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Asymptotics of free fermions in a quadratic well at finite temperature and the Moshe–Neuberger–Shapiro random matrix model

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Abstract. We derive the local statistics of the canonical ensemble of free fermions in a quadratic potential well at finite temperature, as the particle number approaches infinity. This free fermion model is equivalent to a random matrix model proposed by Moshe, Neuberger and Shapiro. Limiting behaviors obtained before for the grand canonical ensemble are observed in the canonical ensemble: We have at the edge the phase transition from the Tracy–Widom distribution to the Gumbel distribution via the Kardar–Parisi–Zhang (KPZ) crossover distribution, and in the bulk the phase transition from the sine point process to the Poisson point process.

Résumé. Nous décrivons les statistiques locales de l'ensemble canonique de fermions libres dans un puits de potentiel quadratique à température finie, dans la limite où le nombre de particules tend vers l'infini. Ce modèle de fermions libres est équivalent à un modèle matriciel aléatoire proposé par Moshe, Neuberger et Shapiro. Les comportements à la limite précédemment obtenus pour l'ensemble grand-canonique sont observés dans l'ensemble canonique: Nous avons, au bord de l'ensemble, une transition de phase de la distribution de Tracy–Widom à la distribution de Gumbel via la distribution croisée de Kardar–Parisi–Zhang (KPZ), et dans l'ensemble, une transition de phase du processus ponctuel sinus au processus ponctuel de Poisson.

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1. Introduction

In this paper we consider the spinless free fermions on $\mathbb{R}^1$ in quadratic potential well (aka harmonic oscillators) at finite temperature. This model was defined by Moshe, Neuberger and Shapiro [24] in the 1990's, further studied by Johansson [16] in the 2000's, and very recently considered in the physics literature by Dean, Le Doussal, Majumdar, Schehr et al [10,11,19]. See also [20] for a dynamical version of the model, and [12] for a generalization to other symmetry types.

The most interesting question on this model (later called the MNS model) is the limiting behavior of the fermions at the edge or in the bulk as the number of particles $n \to \infty$. From the physical point of view, the existing result is already rather complete. When the temperature is low enough, the limiting distribution of the rightmost particle is given by the celebrated Tracy–Widom distribution, and when the temperature is high enough, the limiting distribution is given by the Gumbel distribution. At the critical temperature, the limiting distribution is found to be the crossover distribution in the 1-dimensional Kardar–Parisi–Zhang (KPZ) universality class. For particles in the bulk, analogous results are obtained which interpolate between the sine point process and the Poisson point process.

The original version proposed by Moshe, Neuberger and Shapiro is the *canonical ensemble* of the model, but all the asymptotic results available currently in the mathematical literature are for the *grand canonical ensemble* of the model. It is a universally accepted wisdom in statistical physics that the physical properties of the grand canonical ensemble

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are the same as those of the canonical ensemble as the particle number approaches infinity. In the case of the MNS model, the grand canonical ensemble has a special mathematical feature that it is a *determinantal point process*, which makes it easier to analyze mathematically than the canonical ensemble. Currently all results on the MNS model in the mathematics literature deal with the grand canonical ensemble, although several recent works in the physical literature [10,11,19] have considered the canonical ensemble. The goal of this paper is to analyze the canonical ensemble of the MNS model directly, and rigorously prove that the limiting results obtained for the grand canonical ensemble hold for the canonical ensemble as well.

Our purpose is not rigor for rigor's sake. As suggested by the title, the canonical ensemble of the MNS model is associated to a random matrix model (later referred as the MNS random matrix model) whose dimension is equal to the number of particles in the MNS model. Such a relation is not preserved when we move to the grand canonical ensemble. Also the limiting edge distribution of the MNS model occurs in integrable particle systems like the Asymmetric Simple Exclusion Process (ASEP) and the q-Whittaker processes, which are a subclass of the extensively studied Macdonald processes [7], and contain many interacting particle models in the KPZ universality class as specializations. Although the ASEP and the q-Whittaker processes are integrable, they are considerably more difficult than determinantal processes. The similarity between probability models in the KPZ universality class and free fermions at positive temperature has been noticed in [14], but the relation is via determinantal process. We hope that our analysis of the canonical ensemble of the MNS model sheds light on the study of the integrable particle models in the KPZ universality class.

1.1. *q-analogue notation*

Throughout this paper, we use the following q-analogue notations, which converge to their common counterparts as $q \to 1_-$.

The q-Pochhammer symbol is

$$(a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad n = 0, 1, 2, \ldots, \infty. \tag{1}$$

The q-binomial is

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}, \quad 0 \le m \le n. \tag{2}$$

1.2. *Definition of the MNS model*

First recall the one-dimension harmonic oscillator in quantum mechanics. The time-independent Hamiltonian of the free particle in a quadratic potential well is, upon choosing proper units,

$$H = -\frac{\partial^2}{\partial x^2} + \frac{x^2}{4}. \tag{3}$$

The eigenfunctions of the Hamiltonian H defined in (3) are

$$\varphi_k(x) = \left(\frac{1}{\sqrt{2\pi}k!}\right)^{1/2} H_k(x) e^{-x^2/4}, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $H_k(x)$ is the Hermite polynomial, defined to be the monic polynomial of degree k satisfying the orthogonality

$$\int_{-\infty}^{\infty} H_k(x) H_j(x) e^{-x^2/2}\, dx = \sqrt{2\pi}k!\delta_{kj}. \tag{5}$$

The functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ form an orthonormal basis for $L^2(\mathbb{R})$. See [1, Chapter 22] for basic properties of Hermite polynomials. Note that in [1], the polynomial $H_n(x)$ is denoted as $He_n(x)$, while the notation $H_n(x)$ is reserved for a slightly different polynomial, see [1, 22.5.18]. The eigenvalue/energy level for eigenstate $\varphi_k(x)$ is $k + 1/2$ ($k = 0, 1, 2, \ldots$), since

$$H\varphi_k(x) = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right)\varphi_k(x) = \left(k + \frac{1}{2}\right)\varphi_k(x). \tag{6}$$

Suppose n identical fermions are independent harmonic oscillators, or in other words they are free fermions in the quadratic potential well. The fermionic system has eigenstates indexed by $(k_1, k_2, \dots, k_n)$ where $0 \le k_1 < k_2 < \cdots < k_n$ are integers, and the energy level of the eigenstate is $k_1 + k_2 + \cdots + k_n + n/2$. The corresponding eigenfunction is given by the Slater determinant

$$\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_n}(x_1) & \dots & \varphi_{k_n}(x_n) \end{vmatrix}. \tag{7}$$

In this eigenstate, the density function for the n particles is $|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)|^2$.

For a quantum system at temperature T, all eigenstates occur at a certain chance according to the *Boltzmann distribution*, so that the probability for an eigenstate with energy level E to occur is $Z^{-1}e^{-E/(\kappa T)}$ where Z is the normalization constant and κ is the Boltzmann constant [27, Section 6.2], which we assume to be 1 later. Hence for the n-particle canonical ensemble of the MNS model, that is, n free fermions in the quadratic potential well, if the temperature is $T > 0$, and if we denote

$$q = e^{-1/(\kappa T)} = e^{-1/T}, \tag{8}$$

the probability for eigenstate $(k_1, k_2, \dots, k_n)$ to occur is $Z_n(q)^{-1} q^{k_1+\cdots+k_n+n/2}$, where

$$Z_n(q) = \sum_{0 \le k_1 < k_2 < \cdots < k_n} q^{k_1+\cdots+k_n+n/2} = \frac{q^{n^2/2}}{(q;q)_n}. \tag{9}$$

We then have that the density function for the n particles is

$$\begin{aligned} P_n(x_1, \dots, x_n) &= \frac{1}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \left|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)\right|^2 q^{k_1+\cdots+k_n+n/2} \\ &= \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \left|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)\right|^2 q^{k_1+\cdots+k_n}. \end{aligned} \tag{10}$$

The equivalence of the two expressions in (9) may not be obvious, but it is easily proven by induction on n.

The n-particle canonical ensemble of the MNS model at temperature $T = -(\log q)^{-1} > 0$, which is called simply the MNS model if there is no possibility of confusion, is the main topic of this paper. Although it is defined in the language of quantum mechanics, all our analysis is based on the density function (10), so it is harmless to understand the MNS model as a particle model with density (10). We note that in the limit $T \to 0$, the density function $P_n(x_1, \dots, x_n)$ degenerates into $|\Phi_{0,1,\dots,n-1}(x_1, \dots, x_n)|^2$, the density function for the ground state of the quantum system. One readily recognizes that this $T \to 0$ limiting density is the density of eigenvalues of a random matrix in the Gaussian Unitary Ensemble (GUE) [3, Section 2.5], that is, the random Hermitian matrix model defined below in (13). It is then not a surprise that for general $T > 0$, density (10) is also the eigenvalue density function of a random matrix ensemble.

1.3. *MNS random matrix model*

The random matrix model defined by Moshe, Neuberger and Shapiro [24] is an unitarily invariant generalization of the GUE with a continuous parameter. As the parameter varies, the limiting local statistics of the MNS random matrix model interpolate between the sine point process, which is the hallmark of random Hermitian matrices including the GUE, and the Poisson point process.

The space of n-dimensional Hermitian matrices has a natural measure

$$dX = \prod_{i=1}^{n} dx_{ii} \prod_{1 \le j < k \le n} d\Re x_{jk}\, d\Im x_{jk}, \tag{11}$$

where $X = (x_{jk})_{j,k=1}^n$. Let U be a random unitary matrix in $\mathrm{U}(n)$ with respect to the Haar measure. We say that a random Hermitian matrix H is an MNS random matrix if [24, Formulas (1) and (2)] ($[U, H] = UH - HU$ is the commutator)

$$P(H)\,dH = \frac{1}{C(n,b)} e^{-\operatorname{Tr} H^2} \left[\int_{\mathrm{U}(n)} dU e^{-b \operatorname{Tr}([U,H][U,H]^\dagger)} \right] dH$$
$$= \frac{1}{C(n,b)} e^{-(2b+1)\operatorname{Tr} H^2} \left[\int_{\mathrm{U}(n)} dU e^{2b \operatorname{Tr}(UHU^\dagger H)} \right] dH. \tag{12}$$

By comparing the eigenvalue distribution of H and the known density function of free fermions in a quadratic potential well at finite temperature, Moshe, Neuberger and Shapiro observe the following relation.

Proposition 1 ([24, Formula (4)]). *Suppose the n-dimensional Hermitian random matrix is defined by* (12), *and suppose the parameter $b = q/(1-q)^2$ with $q \in (0,1)$. Then the joint probability density function of the eigenvalues of $\sqrt{\frac{1}{2}(1-q)/(1+q)}H$ is the same as the density function $P_n(x_1, \ldots, x_n)$ defined in* (10).

If we denote the $q \to 0$ limit of $2^{-1/2}H$ by X, then X has the density function

$$P(X)\,dX = \frac{1}{2^{n/2}\pi^{n^2/2}} \exp\left(-\frac{1}{2}\operatorname{Tr}(X^2)\right) dX, \tag{13}$$

or equivalently, $X_{ii} = N(0, 1/2)$, $\Re X_{jk} = N(0,1)$, $\Im X_{jk} = N(0,1)$ for $1 \le i \le n$ and $1 \le j < k \le n$, and they are independent. This is the celebrated GUE ensemble in dimension n [3, Section 2.5].

1.4. *Statement of results*

As the particle number $n \to \infty$, we are interested in the limiting distribution of the rightmost particle in the MNS model. The distribution of the position of the rightmost particle,

$$\mathbb{P}_n\big(\max(x_1, \ldots, x_n) \le s\big) = \mathbb{P}_n\big(x_1, \ldots, x_n \in (-\infty, s]\big), \tag{14}$$

is a special case of the *gap probability*, which is the probability $\mathbb{P}_n(x_1, \ldots, x_n \notin A)$ for a measurable set $A \subseteq \mathbb{R}$.

We are also interested in the limiting local statistics of particles in the bulk. The gap probability is not an efficient way to describe the local statistics in the bulk, and we instead compute the limiting *m-correlation functions*, which are defined as

$$R_n^{(m)}(x_1, \ldots, x_m)$$
$$= \lim_{\Delta \to 0} \frac{1}{(\Delta)^m} \mathbb{P}_n\big(\text{there is at least one particle in each } [x_i, x_i + \Delta), i = 1, 2, \ldots, m\big), \tag{15}$$

or equivalently as

$$R_n^{(m)}(x_1, \ldots, x_m) = \frac{n!}{(n-m)!} \int_{\mathbb{R}} dx_n \int_{\mathbb{R}} dx_{n-1} \ldots \int_{\mathbb{R}} dx_{n-m+1} P_n(x_1, \ldots, x_n), \tag{16}$$

where $P_n(x_1, \ldots, x_n)$ is the joint density of particles. Since the eigenvalue distribution of the MNS random matrix model is also given in (10), the gap probability (14) and the m-correlation functions (1.4) are the same for the eigenvalues of the MNS random matrix model.

For the MNS (random matrix) model, the gap probability and m-correlation functions can be explicitly computed by a contour integral.

Theorem 1. *Given the joint distribution $P_n(x_1, \ldots, x_n)$ in* (10) *for n particles, we have the following*:

(a) *The gap probability is*

$$\mathbb{P}_n(x_1, \ldots, x_n \in A) = \frac{1}{2\pi i} \oint_0 F(z) \det\big(I - \mathbf{K}(z;q)\chi_{A^c}\big) \frac{dz}{z}, \tag{17}$$

where

$$F(z) = q^{-n(n-1)/2}(q;q)_n \frac{(-z;q)_\infty}{z^n}, \tag{18}$$

and $\mathbf{K}(z;q)$ *is the integral operator on* $L^2(\mathbb{R})$, *defined by*

$$\mathbf{K}(z;q)(f)(x) = \int_{\mathbb{R}} K(x,y;z;q) f(y)\,dy, \qquad K(x,y;z;q) = \sum_{k=0}^{\infty} \frac{q^k z}{1+q^k z} \varphi_k(x)\varphi_k(y). \tag{19}$$

(b) *The* m*-correlation function is*

$$R_n^{(m)}(x_1,\dots,x_m) = \frac{1}{2\pi i}\oint_0 F(z) \det\big(K(x_i,x_j;z;q)\big)_{i,j=1}^m \frac{dz}{z}, \tag{20}$$

and $K(x_i,x_j;z;q)$ *is defined in* (19).

We note here that a formula equivalent to (20) has appeared recently in the physical literature [11, equation (86)]. We also remark that the kernel (19) with $z=\lambda>0$ is exactly the one which appears in the grand canonical version of the MNS model [16]. This is not at all surprising, since the grand canonical ensemble is the superposition of canonical ensembles. Indeed, using the concept of superposition, it is straightforward to prove Theorem 1 using the known determinantal formulas in the grand canonical ensemble. In Section 2 below, we present a different proof of Theorem 1(a) which does not rely on known results for the grand canonical ensemble. Our reason for presenting this longer proof is two-fold. Firstly, it makes the results of the current paper self-contained (independent of the grand canonical ensemble); and secondly, the identity of operators proved here may be applied to integrable particle models, see the remark in Section 1.5.

In the theory of point processes, gap probabilities and correlation functions are intimately connected, and it is a standard result that knowledge of one implies knowledge of the other. Thus Theorem 1(a) implies 1(b) (and vice-versa). We prove Theorem 1(a) in detail in Section 2.1. The general argument to derive the correlation functions from the gap probabilities is a rather straightforward application of (1.4) together with the inclusion/exclusion principle, and we present a short proof of Theorem 1(b) in Section 2.2 in the case $m=2$.

For the rightmost particle in the MNS model, or equivalently, the largest eigenvalue in the MNS random matrix model, we state the limiting distribution in two regimes. If the parameter q is in a compact subset of $(0,1)$, the limiting distribution is the celebrated Tracy–Widom distribution, whose probability distribution function is defined by the Fredholm determinant of $\mathbf{K}_{\mathrm{Airy}}$, an operator on $L^2(\mathbb{R})$ with kernel $K_{\mathrm{Airy}}(x,y)$:

$$F_{\mathrm{GUE}}(t) = \det(I - \mathbf{P}_t \mathbf{K}_{\mathrm{Airy}} \mathbf{P}_t), \quad \text{and} \quad K_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+r)\,\mathrm{Ai}(y+r)\,dr, \tag{21}$$

where $\mathbf{P}_t$ is the projection operator defined such that $\mathbf{P}_t f(x) = f(x)\chi_{(t,\infty)}(x)$.

If the parameter q is scaled to be close to 1, such that $1-q = \mathcal{O}(n^{-1/3})$ as $n\to\infty$, the limiting distribution is the so-called crossover distribution that occurs in the weak asymmetric limit of models in the Kardar–Parisi–Zhang (KPZ) universality class [2,9,28], and interpolates the Tracy–Widom distribution and the Gumbel distribution [16]. Its probability distribution function is defined by the Fredholm determinant of $\mathbf{K}_{\mathrm{cross}}(c)$, an integral operator on $L^2(\mathbb{R})$ depending on a continuous parameter $c\in\mathbb{R}$, whose kernel is $K_{\mathrm{cross}}(x,y;c)$ given below:

$$F_{\mathrm{cross}}(t;c) = \det\big(I - \mathbf{P}_t \mathbf{K}_{\mathrm{cross}}(c) \mathbf{P}_t\big), \quad \text{and} \quad K_{\mathrm{cross}}(x,y;c) = \int_{-\infty}^{\infty} \frac{e^{-cr}}{1+e^{-cr}} \mathrm{Ai}(x-r)\,\mathrm{Ai}(y-r)\,dr. \tag{22}$$

It is clear that as the parameter $c\to-\infty$, $F_{\mathrm{cross}}(t;c)\to F_{\mathrm{GUE}}(t)$. Our $K_{\mathrm{cross}}(x,y;c)$ is the correlation kernel of the "interpolating process" in [16].

Theorem 2. *Suppose as* $n\to\infty$, s *depends on* n *as*

$$s \equiv s_n = 2\sqrt{n} + tn^{-1/6}. \tag{23}$$

Then we have the following.

(a) *Suppose* $q\in(0,1)$ *is independent of* n,

$$\lim_{n\to\infty} \mathbb{P}_n\big(\max(x_1,\dots,x_n) \le s_n\big) = F_{\mathrm{GUE}}(t). \tag{24}$$

(b) *Suppose* $q = \exp(-cn^{-1/3})$ *depending on* n, *where* $c > 0$ *is a constant*,

$$\lim_{n\to\infty} \mathbb{P}_n\big(\max(x_1, \dots, x_n) \le s_n\big) = F_{\mathrm{cross}}(t; c). \tag{25}$$

For the particles/eigenvalues in the bulk, we also consider their limiting behavior in two regimes. If the parameter q is in a compact subset of $(0, 1)$, the positions of particles in an $\mathcal{O}(n^{-1/2})$ window converge to the sine point process [3, Sections 3.5 and 4.2], with the m-correlation functions defined by the correlation kernel

$$R^{(m)}_{\sin}(x_1, \dots, x_m) = \det\big(K_{\sin}(x_i, x_j)\big)^m_{i,j=1}, \quad \text{where } K_{\sin}(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}. \tag{26}$$

If the parameter is scaled to be close to 1, such that $1 - q = \mathcal{O}(n^{-1})$, the positions of particles in an $\mathcal{O}(n^{-1})$ window converge to a determinantal point process that interpolates the sine process and the Poisson process. The m-correlation functions of this process are defined by the correlation kernel

$$R^{(m)}_{\mathrm{inter}}(x_1, \dots, x_m; a) = \det\big(K_{\mathrm{inter}}(x_i, x_j; a)\big)^m_{i,j=1}, \quad \text{where } K_{\mathrm{inter}}(x, y; a) = \int_0^\infty \frac{\cos(\pi(x-y)t)}{ae^{t^2}+1}\, dt, \tag{27}$$

which depends on a continuous parameter $a > 0$. We note that as $a \to 0_+$, if $x = \xi/\sqrt{-\log a}$ and $y = \eta/\sqrt{-\log a}$, then

$$\lim_{a\to 0_+} K_{\mathrm{inter}}\left(\frac{\xi}{\sqrt{-\log a}}, \frac{\eta}{\sqrt{-\log a}}; a\right) dy = K_{\sin}(\xi, \eta)\, d\eta, \quad \text{for } \xi, \eta \text{ in a compact subset of } \mathbb{R}. \tag{28}$$

Our correlation kernel K_{inter} is the same as the kernel L_c in [16, Theorem 1.9] up to a change of scaling.

Theorem 3.

(a) *Suppose* $n \to \infty$, $q \in (0, 1)$ *is independent of* n, *and* $x_1, \dots, x_m$ *depend on* n *as*

$$x_i = 2x\sqrt{n} + \frac{\pi\xi_i}{(1-x^2)^{1/2}\sqrt{n}}, \quad i = 1, \dots, m, \tag{29}$$

where ξ_i *are constants and* $x \in (-1, 1)$. *Then*

$$\lim_{n\to\infty} \left(\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}\right)^m R^{(m)}_n(x_1, \dots, x_m) = R^{(m)}_{\sin}(\xi_1, \dots, \xi_m). \tag{30}$$

(b) *Suppose* $n \to \infty$, $q = e^{-c/n}$, *and* $x_1, \dots, x_m$ *depend on* n *as*

$$x_i = 2x\sqrt{n} + \frac{\pi\xi_i}{\sqrt{n/c}}, \quad i = 1, \dots, m, \tag{31}$$

where ξ_i *are constants and* $x \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \left(\frac{\pi}{\sqrt{n/c}}\right)^m R^{(m)}_n(x_1, \dots, x_m) = R^{(m)}_{\mathrm{inter}}\left(\xi_1, \dots, \xi_m; \frac{e^{cx^2}}{e^c - 1}\right). \tag{32}$$

Remark 1.

(i) As $q \to 0$, the MNS random matrix model (12) converges to the GUE (13). The Tracy–Widom limit at the edge and the sine limit in the bulk for GUE is a well known result in random matrix theory [3, Chapter 3].

(ii) Our limiting results for the canonical ensemble of the MNS model agree with those obtained in recent physical works [10,11], as well as results for the grand canonical ensemble [16]. Although the canonical ensemble is not a determinantal point process, as $n \to \infty$ its scaling limits at the edge and in the bulk are both determinantal point processes.

(iii) Since the MNS model can be interpreted as a random matrix model, we would like to expect some universality result in the local statistics. However, in the regime $1 - q = \mathcal{O}(n^{-1})$, Theorem 3(b) shows that the limiting local correlation functions depend on x, the limiting position of the particles. This is different from most other random matrix models, and is a feature which was not observed in earlier studies of the grand canonical ensemble [16], although the kernel $K_{\mathrm{inter}}(x_i, x_j; \frac{e^{cx^2}}{e^c-1})$ is a specialization of the one obtained recently in [11, equation (274)] for free fermions in d dimensions with general potentials.

We note that the 1-correlation function yields the empirical probability density function $\rho_n(x)$, since

$$\rho_n(x) = \frac{1}{n} R_n^{(1)}(x). \tag{33}$$

From (30) we obtain that if q is fixed, then the limiting empirical probability density function is

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n(2\sqrt{n}x) = \frac{2}{\pi}\sqrt{1-x^2}, \quad x \in (-1, 1). \tag{34}$$

Here we use the simple property that $K_{\sin}(x, x) = 1$. This shows that the limiting empirical probability density of the eigenvalues is the semicircle law, the same as that of the GUE random matrix. On the other hand,

$$K_{\text{inter}}(x, x; a) = \frac{-\sqrt{\pi}}{2} \operatorname{Li}_{1/2}\left(-a^{-1}\right), \tag{35}$$

where $\operatorname{Li}_{1/2}$ is the polylogarithm [26, 25.12.11]. Hence if $q = e^{-c/n}$,

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n(2\sqrt{n}x) = \frac{-1}{\sqrt{\pi c}} \operatorname{Li}_{1/2}\left(e^{-cx^2} - e^{c(1-x^2)}\right). \tag{36}$$

This limiting distribution on the right-hand side of (36) is supported on $\mathbb{R}$, but as $c \to +\infty$, it converges to the semicircle law on the right-hand side of (34) which is supported on $[-1, 1]$. The limiting empirical probability density function (36) agrees with [10, Formula (8)]. The asymptotics of $\operatorname{Li}_{1/2}$ can be found in [35].

1.5. *Generalizations and related models*

The MNS model has a dynamical generalization, which is a time-periodic nonintersecting Ornstein-Uhlenbeck process, proposed by Johansson [16, Section 1.2] and then studied by [20]. Our method can be applied to find the multi-time correlation functions for this process, see the arXiv version of our paper [23].

Here we want to remark the formal similarity between the contour integral formulas in Theorem 1 and the integral formula for correlation functions in the nonintersecting Brownian motions on a circle [22], in which particles have a space period, instead of a time period.

A variation of the identity of operators in Lemma 1 below can be used to convert the "Cauchy-type" and "Mellin-Barnes type" formulas in integrable particle models like ASEP [32,33], q-Whittaker process [7], q-Totally Asymmetric Simple Exclusion Process (q-TASEP) [6,8,13,15], q-Totally Asymmetric Zero Range Process (q-TAZRP) [18,21,29,34]. See the arXiv version of this paper [23] for detail.

2. Proof of Theorem 1

Here we present a proof of Theorem 1(a) which is independent of known results for the grand canonical ensemble. Then Theorem 1(b) follows from the general theory of point processes, and we present a short proof in the case $m = 2$. The extension to general m is straightforward.

2.1. *Gap probability*

Let $A \subseteq \mathbb{R}$ be a measurable set. We consider the probability that all the n particles are in A, which we denote by $\mathbb{P}_n(x_1, \dots, x_n \in A)$. We have

$$\begin{aligned} \mathbb{P}_n(x_1, \dots, x_n \in A) &= \int_A \cdots \int_A P_n(x_1, \dots, x_n)\, dx_1 \cdots dx_n \\ &= \frac{q^{n/2}}{Z_n(q)} \int_A \cdots \int_A \sum_{0 \le k_1 < k_2 < \cdots < k_n} \left|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)\right|^2 q^{k_1+\cdots+k_n}\, dx_1 \cdots dx_n \\ &= \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \int_A \cdots \int_A \left|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)\right|^2 q^{k_1+\cdots+k_n}\, dx_1 \cdots dx_n. \end{aligned} \tag{37}$$

Note that by the Andréif formula [4],

$$\begin{aligned}
&\int_A \cdots \int_A \begin{vmatrix} \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \end{vmatrix}^2 q^{k_1+\cdots+k_n} \, dx_1 \cdots dx_n \\
&\quad = n! q^{k_1+\cdots+k_n} \det\left(\langle \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A\right)_{i,j=1}^n \\
&\quad = n! \det\left(\langle q^{k_i} \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A\right)_{i,j=1}^n, \qquad (38)
\end{aligned}$$

where

$$\langle f(x), g(x) \rangle_A = \int_A f(x) g(x) \, dx. \qquad (39)$$

Hence

$$\mathbb{P}_n(x_1, \dots, x_n \in A) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \det\left(\langle q^{k_i} \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A\right)_{i,j=1}^n. \qquad (40)$$

Recall the integral operator $\mathbf{K}(z; q)$ defined in (20). We now introduce another integral operator $\mathbf{M}(q)$ acting on $L^2(\mathbb{R})$, depending on the parameter $q \in (0, 1)$. It is defined by

$$\mathbf{M}(q)(f)(x) = \int_{\mathbb{R}} M(x, y; q) f(y) \, dy, \qquad M(x, y; q) = \sum_{k=0}^{\infty} q^k \varphi_k(x) \varphi_k(y). \qquad (41)$$

Let $A \subseteq \mathbb{R}$ be a measurable set, and let χ_A be the projection onto $L^2(A)$. It is straightforward to check by definition that $\mathbf{M}(q)$ and $\mathbf{K}(z; q)$ are trace class operators for $0 < q < 1$, and then $\mathbf{M}(q)\chi_A$ and $\mathbf{K}(z; q)\chi_{A^c}$ are also trace class operators [30]. Hence the Fredholm determinants $\det(I + z\mathbf{M}(q)\chi_A)$ and $\det(I - \mathbf{K}(z; q)\chi_{A_c})$ are well defined. We have the following relation between $\mathbf{M}(q)$ and $\mathbf{K}(z; q)$.

Lemma 1. *Let $q \in (0, 1)$. For any $z \in \mathbb{C}$, and for any measurable $A \subseteq \mathbb{R}$, the following identity holds*:

$$\left(I + z\mathbf{M}(q)\chi_A\right) = \left(I + z\mathbf{M}(q)\right)\left(I - \mathbf{K}(z; q)\chi_{A^c}\right). \qquad (42)$$

Hence

$$\det\left(I + z\mathbf{M}(q)\chi_A\right) = \det\left(I + z\mathbf{M}(q)\right) \det\left(I - \mathbf{K}(z; q)\chi_{A^c}\right). \qquad (43)$$

Proof. Since the Hermite functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ form an orthonormal basis for $L^2(\mathbb{R})$, it is easy to see that

$$\mathbf{M}\left(q^k\right) = \mathbf{M}(q)^k. \qquad (44)$$

We define the resolvent operator $\mathbf{R}(z; q)$ by

$$I - \mathbf{R}(z; q) = \left(I + z\mathbf{M}(q)\right)^{-1}. \qquad (45)$$

If $|z| < 1$, we have that $\mathbf{R}(z; q)$ is a well-defined integral operator and

$$\mathbf{R}(z; q) = -\sum_{l=1}^{\infty} \left(-z\mathbf{M}(q)\right)^l. \qquad (46)$$

Assuming for now that $|z|< 1$, and using the fact that the functions $\varphi_k(x)$ are uniformly bounded in k and x (see, e.g. [1, 22.14.17]), we have that uniformly for all $x, y \in \mathbb{R}$

$$\begin{aligned} K(x,y;z;q) &= \sum_{k=0}^{\infty} q^k z\varphi_k(x)\varphi_k(y)\sum_{l=0}^{\infty}(-1)^l z^l q^{lk} \\ &= \sum_{l=1}^{\infty}(-1)^{l+1}z^l\sum_{k=0}^{\infty} q^{kl}\varphi_k(x)\varphi_k(y) \\ &= \sum_{l=1}^{\infty}(-1)^{l+1}z^l M\big(x,y;q^l\big). \end{aligned} \tag{47}$$

This implies that

$$\mathbf{K}(z;q) = \mathbf{R}(z;q), \tag{48}$$

for all $|z| < 1$. Using the identity $\mathbf{K}(z;q)\chi_{A^c} = \mathbf{R}(z;q)\chi_{A^c}$ we find

$$\begin{aligned} \big(I + z\mathbf{M}(q)\big)\big(I - \mathbf{K}(z;q)\chi_{A^c}\big) &= I + z\mathbf{M}(q) - \mathbf{R}(z;q)\chi_{A^c} - \mathbf{M}(q)\mathbf{R}(z;q)\chi_{A^c} \\ &= I + z\mathbf{M}(q)\chi_A + \big(z\mathbf{M}(q) - \mathbf{R}(z;q) - \mathbf{M}(q)\mathbf{R}(z;q)\big)\chi_{A^c} \\ &= I + z\mathbf{M}(q)\chi_A, \end{aligned} \tag{49}$$

where in the last step we use $z\mathbf{M}(q) - \mathbf{R}(z;q) - \mathbf{M}(q)\mathbf{R}(z;q) = 0$, which is a consequence of (45). Hence we prove (42) in the case $|z| < 1$. Since the integral operator $\mathbf{K}(z;q)$ is well defined for all $z \in \mathbb{C}$, by analytic continuation (42) holds for all $z \in \mathbb{C}$. □

We expand the Fredholm determinant $\det(I + z\mathbf{M}(q)\chi_A)$ into a series of multiple integrals by [30, Theorem 3.10], and then simplify it by the Cauchy–Binet identity as follows.

$$\begin{aligned} \det\big(I + z\mathbf{M}(q)\chi_A\big) &= 1 + \frac{z}{1!}\int_A M(x,x;q)\,dx + \frac{z^2}{2!}\int_A dx_1\int_A dx_2 \det\big(M(x_i,x_j;q)\big)_{i,j=1}^2 + \cdots \\ &= 1 + z\Bigg(\sum_{0\le k_1}\big\langle q^{k_1}\varphi_{k_1},\varphi_{k_1}\big\rangle_A\Bigg) \\ &\quad + z^2\left(\sum_{0\le k_1<k_2}\begin{vmatrix}\langle q^{k_1}\varphi_{k_1},\varphi_{k_1}\rangle_A & \langle q^{k_1}\varphi_{k_1},\varphi_{k_2}\rangle_A \\ \langle q^{k_2}\varphi_{k_2},\varphi_{k_1}\rangle_A & \langle q^{k_2}\varphi_{k_2},\varphi_{k_2}\rangle_A\end{vmatrix}\right) \\ &\quad + z^3\left(\sum_{0\le k_1<k_2<k_3}\begin{vmatrix}\langle q^{k_1}\varphi_{k_1},\varphi_{k_1}\rangle_A & \langle q^{k_1}\varphi_{k_1},\varphi_{k_2}\rangle_A & \langle q^{k_1}\varphi_{k_1},\varphi_{k_3}\rangle_A \\ \langle q^{k_2}\varphi_{k_2},\varphi_{k_1}\rangle_A & \langle q^{k_2}\varphi_{k_2},\varphi_{k_2}\rangle_A & \langle q^{k_2}\varphi_{k_2},\varphi_{k_3}\rangle_A \\ \langle q^{k_3}\varphi_{k_3},\varphi_{k_1}\rangle_A & \langle q^{k_3}\varphi_{k_3},\varphi_{k_2}\rangle_A & \langle q^{k_3}\varphi_{k_3},\varphi_{k_3}\rangle_A\end{vmatrix}\right) \\ &\quad + \cdots. \end{aligned} \tag{50}$$

With the help of (9), (10) and (40) we thus find

$$\det\big(I + z\mathbf{M}(q)\chi_A\big) = 1 + \sum_{n=1}^{\infty}\frac{z^n}{q^{n(n-1)/2}(q;q)_n}\mathbb{P}_n(x_1,\dots,x_n \in A), \tag{51}$$

and arrive at the formula for any dimension n,

$$\mathbb{P}_n(x_1,\dots,x_n \in A) = q^{n(n-1)/2}(q;q)_n\frac{1}{2\pi i}\oint_0 \det\big(I + z\mathbf{M}(q)\chi_A\big)\frac{dz}{z^{n+1}}. \tag{52}$$

In order to do asymptotic analysis it is convenient to work with the operator $\mathbf{K}(z;q)$ rather than $\mathbf{M}(q)$. Since the operator $\mathbf{M}(q)$ is diagonalized by $\{\varphi_k\}$, the determinant is simple to compute:

$$\det\bigl(I+z\mathbf{M}(q)\bigr)=\prod_{k=0}^{\infty}\bigl(1+q^k z\bigr)=(-z;q)_\infty. \tag{53}$$

Thus substituting (43) and (53) into (52), we obtain the formula (17) and prove Theorem 1(a). In particular, when $A=(-\infty,s]$, (14) implies

$$\mathbb{P}_n\bigl(\max(x_1,\dots,x_n)\le s\bigr)=q^{-n(n-1)/2}(q;q)_n\frac{1}{2\pi i}\oint_0\frac{(-z;q)_\infty}{z^n}\det\bigl(I-\mathbf{K}(z;q)\chi_{(s,\infty)}\bigr)\frac{dz}{z}. \tag{54}$$

2.2. *Correlation functions*

We now prove Theorem 1(b) assuming the result 1(a). We present the proof of (20) for $m=2$, but the proof is nearly identical for any positive integer m. Fix $x_1,x_2\in\mathbb{R}$ and $\Delta>0$, and introduce the notations

$$A_i^\Delta=[x_i,x_i+\Delta),\qquad G_i^\Delta=\bigl\{\text{there are no particles in } A_i^\Delta\bigr\}. \tag{55}$$

We will use the definition (1.4) for the m-correlation function, and note that

$$\begin{aligned}\mathbb{P}_n\bigl(\text{there is at least one particle in each } [x_i,x_i+\Delta\bigr), i=1,2)&=1-\mathbb{P}_n\bigl(G_1^\Delta\cup G_2^\Delta\bigr)\\&=1-\bigl[\mathbb{P}_n\bigl(G_1^\Delta\bigr)+\mathbb{P}_n\bigl(G_2^\Delta\bigr)-\mathbb{P}_n\bigl(G_1^\Delta\cap G_2^\Delta\bigr)\bigr].\end{aligned} \tag{56}$$

Using the formula (17) for the gap probabilities and expanding the Fredholm determinants as series, this is

$$\begin{aligned}1-\frac{1}{2\pi i}\oint_0\frac{F(z)}{z}\biggl[&1+\int_{A_1^\Delta}K(y,y)\,dy+\frac{1}{2!}\int_{A_1^\Delta}\int_{A_1^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2+\mathcal{O}\bigl(\Delta^3\bigr)\\&+1+\int_{A_2^\Delta}K(y,y)\,dy+\frac{1}{2!}\int_{A_2^\Delta}\int_{A_2^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2+\mathcal{O}\bigl(\Delta^3\bigr)\\&-1-\int_{A_1^\Delta\cup A_2^\Delta}K(y,y)\,dy-\frac{1}{2!}\int_{A_1^\Delta}\int_{A_1^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2\\&-\frac{1}{2!}\int_{A_2^\Delta}\int_{A_2^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2\\&-\frac{1}{2!}\int_{A_1^\Delta}\int_{A_2^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2-\frac{1}{2!}\int_{A_2^\Delta}\int_{A_1^\Delta}\det\bigl[K(y_i,y_j)\bigr]_{i,j=1}^2\,dy_1\,dy_2+\mathcal{O}\bigl(\Delta^3\bigr)\biggr]\,dz,\end{aligned} \tag{57}$$

where for brevity we have used $K(y_1,y_2)\equiv K(y_1,y_2;z;q)$. Noting all of the cancellations and the fact that $\frac{1}{2\pi i}\oint_0\frac{F(z)}{z}\,dz=\mathbb{P}_n(\text{all particles are in }\mathbb{R})=1$ we find

$$\begin{aligned}&\mathbb{P}_n\bigl(\text{there is at least one particle in each } [x_i,x_i+\Delta x), i=1,2\bigr)\\&\quad=\frac{1}{2\pi i}\oint_0\frac{F(z)}{z}\biggl[\int_{A_1^\Delta}\int_{A_2^\Delta}\det\bigl[K(y_1,y_2)\bigr]_{i,j=1}^2\,dy_1\,dy_2+\mathcal{O}\bigl(\Delta^3\bigr)\biggr]\,dz,\end{aligned} \tag{58}$$

from which it immediately follows

$$\begin{aligned}&\lim_{\Delta\to0}\frac{\mathbb{P}_n(\text{there is at least one particle in each } [x_i,x_i+\Delta x), i=1,2)}{\Delta^2}\\&\quad=\frac{1}{2\pi i}\oint_0\frac{F(z)}{z}\det\bigl[K(x_1,x_2)\bigr]_{i,j=1}^2\,dz.\end{aligned} \tag{59}$$

This proves (20) in the case $m=2$ and $x_1\ne x_2$. The extension to the general case is straightforward.

3. Proof of Theorem 2

Our starting point is formula (54), the special case of (17) with $A=(-\infty,s]$. After the change of variable

$$w=q^n z, \tag{60}$$

formula (54) becomes

$$\mathbb{P}_n\big(\max(x_1,\dots,x_n)\le s\big)=q^{n(n+1)/2}(q;q)_n\frac{1}{2\pi i}\oint_0\big(-q^{-n}w;q\big)_\infty \det\big(I-\mathbf{P}_s\mathbf{K}\big(q^{-n}w;q\big)\mathbf{P}_s\big)\frac{dw}{w^{n+1}}, \tag{61}$$

where $\mathbf{P}_s$ is the projection onto $L^2(s,\infty)$. It is straightforward to see that

$$\big(-q^{-n}w;q\big)_\infty=(-w;q)_\infty w^n q^{-n(n+1)/2}(-q/w;q)_n. \tag{62}$$

Thus we have that the integral in (61) can be written as

$$\frac{1}{2\pi i}\oint_0(q;q)_n(-w;q)_\infty(-q/w;q)_n\det\big(I-\mathbf{P}_s\mathbf{K}\big(q^{-n}w;q\big)\mathbf{P}_s\big)\frac{dw}{w}. \tag{63}$$

By the triple product identity [5, Theorem 10.4.1]

$$(-w;q)_\infty(-q/w;q)_\infty(q;q)_\infty=\sum_{k=-\infty}^{\infty}q^{\frac{k(k-1)}{2}}w^k, \tag{64}$$

the integral in (61) is written as

$$\frac{(q;q)_n}{(q;q)_\infty}\frac{1}{2\pi i}\oint_0\left(\sum_{k=-\infty}^{\infty}q^{\frac{k(k-1)}{2}}w^k\right)\frac{(-q/w;q)_n}{(-q/w;q)_\infty}\det\big(I-\mathbf{P}_s\mathbf{K}\big(q^{-n}w;q\big)\mathbf{P}_s\big)\frac{dw}{w}. \tag{65}$$

We take the contour in (65) as $|w|=\sqrt{q}$ and make the change of variable $w=\sqrt{q}e^{i\pi\theta}$. Then (65) becomes

$$\frac{1}{2}\int_{-1}^{1}\left(\sum_{k=-\infty}^{\infty}q^{k^2/2}e^{ik\pi\theta}\right)\det\big(I-\mathbf{P}_s\mathbf{K}\big(q^{-n+1/2}e^{i\pi\theta};q\big)\mathbf{P}_s\big)F_n(\theta;q)\,d\theta, \tag{66}$$

where

$$F_n(\theta;q)=\frac{(q;q)_n}{(q;q)_\infty}\frac{(-\sqrt{q}e^{-i\pi\theta};q)_n}{(-\sqrt{q}e^{-i\pi\theta};q)_\infty}. \tag{67}$$

3.1. *Preliminary estimates of $\tilde{K}_n(x,y)$*

In what follows we will need to compute the limit of the Fredholm determinant in the integrand of (66) as $n\to\infty$ in the scaling limit $s=s_n=2\sqrt{n}+tn^{-1/6}$ for $t\in\mathbb{R}$. In this scaling

$$\det\big(I-\mathbf{P}_s\mathbf{K}\big(q^{-n+1/2}e^{i\pi\theta};q\big)\mathbf{P}_s\big)=\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t), \tag{68}$$

where $\tilde{\mathbf{K}}=\tilde{\mathbf{K}}(\theta)$ has the kernel

$$\begin{aligned}\tilde{K}_n(x,y)&=\tilde{K}_n(x,y;\theta)\\&:=n^{-1/6}K\big(2\sqrt{n}+xn^{-1/6},2\sqrt{n}+yn^{-1/6}\big)\\&=n^{-1/6}\sum_{k=0}^{\infty}c_k\varphi_k\big(2\sqrt{n}+xn^{-1/6}\big)\varphi_k\big(2\sqrt{n}+yn^{-1/6}\big),\end{aligned} \tag{69}$$

where

$$c_k = c_k(\theta) := \frac{e^{\pi i\theta} q^{k-n+1/2}}{1 + e^{\pi i\theta} q^{k-n+1/2}} = \frac{e^{\pi i\theta/2} \sqrt{q}^{k-n+1/2}}{2\cosh(\frac{k-n+1/2}{2}\log q + \frac{i\pi\theta}{2})}, \tag{70}$$

with the dependence on θ suppressed if there is no chance of confusion.

We need to compute the $n \to \infty$ limit of $\tilde{K}_n(x, y)$ for x, y in a compact subset of $\mathbb{R}$, and show that $\tilde{K}_n(x, y)$ vanishes exponentially fast as $\max(x, y) \to +\infty$ and $\min(x, y)$ is bounded below. We will use the following global approximation formula for φ_k, which is from [25, Section 11.4, Exercises 4.2 and 4.3]. For x in a compact subset of $(-1, +\infty)$ and $k = 0, 1, 2, \dots$ uniformly,

$$(k+1/2)^{1/12}\varphi_k(2\sqrt{k+1/2}x) = 2^{1/6}\left(\frac{\zeta(x)}{x^2-1}\right)^{1/4}\left(\mathrm{Ai}\big((2k+1)^{2/3}\zeta(x)\big) + \varepsilon_k(x)\right) \times \left(1 + \mathcal{O}\big((k+1/2)^{-1}\big)\right), \tag{71}$$

such that

(i) the $1 + \mathcal{O}((k+1/2)^{-1})$ factor depends on k only;

(ii) $\zeta(x)$ is a continuous, differentiable and monotonically increasing function on $(-1, +\infty)$. Moreover, it is bounded below as $x \to -1_+$ and has x^2 growth as $x \to +\infty$. The explicit formula of $\zeta(x)$ is given in [25, Section 11.4, Exercise 4.2]. Around 1, it satisfies

$$\zeta(1) = 0 \quad \text{and} \quad \zeta'(1) = 2^{1/3}; \tag{72}$$

(iii) $\varepsilon_k(x)$ is defined in [25, Section 11.4, Exercise 4.2], where it is denoted as $\varepsilon(x)$. From [25, Section 11.2], we have the estimate uniform in k and x,

$$\left|\varepsilon(x)\right| = \begin{cases} \mathcal{O}((k+1/2)^{-7/6}(-\zeta(x))^{-1/4}) & \text{if } x \in (-1, 1], \\ \mathcal{O}((k+1/2)^{-7/6}\zeta(x)^{-1/4}\exp(-\frac{2}{3}(2k+1)\zeta(x)^{3/2})) & \text{if } x \in [1, +\infty). \end{cases} \tag{73}$$

To use estimate (71), we also need that by [25, Sections 11.1-2], especially [25, Formulas (2.05), (2.13) and (2.15) in Chapter 11],

$$\left|\mathrm{Ai}(x)\right| \le f(x), \quad \text{where } f(x) = \begin{cases} \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\frac{2}{3}x^{3/2}} & x > 1, \\ 1 & -1 \le x \le 1, \\ \lambda^{1/2}\pi^{-1/2}(-x)^{-1/4} & x < -1, \end{cases} \tag{74}$$

with the constant $\lambda = 1.04\dots$.

Below we provide computational results that are used in the proof of both part (a) and part (b) of Theorem 2. Note that we use C to denote a large enough positive constant and γ a small enough positive constant. It is harmless to assume $C = 1000$ and $\gamma = 1/10$.

x, y in a compact subset

First we consider the case that $x, y \in [-M/2, M/2]$ where M is a positive constant. Without loss of generality, we assume that $Mn^{1/3}$ is an integer, and then write

$$\tilde{K}_n(x, y) = K_n^{(1,M)}(x, y) + K_n^{(2,M)}(x, y) + K_n^{(3,M)}(x, y), \tag{75}$$

where

$$K_n^{(1,M)}(x, y) = n^{-1/6}\sum_{k=0}^{n-Mn^{1/3}-1} c_k\varphi_k\big(2\sqrt{n} + xn^{-1/6}\big)\varphi_k\big(2\sqrt{n} + yn^{-1/6}\big), \tag{76}$$

$$K_n^{(2,M)}(x, y) = n^{-1/6}\sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}} c_k\varphi_k\big(2\sqrt{n} + xn^{-1/6}\big)\varphi_k\big(2\sqrt{n} + yn^{-1/6}\big), \tag{77}$$

$$K_n^{(3,M)}(x, y) = n^{-1/6}\sum_{k=n+Mn^{1/3}+1}^{\infty} c_k\varphi_k\big(2\sqrt{n} + xn^{-1/6}\big)\varphi_k\big(2\sqrt{n} + yn^{-1/6}\big). \tag{78}$$

The following estimates on the coefficients c_k are uniform in k and θ:

$$c_k = \begin{cases} 1+\mathcal{O}(q^{Mn^{1/3}}), & k=0,\dots,n-Mn^{1/3}-1, \\ \mathcal{O}(q^{l+Mn^{1/3}}), & k=n+Mn^{1/3}+l, l=1,2,\dots. \end{cases} \tag{79}$$

With the estimates (79) for c_k and (71) for φ_k, it follows that

$$\left|K_n^{(3,M)}(x,y)\right| \le n^{-1/6} C \sum_{l=1}^{\infty} q^{l+Mn^{1/3}} \left(n+Mn^{1/3}+l+1/2\right)^{-1/6} \le C\frac{n^{-1/3}}{1-q} q^{Mn^{1/3}}, \tag{80}$$

where C is a constant independent of n, x, y, M, θ and q. Similarly,

$$\begin{aligned} \left|K_n^{(1,M)}(x,y)\right| \le n^{-1/6} C &\sum_{k=0}^{n-Mn^{1/3}-1} (k+1/2)^{-1/6} \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n}+xn^{-1/3}}{2\sqrt{k+1/2}}\right)^{3/2}\right) \\ &\times \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n}+yn^{-1/3}}{2\sqrt{k+1/2}}\right)^{3/2}\right), \end{aligned} \tag{81}$$

where C is independent of n, x, y, M, θ and q. After some calculation, the sum $K_n^{(1,M)}(x,y)$ is estimated as

$$\left|K_n^{(1,M)}(x,y)\right| \le C\exp\left(-\gamma M^{3/2}\right), \tag{82}$$

where C and γ are independent of n, x, y, M, θ and q.

The approximation of $K_n^{(2,M)}(x,y)$ depends on θ and will be given later.

$x \to +\infty$ and y is bounded below

Let M be the same as above and $N > M$ be a large positive constant, and without loss of generality assume that $Nn^{1/3}$ is an integer. Suppose $x \ge 2N$ and $y \ge -M/2$. We write

$$\tilde{K}_n(x,y) = K_n^{(4,M,N)}(x,y) + K_n^{(2,M)}(x,y) + K_n^{(5,N)}(x,y), \tag{83}$$

where $K_n^{(2,M)}(x,y)$ is defined in (77), and

$$K_n^{(4,M,N)}(x,y) = n^{-1/6} \sum_{\substack{0\le k\le n-Mn^{1/3}-1 \\ \text{or } n+Mn^{1/3}+1\le k\le n+Nn^{1/3}}} c_k\varphi_k\left(2\sqrt{n}+xn^{-1/6}\right)\varphi_k\left(2\sqrt{n}+yn^{-1/6}\right), \tag{84}$$

$$K_n^{(5,N)}(x,y) = n^{-1/6} \sum_{k=n+Nn^{1/3}+1}^{\infty} c_k\varphi_k\left(2\sqrt{n}+xn^{-1/6}\right)\varphi_k\left(2\sqrt{n}+yn^{-1/6}\right). \tag{85}$$

Similar to (80), we have the estimate

$$\left|K_n^{(5,N)}(x,y)\right| \le C\frac{n^{-1/3}}{1-q} q^{Nn^{1/3}}, \tag{86}$$

where C is independent of n, x, y, N, θ and q. Similarly, like (80) and (81), by the estimate (79) for c_k and (71) for φ_k,

$$\begin{aligned} &\left|K_n^{(4,M,N)}(x,y)\right| \\ &\quad \le n^{-1/6} C \sum_{\substack{0\le k\le n-Mn^{1/3}-1 \\ \text{or } n+Mn^{1/3}+1\le k\le n+Nn^{1/3}}} (k+1/2)^{-1/6} \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n}+xn^{-1/3}}{2\sqrt{k+1/2}}\right)^{3/2}\right) \\ &\quad \le C\exp\left(-\gamma N^{3/2}\right), \end{aligned} \tag{87}$$

where C and γ are independent of n, x, y, M, N, θ and q. Note that in (87) the estimate of $\varphi_k(2\sqrt{n}+xn^{-1/6})$ is the same as in (81), while the estimate of $\varphi_k(2\sqrt{n}+yn^{-1/6})$ is roughly $\mathcal{O}((k+1/2)^{-1/12})$, as in (80).

Below we prove Theorem 2. We first give full detail for part (b), and then show that a simplified argument works for part (a). The technical core is the estimate of $K_n^{(2,M)}(x,y)$.

3.2. *Gap probability for the rightmost particle*: $q = e^{-cn^{-1/3}}$

Now consider the scaling $q = e^{-cn^{-1/3}}$ for some $c > 0$. We begin with the following lemma on the asymptotics of the q-Pochhammer symbols appearing in (66).

Lemma 2. *For $q = e^{-cn^{-1/3}}$, we have the estimate uniformly for $\theta \in [-1, 1]$:*

$$\begin{aligned}
&\frac{(q;q)_n}{(q;q)_\infty} = 1 + \mathcal{O}\big(n^{1/3} e^{-cn^{2/3}}\big),\\
&\frac{(-\sqrt{q}e^{-\pi i\theta};q)_n}{(-\sqrt{q}e^{-\pi i\theta};q)_\infty} = 1 + \mathcal{O}\big(n^{1/3} e^{-cn^{2/3}}\big).
\end{aligned} \tag{88}$$

Thus uniformly for $\theta \in [-1, 1]$, the function $F_n(\theta; q)$ defined in (67) satisfies

$$F_n(\theta;q) = 1 + \mathcal{O}\big(n^{1/3} e^{-cn^{2/3}}\big). \tag{89}$$

Proof. We only prove the second equation of (88). We have

$$\frac{(\sqrt{q}e^{-\pi i\theta};q)_n}{(\sqrt{q}e^{-\pi i\theta};q)_\infty} = \frac{1}{\prod_{k=0}^{\infty}(1 - e^{-\pi i\theta} q^{k+n+1/2})}, \tag{90}$$

thus

$$\begin{aligned}
\left|\log \frac{(\sqrt{q}e^{-\pi i\theta};q)_n}{(\sqrt{q}e^{-\pi i\theta};q)_\infty}\right| &\le \sum_{k=0}^{\infty} \big|\log\big(1 - e^{-\pi i\theta} q^{k+n+1/2}\big)\big|\\
&< \frac{2q^n}{1-q} = \frac{2e^{-cn^{2/3}}}{1 - e^{-cn^{-1/3}}} = \frac{2e^{-cn^{2/3}}}{cn^{-1/3}}\big(1 + \mathcal{O}\big(n^{-1/3}\big)\big).
\end{aligned} \tag{91}$$

The result is obtained by exponentiating. □

Also note that the Poisson summation formula gives

$$\sum_{k=-\infty}^{\infty} e^{\frac{-cn^{-1/3}k^2}{2}} e^{k\pi i\theta} = n^{1/6}\sqrt{2\pi c^{-1}} \sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}. \tag{92}$$

Applying formulas (68), (88) and (92) to the integral (66), we find that (66) becomes

$$n^{1/6} \int_{-1}^{1} \sqrt{\frac{\pi}{2c}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right) \det\big(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta) \mathbf{P}_t\big) F_n(\theta;q)\, d\theta. \tag{93}$$

Fix a small $\epsilon > 0$. We plug the formula (92) into (93), use the estimates in Lemma 2, and split the integral (93) into two parts, I_1 and I_2, where

$$I_1 = n^{1/6} \int_{-1+\epsilon}^{1-\epsilon} \sqrt{\frac{\pi}{2c}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right) \det\big(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta) \mathbf{P}_t\big) F_n(\theta;q)\, d\theta, \tag{94}$$

$$I_2 = n^{1/6} \int_{1-\epsilon}^{1+\epsilon} \sqrt{\frac{\pi}{2c}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right) \det\big(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta) \mathbf{P}_t\big) F_n(\theta;q)\, d\theta. \tag{95}$$

In order to evaluate these integrals as $n \to \infty$, we need some estimates on the determinant $\det(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta) \mathbf{P}_t)$ which are uniform in θ. These are given in the following lemma.

Lemma 3.

(a) *For $\theta \in (-1+\epsilon, 1-\epsilon)$, the determinant $\det(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta)\mathbf{P}_t)$ is bounded uniformly in θ as $n \to \infty$. Furthermore it has the limit*

$$\lim_{n\to\infty} \det\bigl(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta)\mathbf{P}_t\bigr) = \det\bigl(I - \mathbf{P}_t \mathbf{K}_{\mathrm{cross}}(c;\theta)\mathbf{P}_t\bigr), \tag{96}$$

where $\mathbf{K}_{\mathrm{cross}}(c;\theta)$ is the integral operator on $L^2(\mathbb{R})$ with kernel

$$K_{\mathrm{cross}}(x,y;c;\theta) = \int_{-\infty}^{\infty} \frac{e^{i\pi\theta} e^{-cr}}{1+e^{i\pi\theta}e^{-cr}} \operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr. \tag{97}$$

(b) *There exists a positive constant $\tilde{C}$ such that for all $n \in \mathbb{N}$ and all $\theta \in (1-\epsilon, 1+\epsilon)$,*

$$\bigl|\det\bigl(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta)\mathbf{P}_t\bigr)\bigr| \le \exp\bigl(\bigl(\tilde{C}e^{-ct}\log n\bigr)^2 + \tilde{C}e^{-ct}\log n\bigr). \tag{98}$$

Given the results of this lemma, it is fairly straightforward to prove Theorem 2(b). Consider I_1 first. Clearly as $n \to \infty$ the dominant term in the infinite sum is $k = 0$, and we have

$$I_1 = n^{1/6}\sqrt{\frac{\pi}{2c}} \int_{-1+\epsilon}^{1-\epsilon} e^{-\frac{n^{1/3}\pi^2\theta^2}{2c}} \det\bigl(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta)\mathbf{P}_t\bigr)\bigl(1 + \mathcal{O}\bigl(e^{-\frac{n^{1/3}\pi^2}{2c}}\bigr)\bigr)\,d\theta. \tag{99}$$

Since the Fredholm determinant in the integrand has a limit as $n \to \infty$, we can use Laplace's method to evaluate the integral as $n \to \infty$. The integral I_1 is localized close to $\theta = 0$, and Laplace's method immediately gives

$$\begin{aligned}\lim_{n\to\infty} I_1 &= \lim_{n\to\infty} \det\bigl(I - \mathbf{P}_t \tilde{\mathbf{K}}(\theta = 0)\mathbf{P}_t\bigr) \\ &= \det\bigl(I - \mathbf{P}_t \mathbf{K}_{\mathrm{cross}}(c;\theta = 0)\mathbf{P}_t\bigr).\end{aligned} \tag{100}$$

Noting that $\mathbf{K}_{\mathrm{cross}}(c;\theta = 0) \equiv \mathbf{K}_{\mathrm{cross}}(c)$ defined in (22), we find

$$\lim_{n\to\infty} I_1 = \det\bigl(I - \mathbf{P}_t \mathbf{K}_{\mathrm{cross}}(c)\mathbf{P}_t\bigr). \tag{101}$$

It remains only to show that $\lim_{n\to\infty} I_2 = 0$. This follows immediately from (95) and (98), since the infinite sum in (95) is vanishing like the exponent of a power of n whereas the determinant is growing at most as the exponent of a power of $\log n$. This completes the proof of Theorem 2(b), provided that Lemma 3 is true. The remainder of this subsection is dedicated to the proof of this lemma.

Proof of Lemma 3(a). We use the expression for the kernel $\tilde{K}_n(x,y)$ in (75) and (76)–(78) for the pointwise approximation as x, y in a compact subset of $\mathbb{R}$. In the scaling $q = e^{-cn^{-1/3}}$, the estimate (80) becomes

$$\bigl|K_n^{(3,M)}(x,y)\bigr| \le Cc^{-1}e^{-cM}. \tag{102}$$

Combined with (82) we see that as M becomes large, $K_n^{(1,M)}(x,y)$ and $K_n^{(3,M)}(x,y)$ vanish, and the dominant contribution should come from $K_n^{(2,M)}(x,y)$. In the sum $K_n^{(2,M)}(x,y)$, we denote $r = n^{-1/3}(k-n)$ and write the sum as

$$\begin{aligned}&K_n^{(2,M)}(x,y) \\ &\quad = n^{-1/6} \sum_{r\in\{n^{-1/3}\mathbb{Z}\cap[-M,M]\}} \frac{e^{\pi i\theta}q^{1/2+rn^{1/3}}}{1+e^{\pi i\theta}q^{1/2+rn^{1/3}}} \varphi_{n+rn^{1/3}}\bigl(2\sqrt{n} + xn^{-1/6}\bigr)\varphi_{n+rn^{1/3}}\bigl(2\sqrt{n} + yn^{-1/6}\bigr) \\ &\quad = \int_{-M}^{M} \frac{e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}}{1+e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}} n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}\bigl(2\sqrt{n} + xn^{-1/6}\bigr) n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}\bigl(2\sqrt{n} + yn^{-1/6}\bigr)\,dr.\end{aligned} \tag{103}$$

From (71), we find that

$$\lim_{n\to\infty} n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}\bigl(2\sqrt{n} + n^{-1/6}x\bigr) = \operatorname{Ai}(x-r), \tag{104}$$

thus the integrand in (103) has the pointwise limit

$$\frac{e^{i\pi\theta}e^{-cr}}{1+e^{i\pi\theta}e^{-cr}}\operatorname{Ai}(x-r)\operatorname{Ai}(y-r), \tag{105}$$

and the bounded convergence theorem gives

$$\lim_{n\to\infty} K_n^{(2,M)}(x,y) = \int_{-M}^{M} \frac{e^{i\pi\theta}e^{-cr}}{1+e^{i\pi\theta}e^{-cr}}\operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr. \tag{106}$$

Since both $K_n^{(1,M)}(x,y)$ and $K_n^{(3,M)}(x,y)$ are bounded in n and vanish as $M\to\infty$, we now take $M\to\infty$ and obtain

$$\lim_{n\to\infty} \tilde{K}_n(x,y) = \lim_{M\to\infty}\lim_{n\to\infty} K_n^{(2,M)}(x,y) = \int_{-\infty}^{\infty} \frac{e^{i\pi\theta}e^{-cr}}{1+e^{i\pi\theta}e^{-cr}}\operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr, \tag{107}$$

which is the kernel of a trace-class operator for all $\theta\in(-1+\epsilon,1-\epsilon)$.

We have proved the pointwise convergence of the kernels in the determinant, and actually the convergence in (107) is uniform if x, y are in a compact subset of $\mathbb{R}$. To prove the determinant convergence (96), we will need estimates on the kernel $\tilde{K}_n(x,y)$ as $\max(x,y)\to\infty$. The estimates (86) and (87) imply that, if $q=e^{-cn^{-1/3}}$ and $y\ge t$, then for all $x>4\max(-t,1)$, we take $M=2\max(-t,1)$ and $N=x$, and have

$$\left|K_n^{(4,2\max(-t,1),x)}(x,y)\right| \le Ce^{-\gamma x^{3/2}} \quad \text{and} \quad \left|K_n^{(5,x)}(x,y)\right| \le Cc^{-1}e^{-cx}, \tag{108}$$

with constants C and γ independent of n. Using the method of estimating $K_n^{(4,M,N)}(x,y)$ in (87), we have a similar estimate for $K_n^{(2,2\max(-t,1))}(x,y)$, provided that $\theta\in(-1+\epsilon,1-\epsilon)$:

$$\left|K_n^{(2,2\max(-t,1))}(x,y)\right| \le Ce^{-\gamma x^{3/2}}. \tag{109}$$

Combining (108) and (109) we obtain the uniform estimate for $x,y\ge t$

$$\left|\tilde{K}_n(x,y)\right| \le \tilde{C}e^{-cx}, \tag{110}$$

where the constant $\tilde{C}$ depends on t and c, but independent of n.

The Fredholm determinant $\det(I-\mathbf{P}_s\mathbf{K}(q^{-n+1/2}e^{i\pi\theta};q)\mathbf{P}_s) = \det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)$ is given by the series

$$\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t) = 1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_t^{\infty}dx_1\cdots\int_t^{\infty}dx_k \det\big(\tilde{K}_n(x_i,x_j)\big)_{i,j=1}^k. \tag{111}$$

Each of the determinants in this series can be estimated using (110) along with Hadamard's inequality, giving

$$\left|\det\big(\tilde{K}(x_i,x_j)\big)_{i,j=1}^k\right| \le k^{k/2}\tilde{C}^k\prod_{i=1}^k e^{-cx_i}, \tag{112}$$

so each term in (111) is bounded by

$$\begin{aligned}\left|\frac{(-1)^k}{k!}\int_t^{\infty}dx_1\cdots\int_t^{\infty}dx_k \det\big(\tilde{K}_n(x_i,x_j)\big)_{i,j=1}^k\right| &\le \frac{k^{k/2}}{k!}\tilde{C}^k\int_t^{\infty}dx_1e^{-cx_1}\cdots\int_t^{\infty}dx_ke^{-cx_k}\\ &\le \frac{k^{-k/2}}{k!}\big(\tilde{C}c^{-1}e^{-ct}\big)^k.\end{aligned} \tag{113}$$

Thus the series (111) is dominated by an absolutely convergent series, and the dominated convergence theorem gives that the sum converges to the term-by-term limit. This is exactly $\det(I-\mathbf{P}_t\mathbf{K}_{\text{cross}}(c;\theta)\mathbf{P}_t)$, since the integrands are dominated by an absolutely integrable function according to (112), so the dominated convergence theorem implies that each term converges to the corresponding term in the series for $\det(I-\mathbf{P}_t\mathbf{K}_{\text{cross}}(c;\theta)\mathbf{P}_t)$. This completes the proof of Lemma 3(a). □

Proof of Lemma 3(b). Our estimate of $\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)$ for θ close to 1 is based on the identity (see [30, Theorem 9.2(d)])

$$\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)=\det{}_2(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)e^{\operatorname{Tr}\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t}, \tag{114}$$

where $\det_2$ is defined in [30, Chapter 9]. The $\det_2$ functional can be estimated using the Hilbert–Schmidt norm, see [30, Theorem 9.2(b)]. In particular we have

$$\left|\det{}_2(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)\right|\le\exp\left(\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2\right), \tag{115}$$

where $\|\cdot\|_2$ represents the Hilbert–Schmidt norm. Combining this inequality with (114), we have

$$\left|\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)\right|\le\exp\left(\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2\right)e^{|\operatorname{Tr}\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t|}, \tag{116}$$

and we are left to estimate the trace and the Hilbert–Schmidt norms of $\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t$.

We begin by estimating the kernel $\tilde{K}_n(x,y)$ for $\theta\in(1-\epsilon,1+\epsilon)$. Since (102), (82) and (108) still hold for $\theta\in(1-\epsilon,1+\epsilon)$, we concentrate on $K_n^{(2,M)}(x,y)$. Let us estimate this sum. Using (71) we obtain the following estimate, which is uniform for x,y in compact sets and $n-Mn^{1/3}<k<n+Mn^{1/3}$:

$$\begin{aligned}&\varphi_k\left(2\sqrt{n}+xn^{-1/6}\right)\varphi_k\left(2\sqrt{n}+yn^{-1/6}\right)\\&\quad=n^{-1/6}\operatorname{Ai}\left(x-(k-n)/\left(2n^{1/3}\right)\right)\operatorname{Ai}\left(y-(k-n)/\left(2n^{1/3}\right)\right)\left(1+\mathcal{O}\left(n^{-2/3}\right)\right).\end{aligned} \tag{117}$$

The kernel $K_n^{(2,M)}(x,y)$ is thus estimated as

$$\left|K_n^{(2,M)}(x,y)\right|\le Cn^{-1/3}\sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}}\left|c_k(\theta)\operatorname{Ai}\left(x-(k-n)/\left(2n^{1/3}\right)\right)\operatorname{Ai}\left(y-(k-n)/\left(2n^{1/3}\right)\right)\right| \tag{118}$$

for some constant C which is independent of n, M and θ. We therefore need to estimate the coefficients c_k, and it is convenient to estimate the real and imaginary parts separately. They are

$$\Re c_{n+j}(\theta)=\frac{\cos(\pi\theta)+q^{j+1/2}}{2\cos(\pi\theta)+q^{j+1/2}+q^{-j-1/2}},\qquad\Im c_{n+j}(\theta)=\frac{\sin(\pi\theta)}{2\cos(\pi\theta)+q^{j+1/2}+q^{-j-1/2}}. \tag{119}$$

To estimate the imaginary part, note that $\Im c_{n+j}(1)=0$, but c_{n+j} becomes large in a neighborhood of $\theta=1$ when q is close to 1. In this neighborhood the critical points of $\Im c_{n+j}(\theta)$ are found to be at

$$\theta=1\pm\arcsin\left(\frac{q^{-j-1/2}-q^{j+1/2}}{q^{-j-1/2}+q^{j+1/2}}\right), \tag{120}$$

where $|\Im c_{n+j}(\theta)|$ attains the maximum. Plugging these critical points into $\Im c_{n+j}(\theta)$ we find the maximum value of $|\Im c_{n+j}(\theta)|$, obtaining

$$\left|\Im c_{n+j}(\theta)\right|=\frac{1}{|1-q^{-2j-1}|}. \tag{121}$$

Now consider the real part of c_k. The maximum value of $|\Re c_k|$ is attained at $\theta=1$. At this point we have

$$|\Re c_{n+j}|=\frac{1}{|1-q^{-j-1/2}|}. \tag{122}$$

Combining (121) and (122) with (118) we obtain the estimate for x,y in a compact set and n large enough

$$\left|K_n^{(2,M)}(x,y)\right|\le C\sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}}\frac{|\operatorname{Ai}(x-(k-n)/(2n^{1/3}))\operatorname{Ai}(y-(k-n)/(2n^{1/3})))|}{2c|k-n|+1}=\tilde{C}\log n, \tag{123}$$

where $\tilde{C}$ is a positive constant depending on M, c but not n, θ. Now consider the behavior of $\tilde{K}_n(x,y)$ as $x\to\infty$ when $\theta\in(1-\epsilon,1+\epsilon)$. The estimates (108) still hold here. The estimate (109) needs to be modified slightly for $\theta\in$

$(1-\epsilon, 1+\epsilon)$. Since the dependence of $K_n^{(2,M)}(x,y)$ on θ comes entirely from the coefficients c_k, and the dependence on x and y comes entirely from the Hermite functions, we can combine the analysis leading to (123) with (109) to obtain the estimate

$$\left|K_n^{(2,2\max(-t,1))}(x,y)\right| \le Ce^{-\gamma x^{3/2}}\log n, \tag{124}$$

for $\theta \in (1-\epsilon, 1+\epsilon)$, where once again C and γ are constants independent of n. Analogous to (110), we therefore have the uniform estimate for all $x, y \ge t$

$$\left|\tilde{K}_n(x,y)\right| \le \tilde{C}e^{-cx}\log n, \tag{125}$$

where $\tilde{C}$ depends on t and c, but not n, θ. The trace of $\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t$ can thus be estimated as

$$|\operatorname{Tr}\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t| \le \int_t^\infty \left|\tilde{K}_n(x,x)\right| dx \le \tilde{C}\log n\int_t^\infty e^{-cx}\,dx = \tilde{C}c^{-1}e^{-ct}\log n, \tag{126}$$

and the Hilbert–Schmidt norm is estimated as

$$\begin{aligned}\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2 &= \int_t^\infty\int_t^\infty \left|\tilde{K}_n(x,y)\right|^2 dy\,dx \\ &\le \tilde{C}^2(\log n)^2\int_t^\infty\int_t^\infty e^{-cx}e^{-cy}\,dy\,dx = \left(\tilde{C}c^{-1}e^{-ct}\log n\right)^2.\end{aligned} \tag{127}$$

Combining this with (116), (126), and (127) we obtain (98). This completes the proof of Lemma 3(b) (with $\tilde{C}c^{-1}$ replaced by $\tilde{C}$). □

3.3. *Gap probability for the rightmost particle*: *Fixed* $q \in (0,1)$

Let $q \in (0,1)$ be fixed. Then we have the following lemma.

Lemma 4. *Let* $s \equiv s_n = 2\sqrt{n} + tn^{-1/6}$. *The following holds uniformly for all* $\theta \in [-1,1]$.

$$\det\left(I - \mathbf{P}_s\mathbf{K}\left(q^{-n+1/2}e^{i\pi\theta}; q\right)\mathbf{P}_s\right) = \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_t) + o(1). \tag{128}$$

Sketch of proof. In the sum $K_n^{(2,M)}(x,y)$ given by (103), formula (104) implies that the piecewise constant function in the integrand of (103) has the pointwise limit

$$\mathrm{Ai}(x-r)\,\mathrm{Ai}(y-r)\,\chi_{[-M,0]}(r). \tag{129}$$

The bounded convergence theorem then implies that

$$\begin{aligned}\lim_{n\to\infty} K_n^{(2,M)}(x,y) &= \int_{-M}^0 \mathrm{Ai}(x-r)\,\mathrm{Ai}(y-r)\,dr \\ &= \int_0^M \mathrm{Ai}(x+r)\,\mathrm{Ai}(y+r)\,dr.\end{aligned} \tag{130}$$

Since M was arbitrary we can take it to infinity, in which case $K_n^{(1,M)}(x,y)$ and $K_n^{(3,M)}(x,y)$ vanish by (80) and (82), leaving

$$\lim_{n\to\infty} K_n(x,y) = \lim_{M\to\infty}\lim_{n\to\infty} K_n^{(2,M)}(x,y) = \int_0^\infty \mathrm{Ai}(x+r)\,\mathrm{Ai}(y+r)\,dr, \tag{131}$$

which is the kernel of $\mathbf{K}_{\mathrm{Airy}}$.

To prove the convergence of the Fredholm determinant, we need to control the vanishing of $\tilde{K}_n(x,y)$ as $\max(x,y) \to \infty$. Since the procedure is the same as the proof of Lemma 3(a), we omit the detailed verification. We only note that the proof works for all $\theta \in [-1,1]$, since the coefficients $c_k(\theta)$ are uniformly bounded even if θ is around ± 1. □

As $n \to \infty$, we have the very fast convergence analogous to Lemma 2

$$\frac{(-q/w;q)_n}{(-q/w;q)_\infty} = 1 + \mathcal{O}(q^n), \qquad \frac{(q;q)_n}{(q;q)_\infty} = 1 + \mathcal{O}(q^n). \tag{132}$$

Combining this fact with Lemma 4, we see that the integral (66) is

$$\mathbb{P}_n\big(\max(x_1,\dots,x_n) \le s\big) = \frac{1}{2}\int_{-1}^{1}\left(\sum_{k=-\infty}^{\infty} q^{k^2/2} e^{ik\pi\theta}\right)\det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_t)\big(1+o(1)\big)\,d\theta. \tag{133}$$

After integrating, the only nonvanishing term in the infinite sum is $k=0$, thus we find

$$\begin{aligned}\mathbb{P}_n\big(\max(x_1,\dots,x_n) \le s\big) &= \frac{1}{2}\int_{-1}^{1}\det(I - \mathbf{P}_s\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_s)\big(1+o(1)\big)\,d\theta \\ &= \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_t)\big(1+o(1)\big).\end{aligned} \tag{134}$$

This proves part (a) of Theorem 2.

4. Proof of Theorem 3

As in the proof of Theorem 2, we give the detail in part (b), and then show that a simplified argument works for part (a). Also for notational simplicity we only consider the 2-correlation function. The generalization to m-correlation function is straightforward.

4.1. *Correlation functions for the bulk particles*: $q = e^{-c/n}$

We assume the contour in (20) is

$$\Gamma = \left\{|z| = q^{-n} - 1 + \frac{\delta_n}{n} = e^c - 1 + \frac{\delta_n}{n}\right\}, \tag{135}$$

such that $|\delta_n| < 1$ and there exists $\epsilon(q) > 0$ independent of n and

$$\big|1 - q^k\big(q^{-n} - 1 + \delta_n/n\big)\big| > \epsilon(q)/n \tag{136}$$

for all $k \ge 0$. The term δ_n in the definition of Γ keeps Γ away from poles at $-q^{-k}$. For notational simplicity, we assume $\delta_n = 0$ later in this section.

We compute the asymptotics of $F(z)$ and $K(x,y;z;q)$ separately, and then prove Theorem 2(b).

For the asymptotics of $F(z)$, we have the following estimate:

Lemma 5. *Let $\epsilon > 0$ be a small constant independent of n.*

(a) *If $z \in \Gamma$ and $|z - (e^c - 1)| < \epsilon$, then there exist $\delta > 0$ and $C > 0$ such that*

$$\big|F(z)\big| < C\frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c-1)}}\exp\big(-n\delta\big|z - \big(e^c - 1\big)\big|^2\big), \tag{137}$$

and if $|z - (e^c - 1)| < n^{-2/5}$, then

$$\frac{F(z)}{e^c - 1} = \frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c-1)}}\exp\left(\frac{n(z-(e^c-1))^2}{2ce^c(e^c-1)}\right)\big(1 + \mathcal{O}\big(n^{-1/5}\big)\big). \tag{138}$$

(b) *If $z \in \Gamma$ and $|z - (e^c - 1)| \ge \epsilon$, then there exists $\delta > 0$ such that for large enough n,*

$$\big|F(z)\big| < e^{-\delta n}. \tag{139}$$

Proof. We write

$$\frac{1}{n}\log F(z)=\frac{1}{n}\log\bigl(q^{-n(n-1)/2}(q;q)_n\bigr)-\log z+\int_0^\infty \log\bigl(1+e^{-c\lfloor nx\rfloor/n}z\bigr)\,dx. \tag{140}$$

Unless z is very close to the negative real line, $n^{-1}\log F(z)$ is approximated by

$$\frac{1}{n}\log F(z)=G_n(z)+\mathcal{O}\bigl(n^{-1}\bigr)\quad \text{if } \arg z\in\bigl(-\pi+\epsilon',\pi-\epsilon'\bigr), \tag{141}$$

where ϵ' is any positive constant and

$$G_n(z)=\frac{1}{n}\log\bigl(q^{-n(n-1)/2}(q;q)_n\bigr)-\log z+\int_0^\infty \log\bigl(1+e^{-cx}z\bigr)\,dx. \tag{142}$$

Hence by differentiation, we have that for $|z|=e^c-1$ and $\arg z\in(-\pi+\epsilon',\pi-\epsilon')$,

$$\frac{1}{n}\frac{d}{dz}\log F(z)=G_n'(z)+\mathcal{O}\bigl(n^{-1}\bigr)=-\frac{1}{z}+\int_0^\infty \frac{e^{-cx}}{1+ze^{-cx}}\,dx+\mathcal{O}\bigl(n^{-1}\bigr), \tag{143}$$

$$\frac{1}{n}\frac{d^2}{dz^2}\log F(z)=G_n''(z)+\mathcal{O}\bigl(n^{-1}\bigr)=\frac{1}{z^2}-\int_0^\infty \frac{e^{-2cx}}{(1+ze^{-cx})^2}\,dx+\mathcal{O}\bigl(n^{-1}\bigr), \tag{144}$$

and furthermore

$$\frac{1}{n}\frac{d}{dz}\log F(z)\bigg|_{z=e^c-1}=\mathcal{O}\bigl(n^{-1}\bigr),\qquad \frac{1}{n}\frac{d^2}{dz^2}\log F(z)\bigg|_{z=e^c-1}=\frac{1}{c}\frac{1}{e^c(e^c-1)}+\mathcal{O}\bigl(n^{-1}\bigr). \tag{145}$$

Hence $z=e^c-1$ is a saddle point for $\log F(z)$, and as z moves away from the saddle point e^c-1, $|F(z)|$ decreases rapidly, provided that z is in the vicinity of the saddle point. Actually, for z on Γ but not in the vicinity of e^c-1, note that $|z^{-n-1}|$ is a constant for $z\in\Gamma$ while $|1+q^kz|$ decreases as $\arg z$ changes from 0 to $\pm\pi$, $|F(z)|$ decreases as $\arg z$ changes from 0 to $\pm\pi$.

The remaining task is to evaluate $F(e^c-1)$ as $n\to\infty$. Although a direct computation is possible, it is difficult due to the evaluation of $(q;q)_n$ with q close to 1. Instead, we take an indirect approach.

In the gap probability formula (17), if we take $A=\mathbb{R}$, we have that the probability on the left-hand side is 1, and Fredholm determinant on the right-hand side is trivially 1, so we have

$$\frac{1}{2\pi i}\oint_\Gamma F(z)\frac{dz}{z}=1. \tag{146}$$

By the asymptotic properties of $F(z)$ discussed above, we apply the steepest-descent analysis, and have that

$$\frac{1}{2\pi i}\int_{-\infty\cdot i}^{\infty\cdot i} F\bigl(e^c-1\bigr)e^{\frac{w^2}{2ce^c(e^c-1)}}\frac{dw}{\sqrt{n}(e^c-1)}=1+\mathcal{O}\bigl(n^{-1}\bigr), \tag{147}$$

and then

$$\frac{F(e^c-1)}{e^c-1}=\frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c-1)}}\bigl(1+\mathcal{O}\bigl(n^{-1}\bigr)\bigr). \tag{148}$$

Hence the lemma is proved. $\square$

We compute the asymptotics of $K(x_1,x_2;z;q)$ with the scaling

$$x_1=2x\sqrt{n}+\frac{\pi\xi}{\sqrt{n/c}},\qquad x_2=2x\sqrt{n}+\frac{\pi\eta}{\sqrt{n/c}}, \tag{149}$$

where $x\in\mathbb{R}$ is fixed and ξ, η in a compact subset of $\mathbb{R}$. The result we need is as follows.

Lemma 6. *Let $\epsilon>0$ be a small constant independent of n. In both parts of the lemma we assume $q=e^{-c/n}$ and x_1, x_2 are as in* (149).

(a) *If* $z \in \Gamma$ *and* $|z-(e^c-1)|<\epsilon$, *then*

$$\lim_{n\to\infty}\frac{\sqrt{c}}{\sqrt{n}}K(x_1,x_2;z;q)=K_{\text{inter}}(\xi,\eta;x;c;z), \tag{150}$$

where

$$K_{\text{inter}}(\xi,\eta;x;c;z)=\frac{1}{\pi}\int_0^\infty\frac{z}{e^{u^2}e^{cx^2}+z}\cos\bigl(\pi u(\xi-\eta)\bigr)\,du. \tag{151}$$

(b) *If* $z \in \Gamma$ *and* $|z-(e^c-1)|\geq\epsilon$, *then there exists* $C>0$ *such that for large enough* n,

$$\bigl|K(x_1,x_2;z;q)\bigr|<Cn^2. \tag{152}$$

Here we note that

$$K_{\text{inter}}\bigl(\xi,\eta;x;c;e^c-1\bigr)=K_{\text{inter}}\left(\xi,\eta;\frac{e^{cx^2}}{e^c-1}\right). \tag{153}$$

Proof of Lemma 6(a). We concentrate on the case $x>0$. The argument for the $x<0$ case is the same, since φ_k are even or odd functions, depending on the parity of k. The case $x=0$ requires some modification, and we discuss it in Remark 2.

Recall that $K(x_1,x_2;z;q)$ is an infinite linear combination of $\varphi_k(x_1)\varphi_k(x_2)$ with $k\geq 0$. Let $\epsilon>0$ be a small constant. Then we divide $K(x_1,x_2;z;q)$ into four parts as follows:

$$K^{\text{sup}}(x_1;x_2;z;q)=\sum_{k>n(x^2+\epsilon)}^{\infty}\frac{q^kz}{1+q^kz}\varphi_k(x_1)\varphi_k(x_2), \tag{154}$$

$$K^{\text{mid}}(x_1;x_2;z;q)=\sum_{n(x^2-\epsilon)<k\leq n(x^2+\epsilon)}\frac{q^kz}{1+q^kz}\varphi_k(x_1)\varphi_k(x_2), \tag{155}$$

$$K^{\text{sub}}(x_1;x_2;z;q)=\sum_{n\epsilon<k\leq n(x^2-\epsilon)}\frac{q^kz}{1+q^kz}\varphi_k(x_1)\varphi_k(x_2), \tag{156}$$

$$K^{\text{res}}(x_1;x_2;z;q)=\sum_{0\leq k\leq n\epsilon}\frac{q^kz}{1+q^kz}\varphi_k(x_1)\varphi_k(x_2). \tag{157}$$

Below we show that as $n\to\infty$, for all small enough $\epsilon>0$, there exists $C>0$ that is independent of ϵ, such that

$$\left|\frac{\sqrt{c}}{\sqrt{n}}K^{\text{sup}}(x_1,x_2;z;q)-\frac{1}{\pi}\int_{\sqrt{\epsilon c}}^\infty\frac{z}{e^{u^2}e^{cx^2}+z}\cos\bigl(\pi u(\xi-\eta)\bigr)\,du\right|<C\sqrt{\epsilon}, \tag{158}$$

$$\begin{aligned}&\left|\frac{\sqrt{c}}{\sqrt{n}}K^{\text{mid}}(x_1,x_2;z;q)\right|<C\sqrt{\epsilon},\\ &\left|\frac{\sqrt{c}}{\sqrt{n}}K^{\text{sub}}(x_1,x_2;z;q)\right|=o(1),\\ &\left|\frac{\sqrt{c}}{\sqrt{n}}K^{\text{res}}(x_1,x_2;z;q)\right|=o(1).\end{aligned} \tag{159}$$

By taking $\epsilon\to 0$ in the inequalities above, we prove (150). Below we prove the four results. For notational simplicity, when we prove the three estimates in (159), we only consider the case that $x_1=x_2=2x\sqrt{n}$.

First we prove (158). By [31, Formula 8.22.12], for $k>n(x^2+\epsilon)$, we have

$$\begin{aligned}\sqrt{\pi}k^{1/4}\varphi_k(2x\sqrt{n})&=\sin(\phi_k)^{-1/2}\sin\left[\frac{2k+1}{4}\bigl(\sin(2\phi_k)-2\phi_k\bigr)+\frac{3\pi}{4}\right]+\mathcal{O}\bigl(n^{-1}\bigr)\\ &=\left(1-\frac{x^2n}{k+\frac{1}{2}}\right)^{-1/4}\sin\bigl[-(2k+1)\theta_k\bigr]+\mathcal{O}\bigl(n^{-1}\bigr),\end{aligned} \tag{160}$$

where

$$\phi_k = \arccos\left(x\sqrt{\frac{n}{k+1/2}}\right), \qquad \theta_k = -(2k+1)\int_{x\sqrt{\frac{n}{k+1/2}}}^{1} \sqrt{1-t^2}\,dt + \frac{3\pi}{4}. \tag{161}$$

If x_1 is as specified in (149), then

$$\sqrt{\pi}k^{1/4}\varphi_k(x_1) = \left(1 - \frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/4} \sin\left(\theta_k + x\pi\xi\sqrt{c}\sqrt{\frac{k+1/2}{x^2 n} - 1}\right) + \mathcal{O}\left(n^{-1}\right), \tag{162}$$

and also have an analogous formula for $\varphi_k(x_2)$ with x_2 specified in (149). Then we have

$$\begin{aligned}
&\pi k^{1/2}\varphi_k(x_1)\varphi_k(x_2) \\
&\quad = \left(1 - \frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2} \\
&\qquad \times \frac{1}{2}\left[\cos\left(\pi\sqrt{c}(\xi-\eta)\sqrt{\frac{k+1/2}{n} - x^2}\right) - \cos\left(2\theta_k + \pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n} - x^2}\right)\right] + \mathcal{O}\left(n^{-1}\right).
\end{aligned} \tag{163}$$

Now we define

$$K^{\sup,1}(x_1, x_2; z; q) = \sum_{k>n(x^2+\epsilon)}^{\infty} \frac{q^k z}{1+q^k z}\frac{k^{-1/2}}{2\pi}\left(1 - \frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2} \cos\left(\pi\sqrt{c}(\xi-\eta)\sqrt{\frac{k+1/2}{n} - x^2}\right), \tag{164}$$

and

$$\begin{aligned}
&K^{\sup,2}(x_1, x_2; z; q) \\
&\quad = K^{\sup}(x_1, x_2; z; q) - K^{\sup,1}(x_1, x_2; z; q) \\
&\quad = \sum_{k>n(x+\epsilon)}^{\infty} \frac{q^k z}{1+q^k z}\left[\frac{k^{-1/2}}{2\pi}\left(1 - \frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2} \cos\left(2\theta_k + \pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n} - x^2}\right) + \mathcal{O}(1)\right].
\end{aligned} \tag{165}$$

It is not hard to see that if $\arg(z) \in (-\pi+\epsilon', \pi-\epsilon')$ for $\epsilon' > 0$, then

$$\begin{aligned}
&\frac{\sqrt{c}}{\sqrt{n}}K^{\sup,1}(x_1, x_2; z; q) \\
&\quad = \frac{\sqrt{c}}{2\pi}\int_{x^2+\epsilon}^{\infty} \frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\frac{1}{\sqrt{\kappa}}\left(1 - \frac{x^2}{\kappa}\right)^{-1/2} \cos\left(\pi\sqrt{c}\sqrt{\kappa - x^2}(\xi-\eta)\right)d\kappa + \mathcal{O}\left(n^{-1}\right) \\
&\quad = \frac{\sqrt{c}}{2\pi}\int_{\epsilon}^{\infty} \frac{z}{e^{ct}e^{cx^2}+z}\cos\left(\pi\sqrt{tc}(\xi-\eta)\right)\frac{dt}{\sqrt{t}} + \mathcal{O}\left(n^{-1}\right) \\
&\quad = \frac{1}{\pi}\int_{\sqrt{\epsilon c}}^{\infty} \frac{z}{e^{u^2}e^{cx^2}+z}\cos\left(\pi u(\xi-\eta)\right)du + \mathcal{O}\left(n^{-1}\right).
\end{aligned} \tag{166}$$

On the other hand, we need to show that

$$\left|\frac{c}{\sqrt{n}}K^{\sup,2}(x_1, x_2; z; q)\right| = o(1). \tag{167}$$

Since $q^k z/(1+q^k z) = \mathcal{O}(e^{-ck/n})$, although $K^{\sup,2}$ is defined by an infinite sum in (165), it suffices to show that for any $\epsilon > 0$ and $N > x$, as $n \to \infty$,

$$\sum_{n(x+\epsilon)<k<nN} \frac{q^k z}{1+q^k z}\frac{k^{-1/2}}{2\pi}\left(1 - \frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2} \cos\left(2\theta_k + \pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n} - x^2}\right) = o(\sqrt{n}). \tag{168}$$

We note that if we sum up the absolute values of the terms in (168), the result is $\mathcal{O}(\sqrt{n})$. For any $k > n(x^2+\epsilon)$,

$$\theta_k - \theta_{k-1} = \arcsin(x\sqrt{n/k}) - \frac{\pi}{2} + \mathcal{O}(n^{-1}), \tag{169}$$

hence the terms in (168) has cancellations. It is not hard to see that the cancellations make the left-hand side of (168) to be $o(\sqrt{n})$.

The approximations (166) and (167) imply (158).

Next we prove the estimates (159) in the special case $x_1 = x_2 = 2x\sqrt{n}$. The analysis is nearly identical for general ξ and η.

To prove the first estimate, we use the approximation formula (71). For $n(x^2-\epsilon) < k \le n(x^2+\epsilon)$, and x in a compact subset of $(-1,+\infty)$,

$$k^{1/12}\varphi_k(2\sqrt{k+1/2}x) = 2^{1/6}\left(\frac{\zeta(x)}{x^2-1}\right)^{1/4}\mathrm{Ai}\big((2k+1)^{2/3}\zeta(x)\big) + \mathcal{O}(n^{-1}). \tag{170}$$

Hence we have

$$k^{1/12}\varphi_k(2x\sqrt{n}) = 2^{1/6}\left(\frac{\zeta(x_k)}{x_k^2-1}\right)^{1/4}\mathrm{Ai}\big((2k+1)^{2/3}\zeta(x_k)\big) + \mathcal{O}(n^{-1}), \quad \text{where } x_k = \sqrt{\frac{x^2 n}{k+1/2}}. \tag{171}$$

Hence using the estimate (74) of Airy function, we have that if $\arg z \in (-\pi+\delta, \pi-\delta)$ and n is large enough, the first inequality of (159) is proved by the estimate

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\left|K^{\mathrm{mid}}(2x\sqrt{n}, 2x\sqrt{n}; z)\right| \\
&= \left(\frac{2}{n}\right)^{1/3}\left|\int_{x^2-\epsilon}^{x^2+\epsilon}\frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\left(\frac{\zeta(\frac{x}{\sqrt{\kappa}})}{\frac{x^2}{\kappa}-1}\right)^{1/2}\mathrm{Ai}\left((2n)^{2/3}\zeta\left(\frac{x}{\sqrt{\kappa}}\right)\right)^2 d\kappa + \mathcal{O}(n^{-1})\right| \\
&\le \left(\frac{2}{n}\right)^{1/3}\int_{x^2-\epsilon}^{x^2+\epsilon}\left|\frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\right| 2^{1/3}\left(\frac{\zeta(\frac{x}{\sqrt{\kappa}})}{\frac{x^2}{\kappa}-1}\right)^{1/2} f\left((2n)^{2/3}\zeta\left(\frac{x}{\sqrt{\kappa}}\right)\right)^2 d\kappa \\
&\le C\sqrt{\epsilon},
\end{aligned} \tag{172}$$

where f is defined in (74), $\zeta(x)$ has the behavior close to 1 given in (72), and $C > 0$ is independent of n and ϵ.

To prove the second estimate, By [31, Formula 8.22.13], for $n\epsilon < k \le n(x^2-\epsilon)$, we have

$$\begin{aligned}
\sqrt{\pi}k^{1/4}\varphi_k(2x\sqrt{n}) &= \frac{1}{2}\sinh(\phi_k)^{-1/2}\exp\left[\frac{2k+1}{4}\big(2\phi_k - \sinh(2\phi_k)\big)\right]\big(1+\mathcal{O}(n^{-1})\big) \\
&= \frac{1}{2}\left(\frac{x^2 n}{k+\frac{1}{2}} - 1\right)^{-1/4}\exp\left[-(2k+1)\int_1^{x\sqrt{\frac{n}{k+1/2}}}\sqrt{t^2-1}\,dt\right]\big(1+\mathcal{O}(n^{-1})\big),
\end{aligned} \tag{173}$$

where

$$\phi_k = \operatorname{arccosh}\left(x\sqrt{\frac{n}{k+1/2}}\right). \tag{174}$$

It is clear that

$$\left|\varphi_k(2x\sqrt{n})\right| < e^{-\epsilon' n} \tag{175}$$

for all $n\epsilon < k \le n(x^2-\epsilon)$, where $\epsilon' > 0$ is a constant depending on ϵ and c. This estimate implies the second inequality of (159) with $x_1 = x_2 = 2x\sqrt{n}$.

Finally, by the estimate of Hermite polynomials provided in [25, Section 11.4, Exercises 4.2 and 4.3], we have that

$$\left|\varphi_k(2x\sqrt{n})\right| < e^{-\epsilon'' n} \tag{176}$$

for all $k \le n\epsilon$, where $\epsilon'' > 0$ depends on c only, provided that ϵ is small enough. This estimate implies the last inequality of (159) with $x = y = 2x\sqrt{n}$. Here we note that the result in [25, Section 11.4, Exercises 4.2 and 4.3] is valid even for very small k, like $k = 1, 2, \dots$, except for $k = 0$. But it is obvious that when $k = 0$, (176) holds. □

Remark 2. The case $x = 0$ is different, because K^{sub} is not longer meaningful, and K^{mid} and K^{res} need to be combined. The asymptotic analysis becomes easier, since $\varphi_k(\xi/\sqrt{n})$ has limiting formulas simpler than (160), (162), and (173), see [1, 22.15.3–4]. We omit the detail, because a similar computation is done in [16, Proof of Theorem 1.9].

Proof of Lemma 6(b). The difficulty is that when $\arg z$ is close to $\pm\pi$, the denominator $1 + q^k z$ appearing in $K(x_1, x_2; z; q)$ can be close to zero. But since $|z| = q^{-n} - 1 + \delta_n/n = e^c - 1 + \delta_n/n$, and we assume that $\delta_n = 0$, for $k \ge n$

$$\left|1 + q^k z\right| \ge 1 - q^n\left(q^{-n} - 1\right) \ge q^n = e^{-c} > 0, \tag{177}$$

and the denominator is not close to zero. Then by the estimates that we use in the proof of part (a), we have for all $z \in \Gamma$,

$$\sum_{k \ge n}^{\infty} \frac{q^k z}{1 + q^k z}\varphi_k(x_1)\varphi_k(x_2) = \mathcal{O}\left(n^{1/2}\right). \tag{178}$$

On the other hand, for $k < n$, by assumption (136) we have $|1 + q^k z| \ge \epsilon(q)/n$, and then by the uniform boundedness of the Hermite functions,

$$\left|\sum_{k=0}^{n} \frac{q^k z}{1 + q^k z}\varphi_k(x_1)\varphi_k(x_2)\right| < \epsilon(q)n^2. \tag{179}$$

The combination of (178) and (179) implies (152), and then finish the proof. □

Proof of Theorem 3(b) for 2-correlation function. Using the estimates in Lemmas 5 and 6, we have that the integral in (20) concentrates in the vicinity of the saddle point $z = e^c - 1$, and more precisely, in the region $|z - (e^c - 1)| = \mathcal{O}(n^{-1/2})$. A straightforward application of the Laplace method yields that if x_1, x_2 depend on ξ, η as in (149), using (147),

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{\pi}{\sqrt{n/c}}\right)^2 R_n^{(2)}(x_1, x_2) \\
&= \lim_{n\to\infty} \frac{1}{2\pi i}\oint_0 F(z)\begin{vmatrix} \frac{\pi\sqrt{c}}{\sqrt{n}}K(x_1, x_2; z) & \frac{\pi\sqrt{c}}{\sqrt{n}}K(x_1, x_2; z) \\ \frac{\pi\sqrt{c}}{\sqrt{n}}K(x_2, x_1; z) & \frac{\pi\sqrt{c}}{\sqrt{n}}K(x_2, x_2; z)\end{vmatrix}\frac{dz}{z} \\
&= \lim_{n\to\infty} \frac{1}{2\pi i}\int_{-\infty\cdot i}^{\infty\cdot i} \frac{F(e^c - 1)}{e^c - 1}e^{\frac{w^2}{2ce^c(e^c-1)}}\begin{vmatrix} K_{\mathrm{inter}}(\xi, \xi; c; x; e^c - 1) & K_{\mathrm{inter}}(\xi, \eta; x; c; e^c - 1) \\ K_{\mathrm{inter}}(\eta, \xi; c; x; e^c - 1) & K_{\mathrm{inter}}(\eta, \eta; x; c; e^c - 1)\end{vmatrix}\frac{dw}{\sqrt{n}} \\
&= \begin{vmatrix} K_{\mathrm{inter}}(\xi, \xi; x; c; e^c - 1) & K_{\mathrm{inter}}(\xi, \eta; x; c; e^c - 1) \\ K_{\mathrm{inter}}(\eta, \xi; x; c; e^c - 1) & K_{\mathrm{inter}}(\eta, \eta; x; c; e^c - 1)\end{vmatrix}.
\end{aligned} \tag{180}$$

By (153) we prove the 2-correlation function formula in Theorem 3(b). □

4.2. *Correlation functions for the bulk particles: Fixed $q \in (0, 1)$*

We let q be in a compact subset of $(0, 1)$. We assume that the contour in (20) is $|z| = q^{-n+1/2}$, and take the change of variable like in (60)

$$w = q^n z \quad \text{with } |w| = \sqrt{q}. \tag{181}$$

Then analogous to (65), we write the $m=2$ case of (20) as

$$\begin{aligned}
R_n^{(2)}(x_1,x_2) &= \frac{q^{n/2}}{Z_n(q)} q^{n^2} q^{-\frac{n(n+1)}{2}} \frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\prod_{k=0}^{\infty}(1+q^k w)\right)\left(\prod_{k=1}^{n}(1+q^k w^{-1})\right) \\
&\quad\times \begin{vmatrix} K(x_1,x_1;q^{-n}w;q) & K(x_1,x_2;q^{-n}w;q) \\ K(x_2,x_1;q^{-n}w;q) & K(x_2,x_2;q^{-n}w;q)\end{vmatrix} \\
&= \frac{(q;q)_n}{(q;q)_\infty}\frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}} w^k\right)\frac{(-q/w;q)_n}{(-q/w;q)_\infty} \\
&\quad\times \begin{vmatrix} K(x_1,x_1;q^{-n}w;q) & K(x_1,x_2;q^{-n}w;q) \\ K(x_2,x_1;q^{-n}w;q) & K(x_2,x_2;q^{-n}w;q)\end{vmatrix},
\end{aligned} \tag{182}$$

where we make use of identity (64). Next we find the asymptotics of $K(x_i,x_j;q^{-n}w;q)$. We write

$$K(x_i,x_j;q^{-n}w;q) = K_n^{(0)}(x_i,x_j) + K^{(1)}(x_i,x_j;q^{-n}w;q) - K^{(2)}(x_i,x_j;q^{-n}w;q), \tag{183}$$

where

$$K^{(0)}(x_i,x_j) = \left(\sum_{k=0}^{n-1}\varphi_k(x_i)\varphi_k(x_j)\right), \tag{184}$$

$$K^{(1)}(x_i,x_j;q^{-n}w;q) = \left(\sum_{k=0}^{\infty}\frac{q^k w}{1+q^k w}\varphi_{n+k}(x_i)\varphi_{n+k}(x_j)\right), \tag{185}$$

$$K^{(2)}(x_i,x_j;q^{-n}w;q) = \left(\sum_{k=1}^{n}\frac{q^k w^{-1}}{1+q^k w^{-1}}\varphi_{n-k}(x_i)\varphi_{n-k}(x_j)\right). \tag{186}$$

It is well known that $K_n^{(0)}(x_i,x_j)$ is the correlation kernel of n-dimensional GUE random matrix, and for

$$x_i = 2\sqrt{n}x + \frac{\pi\xi_i}{(1-x^2)^{1/2}\sqrt{n}}, \qquad x_j = 2\sqrt{n}x + \frac{\pi\xi_j}{(1-x^2)^{1/2}\sqrt{n}}, \quad \text{where } x\in(-1,1), \tag{187}$$

we have [3, Chapter 3]

$$\lim_{n\to\infty}\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}K_n^{(0)}(x_i,x_j) = K_{\sin}(\xi_i,\xi_j) := \frac{\sin(\pi(\xi_i-\xi_j))}{\pi(\xi_i-\xi_j)}. \tag{188}$$

To estimate $K^{(1)}$ and $K^{(2)}$, it suffices to use the rough estimate from [1, 22.14.17], we have $|\varphi_n(x)| \le \frac{\kappa}{2^{1/4}\pi^{1/4}}$, where $\kappa \approx 1.086435$. Then we have

$$\left|K^{(1)}(x_i,x_j;q^{-n}w;q)\right| \le \sum_{k=0}^{\infty}\left|\frac{q^k w}{1+q^k w}\right|\frac{\kappa^2}{\sqrt{2\pi}} < \sum_{k=0}^{\infty}\frac{q^k}{1-\sqrt{q}}\frac{\kappa^2}{\sqrt{2\pi}} < \frac{1}{(1-q)(1-\sqrt{q})}. \tag{189}$$

Similarly, we also have

$$\left|K^{(2)}(x_i,x_j;q^{-n}w;q)\right| < \frac{1}{(1-q)(1-\sqrt{q})}. \tag{190}$$

Hence we have that uniformly in w on the circle $|w|=\sqrt{q}$

$$\lim_{n\to\infty}\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}K(x_i,x_j;q^{-n}w;q) = K_{\sin}(\xi_i,\xi_j). \tag{191}$$

Using the very fast convergence (132), we have

$$\lim_{n\to\infty}\left(\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}\right)^2 R_n^{(2)}(x_1,x_2) = \frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}} w^k\right)\bigl(1+o(1)\bigr)\det\bigl(K_{\sin}(\xi_i,\xi_j)\bigr)_{i,j=1}^2$$
$$= \det\bigl(K_{\sin}(\xi_i,\xi_j)\bigr)_{i,j=1}^2. \tag{192}$$

Hence Theorem 3(b) is proved for the 2-correlation function case.

Remark 3. The argument in this section also occurs in [17, Proposition 3.7].

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