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Davie's type uniqueness for a class of SDEs with jumps

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Abstract. A result of A.M. Davie (Int. Math. Res. Not. **24** (2007) rnm124) states that a multidimensional stochastic equation $dX_t = b(t, X_t)\,dt + dW_t$, $X_0 = x$, driven by a Wiener process $W = (W_t)$ with a coefficient b which is only bounded and measurable has a unique solution for almost all choices of the driving Wiener path. We consider a similar problem when W is replaced by a Lévy process $L = (L_t)$ and b is β-Hölder continuous in the space variable, $\beta \in (0, 1)$. We assume that L_1 has a finite moment of order θ, for some $\theta > 0$. Using a new càdlàg regularity result for strong solutions, we prove that strong existence and uniqueness for the SDE together with L^p-Lipschitz continuity of the strong solution with respect to x imply a Davie's type uniqueness result for almost all choices of the Lévy path. We apply this result to a class of SDEs driven by non-degenerate α-stable Lévy processes, $\alpha \in (0, 2)$ and $\beta > 1 - \alpha/2$.

Résumé. Un résultat de A.M. Davie (Int. Math. Res. Not. **24** (2007) rnm124) établit qu'une équation stochastique multidimensionnelle $dX_t = b(t, X_t)\,dt + dW_t$, $X_0 = x$, dirigée par un processus de Wiener $W = (W_t)$ avec un coefficient b qui est seulement borné et mesurable admet une unique solution pour presque tout choix de la trajectoire du processus W la dirigeant. Nous considérons un problème similaire lorsque W est remplacé par un processus de Lévy $L = (L_t)$ et b est β-Hölder continu en espace. Nous supposons que L_1 a un moment fini d'ordre θ pour un certain $\theta > 0$. En utilisant un nouveau résultat de régularité càdlàg, nous prouvons que l'existence et unicité forte pour l'EDS, associées à une L^p-Lipschitz continuité de la solution forte par rapport à x, impliquent une unicité de type Davie pour presque tout choix de la trajectoire de Lévy. Nous appliquons ce résultat à une classe d'EDS dirigées par un processus de Lévy α-stable non dégénéré pour $\alpha \in (0, 2)$ et $\beta > 1 - \alpha/2$.

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Keywords: Stochastic differential equations; Lévy processes; Path-by-path uniqueness; Hölder continuous drift

1. Introduction

In [8] A.M. Davie has proved that a SDE $dX_t = b(t, X_t)\,dt + dW_t$, $X_0 = x \in \mathbb{R}^d$, driven by a Wiener process W and having a coefficient b which is only bounded and measurable has a unique solution for almost all choices of the driving Wiener path. This type of uniqueness is also called *path-by-path uniqueness*. In other words, adding a single path of a Wiener process $W = (W_t) = (W_t)_{t\geq 0}$ regularizes a singular ODE whose right-hand side b is only bounded and measurable.

We consider a similar uniqueness problem for SDEs driven by Lévy noises with Hölder continuous drift term b, i.e., we deal with

$$X_t(\omega) = x + \int_s^t b\big(r, X_r(\omega)\big)\,dr + L_t(\omega) - L_s(\omega), \quad t \in [s, T], \tag{1.1}$$

where $T > 0$, $s \in [0, T]$, $x \in \mathbb{R}^d$, $d \geq 1$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable, bounded and β-Hölder continuous in the x-variable, uniformly in t, $\beta \in (0, 1]$. Moreover $L = (L_t)$ is a d-dimensional Lévy process defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\omega \in \Omega$ (see Section 2; recall that $L_0 = 0$, P-a.s.). Suppose that $E[|L_1|^\theta] < \infty$ for some $\theta > 0$ (cf. Hypothesis 2). Assuming that, for any $x \in \mathbb{R}^d$, $s \in [0, T]$, strong existence and uniqueness hold for (1.1) together with L^p-Lipschitz continuity of the strong solution $(X_t^{s,x})$ with respect to x, i.e.,

$$\sup_{s\in[0,T]} E\Big[\sup_{s\le r\le T} \big|X_r^{s,x} - X_r^{s,y}\big|^p\Big] \le C|x-y|^p, \quad x, y \in \mathbb{R}^d, p \in [2, \infty) \tag{1.2}$$

(cf. Hypothesis 1 and Section 2) we prove the following result (cf. Theorem 5.1)

Theorem 1.1. *Assume Hypotheses 1 and 2. There exists an event $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that for any $\omega \in \Omega'$, $x \in \mathbb{R}^d$, the integral equation*

$$f(t) = x + \int_0^t b\big(r, f(r) + L_r(\omega)\big)\,dr, \quad t \in [0, T], \tag{1.3}$$

has exactly one solution f in $C([0, T]; \mathbb{R}^d)$.

The assumptions and the uniqueness property are clear when $\beta = 1$ (the Lipschitz case). When $\beta \in (0, 1)$ the result is a special case of assertion (v) in Theorem 5.1 which also considers $s \neq 0$. It turns out that $f(t) = \phi(0, t, x, \omega) - L_t(\omega)$, $t \in [0, T]$, where $(\phi(s, t, x, \cdot))$ is a particular strong solution to (1.1). In Section 6 we will apply the previous theorem to a class of SDEs driven by non-degenerate α-stable type Lévy processes, $\alpha \in (0, 2)$, assuming as in [23] that $\beta \in (1 - \frac{\alpha}{2}, 1)$. Note that we can also treat locally Hölder drifts $b(x)$ by a localization procedure (see Corollaries 5.4 and 5.5). These uniqueness results seem to be new even in dimension one. For instance, one can consider

$$dX_t = \sqrt{|X_t|}\,dt + dL_t^{(\alpha)}, \qquad X_0 = x \in \mathbb{R},$$

with a symmetric α-stable process $L^{(\alpha)} = (L_t^{(\alpha)})$, $\alpha > 1$, and prove that for almost all $\omega \in \Omega$ there exists at most one solution for (1.3) with $b(r, x) = \sqrt{|x|}$ and $L = L^{(\alpha)}$.

As already mentioned when $L = W$ is a standard Wiener process, Theorem 1.1 is a special case of Theorem 1.1 in [8]. Recall that Davie's uniqueness is stronger then the usual pathwise uniqueness considered in the literature on SDEs (cf. Remark 2.2 and see also [10]). Pathwise uniqueness deals with solutions which are adapted stochastic processes and does not consider solutions corresponding to single paths $(L_t(\omega))_{t\in[0,T]}$. When $L = W$ several results on strong existence and pathwise uniqueness are known for the SDE (1.1) with very irregular drift b: the seminal paper [35] deals with b as in the Davie's result; further recent results consider b which is only locally in some L^p-spaces (see also [13,18] and [9]).

When L is a stable type Lévy process, the SDE (1.1) with a Hölder continuous and bounded drift b and its associated integro-differential generator $\mathcal{L}_b$ (cf. (6.8)) has received a lot of attention (see, for instance, [3,6,23,24,31,32,34] and the references therein). On this respect in Theorem 3.2 of [34] the authors proved that when $d = 1$ and L is a symmetric α-stable process, $\alpha \in (0, 1)$, pathwise uniqueness may fail even with a β-Hölder continuous b if $\alpha + \beta < 1$.

Let us come back to Davie's theorem. The proof in [8] is self-contained but very technical; it relies on explicit computations with Gaussian kernels. An alternative approach to the Davie uniqueness result has been proposed in [30] (see in particular Theorems 1.1 and 3.1 in [30]). This approach uses the flow property of strong solutions of SDEs driven by the Wiener process. Beside [8] our work has been inspired by Theorem 3.1 in [30] which deals with drifts b possibly unbounded in time and such that $b(t, \cdot)$ is Hölder continuous. We mention that applications of Davie's uniqueness to Euler approximations for (1.1) are given in Section 4 of [8].

In our proof we use L^p-estimates (1.2) which are well-known when $L = W$ (they can be easily deduced from Section 2 in [11]). They are even true for more general drifts b (i.e., $b \in L^q(0, T; L^p(\mathbb{R}^d; \mathbb{R}^d))$, $d/p + 2/q < 1$, $p \ge 2$, $q > 2$, see formula (5.9) and Proposition 5.2 in [9]). Moreover, when L is a symmetric non-degenerate α-stable process, $b(t, x) = b(x)$, $\alpha \ge 1$ and $\beta \in (1 - \frac{\alpha}{2}, 1]$, such estimates follow by Theorem 4.3 in [23] (see Theorem 6.6 for a more general case).

By the L^p-estimates (1.2), passing through different modifications (see Sections 3 and 4), we finally obtain a suitable strong solution $\phi(s, t, x, \omega)$ (see Theorem 5.1) which solves (1.1) for any $\omega \in \Omega'$, for some almost sure event Ω' which is independent on s, t and x. Such solution ϕ is used to prove uniqueness of (1.3) (see the proof of (v) of

Theorem 5.1). We also establish càdlàg regularity of ϕ with respect to s, uniformly in $t \in [0, T]$ and x, when x varies in compact sets of $\mathbb{R}^d$. This result seems to be new even when $d = 1$ and b is Lipschitz continuous if L is not the Wiener process W (when $L = W$, the continuous dependence on s, uniformly in x, has been proved in Section 2 of [14] for SDEs with Lipschitz coefficients). We also prove the continuous dependence of $\phi(s, t, x, \omega)$ with respect to x and the flow property, for any $\omega \in \Omega'$ (see assertions (iii) and (iv) in Theorem 5.1). There are recent papers on the flow property for solutions to SDEs with jumps (see, for instance, [6,21,24] and the references therein). However they do not prove the previous assertions on ϕ.

Remark that when $L = W$ and $b(t, \cdot)$ is Hölder continuous as in (1.1), proving the existence of a regular strong solution like ϕ is easier. Indeed in such case one can use the well-known Kolmogorov–Chentsov continuity test to get a continuous dependence on (s, t, x). More precisely, when $L = W$, we can apply the Zvonkin method of [35] or the related Itô-Tanaka trick of [11] and, using a suitable regular solution $u(t, x)$ of a related Kolmogorov equation (cf. Section 6.2), find that the process $(u(t, X_t^x))$ solves an auxiliary SDE with Lipschitz continuous coefficients. On this auxiliary equation one can perform the Kolmogorov–Chentsov test as in [19] and finally obtain the required regular modification of the strong solution. To get our regular strong solution ϕ we do not pass through an auxiliary SDE but work directly on (1.1) using first a result in [14] and then a càdlàg criterion given in [4]. We apply this criterion to a suitable stochastic process with values in a space of continuous functions defined on $\mathbb{R}^d$ (see Theorem 4.4). This approach could be also useful to study regularity properties of solutions to SDEs with multiplicative noise.

In Section 6 we apply Theorem 5.1 to a class of SDEs driven by non-degenerate α-stable type Lévy processes, using also results in [23] and [24]. In particular we prove a Davie's type uniqueness result for (1.1) when L is a standard rotationally invariant α-stable process, $\alpha \in (0, 2)$ and $\beta \in (1 - \frac{\alpha}{2}, 1]$. The generator of L is the well-known fractional Laplacian $-(-\triangle)^{\alpha/2}$. To cover the case $\alpha \in (0, 1)$ we also need an analytic result proved in [31] (cf. Remark 5.5 in [24]). When $\alpha \in [1, 2)$ and $\beta \in (1 - \frac{\alpha}{2}, 1]$ we can treat more general non-degenerate α-stable type processes like relativistic and truncated stable processes and some tempered stable processes (cf. [24] with the references therein and see Examples 6.2). When $\alpha \in [1, 2)$ we can also consider the singular α-stable process $L = (L_t)$, $L_t = (L_t^1, \dots, L_t^d)$, $t \geq 0$, where $L^1, \dots, L^d$ are independent one-dimensional symmetric α-stable processes; well-posedness of SDEs driven by this process has recently received particular attention (see, for instance, [2,6,23,24, 38]).

2. Notations and assumptions

We fix basic notations. We refer to [17,20,28] and [1] for more details on Lévy processes with values in $\mathbb{R}^d$. By $\langle x, y\rangle$ (or $x \cdot y$) we denote the euclidean inner product between x and $y \in \mathbb{R}^d$, for $d \geq 1$; further $|x| = (\langle x, x\rangle)^{1/2}$. If $H \subset \mathbb{R}^d$ we denote by 1_H its indicator function. The Borel σ-algebra of a Borel set $C \subset \mathbb{R}^k$, $k \geq 1$, is indicated by $\mathcal{B}(C)$. Similarly if (S, d) is a metric space we denote its Borel σ-algebra by $\mathcal{B}(S)$. We consider a complete probability space $(\Omega, \mathcal{F}, P)$. The expectation with respect to P is indicated with E. If $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, a random variable $X : \Omega \to S$ with values in a metric space (S, d) which is measurable from $(\Omega, \mathcal{G})$ into $(S, \mathcal{B}(S))$ is called $\mathcal{G}$-measurable. Similarly a function $l : [0, T] \times \Omega \to S$ is $\mathcal{B}([0, T]) \times \mathcal{F}$-measurable if l is measurable with respect to the product σ-algebra $\mathcal{B}([0, T]) \times \mathcal{F}$.

In the sequel we often need to specify the possible dependence of events of probability one from some parameters. Recall that a set $\Omega' \subset \Omega$ is an *almost sure event* if $\Omega' \in \mathcal{F}$ and $P(\Omega') = 1$. To stress that Ω' possibly depends also on a parameter λ we write Ω'_λ (the almost sure event Ω'_λ may change from one proposition to another); for instance the notation $\Omega_{s,x}$ means that the almost sure event $\Omega_{s,x}$ possibly depends also on s and x. We say that a property involving random variables holds on an almost sure event Ω' to indicate that such property holds for any $\omega \in \Omega'$ (i.e., such property holds P-a.s.).

A d-dimensional stochastic process $L = (L_t) = (L_t)_{t\geq 0}$, $d \geq 1$, defined on $(\Omega, \mathcal{F}, P)$ is a *Lévy process* if it has independent and stationary increments, càdlàg paths (i.e., P-a.s., each mapping $t \mapsto L_t(\omega)$ is càdlàg from $[0, \infty)$ into $\mathbb{R}^d$; we denote by $L_{s-}(\omega)$ the left-limit in $s > 0$) and $L_0 = 0$, P-a.s.

Similarly to Chapter II in [19] and Chapter V in [17] we define for $0 \leq s < t < \infty$ the σ-algebra $\mathcal{F}_{s,t}^L$ as the completion of the σ-algebra generated by the random variables $L_r - L_s$, $r \in [s, t]$. We also set $\mathcal{F}_{0,t}^L = \mathcal{F}_t^L$. Since L has independent increments we have that $L_v - L_u$ is independent of $\mathcal{F}_u^L$ for $0 \leq u < v$. Note that $(\Omega, \mathcal{F}, (\mathcal{F}_t^L)_{t\geq 0}, P)$

is an example of stochastic basis which satisfies the usual assumptions (see [1], page 72). Given a Lévy process L there exists a unique function $\psi : \mathbb{R}^d \to \mathbb{C}$ such that

$$E\big[e^{i\langle h, L_t\rangle}\big] = e^{-t\psi(h)}, \quad h \in \mathbb{R}^d, t \geq 0;$$

ψ is called the *exponent* of L. The Lévy–Khintchine formula for ψ states that

$$\psi(h) = \frac{1}{2}\langle Qh, h\rangle - i\langle a, h\rangle - \int_{\mathbb{R}^d} \big(e^{i\langle h, y\rangle} - 1 - i\langle h, y\rangle 1_{\{|y|\leq 1\}}(y)\big)\nu(dy), \tag{2.1}$$

$h \in \mathbb{R}^d$, where Q is a symmetric non-negative definite $d \times d$-matrix, $a \in \mathbb{R}^d$ and ν is a σ-finite (Borel) measure on $\mathbb{R}^d$, such that $\int_{\mathbb{R}^d}(1 \wedge |y|^2)\nu(dy) < \infty$, $\nu(\{0\}) = 0$ $(1 \wedge |y|^2 = \min(1, |y|^2))$; ν is the *Lévy measure* (or intensity measure) of L. The triplet (Q, ν, a) uniquely identifies the law of L (see Proposition 9.8 in [28] or Corollary 2.4.21 in [1]). It is called *generating triplet* (or characteristics) of the Lévy process L.

Given two stochastic processes $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, P)$ and with values in a metric space (S, d), we say that X is a *modification or version* of Y if for any $t \in [0, T]$, $X_t = Y_t$, P-a.s.; if in addition both X and Y have càdlàg paths then, $P(X_t = Y_t, t \in [0, T]) = P(X_t = Y_t, \text{for any } t \in [0, T]) = 1$.

Let $L = (L_t)$ be a d-dimensional Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, let $s \in [0, T]$ and $x \in \mathbb{R}^d$ and consider the SDE

$$dX_t = b(t, X_t)\,dt + dL_t, \quad s \leq t \leq T, \qquad X_s = x, \tag{2.2}$$

with $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ which is a locally bounded Borel function.

According to [19,20] and [33] we say that an $\mathbb{R}^d$-valued stochastic process $U^{s,x} = (U_t^{s,x}) = (U_t^{s,x})_{t\in[s,T]}$ defined on $(\Omega, \mathcal{F}, P)$ is a *strong solution* to (2.2) starting from x at time s if, for any $t \in [s, T]$, the random variable $U_t^{s,x} : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_{s,t}^L$-measurable; further we require that there exists an almost sure event $\Omega_{s,x}$ (possibly depending also on s and x but independent of t) such that the following conditions hold for any $\omega \in \Omega_{s,x}$: (i) the map: $t \mapsto U_t^{s,x}(\omega)$ is càdlàg on $[s, T]$; (ii) we have

$$U_t^{s,x}(\omega) = x + \int_s^t b\big(r, U_r^{s,x}(\omega)\big)\,dr + L_t(\omega) - L_s(\omega), \quad t \in [s, T]; \tag{2.3}$$

(iii) the path $t \mapsto L_t(\omega)$ is càdlàg and $L_0(\omega) = 0$.

Given a strong solution $U^{s,x}$ we set for any $0 \leq t \leq s$, $U_t^{s,x} = x$ on Ω.

Let us recall some function spaces used in the paper. We consider $C_b(\mathbb{R}^d; \mathbb{R}^k)$, for integers $k, d \geq 1$, as the Banach space of all continuous and bounded functions $g : \mathbb{R}^d \to \mathbb{R}^k$ endowed with the supremum norm $\|g\|_0 = \|g\|_{C_b} = \sup_{x\in\mathbb{R}^d} |g(x)|$, $g \in C_b(\mathbb{R}^d; \mathbb{R}^k)$. Moreover, $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$, $\beta \in (0, 1]$, is the subspace of all β-Hölder continuous functions g, i.e., g verifies

$$[g]_{C_b^{0,\beta}} = [g]_\beta := \sup_{x\neq x'\in\mathbb{R}^d} \big(\big|g(x) - g\big(x'\big)\big|\big|x - x'\big|^{-\beta}\big) < \infty$$

(when $\beta = 1$, g is Lipschitz continuous). If $\beta = 0$ we set $C_b^{0,0}(\mathbb{R}^d; \mathbb{R}^k) = C_b(\mathbb{R}^d; \mathbb{R}^k)$. If $\beta \in (0, 1)$ we also write $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k) = C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$; note that $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k)$ is a Banach space with the norm $\|\cdot\|_{C_b^{0,\beta}} = \|\cdot\|_\beta = \|\cdot\|_0 + [\cdot]_\beta$, $\beta \in (0, 1]$. If $\mathbb{R}^k = \mathbb{R}$, we set $C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^k) = C_b^{0,\beta}(\mathbb{R}^d)$ (a similar convention is also used for other function spaces). A function $g \in C_b(\mathbb{R}^d; \mathbb{R}^k)$ belongs to $C_b^1(\mathbb{R}^d; \mathbb{R}^k)$ if it is differentiable on $\mathbb{R}^d$ and its Fréchet derivative $Dg \in C_b(\mathbb{R}^d; \mathbb{R}^{dk})$. If $\beta \in (0, 1)$, a function $g \in C_b^1(\mathbb{R}^d; \mathbb{R}^k)$ belongs to $C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$ if $Dg \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^{dk})$. The space $C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$ is a Banach space endowed with the norm $\|g\|_{1+\beta} = \|g\|_{C_b^{1+\beta}} = \|g\|_0 + [Dg]_\beta$, $g \in C_b^{1+\beta}(\mathbb{R}^d; \mathbb{R}^k)$. $C_b^\infty(\mathbb{R}^d; \mathbb{R}^k)$ is the space of all infinitely differentiable functions from $\mathbb{R}^d$ into $\mathbb{R}^k$ with all bounded derivatives. Finally $g \in C_b^\infty(\mathbb{R}^d)$ belongs to $C_0^\infty(\mathbb{R}^d)$ if g has compact support. Given a bounded open set $B \subset \mathbb{R}^d$ we can define similar Banach spaces $C^\beta(B)$ and $C^{1+\beta}(B)$ with norms $\|\cdot\|_{C^\beta(B)}$ and $\|\cdot\|_{C^{1+\beta}(B)}$, $\beta \in (0, 1)$.

We usually require that the drift b belongs to $L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta\in[0,1]$. This means that $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ is Borel measurable and bounded, $b(t,\cdot)\in C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d)$, $t\in[0,T]$, and $[b]_{\beta,T}=\sup_{t\in[0,T]}[b(t,\cdot)]_{C_b^{0,\beta}}<\infty$.

Set $\|b\|_{\beta,T}=[b]_{\beta,T}+\|b\|_0$, $\|b\|_0=\sup_{t\in[0,T],x\in\mathbb{R}^d}|b(t,x)|$ if $\beta\in(0,1]$ and $\|b\|_{0,T}=\|b\|_0$, $\beta=0$. Note that $(L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))),\|\cdot\|_{\beta,T})$ is a Banach space. We will also use

$$G_0=C\big([0,T];\mathbb{R}^d\big) \tag{2.4}$$

to denote the separable Banach space consisting of all continuous functions $f:[0,T]\to\mathbb{R}^d$, endowed with the usual supremum norm $\|\cdot\|_{G_0}$.

Let us formulate our assumptions on (1.1) when $b\in L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta\in[0,1]$. Note that, possibly changing $b(t,x)$ with $b(t,x)+a$, to study the SDE (1.1) we may always assume that in the generating triplet (Q,ν,a) we have

$$a=0. \tag{2.5}$$

In (1.1) we deal with a Lévy process L defined on $(\Omega,\mathcal{F},P)$ and $b\in L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$ which satisfy

Hypothesis 1.

(i) *For any $s\in[0,T]$ and $x\in\mathbb{R}^d$ on $(\Omega,\mathcal{F},P)$ there exists a strong solution $(U_t^{s,x})_{t\in[0,T]}$ to (2.2).*
(ii) *Let $s\in[0,T]$. Given any two strong solutions $(U_t^{s,x})_{t\in[0,T]}$ and $(U_t^{s,y})_{t\in[0,T]}$ defined on $(\Omega,\mathcal{F},P)$ which both solve (2.2) with respect to L and b (starting from x and $y\in\mathbb{R}^d$, respectively, at time s) we have, for any $p\geq 2$,*

$$\sup_{s\in[0,T]}E\Big[\sup_{s\le t\le T}\big|U_t^{s,x}-U_t^{s,y}\big|^p\Big]\le C(T)|x-y|^p,\quad x,y\in\mathbb{R}^d, \tag{2.6}$$

with $C(T)=C((\nu,Q,0),\|b\|_{\beta,T},d,\beta,p,T)>0$ independent of s, x and y.

The previous hypothesis holds clearly for any Lévy process L if $\beta=1$ (the Lipschitz case). Next we consider the Lévy measure ν associated to the large jump parts of L.

Hypothesis 2. *There exists $\theta>0$ such that $\int_{\{|x|>1\}}|x|^\theta\nu(dx)<\infty$.*

Remark 2.1. By Theorems 25.3 and 25.18 in [28] the following three conditions are equivalent:

(a) $\int_{\{|x|>1\}}|x|^\theta\nu(dx)<\infty$ for some $\theta>0$;
(b) $E[|L_t|^\theta]<\infty$ for some $t>0$;
(c) $E[\sup_{s\in[0,t]}|L_s|^\theta]<\infty$ for any $t>0$.

Note also that $\int_{\{|x|>1\}}|x|^\theta\nu(dx)<\infty$ holds for some $\theta>0$ then $\int_{\{|x|>1\}}|x|^{\theta'}\nu(dx)<\infty$ for any $\theta'\in(0,\theta]$.

Remark 2.2. We present here for the sake of completeness some general concepts about solutions of SDEs (cf. [31] for more details). We will not use these notions in the sequel. Let the initial time $s=0$. A weak solution to (1.1) with initial condition $x\in\mathbb{R}^d$ is a tuple $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},P,L,X)$, where $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},P)$ is a stochastic basis on which it is defined a Lévy process L and a càdlàg $(\mathcal{F}_t)$-adapted $\mathbb{R}^d$-valued process $X=(X_t)$ which solves (1.1) P-a.s. A weak solution X which is $(\mathcal{F}_t^L)$-adapted is called strong solution. One say that pathwise uniqueness holds for (1.1) if given two weak solutions X and Y (starting from $x\in\mathbb{R}^d$) and defined on the same stochastic basis (with respect to the same L) then P-a.s. we have $X_t=Y_t$, for any $t\in[0,T]$.

3. Preliminary results on strong solutions

Consider (2.2) with $b\in L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta\in[0,1]$, and suppose that L defined on $(\Omega,\mathcal{F},P)$ and b satisfy Hypothesis 1.

Let $s \in [0,T]$, $x \in \mathbb{R}^d$. We start with a strong solution $(\tilde{X}_t^{s,x})_{t\in[0,T]}$ to (2.2) defined on $(\Omega, \mathcal{F}, P)$ and introduce the d-dimensional process $\tilde{Y}^{s,x} = (\tilde{Y}_t^{s,x})_{t\in[0,T]}$,

$$\tilde{Y}_t^{s,x} = \tilde{X}_t^{s,x} - (L_t - L_s), \quad t \geq s. \tag{3.1}$$

Note that on some almost sure event $\Omega_{s,x}$ (independent of t) we have

$$\tilde{Y}_t^{s,x} = x + \int_s^t b\big(r, \tilde{Y}_r^{s,x} + (L_r - L_s)\big)\,dr, \quad t \geq s, \tag{3.2}$$

and $\tilde{Y}_t^{s,x} = x$ on Ω if $t \leq s$. It follows that $(\tilde{Y}_t^{s,x})_{t\in[0,T]}$ has *continuous paths*.

Let us fix $s \in [0,T]$ and $x \in \mathbb{R}^d$. We modify the process $\tilde{Y}^{s,x}$ only on $\Omega \setminus \Omega_{s,x}$ by setting $\tilde{Y}_t^{s,x}(\omega) = x$, for $t \in [0,T]$, if $\omega \notin \Omega_{s,x}$ (we still denote by $\tilde{Y}^{s,x}$ such new process).

We find that $\tilde{Y}_\cdot^{s,x}(\omega) \in G_0 = C([0,T];\mathbb{R}^d)$, for any $\omega \in \Omega$. Moreover (cf. (2.4)) it is easy to check that

$$\tilde{Y}^{s,x} = \tilde{Y}_\cdot^{s,x} \quad \text{is a random variable with values in } G_0. \tag{3.3}$$

Now, for each fixed $s \in [0,T]$, we will construct a suitable modification of the random field $(\tilde{Y}^{s,x})_{x\in\mathbb{R}^d}$ with values in G_0. We need the following special case of Theorem 1.1 of [14]. It is a generalized Garsia–Rodemich–Rumsey type lemma.

Theorem 3.1 ([14]). *Let (M,ρ) be a separable metric space and $(\Omega,\mathcal{F},P)$ be a probability space. Let $\psi : \Omega \times \mathbb{R}^d \to M$ be a $\mathcal{F}\times\mathcal{B}(\mathbb{R}^d)$-measurable map such that $\psi(\omega,\cdot)$ is continuous on $\mathbb{R}^d$, for each $\omega \in \Omega$, and there exists $c > 0$ and $p > 2d$ for which $E[(\rho(\psi(\cdot,x),\psi(\cdot,y)))^p] \leq c|x-y|^p$, $x,y \in \mathbb{R}^d$. Then, for any $\omega \in \Omega$, $x,y \in \mathbb{R}^d$,*

$$\rho\big(\psi(\omega,x),\psi(\omega,y)\big) \leq Y(\omega)|x-y|^{1-\frac{2d}{p}}\big[\big(|x|\vee|y|\big)^{\frac{2d+1}{p}} \vee 1\big], \tag{3.4}$$

where $Y : \Omega \to [0,\infty]$ is the following p-integrable random variable:

$$Y(\omega) = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\rho(\psi(\omega,x),\psi(\omega,y)))^p}{|x-y|^p} f(x)f(y)\,dx\,dy\right)^{1/p}, \quad \omega \in \Omega,$$

with $f(x) = c(d,p)([|x|^d[(\log(|x|)\vee 0)^2]\vee 1)^{-1}$, $x \neq 0$, for some constant $c(d,p) > 0$.

In Theorem 1.1 of [14] $f(x)$ is just defined as $([|x|^d[(\log(|x|)\vee 0)^2]\vee 1)^{-1}$. Moreover $Y(\omega) = c_3(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\rho(\psi(\omega,x),\psi(\omega,y)))^p}{|x-y|^p} f(x)f(y)\,dx\,dy)^{1/p}$.

Lemma 3.2. *Consider (2.2) with $b \in L^\infty(0,T;C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta \in [0,1]$, and suppose that L defined on $(\Omega,\mathcal{F},P)$ and b satisfy Hypothesis 1. Let us fix $s \in [0,T]$ and consider the random field $\tilde{Y}^s = (\tilde{Y}^{s,x})_{x\in\mathbb{R}^d}$ with values in G_0 (see (3.3)). We have*:

(i) *There exists a continuous version $Y^s = (Y^{s,x})_{x\in\mathbb{R}^d}$ with values in G_0 (i.e., for any $x \in \mathbb{R}^d$, $Y^{s,x} = \tilde{Y}^{s,x}$ in G_0 on some almost sure event).*

(ii) *For any $p > 2d$ there exists a random variable $U_{s,p}$ with values in $[0,\infty]$ such that, for any $\omega \in \Omega$, $x,y \in \mathbb{R}^d$,*

$$\big\|Y^{s,x}(\omega) - Y^{s,y}(\omega)\big\|_{G_0} \leq U_{s,p}(\omega)\big[\big(|x|\vee|y|\big)^{\frac{2d+1}{p}} \vee 1\big]|x-y|^{1-2d/p}. \tag{3.5}$$

Moreover, with the same constant $C(T)$ appearing in (2.6),

$$\sup_{s\in[0,T]} E\big[U_{s,p}^p\big] \leq C(d)C(T) < \infty, \tag{3.6}$$

where $C(d) = (\int_{\mathbb{R}^d} f(x)\,dx)^2$ (hence $U_{s,p}$ is finite on some almost sure event possibly depending on s and p).

(iii) *On some almost sure event* Ω'_s (*independent of* t *and* x) *we have*

$$Y_t^{s,x} = x + \int_s^t b\big(r, Y_r^{s,x} + (L_r - L_s)\big)\,dr, \quad t \ge s, x \in \mathbb{R}^d \tag{3.7}$$

(*where* $Y_t^{s,x}(\omega) = (Y_\cdot^{s,x}(\omega))(t)$, $t \in [0,T]$).

Proof. (i) Using (2.6) we can apply the Kolmogorov–Chentsov continuity test as in [15], page 57, and obtain a continuous version Y^s of $\tilde{Y}^s$. The classical proof given in [15] uses the Borel-Cantelli lemma; by such proof it is easy to show that an analogous of (2.6) holds for Y^s, i.e., for $p \ge 2$, $x, y \in \mathbb{R}^d$,

$$\sup_{s\in[0,T]} E\big[\big\|Y^{s,x} - Y^{s,y}\big\|_{G_0}^p\big] = \sup_{s\in[0,T]} E\big[\big\|\tilde{Y}^{s,x} - \tilde{Y}^{s,y}\big\|_{G_0}^p\big] \le C(T)|x-y|^p. \tag{3.8}$$

(ii) As in Theorem 3.1 we consider the random variables

$$U_{s,p}(\omega) = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{\|Y^{s,x}(\omega) - Y^{s,y}(\omega)\|_{G_0}}{|x-y|}\right)^p f(x) f(y)\,dx\,dy\right)^{1/p},$$

$\omega \in \Omega$, $p > 2d$ and $s \in [0,T]$. By (3.8) and Theorem 3.1 we obtain (3.5) and (3.6).

(iii) We start from equation (3.2) involving the process $(\tilde{Y}^{s,x})$. Since for some almost sure event $\Omega'_{s,x} \subset \Omega_{s,x}$, we have $Y_t^{s,x}(\omega) = \tilde{Y}_t^{s,x}(\omega)$, $\omega \in \Omega'_{s,x}$, $t \in [0,T]$, we obtain from (3.2)

$$Y_t^{s,x}(\omega) = x + \int_s^t b\big(r, Y_r^{s,x}(\omega) + \big(L_r(\omega) - L_s(\omega)\big)\big)\,dr,$$

for any $s \in [t,T]$, $x \in \mathbb{Q}^d$, $\omega \in \Omega'_s = \bigcap_{x\in\mathbb{Q}^d} \Omega'_{s,x}$. Note also that by (i) the function: $x \mapsto Y^{s,x}(\omega)$ is continuous for all $\omega \in \Omega$. Take now $x \in \mathbb{R}^d$ and let $(x_n) \subset \mathbb{Q}^d$ be a sequence converging to x. It follows from the continuity of $b(r,\cdot)$ and the dominated convergence theorem that, for any $t \ge s$, on Ω'_s we have:

$$\begin{aligned} Y_t^{s,x} &= \lim_{n\to\infty} Y_t^{s,x_n} = \lim_{n\to\infty} x_n + \lim_{n\to\infty}\int_s^t b\big(r, Y_r^{s,x_n} + (L_r - L_s)\big)\,dr \\ &= x + \int_s^t b\big(r, Y_r^{s,x} + (L_r - L_s)\big)\,dr \end{aligned}$$

and this shows the assertion. □

Let $s \in [0,T]$. According to the previous result starting from $Y^s = (Y^{s,x})_{x\in\mathbb{R}^d}$ we can define random variables $X_t^{s,x} : \Omega \to \mathbb{R}^d$ as follows: $X_t^{s,x} = x$ if $t \le s$ and

$$X_t^{s,x} = Y_t^{s,x} + (L_t - L_s), \quad s, t \in [0,T], x \in \mathbb{R}^d, s \le t. \tag{3.9}$$

By the properties of $Y^{s,x}$ we get $P(\tilde{X}_t^{s,x} = X_t^{s,x}, t \in [0,T]) = 1$, for any $x \in \mathbb{R}^d$ (cf. (3.1)). Moreover, using also (3.7), we find that for some almost sure event Ω'_s (independent of x and t) the map: $t \mapsto X_t^{s,x}(\omega)$ is càdlàg on $[0,T]$, for any $\omega \in \Omega'_s$, $x \in \mathbb{R}^d$, and on Ω'_s we have

$$X_t^{s,x} = x + \int_s^t b\big(r, X_r^{s,x}\big)\,dr + L_t - L_s, \quad s \le t \le T, x \in \mathbb{R}^d. \tag{3.10}$$

Thus $(X_t^{s,x})_{t\in[0,T]}$ is a particular *strong solution* to (2.2). By Lemma 3.2 we also have, for any $s \in [0,T]$, $x \in \mathbb{R}^d$, on Ω

$$\lim_{y\to x}\sup_{t\in[0,T]}\big|X_t^{s,x} - X_t^{s,y}\big| = 0. \tag{3.11}$$

We can prove the following flow property.

Lemma 3.3. *Under the same assumptions of Lemma 3.2 consider the strong solution $(X_t^{s,x})_{t\in[0,T]}$ defined in (3.9). Let $0 \le s < u \le T$. There exists an almost sure event $\Omega_{s,u}$ (independent of $t \in [u,T]$ and $x \in \mathbb{R}^d$) such that for $\omega \in \Omega_{s,u}$, $x \in \mathbb{R}^d$, we have*

$$X_t^{s,x}(\omega) = X_t^{u,X_u^{s,x}(\omega)}(\omega), \quad t \in [u,T], x \in \mathbb{R}^d. \tag{3.12}$$

Proof. Let us fix $s,u \in [0,T]$, $s < u$, and $x \in \mathbb{R}^d$. We introduce the process $(V_t^x)_{0\le t\le T}$ on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$:

$$V_t^x(\omega) = \begin{cases} X_t^{s,x}(\omega) & \text{for } 0 \le t \le u, \\ X_t^{u,X_u^{s,x}(\omega)}(\omega) & \text{for } u < t \le T, \end{cases} \quad \omega \in \Omega.$$

In order to prove (3.12) we will show that (V_t^x) is strong solution to (2.2) for $t \ge s$. Then by uniqueness we will get the assertion.

It is easy to prove that (V_t^x) has càdlàg paths. More precisely, by (3.7) on some almost sure event $\Omega_s' \cap \Omega_u'$ (independent of x) we have that $t \mapsto V_t^x(\omega)$ is càdlàg on $[0,T]$ (note also that, for any $\omega \in \Omega_s' \cap \Omega_u'$, $z \in \mathbb{R}^d$, $\lim_{t\to u^+} X_t^{u,z}(\omega) = z$).

Moreover, for any $x \in \mathbb{R}^d$ and $t \ge s$, the random variable V_t^x is $\mathcal{F}_{s,t}^L$-measurable. The assertion is clear if $t \le u$. Let us consider the case when $t > u$. First $X_u^{s,x}$ is $\mathcal{F}_{s,t}^L$-measurable. Define $F_{t,u}(z,\omega) = X_t^{u,z}(\omega)$, $z \in \mathbb{R}^d$, $\omega \in \Omega$. The mapping $F_{t,u}$ is clearly $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_{s,t}^L$-measurable on $\mathbb{R}^d \times \Omega$ and $F_{t,u}(\cdot,\omega)$ is continuous on $\mathbb{R}^d$, for any $\omega \in \Omega$, by (3.11). It follows that also the map: $\omega \mapsto F_{t,u}(X_u^{s,x}(\omega),\omega)$ is $\mathcal{F}_{s,t}^L$-measurable.

It is clear that (V_t^x) solves (3.10) on Ω_s' when $s \le t \le u$ (recall (3.7)). Let us consider the case when $t \ge u$. According to (3.10) we know that on Ω_u' we have

$$X_t^{u,X_u^{s,x}} = X_u^{s,x} + \int_u^t b\big(r, X_r^{u,X_u^{s,x}}\big)\,dr + L_t - L_u, \quad t \ge u. \tag{3.13}$$

Hence on $\Omega_u' \cap \Omega_s'$ we have for $t \ge u$

$$\begin{aligned} V_t^x = X_t^{u,X_u^{s,x}} &= x + \int_s^u b\big(r, X_r^{s,x}\big)\,dr + L_u - L_s \\ &\quad + \int_u^t b\big(r, X_r^{u,X_u^{s,x}}\big)\,dr + L_t - L_u = x + \int_s^t b\big(r, V_r^x\big)\,dr + L_t - L_s. \end{aligned}$$

It follows that (V_t^x) solves (3.10) on $\Omega_s' \cap \Omega_u'$ when $s \le t \le T$. By Hypothesis 1 we infer that, for any $x \in \mathbb{R}^d$, on some almost sure event $\Omega_{s,u,x}$ we have that $V_t^x = X_t^{s,x}$, $t \in [s,T]$. In particular we get $V_t^x = X_t^{s,x}$, $t \in [u,T]$ and this proves (3.12) at least on an almost sure event $\Omega_{s,u,x}$.

To remove the dependence on x in the almost sure event, we note that the mapping: $x \mapsto V_t^x(\omega)$ is continuous from $\mathbb{R}^d$ into $\mathbb{R}^d$, for any $\omega \in \Omega$, $t \in [0,T]$ (see (3.11)). Arguing as in the final part of the proof of Lemma 3.2 we obtain that $X_t^{s,x}(\omega) = V_t^x(\omega)$, for $t \in [u,T]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega_{s,u} = \bigcap_{x\in\mathbb{Q}^d} \Omega_{s,u,x}$. This proves (3.12). □

Following [26] page 169 (see also Problem 48 in [26]) we introduce the space $C(\mathbb{R}^d; G_0)$ consisting of all continuous functions from $\mathbb{R}^d$ into $G_0 = C([0,T];\mathbb{R}^d)$ endowed with the compact-open topology (or the topology of the uniform convergence on compact sets). This is a complete metric space endowed with the following metric:

$$d_0(f,g) = \sum_{N\ge1} \frac{1}{2^N} \frac{\sup_{|x|\le N} \|f(x) - g(x)\|_{G_0}}{1 + \sup_{|x|\le N} \|f(x) - g(x)\|_{G_0}}, \quad f,g \in C\big(\mathbb{R}^d; G_0\big). \tag{3.14}$$

It is well-know that $C(\mathbb{R}^d; G_0)$ is also *separable* (see, for instance, [16]; on the other hand $C_b(\mathbb{R}^d; G_0)$ is not separable). We will also consider the following projections

$$\pi_x : C\big(\mathbb{R}^d; G_0\big) \to G_0, \quad \pi_x(f) = f(x) \in G_0, \quad x \in \mathbb{R}^d, f \in C\big(\mathbb{R}^d; G_0\big) \tag{3.15}$$

(each π_x is a continuous map). According to Lemma 3.2 for any $s \in [0, T]$ the random field $(Y^{s,x})_{x\in\mathbb{R}^d}$ has continuous paths. It is not difficult to prove that, for any $s \in [0, T]$, the mapping:

$$\omega \mapsto Y^s(\omega) = Y^{s,\cdot}(\omega) \tag{3.16}$$

is measurable from $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $C(\mathbb{R}^d; G_0)$. Indeed thanks to the separability of $C(\mathbb{R}^d; G_0)$ to check the measurability it is enough to prove that counter-images of balls $B_r(f_0) = \{f \in C(\mathbb{R}^d; G_0) : \sum_{N\ge 1} \frac{1}{2^N} \times \frac{\sup_{\{|x|\le N, x\in\mathbb{Q}^d\}} \|f(x)-f_0(x)\|_{G_0}}{1+\sup_{\{|x|\le N, x\in\mathbb{Q}^d\}} \|f(x)-f_0(x)\|_{G_0}} < r\}$, $r > 0$, $f_0 \in C(\mathbb{R}^d; G_0)$, are events in Ω.

In the sequel we will set $Y = (Y^s)_{s\in[0,T]}$ to denote the previous stochastic process with values in $C(\mathbb{R}^d; G_0)$ and defined on $(\Omega, \mathcal{F}, P)$.

4. A version of the solution which is càdlàg with respect to the initial time *s*

In Theorem 4.4 we will prove the existence of a càdlàg modification Z of the process $Y = (Y^s)_{s\in[0,T]}$ with values in $C(\mathbb{R}^d; G_0)$ (cf. (3.16)). In particular Z is a modification of Y which is càdlàg in s uniformly in x, when x varies on compact sets of $\mathbb{R}^d$. In Lemma 4.5 we will study important properties of Z. Before discussing on càdlàg modifications we recall a standard definition.

A process $X = (X_t)_{t\in[0,T]}$ defined on $(\Omega, \mathcal{F}, P)$ with values in a metric space (S, d) is *stochastically continuous (or continuous in probability)* if for any $t_0 \in [0, T]$, X_t converges to X_{t_0} in probability (see [12] for more details).

Important results on càdlàg modifications for stochastic processes were given by Gikhman and Skorokhod (see Section III.4 in [12]). We will use a recent result given in Theorem 4.2 of [4]. In contrast with [12] the proof of this theorem does not require the separability of the stochastic process. It is stated in [4] for stochastic processes (X_t) when $t \in [0, 1]$. However a simple rescaling argument shows that it holds when $t \in [0, T]$, for any $T > 0$.

Theorem 4.1 ([4]). *Let $X = (X_t)_{t\in[0,T]}$ be a stochastically continuous process defined on a complete probability space and with values in a complete metric space (S, d). Let $0 \le s < t < u \le T$ and define $\triangle(s, t, u) = d(X_s, X_t) \wedge d(X_t, X_u)$. A sufficient assumption in order that X has a modification with càdlàg paths is the following one: there exist non-negative real functions δ and x_0 (δ is non-decreasing and continuous on $[0, T]$, $\delta(0) = 0$, and x_0 is decreasing and integrable on $(0, T]$) such that the following conditions hold, for any $0 \le s < t < u \le T$, $M > 0$,*

$$E\big[\triangle(s, t, u) 1_{\triangle(s,t,u)\ge M}\big] \le \delta(u - s) \int_0^{P(\triangle(s,t,u)\ge M)} x_0(r)\, dr, \tag{4.1}$$

$$\int_0^1 \left(u^{-1} \int_0^u x_0(r)\, dr\right) \frac{\delta(u)}{u}\, du < \infty. \tag{4.2}$$

The next result follows easily (cf. Section III.4 in [12]).

Corollary 4.2. *Let $X = (X_t)_{t\in[0,T]}$ be a stochastically continuous process with values in a complete metric space (S, d). A sufficient condition in order that X has a càdlàg modification is the following one: there exists $q > 1/2$ and $r > 0$ such that, for any $0 \le s < t < u \le T$, we have*

$$E\big[d(X_s, X_t)^q \cdot d(X_t, X_u)^q\big] \le C|u - s|^{1+r}. \tag{4.3}$$

Proof. In order to apply Theorem 4.1 we introduce $x_0(h) = \frac{2q-1}{2q} h^{-1/2q}$, $h \in (0, T]$. Let us fix $0 \le s < t < u$ and $M > 0$. Noting that for $a, b \ge 0$ we have $a \wedge b \le \sqrt{a}\sqrt{b}$. We find by the Hölder inequality

$$\begin{aligned} E\big[\triangle(s, t, u) 1_{\triangle(s,t,u)\ge M}\big] &\le \big(E\big[\triangle(s, t, u)^{2q}\big]\big)^{1/2q} \big(P\big(\triangle(s, t, u) \ge M\big)\big)^{\frac{2q-1}{2q}} \\ &\le E\big[d(X_s, X_t)^q \cdot d(X_t, X_u)^q\big]^{1/2q} \int_0^{P(\triangle(s,t,u)\ge M)} x_0(r)\, dr \\ &\le C^{1/2q} |u - s|^{(1+r)/2q} \int_0^{P(\triangle(s,t,u)\ge M)} x_0(r)\, dr. \end{aligned}$$

Setting $\delta(h) = h^{(1+r)/2q}$, $h \in [0,T]$, we see that $\int_0^T \delta(u) u^{-1-\frac{1}{2q}}\, du < \infty$ is equivalent to (4.2); we get the assertion. □

We now prove the stochastic continuity of Y.

Lemma 4.3. *Consider (2.2) with $b \in L^\infty(0,T; C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta \in (0,1]$, and suppose that L and b satisfy Hypotheses 1 and 2. Then the process $Y = (Y^s)$ with values in $C(\mathbb{R}^d; G_0)$ (see (3.16)) is continuous in probability.*

Proof. Let us fix $s \in [0,T]$. We have to prove that

$$\lim_{s'\to s} P\Big(\sup_{|x|\le N}\sup_{t\in[0,T]}\big|Y_t^{s,x} - Y_t^{s',x}\big| > r\Big) = 0, \quad \text{for any } r > 0,\ N \ge 1. \tag{4.4}$$

Indeed this is equivalent to $\lim_{s'\to s} P(d_0(Y^s, Y^{s'}) > r) = 0$, $r > 0$. To this purpose it is enough to check both the left and the right continuity in (4.4). Let us check the right continuity in s (assuming $s \in [0,T)$). The proof of the left-continuity in s can be done in a similar way. Since $C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d) \subset C_b^{0,\beta'}(\mathbb{R}^d;\mathbb{R}^d)$ for $0 < \beta' \le \beta \le 1$ we may suppose that β is sufficiently small; we will assume (cf. Hypothesis 2)

$$\beta(2d+1) < 2d\theta. \tag{4.5}$$

Let $(s_n) \subset\,]s,T]$ with $s_n \to s$. We have to prove that for fixed $N \ge 1$, $\delta > 0$,

$$\lim_{n\to\infty} P\Big(\sup_{|x|\le N}\sup_{t\in[0,T]}\big|Y_t^{s,x} - Y_t^{s_n,x}\big| > \delta\Big) = 0. \tag{4.6}$$

If we show that

$$E\Big[\sup_{0\le t\le T}\sup_{|x|\le N}\big|Y_t^{s,x} - Y_t^{s_n,x}\big|\Big] \to 0 \quad \text{as } n \to \infty \tag{4.7}$$

then (4.6) follows. Let us fix $n \ge 1$ and consider the random variable $J_{t,x,n,s} = |Y_t^{s,x} - Y_t^{s_n,x}|$. If $t \le s$ we find $J_{t,x,n,s} = 0$. If $s \le t \le s_n$ then, for any $x \in \mathbb{R}^d$, on some almost sure event Ω_{s,s_n} (independent of x and t; see (3.7))

$$J_{t,x,n,s} = \left|\int_s^t b\big(r, Y_r^{s_n,x} + (L_r - L_s)\big)\, dr\right| \le \|b\|_0 |t-s| \le \|b\|_0 |s - s_n|.$$

Hence in order to get (4.7) we need to prove that

$$E\Big[\sup_{s_n\le t\le T}\sup_{|x|\le N}\big|Y_t^{s,x} - Y_t^{s_n,x}\big|\Big] \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

Let $t \ge s_n$. We have on Ω_{s,s_n}

$$\begin{aligned}\sup_{|x|\le N}\big|Y_t^{s,x} - Y_t^{s_n,x}\big| &\le \sup_{|x|\le N}\left|\int_s^t b\big(r, X_r^{s,x}\big)\, dr - \int_{s_n}^t b\big(r, X_r^{s_n,x}\big)\, dr\right| \\ &\le 2|s-s_n|\,\|b\|_0 + \sup_{|x|\le N}\int_{s_n}^t \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big|\, dr. \end{aligned}\tag{4.9}$$

By Lemma 3.3 on some almost sure event $\Omega'_{s,s_n} \subset \Omega_{s,s_n}$ (independent of x and r) we have for $r \in [s_n, T]$

$$\begin{aligned}\sup_{|x|\le N}\big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big| &= \sup_{|x|\le N}\big|b\big(r, X_r^{s_n, X_{s_n}^{s,x}}\big) - b\big(r, X_r^{s_n,x}\big)\big| \le [b]_{\beta,T}\sup_{|x|\le N}\sup_{r\in[0,T]}\big|X_r^{s_n,X_{s_n}^{s,x}} - X_r^{s_n,x}\big|^\beta \\ &= [b]_{\beta,T}\sup_{|x|\le N}\big\|Y^{s_n,X_{s_n}^{s,x}} - Y^{s_n,x}\big\|_{G_0}^\beta.\end{aligned}$$

By Lemma 3.2 with $p = 4d$, setting $U_{s'} = U_{s',p}$, $s' \in [0, T]$, we get

$$\sup_{|x|\le N} \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big|$$
$$\le [b]_{\beta,T}\big[\big(|x| \vee \big|X_{s_n}^{s,x}\big|\big)^{\frac{2d+1}{4d}} \vee 1\big]^{\beta} U_{s_n}^{\beta} \sup_{|x|\le N}\big|x - X_{s_n}^{s,x}\big|^{\beta/2}. \tag{4.10}$$

Noting that, for $|x| \le N$, $n \ge 1$, $|X_{s_n}^{s,x}| \le N + 2T\|b\|_0 + |L_{s_n} - L_s|$ we obtain on Ω'_{s,s_n}

$$\sup_{|x|\le N} \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big| \le [b]_{\beta,T} V_{s,s_n,N}^{\beta} \sup_{|x|\le N}\big|x - X_{s_n}^{s,x}\big|^{\beta/2}, \tag{4.11}$$

$r \in [s_n, T]$, where we have introduced the random variables

$$V_{s,s',N} = \big[N^{\frac{2d+1}{4d}} + \big(2T\|b\|_0\big)^{\frac{2d+1}{4d}} + |L_{s'} - L_s|^{\frac{2d+1}{4d}}\big]U_{s'}, \tag{4.12}$$

$0 \le s < s' \le T$. By Remark 2.1 and (4.5) we know that, for any $n \ge 1$,

$$E\big[|L_{s_n} - L_s|^{\frac{\beta(2d+1)}{2d}}\big] = E\big[|L_{s_n - s}|^{\frac{\beta(2d+1)}{2d}}\big] \le E\Big[\sup_{s\in[0,T]} |L_s|^{\frac{\beta(2d+1)}{2d}}\Big] < \infty,$$

since $E[\sup_{r\in[0,T]} |L_r|^{\theta}] < \infty$. Using also that $\sup_{r\in[0,T]} E[U_{r,p}^{2\beta}] = k' < \infty$ (see (3.6)) we obtain by the Cauchy–Schwarz inequality

$$\sup_{0\le s<s'\le T} E\big[V_{s,s',N}^{\beta}\big] = k_0 < \infty \tag{4.13}$$

(k_0 also depends on N). Let us revert to (4.11). Since

$$\big|X_{s_n}^{s,x} - x\big| \le \int_s^{s_n} \big|b\big(r, X_r^{s,x}\big)\big|\,dr + |L_{s_n} - L_s| \le \|b\|_0|s - s_n| + |L_{s_n} - L_s|, \tag{4.14}$$

for any $x \in \mathbb{R}^d$, $n \ge 1$, we obtain for $r \in [s_n, T]$

$$\sup_{|x|\le N} \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big| \le [b]_{\beta,T} V_{s,s_n,N}^{\beta}\big(\|b\|_0|s - s_n| + |L_{s_n} - L_s|\big)^{\beta/2} \tag{4.15}$$

and so (cf. (4.9))

$$\sup_{|x|\le N} \int_{s_n}^{t} \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big|\,dr \le T[b]_{\beta,T} V_{s,s_n,N}^{\beta}\big(\|b\|_0|s - s_n| + |L_{s_n} - L_s|\big)^{\beta/2}.$$

Let us define the random variables $Z_n = \|b\|_0|s - s_n| + |L_{s_n} - L_s|$. By the stochastic continuity of L we know that

$$\lim_{n\to\infty} P(Z_n > \delta) = 0, \quad \delta > 0. \tag{4.16}$$

Using (4.9) on an almost sure event Ω'_{s,s_n}, for any $\delta > 0$, we have

$$\sup_{s_n\le t\le T} \sup_{|x|\le N} \big|Y_t^{s,x} - Y_t^{s_n,x}\big|$$
$$\le 2|s - s_n|\|b\|_0 + (1_{\{Z_n\le\delta\}} + 1_{\{Z_n>\delta\}}) \cdot \sup_{|x|\le N} \int_{s_n}^{t} \big|b\big(r, X_r^{s,x}\big) - b\big(r, X_r^{s_n,x}\big)\big|\,dr$$
$$\le T1_{\{Z_n\le\delta\}}[b]_{\beta,T} V_{s,s_n,N}^{\beta}\delta^{\beta/2} + 2T\|b\|_0 1_{\{Z_n>\delta\}} + 2|s - s_n|\|b\|_0.$$

Applying the expectation and using (4.13) we arrive at

$$E\Big[\sup_{s_n \le t \le T} \sup_{|x| \le N} \big|Y_t^{s,x} - Y_t^{s_n,x}\big|\Big]$$
$$\le 2|s - s_n| \|b\|_0 + k_0 T [b]_{\beta,T} \delta^{\beta/2} + 2T \|b\|_0 P(Z_n > \delta).$$

Now, using (4.16), we obtain easily (4.8) and this completes the proof. □

In the next result we need the Lévy-Itô formula. To this purpose we recall the definition of Poisson random measure N: $N((0,t] \times H) = \sum_{0<s\le t} 1_H(\triangle L_s)$ for any Borel set H in $\mathbb{R}^d \setminus \{0\}$; $\triangle L_s = L_s - L_{s-}$ denotes the jump size of L at time $s > 0$. The Lévy-Itô decomposition of the given Lévy process L on $(\Omega, \mathcal{F}, P)$ with generating triplet $(\nu, Q, 0)$ (see Section 19 in [28] or Theorem 2.4.16 in [1]) asserts that there exists a Q-Wiener process $B = (B_t)$ on $(\Omega, \mathcal{F}, P)$ independent of N with covariance matrix Q (cf. (2.1)) such that on some almost sure event Ω' we have

$$L_t = A_t + B_t + C_t, \quad t \ge 0, \text{ where} \tag{4.17}$$

$$A_t = \int_0^t \int_{\{|x|\le 1\}} x \tilde{N}(ds, dx), \qquad C_t = \int_0^t \int_{\{|x|>1\}} x N(ds, dx); \tag{4.18}$$

$\tilde{N}$ is the compensated Poisson measure (i.e., $\tilde{N}(dt, dx) = N(dt, dx) - dt\,\nu(dx)$).

Theorem 4.4. *Under the same assumptions of Lemma 4.3 consider the process $Y = (Y^s)$ with values in $C(\mathbb{R}^d; G_0)$ (see (3.16)). There exists a modification $Z = (Z^s)$ of Y with càdlàg paths.*

Proof. To prove the assertion we will apply Corollary 4.2. We already know by Lemma 4.3 that Y is continuous in probability.

In the proof we will use the fact that $\int_{\{|x|>1\}} |x|^\theta \nu(dx) < \infty$ for some $\theta \in (0, 1)$. This is not restrictive according to Remark 2.1. We proceed in some steps.

Step I. We establish simple moment estimates for the Lévy process L, using the Ito–Lévy decomposition (4.18).

Using basic properties of the martingales (A_t) and (B_t) we obtain

$$E|B_t|^2 = C_Q t, \qquad E|A_t|^2 = t \int_{\{|x|\le 1\}} |x|^2 \nu(dx), \quad t \ge 0 \tag{4.19}$$

Now we concentrate on the compound Poisson process $C = (C_t)$; on some almost sure event Ω' we have

$$|C_t|^\theta = \Big|\sum_{0<s\le t} \triangle L_s 1_{\{|\triangle L_s|>1\}}\Big|^\theta \le \sum_{0<s\le t} |\triangle L_s|^\theta 1_{\{|\triangle L_s|>1\}},$$

since the random sum is finite for any $\omega \in \Omega'$ and $\theta \le 1$. Let $f_0(x) = 1_{\{|x|>1\}}(x)|x|^\theta$, $x \in \mathbb{R}^d$; using a well-know result (cf. pages 145 and 150 in [17] or Section 2.3.2 in [1]) we get

$$E\Big[\sum_{0<s\le t} |\triangle L_s|^\theta 1_{\{|\triangle L_s|>1\}}\Big] = E\Big[\int_0^t \int_{\{|x|>1\}} |x|^\theta N(ds, dx)\Big]$$
$$= \int_{\mathbb{R}^d} f_0(x)\nu(dx) = \int_{\{|x|>1\}} |x|^\theta \nu(dx)$$

and so

$$E|C_t|^\theta \le t \int_{\{|x|>1\}} |x|^\theta \nu(dx) = c_0 t, \quad t \ge 0. \tag{4.20}$$

Step II. Let $0 \le s < s' \le T$. Similarly to the proof of Lemma 4.3 in this step we establish estimates for the random variable $J_{t,x,s,s'} = |Y_t^{s,x} - Y_t^{s',x}|$.

If $t \le s$ we have $J_{t,x,s,s'} = 0$, $x \in \mathbb{R}^d$. If $s \le t \le s'$ then, for any $x \in \mathbb{R}^d$, on some almost sure event $\Omega_{s,s'}$ (independent of t and x) we find

$$\left|Y_t^{s,x} - Y_t^{s',x}\right| \le \|b\|_0 |t-s| \le \|b\|_0 \left|s - s'\right|.$$

Let $t \ge s'$ and $N \ge 1$. We have (cf. (4.9))

$$\sup_{|x|\le N} \left|Y_t^{s,x} - Y_t^{s',x}\right| \le 2\left|s - s'\right| \|b\|_0 + \sup_{|x|\le N} \int_{s'}^t \left|b\left(r, X_r^{s,x}\right) - b\left(r, X_r^{s',x}\right)\right| dr. \tag{4.21}$$

Moreover, there exists an almost sure event $\Omega'_{s,s'} \subset \Omega_{s,s'}$ such that on $\Omega'_{s,s'}$ we have for $r \in [s', T]$

$$\begin{aligned}\sup_{|x|\le N} \left|b\left(r, X_r^{s,x}\right) - b\left(r, X_r^{s',x}\right)\right| &= \sup_{|x|\le N} \left|b\left(r, X_r^{s', X_{s'}^{s,x}}\right) - b\left(r, X_r^{s',x}\right)\right| \\ &\le [b]_{\beta,T} \sup_{|x|\le N} \left\|Y^{s', X_{s'}^{s,x}} - Y^{s',x}\right\|_{G_0}^{\beta}.\end{aligned}$$

Now we use Lemma 3.2 with $p \ge 32d$ *to be fixed* and get, for any $r \in [s', T]$ on $\Omega'_{s,s'}$ (cf. (4.10) and (4.11))

$$\begin{aligned}&\sup_{|x|\le N} \left|b\left(r, X_r^{s,x}\right) - b\left(r, X_r^{s',x}\right)\right| \\ &\quad \le [b]_{\beta,T} \left[\left(|x| \vee \left|X_{s'}^{s,x}\right|\right)^{\frac{2d+1}{p}} \vee 1\right]^{\beta} U_{s',p}^{\beta} \sup_{|x|\le N} \left|x - X_{s'}^{s,x}\right|^{\beta(1-\frac{2d}{p})}\end{aligned}$$

and so

$$\begin{aligned}&\sup_{|x|\le N} \left|b\left(r, X_r^{s,x}\right) - b\left(r, X_r^{s',x}\right)\right| \le [b]_{\beta,T} V_{s,s',N,p}^{\beta} \sup_{|x|\le N} \left|x - X_{s'}^{s,x}\right|^{\beta(1-\frac{2d}{p})}, \\ &V_{s,s',N,p} = \left[N^{\frac{2d+1}{p}} + \left(2T\|b\|_0\right)^{\frac{2d+1}{p}} + |L_{s'} - L_s|^{\frac{2d+1}{p}}\right] U_{s',p},\end{aligned} \tag{4.22}$$

Coming back to (4.21) we find for $t \ge s'$ on $\Omega'_{s,s'}$

$$\begin{aligned}&\sup_{s'\le t\le T} \sup_{|x|\le N} \left|Y_t^{s,x} - Y_t^{s',x}\right| \\ &\quad \le 2\left|s - s'\right| \|b\|_0 + T 1_{\{\sup_{|x|\le N} |X_{s'}^{s,x} - x| \le c_0 |s-s'|^{1/8}\}} [b]_{\beta,T} V_{s,s',N,p}^{\beta} \sup_{|x|\le N} \left|x - X_{s'}^{s,x}\right|^{\beta(1-\frac{2d}{p})} \\ &\qquad + 2T\|b\|_0 1_{\{\sup_{|x|\le N} |X_{s'}^{s,x} - x| > c_0 |s-s'|^{1/8}\}},\end{aligned}$$

with $c_0 > 0$ such that $c_0 \rho^{1/8} - \|b\|_0 \rho \ge \rho^{1/8}$, for any $\rho \in [0, T]$. We obtain on $\Omega'_{s,s'}$

$$\begin{aligned}\sup_{|x|\le N} \sup_{t\in[0,T]} \left|Y_t^{s,x} - Y_t^{s',x}\right| &= \sup_{|x|\le N} \left\|Y^{s,x} - Y^{s',x}\right\|_{G_0} \\ &\le C_1\left|s - s'\right| + C_1 V_{s,s',N,p}^{\beta} \left|s - s'\right|^{\frac{\beta}{8}(1-\frac{2d}{p})} \\ &\quad + C_1 1_{\{\sup_{|x|\le N} |X_{s'}^{s,x} - x| > c_0 |s-s'|^{1/8}\}},\end{aligned} \tag{4.23}$$

with $C_1 = 2(T \vee 1)\|b\|_{\beta,T} c_0^\beta$. Since, for any $x \in \mathbb{R}^d$, $|X_{s'}^{s,x} - x| \le |s'-s|\|b\|_0 + |L_{s'} - L_s|$ and, moreover, $c_0|s-s'|^{1/8} - \|b\|_0|s-s'| \ge |s-s'|^{1/8}$, we find on $\Omega'_{s,s'}$

$$\sup_{|x|\le N} \|Y^{s,x} - Y^{s',x}\|_{G_0} \le C_1\big(|s-s'| + V^\beta_{s,s',N,p}|s-s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} + 1_{\{|L'_s - L_s| > |s-s'|^{1/8}\}}\big). \tag{4.24}$$

Note that C_1 is independent of s, s' and N.

Step III. Using (4.24) we provide an estimate for $d_0(Y^s, Y^{s'})$ (cf. (3.14)) when $0 \le s < s' \le T$.

We have (see (4.22))

$$V^\beta_{s,s',N,p} \le \big[N^{\frac{\beta(2d+1)}{p}} + \big(2T\|b\|_0\big)^{\frac{\beta(2d+1)}{p}} + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}}\big] U^\beta_{s',p}$$

and so

$$\begin{aligned} d_0\big(Y^s, Y^{s'}\big) &= \sum_{N\ge1} \frac{1}{2^N} \frac{\sup_{|x|\le N}\|Y^{s,x} - Y^{s',x}\|_{G_0}}{1 + \sup_{|x|\le N}\|Y^{s,x} - Y^{s',x}\|_{G_0}} \\ &\le C_1|s-s'| + C_1 1_{\{|L'_s - L_s| > |s-s'|^{1/8}\}} \\ &\quad + C_1 U^\beta_{s',p}|s-s'|^{\frac{\beta}{8}(1-\frac{2d}{p})} \sum_{N\ge1}\frac{1}{2^N}\big[N^{\frac{\beta(2d+1)}{p}} + \big(2T\|b\|_0\big)^{\frac{\beta(2d+1)}{p}} + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}}\big] \\ &\le C_3\big(|s-s'| + 1_{\{|L'_s - L_s| > |s-s'|^{1/8}\}} + U^\beta_{s',p}|s-s'|^{\frac{\beta}{8}(1-\frac{2d}{p})}\big(1 + |L_{s'} - L_s|^{\frac{\beta(2d+1)}{p}}\big)\big), \end{aligned}$$

where $C_3 = C_3(\beta, T, \|b\|_{\beta,T}, d, p) > 0$. Recall that $p \ge 32d$ has to be fixed.

Step IV. Let now $0 \le s_1 < s_2 < s_3 \le T$ and set

$$\rho = s_3 - s_1.$$

We will apply Corollary 4.2 with $q = 8/\beta$. Let us fix $p \ge 32d$ (i.e., $1 - \frac{2d}{p} \ge 15/16$) such that $\frac{8(2d+1)}{p} < \frac{\theta}{4}$ and introduce the random variable

$$Z = 1 + \sup_{s\in[0,T]} |L_s|^{\frac{8(2d+1)}{p}}.$$

Clearly we have that $|L_{s'} - L_s|^{\frac{8(2d+1)}{p}} \le 2Z$, $0 \le s < s' \le T$. Moreover by Remark 2.1 we know that $E[Z^4] < \infty$. Using Step III and the previous estimates we will check condition (4.3). In the sequel we denote by C_k or c_k positive constants which may depend on $\beta, T, \|b\|_{\beta,T}, \theta$ and d but are independent of s_1, s_2 and s_3. We have

$$\begin{aligned} \Gamma &= E\big[\big(d_0\big(Y^{s_1}, Y^{s_2}\big)\cdot d_0\big(Y^{s_2}, Y^{s_3}\big)\big)^{8/\beta}\big] \\ &\le C_4 E\big[\big(|s_3 - s_1|^{8/\beta} + 1_{\{|L_{s_2} - L_{s_1}| > |s_2 - s_1|^{1/8}\}} + Z U^8_{s_2,p}|s_3 - s_1|^{1-\frac{2d}{p}}\big) \\ &\quad \cdot \big(|s_3 - s_1|^{8/\beta} + 1_{\{|L_{s_3} - L_{s_2}| > |s_3 - s_2|^{1/8}\}} + Z U^8_{s_3,p}|s_3 - s_1|^{1-\frac{2d}{p}}\big)\big]. \end{aligned}$$

We denote by $c_2 \ge 1$ a constant such that $t^{8/\beta} \le c_2 t^{1-\frac{2d}{p}}$, $t \in [0,T]$. We obtain ($\rho = s_3 - s_1$)

$$\begin{aligned} \Gamma &\le c_2^2 C_4 E\big[\big(\rho^{1-\frac{2d}{p}} + 1_{\{|L_{s_2} - L_{s_1}| > |s_2 - s_1|^{1/8}\}} + Z U^8_{s_2,p}\rho^{1-\frac{2d}{p}}\big) \\ &\quad \cdot \big(\rho^{1-\frac{2d}{p}} + 1_{\{|L_{s_3} - L_{s_2}| > |s_3 - s_2|^{1/8}\}} + Z U^8_{s_3,p}\rho^{1-\frac{2d}{p}}\big)\big] \\ &\le C_5(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4), \end{aligned}$$

where $\Gamma_1 = E[1_{\{|L_{s_2}-L_{s_1}|>|s_2-s_1|^{1/8}\}} \cdot 1_{\{|L_{s_3}-L_{s_2}|>|s_3-s_2|^{1/8}\}}]$,

$$\Gamma_2 = \rho^{1-\frac{2d}{p}}\big[P\big(|L_{s_3}-L_{s_2}|>|s_3-s_2|^{1/8}\big)+P\big(|L_{s_2}-L_{s_1}|>|s_2-s_1|^{1/8}\big)\big],$$

$$\Gamma_3 = \rho^{1-\frac{2d}{p}}E\big[1_{\{|L_{s_3}-L_{s_2}|>|s_3-s_2|^{1/8}\}}ZU^8_{s_2,p}+1_{\{|L_{s_2}-L_{s_1}|>|s_2-s_1|^{1/8}\}}ZU^8_{s_3,p}\big],$$

$$\Gamma_4 = \rho^{2(1-\frac{2d}{p})}+\rho^{2(1-\frac{2d}{p})}E\big[ZU^8_{s_2,p}+ZU^8_{s_3,p}+Z^2U^8_{s_2,p}U^8_{s_3,p}\big].$$

It is not difficult to treat Γ_4. Indeed we can use the Cauchy-Schwarz inequality and

$$\sup_{s\in[0,T]} E\big[U^p_{s,p}\big]=k'<\infty \tag{4.25}$$

(see (3.6)) in order to control the expectation in Γ_4. For instance, we have

$$E\big[Z^2U^8_{s_2,p}U^8_{s_3,p}\big]\le E\big[Z^4\big]^{1/2}\Big(\sup_{s\in[0,T]}E\big[U^{32}_{s,p}\big]\Big)^{1/2}<\infty, \tag{4.26}$$

since $E[Z^4]<\infty$ and $p\ge 32d$. We obtain

$$\Gamma_4\le C_6\rho^{2(1-\frac{2d}{p})}=C_6|s_3-s_1|^{2(1-\frac{2d}{p})}\le C_6|s_3-s_1|^{30/16}. \tag{4.27}$$

To estimates the other terms we need to control $P(|L_s|>|s|^{1/8})$, $s\ge 0$. To this purpose we use Step I. We have

$$P\big(|L_s|>s^{1/8}\big)\le P\big(|B_s|>s^{1/8}/3\big)+P\big(|A_s|>s^{1/8}/3\big)+P\big(|C_s|>s^{1/8}/3\big).$$

By Chebychev inequality we get for $s\ge 0$

$$P\big(|L_s|>s^{1/8}\big)\le \frac{9}{s^{1/4}}E\big[|B_s|^2+|A_s|^2\big]+\frac{3^\theta}{s^{\theta/8}}E\big[|C_s|^\theta\big]\le c_3\big(s^{3/4}+s^{1-\frac{\theta}{8}}\big). \tag{4.28}$$

Using (4.28) and (4.25) we can estimate Γ_2 and Γ_3. For instance, since the increments of L are independent and stationary, we find

$$\begin{aligned}\Gamma_2&\le\rho^{1-\frac{2d}{p}}\big[P\big(|L_{s_3-s_2}|>|s_3-s_2|^{1/8}\big)+P\big(|L_{s_2-s_1}|>|s_2-s_1|^{1/8}\big)\big]\\&\le 2c_3\rho^{1-\frac{2d}{p}}\big(\rho^{3/4}+\rho^{1-\frac{\theta}{8}}\big).\end{aligned}$$

We can proceed similary for Γ_3 (see also (4.26)):

$$\begin{aligned}\Gamma_3\le{}&\rho^{1-\frac{2d}{p}}\big(E\big[Z^4\big]\big)^{1/4}\Big(\sup_{s\in[0,T]}E\big[U^{32}_{s,p}\big]\Big)^{1/4}\big[\big(P\big(|L_{s_3-s_2}|>|s_3-s_2|^{1/8}\big)\big)^{1/2}\\&+\big(P\big(|L_{s_2-s_1}|>|s_2-s_1|^{1/8}\big)\big)^{1/2}\big]\le C_8\rho^{1-\frac{2d}{p}}\big(\rho^{3/8}+\rho^{\frac{1}{2}(1-\frac{\theta}{8})}\big).\end{aligned}$$

Note that $(1-\frac{2d}{p})+3/8>5/4$ and $(1-\frac{2d}{p})+\frac{1}{2}(1-\frac{\theta}{8})>5/4$. We get

$$\Gamma_2+\Gamma_3\le C_9\rho^{\frac{5}{4}}=C_9|s_3-s_1|^{5/4}. \tag{4.29}$$

Finally we consider

$$\begin{aligned}\Gamma_1&\le P\big(|L_{s_3-s_2}|>|s_3-s_2|^{1/8}\big)\cdot P\big(|L_{s_2-s_1}|>|s_2-s_1|^{1/8}\big)\\&\le 2c_3\big(\rho^{3/2}+\rho^{2(1-\frac{\theta}{8})}\big)\le c_4|s_3-s_1|^{3/2}.\end{aligned} \tag{4.30}$$

Collecting together estimates (4.27), (4.29) and (4.30) we arrive at

$$E\big[\big(d_0\big(Y^{s_1}, Y^{s_2}\big) \cdot d_0\big(Y^{s_2}, Y^{s_3}\big)\big)^{8/\beta}\big] \le C_0 |s_3 - s_1|^{5/4}$$

and this finishes the proof. □

Taking into account Theorem 4.4 and using the projections π_x (see (3.15)), in the sequel we write, for $x \in \mathbb{R}^d$, $s, t \in [0, T]$,

$$Z^s = \big(Z^{s,x}\big)_{x\in\mathbb{R}^d}, \quad \text{with } \pi_x\big(Z^s\big) = Z^{s,x} \in G_0. \tag{4.31}$$

Recall that on some almost sure event Ω_s, $Y^{s,x} = Z^{s,x}$, $s \in [0, T]$, $x \in \mathbb{R}^d$ (cf. (3.16)).

Lemma 4.5. *Under the same assumptions of Lemma 4.3 consider the càdlàg process Z with values in $C(\mathbb{R}^d; G_0)$ of Theorem 4.4. The following statements hold*: (i) *There exists an almost sure event Ω_1 (independent of s, t and x) such that for any $\omega \in \Omega_1$, we have that $t \mapsto L_t(\omega)$ is càdlàg, $L_0(\omega) = 0$ and $s \mapsto Z^s(\omega)$ is càdlàg; further, for any $\omega \in \Omega_1$,*

$$Z_t^{s,x}(\omega) = x + \int_s^t b\big(r, Z_r^{s,x}(\omega) + L_r(\omega) - L_s(\omega)\big)\, dr, \quad s, t \in [0, T], s \le t, x \in \mathbb{R}^d.$$

Moreover, for $s \le t$, the r.v. $Z_t^{s,x}$ is $\mathcal{F}_{s,t}^L$-measurable (if $t \le s$, $Z_t^{s,x} = x$).

(ii) *There exists an almost sure event Ω_2 and a $\mathcal{B}([0, T]) \times \mathcal{F}$-measurable function $V_n : [0, T] \times \Omega \to [0, \infty]$, such that $\int_0^T V_n(s, \omega)\, ds < \infty$, for any integer $n > 2d$, $\omega \in \Omega_2$, and, further, the following inequality holds on Ω_2*

$$\sup_{t\in[0,T]} \big|Z_t^{s,x} - Z_t^{s,y}\big| \le |x - y|^{\frac{n-2d}{n}} \big[\big(|x| \vee |y|\big)^{\frac{2d+1}{n}} \vee 1\big] V_n(s, \cdot), \quad x, y \in \mathbb{R}^d, s \in [0, T]. \tag{4.32}$$

(iii) *There exists an almost sure event Ω_3 such that for any $\omega \in \Omega_3$ we have*

$$Z_t^{s,x}(\omega) + L_u(\omega) - L_s(\omega) = Z_t^{u, Z_u^{s,x}(\omega) + L_u(\omega) - L_s(\omega)}(\omega), \tag{4.33}$$

for any $s, u, t \in [0, T]$, $0 \le s < u \le T$, $x \in \mathbb{R}^d$.

Proof. (i) On some almost sure event Ω_s' (independent of t and x) we know that $(Y_t^{s,x})$ verifies the SDE (3.7) for any $x \in \mathbb{R}^d$ and $t \in [s, T]$. Moreover $Y_t^{s,x} = x$, $t < s$.

On the other hand on some almost sure event Ω_s we have $Y^{s,x} = \pi_x(Y^s) = \pi_x(Z^s)$, for any $x \in \mathbb{R}^d$, see (4.31). Using (Z^s), we can rewrite (3.7) on the event $\Omega_1 = \bigcap_{r\in\mathbb{Q}\cap[0,T]}(\Omega_r' \cap \Omega_r)$ as follows:

$$\big[\pi_x\big(Z^s\big)\big]_t = x + \int_s^t b\big(r, \big[\pi_x\big(Z^s\big)\big]_r + (L_r - L_s)\big)\, dr, \tag{4.34}$$

for any $s \in \mathbb{Q} \cap [0, T]$, $t \in [s, T]$, $x \in \mathbb{R}^d$.

Note that by Theorem 4.4 for any $\omega \in \Omega$ and any sequence $s_n \to s^+$ we have $d_0(Z^s(\omega), Z^{s_n}(\omega)) \to 0$ as $n \to \infty$. Take now $s \in [0, T)$ and let $(s_n) \subset \mathbb{Q} \cap [0, T]$ be a sequence monotonically decreasing to s. By the dominated convergence theorem and the right-continuity of L we have on Ω_1, for any $t > s$, $x \in \mathbb{R}^d$,

$$\begin{aligned}\big[\pi_x\big(Z^s\big)\big]_t &= \lim_{n\to\infty} \big[\pi_x\big(Z^{s_n}\big)\big]_t = x + \lim_{n\to\infty} \int_s^t 1_{\{r>s_n\}} b\big(r, \big[\pi_x\big(Z^{s_n}\big)\big]_r + (L_r - L_{s_n})\big)\, dr \\ &= x + \int_s^t b\big(r, \big[\pi_x\big(Z^s\big)\big]_r + (L_r - L_s)\big)\, dr\end{aligned}$$

and we get the assertion.

(ii) Since on Ω we have $Y^{s,x} = \pi_x(Y^s)$ we obtain by (2.6) and (3.9), for any $p \ge 2$,

$$\sup_{s\in[0,T]} E\Big[\sup_{s\le t\le T}\big|X_t^{s,x} - X_t^{s,y}\big|^p\Big] = \sup_{s\in[0,T]} E\Big[\sup_{0\le t\le T}\big|Y_t^{s,x} - Y_t^{s,y}\big|^p\Big]$$
$$= \sup_{s\in[0,T]} E\big[\big\|\pi_x(Z^s) - \pi_y(Z^s)\big\|_{G_0}^p\big] \le C(T)|x-y|^p, \quad x, y \in \mathbb{R}^d. \tag{4.35}$$

Let $s \in [0,T]$ and consider the random field $(\pi_x(Z^s))_{x\in\mathbb{R}^d}$ with values in G_0. Applying Theorem 3.1 with $\psi(x,\omega) = \pi_x(Z^s)(\omega)$ we obtain from (4.35) for $p > 2d$ similarly to (3.5): there exists a $V_p(s,\omega) \in [0,\infty]$ such that, for any $\omega \in \Omega$, $x, y \in \mathbb{R}^d$, $s \in [0,T]$,

$$\big\|\pi_x(Z^s)(\omega) - \pi_y(Z^s)(\omega)\big\|_{G_0} \le \big[\big(|x| \vee |y|\big)^{\frac{2d+1}{p}} \vee 1\big] V_p(s,\omega)|x-y|^{1-2d/p}, \tag{4.36}$$

with $V_p(s,\omega) = (\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\frac{\|\pi_x(Z^s)(\omega)-\pi_y(Z^s)(\omega)\|_{G_0}}{|x-y|})^p f(x)f(y)\,dx\,dy)^{1/p}$, $\omega \in \Omega$, $s \in [0,T]$ (f is defined in Theorem 3.1). Since the map: $(s,x,\omega) \mapsto \pi_x(Z^s)(\omega)$ is $\mathcal{B}([0,T]\times\mathbb{R}^d)\times\mathcal{F}$-measurable with values in G_0, it follows that the real map:

$$(s,x,y,\omega) \mapsto \big\|\pi_x(Z^s)(\omega) - \pi_y(Z^s)(\omega)\big\|_{G_0}|x-y|^{-1}1_{\{x\ne y\}}$$

is $\mathcal{B}([0,T]\times\mathbb{R}^{2d})\times\mathcal{F}$-measurable. By the Fubini theorem we deduce that also $V_p : [0,T]\times\Omega \to [0,\infty]$ is $\mathcal{B}([0,T])\times\mathcal{F}$-measurable. Hence we can consider the random variable $\omega \mapsto \int_0^T V_p(s,\omega)\,ds$ (with values in $[0,\infty]$). Since, with the same constant $C(T)$ appearing in (2.6),

$$\sup_{s\in[0,T]} E\big[\big|V_p(s,\cdot)\big|^p\big] \le C(d)\cdot C(T), \tag{4.37}$$

we find $E[(\int_0^T V_p(s,\cdot)\,ds)^p] \le T^{p-1}\int_0^T E[(V_p(s,\cdot))^p]\,ds \le T^{2p-1}c(d)C(T) < \infty$. It follows that, for any $p > 2d$, there exists an almost sure event Ω_p such that

$$\int_0^T V_p(s,\omega)\,ds < \infty, \quad \omega \in \Omega_p. \tag{4.38}$$

Let $p = n$. We find, for any $n > 2d$, $\int_0^T V_n(s,\omega)\,ds < \infty$, when $\omega \in \Omega_2 = \bigcap_{n>2d}\Omega_n$.

Writing (4.36) for $\omega \in \Omega_2$ and $n > 2d$ we find the assertion.

(iii) First note that the statement of Lemma 3.3 can be rewritten in term of the process $Y^{s,x}$ (see (3.9)) as follows: for any $0 \le s < u \le T$ there exists an almost sure event $\Omega_{s,u}$ (independent of t and x) such that, for any $\omega \in \Omega_{s,u}$, we have

$$Y_t^{s,x}(\omega) + L_u(\omega) - L_s(\omega) = Y_t^{u,Y_u^{s,x}(\omega)+L_u(\omega)-L_s(\omega)}(\omega), \quad \text{for } t \in [u,T], x \in \mathbb{R}^d. \tag{4.39}$$

Since (Z^s) is a modification of (Y^s) (see Theorem 4.4) we know that on some almost sure event $\Omega''_{s,u} \subset \Omega_{s,u}$ identity (4.39) holds when $(Y^{s,x})$ is replaced by $(Z^{s,x})$.

Let us fix $u \in (0,T]$. We know that (4.39) holds for $(Z^{s,x})$ when $t \in [u,T]$, $x \in \mathbb{R}^d$ and $s \in [0,u)\cap\mathbb{Q}$ if $\omega \in \Omega_u = \bigcap_{s\in[0,u)\cap\mathbb{Q}}(\Omega''_{s,u} \cap \Omega_1)$. Using that $(Z^{s,x})$ with values in G_0 is in particular right-continuous in s, uniformly in x, when x varies in compact sets of $\mathbb{R}^d$, it easy to check that (4.33) holds, for any $0 \le s < u \le T$, $x \in \mathbb{R}^d$, $t \in [u,T]$, when $\omega \in \Omega_u$.

Let us define $\Omega_3 = \bigcap_{u\in\mathbb{Q}\cap[0,T)}\Omega_u$; fix any $s, u_0 \in [0,T]$, $x \in \mathbb{R}^d$, with $0 \le s < u_0 \le T$; we consider $\omega \in \Omega_3$ and prove that (4.33) holds for any $t \in [u_0,T]$.

If $t = u_0$ the assertion holds. Let us suppose that $t \in (u_0,T]$. We can find a sequence $(u_j) \in (u_0,t)\cap\mathbb{Q}$ such that $u_j \to u_0^+$. Since for any $j \ge 1$ we have

$$Z_t^{s,x}(\omega) + L_{u_j}(\omega) - L_s(\omega) = Z_t^{u_j,Z_{u_j}^{s,x}(\omega)+L_{u_j}(\omega)-L_s(\omega)}(\omega), \tag{4.40}$$

we can pass to the limit as $j \to \infty$ in both sides of the previous formula (taking also into account that $Z^{s,x}_{u_j}(\omega) + L_{u_j}(\omega) - L_s(\omega)$ belongs to a compact set $K_{x,s,\omega} \subset \mathbb{R}^d$ for any $j \geq 1$) and find that (4.40) holds when u_j is replaced by u_0. The proof of (4.33) is complete. □

5. A Davie's type uniqueness result

Assertion (v) of the next theorem gives a Davie's type uniqueness result for SDE (1.1). The other assertions collect results of Section 4 (see in particular Theorem 4.4 and Lemma 4.5). These are used to prove the uniqueness property (v). We refer to Corollaries 5.4 and 5.5 for the case when $b(t,\cdot)$ is only locally Hölder continuous.

We stress that all the next statements (i)–(v) hold when ω belongs to an almost sure event Ω' (independent of s, $t \in [0,T]$, $s_0 \in [0,T)$ and $x \in \mathbb{R}^d$).

Theorem 5.1. *Let us consider the SDE (1.1) with $b \in L^\infty(0,T; C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta \in (0,1]$, and suppose that L and b satisfy Hypotheses 1 and 2. Then there exists a function $\phi(s,t,x,\omega)$,*

$$\phi : [0,T] \times [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d, \tag{5.1}$$

which is $\mathcal{B}([0,T] \times [0,T] \times \mathbb{R}^d) \times \mathcal{F}$-measurable and such that $(\phi(s,t,x,\cdot))_{t\in[0,T]}$ is a strong solution of (1.1) starting from x at time s. Moreover, there exists an almost sure event Ω' such that the following assertions hold for any $\omega \in \Omega'$.

(i) *For any $x \in \mathbb{R}^d$, the mapping: $s \mapsto \phi(s,t,x,\omega)$ is càdlàg on $[0,T]$ (uniformly in t and x), i.e., let $s \in (0,T)$ and consider sequences (s_k) and (r_n) such that $s_k \to s^-$ and $r_n \to s^+$; we have, for any $M > 0$,*

$$\lim_{n\to\infty} \sup_{|x|\leq M} \sup_{t\in[0,T]} \left|\phi(r_n,t,x,\omega) - \phi(s,t,x,\omega)\right| = 0, \tag{5.2}$$

$$\lim_{k\to\infty} \sup_{|x|\leq M} \sup_{t\in[0,T]} \left|\phi(s_k,t,x,\omega) - \phi(s-,t,x,\omega)\right| = 0$$

(similar conditions hold when $s = 0$ and $s = T$).

(ii) *For any $x \in \mathbb{R}^d$, $s \in [0,T]$, $\phi(s,t,x,\omega) = x$ if $0 \leq t \leq s$, and*

$$\phi(s,t,x,\omega) = x + \int_s^t b\big(r,\phi(s,r,x,\omega)\big)\,dr + L_t(\omega) - L_s(\omega), \quad t \in [s,T]. \tag{5.3}$$

(iii) *For any $s \in [0,T]$, the function $x \mapsto \phi(s,t,x,\omega)$ is continuous in x uniformly in t. Moreover, for any integer $n > 2d$, there exists a $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable function $V_n : [0,T] \times \Omega \to [0,\infty]$ such that $\int_0^T V_n(s,\omega)\,ds < \infty$ and*

$$\begin{aligned}&\sup_{t\in[0,T]} \left|\phi(s,t,x,\omega) - \phi(s,t,y,\omega)\right| \\ &\quad \leq V_n(s,\omega)|x-y|^{\frac{n-2d}{n}}\left[\big(|x| \vee |y|\big)^{\frac{2d+1}{n}} \vee 1\right], \quad x,y \in \mathbb{R}^d, n > 2d, s \in [0,T].\end{aligned} \tag{5.4}$$

(iv) *For any $0 \leq s < r \leq t \leq T$, $x \in \mathbb{R}^d$, we have*

$$\phi(s,t,x,\omega) = \phi\big(r,t,\phi(s,r,x,\omega),\omega\big). \tag{5.5}$$

(v) *Let $s_0 \in [0,T)$, $\tau = \tau(\omega) \in (s_0,T]$ and $x \in \mathbb{R}^d$. If a measurable function $g : [s_0,\tau) \to \mathbb{R}^d$ solves the integral equation*

$$g(t) = x + \int_{s_0}^t b\big(r,g(r)\big)\,dr + L_t(\omega) - L_{s_0}(\omega), \quad t \in [s_0,\tau), \tag{5.6}$$

then we have $g(r) = \phi(s_0,r,x,\omega)$, for $r \in [s_0,\tau)$.

Proof. Let us consider the process $Z = (Z^s)_{s\in[0,T]}$ of Theorem 4.4 with values in $C(\mathbb{R}^d; G_0)$. Recall the notation $Z_t^{s,x} = \pi_x(Z^s)(t)$ (see (3.15)). We define for $\omega \in \Omega$, $s, t \in [0, T]$, $x \in \mathbb{R}^d$:

$$\phi(s,t,x,\omega) = Z_t^{s,x}(\omega) + L_t(\omega) - L_s(\omega), \quad \text{if } s \le t, \tag{5.7}$$

and $\phi(s,t,x,\omega) = x$ if $s > t$. The fact that, for any $0 \le s < t \le T$, $x \in \mathbb{R}^d$, the random variable $\phi(s,t,x,\cdot)$ is $\mathcal{F}_{s,t}^L$-measurable follows from Theorem 4.4 and (i) in Lemma 4.5. We also define

$$\Omega' = \Omega_1 \cap \Omega_2 \cap \Omega_3,$$

where the almost sure events Ω_k, $k = 1, 2, 3$, are considered in Lemma 4.5.

Assertions (i), (ii), (iii), (iv) follow directly from Theorem 4.4 and Lemma 4.5. More precisely, (i) and (ii) follow from the first assertion of Lemma 4.5 since (Z^s) takes values in $C(\mathbb{R}^d; G_0)$ with càdlàg paths. Assertions (iii) and (iv) follow respectively from the second and third assertion of Lemma 4.5.

(v) Let $\omega \in \Omega'$ be fixed and let $g : [s_0, \tau[\to \mathbb{R}^d$ be a solution to the integral equation (5.6) corresponding to ω. Let us fix $t \in (s_0, \tau)$.

We introduce an auxiliary function $f : [s_0, t] \to \mathbb{R}^d$ which is similar to the one used in proof of Theorem 3.1 in [30],

$$f(s) = \phi\big(s, t, g(s), \omega\big), \quad s \in [s_0, t]. \tag{5.8}$$

We will show that f is constant on $[s_0, t]$. Once this is proved we can deduce that $f(t) = f(s_0)$ and so we find $g(t) = \phi(s_0, t, x, \omega)$ which shows the assertion since t is arbitrary. In the sequel we proceed in three steps.

I step. We establish some estimates for $|g(r) - \phi(u, r, g(u), \omega)|$ when $s_0 \le u \le r \le t$.

Since

$$g(r) = x + \int_{s_0}^{u} b\big(p, g(p)\big)\,dp + \big(L_u(\omega) - L_{s_0}(\omega)\big) + \int_u^r b\big(p, g(p)\big)\,dp + \big(L_r(\omega) - L_u(\omega)\big),$$

we obtain

$$\begin{aligned}
&\big|g(r) - \phi\big(u, r, g(u), \omega\big)\big| \\
&\quad\le \bigg|g(u) + \int_u^r b\big(p, g(p)\big)\,dp + \big(L_r(\omega) - L_u(\omega)\big) - g(u) \\
&\qquad - \int_u^r b\big(p, \phi\big(u, p, g(u), \omega\big)\big)\,dp - \big(L_r(\omega) - L_u(\omega)\big)\bigg| \\
&\quad\le \int_u^r \big|b\big(p, g(p)\big) - b\big(p, \phi\big(u, p, g(u), \omega\big)\big)\big|\,dp \le 2\|b\|_0 |r - u|.
\end{aligned}$$

Now using the Hölder continuity of b:

$$\begin{aligned}
\big|g(r) - \phi\big(u, r, g(u), \omega\big)\big| &\le \int_u^r \big|b\big(p, g(p)\big) - b\big(p, \phi\big(u, p, g(u), \omega\big)\big)\big|\,dp \\
&\le [b]_{\beta,T} \int_u^r \big|g(p) - \phi\big(u, p, g(u), \omega\big)\big|^\beta\,dp \\
&\le \big(2\|b\|_0\big)^\beta [b]_{\beta,T} \int_u^r |p - u|^\beta\,dp \le \big(2\|b\|_0\big)^\beta [b]_{\beta,T} |r - u|^{1+\beta}.
\end{aligned} \tag{5.9}$$

II step. We prove that f defined in (5.8) is continuous on $[s_0, t]$.

We first show that it is right-continuous on $[s_0, t)$. Let us fix $s \in [s_0, t)$ and consider a sequence (s_n) such that $s_n \to s^+$. We prove that $f(s_n) \to f(s)$ as $n \to \infty$. Note that $|g(r)| \le M_0$, $r \in [s_0, \tau)$, where $M_0 = |x| + T\|b\|_0 + C(\omega)$.

We have

$$\left|f(s_n)-f(s)\right| \le \left|\phi\big(s_n,t,g(s_n),\omega\big)-\phi\big(s,t,g(s_n),\omega\big)\right| \\ +\left|\phi\big(s,t,g(s_n),\omega\big)-\phi\big(s,t,g(s),\omega\big)\right| \le J_n+I_n,$$

where $I_n = |\phi(s,t,g(s_n),\omega)-\phi(s,t,g(s),\omega)|$ and

$$J_n=\sup_{|x|\le M_0}\sup_{t\in[0,T]}\left|\phi(s_n,t,x,\omega)-\phi(s,t,x,\omega)\right|.$$

Since $g(s_n)\to g(s)$ by the right continuity of g we obtain that $\lim_{n\to\infty} I_n=0$ thanks to (5.4). Moreover $\lim_{n\to\infty} J_n = 0$ thanks to (5.2).

Let us show that f is left-continuous on $(s_0,t]$. We fix $s\in(s_0,t]$ and consider a sequence $(s_k)\subset(s_0,s)$ such that $s_k\to s$. We prove that $f(s_k)\to f(s)$ as $k\to\infty$. Using the flow property (iv) we find

$$\left|f(s_k)-f(s)\right| = \left|\phi\big(s_k,t,g(s_k),\omega\big)-\phi\big(s,t,g(s),\omega\big)\right| \\ = \left|\phi\big(s,t,\phi\big(s_k,s,g(s_k),\omega\big),\omega\big)-\phi\big(s,t,g(s),\omega\big)\right|.$$

By I step we know that

$$\left|\phi\big(s_k,s,g(s_k),\omega\big)-g(s)\right| \le 2\|b\|_0|s_k-s| \tag{5.10}$$

which tends to 0 as $k\to\infty$. Using (5.10) and the continuity property (iii) we obtain the claim since

$$\lim_{k\to\infty}\left|\phi\big(s,t,\phi\big(s_k,s,g(s_k),\omega\big),\omega\big)-\phi\big(s,t,g(s),\omega\big)\right|=0.$$

III step. We prove that f is constant on $[s_0,t]$.

We will use the following well known lemma (see, for instance, pages 239–240 in [36]): *Let S be a real Banach space and consider a continuous mapping $F:[a,b]\subset\mathbb{R}\to S$, $b>a$. Suppose that for any $h\in(a,b]$ there exists the left derivative*

$$\frac{d^-F}{dh}(h)=\lim_{h'\to h^-}\frac{F(h')-F(h)}{h'-h} \tag{5.11}$$

and this derivative is identically zero on $(a,b]$. Then F is constant.

Note that by considering continuous linear functionals on S one may reduce the proof of the lemma to the one of a real analysis result.

To apply the previous lemma with $[s_0,t]=[a,b]$ we first extend our function f to $[s_0,\infty)$ by setting $f(r)=f(t)$ for $r\ge t$. Then set $S=L^1([0,t];\mathbb{R}^d)$ and define $F:[s_0,t]\to S$ as follows: $F(h)=f(\cdot+h)\in S$, $h\in[s_0,t]$, i.e., $F(h)(r)=f(r+h)$, $r\in[0,t]$.

If we prove that the mapping F is constant then we deduce (taking $h=s_0$ and $h=t$) that $f(s_0+\cdot)=f(t+\cdot)=f(t)$ in S. However, since f is continuous this implies that f is constant and finishes the proof.

The continuity of F, i.e., for any $h\in[s_0,t]$, we have

$$\lim_{h'\to h}\left\|F(h)-F\big(h'\big)\right\|_S=\lim_{h'\to h}\int_0^t\left|f(r+h)-f\big(r+h'\big)\right|dr=0,$$

is clear, using the continuity of f. Let us prove that the left derivative of F is identically zero on $(s_0,t]$.

Using the flow property (iv) we find, for $h,h'\in[s_0,t]$, $h'<h$ and $0\le r\le t-h$,

$$\left|f(r+h)-f\big(r+h'\big)\right| \\ = \left|\phi\big(r+h,t,g(r+h),\omega\big)-\phi\big(r+h,t,\phi\big(r+h',r+h,g\big(r+h'\big),\omega\big),\omega\big)\right|. \tag{5.12}$$

Using (5.12) and changing variable we obtain (recall that $f(r) = f(t)$, $r \geq t$)

$$\begin{aligned}
&\int_0^t \big|f(r+h) - f\big(r+h'\big)\big|\,dr \\
&\quad = \int_0^{t-h} \big|\phi\big(r+h, t, g(r+h), \omega\big) - \phi\big(r+h, t, \phi\big(r+h', r+h, g\big(r+h'\big), \omega\big), \omega\big)\big|\,dr \\
&\qquad + \int_{t-h}^{t-h'} \big|f(t) - f\big(r+h'\big)\big|\,dr \\
&\quad = \int_h^t \big|\phi\big(p, t, g(p), \omega\big) - \phi\big(p, t, \phi\big(p+h'-h, p, g\big(p+h'-h\big), \omega\big), \omega\big)\big|\,dp \\
&\qquad + \int_{t-h}^{t-h'} \big|f(t) - f\big(r+h'\big)\big|\,dr.
\end{aligned} \tag{5.13}$$

In order to estimate $\|F(h) - F(h')\|_S$ let us denote by λ_f the modulus of continuity of f. Since in the last integral $t - h + h' \leq r + h' \leq t$ we have the estimate

$$\int_{t-h}^{t-h'} \big|f(t) - f\big(r+h'\big)\big|\,dr \leq \big|h-h'\big|\lambda_f\big(\big|h-h'\big|\big)$$

and $\lim_{r\to 0^+} \lambda_f(r) = 0$. Taking into account that there exists a constant $N_0 = N_0(x, T, \|b\|_0, \omega) \geq 1$ such that

$$\big|g(r)\big| + \big|\phi\big(r, u, g(r), \omega\big)\big| \leq N_0, \quad s_0 \leq r \leq u \leq T,$$

we find for $p \in [h, t]$, $n > 2d$ (see (5.4) and (5.9))

$$\begin{aligned}
&\big|\phi\big(p, t, g(p), \omega\big) - \phi\big(p, t, \phi\big(p+h'-h, p, g\big(p+h'-h\big), \omega\big), \omega\big)\big| \\
&\quad \leq V_n(p, \omega)\big|g(p) - \phi\big(p+h'-h, p, g\big(p+h'-h\big), \omega\big)\big|^{\frac{n-2d}{n}} N_0^{\frac{2d+1}{n}} \\
&\quad \leq \big(2\|b\|_0\big)^{\beta(\frac{n-2d}{n})}[b]_{\beta,T}^{\frac{n-2d}{n}} V_n(p, \omega)\big|h'-h\big|^{(1+\beta)(\frac{n-2d}{n})} N_0^{\frac{2d+1}{n}}.
\end{aligned}$$

Recall that $V_n(p, \omega) \in [0, \infty]$ but $\int_0^T V_n(p, \omega)\,dp < \infty$. Using the previous inequality and (5.13) we obtain for $h, h' \in [s_0, t]$, $h' < h$

$$\begin{aligned}
&\int_0^t \big|f(r+h) - f\big(r+h'\big)\big|\,dr \\
&\quad \leq C_0\big|h'-h\big|^{(1+\beta)(\frac{n-2d}{n})} \int_0^T V_n(p, \omega)\,dp + \big|h-h'\big|\lambda_f\big(\big|h-h'\big|\big),
\end{aligned} \tag{5.14}$$

where $C_0 = C_0(\beta, \|b\|_{\beta,T}, \omega, T, x, n, d) > 0$. Now we choose n large enough such that $(1+\beta)(\frac{n-2d}{n}) > 1$. Dividing by $|h - h'|$ and passing to the limit as $h' \to h^-$ in (5.14) we find

$$\lim_{h'\to h^-} \frac{1}{|h-h'|}\big\|F(h) - F\big(h'\big)\big\|_{L^1([0,t];\mathbb{R}^d)} = 0.$$

This shows that there exists the left derivative of F in each $h \in (s_0, t]$ and this derivative is identically zero on $(s_0, t]$. By the lemma mentioned at the beginning of III step we obtain that F is constant. Thus f is constant on $[s_0, t]$ and this finishes the proof. □

Remark 5.2. Note that if $g : [s_0, \tau] \to \mathbb{R}^d$, $\tau = \tau(\omega) \in (s_0, T]$, solves (5.6) on $[s_0, \tau]$ then we have $g(\tau) = \phi(s_0, \tau, x, \omega)$, $\omega \in \Omega'$. Indeed applying (v) on $[s_0, \tau)$ we can use that $\int_{s_0}^{\tau} b(r, g(r))\,dr = \int_{s_0}^{\tau} b(r, \phi(s_0, r, x, \omega))\,dr$.

Remark 5.3. It is a natural question if one can improve (5.4) in Theorem 5.1. A possible stronger assertion could be the following one: for each $\alpha \in (0,1)$ and $N \in \mathbb{R}$ one can find $C(\alpha, T, N, \omega) < \infty$ such that, for any $x, y \in \mathbb{R}^d$, $|x|$, $|y| < N$,

$$\sup_{s \in [0,T]} \sup_{t \in [s,T]} \left| \phi(s,t,x,\omega) - \phi(s,t,y,\omega) \right| \leq C(\alpha, T, N, \omega)|x-y|^{\alpha}, \quad \omega \in \Omega'. \tag{5.15}$$

This condition is stated as property 4 in Proposition 2.3 of [30] for SDEs (1.1) when L is a Wiener process and $b \in L^q([0,T]; L^p(\mathbb{R}^d))$, $d/p + 2/q < 1$.

Assuming $b \in L^\infty(0,T; C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$ we do not expect that (5.15) holds in general when L and b satisfy Hypotheses 1 and 2. Remark that a basic strategy to get (5.15) when L is a Wiener process is to use the Kolmogorov–Chentsov test to obtain a Hölder continuous dependence on (s,t,x); one cannot use this approach when L is a discontinuous process. Finally note that the proof of (5.15) given in [30] is not complete ((5.15) does not follow directly from estimate (4) in page 5 of [30] applying the Kolmogorov–Chentsov test).

Now we present two corollaries of Theorem 5.1 which deal with SDEs (1.1) with possibly unbounded b.

When $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable and satisfies, for any $\eta \in C_0^\infty(\mathbb{R}^d)$, $b \cdot \eta \in L^\infty(0,T; C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$ we say that $b \in L^\infty(0,T; C_{loc}^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$. By a localization procedure we get

Corollary 5.4. *Let $b \in L^\infty(0,T; C_{loc}^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$, $\beta \in (0,1]$, and suppose that, for any $\eta \in C_0^\infty(\mathbb{R}^d)$, the Lévy process L and $b \cdot \eta$ satisfy Hypotheses 1 and 2.*

Then there exists an almost sure event Ω'' such that, for any $\omega'' \in \Omega''$, $x \in \mathbb{R}^d$, $s_0 \in [0,T)$ and $\tau = \tau(\omega'') \in (s_0, T]$, if $g_1, g_2 : [s_0, \tau) \to \mathbb{R}^d$ are càdlàg solutions of (5.6) when $\omega = \omega''$, starting from x, then $g_1(r) = g_2(r)$, $r \in [s_0, \tau)$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\varphi = 1$ on $\{|x| \leq 1\}$ and $\varphi(x) = 0$ if $|x| > 2$. Set $b_n(t,x) = b(t,x)\varphi(\frac{x}{n})$, $t \in [0,T]$, $x \in \mathbb{R}^d$ and $n \geq 1$. Consider for each n an almost sure event Ω'_n related to $b_n \in L^\infty(0,T; C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d))$ by Theorem 5.1; set $\Omega'' = \bigcap_{n \geq 1} \Omega'_n$. Suppose that g_1, g_2 are solutions of (5.6) for a fixed $\omega'' \in \Omega''$. Let $\tau_k^{(n)} = \tau_k^{(n)}(\omega'') = \inf\{t \in [s_0, \tau) : |g_k(t)| \geq n\}$, $k = 1, 2$ (if $|g_k(s)| < n$, for any $s \in [s_0, \tau)$ then we set $\tau_k^{(n)} = \tau$). Define $\tau^{(n)} = \tau_1^{(n)} \wedge \tau_2^{(n)}$ and note that on $\Omega''\tau^{(n)} \uparrow \tau$ as $n \to \infty$. Since on $[s_0, \tau^{(n)}(\omega''))$ both g_1 and g_2 solve an equation like (5.6) with b replaced by b_n and $\omega = \omega''$ we can apply (v) of Theorem 5.1 and conclude that $g_1 = g_2$ on $[s_0, \tau^{(n)}(\omega''))$. Since this holds for any $n \geq 1$ we get that $g_1 = g_2$ on $[s_0, \tau(\omega''))$. □

Next we construct ω by ω strong solutions to (1.1) when b is possibly unbounded. To simplify we deal with the initial time $s = 0$.

Corollary 5.5. *Suppose that L and b verify the assumptions of Corollary 5.4. Moreover assume that*

$$\left|b(t,x)\right| \leq C\left(1 + |x|\right), \quad x \in \mathbb{R}^d, t \in [0,T], \tag{5.16}$$

for some constant $C > 0$. Let $x \in \mathbb{R}^d$ and $s = 0$. Then there exists a (unique) strong solution to (1.1) starting from x.

Proof. We know that $t \mapsto L_t(\omega)$ is càdlàg for any $\omega \in \Omega'$, where Ω' is an almost sure event. When $\omega \in \Omega'$ a standard argument based on the Ascoli–Arzela theorem shows that there exists a continuous solution $v = v(\cdot, \omega)$ to $v(t) = x + \int_0^t b(s, v(s) + L_s(\omega))\,ds$ on $[0,T]$. We define $v(t,\omega) = 0$, if $\omega \notin \Omega'$, $t \in [0,T]$. By using the function φ as in the proof of Corollary 5.4 we introduce $b_n(t,x) = b(t,x)\varphi(\frac{x}{n})$, $t \in [0,T]$, $x \in \mathbb{R}^d$ and $n \geq 1$. According to Theorem 5.1 for each n there exists a function ϕ_n as in (5.1) and an almost sure event Ω'_n corresponding to b_n such that assertions (i)–(v) hold. Set $\Omega'' = (\bigcap_{n \geq 1} \Omega'_n) \cap \Omega'$.

Define $g(t,\omega) = v(t,\omega) + L_t(\omega)$, $t \in [0,T]$, $\omega \in \Omega$, and set $\tau^{(n)} = \tau^{(n)}(\omega) = \inf\{t \in [0,T) : |g(t,\omega)| \geq n\}$ (if $|g(s,\omega)| < n$, for any $s \in [0,T)$ then we set $\tau^{(n)}(\omega) = T$). Note that on Ω'' we have $\tau^{(n)} \uparrow T$ as $n \to \infty$.

Let $\omega \in \Omega''$ and $n \geq 1$. Since $g(\cdot, \omega)$ on $[0, \tau^{(n)}(\omega))$ solves an equation like (5.6) with $s_0 = 0$ and b replaced by b_{n+k}, $k \geq 0$, we can apply (v) of Theorem 5.1 and get that $g(t,\omega) = \phi_{n+k}(0,t,x,\omega)$, for any $t \in [0, \tau^{(n)}(\omega))$, $k \geq 0$.

Since $\tau^{(n)} \uparrow T$ we deduce that, uniformly on compact sets of $[0, T)$, for any $\omega \in \Omega''$, we have $\lim_{n\to\infty} \phi_n(0, t, x, \omega) = g(t, \omega)$. It follows that $g(t, \cdot)$ is $\mathcal{F}_t^L$-measurable, for any $t \in [0, T)$. By setting $g(T, \omega) = x + \int_0^T b(r, g(r, \omega))\, dr + L_T(\omega)$, we get that $(g(t, \cdot))$ is a strong solution on $[0, T]$. □

Remark 5.6. The previous condition (5.16) can be relaxed, by requiring that, for fixed $x \in \mathbb{R}^d$, $s = 0$ and $\omega \in \Omega'$ (Ω' is an almost sure event) there exists a continuous solution to the integral equation $v(t) = x + \int_0^t b(s, v(s) + L_s(\omega))\, ds$ on $[0, T]$. The assertion about existence and uniqueness of a strong solution starting from x remains true.

6. Uniqueness for SDEs driven by stable Lévy processes

In this section using also results from [23] and [24] we show that Theorem 5.1 can be applied to a class of SDEs driven by non-degenerate α-stable type processes L. Let $s \geq 0$, we are considering

$$X_t(\omega) = x + \int_s^t b\big(X_u(\omega)\big)\, du + L_t(\omega) - L_s(\omega), \tag{6.1}$$

$x \in \mathbb{R}^d$, $d \geq 1$, $t \geq s$, where $b \in C_b^{0,\beta}(\mathbb{R}^d, \mathbb{R}^d)$, $\beta \in [0, 1]$. We deal with *pure-jump Lévy process L* (without drift term), i.e., we assume that the generating triplet is $(\nu, 0, 0)$ (i.e., $Q = 0$ and $a = 0$ as in (2.5)). To state our assumptions on L we use the convolution semigroup (P_t) associated to L (or to its Lévy measure ν) and acting on $C_b(\mathbb{R}^d)$, i.e., $P_t : C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$, $t \geq 0$,

$$P_t f(x) = E\big[f(x + L_t)\big] = \int_{\mathbb{R}^d} f(x + z)\mu_t(dz), \quad t > 0,\ f \in C_b\big(\mathbb{R}^d\big),\ x \in \mathbb{R}^d,$$

where μ_t is the law of L_t, and $P_0 = I$ (cf. [28] or [1]). The generator $\mathcal{L}$ of (P_t) is

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d} \big(g(x + y) - g(x) - 1_{\{|y|\leq 1\}}\langle y, Dg(x)\rangle\big)\nu(dy), \quad x \in \mathbb{R}^d, \tag{6.2}$$

with $g \in C_0^\infty(\mathbb{R}^d)$ (see Section 6.7 in [1] and Section 31 in [28]). We now consider the Blumenthal-Getoor index $\alpha_0 = \alpha_0(\nu)$ (see [5]):

$$\alpha_0 = \inf\left\{\sigma > 0 : \int_{\{|y|\leq 1\}} |y|^\sigma\, \nu(dy) < \infty\right\}; \tag{6.3}$$

we always have $\alpha_0 \in [0, 2]$. In the sequel we require that $\alpha_0 \in (0, 2)$. Moreover, in Section 6.2, we use the following assumption on the Lévy measure ν.

Hypothesis 3. *Let* $\alpha_0 \in (0, 2)$. *The convolution semigroup* (P_t) *verifies*: $P_t(C_b(\mathbb{R}^d)) \subset C_b^1(\mathbb{R}^d)$, $t > 0$, *and, moreover, there exists* $c_{\alpha_0} = c_{\alpha_0}(\nu) > 0$ *such that*

$$\sup_{x\in\mathbb{R}^d} \big|DP_t f(x)\big| \leq c_{\alpha_0} t^{-\frac{1}{\alpha_0}} \cdot \sup_{x\in\mathbb{R}^d} \big|f(x)\big|, \quad t \in (0, 1],\ f \in C_b\big(\mathbb{R}^d\big). \tag{6.4}$$

Note that Hypothesis 3 implies both Hypotheses 1 and 2 in [24] (taking $\alpha = \alpha_0$). Indeed since $\alpha_0 \in (0, 2)$ we have $\int_{\{|x|\leq 1\}} |y|^\sigma\, \nu(dy) < \infty$, for $\sigma > \alpha_0$. To check the validity of the gradient estimate (6.4) we only mention a criterion which is given in [24]; it is based on Theorem 1.3 in [29].

Theorem 6.1. *Let L be a pure-jump Lévy process. A sufficient condition in order that* (6.4) *holds with* α_0 *replaced by* $\gamma \in (0, 2)$ *is the following one*: *the Lévy measure* ν *of L verifies*: $\nu(B) \geq \nu_1(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, *where* ν_1 *is a Lévy measure on* $\mathbb{R}^d$ *such that its corresponding symbol* $\psi_1(h) = -\int_{\mathbb{R}^d} (e^{i\langle h,y\rangle} - 1 - i\langle h, y\rangle 1_{\{|y|\leq 1\}}(y))\nu_1(dy)$, *satisfies, for some positive constants* c_1, c_2 *and* M,

$$c_1|x|^\gamma \leq Re\psi_1(x) \leq c_2|x|^\gamma, \quad \text{when } |x| > M. \tag{6.5}$$

Examples 6.2. The next examples of α-stable type Lévy processes are also considered in [24]. It is easy to check that in each example $\alpha_0 = \alpha \in (0, 2)$. Thanks to Theorem 6.1 also (6.4) holds in each example.

Consider the following Lévy measure $\tilde{\nu}$:

$$\tilde{\nu}(B) = \int_0^r \frac{dt}{t^{1+\alpha}} \int_S 1_B(t\xi)\mu(d\xi), \quad B \in \mathcal{B}(\mathbb{R}^d) \tag{6.6}$$

(cf. Example 1.5 of [29] with the index β of [29] which is equal to ∞). Here $r > 0$ is fixed; μ is a non-degenerate finite non-negative measure on $\mathcal{B}(\mathbb{R}^d)$ with support on the unit sphere S (non-degeneracy of μ is equivalent to say that its support is not contained in a proper linear subspace of $\mathbb{R}^d$), $\alpha \in (0, 2)$. The Lévy measure $\tilde{\nu}$ verifies Hypothesis 3 since its symbol $\tilde{\psi}$ verifies (6.5) with $\gamma = \alpha$. This was already remarked in page 1146 of [29]. We only note that, if $h \neq 0$, we have

$$Re\tilde{\psi}(h) = \int_0^r \frac{dt}{t^{1+\alpha}} \int_S \left[1 - \cos\left(\left\langle \frac{h}{|h|}, t|h|\xi \right\rangle\right)\right]\mu(d\xi).$$

By changing variable $s = t|h|$ after some computations one arrives at (6.5).

Moreover Hypothesis 2 holds. Note that $\int_{\{|x|>1\}} |y|^\theta \tilde{\nu}(dy) < \infty$, $\theta \in (0, \alpha)$. Using also $\tilde{\nu}$ we find that the next examples of Lévy processes verify Hypotheses 2 and 3.

(i) *L is a non-degenerate symmetric α-stable process* (see, for instance, [28] and the references therein). In this case $\nu(B) = \int_0^\infty \frac{dt}{t^{1+\alpha}} \int_S 1_B(t\xi)\mu(d\xi)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $\alpha \in (0, 2)$, where μ is as in (6.6). A standard rotationally invariant α-stable process L belongs to this class since its Lévy measure has density $\frac{c}{|x|^{d+\alpha}}$ (with respect to the Lebesgue measure in $\mathbb{R}^d$).

(ii) *L is a α-stable temperated process of special form.* Here

$$\nu(B) = \int_0^\infty \frac{e^{-t}\,dt}{t^{1+\alpha}} \int_S 1_B(t\xi)\mu(d\xi), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where μ is as in (6.6), $\alpha \in (0, 2)$.

Note that in (i) and (ii) we have $\nu(B) \geq e^{-1}\tilde{\nu}(B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, where $\tilde{\nu}$ is given in (6.6) with $r = 1$.

(iii) *L is a truncated α-stable process.* In this case $\nu(B) = c \int_{\{|x|\leq 1\}} \frac{1_B(x)}{|x|^{d+\alpha}}\,dx$, $B \in \mathcal{B}(\mathbb{R}^d)$, $\alpha \in (0, 2)$.

(iv) *L is a relativistic α-stable process* (cf. [27] and see the references therein). Here $\psi(h) = (|h|^2 + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$, for some $m > 0$, $\alpha \in (0, 2)$, $h \in \mathbb{R}^d$, and so (6.4) holds. Moreover by Lemma 2 in [27] we know that ν has the density $C_{\alpha,d}|x|^{-d-\alpha}e^{-m^{1/\alpha}|x|} \cdot \phi(m^{1/\alpha}|x|)$, $x \neq 0$, with $0 \leq \phi(s) \leq c_{\alpha,d,m}(s^{\frac{d-1+\alpha}{2}} + 1)$, $s \geq 0$. Hence $\alpha = \alpha_0$ and also Hypothesis 2 holds for any $\theta > 0$.

6.1. *Preliminary results on strong existence and uniqueness by using solutions of Kolmogorov equations*

We first present results on strong existence and uniqueness for (6.1) when $s = 0$ which are special cases of Lemma 5.2 and Theorem 5.3 in [24]. Then we study L^p-dependence from the initial condition x following Theorem 4.3 in [23]. Finally in Theorem 6.6 we will consider the general case when $s \in [0, T]$.

All these theorems do not require the gradient estimates (6.4). However they assume the Blumenthal–Getoor index $\alpha_0 \in (0, 2)$, $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ and classical solvability of the following Kolmogorov type equation:

$$\lambda u(x) - \mathcal{L}u(x) - Du(x)b(x) = b(x), \quad x \in \mathbb{R}^d, \tag{6.7}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is given in (6.1), $\mathcal{L}$ in (6.2) and $\lambda > 0$; the equation is intended componentwise, i.e., $u : \mathbb{R}^d \to \mathbb{R}^d$ and, setting $\mathcal{L}_b = \mathcal{L} + b(x) \cdot D$,

$$\lambda u_k(x) - \mathcal{L}_b u_k(x) = b_k(x), \quad k = 1, \dots, d, \tag{6.8}$$

with $u(x) = (u_k(x))_{k=1,\dots,d}$ and $b(x) = (b_k(x))_{k=1,\dots,d}$. The approach to get strong uniqueness passing through solutions to (6.7) is similar to the one used in Section 2 of [11] (see also [35]).

Remark that $\mathcal{L}g(x)$ in (6.2) is well defined even for $g \in C_b^{1+\gamma}(\mathbb{R}^d)$ if $\alpha_0 < 1+\gamma$ and $\gamma \in [0,1)$ (cf. formula (13) in [24]). Indeed when $|y| \le 1$ we can use the bound $|g(y+x) - g(x) - y \cdot Dg(x)| \le [Dg]_\gamma |y|^{1+\gamma}$, $x \in \mathbb{R}^d$.

In addition $\mathcal{L}g \in C_b(\mathbb{R}^d)$ when $g \in C_b^{1+\gamma}(\mathbb{R}^d)$ and $1+\gamma > \alpha_0$. The next result is stated in Theorem 5.3 of [24] in a more general form which also shows the differentiability of solutions with respect to x and the homeomorphism property.

Theorem 6.3. *Let L be any Lévy process on $(\Omega, \mathcal{F}, P)$ with generating triplet $(\nu, 0, 0)$ such that $\alpha_0 = \alpha_0(\nu) \in (0,2)$ (see (6.3)) and let $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ in (6.1). Suppose that, for some $\lambda > 0$, there exists $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$, $\gamma \in (0,1)$ and $2\gamma > \alpha_0$, which solves (6.7). Moreover, assume $\|Du_\lambda\|_0 < 1/3$.*

Then on $(\Omega, \mathcal{F}, P)$, for any $x \in \mathbb{R}^d$, there exists a pathwise unique strong solution $(X_t^x)_{t\ge 0}$ to (6.1) when $s = 0$.

Next we formulate a special case of Lemma 5.2 in [24]. It uses the stochastic integral against the compensated Poisson random measure $\tilde{N}$ (see, for instance, [20]).

Lemma 6.4. *Under the same hypotheses of Theorem 6.3 let $T > 0$ and suppose that $(X_t^x)_{t\in[0,T]}$ is a strong solution of (6.1) on $[0,T]$ when $s = 0$ (starting from $x \in \mathbb{R}^d$), then, using u_λ of Theorem 6.3, we have, P-a.s., for any $t \in [0,T]$,*

$$\begin{aligned} &u_\lambda\big(X_t^x\big) - u_\lambda(x) \\ &\quad = x + L_t - X_t^x + \lambda \int_0^t u_\lambda\big(X_s^x\big)\,ds + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \big[u_\lambda\big(X_{s-}^x + y\big) - u_\lambda\big(X_{s-}^x\big)\big]\tilde{N}(ds,dy). \end{aligned} \tag{6.9}$$

Proof. The assertion is stated in Lemma 5.2 of [24] for weak solutions $(X_t^x)_{t\ge0}$ with the condition $1+\gamma > \alpha_0$, $\gamma \in (0,1]$. Clearly such lemma works also for strong solutions $(X_t^x)_{t\in[0,T]}$ which solves (6.1) on $[0,T]$ (the proof is based on Itô's formula for $u_\lambda(X_t^x)$); further the condition $2\gamma > \alpha_0$ of Theorem 6.3 implies $1+\gamma > \alpha_0$. □

To prove Davie's uniqueness for (6.1) we need the following L^p-continuity of the solutions w.r.t. initial conditions.

Theorem 6.5. *Under the same hypotheses of Theorem 6.3 let $T > 0$, $s = 0$, and consider two strong solutions $(X_t^x)_{t\in[0,T]}$ and $(X_t^y)_{t\in[0,T]}$ of (6.1) on $[0,T]$ which are defined on $(\Omega, \mathcal{F}, P)$, starting from x and $y \in \mathbb{R}^d$ respectively. For any $t \in [0,T]$, $p \ge 2$, we have*

$$E\Big[\sup_{0\le s\le t} \big|X_s^x - X_s^y\big|^p\Big] \le C(t)|x-y|^p, \tag{6.10}$$

with $C(t) = C(t, \nu, p, \lambda, d, \gamma, \|u_\lambda\|_{C_b^{1+\gamma}}) > 0$ which is independent of x and y; here u_λ is as in Theorem 6.3 (further $C(t, \nu, p, \lambda, d, \gamma, \cdot)$ is increasing).

Proof. The proof follows the one of (i) in Theorem 4.3 of [23]. We only give a sketch of the proof here. We set $X = X^x$, $Y = X^y$ and $u = u_\lambda$. We have from Lemma 6.4, P-a.s., using that $\|Du\|_0 \le 1/3$, $|X_t - Y_t| \le \frac{3}{2}(\Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t) + \Gamma_4)$, where

$$\Gamma_1(t) = \bigg|\int_0^t \int_{\{|z|>1\}} \big[u(X_{s-}+z) - u(X_{s-}) - u(Y_{s-}+z) + u(Y_{s-})\big]\tilde{N}(ds,dz)\bigg|,$$

$$\Gamma_2(t) = \lambda \int_0^t \big|u(X_s) - u(Y_s)\big|\,ds,$$

$$\Gamma_3(t) = \bigg|\int_0^t \int_{\{|z|\le1\}} \big[u(X_{s-}+z) - u(X_{s-}) - u(Y_{s-}+z) + u(Y_{s-})\big]\tilde{N}(ds,dz)\bigg|,$$

$\Gamma_4 = |u(x) - u(y)| + |x - y| \le \frac{4}{3}|x - y|$. Remark that, P-a.s.,

$$\sup_{0\le r\le t} |X_r - Y_r|^p \le C_1|x - y|^p + C_1 \sum_{j=1}^{3} \sup_{0\le r\le t} \Gamma_j(r)^p.$$

By the Hölder inequality, $\sup_{0\le r\le t} \Gamma_2(r)^p \le C_2 t^{p-1} \int_0^t \sup_{0\le s\le r} |X_s - Y_s|^p \, dr$, where $C_2 = C_2(p, \lambda, \|u_\lambda\|_{C_b^{1+\gamma}})$. To estimate Γ_1 and Γ_3 we use L^p-estimates for stochastic integrals against $\tilde{N}$ (cf. [20], Theorem 2.11, or the proof of Proposition 6.6.2 in [1]).

We find, since $|u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})| \le \frac{2}{3}|X_{s-} - Y_{s-}|$, setting $A = \{|z| > 1\}$,

$$\begin{aligned}
&E\Big[\sup_{0\le r\le t} \Gamma_1(r)^p\Big] \\
&\quad \le C_3 E\bigg[\bigg(\int_0^t ds \int_A \big|u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})\big|^2 \nu(dy)\bigg)^{p/2}\bigg] \\
&\qquad + C_3 E \int_0^t ds \int_A \big|u(X_{s-} + z) - u(Y_{s-} + z) + u(Y_{s-}) - u(X_{s-})\big|^p \nu(dy) \\
&\quad \le C_4\big(1 + t^{p/2-1}\big) \int_0^t E\Big[\sup_{0\le r\le s} |X_r - Y_r|^p\Big] ds,
\end{aligned}$$

where $C_3 = \int_{\{|z|>1\}} \nu(dz) + (\int_{\{|z|>1\}} \nu(dz))^{p/2}$. To treat Γ_3 we need the hypothesis $2\gamma > \alpha_0$. By L^p-estimates of stochastic integrals and using Lemma 4.1 in [23] we get

$$\begin{aligned}
E\Big[\sup_{0\le r\le t} \Gamma_3(r)^p\Big] &\le C_5 \|u\|^p_{C_b^{1+\gamma}} E\bigg[\bigg(\int_0^t dr \int_{\{|z|\le 1\}} |X_r - Y_r|^2 |z|^{2\gamma} \nu(dz)\bigg)^{p/2}\bigg] \\
&\quad + C_5 \|u\|^p_{C_b^{1+\gamma}} E \int_0^t |X_r - Y_r|^p \, dr \int_{\{|z|\le 1\}} |z|^{\gamma p} \nu(dz).
\end{aligned}$$

Note that $\int_{\{|z|\le 1\}} |z|^{p\gamma} \nu(dz) < \infty$, since $p \ge 2$ and $2\gamma > \alpha_0$. Collecting the previous estimates, we arrive at

$$E\Big[\sup_{0\le r\le t} |X_r - Y_r|^p\Big] \le C_6|x - y|^p + C_6\big(1 + t^{p-1}\big) \int_0^t E\Big[\sup_{0\le r\le s} |X_r - Y_r|^p\Big] ds,$$

$C_6 = C_6(\nu, p, \lambda, d, \gamma) > 0$. By the Gronwall lemma we obtain the assertion with $C(t) = C_6 \exp(C_6(1 + t^{p-1}))$. □

As a consequence of the previous results we get

Theorem 6.6. *Under the same hypotheses of Theorem 6.3 let $T > 0$ and $s \in [0, T]$. Then, for any $x \in \mathbb{R}^d$, there exists a pathwise unique strong solution $\tilde{X}^{s,x} = (\tilde{X}_t^{s,x})_{t\in[0,T]}$ to (6.1) on $(\Omega, \mathcal{F}, P)$ (recall that $\tilde{X}_t^{s,x} = x$ for $t \le s$). Moreover if $U^{s,x}$ and $U^{s,y}$ are two strong solutions on $[0, T]$ defined on $(\Omega, \mathcal{F}, P)$ and starting at x and y, then we have, for $p \ge 2$,*

$$\sup_{s\in[0,T]} E\Big[\sup_{s\le t\le T} \big|U_t^{s,x} - U_t^{s,y}\big|^p\Big] \le C(T)|x - y|^p, \quad x, y \in \mathbb{R}^d, \tag{6.11}$$

where $C(T) = C(T, \nu, p, \lambda, d, \gamma, \|u_\lambda\|_{C_b^{1+\gamma}}) > 0$ as in (6.10).

Proof. *Existence.* Let us fix $s \in [0, T]$ and consider the new process $L^{(s)} = (L_t^{(s)})$ on $(\Omega, \mathcal{F}, P)$, $L_t^{(s)} = L_{s+t} - L_s$, $t \ge 0$. This is a Lévy process with the same generating triplet of L and is independent of $\mathcal{F}_s^L$ (see Proposition 10.7 in

[28]). According to Theorem 6.3 there exists a unique strong solution to

$$X_t = x + \int_0^t b(X_r)\,dr + L_t^{(s)}, \quad t \ge 0, \tag{6.12}$$

which we denote by $(X^x_{t,L^{(s)}})$ to stress its dependence on $L^{(s)}$. Note that, for any $t \ge 0$, $X^x_{t,L^{(s)}}$ is measurable with respect to $\mathcal{F}_t^{L^{(s)}} = \mathcal{F}^L_{s,t+s}$. Let us define a new process with càdlàg paths $(\tilde{X}_t^{s,x})_{t\in[0,T]}$,

$$\tilde{X}_t^{s,x} = X^x_{t-s,L^{(s)}}, \quad \text{for } s \le t \le T; \qquad \tilde{X}_t^{s,x} = x, \quad 0 \le t \le s. \tag{6.13}$$

Writing $V_t = \tilde{X}_t^{s,x}$, $t \in [0,T]$, to simplify notation, we note that V_t is $\mathcal{F}^L_{s,t}$-measurable, $t \ge s$. Moreover it solves equation (6.1); indeed, for $t \in [s,T]$,

$$V_t = X^x_{t-s,L^{(s)}} = x + \int_0^{t-s} b\big(X^x_{r,L^{(s)}}\big)\,dr + L_t - L_s = x + \int_s^t b(V_r)\,dr + L_t - L_s.$$

Uniqueness. Let $(U_t^{s,x})$ be another strong solution. We have, P-a.s., for $s \le t \le T$,

$$U^{s,x}_{t-s+s} = x + \int_s^t b\big(U_r^{s,x}\big)\,dr + L_t - L_s = x + \int_0^{t-s} b\big(U^{s,x}_{r+s}\big)\,dr + L_t - L_s = x + \int_0^{t-s} b\big(U^{s,x}_{r+s}\big)\,dr + L^{(s)}_{t-s}.$$

Hence $(U^{s,x}_{r+s})_{r\in[0,T-s]}$ solves (6.12) on $[0,T-s]$. By (6.10) we get

$$P\big(U^{s,x}_{r+s} = X^x_{r,L^{(s)}}, r \in [0,T-s]\big) = P\big(U^{s,x}_{r+s} = \tilde{X}^{s,x}_{r+s}, r \in [0,T-s]\big) = 1.$$

This shows the assertion.

L^p-estimates. We have for any fixed $s \in [0,T]$, $p \ge 2$, $E[\sup_{s\le t\le T} |U_t^{s,x} - U_t^{s,y}|^p] = E[\sup_{s\le t\le T} |X^x_{t-s,L^{(s)}} - X^y_{t-s,L^{(s)}}|^p]$ by uniqueness. Using (6.10) we get

$$\begin{aligned}\sup_{s\in[0,T]} E\Big[\sup_{s\le t\le T}\big|U_t^{s,x} - U_t^{s,y}\big|^p\Big] &= \sup_{s\in[0,T]} E\Big[\sup_{s\le t\le T}\big|X^x_{t-s,L^{(s)}} - X^y_{t-s,L^{(s)}}\big|^p\Big] \\ &\le \sup_{s\in[0,T]} E\Big[\sup_{t\in[0,T]}\big|X^x_{t,L^{(s)}} - X^y_{t,L^{(s)}}\big|^p\Big] \le C(T)|x-y|^p.\end{aligned}$$

□

6.2. *A Davie's type uniqueness result when $\alpha_0 \in [1,2)$*

Here we prove a Davie's type uniqueness result for (6.1) (cf. Theorem 5.1). We consider a Lévy process L with generating triple $(\nu,0,0)$ satisfying Hypotheses 2 and 3 with the Blumenthal-Getoor index $\alpha_0 \in [1,2)$ (see (6.3)). Moreover we assume as in [23] and [24] that $b \in C_b^{0,\beta}(\mathbb{R}^d,\mathbb{R}^d)$ with $\beta \in (1-\frac{\alpha_0}{2},1]$.

To check Hypothesis 1 we will use Theorem 6.6 and the following purely analytic result (see Theorem 4.3 in [24]; its proof follows the one in Theorem 3.4 of [23]). Note that the next hypothesis $\alpha_0+\beta<2$ could be dropped. Moreover, to simplify we have only considered the case $\lambda \ge 1$ instead of $\lambda > 0$.

Theorem 6.7. *Assume Hypothesis 3 with $\alpha_0 = \alpha_0(\nu) \ge 1$. Let $0<\beta<1$ with $\alpha_0+\beta \in (1,2)$ and consider $\mathcal{L}$ in (6.2). Then, for any $\lambda \ge 1$, $f \in C_b^\beta(\mathbb{R}^d)$, there exists a unique solution $w_\lambda \in C_b^{\alpha_0+\beta}(\mathbb{R}^d)$ to*

$$\lambda w(x) - \mathcal{L}w(x) - b(x)\cdot Dw(x) = f(x), \quad x \in \mathbb{R}^d. \tag{6.14}$$

Moreover, there exists $C_0 = C_0(\alpha_0(\nu), d, \beta, \|b\|_{C_b^\beta}, \nu) > 0$ such that

$$\lambda\|w_\lambda\|_0 + [Dw_\lambda]_{C_b^{\alpha_0+\beta-1}} \le C_0\|f\|_{C_b^\beta}, \quad \lambda \ge 1. \tag{6.15}$$

Finally, we have $\|Dw_\lambda\|_0 < 1/3$, for any $\lambda \ge \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0(\nu), \beta, \nu) \ge 1$.

Proof. We only make some comments on C_0 and λ_0. Let us first consider C_0. To see that $C_0 = C_0(\alpha_0(\nu), d, \beta, \|b\|_{C_b^\beta}, \nu)$ we look into the proof of Theorem 4.3 in [24]. In such proof the Schauder estimates (6.15) are first established as apriori estimates by a localization procedure. This method is based on Schauder estimates already proved in the constant coefficients case, i.e., when $b(x) = k$, $x \in \mathbb{R}^d$ (see Theorem 4.2 in [24]). The Schauder constant C_0 depends on the Schauder constant c appearing in formula (16) of Theorem 4.2 in [24] when $\lambda \ge 1$. Such constant c depends on $\alpha_0(\nu), \beta, d$ and also on the constant c_{α_0} of the gradient estimates (6.4) (see, in particular, estimates (18)–(21) in the proof of Theorem 4.2 in [24]).

Let us consider λ_0. Recall the simple estimate $\|Dw_\lambda\|_0 \le N[Dw_\lambda]_{C_b^{\alpha_0+\beta-1}}^{\frac{1}{\alpha_0+\beta}} \|w_\lambda\|_0^{\frac{\alpha_0+\beta-1}{\alpha_0+\beta}}$, where $N = N(\alpha_0, \beta, d)$ (cf. the proof of Theorem 3.4 in [23]). By (6.15) we get $\|Dw_\lambda\|_0 \le NC_0\lambda^{-\frac{\alpha_0+\beta-1}{\alpha_0+\beta}} \|f\|_{C_b^\beta}$, $\lambda \ge 1$, and the assertion follows by choosing $\lambda_0 > 1 \vee (3NC_0)^{\frac{\alpha_0+\beta}{\alpha_0+\beta-1}}$. □

Currently we do not know if the statements in Theorem 6.7 hold also when $\alpha_0 \in (0,1)$ (maintaining all the other assumptions).

Now we apply Theorem 5.1 to get Davie's type uniqueness for the SDE (6.1).

Theorem 6.8. *Let L be a d-dimensional Lévy process on $(\Omega, \mathcal{F}, P)$ with generating triple $(\nu, 0, 0)$ satisfying Hypothesis 3 with $\alpha_0 \in [1,2)$. Suppose also that $\int_{\{|x|>1\}} |y|^\theta \nu(dy) < \infty$, for some $\theta > 0$. Let us consider (6.1) with $b \in C_b^{0,\beta}(\mathbb{R}^d; \mathbb{R}^d)$ and $\beta \in (1 - \frac{\alpha_0}{2}, 1]$.*

Then L and b satisfy Hypotheses 1 and 2 and, for any $T > 0$, there exists a function ϕ as in Theorem 5.1 such that assertions (i)–(v) *hold on some almost sure event Ω'.*

Proof. When $\beta = 1$ Hypothesis 1 is clearly satisfied. Let us consider $\beta \in (1 - \frac{\alpha_0}{2}, 1)$. Since $C_b^{\beta'}(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ when $0 < \beta \le \beta' \le 1$, we may assume that $1 - \frac{\alpha_0}{2} < \beta < 2 - \alpha_0$. To verify Hypothesis 1 we use Theorems 6.7 and 6.6. By Theorem 6.7 we have a solution $u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (6.7) with $\gamma = \alpha_0 - 1 + \beta \in (0,1)$ for any $\lambda \ge 1$. Note that $2\gamma = 2\alpha_0 - 2 + 2\beta > \alpha_0$. Choosing $\lambda = \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0(\nu), \beta)$ we obtain that also $\|Du_\lambda\| < 1/3$ holds.

Using Theorem 6.6 we can check the validity of (2.6). Note that the constant $C(T)$ appearing in (6.11) depends on T, ν, p, $\alpha_0(\nu)$, λ, d, γ and $\|u_\lambda\|_{C_b^{1+\gamma}}$. However by Theorem 6.7 $\gamma = \alpha_0 - 1 + \beta$, $\lambda = \lambda_0(d, \|b\|_{C_b^\beta}, \alpha_0, \beta)$ and $\|u_\lambda\|_{C_b^{1+\gamma}} = \|u_\lambda\|_{C_b^{\alpha_0+\beta}} \le N(\alpha_0, \beta, d)C_0\|b\|_{C_b^\beta}$ where C_0 appears in the Schauder estimates (6.15). It follows that $C(T)$ in (6.11) has the right dependence on d, p, β, ν, $\|b\|_{C_b^\beta}$ and T as in (2.6). To finish the proof we apply Theorem 5.1 since Hypotheses 1 and 2 hold. □

Remark 6.9. Theorem 6.8 shows that under suitable assumptions on L and b Davie's uniqueness (or path-by-path uniqueness) holds for the SDE (1.1). Moreover, the unique strong solution is given by a function ϕ which satisfies all the assertions of Theorem 5.1, including (5.2) and (5.4), for any $\omega \in \Omega'$, where Ω' is an almost sure event independent of s, t and x. There are no similar results in the literature on stochastic flows for SDEs (1.1) driven by stable type processes (cf. [23,24] and the recent paper [6] which contains the most general available results about existence and C^1-regularity of stochastic flow).

6.3. *Davie's type uniqueness when $\alpha_0 = \alpha \in (0,1)$*

Here we only consider the SDE (6.1) when $L = L_\alpha$ is a symmetric rotationally invariant α-stable process with $\alpha \in (0,1)$ (the case of $\alpha \in [1,2)$ is already treated in Theorem 6.8). For each $\alpha \in (0,1)$ its Lévy measure $\nu = \nu_\alpha$ has density $\frac{c_{\alpha,d}}{|y|^{d+\alpha}}$, $y \neq 0$, and its generator $\mathcal{L} = \mathcal{L}^{(\alpha)}$ (see (6.2)) coincides with the fractional Laplacian $-(-\triangle)^{\alpha/2}$ (see Example 32.7 in [28]). Note that, for any $g \in C_b^1(\mathbb{R}^d)$, the mapping:

$$x \mapsto \mathcal{L}g(x) = c_{\alpha,d} \int_{\mathbb{R}^d} \frac{g(x+y) - g(x)}{|y|^{d+\alpha}} \, dy \quad \text{belongs to } C_b(\mathbb{R}^d). \tag{6.16}$$

Clearly $\alpha = \alpha_0$ (see (3.1)). Using Theorem 6.6 of the previous section together with Theorem 6.11 we can apply Theorem 5.1 and obtain

Theorem 6.10. *Let L be a d-dimensional symmetric rotationally invariant α-stable process with $\alpha \in (0,1)$ defined on $(\Omega, \mathcal{F}, P)$. Let us consider the SDE (6.1) with $b \in C_b^{0,\beta}(\mathbb{R}^d;\mathbb{R}^d)$ and $\beta \in (1-\frac{\alpha}{2}, 1]$.*

Then L and b satisfy Hypotheses 1 and 2 and, for any $T > 0$, there exists a function ϕ as in Theorem 5.1 such that assertions (i)–(v) *hold on some almost sure event Ω'.*

We first state a result which is related to Theorem 6.7. It shows sharp $C_b^{\alpha+\beta}$-regularity of solutions to (6.14). The proof is based on Theorem 1.1 in [31].

Theorem 6.11. *Let us consider the fractional Laplacian $\mathcal{L}$ given in (6.16) with $\alpha \in (0,1)$. Let $\beta \in (0,1)$ such that $\alpha + \beta > 1$. Then, for any $\lambda \geq 1$, $f \in C_b^{\beta}(\mathbb{R}^d)$, there exists a unique solution $w = w_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ to (6.14). Moreover, there exists $C_0 = C_0(\alpha, d, \beta, \|b\|_{C_b^\beta}) > 0$ such that*

$$\lambda \|w_\lambda\|_0 + [Dw_\lambda]_{C_b^{\alpha+\beta-1}} \leq C_0 \|f\|_{C_b^\beta}, \quad \lambda \geq 1. \tag{6.17}$$

Finally, we have $\|Dw_\lambda\|_0 < 1/3$, for any $\lambda \geq \lambda_0$, with $\lambda_0(d, \|b\|_{C_b^\beta}, \alpha, \beta) \geq 1$.

Proof. The uniqueness follows by the maximum principle (see Proposition 3.2 in [23] or Proposition 4.1 in [24]) which states that $\lambda \|w_\lambda\|_0 \leq \|f\|_0$. Let $\mathcal{L}_b$ be the fractional Laplacian $\mathcal{L}$ plus the drift b (i.e., $\mathcal{L}_b = \mathcal{L} + b \cdot D$). The proof proceeds in some steps.

I step. Let $\lambda \geq 1$. We provide apriori estimates for classical C_b^1-solutions u to $\lambda u - \mathcal{L}_b u = f$ on $\mathbb{R}^d$ (with $f \in C_b^\beta(\mathbb{R}^d)$, $b \in C_b^\beta(\mathbb{R}^d;\mathbb{R}^d)$ and $\alpha + \beta > 1$).

Let $u = u_\lambda \in C_b^1(\mathbb{R}^d)$ be a solution to $\lambda u - \mathcal{L}_b u = f$ on $\mathbb{R}^d$; in the sequel we will consider open balls $B_r(x_0)$ of center $x_0 \in \mathbb{R}^d$ and radius $r > 0$. Let $x_0 \in \mathbb{R}^d$. One can define $v(x) = u(x + x_0)$, $x \in \mathbb{R}^d$. Since $\mathcal{L}v(x) = \mathcal{L}u(x + x_0)$, $x \in \mathbb{R}^d$, we get that $v \in C_b^1(\mathbb{R}^d)$ solves $\lambda v - \mathcal{L}_{b_0} v = f_0$ on $\mathbb{R}^d$ where $\mathcal{L}_{b_0}$ has the drift $b_0(\cdot) = b(\cdot + x_0)$ and $f_0(\cdot) = f(\cdot + x_0)$.

Setting $\tilde{v}(t,x) = e^{\lambda t} v(x)$, $\tilde{f}_0(t,x) = e^{\lambda t} f_0(x)$, $t \in [-1,0]$, $x \in \mathbb{R}^d$, we see that $\tilde{v}$ is a bounded solution of

$$\partial_t \tilde{v} - \mathcal{L}_{b_0} \tilde{v} = \tilde{f}_0 \quad \text{on } [-1,0] \times B_1(0)$$

according to the definition of viscosity solution given at the beginning of Section 3.1 in [31]. Hence we can apply Theorem 1.1 in [31] to $\tilde{v}$. Recall that in the Silvestre notations his $s \in (0,1)$ is our $\alpha/2$ and his $\alpha \in (0, 2s)$ corresponds with our $\alpha + \beta - 1$. We deduce by [31] that $\tilde{v}(t,\cdot) \in C^{\alpha+\beta}(B_{1/2}(0))$ and moreover

$$\begin{aligned} \|v\|_{C^{\alpha+\beta}(B_{1/2}(0))} &= \|\tilde{v}\|_{L^\infty([-1/2,0];C^{\alpha+\beta}(B_{1/2}(0))} \\ &\leq C_2\big(\|\tilde{v}\|_{L^\infty([-1,0]\times\mathbb{R}^d)} + \|\tilde{f}\|_{L^\infty([-1,0];C^\beta(B_{1/2}(0)))}\big) = C_2\big(\|v\|_0 + \|f_0\|_{C_b^\beta(\mathbb{R}^d)}\big), \end{aligned}$$

where C_2 depends only on $\|b_0\|_{C_b^\beta(\mathbb{R}^d;\mathbb{R}^d)} = \|b\|_{C_b^\beta(\mathbb{R}^d;\mathbb{R}^d)}$, α and d and is independent of λ. Thus we get that $u_\lambda \in C^{\alpha+\beta}(B_{1/2}(x_0))$ with a bound for the $C^{\alpha+\beta}$-norm of u_λ on $B_{1/2}(x_0)$ by the quantity $C_2(\|u_\lambda\|_0 + \|f\|_{C_b^\beta(\mathbb{R}^d)})$. Since C_2 is independent on x_0 it is clear that we have $u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ (cf. for instance page 434 in [23]) and the following estimate holds with $C_3 = C_3(\|b\|_{C_b^\beta}, \alpha, d, \beta) > 0$

$$\|u_\lambda\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq C_3\big(\|u_\lambda\|_0 + \|f\|_{C_b^\beta(\mathbb{R}^d)}\big).$$

By Proposition 3.2 in [23] we know that $\lambda \|u_\lambda\|_0 \leq \|f\|_0$. Hence we arrive at

$$\|u_\lambda\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \leq 2C_3 \|f\|_{C_b^\beta(\mathbb{R}^d)}, \quad \lambda \geq 1. \tag{6.18}$$

II step. Let $\lambda \geq 1$. We show the existence of a C_b^1-solution to $\lambda w - \mathcal{L}_b w = \tilde{f}$ when $b \in C_b^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $\tilde{f} \in C_b^\infty(\mathbb{R}^d)$.

To construct the solution we use a probabilistic method (for an alternative vanishing viscosity method see Section 3.2 in [31]). Let (X_t^x) be the solution of $dX_t = b(X_t)\,dt + dL_t$, $X_0 = x \in \mathbb{R}^d$ and consider the associated Markov semigroup (R_t), i.e., $R_t l(x) = E[l(X_t^x)]$, $t \geq 0$, $x \in \mathbb{R}^d$, $l \in UC_b(\mathbb{R}^d)$ ($UC_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ denotes the Banach space of all uniformly continuous and bounded functions endowed with the sup-norm). Differentiating with respect to x under the expectation (using the derivative of X_t^x with respect to x, cf. [37]) it is straightforward to prove that $R_t g \in C_b^1(\mathbb{R}^d)$, for any $t \geq 0$ and $g \in C_b^1(\mathbb{R}^d)$. For the given $\tilde{f} \in C_b^\infty(\mathbb{R}^d)$ we define

$$\tilde{w}(x) = \tilde{w}_\lambda(x) = \int_0^\infty e^{-\lambda t} R_t \tilde{f}(x)\,dt, \quad x \in \mathbb{R}^d. \tag{6.19}$$

It is clear that $\tilde{w} \in C_b(\mathbb{R}^d)$. We now show that $\tilde{w} \in C_b^1(\mathbb{R}^d)$ and solves our equation. To this purpose we first prove that for $t > 0$

$$\sup_{x\in\mathbb{R}^d} \left| DR_t \tilde{f}(x) \right| \leq c\big(\alpha, \beta, \|Db\|_0\big)(t \wedge 1)^{(\beta-1)/\alpha} \|\tilde{f}\|_{C_b^\beta(\mathbb{R}^d)}. \tag{6.20}$$

Once this estimate is proved, differentiating under the integral sign in (6.19) we obtain that $w \in C_b^1(\mathbb{R}^d)$ since $\alpha + \beta > 1$. Let us fix $t \in (0,1]$. By Theorem 1.1 in [37] we know in particular that

$$\|DR_t g\|_0 = \sup_{x\in\mathbb{R}^d} \left| DR_t g(x) \right| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} \|g\|_0, \quad g \in C_b^1\big(\mathbb{R}^d\big).$$

Using the total variation norm as in Lemma 7.1.5 of [7] we deduce that $R_t l$ is Lipschitz continuous for any $l \in UC_b(\mathbb{R}^d)$ and moreover $|R_t l(x) - R_t l(y)| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} |x-y| \|l\|_0$, $x, y \in \mathbb{R}^d$. By Theorem 1.1 in [37], for any $g \in C_b^1(\mathbb{R}^d)$, we can write the directional derivative of $R_t g$ along $h \in \mathbb{R}^d$ as follows:

$$D_h R_t g(x) = E\big[g\big(X_t^x\big) J(t,x,h)\big], \quad x \in \mathbb{R}^d, \tag{6.21}$$

where $J(t,x,h)$ is a suitable random variable such that $(E|J(t,x,h)|^2)^{1/2} \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} |h|$, for any $x \in \mathbb{R}^d$. Let again $l \in UC_b(\mathbb{R}^d)$. Using mollifiers we can consider an approximating sequence $(g_n) \subset C_b^\infty(\mathbb{R}^d)$ such that $\|g_n - l\|_0 \to 0$ as $n \to \infty$. Using (6.21) when g is replaced by g_n and passing to the limit it is not difficult to prove that $R_t l \in C_b^1(\mathbb{R}^d)$ and moreover (6.21) holds when g is replaced by l (cf. page 480 in [25]).

We have found that $R_t : UC_b(\mathbb{R}^d) \to C_b^1(\mathbb{R}^d)$ is a linear and bounded operator and

$$\left| DR_t l(x) \right| \leq c(\alpha) e^{\|Db\|_0} t^{-1/\alpha} \|l\|_0,$$

for $x \in \mathbb{R}^d$, $l \in UC_b(\mathbb{R}^d)$. Moreover, $R_t : C_b^1(\mathbb{R}^d) \to C_b^1(\mathbb{R}^d)$ is linear and bounded and $|DR_t g(x)| \leq e^{\|Db\|_0} \|Dg\|_0$, for $x \in \mathbb{R}^d$, $g \in C_b^1(\mathbb{R}^d)$. To prove such estimate we fix $h \in \mathbb{R}^d$ and differentiate $R_t g(x)$ with respect to x along the direction h. One can show that

$$D_h E\big[g\big(X_t^x\big)\big] = E\big[Dg\big(X_t^x\big)\eta_t\big], \tag{6.22}$$

where $\eta_t = D_h X_t^x$ solves $\eta_t = h + \int_0^t Db(X_s^x)\eta_s\,ds$, $t \geq 0$, P-a.s. Note that $|D_h X_t^x| \leq |h| e^{\|Db\|_0 t}$ by the Gronwall lemma (cf. page 1211 in [37]).

By interpolation techniques we know that $(UC_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))_{\beta,\infty} = C_b^\beta(\mathbb{R}^d)$, for $\beta \in (0,1)$ (cf. [22], Chapter 1, and the proof of Theorem 3.3 in [23]); it follows that for any $t \in (0,1]$ we have that $R_t : C_b^\beta(\mathbb{R}^d) \to C_b^1(\mathbb{R}^d)$ is linear and bounded and $|DR_t f(x)| \leq c(\alpha,\beta) e^{\|Db\|_0} t^{(\beta-1)/\alpha} \|f\|_{C_b^\beta}$, for any $x \in \mathbb{R}^d$, $f \in C_b^\beta(\mathbb{R}^d)$.

We have verified (6.20) when $t \in (0,1]$. If $t > 1$ we use a standard argument based on the semigroup property and get, for any $x \in \mathbb{R}^d$, $|DR_t \tilde{f}(x)| = |DR_1(R_{t-1}\tilde{f})(x)| \leq c(\alpha) e^{\|Db\|_0} \|R_{t-1}\tilde{f}\|_0 \leq c(\alpha) e^{\|Db\|_0} \|\tilde{f}\|_0$. Thus (6.20) holds and we know that $\tilde{w} \in C_b^1(\mathbb{R}^d)$. To prove that $\tilde{w}$ is a solution we first establish the identity

$$\partial_t(R_t \tilde{f})(s,x) = R_s(\mathcal{L}_b \tilde{f})(x) = \mathcal{L}_b(R_s \tilde{f})(x), \quad s \geq 0, x \in \mathbb{R}^d. \tag{6.23}$$

By using Ito's formula (see [20], Section 2.3) and taking the expectation we find $E[\tilde{f}(X_{s+h}^x)] - E[\tilde{f}(X_s^x)] = \int_s^{s+h} E[(\mathcal{L}_b\tilde{f})(X_r^x)]\,dr$, for $h \in \mathbb{R}$ such that $s+h>0$. It follows that, for $x \in \mathbb{R}^d$,

$$\partial_t(R_t\tilde{f})(s,x) = \lim_{h\to 0} h^{-1}\big(R_{s+h}\tilde{f}(x) - R_s\tilde{f}(x)\big) = R_s(\mathcal{L}_b\tilde{f})(x), \quad s>0, \quad \text{and} \tag{6.24}$$

$$\lim_{h\to 0^+} h^{-1}\big(R_h\tilde{f}(x) - \tilde{f}(x)\big) = \mathcal{L}_b\tilde{f}(x). \tag{6.25}$$

If $s>0$ by (6.25) we get $\lim_{h\to 0^+}\frac{R_h(R_s\tilde{f})(x)-R_s\tilde{f}(x)}{h} = \mathcal{L}_b(R_s\tilde{f})(x)$ when $\tilde{f}$ in (6.25) is replaced by $R_s\tilde{f}$. By the semigroup law, the last limit and (6.24) coincide and so (6.23) holds. To check that $\tilde{w}$ verifies $\lambda\tilde{w} - \mathcal{L}_b\tilde{w} = \tilde{f}$ we use (6.20) and (6.23). First by the Fubini theorem we have

$$\mathcal{L}_b\tilde{w}(x) = \int_0^\infty e^{-\lambda t}\mathcal{L}_b(R_t\tilde{f})(x)\,dt = \int_0^\infty e^{-\lambda t}R_t(\mathcal{L}_b\tilde{f})(x)\,dt.$$

By (6.23) it follows that, for any $x \in \mathbb{R}^d$, $\mathcal{L}_b\tilde{w}(x) = \int_0^\infty e^{-\lambda t}\frac{d}{dt}(R_t\tilde{f}(x))\,dt$. Integrating by parts, we get the assertion.

III step. Let $\lambda \ge 1$. We prove the existence of a $C_b^{\alpha+\beta}$-solution to $\lambda w - \mathcal{L}_b w = f$ on $\mathbb{R}^d$ when $b \in C_b^\beta(\mathbb{R}^d;\mathbb{R}^d)$ and $f \in C_b^\beta(\mathbb{R}^d)$, $\alpha+\beta>1$, and show (6.17).

Using convolution with mollifiers and possibly passing to subsequences (see, for instance, page 431 in [23]) one can consider operators $\mathcal{L}_{b_n}$ with drifts $b_n \in C_b^\infty(\mathbb{R}^d;\mathbb{R}^d)$ such that $\|b_n\|_{C_b^\beta} \le \|b\|_{C_b^\beta}$, $n \ge 1$, and $b_n \to b$ in $C^{\beta'}(K;\mathbb{R}^d)$ for any compact set $K \subset \mathbb{R}^d$ and $\beta' \in (0,\beta)$. Similarly one can construct $(f_n) \subset C_b^\infty(\mathbb{R}^d)$ such that $\|f_n\|_{C_b^\beta} \le \|f\|_{C_b^\beta}$, $n \ge 1$, and $f_n \to f$ in $C^{\beta'}(K)$ for any compact set $K \subset \mathbb{R}^d$ and $\beta' \in (0,\beta)$. By II step there exist C_b^1-solutions w_n to $\mathcal{L}_{b_n}w_n = \lambda w_n - f_n$, $n \ge 1$. By Step I we know that $w_n \in C_b^{\alpha+\beta}(\mathbb{R}^d)$, $n \ge 1$, with the estimate

$$\|w_n\|_{C_b^{\alpha+\beta}(\mathbb{R}^d)} \le 2C_3\|f\|_{C_b^\beta(\mathbb{R}^d)} \tag{6.26}$$

($C_3 = C_3(\|b\|_{C_b^\beta}, \alpha, \beta, d)$ is independent of λ and n). Possibly passing to a subsequence still denoted with (w_n), we have that $w_n \to w$ in $C^{\alpha+\beta'}(K)$, for any compact set $K \subset \mathbb{R}^d$ with $\beta'>0$ such that $1<\alpha+\beta'<\alpha+\beta$. Moreover, (6.26) holds with w_n replaced by w. We can easily pass to the limit in each term of $\lambda w_n(x) - \mathcal{L}w_n(x) - b_n(x)\cdot Dw_n(x) = f_n(x)$ as $n \to \infty$ and obtain that w solves our equation.

IV step. We prove the final assertion.

We already know that there exists a unique solution $w_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ and that (6.17) holds. To complete the proof we argue as in the final part of the proof of Theorem 6.7. By the interpolatory estimate $\|Dw_\lambda\|_0 \le N(\alpha,\beta,d)[Dw_\lambda]_{C_b^{\alpha+\beta-1}}^{\frac{1}{\alpha+\beta}}\|w_\lambda\|_0^{\frac{\alpha+\beta-1}{\alpha+\beta}}$, we obtain easily that $\|Dw_\lambda\|_0 < 1/3$ for $\lambda \ge \lambda_0(d,\|b\|_{C_b^\beta},\alpha,\beta)$. □

Proof of Theorem 6.10. As in the proof of Theorem 6.8 we verify the assumptions of Theorem 5.1. Note that Hypothesis 2 holds since $\int_{\{|x|>1\}}\frac{|y|^\theta}{|y|^{d+\alpha}}\,dy < \infty$, for any $\theta \in (0,\alpha)$. In order to check Hypothesis 1 we argue as in the proof of Theorem 6.8 (using Theorems 6.11 and 6.6; recall that $\alpha = \alpha_0$). The proof is complete. □

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