Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2020, Vol. 56, No. 2, 1203–1229
https://doi.org/10.1214/19-AIHP999

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Trees within trees II: Nested fragmentations

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Received 17 July 2018; revised 5 April 2019; accepted 29 April 2019

Abstract. Similarly as in (*Electron. J. Probab.* **23** (2018)) where nested coalescent processes are studied, we generalize the definition of partition-valued homogeneous Markov fragmentation processes to the setting of nested partitions, i.e. pairs of partitions (ζ, ξ) where ζ is finer than ξ. As in the classical univariate setting, under exchangeability and branching assumptions, we characterize the jump measure of nested fragmentation processes, in terms of erosion coefficients and dislocation measures. Among the possible jumps of a nested fragmentation, three forms of erosion and two forms of dislocation are identified – one being specific to the nested setting and relating to a bivariate paintbox process.

Résumé. Poursuivant l'idée de (*Electron. J. Probab.* **23** (2018)) où les processus de coalescence emboîtés sont étudiés, nous étendons ici la définition des processus de fragmentation markoviens homogènes aux processus de fragmentation à valeurs dans les partitions emboîtées, c'est-à-dire les paires de partitions (ζ, ξ) telles que ζ soit plus fine que ξ. Comme dans le contexte classique (dit univarié), sous des hypothèses d'échangeabilité et de branchement, nous caractérisons la mesure de saut des processus de fragmentation emboîtés en termes de coefficients d'érosion et de mesures de dislocation. Les sauts d'une fragmentation emboîtée peuvent être de plusieurs natures différentes : nous distinguons trois formes d'érosions et deux formes de dislocations, l'une d'elles étant spécifique au contexte des partitions emboîtées et étant générée par un processus de pots de peinture bivarié.

MSC: 60G09; 60G57; 60J25; 60J35; 60J75; 92D15

Keywords: Fragmentations; Exchangeable; Partition; Random tree; Coalescent; Population genetics; Gene tree; Species tree; Phylogenetics; Evolution

1. Introduction

Evolutionary biology aims at tracing back the history of species, by identifying and dating the relationships of ancestry between past lineages of extant individuals. This information is usually represented by a tree or phylogeny [16,23], species corresponding to leaves of the tree and speciation events (point in time where several species descend from a single one) corresponding to internal nodes.

In modern methods, one analyzes genetic data from samples of individuals to statistically infer their phylogenetic tree. Probabilistic tree models have been well-developed in the last decades – either from individual-based population models like the classical Wright–Fisher model [2,10,15,23], or from forward-in-time branching processes, where the branching particles are species (see for instance Aldous's Markov branching models [1] and the surrounding literature [6,7,11,13]) – allowing for inference from genetic data. A challenge is that trees inferred from different parts of the genome generally fail to coincide, each of them being understood as an alteration of a "true" underlying phylogeny (which we call the *species tree*).

To understand the relation between *gene trees* and the species tree, our goal is to identify a class of Markovian models coupling the evolution of both trees, making the assumption that in general, several gene lineages coexist within the same species, and at speciation events one or several gene lineages diverge from their neighbors to form a new species, i.e. we model the problem as a *tree within a tree* [9,18–20], or *nested tree*. See Figure 1 for an instance of a simple nested genealogy where discrepancies arise between the resulting gene tree and species tree.

Recent research aims at defining mathematical processes giving rise to such nested trees, generalizing several well-studied univariate (we will sometime use this term as opposed to *nested*) processes. Some work in progress involves a nested version [5,17] of the Kingman coalescent [14] (considered the neutral model for evolution, appearing as a scaling

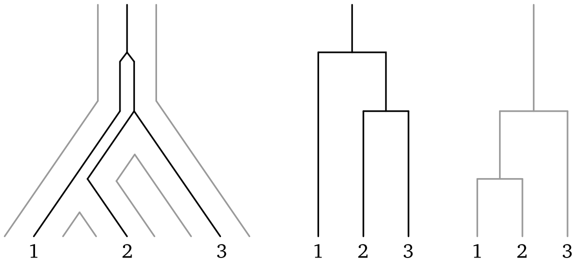

Fig. 1. Example of a nested tree where the gene tree (in black) does not coincide with the species tree (in gray).

limit of many individual-based population models). In [4] we study a nested generalization of Λ-coalescent processes [3,21,22] and characterize their distribution. Our present goal is to generalize the forward-in-time branching models originated from Aldous [1]. His assumptions (which will be formally defined for our context in Section 3) are basically that the random process of evolution is homogeneous in time and that the law of the process is invariant under both relabeling and resampling of individuals (we then say the process is *exchangeable* and *sampling consistent*). We are interested in the partition-valued processes satisfying these assumptions, i.e. the so-called fragmentation processes [3, 13], and in this article we generalize their definition to *nested partition-valued* processes to model jointly a gene tree within a species tree.

Crane [7] also generalizes Aldous's Markov branching models to study the gene tree/species tree problem but uses a different approach to the one we use here. Indeed, his model is such that first the entire species tree $\mathbf{t}$ is drawn according to some probability, and then the gene tree $\mathbf{t}'$ is constructed thanks to a generalized Markov branching model that depends on $\mathbf{t}$. In the meantime, our goal is to characterize the class of models in which there is a joint Markov branching construction of both the gene tree and the species tree, under the assumptions of exchangeability and sampling consistency.

In particular our main result Theorem 14, which will be formally stated in Section 5, shows that nested fragmentation processes satisfying natural branching properties are uniquely characterized by

- three *erosion parameters* $c_{\text{out}}, c_{\text{in},1}$ and $c_{\text{in},2}$ (rates at which a unique lineage can fragment out of its mother block, in three different situations);
- two *dislocation measures* ν_{out} and ν_{in} that are Poissonian intensities of how blocks instantaneously fragment into several new blocks with macroscopic frequencies.

The article is organized as follows. Section 2 introduces some notation used throughout the paper, and the definition of nested fragmentations. We also recall some results in the univariate case which we seek to generalize to the nested case. In Section 3 we study our so-called *strong exchangeability* assumption, and show its relation to a *projective Markov property*, in order to define characteristic kernels of nested fragmentation processes. In Section 4 we use the so-called *outer branching property*, simplifying the representation of characteristic kernels of fragmentations, and giving a natural Poissonian construction of such processes. Focusing on the *inner branching property*, Section 5 is dedicated to the full characterization of the semi-group of nested fragmentations, in terms of *erosion* and *dislocation measures*. It is shown that dislocations, similarly as in the univariate case, can be understood as (bivariate) paintbox processes. Finally Section 6 briefly shows how our main result, Theorem 14, translates in simpler terms when we make the classical biological assumption that all splits are binary.

2. Definitions and examples

2.1. *Definitions*, *notation*

For a set S, write $\mathcal{P}_S$ for the set of partitions of S:

$$\mathcal{P}_S := \left\{\pi \subset \mathfrak{P}(S) \setminus \{\varnothing\},\ \forall A \neq B \in \pi, A \cap B = \varnothing \text{ and } \bigcup_{A \in \pi} A = S\right\},$$

where $\mathfrak{P}(S)$ denotes the power set of S. Throughout the paper, whenever a subset $\pi' \subset \mathfrak{P}(S)$ is defined in a way such that $\pi' = \pi \cup \{\varnothing\}$ for a certain $\pi \in \mathcal{P}_S$, we will implicitly identify π' and π to avoid the formal and cumbersome notation $\pi' \setminus \{\varnothing\}$.

For S, S' two sets, $\pi \in \mathcal{P}_S$ and $\sigma : S' \to S$ an *injection*, we write

$$\pi^\sigma := \{\sigma^{-1}(A), A \in \pi\},$$

and if μ is a measure on $\mathcal{P}_S$ then we write μ^σ for the push-forward of μ by the map $\pi \mapsto \pi^\sigma$.

Note that if $S'' \xrightarrow{\tau} S' \xrightarrow{\sigma} S$ are injections, then we have $\pi^{\sigma\tau} = (\pi^\sigma)^\tau$, and $\mu^{\sigma\tau} = (\mu^\sigma)^\tau$.

For $S' \subset S$, there is a natural surjective map $r_{S,S'} : \mathcal{P}_S \to \mathcal{P}_{S'}$ called the restriction, defined by

$$r_{S,S'}(\pi) = \pi_{|S'} := \{A \cap S', A \in \pi\}.$$

Note that $\pi_{|S'} = \pi^\sigma$ for $\sigma : S' \to S, x \mapsto x$ the canonical injection.

There is always a partial order on $\mathcal{P}_S$, denoted by $\preceq$ and defined as:

$$\pi \preceq \pi' \quad \text{if } \forall (A, B) \in \pi \times \pi', A \cap B \neq \varnothing \Rightarrow A \subset B,$$

that is $\pi \preceq \pi'$ if π is finer than π'. From now on, we prefer to write ζ or ξ for partitions and π for pairs of partitions. Also, throughout the paper we will say if $\zeta \preceq \xi$ that the pair (ζ, ξ) is nested. Let us introduce the space of pairs of nested partitions,

$$\mathcal{P}_S^{2,\preceq} := \{(\zeta, \xi) \in \mathcal{P}_S^2, \zeta \preceq \xi\},$$

which we equip with a partial order $\preceq$ defined naturally as

$$(\zeta, \xi) \preceq (\zeta', \xi') \quad \text{if } \zeta \preceq \zeta' \text{ and } \xi \preceq \xi'.$$

We will use $\mathbf{0}_S$ or sometimes, with some abuse of notation, $\mathbf{0}$ when the context is clear, to denote the partition of S into singletons. Similarly, we will denote by $\mathbf{1}_S$ or $\mathbf{1}$ the partition in one block $\{S\}$. For $S' \subset S$ and $\pi = (\zeta, \xi) \in \mathcal{P}_S^{2,\preceq}$, we define naturally the restriction

$$\pi_{|S'} := (\zeta_{|S'}, \xi_{|S'}) \in \mathcal{P}_{S'}^{2,\preceq}.$$

Let us now define, for $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$ and $[\infty] := \mathbb{N}$, and for $n \in \mathbb{N} \cup \{\infty\}$:

$$\mathcal{P}_n := \mathcal{P}_{[n]} \quad \text{and} \quad \mathcal{P}_n^{2,\preceq} := \mathcal{P}_{[n]}^{2,\preceq}$$

We will generally label the blocks of a partition $\xi = \{\xi_1, \xi_2, \dots\}$, in the unique way such that

$$\min \xi_1 < \min \xi_2 < \cdots$$

The space $\mathcal{P}_\infty^{2,\preceq}$ is endowed with a distance d which makes it compact, defined as follows:

$$d(\pi, \pi') = (\sup\{n \in \mathbb{N}, \pi_{|[n]} = \pi_{|[n]}\})^{-1},$$

with the convention $(\sup \mathbb{N})^{-1} = 0$. Note that the same expression can be used to define a distance on $\mathcal{P}_\infty$, making it a compact space as well.

For $k \le n \le \infty$, $\sigma : [k] \to [n]$ an injection and $\pi = (\zeta, \xi) \in \mathcal{P}_n^{2,\preceq}$, we write

$$\pi^\sigma := (\zeta^\sigma, \xi^\sigma) \in \mathcal{P}_k^{2,\preceq}.$$

A key property of the space $\mathcal{P}_\infty^{2,\preceq}$ is that for any $n \in \mathbb{N}$, and any $\pi \in \mathcal{P}_n^{2,\preceq}$, there is a $\pi^\star \in \mathcal{P}_\infty^{2,\preceq}$ satisfying:

- $\pi^\star_{|[n]} = \pi$;
- for any $\pi' \in \mathcal{P}_\infty^{2,\preceq}$ such that $\pi'_{|[n]} = \pi$, there is an injection $\sigma : \mathbb{N} \to \mathbb{N}$ which satisfies $\sigma_{|[n]} = \mathrm{id}_{[n]}$ and $(\pi^\star)^\sigma = \pi'$.

Indeed, it is clear that one can choose a $\pi^\star = (\zeta^\star, \xi^\star)$ such that $\pi^\star_{|[n]} = \pi$, such that $\zeta^\star$ has infinitely many infinite blocks and no finite blocks, $\xi^\star$ has infinitely many blocks, and each of them contains infinitely many distinct blocks of $\zeta^\star$. This partition immediately satisfies the required property. We will call any such $\pi^\star$ a *universal element of* $\mathcal{P}_\infty^{2,\preceq}$ *with initial part* π whenever we need to use one.

A measure μ on $\mathcal{P}_n$ or on $\mathcal{P}_n^{2,\preceq}$ is said to be *exchangeable* if for any permutation $\sigma : [n] \to [n]$, we have

$$\mu^\sigma = \mu.$$

A random variable Π taking values in $\mathcal{P}_n$ or in $\mathcal{P}_n^{2,\preceq}$ is said to be exchangeable if for any permutation $\sigma : [n] \to [n]$, we have

$$\Pi^\sigma \stackrel{(d)}{=} \Pi,$$

that is if its distribution is exchangeable. Similarly, a random process $(\Pi(t), t \geq 0)$ taking values in $\mathcal{P}_n$ or in $\mathcal{P}_n^{2,\preceq}$ is said to be exchangeable if for any initial state π_0 and any permutation $\sigma : [n] \to [n]$, we have

$$\big(\Pi(t)^\sigma, t \geq 0\big) \text{ under } \mathbb{P}_{\pi_0} \quad \overset{(d)}{=} \quad \big(\Pi(t), t \geq 0\big) \text{ under } \mathbb{P}_{\pi_0^\sigma}, \tag{1}$$

where $\mathbb{P}_\pi$ is the distribution of the process started from π.

Finally, a measure or a random process with values in $\mathcal{P}_\infty$ or $\mathcal{P}_\infty^{2,\preceq}$ will be called *strongly exchangeable* if its distribution is invariant under the action of *injections* $\mathbb{N} \to \mathbb{N}$. Note that while it is easily checked that for measures the two properties are equivalent, for processes this is a strictly stronger assumption than being exchangeable. Indeed, since the number of blocks of a partition is invariant under the action of permutations but not under the action of injections, one can define exchangeable Markov jump processes $(\Pi(t), t \geq 0)$ with jump rates depending on the total number of blocks of $\Pi(t)$, preventing strong exchangeability. The reason we prefer to assume strong exchangeability is the following. Consider a strong exchangeable process Π (say with values in $\mathcal{P}_\infty^{2,\preceq}$) and a universal initial state π. Then for any $\pi' \in \mathcal{P}_\infty^{2,\preceq}$, there is an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\pi' = \pi^\sigma$, so strong exchangeability (1) ensures us that if $\Pi \sim \mathbb{P}_\pi$, then $\Pi^\sigma \sim \mathbb{P}_{\pi'}$. In other words, the process Π under $\mathbb{P}_\pi$ – i.e. started from π – is a coupling of all possible distributions $\mathbb{P}_{\pi'}$, for $\pi' \in \mathcal{P}_\infty^{2,\preceq}$, which will often be convenient.

In the following we only consider time-homogeneous Markov processes. We can now define nested fragmentation processes in a way that extends naturally the definition of fragmentation processes in the univariate case.

Definition 1. Let $\Pi = (\Pi(t), t \geq 0) = ((\zeta(t), \xi(t)), t \geq 0)$ be a Markov process with values in $\mathcal{P}_\infty^{2,\preceq}$. We say Π is a *nested fragmentation process* if:

(i) Π is strongly exchangeable, with nonincreasing càdlàg sample paths.
(ii) *Outer branching property.* For any initial state $\pi = (\zeta, \xi)$ with $\xi = \{\xi_1, \xi_2, \dots\}$ and given bijections $\sigma_i : [\#\xi_i] \to \xi_i$, where $\#\xi_i$ denotes the cardinality of block ξ_i, the processes

$$\big(\big(\Pi^{\sigma_i}(t), t \geq 0\big), i \geq 1\big)$$

are mutually independent under $\mathbb{P}_\pi$.
(iii) *Inner branching property.* The process $(\zeta(t), t \geq 0)$, with values in $\mathcal{P}_\infty$, is a homogeneous univariate fragmentation process, as in [3, Definition 3.2].

In words, the branching properties (ii) and (iii) imply that different blocks at a given time undergo independent fragmentations in the future. Throughout the rest of the paper, unless stated otherwise, we consider an alternative, more convenient definition, which we will prove to be equivalent to Definition 1, and whose idea is the following: distinct blocks fragment at distinct times.

Definition 1′. Let $\Pi = (\Pi(t), t \geq 0) = ((\zeta(t), \xi(t)), t \geq 0)$ be a Markov process with values in $\mathcal{P}_\infty^{2,\preceq}$. We say Π is a *nested fragmentation process* if:

(i) Π is strongly exchangeable, with nonincreasing càdlàg sample paths.
(ii′) Π satisfies the *outer branching property*:

> Almost surely for all t such that $\Pi(t-) \neq \Pi(t)$, there is a unique block $B \in \xi(t-)$ such that $\Pi(t-)_{|B} \neq \Pi(t)_{|B}$.

(iii′) Π satisfies the *inner branching property*:

> Almost surely for all t such that $\zeta(t-) \neq \zeta(t)$, there is a unique block $B \in \zeta(t-)$ such that $\zeta(t-)_{|B} \neq \zeta(t)_{|B}$.

Note that we will show in Section 3 that a nested fragmentation process according to Definition 1 satisfies also Definition 1′, and then in Corollary 15 it will appear the converse is true.

Before describing our results in the setting of nested fragmentations, let us recall the concepts of mass partitions and paintbox processes in the univariate setting. These ideas, which will ultimately be extended to the nested case, are paramount in understanding the possible transitions of fragmentation processes.

2.2. *Univariate results, mass partitions*

Random exchangeable partitions $\pi \in \mathcal{P}_\infty$ and their relation to random mass partitions is well known [see 3, Chapter 2]. We denote the space of mass partitions by

$$\mathscr{P}_{\mathrm{m}} := \left\{ \mathbf{s} = (s_1, s_2, \dots) \in [0,1]^{\mathbb{N}}, s_1 \geq s_2 \geq \dots, \sum_k s_k \leq 1 \right\}. \tag{2}$$

For $\mathbf{s} \in \mathscr{P}_{\mathrm{m}}$, one defines an exchangeable distribution on $\mathcal{P}_\infty$, by the following so-called *paintbox construction*:

- for $k \geq 0$, define $t_k = \sum_{k'=1}^{k} s_{k'}$, with $t_0 = 0$ by convention.
- let $(U_i, i \geq 1)$ be an i.i.d. sequence of uniform random variables in [0, 1].
- define the random partition $\pi \in \mathcal{P}_\infty$ by setting

$$i \sim^\pi j \quad \iff \quad i = j \quad \text{or} \quad \exists k \geq 1, \quad U_i, U_j \in [t_{k-1}, t_k).$$

Then the distribution of π is exchangeable and is denoted by $\varrho_{\mathbf{s}}$. Notice that the set $\pi_0 := \{[t_{k-1}, t_k), k \geq 1\} \cup \{\{t\}, \sum_{k\geq 1} s_k \leq t \leq 1\}$ is a partition of [0, 1], and that we have $\pi = \pi_0^\sigma$, where $\sigma : \mathbb{N} \to [0,1]$ is the random injection defined by $\sigma : i \mapsto U_i$. Also, note that by definition some blocks are singletons (blocks $\{i\}$ such that $U_i \in [\sum_{k\geq 1} s_k, 1]$), and by construction we have

$$\frac{\#\{i \in [n], \{i\} \in \pi\}}{n} \underset{n\to\infty}{\longrightarrow} s_0 := 1 - \sum_{k\geq 1} s_k.$$

These integers that are singleton blocks are called the *dust* of the random partition π and the last display tells us there is a frequency s_0 of dust.

Conversely, any random exchangeable partition π has a distribution that can be expressed with these paintbox constructions $\varrho_{\mathbf{s}}$. Indeed, π has *asymptotic frequencies*, i.e.

$$|B| := \lim_{n\to\infty} \frac{\#(B \cap [n])}{n} \quad \text{exists a.s. for all } B \in \pi.$$

Let us write $|\pi|^\downarrow \in \mathscr{P}_{\mathrm{m}}$ for the nonincreasing reordering of $(|B|, B \in \pi)$, ignoring the zero terms coming from the dust. It is known [14, Theorem 2] that the conditional distribution of π given $|\pi|^\downarrow = \mathbf{s}$ is $\varrho_{\mathbf{s}}$, so we have

$$\mathbb{P}(\pi \in \cdot) = \int \mathbb{P}\big(|\pi|^\downarrow \in \mathrm{d}\mathbf{s}\big) \varrho_{\mathbf{s}}(\cdot).$$

This means that any exchangeable probability measure on $\mathcal{P}_\infty$ is of the form ϱ_ν where ν is a probability measure on $\mathscr{P}_{\mathrm{m}}$, and

$$\varrho_\nu(\cdot) := \int_{\mathscr{P}_{\mathrm{m}}} \varrho_{\mathbf{s}}(\cdot) \nu(\mathrm{d}\mathbf{s}).$$

Furthermore, Bertoin [3, Theorem 3.1] shows that any exchangeable measure μ on $\mathcal{P}_\infty$ such that

$$\mu\big(\{\mathbf{1}\}\big) = 0 \quad \text{and} \quad \forall n \geq 1, \quad \mu(\pi_{|[n]} \neq \mathbf{1}_{[n]}) < \infty \tag{3}$$

can be written $\mu = c\mathfrak{e} + \varrho_\nu$, where $c \geq 0$, ν is a measure on $\mathscr{P}_{\mathrm{m}}$ satisfying

$$\nu\Big(\big\{(1,0,0,\dots)\big\}\Big) = 0 \quad \text{and} \quad \int_{\mathscr{P}_{\mathrm{m}}} (1 - s_1) \nu(\mathrm{d}\mathbf{s}) < \infty, \tag{4}$$

and $\mathfrak{e}$ is the so-called *erosion measure*, defined by

$$\mathfrak{e} := \sum_{i\in\mathbb{N}} \delta_{\{\{i\}, \mathbb{N}\setminus\{i\}\}}.$$

As a result, each fragmentation process with values in $\mathcal{P}_\infty$ is characterized by its erosion coefficient c and characteristic measure ν, in such a way that its rates can be described as follows:

A block of size n fragments, independently of the other blocks, into a partition with k different blocks of sizes $n_1, n_2, \dots, n_k$ at rate

$$c\mathbb{1}\{k=2, \text{ and } n_1 = 1 \text{ or } n_2 = 1\} + \int_{\mathscr{P}_{\mathrm{m}}} \nu(\mathrm{d}\mathbf{s}) \sum_{\mathbf{i}} s_{i_1}^{n_1} \cdot s_{i_2}^{n_2} \cdots s_{i_k}^{n_k},$$

where s_0 is defined to be $1 - \sum_{i \geq 1} s_i$, and the sum is over the vectors $\mathbf{i} = (i_1, \dots, i_k) \in \{0, 1, \dots\}^k$ such that i_j may be 0 only if $n_j = 1$, and if $j \neq j'$ and $i_j \neq 0$, then $i_{j'} \neq i_j$.

A similar result will be shown in the setting of nested fragmentations.

2.3. *Transitions of nested fragmentation processes*

In this article we show that nested fragmentations are processes for which five different fragmentation events – jumps for the Markov process Π – need to be distinguished. All nested fragmentation processes are entirely characterized by the rates at which those fragmentation events occur. While the main result, Theorem 14, cannot be stated at this time because much notation needs to be introduced first, let us briefly explain what the five typical events of a nested fragmentation are with an example. Assume that the nested fragmentation $\Pi = (\zeta, \xi)$ jumps at time t, with (restricting each partition to $\{1, \dots, 12\}$)

$$\begin{aligned} &\xi(t-) = \{1,4,6\}, \quad \{2,5,7,8,9,10,11,12\}, \quad \{3\} \\ &\zeta(t-) = \{1,4\},\{6\}, \quad \{2,9,10,12\},\{5\},\{7,8\},\{11\}, \quad \{3\}. \end{aligned}$$

Then the five following events may occur:

- *Outer erosion*: Each inner block erodes out of its outer block at a constant rate. For example, if the block $\{7, 8\}$ erodes out of its outer block at time t, then we have

$$\begin{aligned} &\xi(t) = \{1,4,6\}, \quad \{2,5,9,10,11,12\}, \quad \{7,8\}, \quad \{3\} \\ &\zeta(t) = \{1,4\},\{6\}, \quad \{2,9,10,12\},\{5\},\{11\}, \quad \{7,8\}, \quad \{3\}. \end{aligned}$$

  Note that a macroscopic – i.e. non-singleton – inner block can erode out of its outer block. This may seem counterintuitive as erosion is usually seen as a continuous loss of mass, but here the idea is simply that a single inner block – not a macroscopic *proportion* of blocks – separates from its outer block.
- *Inner erosion*: Each integer erodes out of its inner block at a constant rate. For example, if the integer 2 erodes out of its inner block at time t, then we have

$$\begin{aligned} &\xi(t) = \{1,4,6\}, \quad \{2,5,7,8,9,10,11,12\}, \quad \{3\} \\ &\zeta(t) = \{1,4\},\{6\}, \quad \{2\},\{9,10,12\},\{5\},\{7,8\},\{11\}, \quad \{3\}. \end{aligned}$$

- *Inner erosion with creation of new species*: Each integer erodes out of its inner and outer blocks at a constant rate. If the integer 2 erodes out of its inner and outer blocks at time t, then we have

$$\begin{aligned} &\xi(t) = \{1,4,6\}, \quad \{2\}, \quad \{5,7,8,9,10,11,12\}, \quad \{3\} \\ &\zeta(t) = \{1,4\},\{6\}, \quad \{2\}, \quad \{9,10,12\},\{5\},\{7,8\},\{11\}, \quad \{3\}. \end{aligned}$$

- *Outer dislocation*: An outer block can split into two or more outer blocks. Each of the inner blocks then decides, according to a Kingman paintbox procedure [14], which outer block to join. For example, if the outer block containing 2 splits into three outer blocks, then the partitions at time t can be

$$\begin{aligned} &\xi(t) = \{1,4,6\}, \quad \{2,9,10,11,12\}, \quad \{5\}, \quad \{7,8\}, \quad \{3\} \\ &\zeta(t) = \{1,4\},\{6\}, \quad \{2,9,10,12\},\{11\}, \quad \{5\}, \quad \{7,8\}, \quad \{3\}. \end{aligned}$$

  Recall that a paintbox process is a way to draw random exchangeable partitions of a (countable) set I: given a partition of $[0, 1]$ into intervals, throw a sequence $(U_i)_{i \in I}$ of i.i.d. uniform random variables on $[0, 1]$; the blocks of the random partition are composed of the i that lie in the same interval. A paintbox procedure corresponding to the example would be Figure 2.

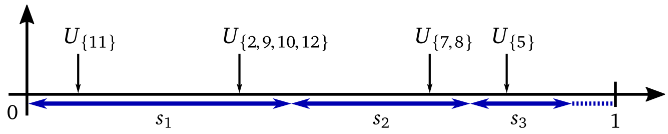

Fig. 2. Usual paintbox process, where the interval partition is composed of three intervals of lengths $s_1 \geq s_2 \geq s_3$.

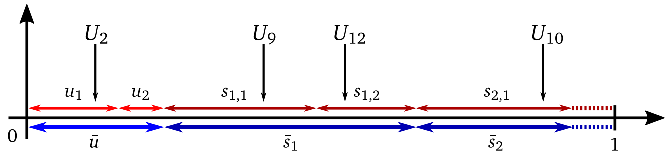

Fig. 3. Bivariate paintbox process, built from two nested interval partitions, the coarser (drawn in blue) with interval lengths $\bar{u}, \bar{s}_1, \bar{s}_2, \ldots$, and the finer (drawn in red) with interval lengths u_1, u_2, $s_{1,1}$, $s_{1,2}$, $s_{2,1}$, etc. In this example, the variable U_2 falls into the distinguished interval with length $\bar{u}$, meaning that the integer 2 remains in its mother outer block. The variables U_9 and U_{12} fall in the same outer interval but in distinct inner intervals so an outer block $\{9, 12\}$ is formed, containing two inner blocks $\{9\}$ and $\{12\}$. Similarly, $\{10\}$ forms a new outer and inner block.

- *Inner dislocation*: An inner block could split into two or more inner blocks, with each of the new inner blocks choosing either to stay in the outer block in which it resided before – its *mother block* –, or move to one of two or more new outer blocks that are created. For example, if the block $\{2, 9, 10, 12\}$ splits into four singletons, with $\{2\}$ choosing to stay in the mother block while the other three integers move to one of two newly created outer blocks. Then the partitions at time t can be

$$\begin{aligned}
\xi(t) &= \{1,4,6\}, \quad \{2,5,7,8,11\}, \quad \{9,12\}, \quad \{10\}, \quad \{3\} \\
\zeta(t) &= \{1,4\},\{6\}, \quad \{2\},\{5\},\{7,8\},\{11\}, \quad \{9\},\{12\}, \quad \{10\}, \quad \{3\}.
\end{aligned}$$

  Note that a bivariate paintbox process is needed to construct inner dislocation events: see Figure 3 for a paintbox corresponding to this example.

Note that not all decreasing transitions are valid. For instance, consider the transition from the initial state above to

$$\begin{aligned}
\xi(t) &= \{1,4\}, \quad \{6\}, \quad \{2,5,7,8,9,10,11,12\}, \quad \{3\} \\
\zeta(t) &= \{1,4\}, \quad \{6\}, \quad \{2\},\{9,10,12\},\{5\},\{7,8\},\{11\}, \quad \{3\},
\end{aligned}$$

where both the inner block $A = \{2, 9, 10, 12\}$ and the outer block $B = \{1, 4, 6\}$ simultaneously undergo fragmentation. In fact since they are not nested ($A \not\subset B$) we will see that this transition is impossible. Also, consider the transition

$$\begin{aligned}
\xi(t) &= \{1,4,6\}, \quad \{2,5,9,10,12\}, \quad \{7,8,11\} \quad \{3\} \\
\zeta(t) &= \{1,4\},\{6\}, \quad \{2\},\{9,10,12\},\{5\}, \quad \{7,8\},\{11\}, \quad \{3\}.
\end{aligned}$$

Now inner block A undergoes fragmentation at the same time as its mother block $B = \{2, 5, 7, 8, 9, 10, 11, 12\}$. However, the transition is invalid because the fragmentation of block B separates, in particular, sites 5 and 7, while neither of them is in A. It will be clear along the proof of Theorem 14 that such events are impossible for nested fragmentation processes (essentially because if such transitions had positive rates, exchangeability would imply that those rates are infinite).

Let us now start the analysis of nested fragmentation processes by exploiting their strong exchangeability property.

3. Projective Markov property – characteristic kernel

The goal of this section is to show that nested fragmentations are processes Π for which the following *projective Markov property* holds:

> For all $n \geq 1$, the process $\Pi^n := (\Pi(t)_{|[n]}, t \geq 0)$ is a continuous-time Markov chain in the finite state space $\mathcal{P}_n^{2,\preceq}$, whose distribution under $\mathbb{P}_\pi$ depends only on $\pi_{|[n]}$.

We already made use of this property in [4, Lemma 3.2] in the context of nested coalescent processes. Here it is exposed in a slightly more general way since we show that for a large class of Markov processes with values in $\mathcal{P}_\infty^{2,\preceq}$ or $\mathcal{P}_\infty$ (not only coalescent or fragmentation processes, but any càdlàg exchangeable process), the projective Markov property is in fact equivalent to strong exchangeability.

Proposition 2. *Let $\Pi = (\Pi(t), t \geq 0)$ be an exchangeable Markov process taking values in $\mathcal{P}_\infty^{2;\preceq}$ or $\mathcal{P}_\infty$ with càdlàg sample paths. The following propositions are equivalent*:

(i) Π *is strongly exchangeable.*
(ii) Π *has the projective Markov property, i.e.* $\Pi^n := (\Pi(t)_{|[n]}, t \geq 0)$ *is a Markov chain for all* $n \in \mathbb{N}$.

Remark 3. Crane and Towsner [8, Theorem 4.26] show that the projective Markov property is equivalent to the Feller property for exchangeable Markov process taking values in a Fraïssé space (i.e. a space satisfying general "stability and universality" assumptions [see 8, Definitions 4.4 to 4.11]). In particular the space of partitions and the space of nested partitions are Fraïssé spaces (the argument essentially being the existence of so-called universal elements $\pi^\star$ defined in Section 2), so for the processes we consider, strong exchangeability is equivalent to the Feller property.

Proof. (i) $\Rightarrow$ (ii): Let $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n^{2;\preceq}$. Fix a universal $\pi^\star \in \mathcal{P}_\infty^{2;\preceq}$ with initial part π. Now take any $\pi_0 \in \mathcal{P}_\infty^{2;\preceq}$ such that $(\pi_0)_{|[n]} = \pi$, and an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma_{|[n]} = \mathrm{id}_{|[n]}$ and $(\pi^\star)^\sigma = \pi_0$. Now we have

$$\begin{aligned}\mathbb{P}_{\pi_0}\big(\Pi^n \in \cdot\big) &= \mathbb{P}_{\pi^\star}\big(\big(\Pi^\sigma\big)^n \in \cdot\big)\\ &= \mathbb{P}_{\pi^\star}\big(\Pi^n \in \cdot\big),\end{aligned}$$

so this distribution depends only on π, which proves that Π^n is a Markov process. Now the assumption that Π has càdlàg sample paths ensures that the process Π^n stays some positive time in each visited state *a.s.* Therefore Π^n is a continuous-time Markov chain.

(ii) $\Rightarrow$ (i): Let $\sigma : \mathbb{N} \to \mathbb{N}$ be an injection. For $n \in \mathbb{N}$, let τ be a permutation of $\mathbb{N}$ such that $\tau_{|[n]} = \sigma_{|[n]}$. This property implies $(\pi^\tau)_{|[n]} = (\pi^\sigma)_{|[n]}$ for any $\pi \in \mathcal{P}_\infty^{2;\preceq}$. We deduce

$$\begin{aligned}\mathbb{P}_\pi\big(\big(\Pi^\sigma\big)^n \in \cdot\big) &= \mathbb{P}_\pi\big(\big(\Pi^\tau\big)^n \in \cdot\big)\\ &= \mathbb{P}_{\pi^\tau}\big(\Pi^n \in \cdot\big)\\ &= \mathbb{P}_{\pi^\sigma}\big(\Pi^n \in \cdot\big),\end{aligned}$$

where the last equality is a consequence of the projective Markov property (the distribution of Π^n under $\mathbb{P}_\pi$ depends only on the initial segment $\pi_{|[n]}$). Since it is true for all n, we have $\mathbb{P}_\pi(\Pi^\sigma \in \cdot) = \mathbb{P}_{\pi^\sigma}(\Pi \in \cdot)$, which proves the property of strong exchangeability. □

Corollary 4. *A nested fragmentation as defined by Definition 1 satisfies the assumptions of Definition 1′.*

Proof. Consider a nested fragmentation process $\Pi = (\zeta, \xi)$ satisfying Definition 1. Note that (i) of Definition 1 implies that Π satisfies the projective Markov property. Fix any initial state $\pi = (\zeta, \xi) \in \mathcal{P}_\infty^{2;\preceq}$ and an integer $n \in \mathbb{N}$, and write $\xi_{|[n]} = \{\xi_1, \xi_2, \dots, \xi_k\}$, for some $1 \leq k \leq n$. Now define bijections $\sigma_i : [\#\xi_i] \to \xi_i$ for each integer $1 \leq i \leq k$. Assumption (ii) and the projective Markov property imply that the processes

$$\big(\big(\Pi^{\sigma_i}(t), t \geq 0\big), 1 \leq i \leq k\big)$$

are mutually independent under $\mathbb{P}_\pi$, and such that Π^{σ_i} has distribution $\Pi^{\#\xi_i}$ started from $\pi^{\sigma_i} = (\zeta^{\sigma_i}, \mathbf{1})$. Independent continuous-time Markov chains have distinct jump times almost surely, so in particular the first jump time T_1^n of Π^n started from $\pi_{|[n]}$ is the first jump time of some Π^{σ_i}, for a unique i. So there is a unique block $B \in \xi(0)_{|[n]} = \xi(T_1^n-)_{|[n]}$ such that $\Pi(T_1^n-)_{|B} \neq \Pi(T_1^n)_{|B}$. By induction and the Markov property applied to successive jumps times $T_1^n, T_2^n, \dots$ of the Markov chain Π^n, it is clear that almost surely, for all $t \geq 0$ such that $\Pi^n(t-) \neq \Pi^n(t)$, there is a unique block $B \in \xi^n(t-)$ such that $\Pi(t-)_{|B} \neq \Pi(t)_{|B}$. Since this is true for all $n \in \mathbb{N}$, the outer branching property as described in (ii′) holds.

It is a result of the univariate theory of fragmentations [3], that (iii) implies (iii′). □

The next proposition is the direct consequence of the projective Markov property in the space $\mathcal{P}_\infty^{2;\preceq}$. It is essentially Lemma 4.1 in [4], from which the proof is easily adapted, the argument being entirely independent from any monotonicity (coalescence or fragmentation) assumption.

Proposition 5. *Let $\Pi = (\Pi(t), t \geq 0)$ be a stochastic process with values in $\mathcal{P}_\infty^{2;\preceq}$ which satisfies the* projective Markov property. *Then Π is a Markov process*, *whose distribution is characterized by a transition kernel K from $\mathcal{P}_\infty^{2;\preceq}$ to $\mathcal{P}_\infty^{2;\preceq}$*

(*i.e.* $K_\pi(\cdot)$ *is a nonnegative measure on* $\mathcal{P}_\infty^{2,\preceq}$ *for all* $\pi \in \mathcal{P}_\infty^{2,\preceq}$ *and* $\pi \mapsto K_\pi(B)$ *is measurable for any* B *Borel set of* $\mathcal{P}_\infty^{2,\preceq}$) *such that*

- *for all* $\pi \in \mathcal{P}_\infty^{2,\preceq}$, *we have* $K_\pi(\{\pi\}) = \infty$,
- *for all* $\pi \in \mathcal{P}_\infty^{2,\preceq}$, $n \in \mathbb{N}$ *and* $\pi' \in \mathcal{P}_n^{2,\preceq} \setminus \{\pi_{|[n]}\}$, *the Markov chain* Π^n *has a transition rate from* $\pi_{|[n]}$ *to* π' *equal to*

$$q^n_{\pi,\pi'} = K_\pi\big(r_n^{-1}\big(\{\pi'\}\big)\big) < \infty,$$

where $r_n(\cdot) = \cdot_{|[n]}$ *denotes the restriction operation.*

This kernel K *will be called the* characteristic kernel *of the process* Π. *Furthermore, if* Π *is exchangeable, then* K *is strongly exchangeable, in the sense that for any* $\pi \in \mathcal{P}_\infty^{2,\preceq}$ *and any injection* $\sigma : \mathbb{N} \to \mathbb{N}$, *we have*

$$K_{\pi^\sigma} = K_\pi^\sigma.$$

Proof. See [4, Lemma 4.1]. □

Remark 6. Note that the transition rates of the Markov chains Π^n are given by the collection of σ-finite measures $K_\pi(\cdot \cap \mathcal{P}_\infty^{2,\preceq} \setminus \{\pi\})$, for $\pi \in \mathcal{P}_\infty^{2,\preceq}$. The value $K_\pi(\{\pi\})$ is irrelevant for the distribution of the process Π, and for uniqueness, we set $K_\pi(\{\pi\}) = \infty$, whereas it is conventional for a transition kernel that this value is taken to be 0. However, setting this value to be infinite is necessary so that strong exchangeability $K_{\pi^\sigma} = K_\pi^\sigma$ holds in general for all *injections* $\sigma : \mathbb{N} \to \mathbb{N}$. Indeed, note that if σ is a bijection, then $K_\pi^\sigma(\{\pi^\sigma\}) = K_\pi(\{\pi\})$, but in general, when σ is an injection, one can have $K_\pi^\sigma(\{\pi^\sigma\}) = K_\pi(\{\pi\}) + a$, where $a > 0$. For instance assume – we will see that it is the case for characteristic kernels of nested fragmentation – that K is such that if $\pi_0 = (\zeta, \xi)$ has at least two outer blocks $B \neq B' \in \xi$, then

$$K_{\pi_0}\big(\big\{\pi_{|B} \neq (\pi_0)_{|B}\big\} \cap \big\{\pi_{|B'} \neq (\pi_0)_{|B'}\big\}\big) = 0.$$

Then if $\sigma : \mathbb{N} \to \mathbb{N}$ is an injection with image $\sigma(\mathbb{N}) \subset B$, then one has

$$K_{\pi_0}\big(\big\{\pi^\sigma = \pi_0^\sigma\big\}\big) \geq K_{\pi_0}\big(\big\{\pi_{|B} = (\pi_0)_{|B}\big\}\big) \geq K_{\pi_0}\big(\{\pi = \pi_0\}\big) + K_{\pi_0}\big(\big\{\pi_{|B'} \neq (\pi_0)_{|B'}\big\}\big),$$

where $K_{\pi_0}(\{\pi_{|B'} \neq (\pi_0)_{|B'}\})$ may be greater than 0 if K is not trivial.

Let us emphasize that the kernel K essentially gives us the infinitesimal generator of the Markov process Π. Indeed, note that the generator G_n of the continuous-time finite-space Markov chain Π^n is then given by

$$\begin{aligned} G_n f(\pi_{|[n]}) &= \sum_{\pi' \in \mathcal{P}_n^{2,\preceq} \setminus \{\pi_{|[n]}\}} q^n_{\pi,\pi'}\big(f\big(\pi'\big) - f(\pi_{|[n]})\big) \\ &= \int_{\mathcal{P}_\infty^{2,\preceq}} K_\pi\big(\mathrm{d}\pi'\big)\big(f\big(\pi'_{|[n]}\big) - f(\pi_{|[n]})\big), \end{aligned}$$

for any function $f : \mathcal{P}_n^{2,\preceq} \to \mathbb{R}$ and $\pi \in \mathcal{P}_\infty^{2,\preceq}$. As an obvious consequence, the generator G of the process Π can be applied to any function $g : \mathcal{P}_\infty^{2,\preceq} \to \mathbb{R}$ of the form $g = f \circ r_n$ for some $n \in \mathbb{N}$ and some function $f : \mathcal{P}_n^{2,\preceq} \to \mathbb{R}$, and is written

$$Gg(\pi) = \int_{\mathcal{P}_\infty^{2,\preceq}} K_\pi\big(\mathrm{d}\pi'\big)\big(g\big(\pi'\big) - g(\pi)\big).$$

4. Outer branching property

In this section we study the outer branching property – as stated in Definition 1′ – to analyze the characteristic kernel of nested fragmentations. The aim is to show that it is entirely characterized by a measure on partitions of $\mathbb{N}^2$ satisfying some invariance property.

4.1. *Simpler kernel*

First, the following proposition expresses that the jump rates from initial states with *a single outer block* are sufficient to characterize the whole process.

Proposition 7. *Let $\Pi = (\Pi(t), t \geq 0) = ((\zeta(t), \xi(t)), t \geq 0)$ be a strongly exchangeable Markov process with values in $\mathcal{P}_\infty^{2,\preceq}$ and nonincreasing càdlàg sample paths. Write K for its exchangeable characteristic kernel.*

If Π satisfies the outer branching property, *then K is characterized by a simpler kernel κ from $\mathcal{P}_\infty$ to $\mathcal{P}_\infty^{2,\preceq}$ which is defined as*

$$\kappa_\zeta(\cdot) := K_{(\zeta,\mathbf{1})}(\cdot),$$

where $\mathbf{1}$ denotes the partition of $\mathbb{N}$ with only one block. The simpler kernel is also strongly exchangeable.

The kernel K is determined by κ in the following way: fix $\pi_0 = (\zeta, \xi) \in \mathcal{P}_\infty^{2,\preceq}$ and for simplicity suppose that all the blocks of ξ are infinite. For all $B \in \xi$, define the injection $\sigma_B : \mathbb{N} \to \mathbb{N}$ as the unique increasing map whose image is B, and $\tau_B : B \to \mathbb{N}$ such that $\sigma_B \circ \tau_B = \mathrm{id}_B$. By definition, $(\pi_0)^{\sigma_B}$ is of the form $(\zeta_B, \mathbf{1})$, with $\zeta_B = \zeta^{\sigma_B}$. Now define f_B as the function which maps $\pi \in \mathcal{P}_\infty^{2,\preceq}$ to the unique $\omega \in \mathcal{P}_\infty^{2,\preceq}$ such that

- $\omega \preceq (\{B, \mathbb{N} \setminus B\}, \{B, \mathbb{N} \setminus B\})$,
- $\omega_{|B} = \pi^{\tau_B}$ *and* $\omega_{|\mathbb{N}\setminus B} = (\pi_0)_{|\mathbb{N}\setminus B}$.

Then for any Borel set $A \subset \mathcal{P}_\infty^{2,\preceq}$, we have

$$K_{\pi_0}(A) = \sum_{B\in\xi} \kappa_{\zeta_B}\big(\big\{f_B(\pi) \in A\big\}\big). \tag{5}$$

Remark 8.

- This proposition shows how K_{π_0} is expressed in terms of the kernel κ only for $\pi_0 = (\zeta, \xi)$ such that all the blocks of ξ are infinite. In fact this is enough to characterize K entirely since if π_0 does not satisfy this property, there exists a nested partition $\pi_0' = (\zeta', \xi')$, where ξ' has infinite blocks, and an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\pi_0 = (\pi_0')^\sigma$. Then we have $K_{\pi_0} = K_{\pi_0'}^\sigma$, which is determined by κ.
- This result implies that different outer blocks undergo independent fragmentations, in other words a nested fragmentation (recall that we only assume Definition $1'$) satisfies (ii) of Definition 1. Indeed, one interprets the sum (5) as: independently for each block $B \in \xi$, $(\pi_0)_{|B}$ is replaced by π^{τ_B} at rate $\kappa_{\zeta_B}(\pi \in \cdot)$, which is a measure which depends only on $(\pi_0)_{|B}$.

Proof. First note that the fact that Π has decreasing sample paths implies that for any $\pi_0 \in \mathcal{P}_\infty^{2,\preceq}$, the support of the measure K_{π_0} is included in $\{\pi \preceq \pi_0\}$. Indeed, since $\{\pi \preceq \pi_0\} = \bigcap_{n\geq 1}\{\pi_{|[n]} \preceq (\pi_0)_{|[n]}\}$, we have

$$K_{\pi_0}\big(\{\pi \not\preceq \pi_0\}\big) = \lim_{n\to\infty} K_{\pi_0}\big(\pi_{|[n]} \not\preceq (\pi_0)_{|[n]}\big),$$

where for any $n \geq 1$, the right-hand side is equal to the (finite) transition rate of the Markov chain Π^n from $(\pi_0)_{|[n]}$ to any π for which $\pi \not\preceq (\pi_0)_{|[n]}$. But Π^n is a decreasing process by assumption, so this rate is zero, so we conclude

$$K_{\pi_0}(\pi \not\preceq \pi_0) = 0 \tag{6}$$

Using the same argument, it is clear that the outer branching property implies that for any $\pi_0 = (\zeta, \xi) \in \mathcal{P}_\infty^{2,\preceq}$, we have

$$K_{\pi_0}\bigg(\bigcup_{B_1\neq B_2\in\xi} \big\{\pi_{|B_1} \neq (\pi_0)_{|B_1} \text{ and } \pi_{|B_2} \neq (\pi_0)_{|B_2}\big\}\bigg) = 0. \tag{7}$$

Now without loss of generality (see Remark 8), suppose that all the blocks of ξ are infinite, and let us define for all $B \in \xi$, the maps σ_B, τ_B and f_B as in the proposition. Equations (6) and (7) imply that for any $B \in \xi$, on the event $\{\pi_{|B} \neq (\pi_0)_{|B}\}$, we have

$$\pi = f_B\big(\pi^{\sigma_B}\big) \quad K_{\pi_0}\text{-a.e.},$$

where f_B is the map defined in the proposition. Then to show (5) for any Borel set $A \subset \mathcal{P}_\infty^{2,\preceq} \setminus \{\pi_0\}$, we have

$$\begin{aligned}
K_{\pi_0}(A) &= K_{\pi_0}\bigg(\bigcup_{B\in\xi}\big(A \cap \big\{\pi_{|B} \neq (\pi_0)_{|B}\big\}\big)\bigg) \\
&= \sum_{B\in\xi} K_{\pi_0}\big(A \cap \big\{\pi_{|B} \neq (\pi_0)_{|B}\big\}\big) \\
&= \sum_{B\in\xi} K_{\pi_0}\big(\big\{f_B\big(\pi^{\sigma_B}\big) \in A\big\} \cap \big\{\pi^{\sigma_B} \neq (\pi_0)^{\sigma_B}\big\}\big) \\
&= \sum_{B\in\xi} K_{(\pi_0)^{\sigma_B}}\big(\big\{f_B(\pi) \in A\big\} \cap \big\{\pi \neq (\pi_0)^{\sigma_B}\big\}\big),
\end{aligned}$$

where we use the strong exchangeability of the kernel K in the last line. Now, note that for every $B \in \xi$, by definition of f_B we have $\{f_B(\pi) \neq \pi_0\} = \{\pi \neq (\pi_0)^{\sigma_B}\}$, therefore $\{f_B(\pi) \in A\} \subset \{\pi \neq (\pi_0)^{\sigma_B}\}$, so one can simply rewrite

$$K_{\pi_0}(A) = \sum_{B\in\xi} K_{(\pi_0)^{\sigma_B}}\big(\big\{f_B(\pi) \in A\big\}\big).$$

In general, if A is a Borel subset of $\mathcal{P}_\infty^{2,\preceq}$ with $\pi_0 \in A$, we have $K_{\pi_0}(A) = \infty$, and for each $B \in \xi$, $K_{(\pi_0)^{\sigma_B}}(\{f_B(\pi) \in A\}) \geq K_{(\pi_0)^{\sigma_B}}(\{f_B(\pi) = \pi_0\}) = K_{(\pi_0)^{\sigma_B}}(\{\pi = (\pi_0)^{\sigma_B}\}) = \infty$, so the equality still holds. Now by definition of σ_B, $(\pi_0)^{\sigma_B}$ is of the form $(\zeta_B, \mathbf{1})$, which concludes the proof that K_{π_0} can be expressed with the simpler kernel κ. Finally, by definition, it is clear that κ inherits strong exchangeability from K. □

Now, to further analyze the *simplified* characteristic kernel κ of an nested fragmentation, we need to introduce some tools, reducing the problem to study exchangeable (with respect to a particular set of injections M) partitions of $\mathbb{N}^2$.

4.2. *M-invariant measures*

Let M be the monoid of functions $\mathbb{N}^2 \to \mathbb{N}^2$ consisting of injective maps of the form

$$(i, j) \longmapsto \big(\sigma(i), \sigma_i(j)\big),$$

where σ and $\sigma_1, \sigma_2, \dots$ are injections $\mathbb{N} \to \mathbb{N}$. Let us write π_{R} for the *rows partition* $\{\{(i, j), j \geq 1\}, i \geq 1\} \in \mathcal{P}_{\mathbb{N}^2}$, which has the property that an injection $\tau : \mathbb{N}^2 \to \mathbb{N}$ is in M if and only if $\pi_{\mathrm{R}}^\tau = \pi_{\mathrm{R}}$.

Note that in $\mathcal{P}_\infty$ any universal element π has the property that κ_π characterize κ entirely, but there is no natural choice for π. The reason for studying partitions of $\mathbb{N}^2$ is that the rows partition π_{R} is a natural universal element of $\mathcal{P}_{\mathbb{N}^2}$. The following proposition shows that one can make sense of a measure essentially defined as "$\kappa_{\pi_{\mathrm{R}}}$", which then characterize κ and therefore the distribution of a nested fragmentation.

Proposition 9. *Let κ be a strongly exchangeable kernel from $\mathcal{P}_\infty$ to $\mathcal{P}_\infty^{2,\preceq}$, and let π_0 denote a universal element of $\mathcal{P}_\infty$, i.e. a partition of $\mathbb{N}$ with infinitely many infinite blocks (and no finite block). Choose a bijection $\sigma : \mathbb{N}^2 \to \mathbb{N}$ such that $\pi_0^\sigma = \pi_{\mathrm{R}}$.*

Then $\mu := \kappa_{\pi_0}^\sigma$ is a measure on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ which is **M**-invariant, *in the sense that for all $\tau \in M$, $\mu = \mu^\tau$. Moreover, μ does not depend on π_0 or σ and the mapping $\kappa \mapsto \mu$ is bijective from the set of strongly exchangeable kernels to the set of M-invariant measures on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$.*

Thinking of κ as the jump kernel of a nested fragmentation process, one can see this measure μ as the measure giving the infinitesimal jump rates from the nested partition $(\pi_{\mathrm{R}}, \mathbf{1})$, where each row of $\mathbb{N}^2$ is an inner block.

Proof. Fix $\tau \in M$ and a Borel set $A \subset \mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$. We need to prove $\mu(\pi^\tau \in A) = \mu(A)$. Consider $\varphi = \sigma \circ \tau \circ \sigma^{-1}$. This map satisfies $\varphi \circ \sigma = \sigma \circ \tau$ and $\pi_0^\varphi = \pi_0$, so we have

$$\begin{aligned}
\mu\big(\pi^\tau \in A\big) &= \kappa_{\pi_0}\big(\pi^{\sigma\circ\tau} \in A\big) \\
&= \kappa_{\pi_0}\big(\pi^{\varphi\circ\sigma} \in A\big)
\end{aligned}$$

$$= \kappa_{\pi_0^\varphi}\big(\pi^\sigma \in A\big)$$
$$= \mu(A).$$

This proves that μ is M-invariant. Let us now prove that μ does not depend on π_0 or σ: fix $\pi_1, \pi_2 \in \mathcal{P}_\infty$ (both with infinitely many infinite blocks and no finite block) and σ_1, σ_2 bijections from $\mathbb{N}^2$ to $\mathbb{N}$ such that $\pi_i^{\sigma_i} = \pi_{\mathrm{R}}$. We need to show

$$\kappa_{\pi_1}\big(\pi^{\sigma_1} \in \cdot\big) = \kappa_{\pi_2}\big(\pi^{\sigma_2} \in \cdot\big).$$

Let φ be a bijection such that $\pi_1^\varphi = \pi_2$. Note that $\pi_{\mathrm{R}}^{\sigma_2^{-1}\circ\varphi^{-1}\circ\sigma_1} = \pi_2^{\varphi^{-1}\circ\sigma_1} = \pi_1^{\sigma_1} = \pi_{\mathrm{R}}$, i.e. $\sigma_2^{-1} \circ \varphi^{-1} \circ \sigma_1 \in M$. Now we have

$$\begin{aligned}
\kappa_{\pi_1}\big(\pi^{\sigma_1} \in \cdot\big) &= \kappa_{\pi_1}\big(\big(\pi^\varphi\big)^{\varphi^{-1}\circ\sigma_1} \in \cdot\big) \\
&= \kappa_{\pi_2}\big(\pi^{\varphi^{-1}\circ\sigma_1} \in \cdot\big) \\
&= \kappa_{\pi_2}\big(\big(\pi^{\sigma_2}\big)^{\sigma_2^{-1}\circ\varphi^{-1}\circ\sigma_1} \in \cdot\big) \\
&= \kappa_{\pi_2}\big(\pi^{\sigma_2} \in \cdot\big),
\end{aligned}$$

where the last equality follows from the M-invariance of $\kappa_{\pi_2}(\pi^{\sigma_2} \in \cdot)$. So μ is well defined and depends only on κ.

We now prove that $\kappa \mapsto \mu$ is bijective. For any injection $\sigma : \mathbb{N} \to \mathbb{N}^2$, we write 2σ for the map

$$2\sigma : \begin{cases} \mathbb{N} & \longrightarrow \mathbb{N}^2 \\ n & \longmapsto 2\sigma(n) = (2i, 2j) \quad \text{where } \sigma(n) = (i, j). \end{cases}$$

Note that for any injection $\sigma : \mathbb{N} \to \mathbb{N}^2$, we have $\pi_{\mathrm{R}}^\sigma = \pi_{\mathrm{R}}^{2\sigma}$. Now let σ_1, σ_2 be any two injections such that $\pi_{\mathrm{R}}^{\sigma_1} = \pi_{\mathrm{R}}^{\sigma_2}$. Then there exists a $\tau \in M$ such that

$$\tau \circ \sigma_1 = 2\sigma_2.$$

Indeed one such τ can be defined in the following way. First let us define an injection $\varphi : \mathbb{N} \to \mathbb{N}$, which will serve as a mapping for rows. For any $i \in \mathbb{N}$, there are two possibilities:

- either there is a $j \in \mathbb{N}$ such that $(i, j) \in \mathrm{im}(\sigma_1)$, and then there is an even integer $i' \in \mathbb{N}$ such that $2\sigma_2(\sigma_1^{-1}(i, j)) = (i', k)$ for some $k \in \mathbb{N}$. This number i' does not depend on j because of the fact that $\pi_{\mathrm{R}}^{\sigma_1} = \pi_{\mathrm{R}}^{\sigma_2}$. Indeed if $j_1, j_2 \in \mathbb{N}$ are such that $(i, j_1), (i, j_2) \in \mathrm{im}(\sigma_1)$, then by definition $\sigma^{-1}(i, j_1)$ and $\sigma^{-1}(i, j_2)$ belong to the same block of $\pi_{\mathrm{R}}^{\sigma_1} = \pi_{\mathrm{R}}^{\sigma_2}$, and so $\sigma_2(\sigma^{-1}(i, j_1))$ and $\sigma_2(\sigma^{-1}(i, j_2))$ belong to the same block of π_{R}. So in that case we can define $\varphi(i) := i'$.
- or $\mathrm{im}(\sigma_1) \cap \{(i, j), j \geq 1\} = \varnothing$, and then we define $\varphi(i) = 2i - 1$.

The map φ is a well-defined injection, and we may now define

$$\tau : \begin{cases} (i, j) \in \mathrm{im}(\sigma_1) & \longmapsto 2\sigma_2(\sigma_1^{-1}(i, j)) \\ (i, j) \notin \mathrm{im}(\sigma_1) & \longmapsto (\varphi(i), 2j - 1) \end{cases}$$

It is easy to check that $\tau \in M$ and that $\tau \circ \sigma_1 = 2\sigma_2$. We can now fix μ an M-invariant measure on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$. Consider a partition $\pi_0 \in \mathcal{P}_\infty$ and an injection $\sigma_0 : \mathbb{N} \to \mathbb{N}^2$ such that $\pi_{\mathrm{R}}^{\sigma_0} = \pi_0$. Now for any other σ_1 such that $\pi_{\mathrm{R}}^{\sigma_1} = \pi_0$, let $\tau \in M$ be such that $\tau \circ \sigma_1 = 2\sigma_0$. By M-invariance of μ, we have

$$\begin{aligned}
\mu\big(\pi^{\sigma_1} \in \cdot\big) &= \mu\big(\pi^{\tau\circ\sigma_1} \in \cdot\big) \\
&= \mu\big(\pi^{2\sigma_0} \in \cdot\big).
\end{aligned}$$

Therefore this measure does not depend on σ_1 but only on π_0, so we may define

$$\kappa_{\pi_0} := \mu\big(\pi^{\sigma_0} \in \cdot\big),$$

which is a measure on $\mathcal{P}_\infty^{2,\preceq}$, for all π_0. Now it remains to check that for any injection $\sigma : \mathbb{N} \to \mathbb{N}$, we have $\kappa_{\pi_0}^\sigma = \kappa_{\pi_0^\sigma}$. But if $\pi_R^{\sigma_0} = \pi_0$, then $\pi_R^{\sigma_0 \circ \sigma} = \pi_0^\sigma$, so

$$\begin{aligned}\kappa_{\pi_0}^\sigma &= \mu\big(\big(\pi^{\sigma_0}\big)^\sigma \in \cdot\big) \\ &= \mu\big(\pi^{\sigma_0\circ\sigma} \in \cdot\big) \\ &= \kappa_{\pi_0^\sigma},\end{aligned}$$

so κ is a strongly exchangeable kernel from $\mathcal{P}_\infty$ to $\mathcal{P}_\infty^{2,\preceq}$, and it is easy to check that the M-invariant measure associated with κ is μ. □

Note that for K a characteristic kernel of a nested fragmentation, we have set (see Remark 6) $K_\pi(\{\pi\}) = \infty$ for any $\pi \in \mathcal{P}_\infty^{2,\preceq}$, which implies that $\mu(\{(\pi_R, \mathbf{1})\}) = \infty$ for the corresponding M-invariant measure. This is only technical and for our processes this value $\mu(\{(\pi_R, \mathbf{1})\})$ has no relevance. Therefore we will from now on abuse notation and identify M-invariant measures on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ with their restriction to $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq} \setminus \{(\pi_R, \mathbf{1})\}$. More precisely, in the rest of the article, we *extend the definition of M-invariance* to all measures μ on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ such that for all $\tau \in M$, μ and μ^τ coincide on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq} \setminus \{(\pi_R, \mathbf{1})\}$. As such, we will now *only* consider M-invariant measures μ satisfying $\mu(\{(\pi_R, \mathbf{1})\}) = 0$.

Putting together Proposition 7 and Proposition 9 gives us:

Theorem 10. *Let $\Pi = (\Pi(t), t \geq 0)$ be a nested fragmentation process. Then its distribution is characterized by a unique M-invariant measure μ on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ satisfying*

$$\begin{aligned}&\mu\big(\pi \not\preceq (\pi_R, \mathbf{1})\big) = 0 \\ &\text{and} \quad \forall n \in \mathbb{N}, \quad \mu\big(\pi_{|[n]^2} \neq (\pi_R, \mathbf{1})_{|[n]^2}\big) < \infty.\end{aligned} \tag{8}$$

The characterization is in the sense that for any $\pi_0, \pi_1 \in \mathcal{P}_\infty$ with infinitely many infinite blocks, for any Borel sets $A \subset \mathcal{P}_{\mathbb{N}^2}^{2,\preceq} \setminus \{(\pi_R, \mathbf{1})\}$ and $B \subset \mathcal{P}_\infty^{2,\preceq} \setminus \{(\pi_1, \mathbf{1})\}$,

$$\mu(A) = \kappa_{\pi_0}^{\sigma_0}(A) \quad \text{and} \quad \kappa_{\pi_1}(B) = \mu^{\sigma_1}(B),$$

where κ is the simplified characteristic kernel of Π, $\sigma_0 : \mathbb{N}^2 \to \mathbb{N}$ is any injection such that $\pi_0^{\sigma_0} = \pi_R$ and $\sigma_1 : \mathbb{N} \to \mathbb{N}^2$ is any injection such that $\pi_R^{\sigma_1} = \pi_1$.

Conversely, for any such measure μ, there is a strongly exchangeable Markov process with values in $\mathcal{P}_\infty^{2,\preceq}$, nonincreasing càdlàg sample paths and the outer branching property with characteristic measure μ.

Remark 11. An explicit construction for the converse part of the theorem is described in the next section (Lemma 12).

4.3. *Poissonian construction*

Consider μ an M-invariant measure on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ satisfying (8), and let Λ be a Poisson point process on $\mathbb{N} \times [0, \infty) \times \mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$ with intensity $\# \otimes \mathrm{d}t \otimes \mu$, where # denotes the counting measure and dt the Lebesgue measure.

Fix $n \in \mathbb{N}$. Because of (8), the points $(k, t, \pi) \in \Lambda$ such that $k \leq n$ and $\pi_{|[n]^2} \neq (\pi_R, \mathbf{1})_{|[n]^2}$ can be numbered

$$\big(k_i^n, t_i^n, \pi_i^n, i \geq 1\big) \quad \text{with } t_1^n < t_2^n < \cdots \text{ and } t_i^n \xrightarrow[i\to\infty]{} \infty.$$

Fix any initial value $\pi_0 \in \mathcal{P}_\infty^{2,\preceq}$. Let us define a process $(\Pi_i^n, i \geq 0)$ with values in $\mathcal{P}_{[n]}^{2,\preceq}$, by $\Pi_0^n = (\pi_0)_{|[n]}$ and by induction, conditional on $\Pi_i^n = (\zeta, \xi)$:

- if ξ has less than k_{i+1}^n blocks, then set $\Pi_{i+1}^n := \Pi_i^n$
- if ξ has a k_{i+1}^n-th block, say B, then let $\tau : B \to [n]^2$ be the injection such that $\tau(k) = (i', j')$ iff $k \in B$ is the j'-th element of the i'-th block of $\zeta_{|B}$.

  Then define Π_{i+1}^n as the only element $\pi \in \mathcal{P}_n^{2,\preceq}$ such that $\pi \preceq \Pi_i^n$, $\pi_{|B} = (\pi_i^n)^\tau$ and $\pi_{|[n]\setminus B} = (\Pi_i^n)_{|[n]\setminus B}$.

Now we define the continuous-time processes $(\Pi^n(t), t \geq 0)$ by

$$\Pi^n(t) := \Pi_i^n \quad \text{iff } t \in \big[t_{i-1}^n, t_i^n\big).$$

Lemma 12. *The processes Π^n built from this Poissonian construction are consistent in the sense that we have for all $m \geq n \geq 1$ and $t \geq 0$,*

$$\Pi^m(t)_{|[n]} = \Pi^n(t).$$

Therefore, for all $t \geq 0$, there is a unique random variable $\Pi(t)$ with values in $\mathcal{P}_\infty^{2,\preceq}$ such that $\Pi(t)_{|[n]} = \Pi^n(t)$ for all n, and the process $(\Pi(t), t \geq 0)$ is a strongly exchangeable Markov process with càdlàg, nonincreasing sample paths, satisfying the outer branching property, and whose characteristic M-invariant measure is μ.

Proof. Choose an integer $n \in \mathbb{N}$ and consider the variable $(k_1^{n+1}, t_1^{n+1}, \pi_1^{n+1})$. It is clear from the definition that $(\Pi_0^{n+1})_{|[n]} = \Pi_0^n$. Now let us show that $(\Pi_1^{n+1})_{|[n]} = \Pi^n(t_1^{n+1})$.

We distinguish two cases:

(1) If $t_1^{n+1} = t_1^n$, then we have necessarily $k_1^{n+1} = k_1^n \leq n$ and $(\pi_1^{n+1})_{|[n]^2} = (\pi_1^n)_{|[n]^2} \neq (\pi_{\mathrm{R}}, \mathbf{1})_{|[n]^2}$. Let us write $\Pi_0^{n+1} = (\zeta^{n+1}, \xi^{n+1})$ and $\Pi_0^n = (\zeta^n, \xi^n)$. Since $(\Pi_0^{n+1})_{|[n]} = \Pi_0^n$, it is clear that the k_1^n-th block of ξ^{n+1} includes the k_1^n-th block of ξ^n, and may at most contain one other element, the integer $n+1$. In other words we have

$$B^{n+1} \cap [n] = B^n,$$

where B^{n+1} and B^n denote those two blocks. Now let us write τ^{n+1}, τ^n for the respective injections in $\mathbb{N}^2$ defined in the construction. Because we defined the injections according to the ordering of the blocks of ζ and with the natural order on $\mathbb{N}$, it is clear that

$$\tau^{n+1}_{|B^n} = \tau^n.$$

Therefore we deduce $((\pi_1^n)^{\tau^{n+1}})_{|B^n} = (\pi_1^n)^{\tau^n}$, which allows us to conclude $(\Pi_1^{n+1})_{|[n]} = \Pi_1^n = \Pi^n(t_1^{n+1})$.

(2) If $t_1^{n+1} < t_1^n$, then we have to further distinguish two possibilities:

(a) $k_1^{n+1} = n+1$. In that case the $(n+1)$-th block of ξ^{n+1} can either be empty or the singleton $\{n+1\}$. Then by definition, we necessarily have $\Pi_1^{n+1} = \Pi_0^{n+1}$, so we can conclude $(\Pi_1^{n+1})_{|[n]} = \Pi_0^n = \Pi^n(t_1^{n+1})$.

(b) $k_1^{n+1} \leq n$, and then necessarily $(\pi_1^{n+1})_{|[n]^2} = (\pi_{\mathrm{R}}, \mathbf{1})_{|[n]^2}$. In that case, let B be the k_1^{n+1}-th block of ξ and $\tau : B \to [n+1]^2$ the injective map defined in the construction. By definition, we have $(\pi_{\mathrm{R}}, \mathbf{1})^\tau = (\zeta, \xi)_{|B}$. Also by definition of τ, for any $k \leq n$, we have $\tau(k) \in [n]^2$. Therefore, we can conclude that

$$\left(\left(\pi_1^{n+1}\right)^\tau\right)_{|B\cap[n]} = \left(\left(\pi_1^{n+1}\right)_{|[n]^2}\right)^{\tau_{|B\cap[n]}} = (\pi_{\mathrm{R}}, \mathbf{1})^{\tau_{|B\cap[n]}} = (\zeta, \xi)_{|B\cap[n]}.$$

This shows that $(\Pi_1^{n+1})_{|[n]} = (\Pi_0^{n+1})_{|[n]}$, which allows us to conclude $(\Pi_1^{n+1})_{|[n]} = \Pi_0^n = \Pi^n(t_1^{n+1})$.

By induction and the strong Markov property of the Poisson point process Λ, this proves that $(\Pi_i^{n+1})_{|[n]} = \Pi^n(t_i^{n+1})$ for all $i \geq 1$, so $\Pi^{n+1}(t)_{|[n]} = \Pi^n(t)$ for all $t \geq 0$, which concludes the first part of the proof.

It remains to show that the process $(\Pi(t), t \geq 0)$ is a strongly exchangeable Markov process with the outer branching property, and whose characteristic M-invariant measure is μ.

First, notice that from the construction, we deduce immediately that for any n, Π^n is a Markov chain, and at any jump time t_i^n, the partitions Π_{i-1}^n and Π_i^n differ at most on one block of ξ, where $\Pi_{i-1}^n = (\zeta, \xi)$. Therefore the distribution of the Markov chain Π^n is given by the transition rates of the form

$$q^n_{\pi_0,\pi_1},$$

with $\pi_0 = (\zeta, \xi) \in \mathcal{P}_\infty^{2,\preceq}$, and with $\pi_1 \preceq (\pi_0)_{|[n]}$ such that, for some $B \in \xi_{|[n]}$, $(\pi_1)_{|[n]\setminus B} = (\pi_0)_{|[n]\setminus B}$ and $(\pi_1)_{|B} \prec (\pi_0)_{|B}$. Now for such π_0, π_1, write $\tau : B \to \mathbb{N}^2$ for the injection such that $\tau(k) = (i, j)$ iff k is the j-th element of the i-th block of $\zeta_{|B}$. By elementary properties of Poisson point processes we have

$$q^n_{\pi_0,\pi_1} = \mu\big(\pi^\tau = (\pi_1)_{|B}\big). \tag{9}$$

Now recall from Proposition 2 that since Π satisfies the projective Markov property and is exchangeable (this is immediate from the M-invariance of μ), Π is strongly exchangeable, with a characteristic kernel K such that with the same notation as in (9),

$$K_{\pi_0}(\pi_{|[n]} = \pi_1) = q^n_{\pi_0,\pi_1}. \tag{10}$$

Now the outer branching property is immediately deduced from the construction of the process, where it is clear that at any jump time, at most one block of the coarser partition is involved. Therefore by Proposition 7, the law of Π is characterized by the simpler kernel κ defined by $\kappa_\zeta = K_{(\zeta,\mathbf{1})}$, for $\zeta \in \mathcal{P}_\infty$. Now putting this together with (10) and (9), since the coarsest partition $\mathbf{1}_{[n]}$ only contains one block $B = [n]$, we have simply

$$\kappa_\zeta(\pi_{|[n]} = \pi_1) = \mu\big((\pi^\tau)_{|[n]} = \pi_1\big),$$

where τ is an injection such that $\pi_{\mathrm{R}}^\tau = \zeta$. In other words with these definitions, the measures κ_ζ and μ^τ coincide on $\mathcal{P}_\infty^{2,\preceq} \setminus \{(\zeta,\mathbf{1})\}$, which shows that μ is the characteristic M-invariant measure of the process Π. □

5. Inner branching property

In the previous section we only exploited the outer branching property of Definition 1′. This section will instead focus on the inner branching property, which will allow us to further the analysis of the M-invariant measure μ appearing in Theorem 10. To introduce the next theorem and main result of this article, let us first give examples of M-invariant measures that give rise to the types of transitions already discussed in Section 2.3.

5.1. *Some examples*

Pure erosion. For $i \geq 1$, let $\xi_{\mathrm{out}}^{(i)}$ be the partition of $\mathbb{N}^2$ with two blocks such that one of them is the i-th line $\{i\} \times \mathbb{N}$, i.e.

$$\xi_{\mathrm{out}}^{(i)} := \big\{\{i\} \times \mathbb{N}, \mathbb{N}^2 \setminus \big(\{i\} \times \mathbb{N}\big)\big\}$$

and define the outer erosion measure $\mathfrak{e}^{\mathrm{out}} := \sum_{i\geq 1} \delta(\pi_{\mathrm{R}}, \xi_{\mathrm{out}}^{(i)})$, where for readability we denote without subscripts $\delta(\zeta,\xi)$ the Dirac measure on (ζ,ξ).

Similarly, for $i, j \geq 1$, we define

$$\zeta_{\mathrm{in}}^{(i,j)} := \big\{\{(i,j)\}\big\} \cup \big\{\big(\{i\} \times \mathbb{N}\big) \setminus \{(i,j)\}\big\} \cup \big\{\{k\} \times \mathbb{N}, k \geq 1, k \neq i\big\},$$
$$\xi_{\mathrm{in}}^{(i,j)} := \big\{\{(i,j)\}, \mathbb{N}^2 \setminus \{(i,j)\}\big\},$$

and the inner erosion measures

$$\mathfrak{e}^{\mathrm{in},1} := \sum_{i,j\geq 1} \delta\big(\zeta_{\mathrm{in}}^{(i,j)}, \mathbf{1}\big) \quad \text{and} \quad \mathfrak{e}^{\mathrm{in},2} := \sum_{i,j\geq 1} \delta\big(\zeta_{\mathrm{in}}^{(i,j)}, \xi_{\mathrm{in}}^{(i,j)}\big).$$

Now, given three real numbers $c_{\mathrm{out}}, c_{\mathrm{in},1}, c_{\mathrm{in},2} \geq 0$, the M-invariant measure $\mu = c_{\mathrm{out}}\mathfrak{e}^{\mathrm{out}} + c_{\mathrm{in},1}\mathfrak{e}^{\mathrm{in},1} + c_{\mathrm{in},2}\mathfrak{e}^{\mathrm{in},2}$ clearly satisfies (8), so by Theorem 10 there exists a fragmentation process having μ as M-invariant measure.

From the construction, we see that the rates of such a process can be described informally as follows:

- any inner block erodes out of its outer block at rate c_{out}, i.e. it does not fragment but forms, on its own, a new outer block.
- any integer erodes out of its inner block at rate $c_{\mathrm{in},1}$, forming a singleton inner block, within the same outer block as its parent.
- any integer erodes out of its inner and outer block at rate $c_{\mathrm{in},2}$, forming singleton inner and outer blocks.

Outer dislocation. Recall the definition of the space of mass partitions $\mathbf{s} = (s_1, s_2, \ldots) \in \mathscr{P}_{\mathrm{m}}$ and of the measures $\varrho_{\mathbf{s}}$ from Section 2.2. We define in a similar way, a collection of probability measure $\widehat{\varrho}_{\mathbf{s}}$ on $\mathcal{P}_{\mathbb{N}^2}^{2,\preceq}$, by constructing $\pi = (\zeta,\xi) \sim \widehat{\varrho}_{\mathbf{s}}$ with the following so-called paintbox procedure:

- for $k \geq 0$, let $t_k := \sum_{k'=1}^{k} s_{k'}$, with $t_0 = 0$ by convention.
- let $U_1, U_2, \ldots$ be a sequence of i.i.d. uniform r.v. on $[0,1]$ and define the random partition $\xi \succeq \pi_{\mathrm{R}}$ on $\mathbb{N}^2$ by

$$(i,j) \sim^{\xi} (i',j') \iff i = i' \quad \text{or} \quad U_i, U_{i'} \in [t_k, t_{k+1}) \quad \text{for a unique } k \geq 0.$$

- $\widehat{\varrho}_{\mathbf{s}}$ is now defined to be the distribution of the random nested partition $\pi = (\pi_{\mathrm{R}}, \xi)$.

Now for ν_{out} a measure on $\mathscr{P}_{\text{m}}$ satisfying (4), we define

$$\widehat{\varrho}_{\nu_{\text{out}}}(\cdot) := \int_{\mathscr{P}_{\text{m}}} \nu_{\text{out}}(\mathrm{d}\mathbf{s})\widehat{\varrho}_{\mathbf{s}}(\cdot).$$

It is straight-forward to check that $\widehat{\varrho}_{\nu_{\text{out}}}$ is an M-invariant measure measure on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ satisfying (8), so there exists a fragmentation process having $\widehat{\varrho}_{\nu_{\text{out}}}$ as M-invariant measure.

In intuitive terms, such a process can be described by saying that the outer blocks independently dislocate *around their inner blocks* with *outer dislocation rate* ν_{out}. In a dislocation event, inner blocks are unchanged, and they are indistinguishable. By construction, each newly created outer block selects a given frequency of inner blocks among those forming the original outer block.

Inner dislocation. The upcoming example is the most complex on our list, exhibiting simultaneous inner and outer fragmentations. However, in construction it is very similar to the previous example, and it should pose no difficulties to get a good intuition of the dislocation mechanics.

Let us first formally define a space which will serve as an analog of the space of mass partitions $\mathscr{P}_{\text{m}}$.

Definition 13. We define a particular space of *bivariate mass partitions*

$$\mathscr{P}_{\text{m},\preceq} \subset [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}^2} \times [0,1] \times [0,1]^{\mathbb{N}}$$

as the subset consisting of elements $\mathbf{p} = ((u_l)_{l\geq 1}, (s_{k,l})_{k,l\geq 1}, \bar{u}, (\bar{s}_k)_{k\geq 1})$ satisfying the following conditions.

$$\begin{aligned}
&u_1 \geq u_2 \geq \cdots \text{ and } \sum_l u_l \leq \bar{u}, \\
&\forall k \geq 1, s_{k,1} \geq s_{k,2} \geq \cdots \text{ and } \sum_l s_{k,l} \leq \bar{s}_k, \\
&\bar{s}_1 \geq \bar{s}_2 \geq \cdots, \\
&\bar{u} + \sum_k \bar{s}_k \leq 1, \\
&\text{if } \bar{s}_k = \bar{s}_{k+1}, \text{ then } \big(l_0 = \inf\{l \geq 1, s_{k,l} \neq s_{k+1,l}\} < \infty\big) \Rightarrow (s_{k,l_0} > s_{k+1,l_0}).
\end{aligned} \tag{11}$$

We claim that $\mathscr{P}_{\text{m},\preceq}$ is Polish with respect to the product topology. Indeed, recall [see e.g. 24, Theorem 2.2.1] that any G_δ subset – i.e. a countable intersection of open sets – of a Polish space is Polish. Now, it is readily checked that every condition in (11) is closed in the compact space $X := [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}^2} \times [0,1] \times [0,1]^{\mathbb{N}}$ except the last one, but the subset of X satisfying this condition can be written

$$\bigcap_{k\geq 1}\left[\{\bar{s}_k \neq \bar{s}_{k+1}\} \cup \left(\bigcap_{l\geq 1}\{\exists i < l, s_{k,i} \neq s_{k+1,i}\} \cup \{s_{k,l} \geq s_{k+1,l}\}\right)\right],$$

so finally $\mathscr{P}_{\text{m},\preceq}$ can be written as a countable intersection of open and closed sets in X, which are all G_δ (recall that closed subsets of any *metrizable* space are G_δ). Therefore considering this topology, $\mathscr{P}_{\text{m},\preceq}$ is Polish and we will have no trouble considering measures on $\mathscr{P}_{\text{m},\preceq}$.

Now, given a fixed $i \geq 1$ and $\mathbf{p} = ((u_l)_{l\geq 1}, (s_{k,l})_{k,l\geq 1}, \bar{u}, (\bar{s}_k)_{k\geq 1}) \in \mathscr{P}_{\text{m},\preceq}$, one can define a random element $\pi^{(i)} = (\zeta^{(i)}, \xi^{(i)}) \in \mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ with the following paintbox procedure:

- for $k \geq 0$, define $\bar{t}_k = \bar{u} + \sum_{k'=1}^{k} \bar{s}_{k'}$.
- for $l \geq 0$, define $t_{\star,l} = \sum_{l'=1}^{l} u_{l'}$.
- for $k \geq 1$ and $l \geq 0$, define $t_{k,l} = \bar{t}_{k-1} + \sum_{l'=1}^{l} s_{k,l'}$.
- write $\pi_0 = (\zeta_0, \xi_0)$ for the unique element of $\mathcal{P}^{2,\preceq}_{[0,1]}$ such that the non-dust blocks of ξ_0 are

  $$[0, \bar{u}) \quad \text{and} \quad [\bar{t}_{k-1}, \bar{t}_k), \quad k \geq 1,$$

  and such that the non-singleton blocks of ζ_0 are

  $$[t_{\star,l-1}, t_{\star,l}), \quad l \geq 1 \quad \text{and} \quad [t_{k,l-1}, t_{k,l}), \quad k, l \geq 1.$$

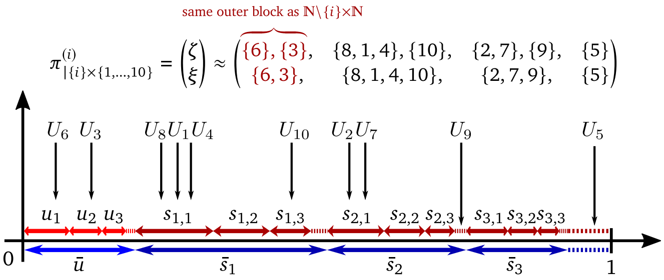

Fig. 4. Paintbox construction of $\pi^{(i)}$.

- let $(U_j, j \geq 1)$ be an i.i.d. sequence of uniform random variables on $[0,1]$.
- define the random element $\pi^{(i)} \in \mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ as the unique element $\pi^{(i)} = (\zeta^{(i)}, \xi^{(i)}) \preceq (\pi_{\mathrm{R}}, \mathbf{1})$ such that
  - $(\zeta^{(i)}, \xi^{(i)})_{|(\mathbb{N}\setminus\{i\})\times\mathbb{N}} = (\pi_{\mathrm{R}}, \mathbf{1})_{|(\mathbb{N}\setminus\{i\})\times\mathbb{N}}$, i.e. only the i-th row may dislocate.
  - On the i-th row, we have

$$(i,j) \sim^{\zeta^{(i)}} (i,j') \iff U_j \sim^{\zeta_0} U_{j'},$$
$$(i,j) \sim^{\xi^{(i)}} (i,j') \iff U_j \sim^{\xi_0} U_{j'},$$

and also

$$(i,j) \sim^{\xi^{(i)}} (i+1,1) \iff U_j \in [0,\bar{u}),$$

where it should be noted that $(i+1,1)$ could be replaced by any element (i',j') with $i' \neq i$.

See Figure 4 for a representation of the bivariate paintbox process. In words, $\pi^{(i)}$ is a random nested partition such that the outer partition $\xi^{(i)}$ has a *distinguished block* containing $(\mathbb{N}\setminus\{i\})\times\mathbb{N}$, which also contains a proportion $\bar{u}$ of elements of the i-th row. Other non-singleton blocks of $\xi^{(i)}$ can be indexed by $k \geq 1$, each containing a proportion $\bar{s}_k$ of elements of the i-th row. The blocks of the inner partition $\zeta^{(i)}$ are the entire rows, except for the i-th row where non-singleton blocks can be indexed by $(\star, l)$ and (k,l) for $k,l \geq 1$, each respectively containing a proportion u_l or $s_{k,l}$ of elements of the i-th row. As the notation suggests, inner blocks with frequency $s_{k,l}$ (resp. u_l) are included in the outer block with frequency $\bar{s}_k$ (resp. $\bar{u}$) on the i-th row.

The distribution of $\pi^{(i)}$ obtained with this construction is a probability on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ that we denote by $\widetilde{\varrho}^{(i)}_{\mathbf{p}}$. We finally define

$$\widetilde{\varrho}_{\mathbf{p}} = \sum_{i\geq 1} \widetilde{\varrho}^{(i)}_{\mathbf{p}}.$$

It is clear from the exchangeability of the sequence $(U_j, j \geq 1)$ that $\widetilde{\varrho}_{\mathbf{p}}$ is M-invariant.

Now consider a measure ν_{in} on $\mathscr{P}_{\mathrm{m},\preceq}$ satisfying

$$\nu_{\mathrm{in}}\big(\{u_1 = 1 \text{ or } s_{1,1} = 1\}\big) = 0, \quad \text{and} \quad \int_{\mathscr{P}_{\mathrm{m},\preceq}} (1-u_1)\nu_{\mathrm{in}}(\mathrm{d}\mathbf{p}) < \infty. \tag{12}$$

Similarly as in the previous example, we define

$$\widetilde{\varrho}_{\nu_{\mathrm{in}}}(\cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}} \widetilde{\varrho}_{\mathbf{p}}(\cdot)\nu_{\mathrm{in}}(\mathrm{d}\mathbf{p}).$$

It is again straight-forward to check that $\widetilde{\varrho}_{\nu_{\mathrm{in}}}$ is an M-invariant measure on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ satisfying (8), so there exists a fragmentation process having $\widetilde{\varrho}_{\nu_{\mathrm{in}}}$ as M-invariant measure.

In intuitive terms, such a process can be described by saying that the inner blocks independently dislocate with *inner dislocation rate* ν_{in}. In a dislocation event, new inner blocks are formed, each with a given proportion of the original block, and regroup, either in the original outer block (with a total proportion $\bar{u}$ with respect to the original inner block) or in newly created outer blocks.

A combination of the above. The mechanisms we discussed in the three proposed examples can be added in a parallel way, each event arising at its own independent rate and events from distinct mechanisms occurring at distinct times. More precisely, for a set of erosion coefficients $c_{\text{out}}, c_{\text{in},1}, c_{\text{in},2} \geq 0$, an outer dislocation measure ν_{out} on $\mathscr{P}_{\text{m}}$ satisfying (4) and an inner dislocation measure ν_{in} on $\mathscr{P}_{\text{m},\preceq}$ satisfying (12), the measure

$$\mu := c_{\text{out}}\mathfrak{e}^{\text{out}} + c_{\text{in},1}\mathfrak{e}^{\text{in},1} + c_{\text{in},2}\mathfrak{e}^{\text{in},2} + \widehat{\varrho}_{\nu_{\text{out}}} + \widetilde{\varrho}_{\nu_{\text{in}}}$$

is a valid M-invariant measure on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ satisfying (8), and thus corresponds to a fragmentation process exhibiting simultaneously all the discussed mechanisms at the rates described above. The main result of this article is to prove that any nested fragmentation process admits such a representation.

5.2. *Characterization of nested fragmentations*

Theorem 14. *Let $\Pi = (\Pi(t), t \geq 0) = ((\zeta(t), \xi(t)), t \geq 0)$ be a nested fragmentation process. Then there are*

- *an outer erosion coefficient $c_{\text{out}} \geq 0$ and two inner erosion coefficients $c_{\text{in},1}, c_{\text{in},2} \geq 0$;*
- *an outer dislocation measure ν_{out} on $\mathscr{P}_{\text{m}}$ satisfying (4);*
- *an inner dislocation measure ν_{in} on $\mathscr{P}_{\text{m},\preceq}$ satisfying (12);*

such that the M-invariant measure μ of the process can be written

$$\mu = c_{\text{out}}\mathfrak{e}^{\text{out}} + c_{\text{in},1}\mathfrak{e}^{\text{in},1} + c_{\text{in},2}\mathfrak{e}^{\text{in},2} + \widehat{\varrho}_{\nu_{\text{out}}} + \widetilde{\varrho}_{\nu_{\text{in}}}.$$

Corollary 15. *Definition 1 is equivalent to Definition 1′.*

Proof. We have shown most of the equivalence in Corollary 4 and Remark 8. What remains is to show that if $\Pi = (\zeta, \xi)$ is a nested fragmentation process according to Definition 1′, then ζ is a homogeneous fragmentation process in $\mathcal{P}_\infty$. Now if μ is given by the expression of the preceding theorem, using the Poissonian construction of Section 4.3 one easily checks that ζ has the same transition rates as a homogeneous fragmentation with erosion coefficient $c = c_{\text{in},1} + c_{\text{in},2}$ and dislocation measure $\nu = \nu_{\text{in}} \circ S^{-1}$, where $S : \mathscr{P}_{\text{m},\preceq} \to \mathscr{P}_{\text{m}}$ is the map given by

$$S(\mathbf{p}) := \text{nonincreasing reordering of } \{u_l, l \geq 1\} \cup \{s_{k,l}, k, l \geq 1\}. \qquad \square$$

The rest of Section 5 is dedicated to proving Theorem 14. Let μ be the M-invariant characteristic measure on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ associated with Π. Recall that π_{R} denotes the *rows partition*, defined by

$$\pi_{\text{R}} = \big\{\{(i,j), j \geq 1\}, i \geq 1\big\}.$$

First, notice that the inner branching property implies that μ-a.e. we have

$$\exists i \in \mathbb{N}, \quad \zeta_{|(\mathbb{N}\setminus\{i\})\times\mathbb{N}} = (\pi_{\text{R}})_{|(\mathbb{N}\setminus\{i\})\times\mathbb{N}},$$

where ζ is the first coordinate in the standard variable $\pi = (\zeta, \xi) \in \mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$. This will enable us to decompose μ further. Let us write

$$\begin{aligned}
&\mu_{\text{out}} := \mu\big(\cdot \cap \{\zeta = \pi_{\text{R}}\}\big),\\
&\text{for } i \in \mathbb{N}, \mu_{\text{in},i} := \mu\big(\{\zeta_{|\{i\}\times\mathbb{N}} \neq \mathbf{1}_{\{i\}\times\mathbb{N}}\} \cap \cdot\big),\\
&\text{such that } \mu_{\text{in}} := \mu\big(\cdot \cap \{\zeta \neq \pi_{\text{R}}\}\big) = \sum_{i\geq 1}\mu_{\text{in},i}\\
&\text{and } \mu = \mu_{\text{out}} + \mu_{\text{in}}.
\end{aligned} \tag{13}$$

On the event $\{\zeta = \pi_{\text{R}}\}$, we have

$$\xi = f\big(\xi^\sigma\big),$$

where $\sigma : \mathbb{N} \to \mathbb{N}^2$ is the injection $i \mapsto (i, 1)$, and $f : \mathcal{P}_\infty \to \mathcal{P}_{\mathbb{N}^2}$ is the map such that $(i,j) \sim^{f(\pi_0)} (i',j') \iff i \sim^{\pi_0} i'$. By M-invariance of μ, the measure

$$\widetilde{\mu}_{\text{out}} := \mu\big(\{\zeta = \pi_{\text{R}}\} \cap \{\xi^\sigma \in \cdot\}\big)$$

is an exchangeable measure on $\mathcal{P}_\infty$, of which μ_{out} is the push-forward by the map $(\pi_{\text{R}}, f(\cdot))$.

Also, note that μ satisfies the σ-finiteness assumption (8), which implies that $\widetilde{\mu}_{\text{out}}$ satisfies (3), showing (see Section 2.2) that it can be decomposed

$$\widetilde{\mu}_{\text{out}} = c_{\text{out}}\mathfrak{e} + \varrho_{\nu_{\text{out}}},$$

where $c_{\text{out}} \geq 0$ and ν_{out} is a measure on $\mathscr{P}_{\text{m}}$ satisfying (4). Thanks to our definitions, this immediately translates into

$$\mu_{\text{out}} = c_{\text{out}}\mathfrak{e}^{\text{out}} + \widehat{\varrho}_{\nu_{\text{out}}},$$

and to prove Theorem 14, it only remains to show that we can write

$$\mu_{\text{in}} = \sum_{i\geq 1} \mu_{\text{in},i} = c_{\text{in},1}\mathfrak{e}^{\text{in},1} + c_{\text{in},2}\mathfrak{e}^{\text{in},2} + \widetilde{\varrho}_{\nu_{\text{in}}}.$$

To that aim, note that by exchangeability we have $\mu_{\text{in},i} = \mu_{\text{in},1}^{\tau_{1,i}}$ where $\tau_{1,i} : \mathbb{N}^2 \to \mathbb{N}^2$ denotes the bijection swapping the first and i-th rows, so the measure $\mu_{\text{in},1}$ is sufficient to recover μ_{in} entirely. Let us examine the distribution of ξ under $\mu_{\text{in},1}$. We claim that μ-a.e. on the event $\{\zeta_{|\{1\}\times\mathbb{N}} \neq \mathbf{1}_{\{1\}\times\mathbb{N}}\}$, the equality $\xi_{|(\mathbb{N}\setminus\{1\})\times\mathbb{N}} = \mathbf{1}_{(\mathbb{N}\setminus\{1\})\times\mathbb{N}}$ holds. Indeed, if this was not the case, by M-invariance we would have

$$a := \mu\big(\zeta_{|\{1\}\times\mathbb{N}} \neq \mathbf{1}_{\{1\}\times\mathbb{N}},\ \text{and } (2,1) \nsim^{\xi} (3,1)\big) > 0.$$

Let us then show that in fact $a = 0$. By M-invariance of μ, we have for any $i \geq 4$,

$$a = \mu\big(\zeta_{|\{i\}\times\mathbb{N}} \neq \mathbf{1}_{\{i\}\times\mathbb{N}},\ \text{and } (2,1) \nsim^{\xi} (3,1)\big),$$

but because of the inner branching property, we have seen that the events $\{\zeta_{|\{i\}\times\mathbb{N}} \neq \mathbf{1}_{\{i\}\times\mathbb{N}}\}$ have μ-negligible intersections. Now we have

$$\begin{aligned}\infty > \mu\big(\pi_{|[3]^2} \neq (\pi_{\text{R}}, \mathbf{1})_{|[3]^2}\big) &\geq \mu\big((2,1) \nsim^{\xi} (3,1)\big)\\ &\geq \mu\Big(\bigcup_{i\geq 4}\big\{\zeta_{|\{i\}\times\mathbb{N}} \neq \mathbf{1}_{\{i\}\times\mathbb{N}},\ \text{and } (2,1) \nsim^{\xi} (3,1)\big\}\Big)\\ &= \sum_{i\geq 4} a.\end{aligned}$$

This shows that necessarily $a = 0$.

Now in order to further study $\mu_{\text{in},1}$ we need to introduce exchangeable partitions on a space with a distinguished element. Results in that direction have been established by Foucart [12], where distinguished exchangeable partitions are introduced and used to construct a generalization of Λ-coalescents modeling the genealogy of a population with immigration. Here we need to define in a similar way distinguished partitions in our bivariate setting. Informally, we will see that in a gene fragmentation, certain resulting gene blocks remain in a distinguished species block, that one can interpret as the mother species.

Definition 16. For $n \in \mathbb{N} \cup \{\infty\}$, we define $[n]_\star := [n] \cup \{\star\}$, where $\star$ is not an element of $\mathbb{N}$. We define $\mathcal{P}_{n,\star}^{2,\preceq}$ as the set of nested partitions $\pi = (\zeta, \xi) \in \mathcal{P}_{[n]_\star}^{2,\preceq}$ such that $\star$ is isolated in the finer partition ζ:

$$\mathcal{P}_{n,\star}^{2,\preceq} := \big\{\pi = (\zeta, \xi) \in \mathcal{P}_{[n]_\star}^{2,\preceq},\ \{\star\} \in \zeta\big\}.$$

We define the action of an injection $\sigma : [n] \to [n]$ on an element $\pi \in \mathcal{P}_{n,\star}^{2,\preceq}$ as the action of the unique extension $\widetilde{\sigma} : [n]_\star \to [n]_\star$ such that $\widetilde{\sigma}(\star) = \star$, and define *exchangeability* for measures on $\mathcal{P}_{n,\star}^{2,\preceq}$ as invariance under the actions of such injections $\sigma : [n] \to [n]$.

Let us come back to the decomposition of $\mu_{\text{in},1}$. We define an injection

$$\tau : \begin{cases} [\infty]_\star & \longrightarrow \mathbb{N}^2 \\ j \in \mathbb{N} & \longmapsto (1, j) \\ \star & \longmapsto (2, 1). \end{cases}$$

Note that here we could have chosen any value $\tau(\star) = (i, j)$ with $i \geq 2$, since μ-a.e. on the event $\{\zeta_{|\{1\}\times\mathbb{N}} \neq \mathbf{1}_{\{1\}\times\mathbb{N}}\}$ those elements are all in the same block of ξ. The argument above shows that on the event $\{\zeta_{|\{1\}\times\mathbb{N}} \neq \mathbf{1}_{\{1\}\times\mathbb{N}}\}$, we have μ-a.e. the equality

$$\pi = (\zeta, \xi) = g(\pi^\tau),$$

where $g : \mathcal{P}^{2,\preceq}_{\infty,\star} \to \mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ is a deterministic function which we can define by: $g(\pi_0)$ is the only $\pi \in \mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ such that

$$\pi^\tau = \pi_0, \qquad \pi \preceq (\pi_\mathrm{R}, \mathbf{1}_{\mathbb{N}^2})$$
$$\text{and} \quad \pi_{|(\mathbb{N}\setminus\{1\})\times\mathbb{N}} = (\pi_\mathrm{R}, \mathbf{1}_{\mathbb{N}^2})_{|(\mathbb{N}\setminus\{1\})\times\mathbb{N}}.$$

Let us now write

$$\widetilde{\mu}_\mathrm{in} := \mu_{\mathrm{in},1}(\pi^\tau \in \cdot). \tag{14}$$

Note that the push-forward of this exchangeable measure on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ by the map g is $\mu_{\mathrm{in},1}$. Also, note that the σ-finiteness assumption (8) and the fact that $\mu_{\mathrm{in},1}$-a.e. we have $\zeta_{|\{1\}\times\mathbb{N}} \neq \mathbf{1}_{\{1\}\times\mathbb{N}}$ imply that $\widetilde{\mu}_\mathrm{in}$ satisfies

$$\widetilde{\mu}_\mathrm{in}(\{\zeta_{|[\infty]} = \mathbf{1}\}) = 0, \quad \text{and} \quad \forall n \geq 1, \quad \widetilde{\mu}_\mathrm{in}(\pi_{|[n]_\star} \neq \pi_n) < \infty \tag{15}$$

where $\pi_n := (\{\{\star\}, [n]\}, \mathbf{1}_{[n]_\star})$ denotes the coarsest partition on $\mathcal{P}^{2,\preceq}_{n,\star}$.

We can summarize the previous discussion in the following lemma.

Lemma 17. *The characteristic M-invariant measure μ of a nested fragmentation process in $\mathcal{P}^{2,\preceq}_\infty$ can be decomposed*

$$\mu = c_\mathrm{out}\mathfrak{e}^\mathrm{out} + \widehat{\varrho}_{\nu_\mathrm{out}} + \mu_\mathrm{in},$$

where $c_\mathrm{out} \geq 0$, ν_out is a measure on $\mathscr{P}_\mathrm{m}$ satisfying (4), and $\mu_\mathrm{in} := \mu(\cdot \cap \{\zeta \neq \pi_\mathrm{R}\})$. Also, there exists an exchangeable measure $\widetilde{\mu}_\mathrm{in}$ on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ which satisfies (15) and such that $\mu_\mathrm{in} = \sum_i \mu^{\tau_{1,i}}_{\mathrm{in},1}$, where

- *$\mu_{\mathrm{in},1}$ is a measure on $\mathcal{P}^{2,\preceq}_{\mathbb{N}^2}$ which is the push-forward of $\widetilde{\mu}_\mathrm{in}$ by the map g defined in the previous paragraph.*
- *$\tau_{1,i} : \mathbb{N}^2 \to \mathbb{N}^2$ is the bijection swapping the first row with the i-th row.*

In the next section, we will develop tools to analyze and further decompose the measure $\widetilde{\mu}_\mathrm{in}$ into terms of erosion and dislocation.

5.3. *Bivariate mass partitions*

Recall our space of bivariate mass partitions defined in Definition 13,

$$\mathscr{P}_{\mathrm{m},\preceq} \subset [0,1]^\mathbb{N} \times [0,1]^{\mathbb{N}^2} \times [0,1] \times [0,1]^\mathbb{N},$$

as the subset consisting of elements $\mathbf{p} = ((u_l)_{l\geq 1}, (s_{k,l})_{k,l\geq 1}, \bar{u}, (\bar{s}_k)_{k\geq 1})$ satisfying conditions (11). We wish to match exchangeable measures on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ and measures on $\mathscr{P}_{\mathrm{m},\preceq}$, and to that aim we need some further definitions. We say that an element $\pi = (\zeta, \xi) \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ has *asymptotic frequencies* if ζ and ξ have asymptotic frequencies, and we write

$$|\pi|^\downarrow = ((u_l)_{l\geq 1}, (s_{k,l})_{k,l\geq 1}, \bar{u}, (\bar{s}_k)_{k\geq 1}) \in \mathscr{P}_{\mathrm{m},\preceq}$$

for the unique – because of the ordering conditions in (11) – element satisfying:

- the block $B \in \xi$ containing $\star$ has asymptotic frequency $|B| = \bar{u}$ and the nonincreasing reordering of the asymptotic frequencies of the blocks of $\zeta \cap B$ is the sequence $(u_l, l \geq 1)$.
- for any other block $B \in \xi$ with a positive asymptotic frequency, there is a $k \in \mathbb{N}$ such that $|B| = \bar{s}_k$ and the nonincreasing reordering of the asymptotic frequencies of the blocks of $\zeta \cap B$ is the sequence $(s_{k,l}, l \geq 1)$.
- the mapping $B \mapsto k$ is injective, and for any k such that $\bar{s}_k > 0$, there is a block $B \in \xi$ such that $|B| = \bar{s}_k$.

5.4. *A paintbox construction for nested partitions*

We first adapt the construction used in our third example of Section 5.1 to our new partition space $\mathcal{P}^{2,\preceq}_{\infty,\star}$. Note that if $\mathbf{p} = ((u_l)_{l\geq 1}, (s_{k,l})_{k,l\geq 1}, \bar{u}, (\bar{s}_k)_{k\geq 1}) \in \mathscr{P}_{\mathrm{m},\preceq}$, then one can define a random element $\pi = (\zeta, \xi) \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ with a paintbox procedure very similar to the one described in the *inner dislocation* example in Section 5.1. For the sake of readability, let us recall the notation and construction:

- for $k \geq 0$, define $\bar{t}_k = \bar{u} + \sum_{k'=1}^{k} \bar{s}_{k'}$.
- for $l \geq 0$, define $t_{\star,l} = \sum_{l'=1}^{l} u_{l'}$.
- for $k \geq 1$ and $l \geq 0$, define $t_{k,l} = \bar{t}_{k-1} + \sum_{l'=1}^{l} s_{k,l'}$.
- write $\pi_0 = (\zeta_0, \xi_0)$ for the unique element of $\mathcal{P}^{2,\preceq}_{[0,1]}$ such that the non-dust blocks of ξ_0 are

  $$[0, \bar{u}) \quad \text{and} \quad [\bar{t}_k, \bar{t}_{k+1}), \quad k \geq 1,$$

  and such that the non-singleton blocks of ζ_0 are

  $$[t_{\star,l-1}, t_{\star,l}), \quad l \geq 1 \quad \text{and} \quad [t_{k,l-1}, t_{k,l}), \quad k, l \geq 1.$$

- let $(U_i, i \geq 1)$ be an i.i.d. sequence of uniform random variables on $[0, 1]$ and define the random injection $\sigma : i \in \mathbb{N} \mapsto U_i \in [0, 1]$.
- finally define the random element $\pi \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ as the unique $\pi = (\zeta, \xi)$ such that $\pi_{|\mathbb{N}} = \pi_0^{\sigma}$, and the block of ξ containing $\star$ is equal to:

  $$\{\star\} \cup \{i \geq 1, U_i < \bar{u}\}.$$

The distribution of π obtained with this construction is a probability on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ that we denote by $\bar{\varrho}_{\mathbf{p}}$. It is clear from the exchangeability of the sequence $(U_i, i \geq 1)$ that $\bar{\varrho}_{\mathbf{p}}$ is exchangeable, and from the strong law of large numbers, that $\bar{\varrho}_{\mathbf{p}}$-a.s., π possesses asymptotic frequencies equal to $|\pi|^{\downarrow} = \mathbf{p}$. For a measure ν on $\mathscr{P}_{\mathrm{m},\preceq}$, we will define a corresponding exchangeable measure $\bar{\varrho}_{\nu}$ on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ by

$$\bar{\varrho}_{\nu}(\cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}} \bar{\varrho}_{\mathbf{p}}(\cdot)\, \nu(\mathrm{d}\mathbf{p}).$$

The following lemma shows that every probability measure on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ is of this form.

Lemma 18. *Let $\pi = (\zeta, \xi)$ be a random exchangeable element of $\mathcal{P}^{2,\preceq}_{\infty,\star}$. Then π has asymptotic frequencies $|\pi|^{\downarrow} \in \mathscr{P}_{\mathrm{m},\preceq}$ a.s. and its distribution conditional on $|\pi|^{\downarrow} = \mathbf{p}$ is $\bar{\varrho}_{\mathbf{p}}$. In other words, we have*

$$\mathbb{P}(\pi \in \cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}} \mathbb{P}\big(|\pi|^{\downarrow} \in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\cdot).$$

Proof. Independently from π, let $(X_i, i \geq 1)$ and $(Y_i, i \geq 1)$ be i.i.d. uniform random variables on $[0, 1]$. Conditional on π, we define a random variable $Z_n \in [0, 1] \times ([0, 1] \cup \{\star\})$ for each $n \in \mathbb{N}$ by

$$Z_n := \begin{cases} (X_{A_n}, Y_{B_n}) & \text{if } \star \nsim^{\xi} n, \\ (X_{A_n}, \star) & \text{if } \star \sim^{\xi} n, \end{cases} \quad \text{where} \quad \begin{cases} A_n := \min\{m \in \mathbb{N}, m \sim^{\zeta} n\} \\ B_n := \min\{m \in \mathbb{N}, m \sim^{\xi} n\}. \end{cases}$$

It is straight-forward that we recover entirely π from the sequence $(Z_n, n \geq 1)$ because we have

$$\begin{aligned} n \sim^{\zeta} m \quad &\iff \quad x(Z_n) = x(Z_m), \\ n \sim^{\xi} m \quad &\iff \quad y(Z_n) = y(Z_m), \\ n \sim^{\xi} \star \quad &\iff \quad y(Z_n) = \star, \end{aligned} \tag{16}$$

where x and y denote respectively the projection maps from $[0, 1] \times ([0, 1] \cup \{\star\})$ to the first and second coordinates. Now, notice that the exchangeability of π implies that the sequence $(Z_n, n \geq 1)$ is an exchangeable sequence of random variables. Then, by an application of de Finetti's theorem, we see that there is a random probability measure P on $[0,1]\times([0,1]\cup\{\star\})$ such that conditional on P, the sequence $(Z_n, n\geq 1)$ is i.i.d. with distribution P.

Now notice that if P is a probability measure on $[0,1]\times([0,1]\cup\{\star\})$, we can define

$$|P|^{\downarrow}=\big((u_l)_{l\geq 1},(s_{k,l})_{k,l\geq 1},\bar{u},(\bar{s}_k)_{k\geq 1}\big)\in\mathscr{P}_{\mathrm{m},\preceq}$$

by setting the following, where everything is numbered in an order compatible with our conditions (11).

- $\bar{u}:=P(y=\star)$.
- $\bar{s}_k:=P(y=y_k)$, where $(y_k, k\geq 1)$ is the injective sequence of points of $[0,1]$ such that $P(y=y_k)>0$.
- $u_l:=P(x=x_{\star,l}, y=\star)$ where $(x_{\star,l}, l\geq 1)$ is the injective sequence of points of $[0,1]$ such that $P(x=x_{\star,l}, y=\star)>0$.
- $s_{k,l}:=P(x=x_{k,l}, y=y_k)$ where $(x_{k,l}, l\geq 1)$ is the injective sequence of points of $[0,1]$ such that $P(x=x_{k,l}, y=y_k)>0$.

It should now be clear that defining with (16) a random $\pi\in\mathcal{P}^{2,\preceq}_{\infty,\star}$ from a sequence $(Z_n, n\geq 1)$ of P-i.i.d. random variables is in fact the same as defining π from a paintbox construction $\bar{\varrho}_{\mathbf{p}}$ with $\mathbf{p}=|P|^{\downarrow}$. Therefore, the distribution of π is given by

$$\mathbb{P}(\pi\in\cdot)=\int_{\mathscr{P}_{\mathrm{m},\preceq}}\mathbb{P}\big(|P|^{\downarrow}\in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\cdot),$$

which concludes the proof since for any $\mathbf{p}$ we have $\bar{\varrho}_{\mathbf{p}}$-a.s. that $|\pi|^{\downarrow}$ exists and is equal to $\mathbf{p}$. □

5.5. *Erosion and dislocation for nested partitions*

As in the standard $\mathcal{P}_\infty$ case, we can decompose any exchangeable measure μ on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ satisfying some finiteness condition similar to (3) in a canonical way. To ease the notation, recall that we define for $n\in\mathbb{N}\cup\{\infty\}$, π_n the maximal element in $\mathcal{P}^{2,\preceq}_{n,\star}$

$$\pi_n:=\big(\big\{\{\star\},[n]\big\},\mathbf{1}_{[n]_\star}\big).$$

We also define two erosion measures $\mathfrak{e}^1$ and $\mathfrak{e}^2$ by

$$\mathfrak{e}^1=\sum_{i\geq 1}\delta_{(\{\{\star\},\{i\},[\infty]\setminus\{i\}\},\mathbf{1}_{[\infty]_\star})},$$

$$\mathfrak{e}^2=\sum_{i\geq 1}\delta_{(\{\{\star\},\{i\},[\infty]\setminus\{i\}\},\{\{i\},[\infty]_\star\setminus\{i\}\})}.$$

Proposition 19. *Let μ be an exchangeable measure on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ satisfying* (15), *namely*

$$\mu\big(\{\zeta_{|[\infty]}=\mathbf{1}\}\big)=0,\quad \textit{and}\quad \forall n\geq 1,\quad \mu(\pi_{|[n]_\star}\neq\pi_n)<\infty.$$

Then there are two real numbers $c_1,c_2\geq 0$ and a measure ν on $\mathscr{P}_{\mathrm{m},\preceq}$ satisfying (12), *namely*

$$\nu\big(\{u_1=1\ or\ s_{1,1}=1\}\big)=0,\quad \textit{and}\quad \int_{\mathscr{P}_{\mathrm{m},\preceq}}(1-u_1)\,\nu(\mathrm{d}\mathbf{p})<\infty$$

such that $\mu=c_1\mathfrak{e}^1+c_2\mathfrak{e}^2+\bar{\varrho}_\nu$. Conversely, any μ of this form is exchangeable and satisfies (15).

Proof. The proof follows closely that of Theorem 3.1 in [3], as our result is a straight-forward extension of it. We first define $\mu_n:=\mu(\cdot\cap\{\pi_{[n]_\star}\neq\pi_n\})$ which is a finite measure, and

$$\overleftarrow{\mu}_n:=\mu_n^{\theta_n},$$

where $\theta_n:\mathbb{N}\to\mathbb{N}$ is the n-shift defined by $\theta_n(i)=i+n$. We can check that $\overleftarrow{\mu}_n$ is an exchangeable measure on $\mathcal{P}^{2,\preceq}_{\infty,\star}$. Indeed let us take $\sigma:\mathbb{N}\to\mathbb{N}$ a permutation, and consider $\tau:\mathbb{N}\to\mathbb{N}$ the permutation defined by

$$\tau:\begin{cases} i\leq n & \longmapsto i\\ i>n & \longmapsto n+\sigma^{-1}(i-n).\end{cases}$$

We have clearly $\tau \circ \theta_n \circ \sigma = \theta_n$ and $\tau_{|[n]} = \mathrm{id}_{[n]}$, so we can use the τ-invariance of μ to conclude

$$\begin{aligned}\overleftarrow{\mu}_n\big(\pi^\sigma \in \cdot\big) &= \mu_n\big(\pi^{\theta_n\circ\sigma} \in \cdot\big) \\ &= \mu\big(\big\{\pi^{\theta_n\circ\sigma} \in \cdot\big\} \cap \{\pi_{|[n]_\star} \neq \pi_n\}\big) \\ &= \mu\big(\big\{\pi^{\tau\circ\theta_n\circ\sigma} \in \cdot\big\} \cap \big\{\big(\pi^\tau\big)_{|[n]_\star} \neq \pi_n\big\}\big) \\ &= \mu\big(\big\{\pi^{\theta_n} \in \cdot\big\} \cap \{\pi_{|[n]_\star} \neq \pi_n\}\big) \\ &= \overleftarrow{\mu}_n(\cdot),\end{aligned}$$

which proves that $\overleftarrow{\mu}_n$ is exchangeable. Since it is also finite, Lemma 18 implies that $|(\pi^{\theta_n})|^\downarrow = |\pi|^\downarrow$ exists μ-a.e. on the event $\{\mu_{|[n]_\star} \neq \pi_n\}$, and that we have

$$\overleftarrow{\mu}_n(\cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}} \mu_n\big(|\pi|^\downarrow \in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\cdot). \tag{17}$$

Now since $\bigcup_n \{\pi_{|[n]_\star} \neq \pi_n\} = \{\pi \neq \pi_\infty\}$ and $\mu(\{\pi = \pi_\infty\}) \le \mu(\{\zeta_{|[\infty]} = \mathbf{1}\}) = 0$, necessarily the existence of $|\pi|^\downarrow \in \mathscr{P}_{\mathrm{m},\preceq}$ holds μ-a.e.

For simplicity, denote by $\mathbf{1} \in \mathscr{P}_{\mathrm{m},\preceq}$ the element $((u_l)_{l\ge1}, (s_{k,l})_{k,l\ge1}, \bar{u}, (\bar{s}_k)_{k\ge1}) \in \mathscr{P}_{\mathrm{m},\preceq}$ with $\bar{u} = u_1 = 1$ (note that $\bar{\varrho}_{\mathbf{1}} = \delta_{\pi_\infty}$), and define $\varphi(\cdot) := \mu(\cdot \cap \{|\pi|^\downarrow \neq \mathbf{1}\})$. Fix $k \in \mathbb{N}$, and consider the measure $\varphi(\pi_{|[k]_\star} \in \cdot)$ on $\mathcal{P}^{2,\preceq}_{k,\star}$. Note that

$$\big\{|\pi|^\downarrow \neq \mathbf{1}\big\} = \bigcup_{n\ge1} \big\{|\pi|^\downarrow \neq \mathbf{1}, \big(\pi^{\theta_k}\big)_{|[n]_\star} \neq \pi_n\big\},$$

where the union is increasing, so one can write

$$\begin{aligned}\varphi(\pi_{|[k]_\star} \in \cdot) &= \mu\big(\{\pi_{|[k]_\star} \in \cdot\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}\big\}\big) \\ &= \lim_{n\to\infty} \mu\big(\{\pi_{|[k]_\star} \in \cdot\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}, \big(\pi^{\theta_k}\big)_{|[n]_\star} \neq \pi_n\big\}\big).\end{aligned} \tag{18}$$

Now let us use invariance of μ under the permutation $\sigma : \mathbb{N} \to \mathbb{N}$ defined by

$$\sigma : \begin{cases} i \in \{1, \dots k\} & \mapsto i + n, \\ i \in \{k+1, \dots, k+n\} & \mapsto i - k, \\ i \ge k + n + 1 & \mapsto i, \end{cases}$$

to obtain

$$\begin{aligned}&\mu\big(\{\pi_{|[k]_\star} \in \cdot\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}, \big(\pi^{\theta_k}\big)_{|[n]_\star} \neq \pi_n\big\}\big) \\ &\quad= \mu\big(\big\{\big(\pi^{\theta_n}\big)_{|[k]_\star} \in \cdot\big\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}, \pi_{|[n]_\star} \neq \pi_n\big\}\big).\end{aligned}$$

Now by definition of μ_n and $\overleftarrow{\mu}_n$, this expression is exactly

$$\mu_n\big(\big\{\big(\pi^{\theta_n}\big)_{|[k]_\star} \in \cdot\big\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}\big\}\big) = \overleftarrow{\mu}_n\big(\{\pi_{|[k]_\star} \in \cdot\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}\big\}\big).$$

Plugging this into (18) and then using (17), we obtain

$$\begin{aligned}\varphi(\pi_{|[k]_\star} \in \cdot) &= \lim_{n\to\infty} \overleftarrow{\mu}_n\big(\{\pi_{|[k]_\star} \in \cdot\} \cap \big\{|\pi|^\downarrow \neq \mathbf{1}\big\}\big) \\ &= \lim_{n\to\infty} \int_{\mathscr{P}_{\mathrm{m},\preceq}\setminus\{\mathbf{1}\}} \mu_n\big(|\pi|^\downarrow \in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\pi_{|[k]_\star} \in \cdot).\end{aligned}$$

Finally, note that the sequence of measures μ_n is increasing and converges to μ, in the sense that $\mu_n(B) \uparrow \mu(B)$ when $n \to \infty$ for any Borel set $B \subset \mathcal{P}^{2,\preceq}_{\infty,\star}$. This allows us to take the limit in the last display:

$$\varphi(\pi_{|[k]_\star} \in \cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}\setminus\{\mathbf{1}\}} \mu\big(|\pi|^\downarrow \in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\pi_{|[k]_\star} \in \cdot).$$

Since this is true for all $k \in \mathbb{N}$, we have

$$\varphi(\cdot) = \int_{\mathscr{P}_{\mathrm{m},\preceq}\setminus\{\mathbf{1}\}} \mu\big(|\pi|^{\downarrow} \in \mathrm{d}\mathbf{p}\big)\bar{\varrho}_{\mathbf{p}}(\cdot) = \bar{\varrho}_{\nu},$$

with $\nu(\cdot) = \mu(\{|\pi|^{\downarrow} \in \cdot\} \cap \{|\pi|^{\downarrow} \neq \mathbf{1}\})$. Now notice that the paintbox construction of the probability measures $\bar{\varrho}_{\mathbf{p}}$ implies that

$$\bar{\varrho}_{\nu}(\pi_{|[n]_\star} \neq \pi_n) = \int_{\mathscr{P}_{\mathrm{m},\preceq}} \nu(\mathrm{d}\mathbf{p})\bigg(1 - \sum_{l\geq 1} u_l^n\bigg),$$

and that since $u_1 \geq u_2 \geq \cdots$ and $\sum_l u_l \leq 1$, we have for $n \geq 2$,

$$1 - u_1 \leq 1 - u_1 \sum_l u_l^{n-1} \leq 1 - \sum_l u_l^n \leq 1 - u_1^n \leq n(1 - u_1).$$

Integrating this with respect to ν, we find that clearly $\bar{\varrho}_{\nu}$ satisfies the right-hand side of (15) iff ν satisfies the right-hand side of (12). For the left-hand side, notice that by construction $\nu(\{u_1 = 1 \text{ or } s_{1,1} = 1\}) = \bar{\varrho}_{\nu}(\{\zeta_{|[\infty]} = \mathbf{1}\}) = 0$.

We now write $\psi(\cdot) := \mu(\cdot \cap \{|\pi|^{\downarrow} = \mathbf{1}\})$ so that $\mu = \varphi + \psi = \bar{\varrho}_{\nu} + \psi$. Take an integer $n \in \mathbb{N}$. We know that $\overleftarrow{\psi}_n(\cdot) := \psi(\{\pi^{\theta_n} \in \cdot\} \cap \{\pi_{|[n]_\star} \neq \pi_n\})$ is a finite exchangeable measure on $\mathcal{P}^{2,\preceq}_{\infty,\star}$ such that $|\pi|^{\downarrow} = \mathbf{1}$ $\overleftarrow{\psi}_n$-a.e. Now recall that $\bar{\varrho}_{\mathbf{1}} = \delta_{\pi_\infty}$. A consequence of Lemma 18 is that $\pi = \pi_\infty$ $\overleftarrow{\psi}_n$-a.e., which in turn implies that ψ-a.e. on the event $\{\pi_{|[n]_\star} \neq \pi_n\}$, we have $\pi^{\theta_n} = \pi_\infty$. Since there is only a finite number of elements $\pi \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ such that $\pi^{\theta_n} = \pi_\infty$, we have

$$\psi\big(\cdot \cap \{\pi_{|[n]_\star} \neq \pi_n\}\big) = \sum_i a_i \delta_{\widehat{\pi}_i},$$

where the sum is finite, and for each i, we have $\widehat{\pi}_i^{\theta_n} = \pi_\infty$. Now suppose we have $\psi(\{\widehat{\pi}\}) > 0$, for a $\widehat{\pi} \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ such that $\widehat{\pi}^{\theta_n} = \pi_\infty$. Let $I(\widehat{\pi}) := \{\widehat{\pi}^{\sigma}, \sigma \text{ permutation}\}$. By the exchangeability of ψ, we have necessarily $\psi(\{\pi\}) = \psi(\{\widehat{\pi}\}) > 0$ for any $\pi \in I(\widehat{\pi})$. Since for any $m \in \mathbb{N}$ we have $\psi(\pi_{|[m]_\star} \neq \pi_m) < \infty$, we deduce

$$\#\big\{\pi \in I(\widehat{\pi}), \pi_{|[m]_\star} \neq \pi_m\big\} \leq \psi(\pi_{|[m]_\star} \neq \pi_m)/\psi\big(\{\widehat{\pi}\}\big) < \infty. \tag{19}$$

We claim that the elements $\widehat{\pi} = (\widehat{\zeta}, \widehat{\xi}) \in \mathcal{P}^{2,\preceq}_{\infty,\star}$ satisfying $\widehat{\pi}^{\theta_n} = \pi_\infty$ and (19) for any m are such that $\widehat{\zeta}$ and $\widehat{\xi}$ have no more than two blocks, and in that case one of the blocks is a singleton. Indeed if $1 \sim 2 \nsim 3 \sim 4$ for $\widehat{\xi}$ or $\widehat{\zeta}$, then the permutations $\sigma_i = (2, i+2)(4, i+4)$, written as a composition of two transpositions, are such that for $i \neq j \geq n$ and $m \geq 3$, $\widehat{\pi}^{\sigma_i} \neq \widehat{\pi}^{\sigma_j}$ and $\widehat{\pi}^{\sigma_i}_{|[m]_\star} \neq \pi_m$. So having two blocks with two or more integers contradicts (19). One can check in the same way that the situation $1 \nsim 2 \nsim 3$ is also contradictory.

Putting everything together, we necessarily have

- either $\widehat{\pi} = (\{\{\star\}, \{i\}, \mathbb{N} \setminus \{i\}\}, \mathbf{1}_{[\infty]_\star})$ for an $i \in \mathbb{N}$,
- or $\widehat{\pi} = (\{\{\star\}, \{i\}, \mathbb{N} \setminus \{i\}\}, \{\{i\}, [\infty]_\star \setminus \{i\}\})$ for an $i \in \mathbb{N}$.

We conclude using the exchangeability of ψ that there exists two real numbers $c_1, c_2 \geq 0$ such that $\psi = c_1\mathfrak{e}^1 + c_2\mathfrak{e}^2$, enabling us to write

$$\mu = \varphi + \psi = \bar{\varrho}_{\nu} + c_1\mathfrak{e}^1 + c_2\mathfrak{e}^2,$$

which concludes the proof. □

Applying this result to $\widetilde{\mu}_{\mathrm{in}}$ implies the existence of $c_{\mathrm{in},1}, c_{\mathrm{in},2} \geq 0$ and ν_{in} a measure on $\mathscr{P}_{\mathrm{m},\preceq}$ satisfying (12) such that

$$\widetilde{\mu}_{\mathrm{in}} = c_{\mathrm{in},1}\mathfrak{e}^1 + c_{\mathrm{in},2}\mathfrak{e}^2 + \bar{\varrho}_{\nu_{\mathrm{in}}}.$$

This concludes the proof of Theorem 14 because with our definitions in Section 5.1, this equality translates into

$$\mu_{\mathrm{in}} = c_{\mathrm{in},1}\mathfrak{e}^{\mathrm{in},1} + c_{\mathrm{in},2}\mathfrak{e}^{\mathrm{in},2} + \widetilde{\varrho}_{\nu_{\mathrm{in}}}.$$

Combining this with Lemma 17, we conclude

$$\mu = c_{\mathrm{out}}\mathfrak{e}^{\mathrm{out}} + c_{\mathrm{in},1}\mathfrak{e}^{\mathrm{in},1} + c_{\mathrm{in},2}\mathfrak{e}^{\mathrm{in},2} + \widehat{\varrho}_{\nu_{\mathrm{out}}} + \widetilde{\varrho}_{\nu_{\mathrm{in}}}.$$

6. Application to binary branching

Consider a nested fragmentation process $(\Pi(t), t \geq 0) = (\zeta(t), \xi(t), t \geq 0)$ with only binary branching. The representation given by Theorem 14 then becomes quite simpler, because the dislocation measures ν_{out} and ν_{in} necessarily satisfy

$$s_1 = 1 - s_2 \quad \nu_{\text{out}}\text{-a.e.}$$

and

$$\begin{cases} & u_1 = 1 - u_2 \\ \text{or} & s_{1,1} = 1 - s_{1,2} \qquad \nu_{\text{in}}\text{-a.e.,} \\ \text{or} & u_1 = 1 - s_{1,1} \end{cases}$$

i.e. their support is the set of mass partitions with only two nonzero terms, and no dust. See Figure 5 for an example of a nested discrete tree illustrating the three possible dislocation events corresponding to ν_{in}.

Therefore, we can decompose ν_{out} and ν_{in} into four measures on [0, 1] defined by

$$\bar{\nu}_{\text{out}}(\cdot) := \nu_{\text{out}}(s_1 \in \cdot) + \nu_{\text{out}}(1 - s_1 \in \cdot)$$
$$\bar{\nu}_{\text{in},1}(\cdot) := \mathbb{1}\{u_1 = 1 - u_2\}\big(\nu_{\text{in}}(u_1 \in \cdot) + \nu_{\text{in}}(1 - u_1 \in \cdot)\big)$$
$$\bar{\nu}_{\text{in},2}(\cdot) := \mathbb{1}\{s_{1,1} = 1 - s_{1,2}\}\big(\nu_{\text{in}}(s_{1,1} \in \cdot) + \nu_{\text{in}}(1 - s_{1,1} \in \cdot)\big)$$
$$\bar{\nu}_{\text{in},3}(\cdot) := \mathbb{1}\{u_1 = 1 - s_{1,1}\}\nu_{\text{in}}(u_1 \in \cdot).$$

Thus defined, and because of the σ-finiteness conditions (4) and (12), those measures satisfy the following

$$\bar{\nu}_{\text{out}}, \bar{\nu}_{\text{in},1} \text{ and } \bar{\nu}_{\text{in},2} \text{ are } (x \mapsto 1 - x)\text{-invariant} \tag{20}$$

$$\int_{[0,1]} \nu(\mathrm{d}x)x(1-x) < \infty, \quad \text{for } \nu \in \{\bar{\nu}_{\text{out}}, \bar{\nu}_{\text{in},1}\} \tag{21}$$

$$\bar{\nu}_{\text{in},2}\big([0,1]\big) < \infty \tag{22}$$

$$\int_{[0,1]} \bar{\nu}_{\text{in},3}(\mathrm{d}x)(1-x) < \infty. \tag{23}$$

For the sake of completeness, let us use those measures to express the transition rates $q^n_{\pi,\pi'}$ of the Markov chain $\Pi^n := (\Pi(t)_{|[n]})$ from one nested partition $\pi = (\zeta, \xi) \in \mathcal{P}_n^{2,\preceq}$ to another $\pi' = (\zeta', \xi') \in \mathcal{P}_n^{2,\preceq} \setminus \{\pi\}$ in the following way:

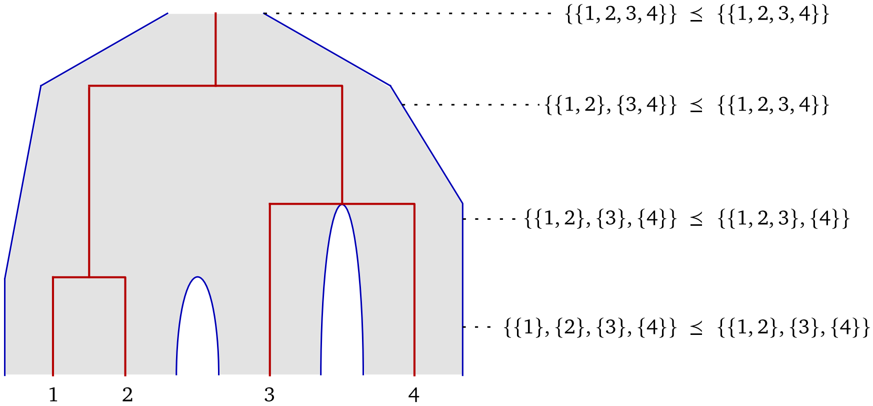

Fig. 5. Binary nested tree exhibiting the three different inner dislocation events. Time flows from top to bottom, and the right-hand side of the picture shows the sequence of nested partitions picked at chosen times between events, in the form $\pi = (\zeta \preceq \xi)$. The first event corresponds to the case $u_1 = 1 - u_2$, where the inner block {1, 2, 3, 4} splits into two blocks {1, 2} and {3, 4} and the outer block remains unchanged. The second dislocation is of the type $u_1 = 1 - s_{1,1}$, that is the block {3, 4} splits into two distinct blocks, one of which (the singleton {3}) stays in the *mother* outer block. The other new inner block {4} forms a new outer block identical to itself. The last and third dislocation is of the type $s_{1,1} = 1 - s_{1,2}$, meaning that {1, 2} splits into {1} and {2}, these two blocks together forming a new outer block, distinct from the mother block – i.e. the one containing {3}.

- If π' cannot be obtained from a binary fragmentation of π, then $q^n_{\pi,\pi'} = 0$.
- If π' can be obtained from a binary fragmentation of π, with $B \in \zeta$ and $C \in \xi$ two blocks of π participating in the fragmentation, but such that $B \not\subset C$, then $q^n_{\pi,\pi'} = 0$.
- Otherwise, let us write $B \subset C$, with $B \in \zeta$ and $C \in \xi$ for (the) two blocks of π participating in the fragmentation, and $B_1, B_2 \in \zeta'$, $C_1, C_2 \in \xi'$ the resulting blocks, chosen in a way that $B_1 \subset C_1$. Note that B or C might not fragment, in which case we let B_2 or C_2 be the empty set $\varnothing$. Now define $X_1 := \#B_1$ and $X_2 := \#B_2$ the cardinality of the resulting blocks of ζ'. Also, we define $Y_1 := \#\zeta'_{|C_1}$ the number of inner blocks in C_1 in the resulting partition π', and similarly $Y_2 := \#\zeta'_{|C_2}$.

With those definitions, the transition rates for the Markov chain Π^n can be written

$$\begin{aligned} q^n_{\pi,\pi'} = {} & c_{\text{out}}\big(\mathbb{1}\{\zeta' = \zeta, Y_1 = 1\} + \mathbb{1}\{\zeta' = \zeta, Y_2 = 1\}\big) \\ & + c_{\text{in},1}\big(\mathbb{1}\{\xi' = \xi, X_1 = 1\} + \mathbb{1}\{\xi' = \xi, X_2 = 1\}\big) \\ & + c_{\text{in},2}\big(\mathbb{1}\{X_1 = Y_1 = 1\} + \mathbb{1}\{B_2 = C_2 \text{ and } X_2 = Y_2 = 1\}\big) \\ & + \mathbb{1}\{\zeta' = \zeta\} \int_{[0,1]} \bar{\nu}_{\text{out}}(dx) x^{Y_1}(1-x)^{Y_2} \\ & + \mathbb{1}\{\xi' = \xi\} \int_{[0,1]} \bar{\nu}_{\text{in},1}(dx) x^{X_1}(1-x)^{X_2} \\ & + \mathbb{1}\{B_1 \cup B_2 = C_1\} \int_{[0,1]} \bar{\nu}_{\text{in},2}(dx) x^{X_1}(1-x)^{X_2} \\ & + \mathbb{1}\{\zeta' = \zeta\} \int_{[0,1]} \bar{\nu}_{\text{in},3}(dx)\big((1-x)^{\#C_1}\mathbb{1}\{Y_1 = 1\} \\ & + (1-x)^{\#C_2}\mathbb{1}\{Y_2 = 1\}\big) \\ & + \mathbb{1}\{\zeta' \neq \zeta\} \int_{[0,1]} \bar{\nu}_{\text{in},3}(dx)\big(x^{X_2}(1-x)^{X_1}\mathbb{1}\{Y_1 = 1\} \\ & + x^{X_1}(1-x)^{X_2}\mathbb{1}\{Y_2 = 1\}\big). \end{aligned} \tag{24}$$

Note that several indicator functions in the last display may be equal to 1 for the same pair (π, π'). This explicit formula allows for computer simulations of binary nested fragmentations, although to that aim it might be simpler to adapt the Poissonian construction (Section 4.3) and use nested partitions of arrays $[n]^2$. Also, one could exactly compute the probability of a given nested tree under different nested fragmentation models, which would be a first step towards statistical inference.

Acknowledgements

I thank the *Center for Interdisciplinary Research in Biology* (Collège de France) for funding, and I am grateful to my supervisor Amaury Lambert for his careful reading and many helpful comments on this project.

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