * \nu$$

by the same method, since the $\vee$ smoothing is just convolution with the kernel K in the case $b = 1$. Here ν is the renewal measure of the proper law η, which under present assumptions satisfies the conditions of Theorem 3.1. In terms of the ν_1 and p of that result we obtain

$$K * r = K * g_1 * \nu_1 + K * g_1 * p.$$

Since $(1/m)\int_{\mathbb{R}} g_1 = C_+$, we can rewrite this as

$$K * (r(\cdot) - C_+) = K * g_1 * \nu_1 + K * g_1 * (p(\cdot) - 1/m). \tag{9.23}$$

Write

$$c(t) := \frac{1}{2\pi} \int_c e^{-i\theta t} \frac{d\theta}{1 - \hat{\eta}(\theta)}$$

and observe that $c(t)$ is $O(e^{-\varepsilon t})$ as $t \to \infty$ and $O(e^{-(\beta-\varepsilon)t})$ as $t \to -\infty$, where $\varepsilon > 0$ is a constant such that all the zeroes of $1 - \hat{\eta}(\theta)$ in D lie in the strip $\varepsilon < -\Im\theta < \beta - \varepsilon$. Since $g_1(t) = O(1)$ as $t \to -\infty$ and $e^{\beta t} g_1(t) \in L^1(\mathbb{R})$, the convolution $g_1 * c$ is well defined and asymptotically is like c in that

$$g_1 * c(t) = \begin{cases} O(e^{-\varepsilon t}), & t \to \infty, \\ O(e^{-(\beta-\varepsilon)t}), & i \to -\infty. \end{cases}$$

Write $p_1(t) := p(t) - 1/m + c(t)$. Then by (3.2) and the above estimates on $c(\cdot)$ we obtain

$$e^{\beta t} p_1(t) = \begin{cases} o(1), & t \to \infty, \\ O(e^{\varepsilon t}), & t \to -\infty. \end{cases}$$

(9.23) becomes

$$K * (r(\cdot) - C_+ + g_1 * c(\cdot)) = K * g_1 * \nu_1 + K * g_1 * p_1.$$

The value of the right-hand side at t is

$$\int_{\mathbb{R}} e^{\beta(t-u)} K * g_1(t-u) e^{\beta u} \nu_1(du) + \int_{\mathbb{R}} e^{\beta(t-u)} p_1(t-u) e^{\beta u} K * g_1(u)\, du.$$

The first integral tends to 0 as $t \to \infty$ because $e^{\beta t} K * g_1(t) \to 0$ and $\int_{\mathbb{R}} e^{\beta u} \nu_1(du) < \infty$. The second integral functions similarly and so

$$K * (C_+ - r - g_1 * c)(t) = o(e^{-\beta t}), \qquad t \to \infty.$$

Now put $r_1(t) := C_+ - r(t) - g_1 * c(t) \mathbf{1}_{t>0}$. Then, for $t > 0$, $K * r_1(t)$ differs from the above left-hand side by $be^{-bt} \int_{-\infty}^{0} e^{bu} g_1 * c(u)\, du$, which is $O(e^{-bt})$ as $t \to \infty$. We conclude, taking the real part, that

$$K * \Re r_1(t) = o(e^{-\beta t}), \qquad t \to \infty. \tag{9.24}$$

The final task is to unpack the convolution with K, by use of Theorem 9.6 below. The function r_1 is bounded on $\mathbb{R}$. The kernel K has $\hat{K}(\theta) = b/(b - i\theta)$ for $\theta \in \mathbb{R}$; hence it satisfies the theorem with $p = 1$ and $a = \infty$. It remains only to check that $f := \Re r_1$ satisfies (9.25) with $p = 1$. Now

$$\begin{aligned} r(x+y) - r(x) &= e^{\kappa(x+y)} P(R > e^{x+y}) - e^{\kappa x} P(R > e^x) \\ &\le (e^{\kappa y} - 1) e^{\kappa x} P(R > e^x) \\ &\le 3\kappa y \cdot 2C_+ \end{aligned}$$

for $x \ge x_1$, say, and $y \le (1/\kappa)\log 4$. So $-r$ satisfies (9.25). Next,

$$g_1 * c(t) = \frac{1}{2\pi}\int_{\mathscr{C}} e^{-i\theta t}\frac{\hat{g}_1(\theta)}{1-\hat{\eta}(\theta)}\,d\theta,$$

which is a finite sum of terms of the form $c_{l,j}t^{j-1}e^{-i\theta_l t}$, where the j are positive integers and the θ_l are in D. It is easy to see that (the real parts of) such terms satisfy (9.25). Hence so does $\Re r_1$. The Tauberian remainder theorem then gives $\Re r_1(t) = O(e^{-\beta t/2})$ as $t \to \infty$, which is (3.8).

(ii) is of course the same result, for $-R$.

(iii) Immediate, via Lemma 9.4. □

THEOREM 9.6 [Beurling–Ganelius Tauberian remainder theorem, Ganelius (1962)]. *Let $K \in L^1(\mathbb{R})$ be such that $\hat{K}$ does not vanish for real arguments, and suppose there exist constants $p > \frac{1}{2}$, $a > 0$, C and a function w holomorphic in the strip $-a < \Im\zeta < 0$, such that*

$$|w'(\zeta)| < C(1+|\zeta|)^{p-1}, \qquad -a < \Im\zeta < 0,$$
$$\lim_{\eta\downarrow 0} w(\xi - i\eta) = 1/\hat{K}(\xi), \qquad \xi \in \mathbb{R}.$$

Let β be constant, with $0 < \beta < a$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded and satisfy the Tauberian condition

(9.25) $f(x) - f(x+y) \le Ae^{-\beta x/(p+1)}, \qquad 0 \le y \le e^{-\beta x/(p+1)},\ x > x_0,$

for some constants A, x_0. Then

$$K * f(x) = O(e^{-\beta x}), \qquad x \to \infty$$

implies

$$f(x) = O(e^{-\beta x/(p+1)}), \qquad x \to \infty.$$

PROOF OF THEOREM 3.3. By Theorem 2.3, (2.10) and (2.11) hold. Suppose $P(M > 0) > 0$ as well as $P(M < 0) > 0$. We use the notation of the proof of Theorem 2.3, Case 2a.

We first check that $E\mathbf{1}_{M>0}M^u < 1$ for $\kappa \le u \le \kappa + \beta$. Because this is a convex function of u and because the assertion holds, by assumption, at the ends of the interval, it must hold within it also.

What we have just proved is that $\tilde{\mu}_+(u) < 1$ for $0 \le u \le \beta$, so $\hat{\mu}_+(\theta) \ne 1$ in $\overline{D}$. Observe that $\mu_+ = p\eta_+$ and $\mu_- = q\eta_-$, hence $\hat{\eta} = \hat{\mu}_+ + \hat{\mu}_-^2/(1-\hat{\mu}_+)$, and we see $\hat{\eta}$ is holomorphic in D and continuous in $\overline{D}$. The same holds for $\hat{\eta}_0 = \hat{\mu}_-/(1-\hat{\mu}_+)$. So we may now prove

$$K * r = K * g_1 * \nu + K * g_{-1} * \eta_0 * \nu$$

similarly to (9.15), and work on each term on the right exactly as in the previous proof.

The function $g_{-1} * \eta_0$ is $O(1)$ at $-\infty$ and has $e^{\beta t}g_{-1} * \eta_0(t) \in L^1(\mathbb{R})$, as required. We conclude again (9.24), where now $r_1(t) = C_+ - r(t) - (g_1 + g_{-1} * \eta_0) * c(t)\mathbf{1}_{t>0}$, and Tauberian remainder theory gives $\Re r_1(t) = O(e^{-\beta t/2})$

as before. Now

$$(g_1 + g_{-1} * \eta_0) * c(t) = \frac{1}{2\pi}\int_{\mathscr{C}} e^{-i\theta t}\frac{\hat{g}_1(\theta) + \hat{g}_{-1}(\theta)\hat{\eta}_0(\theta)}{1 - \hat{\eta}(\theta)}\,d\theta,$$

whence the claimed form of the remainder term on inserting the expressions above for $\hat{\eta}$ and $\hat{\eta}_0$.

The result for $P(R < -t)$ is of course the same, for $-R$ in place of R.

There remains the rather arcane case when $M \le 0$ a.s. In fact the above proof goes through, with μ_+ the zero measure, so that $\eta = \mu_-^{(2)}$ and $\eta_0 = \mu_-$. □

See Schäl (1971), Satz 4.1b, for a somewhat similar use of Stone's decomposition, without explicit rates.

PROOF OF THEOREM 4.1. In (2.16), $\Psi(R)$ is now $Q + MR$ and the left-hand side is the sum of

$$I_1 := \frac{1}{\kappa}\boldsymbol{E}\mathbf{1}_{-Q<MR\le 0}(Q+MR)^\kappa,$$

$$I_2 := \frac{1}{\kappa}\boldsymbol{E}\mathbf{1}_{0<MR\le -Q}(MR)^\kappa,$$

$$I_3 := \frac{1}{\kappa}\boldsymbol{E}\mathbf{1}_{Q>0,\,MR>0}\big((Q+MR)^\kappa - (MR)^\kappa\big)$$

and

$$I_4 := \frac{1}{\kappa}\boldsymbol{E}\mathbf{1}_{0<-Q<MR}\big((MR)^\kappa - (Q+MR)^\kappa\big).$$

In I_1, $0 < Q + MR \le Q^+$, so I_1 is finite by comparison with $E(Q^+)^\kappa$. Similarly I_2 is at most $(1/\kappa)E\mathbf{1}_{Q<0}(-Q)^\kappa$, which is finite. For the other two we need the following elementary inequalities:

$$|x+y|^r \le c_r(|x|^r + |y|^r), \qquad x, y \in \mathbb{R},\ r > 0, \tag{9.26}$$

where $c_r = 2^{r-1} \vee 1$;

$$\big||x|^r - |y|^r\big| \le \begin{cases} |x-y|^r, & 0 < r \le 1,\\ r|x-y|(|x| \vee |y|)^{r-1}, & 1 < r < \infty, \end{cases} \qquad x, y \in \mathbb{R}. \tag{9.27}$$

When $\kappa \le 1$, (9.27) gives $I_3 \le (1/\kappa)E(Q^+)^\kappa < \infty$, while when $\kappa > 1$ it gives

$$\begin{aligned} I_3 &\le \boldsymbol{E}\mathbf{1}_{Q>0,\,MR>0}Q(Q+MR)^{\kappa-1}\\ &\le c_{\kappa-1}E(Q^+)^\kappa + c_{\kappa-1}EQ^+|MR|^{\kappa-1}\\ &= c_{\kappa-1}E(Q^+)^\kappa + c_{\kappa-1}E\big(Q^+|M|^{\kappa-1}\big)E|R|^{\kappa-1}. \end{aligned}$$

Now $EQ^+|M|^{\kappa-1}$ is finite by (2.3), (4.2) and Hölder's inequality, while $E|R|^{\kappa-1}$ is finite by Vervaat (1979), Theorem 5.1.

A similar use of the two cases (9.27) proves I_4 finite. Thus (2.16) is proved, and (2.17) follows similarly because $-R$ solves (1.1) with $(-Q, M)$ in place of (Q, M). Corollary 2.4 now gives all but the last conclusion.

For the nontrivial half of that, let us assume (4.5) and prove $|t|^{\kappa}P(|R| > t)$ is bounded away from 0 for all large t. To the notation of Proposition 4.2 add $m_0 := \operatorname{med} R$ and

$$T_n := R_n + \Pi_n m_0, \qquad n = 1, 2, \dots, \qquad T_0 := m_0,$$

so that $T_n = T_{n-1} + U_n$ where $U_n := \Pi_{n-1}(Q_n - m_0(1 - M_n))$ for $n = 1, 2, \dots$. Now, for $t > |m_0|$,

$$P(|T_n| > t \text{ for some } n \geq 1) \geq P(|U_n| > 2t \text{ for some } n \geq 1),$$

since if $|U_n| > 2t$, then either $|T_{n-1}| > t$ or, if not, then $|T_n| \geq |U_n| - |T_{n-1}| > 2t - t$. Then

$$\begin{aligned} P(|R| > t) &\geq \tfrac{1}{2}P(|T_n| > t \text{ for some } n \geq 1) \quad [\text{by } (4.6)] \\ &\geq \tfrac{1}{2}P(|U_n| > 2t \text{ for some } n \geq 1) \\ &\geq \tfrac{1}{2}P(|Q - m_0(1 - M)| > \varepsilon)P(|\Pi_n| > 2t/\varepsilon \text{ for some } n \geq 1). \end{aligned}$$

By (4.5), ε can be chosen so that $P(|Q - m_0(1 - M)| > \varepsilon) > 0$. So to finish off it suffices to show $P(|\Pi_n| > e^t$ for some $n \geq 1)$ is at least $\delta e^{-\kappa t}$ for all large t, where $\delta > 0$. But this probability is $P(\sup_{n \in \mathbb{N}} V_n > t)$, and standard results on the transient random walk (V_n) apply, notwithstanding that its step $Y = \log|M|$ may have positive probability of being $-\infty$. Let H be the right Wiener–Hopf factor of the law of Y, in other words, the (improper) law of the strict upper ladder height of (V_n):

$$H(dy) := \sum_{n=1}^{\infty} P(V_k \leq 0 (k < n), V_n \in dy), \qquad 0 < y < \infty.$$

Feller [(1971), Chapter XII, (5.13)] proves that

$$P\Big(\sup_n V_n > t\Big) \sim \frac{H(\mathbb{R}^+)}{\beta\kappa} e^{-\kappa t}, \qquad t \to \infty, \tag{9.28}$$

where $\beta := \int_0^\infty t e^{\kappa t} H(dt)$. Now $H(\mathbb{R}^+) > 0$ because $P(Y > 0) > 0$ and β is the expected value of the strict upper ladder height for the "associated" random walk, that with step ${}_aY$ having law

$$P({}_aY \in dy) := e^{\kappa y}P(Y \in dy), \qquad y \in \mathbb{R}.$$

This ladder height is at most the value of one step of the random walk, so its expected value is at most $E_a Y^+$ [cf. Gut (1974), Theorem 2.1(c), with $c = 0$, $r = 1$]. Since $E_a Y^+ = EY^+ e^{\kappa Y}$, finiteness comes from (2.4). So the coefficient of $e^{-\kappa t}$ in (9.28) is positive. □

PROOF OF COROLLARY 4.3. Apply (9.27) to the expression inside the expectation in (4.4). That gives what is claimed when $\kappa \le 1$, while for $\kappa > 1$,

$$C_+ + C_- \le \frac{1}{m} E|Q|(|Q + MR| \vee |MR|)^{\kappa - 1}$$

$$\le \frac{c_{\kappa-1}}{m} E|Q|\left(|Q|^{\kappa-1} + |MR|^{\kappa-1}\right)$$

and (4.7) follows. □

For Proposition 4.6 a lemma is needed.

LEMMA 9.7. *If X and Y are independent with $X \sim \beta_{a,b}$ and $Y \sim \beta_{c,a+b}$, then $Y(1 + X) \sim \beta_{c,b}$.*

PROOF. For $-c < s < b$,

$$EY^s = \Gamma(c+s)\Gamma(a+b-s)/(\Gamma(c)\Gamma(a+b))$$

and

$$E(1+X)^s = \Gamma(b-s)\Gamma(a+b)/(\Gamma(a+b-s)\Gamma(b)),$$

so

$$E(Y(1+X))^s = \Gamma(c+s)\Gamma(b-s)/(\Gamma(c)\Gamma(b))$$

which is the Mellin transform of the $\beta_{c,b}$ law. □

PROOF OF PROPOSITION 4.6. Write $R_1 := R$ and $R_{l+1} := Y_l(1 + R_l)$ for $l = 1, \dots, k-1$. By induction, via the lemma, $R_l \sim \beta_{a_l, b}$ for $l = 1, \dots, k$. Since $Q + MR = Y_k(1 + R_k)$, a final use of the lemma gives $Q + MR \sim \beta_{a_1, b}$. □

PROOF OF THEOREM 4.7. In Theorems 3.2 and 3.3 all that remains to be established is (3.11), for (3.12) is the same assertion but for $(-R, M, -Q)$. The verification of (3.11) follows exactly the verification of (2.16) at the start of the proof of Theorem 4.1, but with $\kappa + \beta$ replacing κ. Finiteness of $E|R|^{\kappa+\beta-1}$ is needed. It holds, similarly to (4.8), because we have insisted $\beta < 1$, so $\|M\|_{\kappa+\beta-1} < 1$. □

PROOF OF PROPOSITION 5.1. Choose c such that $E \log M < -c < 0$. Then with probability 1 for all large n we have $\Pi_n \le e^{-cn}$. The moment assumption on Q implies $\sum_{n=1}^{\infty} P(\log(Q_n \vee 1) > cn/2) < \infty$, so $Q_n \le e^{cn/2}$ for all large n, almost surely. But these facts give that $\lim_{n \to \infty} Z_n(t)$ exists a.s. and does not depend on t. Theorem 2.1 now gives the result. □

PROOF OF THEOREM 5.2. Proposition 5.1 gives existence of the law of R. Now

$$E|((Q \vee MR)^+)^\kappa - ((MR)^+)^\kappa| = E\mathbf{1}_{MR<Q,Q>0}(Q^\kappa - (MR^+)^\kappa)$$
$$\leq E(Q^+)^\kappa < \infty,$$

so Corollary 2.4 gives (2.10) and the formula for C_+. If $Q \leq 0$ a.s., then obviously $C_+ = 0$. Otherwise, choose $c > 0$ so that $P(Q > c) > 0$ and set $N_t := \min\{k: \Pi_{k-1} > t/c\}$. Then, for $t > 0$,

$$\begin{aligned} P(R > t) &= P\left(\bigvee_{k=1}^{\infty} Q_k \Pi_{k-1} > t\right) \\ &\geq \sum_{n=1}^{\infty} P(N_t = n, Q_n > c) \\ &= P(Q > c) \sum_{n=1}^{\infty} P(N_t = n) = P(Q > c) P\left(\bigvee_{k=1}^{\infty} \Pi_{k-1} > t/c\right). \end{aligned}$$

That $P(\bigvee_{k=0}^{\infty}\Pi_k > t)$ is at least $\delta t^{-\kappa}$ for large t now follows as shown in the last part of the proof of Theorem 4.1. So $C_+ > 0$. □

PROOF OF THEOREM 5.3. By the calculation at the beginning of the previous proof, with $\kappa + \beta$ replacing κ, we find that (3.11) holds for the present Ψ. The result follows by Theorem 3.2(iii). □

PROOF OF PROPOSITION 6.1. One may simply copy the proof of Proposition 5.1, making use additionally of the fact that $L_n \leq e^{cn/2}$ for all large n, almost surely. □

PROOF OF THEOREM 6.2. Proposition 6.1 gives existence of the law of R. We first check that

$$E(R^+)^p < \infty, \qquad 0 < p < \kappa. \tag{9.29}$$

Since the operator $^+$ is subadditive,

$$\begin{aligned} (Q + M\max(L, R))^+ &\leq Q^+ + M\max(L^+, R^+) \\ &\leq Q^+ + ML^+ + MR^+, \end{aligned}$$

and then

$$\begin{aligned} \|R^+\|_p &= \|(Q + M\max(L, R))^+\|_p \\ &\leq \|Q^+ + ML^+ + MR^+\|_p \\ &\leq \|Q^+\|_p + \|ML^+\|_p + \|M\|_p \|R^+\|_p, \end{aligned}$$

whence (9.29) since $0 < EM^p < 1$.

We verify (2.16). Its left-hand side is the sum of

$$I_1 := E\mathbf{1}_{-Q<M(R\vee L)\le 0}(Q+M(R\vee L))^{\kappa},$$

$$I_2 := E\mathbf{1}_{0<M(R\vee L)\le -Q}(((MR)^+)^{\kappa},$$

$$I_3 := E\mathbf{1}_{Q>0,\,M(R\vee L)>0}\big((Q+M(R\vee L))^{\kappa}-((MR)^+)^{\kappa}\big)$$

and

$$I_4 := E\mathbf{1}_{0<-Q<M(R\vee L)}|(Q+M(R\vee L))^{\kappa}-((MR)^+)^{\kappa}|.$$

Obviously $I_1 \le E(Q^+)^{\kappa} < \infty$ and $I_2 \le E(Q^-)^{\kappa} < \infty$. Write $I_3 = I_{31} + I_{32}$ and $I_4 = \Sigma_1^4 I_{4i}$ where

$$I_{31} := E\mathbf{1}_{Q>0,\,R>0,\,R>L}\big((Q+MR)^{\kappa}-(MR)^{\kappa}\big),$$

$$I_{32} := E\mathbf{1}_{Q>0,\,L>0,\,R\le L}\big((Q+ML)^{\kappa}-((MR)^+)^{\kappa}\big),$$

$$I_{41} := E\mathbf{1}_{0<-Q<MR,\,R\ge L}\big((MR)^{\kappa}-(Q+MR)^{\kappa}\big),$$

$$I_{42} := E\mathbf{1}_{0<-Q<ML,\,R\le 0}(Q+ML)^{\kappa},$$

$$I_{43} := E\mathbf{1}_{0<-Q<ML,\,0<R<L,\,MR<Q+ML}\big((Q+ML)^{\kappa}-(MR)^{\kappa}\big),$$

$$I_{44} := E\mathbf{1}_{0<-Q<ML,\,0<R<L,\,MR>Q+ML}\big((MR)^{\kappa}-(Q+ML)^{\kappa}\big).$$

Finiteness of I_{31} and I_{41} follows just as for I_3 and I_4 in the proof of Theorem 4.1. For the others,

$$I_{32} \le E\mathbf{1}_{Q>0,\,L>0}(Q+ML)^{\kappa} < \infty,$$

$$I_{42} \le E\mathbf{1}_{L>0}(ML)^{\kappa} < \infty$$

and the latter bound holds also for I_{43} and I_{44}.

With (2.16) established, Corollary 2.4 gives (2.10) and the formula for C_+.

Now suppose also that $Q(c) := Q - c(1-M) \ge 0$ a.s. Since

$$R - c =_L Q(c) + M(L-c) \vee M(R-c),$$

Proposition 6.1 gives

$$R - c =_L \sup\left(R^*(c), \left(\sum_{k=1}^{m} Q_k(c)\Pi_{k-1} + (L_m - c)\Pi_m\right)_{m=1}^{\infty}\right),$$

where $R^*(c) := \Sigma_{k=1}^{\infty} Q_k(c)\Pi_{k-1}$. Suppose first that $P(Q(c)>0)>0$. Then $R^*(c) > 0$ and

$$R^*(c) =_L Q(c) + MR^*(c), \qquad R^*(c) \text{ independent of } (Q(c), M),$$

so Theorem 4.1 applies to $R^*(c)$ and gives that $P(R^*(c)>t) \sim C_+(c)t^{-\kappa}$ where

$$C_+(c) = E\big((Q(c)+MR^*(c))^{\kappa} - (MR^*(c))^{\kappa}\big) > 0.$$

Thus

$$P(R > t) \geq P(R^*(c) > t - c) \geq C_+(c)(1 + o(1))t^{-\kappa}.$$

On the other hand, if $Q(c) = 0$ a.s. then $R - c =_L \bigvee_1^\infty (L_m - c)\Pi_m$ and we have the setup of Section 5 with $Q := M(L - c)$. So, by Theorem 5.2, $C_+ > 0$ if $P(M(L - c) > 0) > 0$. □

PROOF OF THEOREM 6.3. This conforms to previous patterns. Since $\beta < 1$, (9.29) gives $E(R^+)^{\kappa+\beta-1} < \infty$. Retrace the middle part of the proof of Theorem 6.2, with $\kappa + \beta$ replacing κ, to arrive at (3.11). Theorem 3.2(iii) now gives the result. □

PROOF OF PROPOSITION 7.1.

$$\begin{aligned}
&E\left|\left((Q \vee MR)^+\right)^\kappa - \left((MR)^+\right)^\kappa\right| \\
&\quad = E\mathbf{1}_{Q > |MR|}\left(Q^\kappa - \left((MR)^+\right)^\kappa\right) + E\mathbf{1}_{Q < -|MR|}\left((MR)^+\right)^\kappa \\
&\quad \leq E(Q^+)^\kappa + E(Q^-)^\kappa = E|Q|^\kappa < \infty.
\end{aligned}$$

The verification of the other conditions of Corollary 2.4 is similar, so the convergence of $t^\kappa P(\pm R > t)$ follows. For nontriviality, take c so that $P(|Q| > c) > 0$. Then

$$\begin{aligned}
P(|R| > t) &= P(|Q_k \Pi_{k-1}| > t \text{ for some } k) \\
&\geq P(|Q| > c)P(|\Pi_{k-1}| > t/c \text{ for some } k) \\
&\sim P(|Q| > c)\delta t^{-\kappa}, \qquad t \to \infty,
\end{aligned}$$

for some $\delta > 0$, as shown at the end of the proof of Theorem 4.1. □

PROOF OF PROPOSITION 8.1. We have $-\infty \leq E \log M < 0$, and because of this and (8.2) may find c sufficiently large that $E \log(M + N/\sqrt{c}) < 0$. Now

$$\begin{aligned}
\Xi(t) &= Mc\frac{t}{c} + N\sqrt{c}\sqrt{\frac{t}{c}} + Q \\
&\leq (Mc + N\sqrt{c})\max\left(1, \frac{t}{c}\right) + Q \qquad (9.30)\\
&= (M + N/\sqrt{c})\max(t, c) + Q =: \Psi(t),
\end{aligned}$$

say, where Ψ is a particular case of that in Letac's model E [see (6.1)] and satisfies the conditions of Proposition 6.1. So $Z_n(t) := \Xi_1 \circ \cdots \circ \Xi_n(t)$ is a.s. bounded as $n \to \infty$, being nonnegative and bounded above by the a.s.-convergent sequence $\Psi_1 \circ \cdots \circ \Psi_n(t)$.

Further, $Z_n(0)$ is a nondecreasing sequence. For $Z_n(0) = Z_{n-1}(Q_n)$, while $Z_{n+1}(0) = Z_{n-1}(Q_n + N_n\sqrt{Q_{n+1}} + M_nQ_{n+1})$, so it suffices if $Z_{n-1}(t)$ is nondecreasing in t. But that is so, as Z_{n-1} is the composition of nondecreasing functions $\Xi_1, \ldots, \Xi_{n-1}$.

The facts adduced show $Z_n(0)$ converges a.s. to some finite r.v. Z. [The argument above is a variant of that of Loynes (1962), proof of Lemma 1.] We must check that $Z_n(t)$ also converges to Z, whatever t. Write $\Xi_{mn} := \Xi_{m+1} \circ \cdots \circ \Xi_n$ for $m < n$, $\Xi_{nn}(t) := t$. Then since

$$\Xi(t') - \Xi(t) = \left(M + \frac{N}{\sqrt{t'} + \sqrt{t}}\right)(t' - t)$$

we see by induction that

$$Z_n(t') - Z_n(t) = (t' - t) \prod_{m=1}^{n} \left(M_m + \frac{N_m}{\sqrt{\Xi_{mn}(t')} + \sqrt{\Xi_{mn}(t)}}\right),$$

whence

$$(9.31) \quad 0 \le Z_n(t) - Z_n(0) \le t\left(M_n + \frac{N_n}{\sqrt{t}}\right) \prod_{m=1}^{n-1} \left(M_m + \frac{N_m}{2\sqrt{Q_{m+1}}}\right).$$

Each of $\prod_{m=1}^{k}(M_{2m} + \frac{1}{2}N_{2m}/\sqrt{Q_{2m+1}})$ and $\prod_{m=1}^{k}(M_{2m-1} + \frac{1}{2}N_{2m-1}/\sqrt{Q_{2m}})$ is a product of independent terms, and by (8.3) and the strong law they both tend to 0 exponentially fast, a.s. An easy Borel–Cantelli argument shows $M_n + N_n/\sqrt{t} = o(e^{\varepsilon n})$ a.s. as $n \to \infty$, for every $\varepsilon > 0$, so (9.31) does indeed give that $Z_n(t) \to Z$. Letac's principle now yields the result. □

PROOF OF THEOREM 8.2. The expression inside the expectation in (8.4) is nonnegative, so to satisfy (2.16) we merely have to prove the right-hand side of (8.4) finite. Then Corollary 2.4 gives the result apart from its last statement which, though, is immediate.

We first show that $\|S\|_p < \infty$ for all $p < \kappa$. Since M and N have finite κ moments, we may by taking c sufficiently large make $c_0 := \|M + N/\sqrt{c}\|_p$ as close to $\|M\|_p$ as we wish, and so less than 1. Then, with Ψ defined by (9.30),

$$\begin{aligned} \|S\|_p &= \|\Xi(S)\|_p \\ &\le \|\Psi(S)\|_p \\ &\le c_0\|S \vee c\|_p + \|Q\|_p \\ &\le c_0\|S\|_p + c_0\|c\|_p + \|Q\|_p, \end{aligned}$$

whence $\|S\|_p < \infty$.

If $\kappa \le 1$ the right-hand side of (8.4) is by (9.27) at most $E(N\sqrt{S} + Q)^\kappa$ which is in turn at most $EN^\kappa ES^{\kappa/2} + EQ^\kappa < \infty$. If $\kappa > 1$ it is by (9.27) and (9.26) at most

$$\begin{aligned} &E(N\sqrt{S} + Q)(MS + N\sqrt{S} + Q)^{\kappa-1} \\ &\qquad \le c_{\kappa-1}E(N\sqrt{S} + Q)(MS)^{\kappa-1} + c_{\kappa-1}E(N\sqrt{S} + Q)^{\kappa}. \end{aligned}$$

In this last expression the first expectation is $ENM^{\kappa-1}ES^{\kappa-1/2} + EQM^{\kappa-1}ES^{\kappa-1}$, which is finite, while the second may be dealt with similarly after a further use of (9.26). So the right-hand side of (8.4) is finite, as required. □

PROOF OF THEOREM 8.3. Since $\kappa + \beta - 1 < \kappa$ and $(\kappa + \beta)/2 < \kappa$, by the previous proof $ES^{\kappa+\beta-1}$ and $ES^{(\kappa+\beta)/2}$ are finite. Hence, by the same calculation as in the previous proof, $E|(MS + N\sqrt{S} + Q)^{\kappa+\beta} - (MS)^{\kappa+\beta}| < \infty$. The result then follows by Theorem 3.2. □

10. Domains of attraction. A probability law whose upper tail is regularly varying with negative index is in the extremal domain of attraction of a Fisher–Tippett max-stable law. When the upper and lower tails are regularly varying with common index less than 2 and a tail-balance condition is satisfied, the law is also in the domain of attraction for sums of a classical (sum-) stable law. The conclusions in Sections 2–8 can thus be seen as extremal domain-of-attraction results, and, when both tails are treated, as sum-domain-of-attraction results as well.

To take the extremal case, the cumulative maximum of i.i.d. r.v.s with such a law will thus converge, suitably normed, to a nondegenerate limit law. Recent work has succeeded in relaxing the independence assumption here, to various forms of weak or short-range dependence [cf. Leadbetter and Rootzén (1988) and Leadbetter, Lindgren and Rootzén (1983)]. Now the sequence $W_n(t)$ of Theorem 2.1 is likely to have only short-range dependence for the models that our results can handle, because the product $M_1 \cdots M_n$ of i.i.d. r.v.s with a law satisfying the conditions of Lemma 2.2 will almost surely converge to zero geometrically fast. Also, the laws of $W_n(t)$ converge by Letac's principle to the law of R satisfying (2.1), which our results show has upper tail asymptotic to a power. Thus it is to be expected that, in specific models, our results and the extreme-value theory may be applied to prove the sequence $W_n(t)$, suitably normed, convergent in law to a Fisher–Tippett limit.

The case where this programme has been carried out is the random difference equation. In the setup of Section 4 the $W_n(t)$ become S_n generated by

$$S_{n+1} = Q_{n+1} + M_{n+1}S_n, \qquad n = 0, 1, \dots, \tag{10.1}$$

where (Q_n, M_n) for $n = 0, 1, \dots$ are i.i.d. and S_0 has an arbitrary law independent of the sequence $((Q_n, M_n))$. For nonnegative M and Q de Haan, Resnick, Rootzén and de Vries (1989) prove that under the conditions of Theorem 4.1 [i.e., Kesten (1973), Theorem 5],

$$\lim_{n\to\infty} P\big(n^{-1/\kappa} \max(S_1, \dots, S_n) \le x\big) = \exp(-C_+\theta x^{-\kappa}), \qquad x > 0.$$

Here $\theta = \int_1^\infty P(\bigvee_{j=1}^\infty \Pi_j \le y^{-1})\kappa y^{-\kappa-1}\,dy$ where $\Pi_j := M_1 \cdots M_j$. In our Corollary 4.3 there are thus bounds for the C_+ appearing in the limit law and in Corollary 4.4 explicit values when κ is an integer.

The authors apply their results to the autoregressive conditional heteroscedastic (ARCH) sequence of Engle (1982), which is generated by

$$\xi_n := Z_n\sqrt{\alpha + \lambda\xi_{n-1}^2}, \qquad n = 1, 2, \dots,$$

where the Z_n are independent standard Gaussian r.v.s, $\xi_0 \ge 0$ and $\alpha > 0$, $0 < \lambda < 1$ are constants. Thus ξ_n^2 satisfies (10.1) with $(Q_n, M_n) := (\alpha Z_n^2, \lambda Z_n^2)$ and so

$$\lim_{n\to\infty} P\big(n^{-1/(2\kappa)}\max(\xi_1, \dots, \xi_n) \le x\big) = \exp\big(-C_+\theta x^{-2\kappa}\big), \qquad x > 0,$$

where $C_+ = (\kappa m)^{-1}E((Q + MR)^\kappa - (MR)^\kappa)$ and R solves (1.1) with $(Q, M) := (\alpha Z^2, \lambda Z^2)$ and Z standard Gaussian. Here κ is the unique solution in $(0, \infty)$ of $EM^\kappa = 1$, where now $EM^\kappa \equiv \pi^{-1/2}(2\lambda)^\kappa\Gamma(\kappa + \frac{1}{2})$. For fixed $0 < \lambda < 1$ we see EM^κ equals λ at $\kappa = 1$ and tends to ∞ as $\kappa \to \infty$, so equals 1 at some $\kappa \in (1, \infty)$, which is therefore the value of κ required. So κ is the unique solution of

$$\text{(10.2)} \qquad \kappa > 1, \qquad \Gamma\big(\kappa + \tfrac{1}{2}\big) = \sqrt{\pi}/(2\lambda)^\kappa.$$

Next, $m = \lambda^\kappa \log\lambda + \lambda^\kappa E|Z|^{2\kappa}\log Z^2$. In terms of the digamma function $\psi = \Gamma'/\Gamma$,

$$E|Z|^{2\kappa}\log Z^2 = 2^\kappa\pi^{-1/2}\Gamma\big(\kappa + \tfrac{1}{2}\big)\big(\log 2 + \psi\big(\kappa + \tfrac{1}{2}\big)\big),$$

hence m. The formula for C_+ becomes $(\kappa m)^{-1}E((\alpha + \lambda R)^\kappa - (\lambda R)^\kappa)E|Z|^{2\kappa}$, where $E|Z|^{2\kappa} = 2^\kappa\pi^{-1/2}\Gamma(\kappa + \frac{1}{2})$.

All values of $\kappa > 1$ are attainable in (10.2) for some $\lambda \in (0, 1)$, since $\Gamma(\kappa + \frac{1}{2}) > \frac{1}{2}\sqrt{\pi}$ for all $\kappa > 1$. For integer κ one needs to take $\lambda := (1 \cdot 3 \cdot 5 \cdots (2\kappa - 1))^{-1/\kappa}$, which gives the first few values as

κ	2	3	4	5	6	7	8	9	10
λ	0.577	0.406	0.312	0.254	0.214	0.185	0.163	0.145	0.105

As noted above, C_+ can in these cases be found explicitly for the ARCH model.

The rate result, Theorem 4.7, applies, whatever $\kappa > 1$. Now $\hat{\eta}(\theta) = EM^{\kappa+i\theta} = \pi^{-1/2}(2\lambda)^{\kappa+i\theta}\Gamma(\kappa + \frac{1}{2} + i\theta)$. Choose any $\beta \in (0, 1)$ apart from the at most finitely many for which $\hat{\eta}(u - i\beta) = 1$ for some $u \in \mathbb{R}$. Then since η is absolutely continuous, and $EM^{\kappa+\beta} < \infty$, $EQ^{\kappa+\beta} < \infty$, all the conditions of the $M \ge 0$ case of Theorem 4.7 are satisfied, and the limit law of the ξ_n^2 (the law of R) thus satisfies (3.8). However, a numerical study indicates the likely nonexistence of any solutions of $\hat{\eta}(\theta) = 1$ with $0 < -\Im\theta < 1$. Assuming that is so, the contour integral in (3.8) vanishes, leaving only the O bound on rate of approach to C_+.

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