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# ASEP$(q, j)$ converges to the KPZ equation

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**Abstract.** We show that a generalized Asymmetric Exclusion Process called ASEP$(q, j)$ introduced in (*Probab. Theory Related Fields* **166** (2016) 887–933). converges to the Cole–Hopf solution to the KPZ equation under weak asymmetry scaling.

**Résumé.** Nous montrons qu'une généralisation du processus d'exclusion asymétrique appelée ASEP$(q, j)$, introduite dans (*Probab. Theory Related Fields* **166** (2016) 887–933), converge sous faible asymétrie vers la solution de l'équation KPZ au sens de Cole–Hopf.

*MSC:* 60H15; 35R60; 82C22

*Keywords:* ASEP$(q, j)$; Gärtner transformation; KPZ equation

## 1. Introduction

In this paper we study the generalized Asymmetric Exclusion Process called ASEP$(q, j)$ introduced in [6], and show that under the weak asymmetry scaling, it converges to the Cole–Hopf solution to the KPZ equation:

$$\partial_T H = \frac{1}{2}\Delta H + \frac{1}{2}(\partial_X H)^2 + \dot{W}, \tag{1.1}$$

where $\dot{W}$ is the space-time white noise: formally, $\mathbf{E}(\dot{W}_T(X)\dot{W}_S(X')) = \delta(T-S)\delta(X-X')$. Here the Cole–Hopf solution is defined by $H_T(X) = \log \mathscr{Z}_T(X)$ where $\mathscr{Z} \in C([0,\infty), C(\mathbf{R}))$ is the mild solution (see (1.12) below) to the stochastic heat equation (SHE)

$$\partial_T \mathscr{Z} = \frac{1}{2}\Delta \mathscr{Z} + \mathscr{Z}\dot{W}. \tag{1.2}$$

For the standard ASEP model, Bertini and Giacomin [4] proved its convergence in the weak asymmetry regime to the Cole–Hopf solution of the KPZ equation. They assumed near equilibrium initial data, and narrow wedge initial data was treated in [3]. Both of these results rely on the Gärtner transformation [11,13], which is the discrete analogue of Cole–Hopf transformation. Recently there has been a resurgence of interest in showing that a large class of one-dimensional weakly asymmetric interacting particle system (including ASEP) should all converge to the KPZ equation. Besides the work of [3,4] (and previous to the present work), the only other result of this type via Gärtner

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/ Cole–Hopf transform is due to [9], wherein they show KPZ equation convergence for a class of weakly asymmetric non-simple exclusion processes with hopping range at most 3. Another work which was posted slightly after our present article is by Labbé [25,26] who showed that in particular range of scaling regimes the fluctuations of the weakly asymmetric bridges converge to the KPZ equation, also via the method of Gärtner transform.

Another approach to proving KPZ equation limits for particle systems is via energy solutions, and many microscopic models have been shown to converge to energy solutions to the KPZ equation [10,12,14–18], see also the lecture notes [20]. Energy solutions are proved to be unique in [19]. The energy solution method currently only applies in equilibrium and one needs to know the invariant measure as well as other hydrodynamic quantities explicitly. The ASEP$(q, j)$ model considered presently does not have simple product form invariant measures, so it seems to us that the energy solution method does not apply for this model.

There are other types of systems which converge under certain weak scalings of parameters to the KPZ equation. For instance, [1,2] demonstrated KPZ convergence for the free energy of directed polymers with arbitrary disorder distributions in the intermediate disorder regime (also called weak noise scaling). Also, [8] showed that the stochastic higher-spin vertex models introduced by [7] converge to KPZ under a particular weak scaling of their parameter $q \to 1$. The paper [22] proved the convergence of the Sasamoto–Spohn type discretizations ([28]) of the KPZ/stochastic Burgers equation using paracontrolled analysis. We also mention the recent results in the continuum setting by [23] and [24] using regularity structure theory, and by [21] using energy solution in the equilibrium.

The system we focus on in this paper is the ASEP$(q, j)$ which was introduced in [6] as a generalization of ASEP which allows multiple occupancy at each site (i.e., a higher spin version of ASEP). ASEP$(q, j)$ reduces to the usual ASEP when $j = 1/2$. This class of systems was introduced through an algebraic machinery developed to construct particle systems which enjoy a certain self-duality property. The simplest case of self-duality (duality to a one-particle dual system) implies that the expectation of $q$ raised to the current of the system solves the Kolmogorov backward equation for a single particle version of the model (see Lemma 3.1 of [6]). This suggested to us that if we do not take expectations, the same observable might satisfy a discrete stochastic heat equation (SHE). Indeed, after writing this down, we are able to demonstrate such a discrete version of the Cole–Hopf a.k.a. Gärtner transform. We then employ methods similar to that of [4] to ultimately prove convergence of the continuum SHE. We also remark on a similar Gärtner transform structure for the recently introduced ASIP$(q, k)$ [5] but do not provide a proof of convergence to KPZ for that process.

### 1.1. *Definition of the model and the main results*

For $q \in (0, 1)$ and $n \in \mathbf{Z}$, the $q$-number is defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{1.3}$$

satisfying the property $\lim_{q\to 1}[n]_q = n$. We recall the following definition of ASEP$(q, j)$ from [6].

**Definition 1.1.** Fix $q \in (0, 1)$ and a half integer $j \in \mathbf{N}/2$. Let $\tilde{\eta}(x) \in \{0, 1, \dots, 2j\}$ denote the occupation variable, i.e. the number of particles, at site $x \in \mathbf{Z}$. The ASEP$(q, j)$ is a continuous-time Markov process on the state space $\{0, 1 \dots, 2j\}^{\mathbf{Z}} = \{(\tilde{\eta}(x))_{x\in\mathbf{Z}}\}$ defined by the following dynamics: at any given time $t \in [0, \infty)$, a particle jumps from site $x$ to site $x + 1$ at rate[4]

$$\tilde{c}_q^+(\tilde{\eta}, x) = \frac{1}{2[2j]_q} q^{\tilde{\eta}(x)-\tilde{\eta}(x+1)-(2j+1)}\big[\tilde{\eta}(x)\big]_q\big[2j - \tilde{\eta}(x+1)\big]_q$$

and from site $x + 1$ to site $x$ at rate

$$\tilde{c}_q^-(\tilde{\eta}, x) = \frac{1}{2[2j]_q} q^{\tilde{\eta}(x)-\tilde{\eta}(x+1)+(2j+1)}\big[2j - \tilde{\eta}(x)\big]_q\big[\tilde{\eta}(x+1)\big]_q$$

[4] A factor $\frac{1}{2[2j]_q}$ is inserted here (comparing with [6]), which is unimportant but will make the coefficient in front of the Laplacian of the heat equation $\frac{1}{2}$ for convenience, so that we can employ the standard heat kernel estimates.

independently of each other. With $[0]_q = 0$, the property $\tilde{\eta}(x) \in \{0, 1, \dots, 2j\}$ is clearly preserved by the dynamics described in the preceding, and with $\tilde{c}_q^{\pm}(\cdot, \cdot)$ being uniformly bounded, such a process is constructed by the standard procedures as in [27].

Focusing on the fluctuation around density $j$, we define the centered occupation variable $\eta(x) := \tilde{\eta}(x) - j \in \{-j, \dots, j\}$ and the corresponding jumping rate

$$
\begin{aligned}
c_q^+(\eta, x) &= \frac{1}{2[2j]_q} q^{\eta(x)-\eta(x+1)-(2j+1)} \big[j + \eta(x)\big]_q \big[j - \eta(x+1)\big]_q, \\
c_q^-(\eta, x) &= \frac{1}{2[2j]_q} q^{\eta(x)-\eta(x+1)+(2j+1)} \big[j - \eta(x)\big]_q \big[j + \eta(x+1)\big]_q.
\end{aligned}
\tag{1.4}
$$

Under these notations, the ASEP$(q, j)$ has the generator

$$
(\mathcal{L} f)(\eta) = \sum_{x \in \mathbf{Z}} (\mathcal{L}_{x,x+1} f)(\eta), \tag{1.5}
$$

where

$$
(\mathcal{L}_{x,x+1} f)(\eta) = c_q^+(\eta, x)\big(f\big(\eta^{x,x+1}\big) - f(\eta)\big) + c_q^-(\eta, x) n\big(f\big(\eta^{x+1,x}\big) - f(\eta)\big) \tag{1.6}
$$

and $\eta^{x,y}$ is the configuration obtained by moving a particle from site $x$ to site $y$.

For any function $f : \mathbf{Z} \to \mathbf{R}$, define the forward and backward discrete gradients as

$$
\nabla^+ f(x) \stackrel{\text{def}}{=} f(x+1) - f(x), \qquad \nabla^- f(x) \stackrel{\text{def}}{=} f(x-1) - f(x).
$$

Define the height function $h$ so that $\nabla^+ h(x) = \eta(x+1)$. More precisely, let $h_t(0)$ be the net flow of particles from $x = 1$ to $x = 0$ during the time interval $[0, t]$, counting *left-going* particles as positive, and

$$
h_t(x) \stackrel{\text{def}}{=} h_t(0) + \begin{cases} \sum_{0<y\le x} \eta_t(y), & \text{when } x \ge 0, \\ -\sum_{x<y\le 0} \eta_t(y), & \text{when } x < 0. \end{cases} \tag{1.7}
$$

We define the microscopic Hopf–Cole/Gärtner transform of the height function $h_t(x)$ as

$$
Z_t(x) \stackrel{\text{def}}{=} q^{-2h_t(x)+\nu t}, \tag{1.8}
$$

where the term $\nu t$ is to balance the overall linear (in time) growth of $h_t(x)$, with

$$
\nu \stackrel{\text{def}}{=} \left(\frac{[4j]_q}{2[2j]_q} - 1\right) \Big/ \ln q. \tag{1.9}
$$

We linearly interpolate $Z_t(x)$ in $x \in \mathbf{R}$ so that $Z \in D([0, \infty), C(\mathbf{R}))$, the space of $C(\mathbf{R})$-valued, right-continuous-with-left-limits processes.

Turning to our main result, we consider the weakly asymmetric scaling $q = q_\varepsilon = e^{-\sqrt{\varepsilon}}$, $\varepsilon \to 0$, whereby $\nu = \nu_\varepsilon = -2j^2\sqrt{\varepsilon} + O(\varepsilon)$. To indicate this scaling, we denote *parameters* such as $\nu$ by $\nu_\varepsilon$, but for *processes* such as $h_t(x)$ and $M_t(x)$, we often omit the dependence on $\varepsilon$ to simplify notations. Following [4], we consider the following near equilibrium initial conditions:

**Definition 1.2.** Let $\|f_t(x)\|_n \stackrel{\text{def}}{=} (\mathbf{E}|f_t(x)|^n)^{\frac{1}{n}}$ denote the $L^n$-norm. We say a sequence $\{h_0^\varepsilon(\cdot)\}_\varepsilon$ of initial conditions is near equilibrium if, for any $\alpha \in (0, \frac{1}{2})$ and every $n \in \mathbf{N}$ there exist finite constants $C$ and $a$ such that

$$
\big\|Z_0(x)\big\|_n \le C e^{a\varepsilon|x|}, \tag{1.10}
$$

$$
\big\|Z_0(x) - Z_0\big(x'\big)\big\|_n \le C\big(\varepsilon\big|x - x'\big|\big)^\alpha e^{a\varepsilon(|x|+|x'|)}. \tag{1.11}
$$

Recall that $\mathscr{Z}_T(X)$ is the solution to the SHE (1.2) starting from $\mathscr{Z}_0(\cdot) \in C(\mathbf{R})$ if

$$\mathscr{Z}_T = \mathscr{P}_T * \mathscr{Z}_0 + \int_0^T \mathscr{P}_{T-S} * (\mathscr{Z}_S n_S), \tag{1.12}$$

where $\mathscr{P}$ is the standard heat kernel, and the last integral is in Itô sense and $*$ denotes the spatial convolution. Hereafter, we endow the space $D([0,\infty), C(\mathbf{R}))$ with the Skorokhod topology and the space $C(\mathbf{R})$ with the topology of uniform convergence on compact sets, and use $\Rightarrow$ to denote weak convergence of probability laws. Write $\varepsilon_j \stackrel{\text{def}}{=} 2j\varepsilon$ and consider the scaled processes

$$\mathcal{Z}_T^\varepsilon(X) \stackrel{\text{def}}{=} Z_{\varepsilon_j^{-2}T}\big(\varepsilon_j^{-1}X\big) \in D\big([0,\infty), C(\mathbf{R})\big). \tag{1.13}$$

The following is our main theorem.

**Theorem 1.3.** *Let $\mathscr{Z}^{ic} \in C(\mathbf{R})$ and $\mathscr{Z}$ be the unique solution to SHE from $\mathscr{Z}^{ic}$. Given any near equilibrium initial conditions such that $\mathcal{Z}_0^\varepsilon \Rightarrow \mathscr{Z}^{ic}$, as $\varepsilon \to 0$, under the preceding weakly asymmetric scaling, we have that $\mathcal{Z}^\varepsilon \Rightarrow \mathscr{Z}$, as $\varepsilon \to 0$.*

Definition 1.2 (and therefore Theorem 1.3) leaves out an important initial condition, i.e. the step initial condition:

$$\eta_0(x) = j \quad \text{for } x \le 0, \quad \text{and} \quad \eta_0(x) = -j \quad \text{for } x > 0. \tag{1.14}$$

Following [3], we generalize Theorem 1.3 to the following:

**Theorem 1.4.** *Let $\mathscr{Z}^*$ be the unique solution of SHE starting from the delta measure $\delta(\cdot)$, let $\{\eta_0(x)\}_x$ the step initial condition as in (1.14), and let $\mathcal{Z}_T^{*,\varepsilon}(X) := \frac{1}{2\sqrt{\varepsilon}}\mathcal{Z}_T^\varepsilon(X)$. We have that $\mathcal{Z}^{*,\varepsilon} \Rightarrow \mathscr{Z}^*$, as $\varepsilon \to 0$.*

## 1.2. *Proof of Theorems 1.3 and 1.4*

In Section 3, we establish the following moment estimates.

**Proposition 1.5.** *Fix $\bar{T} < \infty$, $n \in \mathbf{N}$, $\alpha \in (0, 1/2)$, and some near equilibrium initial conditions as in Definition 1.2, with the corresponding finite constant $a$. Then, there exists some finite constant $C$ such that*

$$\big\|Z_t(x)\big\|_{2n} \le Ce^{a\varepsilon|x|}, \tag{1.15}$$

$$\big\|Z_t(x) - Z_t\big(x'\big)\big\|_{2n} \le C\big(\varepsilon\big|x-x'\big|\big)^\alpha e^{a\varepsilon(|x|+|x'|)}, \tag{1.16}$$

$$\big\|Z_t(x) - Z_{t'}(x)\big\|_{2n} \le C\big(1 \vee \big|t'-t\big|^{\frac{\alpha}{2}}\big)\varepsilon^\alpha e^{2a\varepsilon|x|}, \tag{1.17}$$

*for all $t, t' \in [0, \varepsilon_j^{-2}\bar{T}]$ and $x, x' \in \mathbf{R}$.*

Applying the argument as in [9, Proof of Proposition 1.4] (see also [4, Proof of Theorem 3.3]), we then have that Proposition 1.5 implies the following tightness result.

**Proposition 1.6.** *For near equilibrium initial conditions, the law of $\{\mathcal{Z}^\varepsilon\}_\varepsilon$ is tight in $D([0,\infty) \times \mathbf{R})$. Moreover, limit points of $\{\mathcal{Z}^\varepsilon\}_\varepsilon$ concentrates on $C([0,\infty) \times C(\mathbf{R}))$.*

With this, in Section 4 we prove the following proposition, which, together with the uniqueness of the SHE, completes the proof of Theorem 1.3.

**Proposition 1.7.** *For near equilibrium initial conditions, any limiting point $\mathscr{Z}$ of $\{\mathcal{Z}^\varepsilon\}_\varepsilon$ solves the SHE.*

Turning to Theorem 1.4, with $\mathcal{Z}^{\varepsilon,*}$ as in Theorem 1.4, we have that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbf{Z}} Z_0^{\varepsilon,*}(x) \to 1.$$

Combining this with the exponential decay (in $|x|$) of $Z_0^{\varepsilon,*}(x)$, one easily obtains $\mathcal{Z}_0^{\varepsilon,*}(\cdot) \Rightarrow \delta(\cdot)$. With this and Theorem 1.3, following the argument of [3, Section 3] Theorem 1.4 is an immediate consequence of the following moment estimates, which we establish in Section 3.

**Proposition 1.8.** *Let $Z_t^*(x) = \frac{1}{2\sqrt{\varepsilon}} Z_t(x)$. For the step initial condition, for any $T < \infty$, $n \geq 1$ and $\alpha \in (0, 1/2)$, there exists $C$ such that*

$$\|Z_t^*(x)\|_{2n} \leq C/\sqrt{\varepsilon^2 t}, \tag{1.18}$$

$$\|Z_t^*(x) - Z_t^*(x')\|_{2n} \leq C(\varepsilon|x - x'|)^\alpha (\varepsilon^2 t)^{-(1+\alpha)/2}, \tag{1.19}$$

*for all $t \in (0, \varepsilon_j^{-2} T]$ and $x, x' \in \mathbf{R}$.*

1.3. *Outline*

The rest of the paper is organized as follows. In Section 2 we show the crucial result that for the ASEP$(q, j)$ model, one can still achieve the discrete Hopf–Cole/Gärtner transform. In Section 3 we prove tightness of the rescaled processes as in the ASEP case in [4], but we use some of the more recent treatments in [9] which simplified the arguments of [4]. In Section 4 we identify the limit as the solution of SHE; which essentially follows the arguments of [4] but in the "key estimate" we provided a proof to the more general case of a crucial cancellation and since [4] was written twenty years ago, we make the proofs slightly more streamlined in our presentation.

## 2. Microscopic SHE

In this section we derive the microscopic Hopf–Cole/Gärtner transform of ASEP$(q, j)$, stated in the following proposition. This discrete level Hopf–Cole transformation was introduced by Gärtner [13], see also [11] by Dittrich and Gärtner.

**Proposition 2.1.**

(a) *For $Z_t(x)$ is defined as in (1.8), we have that*

$$dZ_t(x) = \frac{1}{2}\Delta Z_t(x)\,dt + dM_t(x), \tag{2.1}$$

*where $\Delta f(x) = f(x+1) + f(x-1) - 2f(x)$ denotes the standard discrete Laplacian and $M.(x)$, $x \in \mathbf{Z}$, are martingales*;

(b) *Furthermore, for the martingale term, we have*

$$\frac{d}{n}\langle M(x), M(y)\rangle_t = \mathbf{1}_{\{x=y\}}\left(\frac{4\varepsilon j^2}{[2j]_q} Z_t(x)^2 + \frac{1}{[2j]_q}\nabla^+ Z_t(x)\nabla^- Z_t(x) + o(\varepsilon) Z_t(x)^2\right), \tag{2.2}$$

*where $o(\varepsilon)$ is a term uniformly bounded by constant $C_\varepsilon$ and $C_\varepsilon/\varepsilon \to 0$.*

To simplify notations, throughout this section we *omit* the dependence of parameters (e.g. $q, \nu$) on $\varepsilon$. To prove Proposition 2.1, we note that each jump from $x$ to $x+1$ (resp. from $x+1$ to $x$) decreases (resp. increases) $h(x)$ by 1. Taking into account the factor $q^{\nu t}$ in (1.8), we obtain from (1.5) that

$$\begin{aligned} dZ_t(x) = {} & (q^2 - 1) Z_t(x) c^+(\eta, x)\,dP_t^+(x) + (q^{-2} - 1) Z_t(x) c^-(\eta, x)\,dP_t^-(x) \\ & + Z_t(x)\nu \ln q\,dt, \end{aligned} \tag{2.3}$$

where $\{P_t^+(x)\}_{x\in\mathbf{Z}}$ and $\{P_t^-(x)\}_{x\in\mathbf{Z}}$ are independent Poisson processes with unit rate. Letting $M_t^{\pm}(x) := \int_0^t (c^{\pm}(\eta(s), x)\, dP_s^{\pm}(x) - c^{\pm}(\eta(s), x)\, ds)$ denote the corresponding compensated Poisson processes, which is a martingale, we have that

$$dZ_t(x) = \Omega Z_t(x)\, dt + dM_t(x),$$

where the drift term has coefficient

$$\Omega = \big(q^2-1\big)c^+(\eta,x) + \big(q^{-2}-1\big)c^-(\eta,x) + \nu \ln q \tag{2.4}$$

and the martingales $\{M_t(x)\}_{x\in\mathbf{Z}}$ are defined as

$$M_t(x) = \int_0^t \big(\big(q^2-1\big)Z_s(x)\, dM_s^+(x) + \big(q^{-2}-1\big)Z_s(x)\, dM_s^-(x)\big). \tag{2.5}$$

**Proof Proposition 2.1(a).** With (2.3), proving (2.1) amounts to proving $\Omega Z_t(x) = \frac{1}{2}\Delta Z_t(x)$. First of all, by the definition (1.8) of $Z_t$, we clearly have (omitting the subscript $t$ for simplicity):

$$\Delta Z(x) = \big(q^{-2\eta(x+1)} + q^{2\eta(x)} - 2\big)Z(x). \tag{2.6}$$

On the other hand, by straightforward computation using the definition (2.4) of $\Omega$ and the expression (1.4) of the rates $c^{\pm}$,

$$\begin{aligned}
&2[2j]_q(\Omega - \nu \ln q) \\
&\quad = \big(q^2-1\big)q^{\eta(x)-\eta(x+1)-(2j+1)}\big[j+\eta(x)\big]_q\big[j-\eta(x+1)\big]_q \\
&\qquad + \big(q^{-2}-1\big)q^{\eta(x)-\eta(x+1)+(2j+1)}\big[j-\eta(x)\big]_q\big[j+\eta(x+1)\big]_q \\
&\quad = \big(q^2-1\big)q^{\eta(x)-\eta(x+1)-(2j+1)}\frac{q^{j+\eta(x)}-q^{-(j+\eta(x))}}{q-q^{-1}}\frac{q^{j-\eta(x+1)}-q^{-(j-\eta(x+1))}}{q-q^{-1}} \\
&\qquad + \big(q^{-2}-1\big)q^{\eta(x)-\eta(x+1)+(2j+1)}\frac{q^{j-\eta(x)}-q^{-(j-\eta(x))}}{q-q^{-1}}\frac{q^{j+\eta(x+1)}-q^{-(j+\eta(x+1))}}{q-q^{-1}} \\
&\quad = \frac{1}{q-q^{-1}}\big(\big(q^{2\eta(x)}-q^{-2j}\big)\big(q^{-2\eta(x+1)}-q^{-2j}\big) - \big(q^{2j}-q^{2\eta(x)}\big)\big(q^{2j}-q^{-2\eta(x+1)}\big)\big) \\
&\quad = \frac{1}{q-q^{-1}}\big(q^{-4j}-q^{4j}+q^{2\eta(x)+2j}-q^{2\eta(x)-2j}+q^{2j-2\eta(x+1)}-q^{-2j-2\eta(x+1)}\big) \\
&\quad = [2j]_q\big(q^{2\eta(x)}+q^{-2\eta(x+1)}-2\big) - [4j]_q + 2[2j]_q.
\end{aligned}$$

Comparing this with (2.6) one obtains

$$\Omega Z_t(x) = \frac{1}{2}\Delta Z_t(x) + \big(\nu \ln q + 1 - [4j]_q/\big(2[2j]_q\big)\big)Z_t(x).$$

With this and (1.9), the desired result $\Omega Z_t(x) = \frac{1}{2}\Delta Z_t(x)$ follows. □

**Proof of Proposition 2.1(b).** By the definition (2.5), the bracket process of $M_t$ is

$$\frac{d}{dt}\big\langle M(x), M(y)\big\rangle_t = \mathbf{1}_{\{x=y\}}\big(\big(q^2-1\big)^2 c^+(\eta_t,x) + \big(q^{-2}-1\big)^2 c^-(\eta_t,x)\big)Z_t(x)^2.$$

For ASEP($q$, $j$), substituting $c^{\pm}$ and following similar computations as above, and by independence of $M(x)$ and $M(y)$ for $x \neq y$, one has

$$\begin{aligned}
&\frac{d}{n}\langle M(x), M(y)\rangle_t \\
&= \frac{\mathbf{1}_{\{x=y\}}}{2[2j]_q} Z_t(x)^2 \\
&\times \left( \frac{q^2-1}{q-q^{-1}} \big(q^{2\eta(x)} - q^{-2j}\big)\big(q^{-2\eta(x+1)} - q^{-2j}\big) - \frac{q^{-2}-1}{q-q^{-1}} \big(q^{2j} - q^{2\eta(x)}\big)\big(q^{2j} - q^{-2\eta(x+1)}\big) \right).
\end{aligned}$$

With $q = e^{-\sqrt{\varepsilon}}$, $q^a = 1 - a\sqrt{\varepsilon} + o(\sqrt{\varepsilon})$, for any uniformly bounded variable $a$, we further obtain

$$\begin{aligned}
&\frac{d}{dt}\langle M(x), M(y)\rangle_t \\
&\quad = -\frac{2\varepsilon \mathbf{1}_{\{x=y\}}}{[2j]_q} Z_t(x)^2 \Big( \big(\eta(x)+j\big)\big(\eta(x+1)-j\big) + \big(\eta(x)-j\big)\big(\eta(x+1)+j\big) + o(1) \Big) \\
&\quad = \frac{4\varepsilon \mathbf{1}_{\{x=y\}}}{[2j]_q} Z_t(x)^2 \big( j^2 - \eta(x)\eta(x+1) + o(1) \big).
\end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned}
\nabla^+ Z_t(x) &= \big(q^{-2\eta(x+1)} - 1\big) Z_t(x) = \big(2\eta(x+1)\sqrt{\varepsilon} + o(\sqrt{\varepsilon})\big) Z_t(x), \\
\nabla^- Z_t(x) &= \big(q^{2\eta(x)} - 1\big) Z_t(x) = \big(-2\eta(x)\sqrt{\varepsilon} + o(\sqrt{\varepsilon})\big) Z_t(x),
\end{aligned}$$

from which the desired result (2.2) follows. □

A useful bound on $\frac{d}{dt}\langle M(x), M(x)\rangle_t$ is the following

**Corollary 2.2.** *For $M_t(x)$ as in Proposition 2.1, we have that*

$$\left| \frac{d}{n}\langle M(x), M(y)\rangle_t \right| \leq \mathbf{1}_{\{x=y\}} C\varepsilon Z_t(x)^2 \tag{2.8}$$

*for some finite constant $C$.*

**Proof.** This follows directly from (2.7) and the boundedness of $\eta_t(x)$. □

**Remark 2.3.** The same term $\nabla^+ Z_t(x)\nabla^- Z_t(x)$ as in (2.2) also appears in [4, Equation (3.15)]. The appearance of this term indicates that we will need to adapt the "key estimate" in [4, Lemma 4.8] to our case. Note also that if $j = \frac{1}{2}$, the coefficient of $Z_t(x)^2$ in $\frac{d}{dt}\langle M(x), M(x)\rangle_t$ is nearly $\varepsilon$, the same with [4].

We note here that an alternative approach, based on hydrodynamic limit, was adopted in [9] in place of the "key estimate" of [4]. This hydrodynamic approach, however, does not apply here for ASEP($q$, $j$), for $j > \frac{1}{2}$, due the lack of accessible invariant measures.

## 3. Tightness, proof of Propositions 1.5 and 1.8

**Lemma 3.1.** *Given any $n \in \mathbf{N}$, there exists a finite constant $C$ such that, for any deterministic function $f_s(x, x')$: $[0, \infty) \times \mathbf{Z}^2 \to \mathbf{R}$ and any $t \leq t' \in [0, \infty)$ with $t' - t \geq 1$,*

$$\left\| \int_t^{t'} \sum_{x' \in \mathbf{Z}} f_s\big(x, x'\big)\, dM_s\big(x'\big) \right\|_{2n}^2 \leq C\varepsilon \int_t^{t'} \sum_{x' \in \mathbf{Z}} \bar{f}_s\big(x, x'\big)^2 \big\| Z_s\big(x'\big)^2 \big\|_n \, ds,$$

*where* $\bar f_s(x,x') \overset{\text{def}}{=} \max_{j\in\{0,\frac13,\frac23\}} \sup_{s':\lfloor s'\rfloor_j=\lfloor s\rfloor_j} |f_{s'}(x,x')|$ *and* $\lfloor s\rfloor_j$ *denotes the largest number in* $\mathbf{Z}+j$ *that is smaller than* $s$.

**Proof.** This proof is essentially by [9, Lemma 3.1], which we adapt into our setting. Fix such $t,t'$ and let $R_{t'}(x) := \int_t^{t'} \sum_{x'\in\mathbf{Z}} f_s(x,x')\,dM_s(x')$. By the Burkholder–Davis–Gundy (BDG) inequality,

$$\big\| R_{t'}(x)^2 \big\|_n \le C \big\| \big[R_\cdot(x)\big]_{t'} \big\|_n, \tag{3.1}$$

where $[-]$ denotes the optional quadratic variation, or more explicitly

$$\big[R_\cdot(x)\big]_{t'} = \sum_{x'} \sum_{s\in\mathfrak{T}(x')} f_{s^-}\big(x,x'\big)^2\big(q^{\pm 2}-1\big)^2 Z_{s^-}\big(x'\big)^2,$$

where $\mathfrak{T}(x')$ is the set of $s\in(t,t']$ at which a jump occurs at the site $x'$, and the $\pm$ is dictated by the direction of the jump.

Next, for $j\in\{0,\frac13,\frac23\}$ we partition $(t,t']$ into $k$ subintervals $\mathcal{T}_i=(t_{i-1},t_i]$ where $t=t_0<t_1<\cdots<t_k=t'$ and $\{t_1,\dots,t_{k-1}\}=(t,t')\cap(\mathbf{Z}+j)$ are points with integer gaps. Obviously we can choose a suitable $j\in\{0,\frac13,\frac23\}$ such that $t_1-t$ and $t'-t_{k-1}$ are larger than $1/5$, and we fix this choice of $j$ in the sequel. Using $|q^{\pm2}-1|\le C\sqrt{\varepsilon}$, and replacing $f_s$ and $Z_s$ by their supremum over $\mathcal{T}_i$, we have

$$\big[R_\cdot(x)\big]_{t'} \le C\varepsilon \sum_{i=1}^k \sum_{x'} N_{\mathcal{T}_i}\big(x'\big)\bar f_{t_{i-1}}\big(x,x'\big)^2\Big(\sup_{s\in\mathcal{T}_i} Z_s\big(x'\big)^2\Big),$$

where $N_I(x')$ is the number of jumps at $x'$ during the time interval $I$. Further using

$$Z_{s_1}\big(x'\big) \le e^{2\sqrt{\varepsilon}N_{\mathcal{T}_i}(x')} Z_{s_2}\big(x'\big) \quad \forall s_1,s_2\in\mathcal{T}_i, \tag{3.2}$$

and the fact that $N_{\mathcal{T}_i}(x')$ is stochastically bounded (due to exclusion of particles) by a Poisson random variable with rate being constant times $|\mathcal{T}_i|=t_i-t_{i-1}$, one obtains:

$$\big\|\big[R_\cdot(x)\big]_{t'}\big\|_n \le C\varepsilon \sum_{i=1}^k \sum_{x'} (t_i-t_{i-1})\bar f_{t_{i-1}}\big(x,x'\big)^2 \big\|Z_{t_{i-1}}\big(x'\big)^2\big\|_n.$$

Note that the $n$th moment of a Poisson random variable with mean $\lambda$ is uniformly bounded by $\lambda^n$ only when $\lambda$ is sufficiently large, and this is the reason we chose a suitable $j$ as above. By definition of $\bar f$, one has $\bar f_{t_{i-1}}(x,x') = \bar f_s(x,x')$ for all $s$ such that $\lfloor s\rfloor_j = t_{i-1}$. By the same argument as (3.2) and the stochastic bound on $N_{\mathcal{T}_i}(x')$, one has $\|Z_{t_{i-1}}(x')^2\|_n \le C\|Z_s(x')^2\|_n$ if $\lfloor s\rfloor_j=t_{i-1}$. Therefore, we can replace the sum over $i$ in the right hand side of the above inequality by a time integral and thus obtain the desired bound as claimed in the Lemma. □

Let $R(t)$ be the continuous time random walk on $\mathbf{Z}$, starting from $x=0$, which jump symmetrically $\pm1$ step at rate $\frac12$. Let $p_t(x)=\mathbf{P}(R(t)=x)$ denote the corresponding heat kernel. We rewrite the discrete SHE (2.1) in the following integrated form:

$$Z_t = p_t * Z_0 + \int_0^t p_{t-s} * dM_s, \tag{3.3}$$

where $*$ stands for the discrete convolution: $(f*g)(x) \overset{\text{def}}{=} \sum_{x'\in\mathbf{Z}} f(x-x')g(x')$ for any $x\in\mathbf{Z}$.

**Proof of Proposition 1.5.** Let $I_1$ and $I_2$ denote the first and second terms on the RHS of (3.3), respectively.

We begin by proving (1.15). First, by [9, (A.24)] we have the following bound on the standard heat kernel

$$\big(p_t * e^{a\varepsilon|\cdot|}\big)(x) \le Ce^{a\varepsilon|x|} \quad \text{for } t\le \varepsilon^{-2}\bar T. \tag{3.4}$$

For $I_1$, by the triangle inequality we have $\|I_1(t,x)^2\|_n = \|I_1(t,x)\|_{2n}^2 \le (p_t * \|Z_0(\cdot)\|_{2n}(x))^2$. Combining this with (3.4) and (1.10), we obtain

$$\left\|I_1(t,x)^2\right\|_n \le Ce^{2a\varepsilon|x|}. \tag{3.5}$$

Turning to bounding $I_2$, we assume $t \ge 1$ and apply Lemma 3.1 with $f_s(x,x') = p_{t-s}(x-x')$ to obtain

$$\left\|I_2(t,x)^2\right\|_n \le C\varepsilon \int_0^t \bar{p}_{t-s}^2 * \left\|Z_s^2\right\|_n(x)\, ds,$$

where $\bar{p}$ is the local supremum of $p$ defined as in Lemma 3.1. By $p_t \le Cp_{t'}$ for $|t-t'| \le 1$ and the standard heat kernel estimate $p_t \le Ct^{-\frac{1}{2}}$,

$$\left\|I_2(t,x)^2\right\|_n \le C\varepsilon \int_0^t (t-s)^{-\frac{1}{2}}\big(p_{t-s} * \left\|Z_s^2\right\|_n(x)\big)\, ds, \quad \text{for } t \ge 1.$$

Combining this with (3.5) yields

$$\left\|Z_t^2(x)\right\|_n \le Ce^{2a\varepsilon|x|} + C\varepsilon \int_0^t (t-s)^{-\frac{1}{2}}\big(p_{t-s} * \left\|Z_s^2\right\|_n(x)\big)\, ds. \tag{3.6}$$

The bound (3.6) was derived for $t \ge 1$, but it in fact holds true also for $t \le 1$. This is so because, by (1.10) and (3.2) with $s_2 = 0, s_1 = t$, we already have $\|Z_t^2(x)\|_{2n} \le Ce^{2a\varepsilon|x|}$, for $t \le 1$. With this, iterating this inequality, using the semi-group property $p_s * p_{s'} = p_{s+s'}$ and (3.4), we then arrive at

$$\left\|Z_t^2(x)\right\|_n \le \left(Ce^{2a\varepsilon|x|} + \sum_{j=1}^{\infty} \frac{C^j}{j!}\left(\varepsilon \int_0^t s^{-1/2}\, ds\right)^j e^{2a\varepsilon|x|}\right).$$

With $t \le \varepsilon^{-2}\bar{T}$, the desired result (1.15) follows.

The bound (1.16) is proved analogously. Indeed,

$$\left\|I_1(t,x) - I_1\big(t,x'\big)\right\|_{2n}^2 \le \left(\sum_{\bar{x}} p_t(\bar{x})\left\|Z_0(x-\bar{x}) - Z_0\big(x'-\bar{x}\big)\right\|_{2n}\right)^2.$$

By (1.11), followed again by (3.4), the preceding expression is bounded by

$$\left(\sum_{\bar{x}} p_t(\bar{x})\big(\varepsilon\big|x-x'\big|\big)^{\alpha} e^{a\varepsilon(|x-\bar{x}|+|x'-\bar{x}|)}\right)^2 \le \big(\varepsilon\big|x-x'\big|\big)^{2\alpha} e^{2a\varepsilon(|x|+|x'|)}. \tag{3.7}$$

For $\|I_2(t,x) - I_2(t,x')\|_{2n}^2$, we apply Lemma 3.1 with $f_s(x,\bar{x}) = p_{t-s}(x'-\bar{x}) - p_{t-s}(x-\bar{x})$, use the fact that

$$\big(p_{t-s}\big(x'-\bar{x}\big) - p_{t-s}(x-\bar{x})\big)^2 \le \big|p_{t-s}\big(x'-\bar{x}\big) - p_{t-s}(x-\bar{x})\big|\big(p_{t-s}\big(x'-\bar{x}\big) + p_{t-s}(x-\bar{x})\big)$$

and use the gradient estimate for the heat kernel, for instance [9, (A.13)]:

$$\big|p_{t-s}\big(x'-\bar{x}\big) - p_{t-s}(x-\bar{x})\big| \le C\big(1 \wedge (t-s)^{-\frac{1}{2}-\frac{\alpha}{2}}\big)\big|x-x'\big|^{\alpha}.$$

The rest of the arguments follow in the same way as the proof for (1.15).

Next we prove (1.17). Without lost of generality, we assume $t < t'-1$. For $I_1$, using the semi-group properties $p_{t'} = p_{t'-t} * p_t$ and $\sum_{x_1} p_{t'-t}(x_1) = 1$ we have

$$I_1\big(t',x\big) - I_1(t,x) = \sum_{\bar{x}} p_{t'-t}(x-\bar{x})\big(I_1(t,\bar{x}) - I_1(t,x)\big). \tag{3.8}$$

By (3.7), we have $\|I_1(t,\bar{x})-I_1(t,x)\|_{2n} \le C(\varepsilon|x-\bar{x}|)^\alpha e^{a\varepsilon|x-\bar{x}|}e^{2a\varepsilon|x|}$. Using this and the estimate $\sum_x |x|^\alpha p_{t'-t}(x) \times e^{a\varepsilon|x|} \le C|t'-t|^{\frac{\alpha}{2}}$ in (3.8), one obtains the desired bound on $\|I_1(t,\bar{x})-I_1(t,x)\|_{2n}$.

Next, we write $I_2(t')-I_2(t)$ as the sum of $J_1 = \int_t^{t'} p_{t'-s} * dM_s$ and $J_2 = \int_0^t (p_{t'-s}-p_{t-s}) * dM_s$. For the term $J_1$, we apply Lemma 3.1, followed by the uniform bound (1.15) on $Z$. Regarding the function $\bar{p}^2_{t'-s}$ arising from the application of Lemma 3.1, we apply $p_{t_1} \le Cp_{t_2}$ for $|t_1-t_2| \le 1$ as above, then bound the $L^\infty$ norm of one factor $p_{t'-s}$ using [9, (A.12)], and bound the sum over space of the other factor $p_{t'-s}$ using [9, (A.24)], and finally integrate over $s$ to obtain

$$\|(J_1)^2\|_n \le C\big(\varepsilon^\alpha |t'-t|^{\frac{\alpha}{2}} e^{a\varepsilon|x|}\big)^2, \quad \forall \alpha \in (0,1/2).$$

As for $J_2$, if $t > 1$, applying Lemma 3.1, using $(p_{t'-s}-p_{t-s})^2 \le |p_{t'-s}-p_{t-s}|(p_{t'-s}+p_{t-s})$ followed by the estimate (see for instance [9, (A.10)])

$$|p_{t'}(x)-p_t(x)| \le C\big(1\wedge t^{-\frac{1}{2}-\alpha}\big)(t'-t)^\alpha$$

one obtains the desired bound $\|J_2\|^2_{2n} \le C\varepsilon^{2\alpha}|t'-t|^\alpha e^{2a\varepsilon|x|}$ for any $\alpha \in (0,\frac{1}{2})$. If $t \le 1$, we apply the BDG inequality as in (3.1), and brutally bound the number of jumps as $N_{[0,t]}(x') \le N_{[0,1]}(x')$ which is then stochastically bounded by a Poisson variable of constant rate. We then invoke the uniform bound (1.15) on $Z$, a brutal bound $|p_{t'-s}-p_{t-s}| \le C$, and then [9, (A.24)] to bound $\sum_{x'} |p_{t'-s}-p_{t-s}|(x-x')e^{2a\varepsilon|x'|}$ by $e^{2a\varepsilon|x|}$. This yields the bound $\|J_2\|^2_{2n} \le C\varepsilon^{2\alpha}e^{2a\varepsilon|x|}$ for any $\alpha \in (0,\frac{1}{2})$; by our assumption that $|t'-t| > 1$ the desired bound follows.

Combining all these bounds completes the proof of the proposition. □

**Proof of Proposition 1.8.** With $Z^*_t(x) = \frac{1}{2\sqrt{\varepsilon}} Z_t(x)$, similarly to (3.6), using Lemma 3.1 we have

$$\|(Z^*_t(x))^2\|_n \le C\big(I^*_1(t,x)\big)^2 + C\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\big(p_{t-s} * \|(Z^*_s)^2\|_n\big)(x)\,ds, \tag{3.9}$$

where $I^*_1(t,x) = \frac{1}{\sqrt{\varepsilon}}(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|})(x)$. With $p_t \le \frac{C}{\sqrt{t}}$, we have $|I^*_1(t,x)| \le \frac{C}{\sqrt{\varepsilon^2 t}}$. Using this in (3.9) yields

$$\|(Z^*_t(x))^2\|_n \le \frac{C}{\sqrt{\varepsilon^3 t}}\big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x) + C\varepsilon \int_0^t \frac{1}{\sqrt{t-s}}\big(p_{t-s} * \|(Z^*_s)^2\|_n\big)(x)\,ds. \tag{3.10}$$

Strictly speaking, since Lemma 3.1 is used the process of deriving (3.9), so far we have only derived (3.10) for $t \in [1,\varepsilon^{-2}T]$. To bridge the gap for $t < 1$, using $Z^*_0(x) \le \frac{1}{\sqrt{\varepsilon}}$ and (3.2) for $(s_1,s_2] = (0,1]$, it is easily to see that (3.10) also hold for $t \le 1$. Now, iterate (3.10) using the semi-group property $p_s * p_{s'} = p_{s+s'}$ to obtain

$$\|(Z^*_t(x))^2\|_n \le \frac{C}{\sqrt{\varepsilon^3 t}}\big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x) + \sum_{k=1}^\infty C^k I_k(\varepsilon^2 t)\big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x),$$

where $I_k(T) = \int_{\Delta_k(T)} (S_1 \cdots S_{k+1})^{-1/2} n_1 \cdots dS_k$ and $\Delta_j(T) := \{(S_1,\dots,S_{k+1}) \in (0,\infty)^{k+1} : S_1+\cdots+S_{k+1} = T\}$. With $I_{(k)}(T) = T^{(j-1)/2}\Gamma(1/2)^{k+1}/\Gamma((j+1)/2)$, we have $\sum_{k=1}^\infty C^k I_k(\varepsilon^2 t) \le C$, and consequently

$$\|(Z^*_t(x))^2\|_n \le \frac{C}{\sqrt{\varepsilon^3 t}}\big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x). \tag{3.11}$$

Further using $(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|})(x) \le \frac{C}{\sqrt{\varepsilon t}}$, we conclude the desired bound (1.18).

Turning to proving (1.19), similar to (3.10) we have

$$\|(Z^*_t(x)-Z^*_t(x'))^2\|_n \le \frac{C}{\sqrt{\varepsilon^3 t}}\big|\big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x) - \big(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|}\big)(x')\big| \tag{3.12}$$

$$+ C\big|\varepsilon(x-x')\big|^{2\alpha}\varepsilon^{1-2\alpha}\int_0^t (t-s)^{-1/2-\alpha}\big(p_{t-s} * \|(Z^*_s)^2\|_n\big)(x)\,ds. \tag{3.13}$$

Use $|p_t(x+y) - p_t(y)| \leq Ct^{-1/2-\alpha}|y|^{2\alpha}$ and $\sum_y e^{-2j|y|\sqrt{\varepsilon}} \leq C\varepsilon^{-\frac{1}{2}}$ to bound the RHS of (3.12). As for the term in (3.13), insert (3.11) into (3.13), followed by using the semi-group property $p_{t-s} * p_s = p_t$ and using $(p_t * e^{-2j\sqrt{\varepsilon}|\cdot|})(x) \leq \frac{C}{\sqrt{\varepsilon t}}$. We then obtain

$$\begin{aligned}\big\|\big(Z_t^*(x)\big)^2\big\|_n &\leq C\big(\varepsilon\big|x-x'\big|\big)^{2\alpha}\varepsilon^{-2-2\alpha}t^{-1-\alpha} + C\big|\varepsilon\big(x-x'\big)\big|^{2\alpha}\varepsilon^{-1-2\alpha}t^{-\frac{1}{2}-\alpha} \\ &= \big(\varepsilon\big|x-x'\big|\big)^{2\alpha}\big(C\big(\varepsilon^2 t\big)^{-1-\alpha} + C\big(\varepsilon^2 t\big)^{-\frac{1}{2}-\alpha}\big).\end{aligned}$$

Further using $\varepsilon^2 t \leq T$, we conclude the desired bound (1.19). □

## 4. Identifying the limit, proof of Proposition 1.7

In order to identify the limit of $\mathcal{Z}^\varepsilon$, we recall (for instance [4, Proposition 4.11]) that the mild solution $\mathscr{Z}$ to (1.2) with initial condition $\mathscr{Z}^{\mathrm{ic}}$ is equivalent to the *unique* solution of the martingale problem with initial condition $\mathscr{Z}^{\mathrm{ic}}$, provided that $\|\mathscr{Z}^{\mathrm{ic}}(X)\|_2 \leq Ce^{a|X|}$ for some $C, a > 0$. Also recall that a $C(\mathbf{R}_+, C(\mathbf{R}))$ valued process $Z$ is said to *solve the martingale problem* with initial condition $\mathscr{Z}^{\mathrm{ic}}$ if $Z_0 = \mathscr{Z}^{\mathrm{ic}}$ in distribution, and for all $\bar{T} > 0$, there exists $a \geq 0$ such that

$$\sup_{T\in[0,\bar{T}]}\sup_{X\in\mathbf{R}} e^{-a|X|}\mathbf{E}\big(Z_T(X)^2\big) < \infty \tag{4.1}$$

and for all $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$,

$$N_T(\varphi) \stackrel{\mathrm{def}}{=} (Z_T, \varphi) - (Z_0, \varphi) - \frac{1}{2}\int_0^T \big(Z_S, \varphi''\big)\,dS, \tag{4.2}$$

$$\Lambda_T(\varphi) \stackrel{\mathrm{def}}{=} N_T(\varphi)^2 - \int_0^T \big(Z_S^2, \varphi^2\big)\,dS \tag{4.3}$$

are local martingales. Here, $(\varphi, \psi) \stackrel{\mathrm{def}}{=} \int_{\mathbf{R}} \varphi(X)\psi(X)\,dX$.

**Proof of Proposition 1.7.** By (1.15), any limit point of the family $\mathcal{Z}^\varepsilon$ satisfies (4.1). Since $\mathcal{Z}_0^\varepsilon \Rightarrow \mathscr{Z}^{\mathrm{ic}}$, the initial condition of the martingale problem is also satisfied for any limit point.

Define for all $t \in [0, \varepsilon^{-2}\bar{T}]$, $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$

$$(Z_t, \varphi)_\varepsilon \stackrel{\mathrm{def}}{=} \varepsilon_j \sum_{x\in\mathbf{Z}} \varphi(\varepsilon_j x) Z_t(x).$$

Recall that $\varepsilon_j$ was introduced in (1.13) as $\varepsilon_j = 2j\varepsilon$.

Consider the microscopic analogs of (4.2)–(4.3) as

$$N_T^\varepsilon(\varphi) \stackrel{\mathrm{def}}{=} (Z_{\varepsilon_j^{-2}T}, \varphi)_\varepsilon - (Z_0, \varphi)_\varepsilon - \frac{1}{2}\int_0^{\varepsilon_j^{-2}T} (\Delta Z_s, \varphi)_\varepsilon\,ds, \qquad \Lambda_T^\varepsilon(\varphi) \stackrel{\mathrm{def}}{=} N_T^\varepsilon(\varphi)^2 - \big\langle N_T^\varepsilon(\varphi)\big\rangle. \tag{4.4}$$

Indeed, by Proposition 2.1, $N_T^\varepsilon(\varphi)$ and hence $\Lambda_T^\varepsilon(\varphi)$ are martingales. Further applying (2.2) to calculate $\langle N_T^\varepsilon(\varphi)\rangle$ and using the factor $\mathbf{1}_{\{x=y\}}$ to re-write a double sum as a single sum over lattice sites, we obtain the following expression for $\Lambda_T^\varepsilon(\varphi)$:

$$\Lambda_T^\varepsilon(\varphi) \stackrel{\mathrm{def}}{=} N_T^\varepsilon(\varphi)^2 - \varepsilon_j^2\int_0^{\varepsilon_j^{-2}T} \big(Z_s^2, \varphi^2\big)_\varepsilon\,ds + R_1^\varepsilon(\varphi) + R_2^\varepsilon(\varphi) + R_3^\varepsilon(\varphi), \tag{4.5}$$

where

$$R_1^\varepsilon(\varphi) \stackrel{\text{def}}{=} \varepsilon_j^2\biggl(\frac{2j}{[2j]_q}-1\biggr)\int_0^{\varepsilon_j^{-2}T}\bigl(Z_s^2,\varphi^2\bigr)_\varepsilon\,ds,$$

$$R_2^\varepsilon(\varphi) \stackrel{\text{def}}{=} -\frac{\varepsilon_j}{[2j]_q}\int_0^{\varepsilon_j^{-2}T}\bigl(\nabla^- Z_s\nabla^+ Z_s,\varphi^2\bigr)_\varepsilon\,ds,$$

$$R_3^\varepsilon(\varphi) \stackrel{\text{def}}{=} o\bigl(\varepsilon^2\bigr)\int_0^{\varepsilon_j^{-2}T}\bigl(Z_s^2,\varphi^2\bigr)_\varepsilon\,ds.$$

In (4.4), applying summation by part yields $(\Delta Z_s,\varphi)_\varepsilon = (Z_s,\Delta\varphi)_\varepsilon$. Further, as $\varphi\in C_c^\infty(\mathbf{R})$, we have that $\varepsilon_j^{-2}\Delta\varphi$ converges uniformly to $\varphi''$. By comparing the expressions as in (4.2)–(4.3) and (4.4)–(4.5), it clearly suffices to prove that $\mathbf{E}(R_i^\varepsilon(\varphi))^2\to 0$, for $i=1,2,3$. By the uniform bound (1.15) on $Z$, with $|\frac{2j}{[2j]_q}-1|\le C\varepsilon$, we clearly have $\mathbf{E}(R_i^\varepsilon(\varphi)^2)\to 0$, for $i=1,3$. To control $R_2^\varepsilon(\varphi)^2$, we will follow the "key estimate" in [4]. Indeed, letting $\mathcal{F}_t \stackrel{\text{def}}{=} \sigma(Z_s(x): x\in\mathbf{Z}, s\le t)$ denote the canonical filtration and let

$$U^\varepsilon\bigl(y,s,s'\bigr) \stackrel{\text{def}}{=} \mathbf{E}\bigl(\nabla^- Z_s(y)\nabla^+ Z_s(y)|\mathcal{F}_{s'}\bigr), \tag{4.6}$$

with $R_2^\varepsilon(\varphi)$ defined as in the preceding, we have

$$\mathbf{E}\bigl(R_2^\varepsilon(\varphi)^2\bigr) = \frac{2\varepsilon_j^4}{[2j]_q^2}\int_0^{\varepsilon_j^{-2}T} ds\int_0^s n'\sum_{x,y\in\mathbf{Z}}\varphi(\varepsilon x)^2\varphi(\varepsilon y)^2\mathbf{E}\bigl(\nabla^- Z_{s'}(x)\nabla^+ Z_{s'}(x)U^\varepsilon\bigl(y,s,s'\bigr)\bigr).$$

With $|\nabla^\pm Z_t(x)|\le C\varepsilon^{\frac12}Z_t(x)$, we further obtain

$$\mathbf{E}\bigl(R_2^\varepsilon(\varphi)^2\bigr) \le C\varepsilon^5\int_0^{\varepsilon_j^{-2}T} ds\int_0^s ds'\sum_{x,y\in\mathbf{Z}}\varphi(\varepsilon x)^2\varphi(\varepsilon y)^2\mathbf{E}\bigl(Z_{s'}(x)^2U^\varepsilon\bigl(y,s,s'\bigr)\bigr). \tag{4.7}$$

Note if we simply use $|\nabla^\pm Z_t(x)|\le C\varepsilon^{\frac12}Z_t(x)$ to bound $U^\varepsilon(y,s,s')$ as $|U^\varepsilon(y,s,s')|\le\varepsilon C Z_s^2(y)$, and insert this bound into (4.7), the resulting bound on $\mathbf{E}(R_2^\varepsilon(\varphi)^2)$ is of order $O(1)$ (since the change of time and space variables to macroscopic variables gives $\varepsilon^{-6}$), which is insufficient for our purpose. To obtain the desired bound $\mathbf{E}(R_2^\varepsilon(\varphi)^2)\to 0$, we utilize the smoothing effect of the conditional expectation $\mathbf{E}(\cdot|\mathcal{F}_{s'})$ in (4.6) to show the following

**Lemma 4.1.** *For all $\bar T>0$, $\delta>0$, there are constants $a, C>0$ such that*

$$\sup_{x\in\mathbf{Z}} e^{-a\varepsilon|x|}\mathbf{E}\bigl|U^\varepsilon(x,t,s)\bigr| \le C\varepsilon^{\frac32-\delta}\bigl(\varepsilon^2(t-s)\bigr)^{-\frac12} \tag{4.8}$$

*for all $\sqrt{\varepsilon}\le\varepsilon^2 s<\varepsilon^2 t\le\bar T$ and all $\varepsilon>0$.*

With this, $\mathbf{E}(R_2^\varepsilon(\varphi)^2)\to 0$ follows by standard argument as in [4, Proof of Proposition 4.11]. We omit the details here and prove only Lemma 4.1. □

Proving Lemma 4.1 requires a certain integral identity on the heat kernel $p_t(x)$ as in [4, Lemma A.1]. Here, to shed light on the underlying structure of this identity, we state and prove the following more general identity.

**Lemma 4.2.** *Let $p_t(x)$ be the transition probability of the continuous time symmetric simple random walk on $\mathbf{Z}^d$, with the convention $p_t(x)=0$ for $t<0$. Then one has*

$$\frac1d\sum_{x\in\mathbf{Z}^d}\sum_{n=1}^d\int_{-\infty}^\infty \nabla_n p_{t+s}(x+y)\nabla_n p_{t+s'}\bigl(x+y'\bigr)\,dt = p_{|s-s'|}\bigl(y-y'\bigr), \tag{4.9}$$

*for all* $s, s' \in \mathbf{R}$ *and* $y, y' \in \mathbf{R}^d$, *where*

$$\nabla_n f(x_1, \dots, x_d) \stackrel{\text{def}}{=} f(x_1, \dots, x_n + 1, \dots, x_d) - f(x_1, \dots, x_d).$$

**Proof.** Let $\mathscr{F}_x, \mathscr{F}_t$ denote the Fourier transform operators in the spatial variable and time variable respectively, and let $\mathscr{F}$ denote the Fourier transform operator in both variables. Since $p$ solves $\partial_t p = \frac{1}{2d}\Delta p$ with initial condition $\mathbf{1}_{x=0}$, and $e^{ik\cdot x}$ is the eigenfunction of $\frac{1}{2d}\Delta$ with eigenvalue $\lambda_k \stackrel{\text{def}}{=} \frac{1}{d}\sum_{n=1}^d(\cos k_n - 1)$, we have

$$(\mathscr{F} p)(\omega, k) = \frac{1}{-\lambda_k + i\omega}.$$

The LHS of (4.9) can be written as $\frac{1}{d}\sum_{n=1}^d (\nabla_n p \hat{*} \tilde{\nabla_n p})_{s-s'}(y - y')$ where

$$\tilde{\nabla_n p}(t, x) = \nabla_n p(-t, -x)$$

denotes reflected function, and $\hat{*}$ denotes the space-time convolution, as

$$(f \hat{*} g)_s(y) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \sum_{x\in\mathbf{Z}^d} f_{t+s}(y + x) g_{-t}(-x)\, dt.$$

Therefore the Fourier transform of the LHS of (4.9) is equal to

$$\frac{1}{d}\sum_{n=1}^{d}\left|\frac{e^{ik_n} - 1}{-\lambda_k + i\omega}\right|^2 = \frac{1}{d}\sum_{n=1}^{d}\frac{2 - 2\cos k_n}{\lambda_k^2 + \omega^2}. \tag{4.10}$$

On the other hand, for the RHS of (4.9), one has $(\mathscr{F}_x p_{|t|})(k) = e^{\lambda_k |t|}$. Further take Fourier transform in $t$, one has [5]

$$\mathscr{F}_t\big(\mathscr{F}_x p_{|\cdot|}(\cdot)\big)(\omega, k) = \frac{-2\lambda_k}{\lambda^2 + \omega^2},$$

which is equal to (4.10). □

**Remark 4.3.** A continuous version of (4.9) for $d = 1$ is stated in the recent paper [23, Proof of Lemma 6.11] (up to a factor 2 on the LHS because the heat operator is defined as $\partial_t - \Delta$ therein), and is used to show that the logarithmically divergent renormalization constants add up to a finite constant $c$ and if the KPZ equation is only spatially regularized then $c = 0$.

Now, setting $d = 1$, $s = s' = 0$ and $y, y' \in \{0, -1\}$, one recovers [4, Lemma A.1]

$$\sum_x \int_0^\infty \nabla^+ p_t(x) \nabla^- p_t(x)\, dt = 0, \tag{4.11}$$

and, by using also the Cauchy–Schwartz inequality, we also obtain [4, Lemma A.2]

$$\sum_x \int_0^\infty \big|\nabla^+ p_t(x) \nabla^- p_t(x)\big|\, dt < \prod_{\sigma\in\{+,-\}}\left(\sum_x \int_0^\infty \big(\nabla^\sigma p_t(x)\big)^2\, dt\right)^{\frac{1}{2}} = 1 \cdot 1 = 1. \tag{4.12}$$

[5] Here we use the fact that for any $a > 0$, $\mathscr{F}_t e^{-a|t|} = \frac{2a}{a^2+\omega^2}$.

**Proof of Lemma 4.1.** The proof follows the same argument as in [4, Lemma 4.8].

Let $N_s^t(x) \overset{\text{def}}{=} \int_0^s p_{t-\tau} * dM_\tau$ so that $Z_t(x) = I_t(x) + N_t^t(x)$ where $I_t = p_t * Z_0$. For $s \le r \le t$, one has

$$\mathbf{E}\big(\nabla^- N_r^t(x)\nabla^+ N_r^t(x) \mid \mathcal{F}_s\big) = \nabla^- N_s^t(x)\nabla^+ N_s^t(x) + \mathbf{E}\bigg(\int_s^r K_{t-\tau} * \langle M(\cdot), M(\cdot)\rangle_\tau(x) \,\Big|\, \mathcal{F}_s\bigg), \tag{4.13}$$

where

$$K_t(x) \overset{\text{def}}{=} \nabla^+ p_t(x)\nabla^- p_t(x). \tag{4.14}$$

With $U^\varepsilon(y,t,s)$ defined as in (4.6) and with $\mathbf{E}(N_r^t(x) \mid \mathcal{F}_s) = N_s^t(x)$, one has by (4.13)

$$U^\varepsilon(y,t,s) = \nabla^- I_t(x)\nabla^+ I_t(x) + \nabla^- I_t(x)\nabla^+ N_s^t(x) + \nabla^- N_s^t(x)\nabla^+ I_t(x) + \nabla^- N_s^t(x)\nabla^+ N_s^t(x) + \mathbf{E}\bigg(\int_s^t K_{t-\tau} * d\langle M(\cdot)\rangle_\tau(x) \,\Big|\, \mathcal{F}_s\bigg). \tag{4.15}$$

We bound the $L^1$-norms (i.e. $\mathbf{E}|\cdot|$) of the terms on the RHS. For the first four terms, by the Cauchy-Schwartz inequality one needs only to show

$$\mathbf{E}\big(\nabla^\pm I_t(x)\big)^2, \mathbf{E}\big(\nabla^\pm N_s^t(x)\big)^2 \le C\varepsilon^{\frac{1}{2}}(t-s)^{-\frac{1}{2}}e^{2a\varepsilon|x|}. \tag{4.16}$$

To bound $\nabla^\pm I$, we use (1.10) to obtain

$$\mathbf{E}\big((\nabla^\pm I_t(x))^2\big) = \sum_{y,y'} \nabla^\pm p_t(y)\nabla^\pm p_t(y')\mathbf{E}\big(Z_0(x-y)Z_0(x-y')\big) \le Ce^{2a\varepsilon|x|}\bigg(\sum_y \nabla^\pm p_t(y)e^{a\varepsilon|y|}\bigg)^2.$$

Using [9, (A.26)] with $v=1$, we bound the RHS by $Ce^{2a\varepsilon|x|}t^{-1}$. Further expressing $t^{-1}$ as $t^{-1/2}t^{-1/2}$, and applying $t^{-\frac{1}{2}} < (t-s)^{-\frac{1}{2}}$ and $t^{-1/2} \le \varepsilon^{3/4}$ (since we assume $\varepsilon^2 t \ge \varepsilon^{1/2}$), we obtain desired bound on $\mathbf{E}(\nabla^\pm I)^2$ as in (4.16). Turning to bounding $\mathbf{E}(\nabla^\pm N)^2$, one has

$$\mathbf{E}\big((\nabla^\pm N_s^t(x))^2\big) = \mathbf{E}\int_0^s \sum_y \big(\nabla^\pm p_{t-\tau}\big)^2 * d\langle M\rangle_\tau \le C\int_0^s \Big(\sup_y \big|\nabla^\pm p_{t-\tau}(y)\big|\Big)\bigg(\big|\nabla^\pm p_{t-\tau}\big| * \mathbf{E}\bigg|\frac{d}{d\tau}\langle M\rangle_\tau\bigg|\bigg)(x)\,d\tau.$$

By (2.7) and the uniform bound (1.15), one has $\mathbf{E}|\frac{d}{\tau}\langle M(y)\rangle_\tau| \le C\varepsilon e^{a\varepsilon|y|}$; we then apply the estimates [9, (A.26), (A.28)] with $v=1$ to obtain

$$\mathbf{E}\big((\nabla^\pm N_s^t(x))^2\big) \le C\varepsilon e^{2a\varepsilon|x|}\int_0^s (t-\tau)^{-3/2}\,d\tau.$$

Upon integrating over $\tau$, we obtain the desired bound on $\mathbf{E}(\nabla^\pm N)^2$ as in (4.16).

To bound the last term on the RHS of (4.15), we use the explicit expression of the predictable quadratic variation (2.2) to re-write the last term on the RHS of (4.15) as $I_1 + I_2 + I_3$ where

$$I_1(s,t,x) \overset{\text{def}}{=} \frac{4\varepsilon j^2}{[2j]_q}\sum_y \int_s^t K_{t-\tau}(x-y)\mathbf{E}\big(Z_\tau(y)^2 \mid \mathcal{F}_s\big)\,d\tau,$$

$$I_2(s,t,x) \stackrel{\text{def}}{=} -\frac{1}{[2j]_q}\sum_y \int_s^t K_{t-\tau}(x-y)\mathbf{E}\big(\nabla^- Z_\tau(y)\nabla^+ Z_\tau(y) \mid \mathcal{F}_s\big)\,d\tau,$$

$$I_3(s,t,x) = o(\varepsilon)\sum_y \int_s^t K_{t-\tau}(x-y)\mathbf{E}\big(Z_\tau(y)^2 \mid \mathcal{F}_s\big)\,d\tau.$$

Since $0 \le I_3 \le I_1$ for all $\varepsilon$ small enough, we drop $I_3$ in the following.

To bound $I_1$ we apply the identity (4.11) to obtain

$$I_1(s,t,x) = \frac{4\varepsilon j^2}{[2j]_q}\sum_y \int_s^t K_{t-\tau}(x-y)\mathbf{E}\big(Z_\tau(y)^2 - Z_t(x)^2 \mid \mathcal{F}_s\big)\,d\tau, \\ + \frac{4\varepsilon j^2}{[2j]_q}\mathbf{E}\big(Z_t(x)^2 \mid \mathcal{F}_s\big)\sum_y \int_{t-s}^\infty K_\tau(x-y)\,d\tau.$$

Hence $|I_1(s,t,x)| \le C(I_{11}(s,t,x) + I_{12}(s,t,x))$, where

$$I_{11}(s,t,x) \stackrel{\text{def}}{=} \varepsilon \sum_y \int_s^t K_{t-\tau}(x-y)\mathbf{E}\big(\big|Z_\tau(y)^2 - Z_t(x)^2\big| \mid \mathcal{F}_s\big)\,d\tau, \tag{4.17}$$

$$I_{12}(s,t,x) \stackrel{\text{def}}{=} \varepsilon\mathbf{E}\big(Z_t(x)^2 \mid \mathcal{F}_s\big)\int_{t-s}^\infty \sum_y \big|K_\tau(x-y)\big|\,d\tau. \tag{4.18}$$

With $K$ defined as in the preceding, applying [9, (A.26), (A.28)] with $\upsilon = 1$, we obtain $\sum_y |K_\tau(x-y)| \le C(1 \wedge \tau^{-3/2})$. Using this and the uniform bound (1.15) in (4.18), we obtain the desired bound on $I_{12}$ as

$$\mathbf{E}\big|I_{12}(s,t,x)\big| \le C\varepsilon e^{a\varepsilon|x|}\int_{t-s}^\infty \tau^{-\frac{3}{2}}\,d\tau \le C\varepsilon e^{a\varepsilon|x|}(t-s)^{-\frac{1}{2}}. \tag{4.19}$$

Next, the idea of controlling $I_{11}$ is to use the fact that $K_{t-\tau}(x-y)$ concentrates on values of $(\tau, y)$ which are close to $(t,x)$, and that, thanks to the Hölder estimates (1.16)–(1.17), $|Z_\tau(y)^2 - Z_t(x)^2|$ is small when $(\tau, y) \approx (t,x)$. More precisely, with

$$\big|Z_\tau(y)^2 - Z_t(x)^2\big| \le \big(Z_\tau(y) + Z_t(x)\big)\big(\big|Z_\tau(y) - Z_t(y)\big| + \big|Z_t(y) - Z_t(x)\big|\big)$$

we use the Cauchy–Schwarz inequality and the Hölder estimates (1.16)–(1.17) for $\alpha = \frac{1}{2} - \delta$ to obtain

$$\mathbf{E}\big|Z_\tau(y)^2 - Z_t(x)^2\big| \le C\varepsilon^{\frac{1}{2}-\delta}e^{a\varepsilon(|x|+|y|)}\big(|y-x|^{\frac{1}{2}-\delta} + \big(|t-\tau| \vee 1\big)^{\frac{1}{4}-\delta/2}\big).$$

Inserting this into (4.17), after the change of variables $t - \tau \mapsto \tau$ and $x - y \mapsto y$, we arrive at

$$\mathbf{E}\big|I_{11}(s,t,x)\big| \le C\varepsilon^{\frac{3}{2}-\delta}e^{a\varepsilon|x|}\int_0^{\varepsilon^{-2}\bar{T}} \Big(\sup_y\big|\nabla^+ p_\tau(y)\big|\Big) \\ \times \Big(\sum_y \big|\nabla^- p_\tau(y)\big|e^{a\varepsilon|y|}\big(|y|^{\frac{1}{2}} + \big(|\tau| \vee 1\big)^{\frac{1}{4}}\big)\Big)\,d\tau.$$

Further using [9, (A.26), (A.28)] with $\upsilon = 1$, to bound the terms within the integral, we obtain

$$\mathbf{E}\big|I_{11}(s,t,x)\big| \le C\varepsilon^{\frac{3}{2}-\delta}e^{a\varepsilon|x|}\int_0^{\varepsilon^{-2}\bar{T}} \big(1 \wedge \tau^{-1}\big)\tau^{-1/4}\,d\tau \le C\varepsilon^{\frac{3}{2}-\delta}e^{a\varepsilon|x|}.$$

With $(t-s)^{-1/2} \ge t^{-1/2} \ge \bar{T}^{-1/2}\varepsilon^{-1}$, the desired bound $\mathbf{E}|I_{11}(s,t,x)| \le C\varepsilon^{\frac{1}{2}-\delta}e^{a\varepsilon|x|}(t-s)^{-1/2}$ follows.

So far we have obtained the desired bounds on all the terms on the RHS of (4.15) except for the term $I_2$ from the last term in (4.15); but $I_2$ contains the same conditional expectation on the LHS of (4.15). Define $A_{t,s}$ to be the LHS of (4.8). Collecting the bounds for the terms in (4.15), then multiplying both sides by $e^{-a\varepsilon|x|}$ and taking supremum, one has

$$A_{t,s} \le C\varepsilon^{\frac{1}{2}-\delta}(t-s)^{-\frac{1}{2}} + \sum_y \int_s^t \left|K_{t-\tau}(y)\right| e^{a\varepsilon|y|} A_{\tau,s}\, d\tau, \tag{4.20}$$

where a change of variable $x - y \mapsto y$ is preformed. The desired estimate (4.8) now follows by iterating (4.20) as in [4, Lemma 4.8]. □

## 5. Remarks on ASIP$(q, k)$

The asymmetric inclusion process with parameters $q, k$ (ASIP$(q, k)$ for short) is introduced in [5], which also enjoys a self-duality property similar to that of ASEP$(q, j)$. In this section we apply our methods in Section 2 to derive a microscopic Cole–Hopf transformation of ASIP$(q, k)$, and discuss the possibility of showing convergence to the KPZ equation. Following [5], we consider the process on the finite lattice $\Lambda_L = \{1, \dots, L\}$.

**Definition 5.1.** (ASIP$(q, k)$ on $\Lambda_L$.) Let $q \in (0, 1)$ and $k \in \mathbf{R}_+$ be a positive real number. Denote by $\tilde{\eta}(x) \in \mathbf{N}$ the occupation variable, i.e. the number of particles at site $x \in \Lambda_L$. Note that $\tilde{\eta}(x)$ can be any non-negative integer. The ASIP$(q, k)$ is a continuous-time Markov process on the state space $\mathbf{N}^{\Lambda_L}$ defined by: at any given time $t \in [0, \infty)$, a particle jumps from site $x$ to site $x + 1$ at rate

$$\tilde{c}_q^+(\tilde{\eta}, x) = \frac{1}{2[2k]_q} q^{\tilde{\eta}(x)-\tilde{\eta}(x+1)+(2k-1)} \big[\tilde{\eta}(x)\big]_q \big[2k + \tilde{\eta}(x+1)\big]_q$$

and from site $x + 1$ to site $x$ at rate

$$\tilde{c}_q^-(\tilde{\eta}, x) = \frac{1}{2[2k]_q} q^{\tilde{\eta}(x)-\tilde{\eta}(x+1)-(2k-1)} \big[2k + \tilde{\eta}(x)\big]_q \big[\tilde{\eta}(x+1)\big]_q$$

independently of each other.

As in Definition 1.1 we define the *centered* occupation variable $\eta(x) \overset{\text{def}}{=} \tilde{\eta}(x) + k \in \mathbf{N} + k$ and the corresponding jumping rate

$$c_q^\pm(\eta, x) = \frac{1}{2[2k]_q} q^{\eta(x)-\eta(x+1)\pm(2k-1)} \big[\eta(x) \mp k\big]_q \big[\eta(x+1) \pm k\big]_q. \tag{5.1}$$

Define $\eta^{x,y}$ in the preceding. With these notations, the ASIP$(q, k)$ has the generator

$$(\mathcal{L}f)(\eta) = \sum_{x\in\Lambda_L} (\mathcal{L}_{x,x+1}f)(\eta), \tag{5.2}$$

where $(\mathcal{L}_{x,x+1}f)(\eta) = c_q^+(\eta, x)(f(\eta^{x,x+1}) - f(\eta)) + c_q^-(\eta, x)(f(\eta^{x+1,x}) - f(\eta))$.

**Remark 5.2.** By comparing (1.4) and (5.1), we find that the *generator* of ASEP$(q, k)$ is converted to that of ASIP$(q, j)$ by letting $j \mapsto -k$ (although the domain of the generator is different).

The article [5] raised up the following question.

**Question 1.** *Can the* ASIP$(q, k)$ *be constructed on the entire* $\mathbf{Z}$*?*

Define the processes $h$ and $Z$ in the same way as in (1.7) and (1.8), with respect to the ASIP$(q,k)$ occupation configuration $\eta$. Set

$$\nu \stackrel{\text{def}}{=} \left(\frac{[4k]_q}{2[2k]_q} - 1\right) \Big/ \ln q. \tag{5.3}$$

Parallel with Proposition 2.1, we have

**Proposition 5.3.** *We have that*

$$dZ_t(x) = \frac{1}{2}\Delta Z_t(x)\,dt + dM_t(x), \tag{5.4}$$

*where* $M_\cdot(x)$, $x \in \mathbf{L}$, *are martingales.*

**Proof.** Proceeding as (2.4) and (2.5) we have that

$$n_t(x) = \Omega Z_t(x)\,dt + dM_t(x), \quad \Omega = \sum_{\sigma=\pm} \left(q^{2\sigma} - 1\right) c_q^\sigma(\eta, x), + \nu \ln q, \tag{5.5}$$

where $c_q^\pm$ is now defined as (5.1), and the martingales $\{M_t(x)\}_{x\in\mathbf{L}}$ are defined as (2.5) with the ASIP$(q,k)$ rates (5.1). To compute $\Omega Z_t(x)$, by Remark 5.2, we simply perform the substitution $j \mapsto -k$ in the proof of Proposition 2.1(a), whereby obtaining $[2k]_q(\Omega - \nu \ln q) = [2k]_q \Delta Z(x) - [4k]_q + 2[2k]_q$. With this and (5.3), the statement (5.4) follows. □

We turn to consider the bracket process of ASIP$(q,k)$. As in the proof of Proposition 2.1(b), we compute

$$\begin{aligned}
&\frac{d}{dt}\langle M(x), M(y)\rangle_t \\
&\quad = \mathbf{1}_{\{x=y\}}\left(\left(q^2 - 1\right)^2 c^+(\eta_t, x) + \left(q^{-2} - 1\right)^2 c^-(\eta_t, x)\right) Z_t(x)^2 \\
&\quad = \frac{\mathbf{1}_{\{x=y\}}}{2[2k]_q} Z_t(x)^2 \left(q\left(q^{2\eta(x)} - q^{2k}\right)\left(q^{2k} - q^{-2\eta(x+1)}\right) + q^{-1}\left(q^{2\eta(x)} - q^{-2k}\right)\left(q^{-2k} - q^{-2\eta(x+1)}\right)\right),
\end{aligned}$$

where $q = e^{-\sqrt{\varepsilon}}$. But the occupation variable $\eta(x)$ is unbounded ASIP$(q,k)$, so the argument of Taylor expansion in $\sqrt{\varepsilon}$ in the proof of Proposition 2.1(b) is not useful.

**Question 2.** *Does* ASIP$(q,k)$ *converge to the KPZ equation under the same scaling as studied in* ASEP$(q, j)$*?*

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