

Parametrix construction of the transition probability density of the solution to an SDE driven by α -stable noise¹

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Abstract. Let $L := -a(x)(-\Delta)^{\alpha/2} + (b(x), \nabla)$, where $\alpha \in (0, 2)$, and $a : \mathbb{R}^d \rightarrow (0, \infty)$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Under certain regularity assumptions on the coefficients a and b , we associate with the $C_\infty(\mathbb{R}^d)$ -closure of $(L, C_\infty^2(\mathbb{R}^d))$ a Feller Markov process X , which possesses a transition probability density $p_t(x, y)$. To construct this transition probability density and to obtain the two-sided estimates on it, we develop a new version of the parametrix method, which even allows us to handle the case $0 < \alpha \leq 1$ and $b \neq 0$, i.e. when the gradient part of the generator is not dominated by the jump part.

Résumé. Soit $L := -a(x)(-\Delta)^{\alpha/2} + (b(x), \nabla)$, avec $\alpha \in (0, 2)$, et $a : \mathbb{R}^d \rightarrow (0, \infty)$, $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Sous certaines hypothèses de régularité des coefficients a et b , nous associons à la $C_\infty(\mathbb{R}^d)$ -fermeture de $(L, C_\infty^2(\mathbb{R}^d))$ un processus de Markov fellerien X , possédant une densité de probabilité de transition $p_t(x, y)$. Afin de construire cette densité, et d'en obtenir des bornes supérieures et inférieures, nous développons une nouvelle version de la méthode parametrix, qui permet même de traiter le cas où $0 < \alpha \leq 1$ et $b \neq 0$, c'est-à-dire quand la partie de gradient du générateur n'est pas dominée par la partie de saut.

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1. Introduction

Let $Z^{(\alpha)}$, $\alpha \in (0, 2)$, be a symmetric α -stable process in \mathbb{R}^d ; that is, a Lévy process with the characteristic function

$$\mathbb{E}e^{i(\xi, Z_t^{(\alpha)})} = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.$$

It is well known that the generator $L^{(\alpha)}$ of the semigroup $(P_t^{(\alpha)})_{t \geq 0}$, where

$$P_t^{(\alpha)} f(x) = \mathbb{E}^x f(Z_t^{(\alpha)}),$$

admits on $C_\infty^2(\mathbb{R}^d)$ the representation

$$L^{(\alpha)} f(x) = \text{P.V.} \int_{\mathbb{R}^d} (f(x+u) - f(x)) \frac{c_\alpha}{|u|^{d+\alpha}} du. \quad (1.1)$$

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Here and below we denote by $C_\infty^k(\mathbb{R}^d)$, $k \geq 0$, the space of k times continuously differentiable functions, vanishing at infinity together with their derivatives. The operator $L^{(\alpha)}$ is also called a *fractional Laplacian*, and is denoted by $-(-\Delta)^{\alpha/2}$.

Consider the following ‘‘perturbation’’ of the operator $L^{(\alpha)}$:

$$Lf(x) = a(x)L^{(\alpha)}f(x) + (b(x), \nabla f(x)), \quad f \in C_\infty^2(\mathbb{R}^d), \quad (1.2)$$

where $a(\cdot) > 0$, $b(\cdot) \in \mathbb{R}^d$. When the coefficients $a(\cdot)$ and $b(\cdot)$ do not depend on x , the operator L is just the restriction to $C_\infty^2(\mathbb{R}^d)$ of the generator of the semigroup $\{T_t, t \geq 0\}$, which corresponds to the Lévy process $Z^{(\alpha)}$ re-scaled by a and with drift b .

The general case is much more complicated, and the purpose of this paper is to show that under suitable assumptions on the coefficients and the parameter α , the C_∞ -closure of the operator $(L, C_\infty^2(\mathbb{R}^d))$ is the generator of a semigroup $\{P_t, t \geq 0\}$, which corresponds to a strong Markov process.

Our approach is analytic, and mainly relies on the *parametrix construction* of the ‘‘candidate’’ $p_t(x, y)$ for the transition probability density of the required Markov process. We develop a new version of the parametrix method, which substantially depends on the relation between the regularity of the drift coefficient $b(x)$ and the parameter α and, in particular, allows us to handle the case $0 < \alpha \leq 1$ and $b \neq 0$, i.e. the one where the gradient part of the generator is not dominated by the jump part. To associate the constructed kernel $p_t(x, y)$ with a Markov process in a unique way, we develop a new method, which we believe may be useful in other settings as well. This method relies on the fact that L possesses the positive maximum principle, and exploits a new notion of an *approximate fundamental solution*. We refer the reader to a detailed discussion in Section 2.3, where we also give an overview of available results.

We also consider the probabilistic counterpart to the problem described above. Namely, we consider an SDE driven by $Z^{(\alpha)}$

$$dX_t = b(X_t)dt + \sigma(X_{t-})dZ_t^{(\alpha)}; \quad (1.3)$$

here and below we denote $\sigma(x) = a^{1/\alpha}(x)$. Using the parametrix construction and the fact that the closure of $(L, C_\infty^2(\mathbb{R}^d))$ is the generator of a Markov process, we show that a weak solution to (1.3) is unique and actually coincides with this Markov process. This fact also ensures that the martingale problem for $(L, C_\infty^2(\mathbb{R}^d))$ is well posed. Finally, we provide lower and upper bounds for the transition probability density $p_t(x, y)$.

The paper is organized as follows. In Section 2 we formulate the main results, give the outline of the proofs and an overview of already existing results, comparing them with ours. Section 3 is devoted to the parametrix method for construction of the function $p_t(x, y)$, which is the candidate for being the fundamental solution to the Cauchy problem for $\partial_t - L$. Section 4 is devoted to the relation between the operators P_t and L , in particular, we prove that the family of operators $\{P_t, t \geq 0\}$ forms a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^d)$. Then we prove that the extension $(A, D(A))$ of $(L, C_\infty^2(\mathbb{R}^d))$ is in fact the generator of the semigroup $\{P_t, t \geq 0\}$, and, moreover, that $p_t(\cdot, y) \in D(A)$, and is the fundamental solution to the Cauchy problem for $\partial_t - A$. In Section 5 we prove that the constructed process X is the weak solution to (1.3), and that the martingale problem $(L, C_\infty^2(\mathbb{R}^d))$ is well-posed. In Section 6 we give the estimates on the time derivative $\partial_t p_t(x, y)$ and related auxiliary function appearing in the parametrix construction. In Section 7 we give the proofs of lower and upper bounds on $p_t(x, y)$. Appendices A and B contain some auxiliary results, used in the proofs.

2. The main results: Preliminaries, formulation, and discussion

2.1. Notation and preliminaries

Through the paper we use the following notation.

By $g^{(\alpha)}(x)$ we denote the distribution density of the symmetric α -stable variable $Z_1^{(\alpha)}$. Note that $L^{(\alpha)}$ is a homogeneous operator of the order α and the process $Z^{(\alpha)}$ is self-similar: for any $c > 0$, the process

$$c^{-1/\alpha}Z_{ct}^{(\alpha)}, \quad t \geq 0,$$

has the same law as $Z^{(\alpha)}$. Consequently, the transition probability density of $Z^{(\alpha)}$ equals $t^{-d/\alpha} g^{(\alpha)}(t^{-1/\alpha}(y-x))$. By $C_\infty(\mathbb{R}^d)$ (respectively, $C_b(\mathbb{R}^d)$) we denote the class of continuous functions vanishing at infinity (respectively, bounded); clearly, $C_\infty(\mathbb{R}^d)$ is a Banach space with respect to the sup-norm $\|\cdot\|_\infty$. By $C_\infty^k(\mathbb{R}^d)$ (respectively, $C_b^k(\mathbb{R}^d)$), $k \geq 1$, we denote the class of k -times continuously differentiable functions vanishing at infinity (respectively, bounded) together with their derivatives.

We use the following notation for space and time-space convolutions of functions:

$$(f * g)_t(x, y) := \int_{\mathbb{R}^d} f_t(x, z) g_t(z, y) dz,$$

$$(f \circledast g)_t(x, y) := \int_0^t \int_{\mathbb{R}^d} f_{t-s}(x, z) g_s(z, y) dz ds.$$

As usual, $a \wedge b := \min(a, b)$, $a \vee b := \max(a, b)$. By $|\cdot|$ we denote both the modulus of a real number and the Euclidean norm of a vector. By c and C we denote positive constants, the value of which may vary from place to place. Relation $f \asymp g$ means that

$$cg \leq f \leq Cg.$$

By $\Gamma(\cdot)$, $B(\cdot, \cdot)$ we denote the Euler Gamma- and Beta-functions. Finally, we write L_x to emphasize that the operator L acts on a function $f(x, y)$ with respect to the variable x , i.e., $L_x f(x, y) = Lf(\cdot, y)(x)$.

Recall that a real-valued function $p_t(x, y)$ is said to be the *fundamental solution* to the Cauchy problem for the operator

$$\partial_t - L, \tag{2.1}$$

if for $t > 0$ it is differentiable in t , belongs to the domain of L as a function of x , and satisfies

$$(\partial_t - L_x)p_t(x, y) = 0, \quad t > 0, x, y \in \mathbb{R}^d, \tag{2.2}$$

$$p_t(x, \cdot) \Rightarrow \delta_x, \quad t \rightarrow 0+, x \in \mathbb{R}^d; \tag{2.3}$$

see [33, Definition 2.7.12] in the case of a general pseudo-differential operator, which is the generalization of the corresponding definition (cf. [21], for example) in the parabolic/elliptic setting.

In order to simplify the further exposition, we briefly outline the *parametrix method*, which we use to construct $p_t(x, y)$. Consider *some* approximation $p_t^0(x, y)$ to this function, and denote by $r_t(x, y)$ the residue term with respect to this approximation:

$$p_t(x, y) = p_t^0(x, y) + r_t(x, y). \tag{2.4}$$

Put

$$\Phi_t(x, y) := -(\partial_t - L_x)p_t^0(x, y), \quad t > 0, x, y \in \mathbb{R}^d. \tag{2.5}$$

Recall that $p_t(x, y)$ is supposed to be the fundamental solution for the operator (2.1), hence

$$(\partial_t - L_x)r_t(x, y) = \Phi_t(x, y). \tag{2.6}$$

Recall that if $p_t(x, y)$ is the fundamental solution to (2.2), then one expects the solution to equation (2.6) to be of the form

$$r_t(x, y) = (p \circledast \Phi)_t(x, y).$$

Substituting now in the right-hand side of the above equation representation (2.4) for $p_t(x, y)$, we get the following equation for $r_t(x, y)$:

$$r_t(x, y) = (p^0 \circledast \Phi)_t(x, y) + (r \circledast \Phi)_t(x, y).$$

The formal solution to this equation is given by the convolution

$$r_t(x, y) = (p^0 \circledast \Psi)_t(x, y), \quad (2.7)$$

where Ψ is the sum of \circledast -convolutions of Φ :

$$\Psi_t(x, y) := \sum_{k \geq 1} \Phi_t^{\circledast k}(x, y). \quad (2.8)$$

If the series (2.8) converges and the convolution (2.7) is well defined, we obtain the required function $p_t(x, y)$ in the form

$$p_t(x, y) = p_t^0(x, y) + \sum_{k \geq 1} (p^0 \circledast \Phi^{\circledast k})_t(x, y). \quad (2.9)$$

Clearly, the above argument is yet purely formal; in order to make it rigorous, we need to prove that the parametrix construction is feasible, i.e. that the sum in the right hand side of (2.9) is well defined, and then to associate $p_t(x, y)$ with the initial operator L . We note that the key point to make the entire approach successful is the proper choice of the zero order approximation $p_t^0(x, y)$; see Section 2.3 for more detailed discussion of this point.

2.2. The main results

Our standing assumption on the intensity coefficient $a(x)$ is that it is *strictly positive, bounded* from above and below and *Hölder continuous* with some index $0 < \eta \leq 1$, i.e. there exist $0 < c < C$ such that

$$c \leq a(x) \leq C, \quad |a(x) - a(y)| \leq C|x - y|^\eta, \quad x, y \in \mathbb{R}^d. \quad (2.10)$$

We also assume that the drift coefficient $b(x)$ satisfies the assumption below:

$$b(\cdot) \in C_b(\mathbb{R}^d), \quad |b(x) - b(y)| \leq C|x - y|^\gamma, \quad x, y \in \mathbb{R}^d, \quad (2.11)$$

where $\gamma \in [0, 1]$. We consider three cases; in each of them the Hölder index γ of the drift coefficient is related to the index α .

Case A. $\alpha \in (1, 2)$, $\gamma = 0$.

Case B. $\alpha \in ((1 + \gamma)^{-1}, 2)$, $0 < \gamma < 1$.

Case C. $\alpha \in (0, 2)$, $\gamma = 1$.

Remark 2.1.

(a) Observe that assumptions in the cases **A–C** overlap, but none of the assumptions is implied by the other one: if we denote by $\alpha_A, \alpha_B, \alpha_C$ the infima of α allowed in each of these cases, then we have $\alpha_A = 1$, $\alpha_B = (1 + \gamma)^{-1}$, and $\alpha_C = 0$. Note that the regularity assumption on drift coefficient is weakened from case **A** to case **C**, but on the other hand we have $\alpha_C \leq \alpha_B \leq \alpha_A$. Heuristically, this means that by increasing the regularity of b we can relax the assumption on α , and vice versa. Note also that if we let $\gamma \rightarrow 1$ in the “intermediate case” **B**, we get $\alpha_B \rightarrow 1/2 \neq \alpha_C$, which means that case **C** cannot be obtained from **B** by such a limit procedure.

(b) As we will see below (cf. Proposition 2.1), the constructed function $p_t(x, y)$ is uniquely associated with the operator in (1.2), which implies that if the coefficients $a(x)$ and $b(x)$ are such that some of the cases **A–C** overlap, then different choices of the zero-order approximation provided by (2.15) below give the same outcome $p_t(x, y)$. However, the upper and lower bounds on $p_t(x, y)$ depend on the choice of $p_t^0(x, y)$.

When b is Lipschitz continuous, the Cauchy problem for the ordinary differential equation (ODE)

$$d\chi_t = b(\chi_t) dt, \quad \chi_0 = x, \quad (2.12)$$

admits the *flow of solutions* $\{\chi_t, t \in \mathbb{R}\}$, cf. [11, Theorem 2.1]. Denote by $\{\theta_t = \chi_t^{-1}, t \in \mathbb{R}\}$ the *inverse flow*, i.e. $(\theta_t \circ \chi_t)(x) = (\chi_t \circ \theta_t)(x) = x$, which can be also defined just as the flow of solutions to the Cauchy problem for the ODE

$$d\theta_t = -b(\theta_t) dt, \quad \theta_0 = x. \quad (2.13)$$

In all the results formulated in the sequel, we assume that (2.10), (2.11) hold true, and one of three assumptions which relate α and γ (cases **A–C**) is satisfied. In our first main result we specify in each of the cases **A–C** the choice of the zero order approximation $p_t^0(x, y)$ in the parametrix construction outlined above, and prove that this construction is feasible.

Theorem 2.1. *Let*

$$p_t^0(x, y) := \frac{1}{t^{d/\alpha} a^{d/\alpha}(y)} g^{(\alpha)}\left(\frac{\omega(t, y) - x}{t^{1/\alpha} a^{1/\alpha}(y)}\right), \quad (2.14)$$

where

$$\omega(t, y) := \begin{cases} y, & \text{in case A;} \\ y - tb(y), & \text{in case B;} \\ \theta_t(y), & \text{in case C.} \end{cases} \quad (2.15)$$

Then the following statements hold true.

1. For $t > 0$, $x, y \in \mathbb{R}^d$, the function $p_t(x, y)$ given by (2.9) is well defined, in the sense that the integrals $\Phi^{\otimes k}$ and $p^0 \otimes \Phi^{\otimes k}$ exist, and for every $T > 0$ the series involved in (2.9) converges absolutely on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, i.e.

$$\sum_{k \geq 1} |(p^0 \otimes \Phi^{\otimes k})_t(x, y)| < \infty.$$

2. The function $p_t(x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
3. For any $\kappa \in (0, \alpha \wedge \eta)$ and $T > 0$, the following estimate for $r_t(x, y)$ holds true:

$$|r_t(x, y)| \leq Cp_t^0(x, y)V_t(\omega(t, y) - x), \quad t \in (0, T], x, y \in \mathbb{R}^d, \quad (2.16)$$

where

$$V_t(z) = \begin{cases} t^{\kappa/\alpha} + t^\delta, & \text{if } |z| \leq t^{1/\alpha}, \\ |z|^\kappa + t^\delta, & \text{if } t^{1/\alpha} \leq |z| \leq 1, \\ 1 + t^\delta, & \text{if } |z| \geq 1, \end{cases} \quad (2.17)$$

and

$$\delta := \begin{cases} \kappa, & \text{in cases A and C,} \\ (1 - \frac{1}{\alpha} + \gamma) \wedge (1 - \frac{1}{\alpha} + \frac{\gamma}{\alpha}) \wedge \kappa, & \text{in case B.} \end{cases}$$

Remark 2.2. We have

$$p_t^{(0)}(x, y) \asymp \frac{1}{t^{d/\alpha}} \frac{1}{(1 + t^{-1/\alpha} |\omega(t, y) - x|)^{d+\alpha}} \quad (2.18)$$

(see (3.6) below). Hence (2.16) is equivalent to the following:

$$|r_t(x, y)| \leq R_t(\omega(t, y) - x), \quad t \in (0, T], x, y \in \mathbb{R}^d, \quad (2.19)$$

where

$$R_t(z) = \begin{cases} ct^{-d/\alpha}(t^{\kappa/\alpha} + t^\delta), & \text{if } |z| \leq t^{1/\alpha}, \\ c(|z|^\kappa + t^\delta) \frac{t}{|z|^{d+\alpha}}, & \text{if } t^{1/\alpha} < |z| \leq 1, \\ \frac{ct}{|z|^{d+\alpha}}, & \text{if } |z| > 1. \end{cases} \quad (2.20)$$

The proof of Theorem 2.1 and all the other results in this section formulated below are postponed to subsequent sections. All the subsequent results use the same notation.

Next, we relate the function $p_t(x, y)$ to the initial operator L . To make the structure of the proofs the most transparent, we do this in two steps. First, we prove that $p_t(x, y)$ is a transition probability density of *some* Markov process, and that the $C_\infty(\mathbb{R}^d)$ -generator of the respective semigroup is an extension of $(L, C_\infty^2(\mathbb{R}^d))$.

Theorem 2.2. *Identity*

$$P_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y) dy, \quad f \in C_\infty(\mathbb{R}^d), \quad (2.21)$$

defines a strongly continuous conservative contraction semigroup of non-negative operators on $C_\infty(\mathbb{R}^d)$, which in turn defines a (strong) Feller Markov process X . Every function $f \in C_\infty^2(\mathbb{R}^d)$ belongs to the domain $D(A)$ of the generator A of this semigroup, and

$$A f(x) = L f(x) = a(x) L^{(\alpha)} f(x) + (b(x), \nabla f(x)), \quad f \in C_\infty^2(\mathbb{R}^d);$$

that is, $(A, D(A))$ is an extension of $(L, C_\infty^2(\mathbb{R}^d))$.

In the next theorem we prove that the semigroup (2.21) with $p_t(x, y)$ defined by (2.9) is in fact the *unique* Feller semigroup associated with the operator (1.2).

Theorem 2.3. *The generator $(A, D(A))$ is the closure of $(L, C_\infty^2(\mathbb{R}^d))$ in $C_\infty(\mathbb{R}^d)$.*

The proposition below clarifies the relation between the function $p_t(x, y)$ and the notion of the “fundamental solution,” on which the parametrix construction of $p_t(x, y)$ was based. We formulate and prove this proposition under the additional assumption that in the case **C** the function $b(\cdot)$ is continuously differentiable (no additional assumptions in the cases **A**, **B** are required).

Proposition 2.1. *The real-valued function $p_t(x, y)$ is a fundamental solution to the Cauchy problem for the operator $\partial_t - A$; that is, for $t > 0$ it is differentiable in t , belongs to the domain of A as a function of x , and satisfies (2.3) and the analogue of (2.2) with A_x instead of L_x .*

Our next step is to relate the process X constructed in Theorem 2.2 to the weak solution to the SDE driven by an α -stable noise or, in a closely related terminology, to the solution of the martingale problem for $(L, C_\infty^2(\mathbb{R}^d))$. Namely, the semigroup $\{P_t\}$ corresponding to the process X possesses the Feller property, hence the process X has a càdlàg modification, see [17, Chapter 4, Theorem 2.7]. Denote by \mathbb{P}_x the law of the Markov process X with $X_0 = x$ in the Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ of càdlàg functions $[0, \infty) \rightarrow \mathbb{R}^d$. Recall that a measure \mathbb{P} on $\mathbb{D}([0, \infty), \mathbb{R}^d)$ is called a solution to the *martingale problem* $(L, D(L))$, if for every $f \in D(L)$ the process

$$f(X_t) - \int_0^t L f(X_s) ds, \quad t \geq 0$$

is a martingale with respect to \mathbb{P} , and the martingale problem for $(L, D(L))$ is called *well posed*, if for every $x \in \mathbb{R}^d$ there exists the unique solution to $(L, D(L))$ with $\mathbb{P}(X_0 = x) = 1$.

Theorem 2.4. For every $x \in \mathbb{R}^d$ the SDE (1.3) with the initial condition $X_0 = x$ has a unique weak solution, and the law of this solution in $\mathbb{D}([0, \infty), \mathbb{R}^d)$ equals \mathbb{P}_x .

In addition, the martingale problem $(L, C_\infty^2(\mathbb{R}^d))$ is well posed, and \mathbb{P}_x is its unique solution with the initial condition $X_0 = x$.

Remark 2.3. It is well known that, when both coefficients in equation of (1.3) are Lipschitz continuous, there exists a unique strong solution to (1.3) (see, for example, [28, Theorem IV.9.1], or [23, Theorem IV.3]). In our framework, the coefficient $a(x)$ is assumed to be only Hölder continuous. Up to our knowledge, there are no results on the existence and uniqueness of the strong solution under the assumption of the Hölder continuity of coefficients in equations of type (1.3), see also [3] for the negative example. See, however, [8,48], where under the assumption that $a = 1$ and b is Hölder continuous the existence and uniqueness of the strong solution is shown.

The last two theorems contain explicit estimates, respectively, for $p_t(x, y)$ and for its derivative with respect to the time variable.

Theorem 2.5. We have

$$p_t(x, y) \asymp \frac{1}{t^{d/\alpha}} g^{(\alpha)}\left(\frac{\omega(t, y) - x}{t^{1/\alpha}}\right), \quad t \in (0, T], x, y \in \mathbb{R}^d. \quad (2.22)$$

Finally, in the theorem below we show the continuity of the time derivative $\partial_t p_t(x, y)$, and provide the upper estimate for it. Note that these properties of $\partial_t p_t(x, y)$ are involved in the proofs of Theorem 2.3 and Proposition 2.1; see assertions (4.21), (4.22) below. These properties also have an independent interest, e.g. in the context of estimation of the accuracy of discrete approximation of occupation time functionals; see [22].

Theorem 2.6.

1. There exists a set $\Upsilon \subset (0, \infty) \times \mathbb{R}^d$ of zero Lebesgue measure such that the function $p_t(x, y)$ defined by (2.4)–(2.8) has a derivative

$$\partial_t p_t(x, y), \quad x \in \mathbb{R}^d, (t, y) \notin \Upsilon,$$

which for every fixed $(t, y) \notin \Upsilon$ is continuous in x . Moreover, in the cases **A** and **B** the set Υ is empty, and $\partial_t p_t(x, y)$ is continuous in (t, x, y) .

2. The derivative $\partial_t p_t(x, y)$ possesses the bound

$$|\partial_t p_t(x, y)| \leq C(t^{-1} \vee t^{-1/\alpha}) \frac{1}{t^{d/\alpha}} g^{(\alpha)}\left(\frac{\omega(t, y) - x}{t^{1/\alpha}}\right), \quad x \in \mathbb{R}^d, (t, y) \notin \Upsilon.$$

3. For every $f \in C_\infty(\mathbb{R}^d)$ the function

$$(0, \infty) \ni t \mapsto P_t f \in C_\infty(\mathbb{R}^d)$$

is continuously differentiable, and

$$(\partial_t P_t f)(x) = \int_{\mathbb{R}^d} \partial_t p_t(x, y) f(y) dy.$$

Remark 2.4. For $\lambda > 0$ denote

$$G^{(\lambda)}(x) := (|x| \vee 1)^{-d-\lambda}, \quad x \in \mathbb{R}^d. \quad (2.23)$$

Since $g^{(\alpha)}(x) \asymp G^{(\alpha)}(x)$ (see Proposition 3.2 below), one can replace in the above bounds $g^{(\alpha)}$ by $G^{(\alpha)}$, which gives more explicit estimates. We used $g^{(\alpha)}$ in the estimates for $p_t(x, y)$ and $\partial_t p_t(x, y)$ in order to emphasize the impact of the original α -stable process.

2.3. Overview and discussion

For the description and the background of the parametrix construction of the fundamental solution to a Cauchy problem for parabolic second order PDE's, we refer to the monograph of Friedman [21]; see also the original papers by E. Levi [42] and W. Feller [18]. This construction was extended in [14,15,36,37], to equations with pseudo-differential operators, see also the reference list and an extensive overview in the monograph [16]. In [14,15,36], the “main term” in the pseudo-differential operator is assumed to have the form $a(x)L^{(\alpha)}$ (in our notation) with $\alpha > 1$. In [37] the operator of such a type is treated, and although the case $\alpha \leq 1$ is allowed, in this case the gradient term is not involved in the equation. The list of subsequent and related publications is large, and we cannot discuss it here in details. Let us only mention two recent preprints: [10], where two-sided estimates, more precise than those in [36] are obtained, and [1], where the probabilistic interpretation of the parametrix construction and its application to the Monte-Carlo simulation is developed.

In all the references listed above it is required that either the stability index α satisfies $\alpha > 1$, or the gradient term is not involved in the equation. This is the common assumption in all the references available for us in this direction, with the one important exception given by the recent paper [12], see also [19,20]. In [12], for a Lévy driven SDE with α -stable like noise, the question of *existence* of the distribution density is studied by a different method, based on thoroughly balanced approximation of the initial SDE, Fourier transform based estimates, and discrete integration by parts. Such an approach is applicable for SDE's with the parameter of the noise $\alpha < 1$ and a non-trivial drift, but it does not give proper tools neither for obtaining explicit estimates for this density, nor even for proving the existence and uniqueness of the solution to the initial SDE. Hence, the scope of our approach based on the parametrix construction, differs substantially from that of [12].

Our version of the parametrix construction contains a substantial novelty, which makes it possible to handle the case $\alpha \leq 1$ with non-trivial gradient term. To explain this modified construction in the most transparent way, we took the “jump component” in a relatively simple form $a(x)Z^{(\alpha)}$. Clearly, one can think about considering, for example, α -stable symbol with state dependent spectral measure, see [37]. This, however, leads to additional cumbersome but inessential technicalities, and we prefer not do this in the current paper.

It was already mentioned that the parametrix construction described in Section 2.1 heavily relies on the choice of the “zero-order” approximation $p_t^0(x, y)$. In the case $\alpha > 1$, the first term $a(x)L^{(\alpha)}$ dominates the second term $b(x)\nabla$ in the sense that the symbol $a(x)|\xi|^\alpha$ of the first term grows as $|\xi| \rightarrow \infty$ faster than the modulus of the symbol $ib(x)\xi$ of the second term, see [33] for the detailed explanation. This allows us to chose in the case $\alpha > 1$ the zero-order approximation p^0 in the “classical” way (cf. [21,36]): Take the “principal part” $a(x)L^{(\alpha)}$ of the generator, “freeze” the coefficient $a(x)$ at some point $x = z$, then take the fundamental solution $q_t^z(x, y)$ to the operator $\partial_t - a(z)L^{(\alpha)}$ with this “frozen principal symbol,” and finally put $p_t^0(x, y) := q_t^z(x, y)|_{z=y}$. However, this procedure is not successful in the case $\alpha \leq 1$, and the reason for this is already mentioned: in this case, the operator $a(x)L^{(\alpha)}$ no longer dominates the gradient term, and therefore it can not be treated as the “principal part” of L . One can modify the choice of the zero order approximation in several ways. One of the possible choices is to “freeze” the coefficients in the entire operator L (which is now itself considered as the “principal part”), and to take the fundamental solution $q_t^z(x, y)$ to the Cauchy problem for $\partial_t - a(z)L^{(\alpha)} - (b(z), \nabla)$. This is exactly what we do in case **B**. However, this procedure is restricted by the relation between the parameter of the Hölder continuity γ and α . In case **C** the choice of the zero order approximation is no longer related to the fundamental solution to an equation with frozen coefficients, but instead uses a carefully designed “corrector” $\omega(t, y)$, which allows us to treat all $\alpha \in (0, 2)$. The price of such an approach is the assumption of the Lipschitz continuity of the drift b .

The effect of the interplay between the value of α and the regularity properties required for b , which we have mentioned in Remark 2.1, was observed first in [46,47]. It was shown therein that the parametrix construction is still feasible for (possibly unbounded) $b \in L_p(\mathbb{R}^d)$, $p > d/(\alpha - 1)$, where $\alpha > 1$. In [5] this effect was rediscovered in a stronger form: it is required that b belongs to the Kato class $\mathbb{K}_{d,\alpha-1}$. In [35] this result is extended even further, with b being allowed to be a generalized function equal to the derivative of a measure from $\mathbb{K}_{d,\alpha-1}$.

In general, there is a substantial gap between the problem of constructing a “candidate for being the fundamental solution” (i.e., to prove that relations (2.4)–(2.8) make sense), and the problem of relating the constructed kernel $p_t(x, y)$ to a Markov process. The first way how one can possibly solve this problem was proposed in [36]. It extends the approach from [21], where it is shown in the parabolic setting that $p_t(x, y)$ is twice continuously differentiable in x , and satisfies (2.2) in the classical sense. Note that the domain of the operator L is $C_b^2(\mathbb{R}^d)$. In the α -stable

case the natural upper bound $\partial_{xx}^2 p_t^0(x, y) \leq Ct^{-2/\alpha}$ is strongly singular for small t , and therefore it is difficult to prove using the parametrix construction that $p_t(\cdot, y) \in C_b^2(\mathbb{R}^d)$; thus, one cannot check straightforwardly that the expression (2.2) makes sense. Instead, in [36] the extended domain of $L^{(\alpha)}$ is introduced in terms of “hyper-singular integrals,” and it is proved that $p_t(x, y)$ satisfies (2.2) in the corresponding “extended” sense. Once (2.2) is proved, the required properties of the Markov process associated with $p_t(x, y)$ follow from the positive maximum principle in a rather standard way. Another way to verify the parametrix construction proposed in [37] is to guarantee the required smoothness of $p_t(x, y)$ by using the integration by parts procedure, but this approach seems to be only partially relevant; see Remark 6.1 below.

Partially, one can solve the problem of relating the kernel $p_t(x, y)$ to a Markov process by using some approximation procedure (e.g. [46,47]), or by analysing the perturbation of the resolvent kernels (cf. [5]). However, the most difficult part here is to relate *uniquely* the initial symbol and the Markov process associated with $p_t(x, y)$. This problem was recently solved in [35], in the framework of a singular gradient perturbation of an α -stable generator, in terms of the weak solution to the corresponding SDE. See also [9], where alternatively the martingale problem approach was used. The technique therein is closely related to those introduced (in the diffusive setting) in [4], and apparently strongly relies on the structural assumption that the resolvent, which corresponds to $p_t(x, y)$, is a perturbation of the resolvent of an α -stable process.

We propose a new method for solving this correspondence problem. Our method is based on the notion of the *approximate fundamental solution* to the Cauchy problem for (2.1), see Section 4 and especially the discussion at the beginning of Section 4.2. Combined with a proper “approximate” version of the positive maximum principle, this notion gives a flexible tool both for proving the semigroup properties of $p_t(x, y)$ (Theorem 2.2), and for studying more delicate uniqueness issues (Theorem 2.3). We expect that this method will be well applicable in other situations, where the parametrix construction is feasible; this is the subject of our ongoing research.

Let us briefly discuss another large group of results, focused on the construction of a *semigroup* for a Markov process with a given symbol rather than on the transition probability density $p_t(x, y)$ for it. An approach based on properties of the symbol of the operator and on the Hilbert space methods, is developed in the works of Jacob [30], see also the monograph [31]. It allows to show the existence of the closed extension in $C_\infty(\mathbb{R}^d)$ of a given pseudo-differential operator, and that this extension is the generator of a Feller semigroup. This approach was further developed in [6,7,26,27], and relies on the symbolic calculus approach for the parametrix construction (cf. [40], the original papers [29,55], and see also [32–34] for the detailed treatment).

Finally, we mention the group of results devoted to the well-posedness of the martingale problem for an integro-differential operator of certain type. For different types of perturbations of an α -stable generator, this problem was treated in [2,38,39,43–45,53,54], see also [24,25] for yet another approach for rather wide class of operators.

3. Proof of Theorem 2.1 and continuity properties of P_t

3.1. Function Φ : Evaluation and an upper bound

Our first step in the proof of Theorem 2.1 is to evaluate the kernel Φ and to give an upper bound for it.

For $\lambda \in [0, \alpha]$ we introduce a family of kernels

$$Q_t^{(\lambda)}(x, y) := \left(\left| \frac{\omega(t, y) - x}{t^{1/\alpha}} \right|^\lambda \wedge t^{-\lambda/\alpha} \right) \frac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\frac{\omega(t, y) - x}{t^{1/\alpha}} \right), \tag{3.1}$$

where the function $G^{(\alpha)}(x)$ is defined in (2.23), and $\omega(t, y)$ is defined in (2.15). We remark that since $Q_t^{(\lambda)}(x, y)$ involves $\omega(t, y)$, we have in fact three different families $Q_t^{(\lambda)}(x, y)$, $\lambda \in [0, \alpha]$, which correspond to the cases **A–C**.

Lemma 3.1. *Let $\kappa \in (0, \alpha \wedge \eta)$, $T > 0$. Then*

$$|\Phi_t(x, y)| \leq C(t^{-1+\kappa/\alpha} Q_t^{(\kappa)}(x, y) + t^{-1+\delta} Q_t^{(0)}(x, y)), \quad t \in (0, T], x, y \in \mathbb{R}^d, \tag{3.2}$$

where $\delta = \kappa$ in the cases **A** and **C**, and $\delta = (1 - 1/\alpha + \gamma) \wedge (1 - 1/\alpha + \gamma/\alpha) \wedge \kappa$ in the case **B**.

Before we proceed to the proof, we formulate some auxiliary statements.

Proposition 3.1.

1. For any $\lambda > 0$, $c > 0$ there exists $C > 0$ such that

$$G^{(\lambda)}(cx) \leq CG^{(\lambda)}(x). \quad (3.3)$$

2. For any $\lambda_1 > \lambda_2$ we have

$$G^{(\lambda_1)}(x) \leq G^{(\lambda_2)}(x). \quad (3.4)$$

3. For any $\varepsilon \in (0, \lambda)$,

$$|x|^\varepsilon G^{(\lambda)}(x) \leq CG^{(\lambda-\varepsilon)}(x). \quad (3.5)$$

The proof of Proposition 3.1 is obvious; we omit the details.

Proposition 3.2. For any $\alpha \in (0, 2)$,

$$g^{(\alpha)}(x) \asymp G^{(\alpha)}(x), \quad (3.6)$$

$$|(\nabla g^{(\alpha)})(x)| \leq CG^{(\alpha+1)}(x), \quad (3.7)$$

$$|(\nabla^2 g^{(\alpha)})(x)| \leq CG^{(\alpha+2)}(x), \quad (3.8)$$

$$|(L^{(\alpha)} g^{(\alpha)})(x)| \leq CG^{(\alpha)}(x), \quad (3.9)$$

$$|(\nabla L^{(\alpha)} g^{(\alpha)})(x)| \leq CG^{(\alpha+1)}(x). \quad (3.10)$$

The results stated in Proposition 3.2 are partly known; we defer the discussion and the remaining proofs to Appendix A.

Proof of Lemma 3.1. We consider the cases A–C separately. To improve the readability, here and below we assume that $T > 0$ is fixed and, if it is not stated otherwise, in every formula containing t , x , or y we assume $t \in (0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$.

Case A. Fix $z \in \mathbb{R}^d$, and denote

$$L^z = a(z)L^{(\alpha)};$$

that is, consider “the principal part” of the operator L with the coefficient “frozen” at the point z (cf. the discussion in Section 2.3). Denote by $q_t^z(x, y)$ the transition probability density of the process $Z^{(\alpha)}$ with the time, re-scaled by $a(z)$:

$$q_t^z(x, y) = \frac{1}{t^{d/\alpha} a^{d/\alpha}(z)} g^{(\alpha)}\left(\frac{y-x}{t^{1/\alpha} a^{1/\alpha}(z)}\right).$$

Then $q_t^z(x, y)$ is a fundamental solution to the Cauchy problem for the operator $(\partial_t - L^z)$, and

$$p_t^0(x, y) = q_t^y(x, y).$$

Observe that for every fixed $x, y \in \mathbb{R}^d$ the function $p_t^0(x, y)$ belongs to $C^1(0, \infty)$ as a function of t , and for every fixed $t \in (0, \infty)$ and $y \in \mathbb{R}^d$ it belongs to $C_\infty^2(\mathbb{R}^d)$ as a function of x . Since both ∇ and $L^{(\alpha)}$ are well defined on $C_\infty^2(\mathbb{R}^d)$, the function $\Phi_t(x, y)$ is well defined by (2.5).

Since q^z is a fundamental solution for $\partial_t - L^z$, one has

$$\begin{aligned}\Phi_t(x, y) &= [-(\partial_t - L_x^z)p_t^0(x, y) + (L_x - L_x^z)p_t^0(x, y)]|_{z=y} = (L_x - L_x^z)p_t^0(x, y)|_{z=y} \\ &= (a(x) - a(y))L_x^{(\alpha)}p_t^0(x, y) + (b(x), \nabla_x p_t^0(x, y)) \\ &= (a(x) - a(y))\frac{1}{t^{d/\alpha+1}a^{d/\alpha+1}(y)}(L^{(\alpha)}g^{(\alpha)})\left(\frac{y-x}{t^{1/\alpha}a^{1/\alpha}(y)}\right) \\ &\quad - \frac{1}{t^{(d+1)/\alpha}a^{(d+1)/\alpha}(y)}\left(b(x), (\nabla g^{(\alpha)})\left(\frac{y-x}{t^{1/\alpha}a^{1/\alpha}(y)}\right)\right) =: \Phi_t^1(x, y) + \Phi_t^2(x, y).\end{aligned}\quad (3.11)$$

First we estimate Φ^1 . Since $a(x)$ is bounded from above and away from zero, and is η -Hölder continuous, we have

$$|a(x) - a(y)| \leq c(|x - y|^\eta \wedge 1) \leq c(|x - y|^\kappa \wedge 1), \quad (3.12)$$

where we used that $\kappa < \alpha \wedge \eta$ (recall that by c and C we denote the generic constants, which may vary from place to place). Then by (3.9), (3.3) and (3.1) we obtain

$$|\Phi_t^1(x, y)| \leq C\frac{|y-x|^\kappa \wedge 1}{t^{1+d/\alpha}}G^{(\alpha)}\left(\frac{y-x}{t^{1/\alpha}}\right) \leq Ct^{-1+\kappa/\alpha}Q_t^{(\kappa)}(x, y). \quad (3.13)$$

To estimate Φ^2 we use that the functions $a(x)$ and $b(x)$ are bounded, and $a(x)$ is bounded away from zero. Hence by (3.7), (3.3) and (3.4) we have

$$|\Phi_t^2(x, y)| \leq Ct^{-(d+1)/\alpha}G^{(\alpha+1)}\left(\frac{y-x}{t^{1/\alpha}}\right) \leq Ct^{-1/\alpha}Q_t^{(0)}\left(\frac{y-x}{t^{1/\alpha}}\right). \quad (3.14)$$

Combining estimates (3.13) and (3.14), we obtain the required estimate.

Case B. We fix $z \in \mathbb{R}^d$, and define

$$L^z = a(z)L^{(\alpha)} + (b(z), \nabla);$$

that is, consider the entire operator L as its ‘‘principal part’’ and ‘‘freeze’’ its coefficients at the point z (cf. the discussion in Section 2.3 and the proof in the case A). The fundamental solution $q_t^z(x, y)$ to the Cauchy problem for $(\partial_t - L^z)$ is equal to the transition probability density of the process $Z^{(\alpha)}$ with the time parameter rescaled by $a(z)$, and with the additional constant drift $tb(z)$:

$$q_t^z(x, y) = \frac{1}{t^{d/\alpha}a^{d/\alpha}(z)}g^{(\alpha)}\left(\frac{y-x-tb(z)}{t^{1/\alpha}a^{1/\alpha}(z)}\right). \quad (3.15)$$

Again, we have

$$p_t^0(x, y) = q_t^y(x, y).$$

Since q^ξ is the fundamental solution for $\partial_t - L^\xi$, we have in the same way as in (3.11)

$$\begin{aligned}\Phi_t(x, y) &= [-(\partial_t - L_x^z)p_t^0(x, y) + (L_x - L_x^z)p_t^0(x, y)]|_{z=y} \\ &= (a(x) - a(y))L_x^{(\alpha)}p_t^0(x, y) + (b(x) - b(y), \nabla_x p_t^0(x, y)) \\ &= (a(x) - a(y))\frac{1}{t^{d/\alpha+1}a^{d/\alpha+1}(y)}(L^{(\alpha)}g^{(\alpha)})\left(\frac{y-tb(y)-x}{t^{1/\alpha}a^{1/\alpha}(y)}\right) \\ &\quad + \frac{1}{t^{(d+1)/\alpha}a^{(d+1)/\alpha}(y)}\left(b(y) - b(x), (\nabla g^{(\alpha)})\left(\frac{y-tb(y)-x}{t^{1/\alpha}a^{1/\alpha}(y)}\right)\right) \\ &=: \Phi_t^1(x, y) + \Phi_t^2(x, y).\end{aligned}\quad (3.16)$$

Recall (3.12), and write

$$x - y = (y - tb(y) - x) + tb(y).$$

Then by elementary inequalities $|u + v|^\kappa \leq 2^{\kappa-1}(|u|^\kappa + |v|^\kappa)$, $|u + v|^\kappa \wedge 1 \leq |u|^\kappa \wedge 1 + |v|^\kappa \wedge 1$, and the fact that $b(\cdot)$ is bounded, we obtain

$$|a(x) - a(y)| \leq c(|y - tb(y) - x|^\kappa \wedge 1) + ct^\kappa. \quad (3.17)$$

Then using (3.9), (3.3) and (3.1) we derive

$$|\Phi_t^1(x, y)| \leq Ct^{-1+\kappa/\alpha} Q_t^{(\kappa)}(x, y) + Ct^{-1+\kappa} Q_t^{(0)}(x, y). \quad (3.18)$$

Similar argument can be applied to $\Phi_t^2(x, y)$. Namely, using that $b(\cdot)$ is γ -Hölder continuous and bounded, we get

$$|b(x) - b(y)| \leq c|y - tb(y) - x|^\gamma + ct^\gamma.$$

Then using (3.7) and (3.3)–(3.5) we derive

$$\begin{aligned} |\Phi_t^2(x, y)| &\leq Ct^{-1/\alpha+\gamma/\alpha} \left| \frac{y - tb(y) - x}{t^{1/\alpha}} \right|^\gamma \frac{1}{t^{d/\alpha}} G^{(\alpha+1)} \left(\frac{y - tb(y) - x}{t^{1/\alpha}} \right) \\ &\quad + Ct^{-1/\alpha+\gamma} \frac{1}{t^{d/\alpha}} G^{(\alpha+1)} \left(\frac{y - tb(y) - x}{t^{1/\alpha}} \right) \\ &\leq Ct^{-1/\alpha+\gamma/\alpha} \frac{1}{t^{d/\alpha}} G^{(\alpha+1-\gamma)} \left(\frac{y - tb(y) - x}{t^{1/\alpha}} \right) + Ct^{-1/\alpha+\gamma} \frac{1}{t^{d/\alpha}} G^{(\alpha+1)} \left(\frac{y - tb(y) - x}{t^{1/\alpha}} \right) \\ &\leq Ct^{-1+\zeta} Q_t^{(0)}(x, y), \end{aligned} \quad (3.19)$$

where $\zeta := (1 - 1/\alpha + \gamma) \wedge (1 - 1/\alpha + \gamma/\alpha) > 0$. Thus, we arrive at (3.2).

Case C. In contrast with two previous cases, now we cannot interpret $p_t^0(x, y)$ as a fundamental solution to a Cauchy problem for some operator with “frozen” coefficients. Instead, we use the definition of the flow $\theta_t(y)$ and evaluate Φ directly.

Operators ∇ and $L^{(\alpha)}$ are homogeneous with respective orders 1 and α . From the identity

$$(\partial_t - L^{(\alpha)})[t^{-d/\alpha} g^{(\alpha)}(t^{-1/\alpha} x)] = 0,$$

we derive

$$\begin{aligned} \partial_t p_t^0(x, y) &= \left[a(y) \frac{1}{a^{d/\alpha}(y) t^{d/\alpha}} (L^{(\alpha)} g^{(\alpha)}) \left(\frac{w}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right. \\ &\quad \left. + \left(\partial_t \theta_t(y), \frac{1}{a^{d/\alpha+1}(y) t^{d/\alpha+1}} (\nabla g^{(\alpha)}) \left(\frac{w}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right) \right] \Big|_{w=\theta_t(y)-x}. \end{aligned}$$

On the other hand,

$$\begin{aligned} L_x p_t^0(x, y) &= \left[a(x) \frac{1}{a^{d/\alpha}(y) t^{d/\alpha}} (L^{(\alpha)} g^{(\alpha)}) \left(\frac{w}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right. \\ &\quad \left. - \left(b(x), \frac{1}{a^{d/\alpha+1}(y) t^{d/\alpha+1}} (\nabla g^{(\alpha)}) \left(\frac{w}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right) \right] \Big|_{w=\theta_t(y)-x}. \end{aligned}$$

Since $\partial_t \theta_t(y) = -b(\theta_t(y))$, we finally get

$$\begin{aligned} \Phi_t(x, y) &= (L_x - \partial_t) p_t^0(x, y) \\ &= (a(x) - a(y)) \frac{1}{t^{d/\alpha+1} a^{d/\alpha+1}(y)} (L^{(\alpha)} g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \\ &\quad + \frac{1}{t^{(d+1)/\alpha} a^{(d+1)/\alpha}(y)} \left(b(\theta_t(y)) - b(x), (\nabla g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right) \\ &=: \Phi_t^1(x, y) + \Phi_t^2(x, y). \end{aligned} \tag{3.20}$$

Since b is bounded, we have $|\theta_t(y) - y| \leq ct$. Similarly to (3.17) we have

$$|a(x) - a(y)| \leq c(|\theta_t(y) - x|^\kappa \wedge 1) + ct^\kappa. \tag{3.21}$$

Then using (3.9), (3.3) and (3.4), we get

$$\begin{aligned} |\Phi_t^1(x, y)| &\leq Ct^{-1+\kappa/\alpha} \left(\left| \frac{\theta_t(y) - x}{t^{1/\alpha}} \right|^\kappa \wedge t^{-\kappa/\alpha} \right) \frac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\frac{\theta_t(y) - x}{t^{1/\alpha}} \right) \\ &\quad + Ct^{-1+\kappa} \frac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\frac{\theta_t(y) - x}{t^{1/\alpha}} \right) \\ &= Ct^{-1+\kappa/\alpha} Q_t^{(\kappa)}(x, y) + Ct^{-1+\kappa} Q_t^{(0)}(x, y). \end{aligned}$$

For Φ^2 we have, since $b(x)$ is Lipschitz continuous,

$$|\Phi_t^2(x, y)| \leq Ct^{-d/\alpha} \left| \frac{\theta_t(y) - x}{t^{1/\alpha}} \right| \left| (\nabla g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right|.$$

Thus, using (3.7) and (3.3)–(3.5), we derive

$$|\Phi_t^2(x, y)| \leq C Q_t^{(0)}(x, y).$$

Combining the above estimates, we arrive at (3.2). □

3.2. Convergence of the parametrix series and the estimate for the residue term

To estimate the convolution powers $\Phi_t^{\otimes k}(x, y)$, $k \geq 1$, inductively we first slightly modify the upper bound for Φ obtained in Lemma 3.1. For $\lambda \in [0, \alpha)$, define

$$H_t^{(\lambda)}(x, y) := \left(\left(\left| \frac{\omega(t, y) - x}{t^{1/\alpha}} \right|^\lambda \vee 1 \right) \wedge t^{-\lambda/\alpha} \right) \frac{1}{t^{d/\alpha}} G^{(\alpha)} \left(\frac{\omega(t, y) - x}{t^{1/\alpha}} \right). \tag{3.22}$$

Clearly,

$$Q_t^{(\lambda)}(x, y) \leq H_t^{(\lambda)}(x, y),$$

and therefore a (weaker) analogue of (3.1) with $Q_t^{(\kappa)}(x, y)$, $Q_t^{(0)}(x, y)$ replaced by $H_t^{(\kappa)}(x, y)$, $H_t^{(0)}(x, y)$ holds true. The reason for us to modify (in fact, to weaken) estimate (3.2) in that way is that this form is well designed for further inductive estimation of convolution powers of Φ ; see the detailed discussion of this point in Remark 3.1 below. For possible further reference, we first develop this calculation in a general form, and then apply it for the particular function $\Phi_t(x, y)$ and kernels $H_t^{(\lambda)}(x, y)$.

Definition 3.1. A non-negative kernel $\{H_t(x, y), t > 0, x, y \in \mathbb{R}^d\}$ has a sub-convolution property, if for every $T > 0$ there exists a constant $C_{H,T} > 0$ such that

$$(H_{t-s} * H_s)(x, y) \leq C_{H,T} H_t(x, y), \quad t \in (0, T], s \in (0, t), x, y \in \mathbb{R}^d. \quad (3.23)$$

The kernel $\{H_t(x, y), t > 0, x, y \in \mathbb{R}^d\}$ has a super-convolution property if the sign “ \leq ” in (3.23) is changed to “ \geq .”

Lemma 3.2. Suppose that the function $\Phi_t(x, y)$ satisfies

$$|\Phi_t(x, y)| \leq C_{\Phi,T} (t^{-1+\delta_1} H_t^1(x, y) + t^{-1+\delta_2} H_t^2(x, y)), \quad t \in (0, T], x, y \in \mathbb{R}^d, \quad (3.24)$$

with some $\delta_1, \delta_2 \in (0, 1)$ and some non-negative kernels $H_t^1(x, y), H_t^2(x, y)$. Assume also that the kernels $H_t^i(x, y)$, $i = 1, 2$, satisfy the sub-convolution property with constant $C_{H,T}$, and

$$H_t^1(x, y) \geq H_t^2(x, y). \quad (3.25)$$

Then for every $t \in (0, T], x, y \in \mathbb{R}^d$, the statements below hold true.

(a) For $k \geq 1$,

$$|\Phi_t^{\otimes k}(x, y)| \leq \frac{C_1 C_2^k}{\Gamma(k\zeta)} t^{-1+(k-1)\zeta} (t^{\delta_1} H_t^1(x, y) + t^{\delta_2} H_t^2(x, y)), \quad (3.26)$$

where

$$C_1 = (3(T \vee 1)C_{H,T})^{-1}, \quad C_2 = 3(T \vee 1)C_{\Phi,T}C_{H,T}\Gamma(\zeta), \quad \text{and} \quad \zeta = \delta_1 \wedge \delta_2; \quad (3.27)$$

(b) The series $\sum_{k=1}^{\infty} \Phi_t^{\otimes k}(x, y)$ is absolutely convergent and

$$\left| \sum_{k=1}^{\infty} \Phi_t^{\otimes k}(x, y) \right| \leq C (t^{-1+\delta_1} H_t^1(x, y) + t^{-1+\delta_2} H_t^2(x, y)); \quad (3.28)$$

(c)

$$\left| \left(H^1 \circledast \sum_{k=1}^{\infty} \Phi^{\otimes k} \right)_t (x, y) \right| \leq C t^\zeta H_t^1(x, y). \quad (3.29)$$

Proof. Using the sub-convolution property of $H_t^i(x, y)$ and (3.25), we get

$$\begin{aligned} (H_{t-s}^1 * H_s^1)(x, y) &\leq C_{H,T} H_t^1(x, y), \\ (H_{t-s}^1 * H_s^2)(x, y) &\leq C_{H,T} H_t^1(x, y), \\ (H_{t-s}^2 * H_s^2)(x, y) &\leq C_{H,T} H_t^2(x, y), \end{aligned} \quad (3.30)$$

for every $t \leq T, s < t$.

Observe that (3.26) with $k = 1$ coincides with (3.24). Next, we suppose that (3.26) holds for $k \geq 1$, and show that it also holds for $k + 1$. Using (3.24), (3.26), (3.30), and the sub-convolution property for $H_t^1(x, y)$, we get

$$\begin{aligned}
|\Phi_t^{\otimes(k+1)}(x, y)| &= \left| \int_0^t (\Phi_{t-s}^{\otimes k} * \Phi_s)(x, y) ds \right| \\
&\leq \frac{C_1 C_2^k C_{\Phi, T}}{\Gamma(k\zeta)} \left\{ \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_1} s^{-1+\delta_1} (H_{t-s}^1 * H_s^1)(x, y) ds \right. \\
&\quad + \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_1} s^{-1+\delta_2} (H_{t-s}^1 * H_s^2)(x, y) ds \\
&\quad + \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_2} s^{-1+\delta_1} (H_{t-s}^2 * H_s^1)(x, y) ds \\
&\quad \left. + \int_0^t (t-s)^{-1+(k-1)\zeta} (t-s)^{\delta_2} s^{-1+\delta_2} (H_{t-s}^2 * H_s^2)(x, y) ds \right\} \\
&\leq \frac{(T^{\delta_1 \vee \delta_2 - \zeta} \vee 1) 3C_{\Phi, T} C_1 C_2^k C_{H, T}}{\Gamma(k\zeta)} \mathbf{B}(k\zeta, \zeta) t^{-1+k\zeta} (t^{\delta_1} H_t^1(x, y) + t^{\delta_2} H_t^2(x, y)) \\
&\leq \frac{(1 \vee T) 3C_{\Phi, T} C_1 C_2^k C_{H, T}}{\Gamma(k\zeta)} \mathbf{B}(k\zeta, \zeta) t^{-1+k\zeta} (t^{\delta_1} H_t^1(x, y) + t^{\delta_2} H_t^2(x, y)) \\
&= \frac{C_1 C_2^{k+1}}{\Gamma((k+1)\zeta)} t^{-1+k\zeta} (t^{\delta_1} H_t^1(x, y) + t^{\delta_2} H_t^2(x, y)), \tag{3.31}
\end{aligned}$$

which proves (3.26). By (3.26), the series $\sum_{k=2}^{\infty} \Phi_t^{\otimes(k)}(x, y)$ converge absolutely and

$$\left| \sum_{k=2}^{\infty} \Phi_t^{\otimes k}(x, y) \right| \leq C t^{-1+\zeta} (t^{\delta_1} H_t^1(x, y) + t^{\delta_2} H_t^2(x, y)), \tag{3.32}$$

which gives (3.28). Finally, (3.29) follows from (3.24), (3.28) and (3.30). \square

In the following proposition we collect the properties of the kernels $H_t^{(\lambda)}(x, y)$, $\lambda \geq 0$ which we require to complete the proof of Theorem 2.1. We defer the proof of this proposition to Appendix B.

Proposition 3.3. *In each of the cases A–C, for every $\lambda \in [0, \alpha)$, $T > 0$ the kernel $H_t^{(\lambda)}(x, y)$ satisfies*

$$c \leq \int_{\mathbb{R}^d} H_t^{(\lambda)}(x, y) dy \leq C, \quad t \leq T, \tag{3.33}$$

and possesses the sub- and super-convolution properties.

Remark 3.1. Now we can explain why it is convenient to replace in (3.2) the kernels $Q_t^{(\kappa)}(x, y)$. For $\lambda > 0$ the kernel $Q_t^{(\lambda)}(x, y)$ is equal to zero when $x = \omega(t, y)$, and using this observation we can easily verify that $Q_t^{(\lambda)}(x, y)$ does not satisfy the sub-convolution property. On the contrary, by Proposition 3.3 the kernels $H_t^{(\lambda)}(x, y)$, $\lambda \geq 0$, possess the sub-convolution property, hence we can easily derive the bounds for the convolutions powers $\Phi^{\otimes k}$ using Lemma 3.2. We remark that the sub- and super-convolution properties for the kernels $H_t^{(\lambda)}(x, y)$, $\lambda \geq 0$ are, in a sense, inherited from the convolution identity for the transition probability density of a symmetric α -stable process, which is just the Chapman–Kolmogorov equation for this process:

$$\int_{\mathbb{R}^d} g_{t-s}^{(\alpha)}(z-x) g_s^{(\alpha)}(y-z) dz = g_t^{(\alpha)}(y-x), \quad 0 < s < t.$$

An easier, but less precise way to estimate $\Phi_t^{\otimes k}(x, y)$, $k \geq 1$, dates back to [36], where the kernels $Q_t^{(\kappa)}(x, y)$ were bounded from above by $\frac{C}{t^{d/\alpha}} G^{(\alpha-\kappa)}(\frac{y-x}{t^{1/\alpha}})$. These modified kernels possess the sub-convolution property as well. However, their “tails” are heavier than those of the α -stable density, and therefore such an estimate does not lead to a precise bound for the residue. The latter weak point was resolved in [10], where the (mixed) k th convolutions of $Q_t^{(\kappa)}(x, y)$ and $Q_t^{(0)}(x, y)$, $k \geq 1$, were estimated directly, although in a rather cumbersome way. Our way to estimate the convolutions is motivated by both approaches, and inherits their advantages: by using the sub-convolution property we make the overall proof reasonably transparent, and because the “tails” of the auxiliary kernels $H_t^{(\kappa)}(x, y)$ and $Q_t^{(\kappa)}(x, y)$ are the same, our upper bounds on $\Phi_t^{\otimes k}(x, y)$ coincide with those obtained in [10].

Proof of statements 1 and 3 of Theorem 2.1. We have

$$|\Phi_t(x, y)| \leq C_{\Phi, T} (t^{-1+\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^{-1+\delta} H_t^{(0)}(x, y)), \quad (3.34)$$

which is just (3.2) modified as we have explained above. Then we apply Lemma 3.2 with $\delta_1 = \kappa/\alpha$, $\delta_2 = \delta$, $H_t^1(x, y) = H_t^{(\kappa)}(x, y)$ and $H_t^2(x, y) = H_t^{(0)}(x, y)$, and get

$$|\Phi_t^{\otimes k}(x, y)| \leq \frac{C_1 C_2^k}{\Gamma(k\zeta)} t^{-1+(k-1)\zeta} (t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y)), \quad (3.35)$$

$$|\Psi_t(x, y)| \leq C (t^{-1+\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^{-1+\delta} H_t^{(0)}(x, y)), \quad (3.36)$$

where $\zeta = \delta \wedge (\kappa/\alpha)$. In addition, we have $p_t^0(x, y) \asymp H_t^{(0)}(x, y)$, and therefore by (2.8) and (3.29)

$$|r_t(x, y)| \leq C (t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y)). \quad (3.37)$$

This completes the proof of statement 1. Since

$$H_t^{(\kappa)}(x, y) = \begin{cases} H_t^{(0)}(x, y), & |\omega(t, y) - x| \leq t^{1/\alpha}, \\ \frac{|\omega(t, y) - x|^\kappa}{t^{\kappa/\alpha}} H_t^{(0)}(x, y), & t^{1/\alpha} \leq |\omega(t, y) - x| \leq 1, \\ t^{-\kappa/\alpha} H_t^{(0)}(x, y), & |\omega(t, y) - x| \geq 1, \end{cases}$$

this also implies (2.16) and completes the proof of statement 3. \square

The proof of statement 2 is postponed to the next subsection, where the continuity issues are treated in a unified way.

3.3. Continuity properties of $p_t(x, y)$ and P_t

Proof of statement 2 of Theorem 2.1. In each of the cases **A–C**, one can verify directly and easily that the function $p_t^0(x, y)$ and the corresponding $\Phi_t(x, y)$ are continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. We show by induction the continuity of the functions $\Phi_t^{\otimes k}(x, y)$, $k \geq 2$.

Suppose that $\Phi_t^{\otimes(k-1)}(x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Denote

$$I_R(t, s, x, y) := \int_{B(0, R)} \Phi_{t-s}^{\otimes(k-1)}(x, z) \Phi_s(z, y) dz, \quad (3.38)$$

where $B(0, R)$ is the ball in \mathbb{R}^d centered at 0 with radius R . By the induction assumption, the function under the integral is continuous in t, x, y for $0 < s < t$.

Fix $R_0 > 0$, $\tau > 0$, $T > \tau$, and $\varepsilon \in (0, \tau)$. Then the expression under the integral is uniformly continuous in t, x, y for $s \in [\varepsilon, t - \varepsilon]$, $t \in [\tau, T]$, $x \in B(0, R_0)$, $y \in B(0, R_0)$, which by the dominated convergence theorem implies the continuity of $I_R(t, s, x, y)$ in the same domain, if $R > R_0$.

Denote

$$I(t, s, x, y) = \int_{\mathbb{R}^d} \Phi_{t-s}^{\otimes(k-1)}(x, z) \Phi_s(z, y) dz,$$

and observe that

$$|I(t, s, x, y) - I_R(t, s, x, y)| \leq \int_{\mathbb{R}^d \setminus B(0, R)} |\Phi_{t-s}^{\otimes(k-1)}(x, z) \Phi_s(z, y)| dz \rightarrow 0, \quad R \rightarrow \infty, \quad (3.39)$$

uniformly in $s \in [\varepsilon, t - \varepsilon]$, $t \in [\tau, T]$, $x \in B(0, R_0)$, $y \in B(0, R_0)$. Indeed, for R large enough and $z \in \mathbb{R}^d \setminus B(0, R)$, $y \in B(0, R_0)$, we have $|z - \omega(s, y)| \asymp |z|$, because the function $\omega(s, y)$ is bounded. This implies by (3.34)

$$|\Phi_s(z, y)| \leq C(\varepsilon) |z|^{-d-\alpha+\kappa}$$

for $s \in [\varepsilon, t - \varepsilon]$, $t \in [\tau, T]$, $z \in \mathbb{R}^d \setminus B(0, R)$, $y \in B(0, R_0)$. Since for such s, t, x and z

$$|\Phi_{t-s}^{\otimes(k-1)}(x, y)| \leq C(\varepsilon),$$

convergence (3.39) follows by the dominated convergence theorem. This gives that $I(t, s, x, y)$ is continuous in t, x, y .

Since

$$\Phi_t^{\otimes k}(x, y) = \int_0^t I(t, s, x, y) ds = \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{t-\varepsilon} I(t, s, x, y) ds, \quad (3.40)$$

the same argument yields continuity of $\Phi_t^{\otimes k}(x, y)$. Namely, proceeding in the same way as in (3.31), we derive that for $t \in [\tau, T]$

$$\begin{aligned} |I(t, s, x, y)| &= \left| \int_{\mathbb{R}^d} \Phi_{t-s}^{\otimes(k-1)}(x, z) \Phi_s(z, y) dz \right| \\ &\leq c(t-s)^{-1+(k-1)\zeta} s^{-1+\zeta} (H_t^{(\kappa)}(x, y) + H_t^{(0)}(x, y)) \\ &\leq C(t-s)^{-1+(k-1)\zeta} s^{-1+\zeta}, \quad \zeta = \delta \wedge (\kappa/\alpha). \end{aligned} \quad (3.41)$$

Hence for every $\varepsilon > 0$ the integral in the right hand side of (3.40) is continuous by the dominated convergence theorem. Convergence in (3.40) is uniform on compacts in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$; one can easily prove this using (3.41) and the expressions for $H^{(\kappa)}, H^{(0)}$. This completes the proof of continuity of $\Phi_t^{\otimes k}(x, y)$.

Since the series $\sum_{k=1}^{\infty} \Phi_t^{\otimes k}(x, y)$ converges uniformly on compact subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, the function $\Psi_t(x, y)$ is continuous, as well. Continuity of

$$r_t(x, y) = (p^0 \otimes \Psi)_t(x, y) = p_t(x, y) - p_t^0(x, y)$$

follows by the same argument as that for $\Phi_t^{\otimes k}(x, y)$. □

In the rest of the section we derive the basic properties of the family of the operators P_t , $t > 0$, defined by (2.21). This is used in the further analysis of the function $p_t(x, y)$ obtained via the parametrix construction.

Recall that $p_t^0(x, y) = H_t^{(0)}(x, y)$, hence by (3.37) and (3.33), for every $T > 0$

$$\int_{\mathbb{R}^d} p_t(x, y) dy \leq C, \quad t \in (0, T], x \in \mathbb{R}^d. \quad (3.42)$$

Then the family $P_t f(x)$, $t > 0$, is well defined by (2.21) for any bounded function f . We also put $P_0 f = f$. In order to show that each P_t maps $C_\infty(\mathbb{R}^d)$ into itself and the family $\{P_t, t \geq 0\}$ is strongly continuous at the point $t = 0$, we need the following proposition.

Proposition 3.4. *In each of the cases A–C, for every $f \in C_\infty(\mathbb{R}^d)$*

$$\lim_{|x| \rightarrow \infty} \int_{\mathbb{R}^d} p_t^0(x, y) f(y) dy = 0, \quad t > 0, \quad (3.43)$$

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_t^0(x, y) f(y) dy - f(x) \right| \rightarrow 0, \quad t \rightarrow 0. \quad (3.44)$$

We defer the proof of this proposition to Appendix B.

Lemma 3.3. *In each of the cases A–C of Theorem 2.1, the following properties hold true.*

1. For every $t > 0$, P_t is a bounded operator in $C_\infty(\mathbb{R}^d)$.
2. For every $f \in C_\infty(\mathbb{R}^d)$ we have $\lim_{t \rightarrow 0^+} \|P_t f - f\|_\infty = 0$.

Proof. 1. The proof of continuity of $P_t f$ is repeats the proof of continuity of $p_t(x, y)$, and thus is omitted.

To prove that $P_t f(x)$ vanishes as $|x| \rightarrow \infty$, we use the representation for $p_t(x, y)$, estimate (2.16) on $r_t(x, y)$, and (3.43):

$$\begin{aligned} \left| \int_{\mathbb{R}^d} p_t(x, y) f(y) dy \right| &= \left| \int_{\mathbb{R}^d} (p_t^0(x, y) + r_t(x, y)) f(y) dy \right| \\ &\leq C \int_{\mathbb{R}^d} p_t^0(x, y) |f(y)| dy \rightarrow 0, \quad |x| \rightarrow \infty. \end{aligned}$$

Hence P_t maps $C_\infty(\mathbb{R}^d)$ to $C_\infty(\mathbb{R}^d)$. Clearly, this operator is linear and bounded (its norm is bounded by the constant C from (3.42)).

2. By (3.37) and (3.33),

$$\sup_x \left| \int_{\mathbb{R}^d} r_t(x, y) f(y) dy \right| \leq C(t^{\kappa/\alpha} + t^\delta) \|f\|_\infty \rightarrow 0, \quad t \rightarrow 0. \quad (3.45)$$

Together with (3.44) this gives the required statement. \square

4. Proofs of Theorem 2.2 and Theorem 2.3

If we knew that the function $p_t(x, y)$, constructed in Theorem 2.1, is a fundamental solution to the Cauchy problem for $\partial_t - L$, then the properties of P_t stated in Theorem 2.2 could be obtained in a standard way based on the positive maximum principle, which holds true for the operator L , see [32, Theorem 4.5.13]. We refer to [17, Chapter 4] for the definition and the general results on the positive maximum principle. However, on this way we meet substantial difficulties already when we try to prove that L_x can be applied to $p_t(x, y)$. Recall that the domain of L is $C_\infty^2(\mathbb{R}^d)$. On the other hand, for the function $p_t^0(x, y)$ we have the following bounds, which can be derived from Proposition 3.2, but yet it seems that they can not be improved:

$$|\nabla_x p_t^0(x, y)| \leq C t^{-1/\alpha} H_t^{(0)}(x, y), \quad |\nabla_{xx}^2 p_t^0(x, y)| \leq C t^{-2/\alpha} H_t^{(0)}(x, y). \quad (4.1)$$

Hence the spatial derivatives of $p_t^0(x, y)$ have non-integrable singularities in t near 0, and such a behaviour of $p_t^0(x, y)$ makes it unclear why $p_t(x, y)$ should belong to the domain of L_x .

This difficulty is rather typical. In what follows, we develop an approach which we believe to be well applicable in various situations similar to those explained above. The keystone of this approach is that we use certain *approximate* solution to the Cauchy problem for $\partial_t - L$ instead of the exact one.

4.1. Approximate fundamental solution: Construction and basic properties

For $\varepsilon > 0$ we introduce the auxiliary function

$$p_{t,\varepsilon}(x, y) := p_{t+\varepsilon}^0(x, y) + \int_0^t \int_{\mathbb{R}^d} p_{t-s+\varepsilon}^0(x, z) \Psi_s(z, y) dz ds, \quad (4.2)$$

and define

$$P_{t,\varepsilon} f(x) := \int_{\mathbb{R}^d} p_{t,\varepsilon}(x, y) f(y) dy, \quad t > 0, x \in \mathbb{R}^d, f \in C_\infty(\mathbb{R}^d). \quad (4.3)$$

The additional time shift by positive ε removes the singularity at the point $s = t$, and this is the main reason why $p_{t,\varepsilon}(x, y)$ possesses the following properties:

- (i) $p_{t,\varepsilon}(x, y) \in C^1(0, \infty)$ for any fixed $\varepsilon > 0, x, y \in \mathbb{R}^d$;
- (ii) $p_{t,\varepsilon}(\cdot, y) \in C_\infty^2(\mathbb{R}^d)$ for any fixed $\varepsilon > 0, t > 0, y \in \mathbb{R}^d$;
- (iii) for any $0 < \tau < T$ we have $p_{t,\varepsilon}(x, y) \rightarrow p_t(x, y)$ as $\varepsilon \rightarrow 0$, uniformly in $(t, x, y) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$;
- (iv) for any $0 < \tau < T$ we have

$$q_{t,\varepsilon}(x, y) := (\partial_t - L_x) p_{t,\varepsilon}(x, y) \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

uniformly in $(t, x, y) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

We do not give the detailed proof of these properties, because everywhere below (with the only exception of the proof of Proposition 2.1) we will need the analogues of these properties for $P_{t,\varepsilon} f$, see Lemma 4.1 and Lemma 4.2; the proofs of (i)–(iv) above are completely analogous and omitted.

Properties (iii), (iv) motivate the name *approximate fundamental solution* we use for $p_{t,\varepsilon}(x, y)$: it approximates $p_t(x, y)$ and “satisfies” (2.2) in the approximate sense.

Lemma 4.1.

1. For every $f \in C_\infty(\mathbb{R}^d)$, $\varepsilon > 0$ the function $P_{t,\varepsilon} f(x)$ belongs to $C^1(0, \infty)$ as a function of t and to $C_\infty^2(\mathbb{R}^d)$ as a function of x .
2. For every $f \in C_\infty(\mathbb{R}^d)$, $T > 0$,

$$\|P_{t,\varepsilon} f - P_t f\|_\infty \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (4.4)$$

uniformly in $t \in [0, T]$, and for every $\varepsilon > 0$

$$P_{t,\varepsilon} f(x) \rightarrow 0, \quad |x| \rightarrow \infty \quad (4.5)$$

uniformly in $t \in [0, T]$.

3. For $f \in C_\infty(\mathbb{R}^d)$ we have

$$\lim_{t,\varepsilon \rightarrow 0^+} \|P_{t,\varepsilon} f - f\|_\infty = 0.$$

In the proof of this lemma we use the following proposition.

Proposition 4.1.

1. The derivative $\partial_t p_t^0(x, y)$ exists, and is continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
2. There exists $C > 0$ such that

$$|\partial_t p_t^0(x, y)| \leq C(t^{-1} \vee t^{-1/\alpha}) H_t^{(0)}(x, y), \quad t > 0, x, y \in \mathbb{R}^d.$$

The proof follows directly from the definition of $p_t^0(x, y)$, the properties of $g^{(\alpha)}(x)$ (cf. Proposition 3.2), and the definition of $\omega(t, y)$ in the expression for $p_t^0(x, y)$.

Proof of Proposition 4.1. Statement 1 can be easily derived by using the upper bound (3.36) on $\Psi_t(x, y)$, the upper estimates on $p_t^0(x, y)$, its space derivatives (4.1), time derivatives (see Proposition 4.1), and the dominated convergence theorem.

To prove statement 2, we first observe that the function

$$[0, T] \ni t \mapsto \int_{\mathbb{R}^d} p_t^0(\cdot, y) f(y) dy \in C_\infty(\mathbb{R}^d)$$

is continuous: the continuity at the point $t = 0$ is provided by Proposition 3.4, and the continuity at all the other points easily follows from the continuity of $p_t^0(x, y)$. Then

$$\int_{\mathbb{R}^d} p_{t+\varepsilon}^0(x, y) f(y) dy \rightarrow \int_{\mathbb{R}^d} p_t^0(x, y) f(y) dy, \quad \varepsilon \rightarrow 0,$$

uniformly in $t \in [0, T]$, $x \in \mathbb{R}^d$. This together with estimate (3.36) and the dominated convergence theorem implies statement 2.

The proof of statement 3 is a slight variation of the proof of statement 2 in Lemma 3.3. Namely, by (3.44) we have

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_{t+\varepsilon}^0(x, y) f(y) dy - f(x) \right| \rightarrow 0, \quad t, \varepsilon \rightarrow 0.$$

Hence it is sufficient to show that

$$\sup_x \left| \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Psi_{t-s+\varepsilon}(x, z) p_s^0(z, y) f(y) dy dz ds \right| \rightarrow 0, \quad t, \varepsilon \rightarrow 0. \quad (4.6)$$

Using (3.36), the identity $p_t^0(x, y) = H_t^{(0)}(x, y)$ and the properties of $H_t^{(\kappa)}(x, y)$, $H_t^{(0)}(x, y)$, one can verify this relation similarly to (3.45). \square

Denote

$$Q_{t,\varepsilon} f(x) = (\partial_t - L_x) P_{t,\varepsilon} f(x), \quad f \in C_\infty(\mathbb{R}^d). \quad (4.7)$$

Lemma 4.2. For any $f \in C_\infty(\mathbb{R}^d)$ we have

1.

$$Q_{t,\varepsilon} f(x) \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (4.8)$$

uniformly in $(t, x) \in [\tau, T] \times \mathbb{R}^d$ for every $\tau > 0$, $T > \tau$;

2.

$$\int_0^t Q_{s,\varepsilon} f(x) ds \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (4.9)$$

uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$ for any $T > 0$.

Proof. We have

$$L P_{t,\varepsilon} f(x) = L_x \int_{\mathbb{R}^d} p_{t+\varepsilon}^0(x, y) f(y) dy + L_x \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s+\varepsilon}^0(x, z) \Psi_s(z, y) f(y) dy dz ds; \quad (4.10)$$

note that $P_{t,\varepsilon}f$ and both integrals in the right hand side are C_∞^2 -functions in x (see Lemma 4.1 for the first term; the argument for the second integral is the same). Hence the operator L_x in (4.10) is well applicable. We would like to interchange L_x with the integrals in (4.10), i.e. to write

$$LP_{t,\varepsilon}f(x) = \int_{\mathbb{R}^d} L_x p_{t+\varepsilon}^0(x, y) f(y) dy + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} L_x p_{t-s+\varepsilon}^0(x, z) \Psi_s(z, y) f(y) dy dz ds. \quad (4.11)$$

Recall that L is an integro-differential operator given by (1.2), and observe that the argument based on the dominated convergence theorem allows us to interchange the ‘‘differential part’’ $(b(x), \nabla)$ of this operator with the integrals in (4.10). To do the same with the ‘‘integral part’’ $a(x)L^{(\alpha)}$, recall that

$$L^{(\alpha)}f(x) = \lim_{\varepsilon \rightarrow 0^+} L^{(\alpha,\varepsilon)}f(x), \quad L^{(\alpha,\varepsilon)}f(x) := \int_{|u|>\varepsilon} (f(x+u) - f(x)) \frac{c_\alpha}{|u|^{d+\alpha}} du,$$

and we interchange $a(x)L^{(\alpha,\varepsilon)}$ with the integrals just using the Fubini theorem. On the other hand,

$$\begin{aligned} |L^{(\alpha)}f(x) - L^{(\alpha,\varepsilon)}f(x)| &= \left| \int_{|u|\leq\varepsilon} (f(x+u) - f(x) - 1_{|u|\leq 1}(u, \nabla f)) \frac{c_\alpha}{|u|^{d+\alpha}} du \right| \\ &\leq C \sup_{x \in \mathbb{R}^d} |\nabla^2 f(x)| \int_{|u|\leq\varepsilon} |u|^2 \frac{c_\alpha}{|u|^{d+\alpha}} du. \end{aligned}$$

The integrals in (4.10) and the expressions under these integrals belong to $C_\infty^2(\mathbb{R}^d)$ in x and their second derivatives admit explicit bounds, cf. (4.1) and Lemma 4.1. Hence if we put in the right hand side of (4.10) operators

$$(b(x), \nabla_x) + a(x)L_x^{(\alpha,\varepsilon)},$$

instead of L_x , we get the expressions which tend to the original expressions as $\varepsilon \rightarrow 0$. The same is true for (4.11), and since we already have proved that we can interchange $(b(x), \nabla_x) + a(x)L_x^{(\alpha,\varepsilon)}$ with the integrals, we finally obtain (4.11).

Similarly, using the differentiability of $p_t^0(x, y)$ in t and the upper estimate on the respective derivatives (see Lemma 4.1 above), we derive

$$\begin{aligned} \partial_t P_{t,\varepsilon}f(x) &= \int_{\mathbb{R}^d} \partial_t p_{t+\varepsilon}^0(x, y) f(y) dy + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_t p_{t-s+\varepsilon}^0(x, z) \Psi_s(z, y) f(y) dy dz ds \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\varepsilon^0(x, z) \Psi_t(z, y) f(y) dy dz. \end{aligned} \quad (4.12)$$

Since

$$(L_x - \partial_t) p_t^0(x, y) = \Phi_t(x, y),$$

combining (4.11) and (4.12) we derive

$$\begin{aligned} Q_{t,\varepsilon}f(x) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\varepsilon^0(x, z) \Psi_t(z, y) f(y) dy dz - \int_{\mathbb{R}^d} \Phi_{t+\varepsilon}(x, y) f(y) dy \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi_{t-s+\varepsilon}(x, z) \Psi_s(z, y) f(y) dy dz ds. \end{aligned} \quad (4.13)$$

Since the function Ψ satisfies the equation

$$\Phi_t(x, y) = \Psi_t(x, y) - \int_0^t \int_{\mathbb{R}^d} \Phi_{t-s}(x, z) \Psi_s(z, y) dz ds,$$

we can rewrite $Q_{t,\varepsilon}f(x)$ as follows:

$$\begin{aligned} Q_{t,\varepsilon}f(x) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_\varepsilon^0(x, z) \Psi_t(z, y) dz - \Psi_{t+\varepsilon}(x, y) \right) f(y) dy \\ &\quad + \int_{\mathbb{R}^d} \left(\int_t^{t+\varepsilon} \int_{\mathbb{R}^d} \Phi_{t-s+\varepsilon}(x, z) \Psi_s(z, y) dz ds \right) f(y) dy \\ &=: Q_{t,\varepsilon}^1 f(x) + Q_{t,\varepsilon}^2 f(x). \end{aligned}$$

By the uniform continuity of Ψ on compact subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and estimate (3.36), we have

$$\sup_{t \in [\tau, T], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \Psi_{t+\varepsilon}(x, y) f(y) dy - \int_{\mathbb{R}^d} \Psi_t(x, y) f(y) dy \right| \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Using again the uniform continuity of Ψ on compact subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, relation (3.44) and estimate (3.36), we obtain

$$\begin{aligned} \sup_{t \in [\tau, T], x \in \mathbb{R}^d \times \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\varepsilon^0(x, z) \Psi_t(z, y) dz - \Psi_t(x, y) \right| &\rightarrow 0, \quad \varepsilon \rightarrow 0, \\ \sup_{t \in [\tau, T], x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\varepsilon^0(x, z) \Psi_t(z, y) f(y) dz dy - \int_{\mathbb{R}^d} \Psi_t(x, y) f(y) dy \right| &\rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

This proves (4.8) with $Q_{t,\varepsilon}^1 f(x)$ instead of $Q_{t,\varepsilon} f(x)$. By (3.36) we have

$$|Q_{t,\varepsilon}^1 f(x)| \leq C t^{-1+\zeta}, \quad \zeta = \delta \wedge (\kappa/\alpha),$$

hence (4.9) for $Q_{t,\varepsilon}^1 f(x)$ easily follows from (4.8).

By (3.34), (3.36) and inequality $H^{(0)} \leq H^{(\kappa)}$, we obtain in the same way as in (3.31)

$$\begin{aligned} &\int_t^{t+\varepsilon} \int_{\mathbb{R}^d} |\Phi_{t-s+\varepsilon}(x, z) \Psi_s(z, y)| dz ds \\ &\leq C H_{t+\varepsilon}^{(\kappa)}(x, y) \int_t^{t+\varepsilon} ((t+\varepsilon-s)s)^{-1+\zeta} ds \\ &\leq C H_{t+\varepsilon}^{(\kappa)}(x, y) t^{-1+\zeta} \int_t^{t+\varepsilon} (t+\varepsilon-s)^{-1+\zeta} ds \leq C \varepsilon^\zeta t^{-1+\zeta} H_{t+\varepsilon}^{(\kappa)}(x, y). \end{aligned} \tag{4.14}$$

This immediately gives (4.8) and (4.9) with $Q_{t,\varepsilon}^2 f(x)$ instead of $Q_{t,\varepsilon} f(x)$: we multiply (4.14) by $|f(y)|$ and integrate it either with respect to dy (for (4.8)), or with respect to $dy ds$ (for (4.9)). \square

4.2. Positive maximum principle, applied to the approximate fundamental solution. Proof of Theorem 2.2

The proof of Theorem 2.2 follows from Lemmae 4.3–4.5 given below.

Lemma 4.3. *The operator P_t defined in (2.21) is positivity preserving, i.e. $P_t f \geq 0$ if $f \geq 0$.*

Proof. Take $f \in C_\infty(\mathbb{R}^d)$, $f \geq 0$, and suppose that

$$\inf_{t,x} P_t f(x) < 0. \tag{4.15}$$

Then there exists $T > 0$ such that

$$\inf_{t \leq T, x \in \mathbb{R}^d} P_t f(x) < 0.$$

Then by (4.4) there exist $\eta > 0, \theta > 0, \varepsilon_1 > 0$ such that

$$\inf_{t \leq T, x \in \mathbb{R}^d} (P_{t,\varepsilon} f(x) + \theta t) < -\eta, \quad \varepsilon < \varepsilon_1.$$

Denote

$$u_\varepsilon(t, x) = P_{t,\varepsilon} f(x) + \theta t,$$

and note that by (4.5)

$$u_\varepsilon(t, x) \rightarrow \theta t > 0, \quad |x| \rightarrow \infty,$$

uniformly in $t \in [0, T]$. Hence the above infimum is in fact attained at some point in $[0, T] \times \mathbb{R}^d$; in what follows we fix one such a point for each ε , and denote it by $(t_\varepsilon, x_\varepsilon)$.

Since $f(x) \geq 0$, by statement 2 of Lemma 4.1 there exist $\varepsilon_0 > 0, \tau > 0$ such that

$$P_{t,\varepsilon} f(x) + \theta t \geq -\frac{\eta}{2}, \quad t \leq \tau, \varepsilon < \varepsilon_0, x \in \mathbb{R}^d.$$

Since

$$u_\varepsilon(t_\varepsilon, x_\varepsilon) = \min_{t \in [0, T], x \in \mathbb{R}^d} u_\varepsilon(t, x) < -\eta < -\frac{\eta}{2},$$

we have $t_\varepsilon > \tau$ as soon as $\varepsilon < \varepsilon_0$.

The operator L satisfies the positive maximum principle; that is, if whenever $f \in D(L)$, and $f(x_0) \geq 0$ where $x_0 = \arg \max f(x)$, then $Lf(x_0) \leq 0$, cf. [17, Chapter 4.2]. Therefore

$$L_x u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon, x_\varepsilon)} \geq 0.$$

In addition, for $\varepsilon < \varepsilon_0$ we always have

$$\partial_t u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon, x_\varepsilon)} \leq 0,$$

where the sign “<” may appear only if $t_\varepsilon = T$.

Then

$$(\partial_t - L_x)u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon, x_\varepsilon)} \leq 0. \tag{4.16}$$

On the other hand, since $t_\varepsilon \in [\tau, T]$, $\varepsilon < \varepsilon_0$, we have by the first statement of Lemma 4.2

$$(\partial_t - L_x)u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon, x_\varepsilon)} = \theta + Q_{t_\varepsilon, \varepsilon} f(x_\varepsilon) \rightarrow \theta > 0, \quad \varepsilon \rightarrow 0.$$

This gives a contradiction and shows that (4.15) fails. □

Lemma 4.4. *The family of operators possesses the semigroup property: $P_{t+s} = P_s P_t$.*

Proof. The proof is based on the same argument as the proof of Lemma 4.3, hence we just sketch it. Take $f \in C_\infty(\mathbb{R}^d)$ and assume, for instance, that

$$P_{t+s} f(x) - P_t P_s f(x) < 0 \tag{4.17}$$

for some $s, t > 0, x \in \mathbb{R}^d$. Fix this s , and observe that then for some $\eta > 0, \theta > 0, T > 0$ the function

$$u_\varepsilon(t, x) = P_{t+s, \varepsilon} f(x) - P_{t, \varepsilon} P_s f(x) + \theta t$$

satisfies

$$\inf_{t \leq T, x \in \mathbb{R}^d} u_\varepsilon(t, x) < -\eta.$$

In addition, $u_\varepsilon(t, x) \rightarrow \theta t$, $|x| \rightarrow \infty$, hence the infimum is attained at some point $(t_\varepsilon, x_\varepsilon)$. Finally, $u_\varepsilon(t, x) \rightarrow 0$, $t, \varepsilon \rightarrow 0$ and therefore there exist $\tau > 0, \varepsilon_0 > 0$ such that $t_\varepsilon > \tau$, provided that $\varepsilon \in (0, \varepsilon_0)$. Then, on one hand, (4.16) holds, and on the other hand

$$(\partial_t - L_x)u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon,x_\varepsilon)} = Q_{t_\varepsilon+s,\varepsilon}f(x_\varepsilon) - Q_{t_\varepsilon,\varepsilon}P_s f(x_\varepsilon) + \theta \rightarrow \theta > 0, \quad \varepsilon \rightarrow 0.$$

This gives a contradiction and proves that (4.17) is impossible. Changing f to $-f$, we see that inequality (4.17) with “>” instead of “<” is impossible as well, which completes the proof. \square

Lemma 4.5. *We have*

$$(a) \quad P_t f(x) - f(x) = \int_0^t P_s Lf(x) ds, \quad f \in C_\infty^2(\mathbb{R}^d); \quad (4.18)$$

$$(b) \quad P_t 1 = 1.$$

Proof. We apply the same argument as in the above lemmas. Take $f \in C_\infty^2(\mathbb{R}^d)$, and assume that for every $t > 0$, $x \in \mathbb{R}^d$,

$$P_t f(x) < f(x) + \int_0^t P_s Lf(x) ds. \quad (4.19)$$

Then repeating the above argument, we obtain the functions

$$u_\varepsilon(t, x) = P_{t,\varepsilon}f(x) - f(x) - \int_0^t P_{s,\varepsilon}Lf(x) ds + \theta t, \quad \varepsilon > 0$$

and the points $(t_\varepsilon, x_\varepsilon)$, in which these functions attain their minima on $[0, T] \times \mathbb{R}^d$, such that for some $\tau > 0, \varepsilon_0 > 0$ we have $t_\varepsilon > \tau$, provided before $\varepsilon \in (0, \varepsilon_0)$.

On one hand, for these functions we have (4.16). On the other hand,

$$\begin{aligned} (\partial_t - L_x)u_\varepsilon(t, x) &= Q_{t,\varepsilon}f(x) + Lf(x) - P_{t,\varepsilon}Lf(x) + \int_0^t LP_{s,\varepsilon}Lf(x) ds + \theta \\ &= Q_{t,\varepsilon}f(x) + Lf(x) - P_{t,\varepsilon}Lf(x) + \int_0^t (\partial_s P_{s,\varepsilon}Lf(x) - Q_{s,\varepsilon}Lf(x)) ds + \theta \\ &= Q_{t,\varepsilon}f(x) - \int_0^t Q_{s,\varepsilon}Lf(x) ds + \theta. \end{aligned}$$

Then by Lemma 4.2

$$(\partial_t - L_x)u_\varepsilon(t, x)|_{(t,x)=(t_\varepsilon,x_\varepsilon)} \rightarrow \theta > 0, \quad \varepsilon \rightarrow 0,$$

which gives a contradiction to (4.16) and disproves (4.19). Changing f to $-f$, we complete the proof of statement (a).

To prove statement (b), take $f \in C_\infty^2(\mathbb{R}^d)$ such that $f \equiv 1$ on the unit ball in \mathbb{R}^d , and put $f_k(x) = f(k^{-1}x)$. Then

$$f_k(x) \rightarrow 1, \quad Lf_k(x) \rightarrow 0, \quad k \rightarrow \infty,$$

and both $|f_k|$ and $|Lf_k|$ are bounded by some constant, independent of k . Then using the equality $p_t^0(x, y) = H_t^{(0)}(x, y)$, the estimate (2.16) on the remainder $r_r(x, y)$, and Proposition 3.3, we can apply the dominated convergence theorem and pass to the limit in (4.18) as $k \rightarrow \infty$ in (4.18) with f_k instead f . This finishes the proof of statement (b). \square

Proof of Theorem 2.2. By Lemmas 4.3, 4.4 and the second statement of Lemma 4.5, the family $\{P_t, t \geq 0\}$ forms a strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^d)$, which is positivity preserving. Since the semigroup $\{P_t, t \geq 0\}$ possesses the continuous transition probability density $p_t(x, y)$, the respective Markov process X is strong Feller. Finally, the first statement of Lemma 4.5 implies that the restriction of the generator of $\{P_t, t \geq 0\}$ coincides with L on functions from $C_\infty^2(\mathbb{R}^d)$. \square

4.3. The generator of the semigroup $(P_t)_{t \geq 0}$: Proofs of Theorem 2.3 and Proposition 2.1

In Lemma 4.5 we proved that $(L, C_\infty^2(\mathbb{R}^d))$ is the restriction of $(A, D(A))$. Since A is a closed operator, this yields that $(L, C_\infty^2(\mathbb{R}^d))$ is closable. Let us show that its closure coincides with $(A, D(A))$.

Take $f \in C_\infty(\mathbb{R}^d) \cap D(A)$. Fix $t > 0$, and consider $P_t f$ and $P_{t,\varepsilon} f$. Since $f \in D(A)$, then $P_t f \in D(A)$, and

$$AP_t f = \partial_t P_t f. \quad (4.20)$$

Recall that by statement 1 of Lemma 4.1 we have $P_{t,\varepsilon} f \in C_\infty^2(\mathbb{R}^d)$ and thus by Lemma 4.5 $P_{t,\varepsilon} f \in D(A)$. Hence,

$$AP_{t,\varepsilon} f = LP_{t,\varepsilon} f = \partial_t P_{t,\varepsilon} f.$$

Observe that

- By statement 2 of Lemma 4.1, one has $P_{t,\varepsilon} f \rightarrow P_t f$ in $C_\infty(\mathbb{R}^d)$ as $\varepsilon \rightarrow 0$;
- By statement 1 of Lemma 4.2, one has $(\partial_t - L)P_{t,\varepsilon} f \rightarrow 0$ in $C_\infty(\mathbb{R}^d)$ as $\varepsilon \rightarrow 0$.

Assuming that we know

$$\partial_t P_{t,\varepsilon} \rightarrow \partial_t P_t f \quad \text{in } C_\infty(\mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0, \quad (4.21)$$

we derive

$$LP_{t,\varepsilon} f \rightarrow AP_t f \quad \text{in } C_\infty(\mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0,$$

which implies that $P_t f$ belongs to the domain of the $C_\infty(\mathbb{R}^d)$ -closure of $(L, C_\infty^2(\mathbb{R}^d))$. Consequently, this closure coincides with $(A, D(A))$.

We have proved Theorem 2.3 under the assumption (4.21). We verify this assumption in Lemma 6.4 below.

We also show in Lemma 6.4 that

$$\partial_t p_{t,\varepsilon}(x, y) \rightarrow \partial_t p_t(x, y), \quad \varepsilon \rightarrow 0, \quad (4.22)$$

uniformly on compact subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Using properties (i), (ii) and (iv) of $p_{t,\varepsilon}(\cdot, y) \in C_\infty^2(\mathbb{R}^d)$ (cf. Section 4.1), and applying to this function literally the same argument used in the case of $P_{t,\varepsilon} f(\cdot)$, we derive that $p_t(\cdot, y) \in D(A)$, and $p_t(x, y)$ is a fundamental solution to the Cauchy problem for $\partial_t - A$. \square

5. Proof of Theorem 2.4

Let X be the canonical Markov process which corresponds to the semigroup constructed in Theorem 2.2. Using the Markov property of X , it is easy to deduce from (4.18) and the semigroup property for $p_t(x, y)$ the following: For given $f \in C_\infty^2(\mathbb{R}^d)$, $t_2 > t_1$, and $x \in \mathbb{R}^d$, for any $m \geq 1$, $r_1, \dots, r_m \in [0, t_1]$, and bounded measurable $G : (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ the identity

$$\mathbb{E}_x \left[f(X_{t_2}) - f(X_{t_2}) - \int_{t_1}^{t_2} h_f(X_s) ds \right] G(X_{r_1}, \dots, X_{r_m}) = 0$$

holds true. Here and below \mathbb{E}_x denotes the expectation with respect to the law of the underlying process, starting at $x \in \mathbb{R}^d$. This means that for every $f \in C_\infty^2(\mathbb{R}^d)$ the process

$$M_t^f = f(X_t) - \int_0^t h_f(X_s) ds, \quad t \geq 0, \quad (5.1)$$

is a \mathbb{P}_x -martingale for every $x \in \mathbb{R}^d$; that is, \mathbb{P} is a solution to the martingale problem for $(L, C_\infty^2(\mathbb{R}^d))$.

Operator $(L, C_\infty^2(\mathbb{R}^d))$ is dissipative, which follows from the positive maximum principle, see [17, Lemma 4.2.1], or [32, Lemma 4.5.2]. Next, its closure equals to the generator A of $C_\infty(\mathbb{R}^d)$ -semigroup $\{P_t, t \geq 0\}$, hence for every $\lambda > 0$ the range of the resolvent $(\lambda - L)^{-1}$ in $C_\infty(\mathbb{R}^d)$ is dense. Hence the required uniqueness of the solution to the martingale problem $(L, C_\infty^2(\mathbb{R}^d))$ follows by [17, Theorem 4.4.1].

It follows from the Itô formula, that every weak solution to (1.3) is a solution to the martingale problem for $(L, C_\infty^2(\mathbb{R}^d))$. Since we have already proved that this martingale problem is well posed, this immediately proves the *uniqueness* of the weak solution to (1.3). The proof of the *existence* of a weak solution can be conducted in a standard way, which we outline below.

- Consider the family of equations

$$dX_t^{(n)} = b_n(X_t^{(n)}) dt + \sigma_n(X_t^{(n)}) dZ_t^{(\alpha)} \quad (5.2)$$

with smooth coefficients b_n and σ_n , approximating the coefficients b and σ of (1.3). We can choose the approximations b_n and σ_n such that the functions $a_n = (\sigma_n)^\alpha$, $n \geq 1$ and b_n , $n \geq 1$ the constants satisfy (2.10) and (2.11) with the same constants. Since the coefficients in (5.2) are smooth, there exists a (strong) solution $X_t^{(n)}$ to (5.2). This solution is a strong Markov process, admitting the transition probability density $p_t^{(n)}(x, y)$. Note that under our assumption on the coefficients the upper bound for the residue term in Theorem 2.1 can be achieved uniformly in n , and consequently

$$p_t^{(n)}(x, y) \leq \frac{C}{t^{d/\alpha}} g^{(\alpha)}\left(\frac{\omega_n(t, y) - x}{t^{1/\alpha}}\right), \quad t \in (0, T], x, y \in \mathbb{R}^d, \quad (5.3)$$

where the constant $C > 0$ is independent of n , and $\omega_n(t, y)$ is given by (2.15) with b replaced by b_n , and the flow θ_t replaced by the corresponding flow.

- We show that

$$\mathbb{E}_x[|(X_{t+s}^{(n)}, Z_{t+s}^{(\alpha)}) - (X_t^{(n)}, Z_t^{(\alpha)})|^\beta | \mathcal{F}_t] \leq \rho(s), \quad (5.4)$$

where $\mathcal{F}_t := \sigma\{Z_s^{(\alpha)}, s \leq t\}$, the non-random function $\rho(s)$ tends to 0 as $s \rightarrow 0$, $\beta \in (0, \alpha)$ is some constant. By Theorem 8.6 and Remark 8.7 from [17, Chapter 3] we deduce that the sequence $(X^{(n)}, Z^{(\alpha)})$ is weakly compact in $\mathbb{D}(\mathbb{R}^+, (\mathbb{R}^d)^2)$.

Using the inequality $(u + v)^\beta \leq 2^{\beta-1}(u^\beta + v^\beta)$ for positive u, v and β , and the Markov property of $(X_t^{(n)}, Z_t^{(\alpha)})$ we derive

$$\begin{aligned} & \mathbb{E}_x[|(X_{t+s}^{(n)}, Z_{t+s}^{(\alpha)}) - (X_t^{(n)}, Z_t^{(\alpha)})|^\beta | \mathcal{F}_t] \\ &= \mathbb{E}_{(X_t^{(n)}, Z_t^{(\alpha)})}[|(X_s^{(n)}, Z_s^{(\alpha)}) - (X_0^{(n)}, Z_0^{(\alpha)})|^\beta] \\ &= \mathbb{E}_{(X_t^{(n)}, Z_t^{(\alpha)})}[(|X_s^{(n)} - X_0^{(n)}|^2 + |Z_s^{(\alpha)} - Z_0^{(\alpha)}|^2)^{\beta/2}] \\ &\leq 2^{\beta/2-1} (\mathbb{E}_{X_t^{(n)}}[|X_s^{(n)} - X_0^{(n)}|^\beta] + \mathbb{E}_{Z_t^{(\alpha)}}[|Z_s^{(\alpha)} - Z_0^{(\alpha)}|^\beta]) \\ &\leq C \sup_x \int_{\mathbb{R}^d} |y - x|^\beta \frac{1}{s^{d/\alpha}} g^{(\alpha)}\left(\frac{\omega_n(s, y) - x}{s^{1/\alpha}}\right) dy + C \int_{\mathbb{R}^d} |z|^\beta \frac{1}{s^{d/\alpha}} g^{(\alpha)}\left(\frac{z}{s^{1/\alpha}}\right) dz. \end{aligned}$$

We have

$$\int_{\mathbb{R}^d} |z|^\beta \frac{1}{s^{d/\alpha}} g^{(\alpha)}\left(\frac{z}{s^{1/\alpha}}\right) dz = cs^{\beta/\alpha}.$$

On the other hand, we can decompose $y - x = (\omega_n(s, y) - x) - (\omega_n(s, y) - y)$, and use the inequality $|\omega_n(s, y) - y| \leq Cs$, since the sequence $b_n(\cdot)$, $n \geq 1$ is uniformly bounded. Now simple calculation finally gives (5.4) with $\rho(s) = C(s^{\beta/\alpha} + s^\beta)$.

- By the weak compactness of the sequence $(X^{(n)}, Z^{(\alpha)})$, there exists a weak limit $(\tilde{X}, \tilde{Z}^{(\alpha)})$ of $(X^{(n)}, Z^{(\alpha)})$. By [41, Theorem 2.2, Remark 2.5], this weak limit is a weak solution to (1.3).

6. Time derivative of $p_t(x, y)$. Proof of Theorem 2.6

6.1. Outline

Our goal in this section is to prove the existence of the time derivative $\partial_t p_t(x, y)$, and to give estimates for this derivative. We begin with the outline of our approach, and indicate the main difficulties.

We would like to extend the properties of $p_t^0(x, y)$ stated in Proposition 4.1 to similar properties of $p_t(x, y)$. We have the integral representation

$$p_t(x, y) = p_t^0(x, y) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}^0(x, z) \Psi_s(z, y) dz ds,$$

which is just another form of (2.9); cf. (2.8). However, for the required extension we cannot use this representation, because the upper bound for $|\partial_t p_{t-s}^0(x, z)|$ has a non-integrable singularity $(t-s)^{-1} \vee (t-s)^{-1/\alpha}$ at the point $s = t$. Therefore we rewrite the integral representation for $p_t(x, y)$ in the following way:

$$p_t(x, y) = p_t^0(x, y) + \int_0^{t/2} \int_{\mathbb{R}^d} p_{t-s}^0(x, z) \Psi_s(z, y) dz ds + \int_0^{t/2} \int_{\mathbb{R}^d} p_s^0(x, z) \Psi_{t-s}(z, y) dz ds. \quad (6.1)$$

Using this representation, we avoid the annoying singularities related to $p_t^0(x, y)$, but instead we have to establish the differential properties of Ψ with respect to the time variable. For this we proceed in the way similar to that used in Section 3.2: first we establish the required properties for Φ , then for its convolutions, and finally for Ψ . The minor difficulty which arises is that in case **C** the function $b(x)$ is not supposed to be from the class $C_b^1(\mathbb{R}^d)$, and therefore $\Phi_t(x, y)$ is not continuously differentiable in t . This difficulty is of completely technical nature, and is resolved by choosing a suitable formulation for differentiability property of $\Phi_t(x, y)$ and its convolutions. \square

6.2. Time derivatives of Φ , $\Phi^{\otimes k}$ and Ψ . Proof of the convergence in (4.21) and (4.22)

Consider first the following ‘‘smooth’’ case.

Lemma 6.1. *Assume that either one of cases **A** or **B** of Theorem 2.1 holds true, or case **C** of Theorem 2.1 holds true with an additional assumption $b \in C_b^1(\mathbb{R}^d)$. Then the statements below hold true.*

1. Function $\Phi_t(x, y)$ defined by (2.5) possesses the derivative $\partial_t \Phi_t(x, y)$, which is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
2. For any $\kappa \in (0, \eta \wedge \alpha)$ and $T > 0$, the derivative $\partial_t \Phi_t(x, y)$ possesses the bound

$$|\partial_t \Phi_t(x, y)| \leq C(t^{-1} \vee t^{-1/\alpha})(t^{-1+\kappa/\alpha} H_t^{(\kappa)} + t^{-1+\delta} H_t^{(0)}(x, y)), \quad t \in (0, T], x, y \in \mathbb{R}^d. \quad (6.2)$$

Proof. We give the calculations for the case **C** only; the other cases are similar and simpler. Statement 1 follows directly from the explicit formula (3.20). To prove statement 2 we estimate separately the derivatives of $\Phi_t^1(x, y)$,

$\Phi_t^2(x, y)$ in (3.20). We have

$$\begin{aligned} |\partial_t \Phi_t^1(x, y)| &\leq C |a(x) - a(y)| \left\{ \frac{1}{t^{d/\alpha+2}} \left| (L^{(\alpha)} g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \right. \\ &\quad \left. + \frac{1}{t^{d/\alpha+1+1/\alpha}} \left| (\nabla L^{(\alpha)} g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \right. \\ &\quad \left. + \frac{1}{t^{d/\alpha+2}} \left| (\nabla L^{(\alpha)} g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \left| \frac{\theta_t(y) - x}{t^{1/\alpha}} \right| \right\}, \end{aligned}$$

because $\partial_t \theta_t(y) = -b(\theta_t(y))$, which is bounded. Applying (3.21) with $\kappa \in (0, \eta \wedge \alpha)$, we get by (3.9), (3.10), and (3.3)–(3.5) the estimates

$$\begin{aligned} |\partial_t \Phi_t^1(x, y)| &\leq C |a(y) - a(x)| \left(\frac{1}{t^{d/\alpha+2}} + \frac{1}{t^{d/\alpha+1+1/\alpha}} \right) g^{(\alpha)} \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \\ &\leq C (t^{-2} + t^{-1-1/\alpha}) (t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\kappa H_t^{(0)}(x, y)) \\ &\leq C (t^{-1} \vee t^{-1/\alpha}) (t^{-1+\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^{-1+\kappa} H_t^{(0)}(x, y)). \end{aligned}$$

Similarly,

$$\begin{aligned} |\partial_t \Phi_t^2(x, y)| &\leq C t^{-d/\alpha-1/\alpha} \left| (\nabla g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \\ &\quad + C t^{-d/\alpha-1/\alpha-1} |b(\theta_t(y)) - b(x)| \left| (\nabla g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \\ &\quad + C t^{-d/\alpha-1/\alpha-1} |b(\theta_t(y)) - b(x)| \left| (\nabla^2 g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right| \left| \frac{\theta_t(y) - x}{t^{1/\alpha}} \right| \\ &\quad + C t^{-d/\alpha-2/\alpha} |b(\theta_t(y)) - b(x)| \left| (\nabla^2 g^{(\alpha)}) \left(\frac{\theta_t(y) - x}{t^{1/\alpha} a^{1/\alpha}(y)} \right) \right|, \end{aligned}$$

where we have used that ∇b and $\partial_t \theta_t(y)$ are bounded. Therefore, using the Lipschitz condition for b and (3.7), (3.8), we can write a shorter (and less precise) estimate

$$|\partial_t \Phi_t^2(x, y)| \leq C (t^{-1} \vee t^{-1/\alpha}) H_t^{(0)}(x, y), \quad t \in (0, T], x, y \in \mathbb{R}^d,$$

which combined with the estimate for $\Phi_t^1(x, y)$ completes the proof. \square

Lemma 6.2. *Under the condition of Lemma 6.1, the following statements hold true.*

1. *The functions $\Phi_t^{\otimes k}(x, y)$ and $\Psi_t(x, y)$, defined by (2.5) have derivatives*

$$\partial_t \Phi_t^{\otimes k}(x, y), \quad \partial_t \Psi_t(x, y), \quad t > 0, x, y \in \mathbb{R}^d,$$

continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

2. *For any $\kappa \in (0, \eta \wedge \alpha)$ and $T > 0$ there exist constants C_3, C_4 and C_5 such that*

$$|\partial_t \Phi_t^{\otimes k}(x, y)| \leq \frac{C_3 C_4^k}{\Gamma(k\xi)} (t^{-1} \vee t^{-1/\alpha}) t^{-1+(k-1)\xi} (t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y)), \quad (6.3)$$

$$|\partial_t \Psi_t(x, y)| \leq C_5 (t^{-1} \vee t^{-1/\alpha}) (t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y)), \quad (6.4)$$

for all $t \in (0, T], x, y \in \mathbb{R}^d$.

Proof. Since the proof is similar to that of Lemma 3.2, we only sketch the argument. Let C_1, C_2 be that same as in (3.35) (see also (3.27)). We show that (6.3) holds true with

$$C_3 = \frac{C\Gamma(\zeta)}{C_4} \vee \frac{CC_1}{C_{\Phi,T}} \vee \frac{C_1}{\Gamma(\zeta)}, \quad C_4 = 9(2 \vee 2^{1/\alpha})(T \vee 1)C_{\Phi,T}C_{H,T}\Gamma(\zeta),$$

where the constant $C > 0$ is the one from (6.2).

Split

$$\Phi_t^{\otimes(k+1)}(x, y) = \int_0^{t/2} \int_{\mathbb{R}^d} \Phi_{t-s}^{\otimes k}(x, z) \Phi_s(z, y) dz ds + \int_0^{t/2} \int_{\mathbb{R}^d} \Phi_s^{\otimes k}(x, z) \Phi_{t-s}(z, y) dz ds. \quad (6.5)$$

By induction, it can be shown that each $\Phi_t^{\otimes k}(x, y)$ is continuously differentiable in t , and

$$\begin{aligned} \partial_t \Phi_t^{\otimes(k+1)}(x, y) &= \int_0^{t/2} \int_{\mathbb{R}^d} (\partial_t \Phi^{\otimes k})_{t-s}(x, z) \Phi_s(z, y) dz ds + \int_0^{t/2} \int_{\mathbb{R}^d} \Phi_s^{\otimes k}(x, z) (\partial_t \Phi)_{t-s}(z, y) dz ds \\ &\quad + \int_{\mathbb{R}^d} \Phi_{t/2}^{\otimes k}(x, z) \Phi_{t/2}(z, y) dz. \end{aligned} \quad (6.6)$$

Observe that

$$(t-s)^{-1} \vee (t-s)^{-1/\alpha} \leq (2 \vee 2^{1/\alpha})(t^{-1} \vee t^{-1/\alpha}), \quad s \in (0, t/2).$$

Using this inequality and pulling out from the integrals the term $(2 \vee 2^{1/\alpha})(t^{-1} \vee t^{-1/\alpha})$, we get by the induction assumption

$$\begin{aligned} &\left| \int_0^{t/2} \int_{\mathbb{R}^d} (\partial_t \Phi^{\otimes k})_{t-s}(x, z) \Phi_s(z, y) dz ds \right| \\ &\leq \frac{C_3 C_4^k C_{\Phi,T} (2 \vee 2^{1/\alpha})}{\Gamma(\zeta k)} (t^{-1} \vee t^{-1/\alpha}) \\ &\quad \times \int_0^t \int_{\mathbb{R}^d} (t-s)^{-1+(k-1)\zeta} \left((t-s)^{\kappa/\alpha} H_{t-s}^{(\kappa)}(x, z) + (t-s)^\delta H_{t-s}^{(0)}(x, z) \right) \\ &\quad \times \left(s^{\kappa/\alpha} H_s^{(\kappa)}(z, y) + s^\delta H_s^{(0)}(z, y) \right) dz ds \\ &\leq \frac{3(T \vee 1)C_3 C_4^k C_{\Phi,T} C_{H,T} (2 \vee 2^{1/\alpha})}{\Gamma(\zeta k)} (t^{-1} \vee t^{-1/\alpha}) \mathbf{B}(\zeta k, \zeta) t^{-1+k\zeta} \left(t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right) \\ &= \frac{3(T \vee 1)C_3 C_4^k C_{\Phi,T} C_{H,T} (2 \vee 2^{1/\alpha}) \Gamma(\zeta)}{\Gamma(\zeta(k+1))} (t^{-1} \vee t^{-1/\alpha}) t^{-1+k\zeta} \left(t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right) \\ &\leq \frac{C_3 C_4^{k+1}}{3\Gamma(\zeta(k+1))} (t^{-1} \vee t^{-1/\alpha}) t^{-1+k\zeta} \left(t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right). \end{aligned}$$

In the same fashion, it can be shown that

$$\begin{aligned} &\left| \int_0^{t/2} \int_{\mathbb{R}^d} \Phi_s^{\otimes k}(x, z) (\partial_t \Phi)_{t-s}(z, y) dz ds \right| \\ &\leq \frac{C_3 C_4^{k+1}}{3\Gamma((k+1)\zeta)} (t^{-1} \vee t^{-1/\alpha}) t^{-1+k\zeta} \left(t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right). \end{aligned}$$

For the third term we have by (3.35)

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \Phi_{t/2}^{\otimes k}(x, z) \Phi_{t/2}(z, y) dz \right| \\ & \leq \frac{C_1 C_2^k C_{\Phi, T} C_{H, T}}{\Gamma(k\zeta)} \left(\frac{t}{2} \right)^{-1+(k-1)\zeta} \\ & \quad \times \left\{ \left(2 \left(\frac{t}{2} \right)^{\kappa/\alpha+\delta} + \left(\frac{t}{2} \right)^{2\kappa/\alpha} \right) H_t^{(\kappa)}(x, y) + \left(\frac{t}{2} \right)^{2\delta} H_t^{(0)}(x, y) \right\} \\ & \leq \frac{3(T \vee 1) C_1 C_{\Phi, T} C_{H, T} C_2^k}{\Gamma(k\zeta)} t^{-1+k\zeta} \frac{2^{1-\zeta}}{(2\varepsilon)^{k\zeta}} \left\{ t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right\}. \end{aligned}$$

From the inequality $u^\zeta \leq e^{(1-\varepsilon)u}$, $u > 0$, $\varepsilon \in (0, 1)$, we get the estimate

$$\Gamma((k+1)\zeta) = \int_0^\infty e^{-u} u^{(k+1)\zeta-1} du \leq \varepsilon^{-k\zeta} \Gamma(\zeta k).$$

Without loss of generality we assume that $\varepsilon > 1/2$; then $(2\varepsilon)^{-\zeta k} \leq 1$. Therefore, we arrive at

$$\left| \int_{\mathbb{R}^d} \Phi_{t/2}^{\otimes k}(x, z) \Phi_{t/2}(z, y) dz \right| \leq \frac{C_1 C_4^{k+1}}{3\Gamma(\zeta)\Gamma((k+1)\zeta)} t^{-1+k\zeta} \left\{ t^{\kappa/\alpha} H_t^{(\kappa)}(x, y) + t^\delta H_t^{(0)}(x, y) \right\}.$$

Adding the obtained estimate, we get (6.3) with $k+1$ instead of k .

In addition, for fixed $y \in \mathbb{R}^d$ each term in the sum has a derivative in t , continuous in $(t, x) \in (0, T] \times \mathbb{R}^d$, and by (6.3) the series for the derivative is also uniformly convergent. Thus, Ψ has a derivative in t , which is continuous with respect to $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and satisfies (6.4). \square

In the above proof, in the case **C** we differentiate in t the term

$$b(\theta_t(y))$$

in the expression for $\Phi_t^2(x, y)$. If b does not belong to C^1 , this term may not be continuously differentiable. Nevertheless, it is possible to show that the above result extends in a certain sense to the case when b is only assumed to be Lipschitz continuous.

Lemma 6.3. *In case C of Theorem 2.1, the following statements hold true.*

1. *There exists a set $\Upsilon \subset (0, \infty) \times \mathbb{R}^d$ of zero Lebesgue measure such that the functions $\Phi_t^{\otimes k}(x, y)$, $k \geq 1$, and $\Psi_t(x, y)$ are differentiable in t for every $x \in \mathbb{R}^d$ and $(t, y) \notin \Upsilon$.*
2. *For every $(t, y) \notin \Upsilon$, the time derivatives $\partial_t \Phi_t^{\otimes k}(x, y)$, $k \geq 1$, and $\partial_t \Psi_t(x, y)$ are continuous in $x \in \mathbb{R}^d$ and satisfy (6.3), (6.4).*

Proof. Recall that by the Rademacher theorem (cf. [13, Theorem VII.23.2]) the Lipschitz continuous function b has a gradient a.e. with respect to the Lebesgue measure on \mathbb{R}^d . Denote by Υ_b the exceptional set of zero Lebesgue measure, such that b is differentiable at every point outside Υ_b . Since θ_t is a diffeomorphism of \mathbb{R}^d (see Theorem I.2.3 and the comment in Chapter I, Section 5 from [11]), the set $\Upsilon_{t,b} = \{y : \theta_t(y) \in \Upsilon_b\}$ is again of zero Lebesgue measure. Since $\partial_t \theta_t(y) = -b(\theta_t(y))$, the derivative $\partial_t b(\theta_t(y))$ is well defined for every $y \in \Upsilon_{t,b}$. This derivative is given by

$$\partial_t b(\theta_t(y)) = - \sum_{j=1}^d \partial_j b(\theta_t(y)) b_j(\theta_t(y)),$$

where the partial derivatives $\partial_j b$ are now well defined on Υ_b and bounded, because b is Lipschitz continuous. The term $b(\theta_t(y))$ comes in the expression for Φ in a multiplicative way, and all other terms have derivatives in t , and are

continuous in (t, x, y) . Hence, repeating the calculations from the proof of Lemma 6.1, we get the (part of) required statements for Φ , with the exceptional set

$$\Upsilon^1 = \{(t, y) : y \in \Upsilon_{t,b}\}.$$

Further, it is easy to get by induction the same statements for $\Phi^{\otimes k}$, $k \geq 2$, with the exceptional set

$$\Upsilon = \Upsilon^1 \cup \left\{ (0, \infty) \times \left\{ y : \int_0^\infty 1_{y \in \Upsilon_{s,b}} ds > 0 \right\} \right\}.$$

Indeed, by (6.5)

$$\begin{aligned} \frac{\Phi_{t+\Delta t}^{\otimes(k+1)}(x, y) - \Phi_t^{\otimes(k+1)}(x, y)}{\Delta t} &= \int_0^{t/2} \int_{\mathbb{R}^d} \frac{\Phi_{t+\Delta t-s}^{\otimes k}(x, z) - \Phi_{t-s}^{\otimes k}(x, z)}{\Delta t} \Phi_s(z, y) dz ds \\ &+ \frac{1}{\Delta t} \int_{t/2}^{(t+\Delta t)/2} \int_{\mathbb{R}^d} \Phi_{t+\Delta t-s}^{\otimes k}(x, z) \Phi_s(z, y) dz ds \\ &+ \int_0^{t/2} \int_{\mathbb{R}^d} \Phi_s^{\otimes k}(x, z) \frac{\Phi_{t+\Delta t-s}(z, y) - \Phi_{t-s}(z, y)}{\Delta t} dz ds \\ &+ \frac{1}{\Delta t} \int_{t/2}^{(t+\Delta t)/2} \int_{\mathbb{R}^d} \Phi_s^{\otimes k}(x, z) \Phi_{t+\Delta t-s}(z, y) dz ds. \end{aligned}$$

Observe that if $(t, y) \notin \Upsilon$ then the respective ratios under the first and the third integrals converge ds -a.e. to the derivatives $(\partial_t \Phi^{\otimes k})_{t-s}(x, z)$ and $\partial_t \Phi_{t-s}(z, y)$, and the functions $\Phi_{t+\Delta t-s}^{\otimes k}(x, z)$ and $\Phi_{t+\Delta t-s}(z, y)$ converge, respectively, to $\Phi_{t-s}^{\otimes k}(x, z)$ and $\Phi_{t-s}(z, y)$. Then the convergence of the integrals follows from dominated convergence theorem and estimates (6.3) and (3.35). Hence, the derivative $\partial_t \Phi_t^{\otimes(k+1)}(x, y)$ exists and admits representation (6.6). The bound (6.3) for it follows by induction. Its continuity in x also follows by induction and the dominated convergence theorem.

Similarly, one can obtain the required statement for Ψ . Recall that $\Psi_t(x, y)$ is given by the (uniformly convergent) series, and for each term both its differentiability in t and the bound (6.3) are proved for $(t, y) \notin \Upsilon$. Then by the dominated convergence theorem we get the same properties for the whole sum. To get the continuity with respect to x , we again use the dominated convergence theorem. \square

The estimates on the derivatives we just obtained allow us to verify easily assertions (4.21), (4.22), which play the crucial role in the proofs of Theorem 2.3 and Proposition 2.1.

Lemma 6.4.

1. For any $f \in C_\infty(\mathbb{R}^d)$,

$$\|\partial_t P_{t,\varepsilon} f - \partial_t P_t f\|_\infty \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

uniformly on compact subsets of $(0, \infty)$. Moreover, $\partial_t P_t f(x) = \int_{\mathbb{R}^d} \partial_t p_t(x, y) f(y) dy$.

2. Under the assumptions of Proposition 2.1,

$$\partial_t p_{t,\varepsilon}(x, y) \rightarrow \partial_t p_t(x, y) \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly on compact subsets of $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$;

Proof. The proofs of both statements rely on decomposition (6.1). We prove the first statement; the proof of the second statement is completely similar.

Using (6.1) we have

$$\begin{aligned}
 \partial_t \int_{\mathbb{R}^d} p_{t,\varepsilon}(x, y) f(y) dy &= \int_{\mathbb{R}^d} \partial_t p_{t,\varepsilon}(x, y) f(y) dy \\
 &= \int_{\mathbb{R}^d} \partial_t p_{t+\varepsilon}^0(x, y) f(y) dy \\
 &\quad + \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\partial_t p^0)_{t-s+\varepsilon}(x, z) \Psi_s(z, y) f(y) dz dy ds \\
 &\quad + \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{s+\varepsilon}^0(x, z) (\partial_t \Psi)_{t-s}(z, y) f(y) dz dy ds \\
 &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P_{t/2+\varepsilon}^0(x, z) \Psi_{t/2}(z, y) f(y) dz dy.
 \end{aligned} \tag{6.7}$$

Note that for every positive $t_0 < t_1$

$$p_{t+\varepsilon}^0(x, y) \rightarrow p_t^0(x, y), \quad \partial_t p_{t+\varepsilon}^0(x, y) \rightarrow \partial_t p_t^0(x, y), \quad \varepsilon \rightarrow 0,$$

uniformly in $t \in [t_0, t_1]$, $x, y \in \mathbb{R}^d$. Now the required convergence follows from (6.7) and the bounds for p^0 , Ψ , $\partial_t p^0$, $\partial_t \Psi$ obtained above. \square

6.3. Completion of the proof of Theorem 2.6

Now we can finalize the proof of Theorem 2.6. Again, we consider only the most cumbersome case **C** with b being Lipschitz continuous.

By representation (6.1), the first two statements of the theorem follow from the statements given above on time derivatives of $p_t^0(x, y)$ and $\Psi_t(x, y)$; the proofs are completely analogous to those of Lemma 6.3, and therefore are omitted. To prove statement 3, note that the set Υ constructed in Lemma 6.3 is such that for every fixed $t > 0$ the set $\{y : (t, y) \in \Upsilon\}$ has zero Lebesgue measure. Together with the bounds for $\partial_t p_t(x, y)$ from statement 2, this makes it possible to use the dominated convergence theorem and prove that for given $t > 0$ and $f \in C_\infty(\mathbb{R}^d)$

$$\frac{P_{t+\Delta t} f(x) - P_t f(x)}{\Delta t} \rightarrow \int_{\mathbb{R}^d} \partial_t p_t(x, y) f(y) dy, \quad \Delta t \rightarrow 0,$$

uniformly in $x \in \mathbb{R}^d$, which gives statement 3. \square

Remark 6.1. In the above proof of Theorem 2.6, which is based on (6.1) and the subsequent parametrix-type iteration of convolutions, we are strongly motivated by the idea used in the proof of Theorem 3.1 in [37]. According to this idea, we decompose the \otimes -convolution in two parts in such a way, that after such a decomposition the time derivative is applied to the “least singular” function under the integral, as it was done in (6.1)–(6.6). Unfortunately, we cannot proceed in the same way with the derivative ∂_x unless $p_t^0(x, y)$ depends on t and $x - y$ only. It seems that in this place in [37] there is a mistake hidden in the calculations, because in this part of the proof the respective (space) convolutions are treated as if they only depend on the difference of space arguments, but in fact their structure is more complicated. Therefore we do not use the above argument from [37] for the derivative ∂_x , and develop another way to justify the whole method.

7. Proof of Theorem 2.5

Since the function $V_t(x)$ (cf. (2.17)) is bounded, the upper bound in (2.22) follows just by the definition of $p_t^{(0)}(x, y)$ and (2.4).

Let us prove the lower bound. First we observe that if we manage to prove the lower bound for *some* $T > 0$, then we actually can do that for *all* $T > 0$. This follows directly from the super-convolution property of the kernel $H_t^{(0)}(x, y)$

at the right hand side of (2.22) and the Chapman–Kolmogorov identity (that is, the convolution identity) for $p_t(x, y)$ at the left hand side.

Note that for $|\omega(t, y) - x| \leq t^{1/\alpha}$ we have by (3.37)

$$|r_t(x, y)| \leq C(t^{\kappa/\alpha} + t^\delta)H_t^{(0)}(x, y). \quad (7.1)$$

Therefore, by (2.4) and (7.1) we get

$$p_t(x, y) \asymp t^{-d/\alpha}, \quad t \in (0, T], \quad |\omega(t, y) - x| \leq t^{1/\alpha}. \quad (7.2)$$

Further, by (2.16) and (2.17) there exists $\rho \in (0, 1)$ small, such that if $t^{1/\alpha} \leq |\omega(t, y) - x| \leq \rho$ then

$$|r_t(x, y)| \leq 2^{-1}p_t^{(0)}(x, y). \quad (7.3)$$

This implies

$$p_t(x, y) \geq 2^{-1}p_t^{(0)}(x, y), \quad t \in (0, T], \quad t^{1/\alpha} \leq |\omega(t, y) - x| \leq \rho. \quad (7.4)$$

Let us show that there exists $c > 0$ such that

$$p_t(x, y) \geq \frac{ct}{|\omega(t, y) - x|^{d+\alpha}}, \quad t \in (0, T], \quad |\omega(t, y) - x| > \rho. \quad (7.5)$$

Consider the set

$$D = \{(s, z) : |\omega(t - s, y) - z| < t^{1/\alpha}\} \subset [0, \infty) \times \mathbb{R}^d,$$

and denote

$$\tau = \inf\{s : (s, X_s) \in D\}.$$

If $\tau \leq t/2$, then we have $|\omega(t - \tau, y) - X_\tau| \leq t^{1/\alpha}$, $t - \tau > t/2$, hence by the strong Markov property and (7.2) we have

$$p_t(x, y) \geq \mathbb{E}_x[p_{t-\tau}(X_\tau, y)1_{\tau \leq t/2}] \geq ct^{-d/\alpha}\mathbb{P}_x(\tau \leq t/2).$$

To estimate $\mathbb{P}_x(\tau \leq t/2)$, we introduce another stopping time σ in the following way. Up to now, $T > 0$ was fixed but arbitrary. Now we take another $T_1 > 0$ small enough, so that

$$t^{1/\alpha} \leq \frac{\rho}{3}, \quad |\omega(t - s, y) - \omega(t, y)| < \frac{\rho}{3}, \quad 0 \leq s \leq t \leq T_1, \quad y \in \mathbb{R}^d.$$

Here in the second inequality we have used that

$$\partial_t \omega(t, y) = \begin{cases} 0, & \text{in case A,} \\ -b(y), & \text{in case B,} \\ -b(\theta_t(y)), & \text{in case C,} \end{cases} \quad (7.6)$$

and thus $\partial_t \omega(t, y)$ is bounded. Define

$$\sigma = \inf\left\{s : |X_s - x| \geq \frac{\rho}{3}\right\} \wedge \left(\frac{t}{2}\right),$$

then $\sigma \leq t/2$, and if $t \leq T_1$ for every $s < \sigma$ we have

$$|\omega(t, y) - x| \leq |\omega(t - s, y) - X_s| + |X_s - x| + |\omega(t, y) - \omega(t - s, y)| < |\omega(t - s, y) - X_s| + \frac{2\rho}{3}. \quad (7.7)$$

Since $|\omega(t, y) - x| > \rho$, we have $|\omega(t - s, y) - X_s| > \rho/3 > t^{1/\alpha}$, i.e. $(s, X_s) \notin D$. Hence

$$\{\tau \leq t/2\} \supset \{(\sigma, X_\sigma) \in D\}. \quad (7.8)$$

Take $f \in C_\infty^2(\mathbb{R}^d)$ such that $f(z) \in [0, 1]$,

$$f(z) = \begin{cases} 1, & |z| \leq 2^{-1}t^{-1/\alpha}, \\ 0, & |z| > t^{-1/\alpha}, \end{cases}$$

and for a fixed $t \leq T_1$, $y \in \mathbb{R}^d$ consider the function $F(s, z) = f(\omega(t, y) - z)$. Then by the Itô formula and Doob's optional sampling theorem applied to the bounded stopping time σ , we have

$$\mathbb{E}_x F(\sigma, X_\sigma) = F(0, x) + \mathbb{E}_x \int_0^\sigma (L_x F(s, X_s) + F'_s(s, X_s)) ds,$$

where

$$F'_s(s, X_s) = (\nabla f(\omega(t - s, y)), \partial_s \omega(t - s, y)),$$

see (7.6) for the formula for $\partial_t \omega(t, y)$. Now we recall that

- (i) $F \leq 1$, and $F(\sigma, X_\sigma) = 0$ if $(\sigma, X_\sigma) \notin D$;
- (ii) for every $s < \sigma$, $|X_\sigma - x| < \rho/3$, and therefore by the calculation (7.7) we have $|\omega(t - s, y) - X_s| > t^{1/\alpha}$, which yields that

$$F(s, X_s) = 0, \quad \nabla_x F(s, X_s) = 0, \quad F'_s(s, X_s) = 0.$$

Hence by (7.8) we have

$$\begin{aligned} \mathbb{P}_x(\tau \leq t/2) &\geq \mathbb{E}_x \int_0^\sigma a(X_s) L_x^{(\alpha)} F(s, X_s) ds \\ &= \mathbb{E}_x \left[\int_0^\sigma a(X_s) \int_{|\omega(t-s, y) - (X_s + u)| \leq t^{1/\alpha}} \frac{c_\alpha}{|u|^{d+\alpha}} du ds \right] \\ &\geq c \mathbb{E}_x \int_0^\sigma \frac{t^{d/\alpha}}{|\omega(t-s, y) - X_s|^{d+\alpha}} ds. \end{aligned}$$

Observe that by (7.7) $|\omega(t - s, y) - X_s| \geq 3^{-1}|\omega(t, y) - x|$, hence

$$\mathbb{P}_x(\tau \leq t/2) \geq \frac{Ct^{d/\alpha+1}}{|\omega(t, y) - x|^{d+\alpha}} \mathbb{P}_x\left(\sigma > \frac{t}{4}\right).$$

It is easy to verify that by choosing T_1 small enough we can ensure that $\mathbb{P}_x(\sigma > t/4) > 1/2$ for $t < T_1$, $x \in \mathbb{R}^d$. Summarizing all the calculations above we get the required bound (7.5).

Appendix A: Proof of Proposition 3.2

Estimate (3.6) for the α -stable transition probability density is well known, see, for example, [49–52, 56]; see also [57] for the asymptotic behaviour of an α -stable distribution density in the one-dimensional case.

Inequality (3.7) was proved in [5, Lemma 5]. The proof therein is based on the subordination argument, i.e. on the representation of $Z^{(\alpha)}$ as a Brownian motion with a time change performed by an independent one-sided $\alpha/2$ -stable process. The same approach can be applied to the proof of (3.8); since the proof follows literally the proof of (3.7) in [5, Lemma 5], we omit the details.

Let us show (3.9) and (3.10). Recall that $g_t^{(\alpha)}(y-x) = \frac{1}{t^{d/\alpha}} g^{(\alpha)}\left(\frac{y-x}{t^{1/\alpha}}\right)$ is the transition probability density of $Z^{(\alpha)}$, and therefore

$$L^{(\alpha)} g_t^{(\alpha)}(x) = \partial_t g_t^{(\alpha)}(x) = -\frac{d}{\alpha t^{d/\alpha+1}} g^{(\alpha)}\left(\frac{x}{t^{1/\alpha}}\right) - \frac{1}{\alpha t^{(d+1)/\alpha+1}} \left(x, \nabla g^{(\alpha)}\left(\frac{x}{t^{1/\alpha}}\right)\right). \quad (\text{A.1})$$

Now (3.9) follows from (3.6), (3.7) and (A.1) with $t = 1$. Differentiating (A.1) in x , taking $t = 1$, and applying (3.7) and (3.8), we get (3.10). \square

Appendix B: Proofs of Propositions 3.3 and 3.4

Proof of Proposition 3.3. We prove the sub-convolution property, only: the proof of the super-convolution property is completely analogous and is omitted.

Recall that $H_t^{(\lambda)}(x, y)$ is defined in (3.22), where $\omega(t, y)$ is given in (2.15) for each of the cases **A–C**. In what follows, we fix $\lambda \in [0, \alpha)$, and omit it in the notation, i.e. write $H_t(x, y)$ instead of $H_t^{(\lambda)}(x, y)$. We keep the same notation $H_t(x, y)$ for each of the cases **A–C**, but have in mind, that it is defined according to (2.15).

Define

$$K_t(x) := \left(\left(\left| \frac{x}{t^{1/\alpha}} \right|^\lambda \vee 1 \right) \wedge t^{-\lambda/\alpha} \right) \frac{1}{t^{d/\alpha}} G^{(\alpha)}\left(\frac{x}{t^{1/\alpha}}\right). \quad (\text{B.1})$$

Case A. Note that in case **A** the kernel $H_t(x, y)$ depends on the difference $y - x$ only, which immediately gives (3.33). Let us show the sub-convolution property.

Note that

$$K_t(x) \leq \frac{1}{t^{d/\alpha}} G^{(\alpha-\lambda)}\left(\frac{x}{t^{1/\alpha}}\right), \quad (\text{B.2})$$

and

$$K_t(x) = \frac{1}{t^{d/\alpha}} G^{(\alpha-\lambda)}\left(\frac{x}{t^{1/\alpha}}\right) \quad \text{if } |x| \leq 1. \quad (\text{B.3})$$

On the other hand, by (3.6) we have $\frac{1}{t^{d/\alpha}} G^{(\alpha-\lambda)}\left(\frac{x}{t^{1/\alpha}}\right) \asymp g_{t^{1-\lambda/\alpha}}^{(\alpha-\lambda)}(x)$. Since the function $g_t^{(\alpha-\lambda)}(y-x)$ is the transition probability density of an $(\alpha - \lambda)$ -stable process $Z^{(\alpha-\lambda)}$, it possesses the convolution property; see Remark 3.1. Therefore, if $|x| \leq 1$, we have

$$(K_{t-s} * K_s)(x) \leq C K_{(t-s)^{1-\lambda/\alpha} + s^{1-\lambda/\alpha}}(x)$$

with the constant $C > 0$ depending on α, λ , and d only. Observe that

$$t^{1-\lambda/\alpha} \leq (t-s)^{1-\lambda/\alpha} + s^{1-\lambda/\alpha} \leq 2t^{1-\lambda/\alpha}, \quad 0 \leq s \leq t.$$

Thus, it follows from the explicit representation for $G^{(\alpha-\lambda)}(x)$, (3.3) and (B.3), that

$$(K_{t-s} * K_s)(x) \leq C K_t(x), \quad |x| \leq 1.$$

Consider now the case $|x| > 1$. Split

$$(K_{t-s} * K_s)(x) \leq \left(\int_{|z| \geq |x|/2} + \int_{|x-z| \geq |x|/2} \right) K_{t-s}(z) K_s(x-z) dz.$$

Note that $K_t(x)$ is a monotone function of $|x|$. In addition, it depends on $|x|$ in a piece-wise power-type way, and therefore possesses the same property formulated in statement 1 of Proposition 3.1 for the function $G^{(\lambda)}$. Then

$$K_{t-s}(z) \leq K_{t-s}(x/2) \leq c K_{t-s}(x), \quad |z| \geq |x|/2.$$

For $|x| \geq 1$ we have $K_t(x) = t^{1-\lambda/\alpha} |x|^{-d-\alpha}$, and thus

$$K_{t-s}(x) = (t-s)^{1-\lambda/\alpha} |x|^{-d-\alpha} \leq (t-s)^{1-\lambda/\alpha} t^{-1+\lambda/\alpha} K_t(x) \leq K_t(x), \quad |x| \geq 1.$$

Then for $|x| \geq 1$

$$\begin{aligned} \int_{|z| \geq |x|/2} K_{t-s}(z) K_s(y-z) dz &\leq c K_t(x) \int_{|z| \geq |x|/2} K_s(y-z) dz \\ &\leq c K_t(x) \int_{\mathbb{R}^d} K_s(z') dz' \leq C K_t(x), \end{aligned}$$

where in the last inequality we used (B.2) and (3.36). Similarly,

$$\int_{|y-z| \geq |x|/2} K_{t-s}(z) K_s(y-z) dz \leq C K_t(x), \quad |x| \geq 1.$$

Summarizing the estimates proved above, we derive the required sub-convolution property for $H_t(x, y)$.

Case B. Denote for $q \in [0, 1]$

$$K_t^{(q)}(x, y) = K_t(y - qb(x)t - (1-q)b(y)t - x),$$

where $K_t(x)$ is defined in (B.1). Observe that now $H_t(x, y) = K_t^{(0)}(x, y)$.

Let us prove the following statement: For a given $T > 0$ there exist c, C such that for every $q \in [0, 1]$

$$c K_t^{(q)}(x, y) \leq K_t(y - tb(y) - x) \leq C K_t^{(q)}(x, y), \quad t \in (0, T]. \quad (\text{B.4})$$

We prove only the first inequality, the proof of the second one is completely analogous. Consider two cases: $|x - y| > 2Bt$ and $|x - y| \leq 2Bt$, where $B = \sup_x |b(x)|$. In the first case, we have

$$|y - x - b(y)t| \geq \frac{1}{2}|y - x|, \quad |y - x - qb(x)t - (1-q)b(y)t| \leq \frac{3}{2}|y - x|. \quad (\text{B.5})$$

Then by the analogue of (3.3) for $K_t(x)$ we get the first inequality in (B.4).

Consider the case $|x - y| \leq 2Bt$. Then we have

$$|b(x) - b(y)| \leq ct^\gamma. \quad (\text{B.6})$$

Since in case **B** we have $1 + \gamma > 1/\alpha$, by the triangle inequality we get

$$\begin{aligned} \left| \frac{y - z - tb(y)}{t^{1/\alpha}} \right|^\lambda \vee 1 &\leq C \left(\left| \frac{y - z - tb(y) - qt(b(x) - b(y))}{t^{1/\alpha}} \right|^\lambda \vee 1 \right) + C \\ &\leq C \left(\left| \frac{y - z - tb(y) - qt(b(x) - b(y))}{t^{1/\alpha}} \right|^\lambda \vee 1 \right), \quad t \leq T. \end{aligned}$$

Note that for any $C \geq 1, A \geq 0$

$$(CA) \vee t^{-1/\alpha} \leq C(A \vee t^{-1/\alpha}),$$

hence we can finalize the above estimate in the following way:

$$\frac{K_t^{(0)}(x, y)}{K_t^{(q)}(x, y)} \leq C \frac{g^{(\alpha-\lambda)}(v)}{g^{(\alpha-\lambda)}(u)},$$

where

$$u = \frac{y - x - b(y)t}{t^{1/\alpha}}, \quad v = \frac{y - x - qb(x)t - (1 - q)b(y)t}{t^{1/\alpha}}.$$

Note that the logarithmic derivative of $g^{(\alpha-\lambda)}(x)$ is bounded; see (3.6), (3.7). Then

$$\frac{g^{(\alpha-\lambda)}(v)}{g^{(\alpha-\lambda)}(u)} \leq e^{c|u-v|}, \quad u, v \in \mathbb{R}^d, \quad (\text{B.7})$$

which implies

$$\frac{K_t^{(0)}(x, y)}{K_t^{(q)}(x, y)} \leq C \exp[cq |b(x) - b(y)| t^{-1/\alpha+1}]. \quad (\text{B.8})$$

Since for $|x - y| \leq 2Bt$ we have (B.6), we can estimate the right-hand side of (B.8) by $C \exp[ct^{-1/\alpha+1+\gamma}]$, which is bounded for $t \in [0, T]$ since $\alpha > (1 + \gamma)^{-1}$. This completes the proof of (B.4).

Now we can finalize the proof in case **B**. By (B.4) with $q = 1$,

$$\begin{aligned} \int_{\mathbb{R}^d} H_{t-s}(x, z) H_s(z, y) dz &= \int_{\mathbb{R}^d} K_{t-s}^{(0)}(x, z) K_s^{(0)}(z, y) dz \\ &\leq C \int_{\mathbb{R}^d} K_{t-s}^{(1)}(x, z) K_s^{(0)}(z, y) dz = C \int_{\mathbb{R}^d} K_{t-s}(z - x') K_s(y' - z) dz, \end{aligned} \quad (\text{B.9})$$

where

$$x' = x + (t - s)b(x), \quad y' = y - sb(y).$$

The sub-convolution property of the kernel $K_t(y - x)$ was actually shown in the proof of case **A**, hence

$$\int_{\mathbb{R}^d} H_{t-s}(x, z) H_s(z, y) dz \leq C K_t(y' - x') = C K_t^{(1-s/t)}(x, y). \quad (\text{B.10})$$

Applying (B.4) with $q = 1 - s/t$, we complete the proof of the required sub-convolution property for $H_t(x, y)$ in case **B**. Finally, applying (B.4) with $q = 1$ we get estimates (3.33).

Case C. The scheme of the proof in this case is similar to that one in the case **B**. Denote for $q \in [0, 1]$

$$\tilde{K}_t^{(q)}(x, y) = K_t(\chi_{qt}(\theta_t(y)) - \chi_{qt}(x)).$$

Observe that $\tilde{K}_t^{(0)}(x, y) \equiv K_t(\theta_t(y) - x)$ is equal to the kernel $H_t(x, y)$ in the case **C**. As in the case **B**, let us show that

$$c \tilde{K}^{(q)}(x, y) \leq K_t(\theta_t(y) - x) \leq C \tilde{K}^{(q)}(x, y), \quad t \in (0, T]. \quad (\text{B.11})$$

Suppose first that $b \in C_b^1(\mathbb{R}^d)$. In this case, every $\chi_t(x)$ is differentiable in x , and the derivative $D_t(x) := \nabla_x \chi_t(x)$ satisfies the following linear ODE (cf. [11, Chapter I, (7.12)–(7.14)])

$$\frac{d}{dt} D_t(x) = B(t, x) D_t(x), \quad B(t, x) := (\nabla b)(\chi_t(x)).$$

In addition, $D_0(x)$ is the identity matrix. Similar relations hold true for the inverse flow, since θ_t is the solution to (2.13), which differs from (2.12) by the sign “−.” Then $\nabla_x \theta_t(x) = D_t^{-1}(x)$, where $D_t^{-1}(x)$ is the inverse matrix of $D_t(x)$, and

$$\frac{d}{dt} D_t^{-1}(x) = \tilde{B}(t, x) D_t^{-1}(x), \quad \tilde{B}(t, x) := -(\nabla b)(\theta_t(x)).$$

Hence, we have the following bounds for the matrix norms of $D_t(x)$ and $D_t^{-1}(x)$:

$$\|D_t(x)\| \leq C_{b,T}, \quad \|(D_t(x))^{-1}\| \leq C_{b,T}, \quad t \in (0, T]. \quad (\text{B.12})$$

Note that the constant $C_{b,T}$ depends only on T and on the supremum of the matrix norm of ∇b . Using these inequalities, we derive

$$C_{b,T}^{-1} |\theta_t(y) - x| \leq |\chi_{qt}(\theta_t(y)) - \chi_{qt}(x)| \leq C_{b,T} |\theta_t(y) - x|, \quad q \in [0, 1], t \in (0, T].$$

Then since $\theta_t(y) - x = (\chi_{qt}(\theta_t(y)) - \chi_{qt}(x))|_{q=0}$, we derive (B.11) by the property of K_t (cf. the explanation in the case **A**). Note that

$$\theta_s(y) - \chi_{t-s}(x) = \chi_{t-s}(\theta_t(y)) - \chi_{t-s}(x) = \chi_{(1-s/t)t}(\theta_t(y)) - \chi_{(1-s/t)t}(x).$$

Then we derive (B.9) and (B.10) with $\tilde{K}_{t-s}^{(q)}(\cdot, \cdot)$ instead of $K_t^{(q)}(\cdot, \cdot)$, with $q = 0, 1$ and $1 - s/t$, respectively, and

$$x' = \chi_{t-s}(x), \quad y' = \theta_s(y).$$

Then applying finally (B.11) with $q = 1 - s/t$ we derive the sub-convolution property of $H_t(x, y)$. Applying (B.11) with $q = 1$ we get (3.33), which finalized the proof of the proposition in case **C** if $b \in C^1(\mathbb{R}^d)$.

To handle the Lipschitz case one can approximate b uniformly by a sequence of functions $b_n \in C_b^1(\mathbb{R}^n)$ in such a way that the matrix norms of ∇b_n remain uniformly bounded. \square

Proof of Proposition 3.4. (a) Without loss of generality assume that $f \in C_\infty(\mathbb{R}^d)$ is non-negative. Then in case **A** we have

$$\int_{\mathbb{R}^d} g_t^{(\alpha)}(y-x) f(y) dy = \int_{\mathbb{R}^d} g_t^{(\alpha)}(z) f(z+x) dz \rightarrow 0, \quad |x| \rightarrow \infty.$$

In case **B** we have by (B.4)

$$\begin{aligned} \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - tb(y) - x) f(y) dy &\leq C \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - tb(x) - x) f(y) dy \\ &= C \int_{\mathbb{R}^d} g_t^{(\alpha)}(z) f(z + x + tb(x)) dz \rightarrow 0, \quad |x| \rightarrow \infty, \end{aligned} \quad (\text{B.13})$$

since $b(\cdot)$ is bounded, and $f \in C_\infty(\mathbb{R}^d)$.

Analogously, in case **C** we have

$$\begin{aligned} \int_{\mathbb{R}^d} g_t^{(\alpha)}(\theta_t(y) - x) f(y) dy &\leq C \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - \chi_t(x)) f(y) dy \\ &= C \int_{\mathbb{R}^d} g_t^{(\alpha)}(z) f(z + \chi_t(x)) dz \rightarrow 0, \quad |x| \rightarrow \infty, \end{aligned}$$

because $|\chi_t(x)| = |x + \int_0^t b(\chi_s(x)) ds| \rightarrow \infty, |x| \rightarrow \infty$, since the function $b(\cdot)$ is bounded.

(b) In case **A** the statement follows from the fact that $g_t^{(\alpha)}(y-x)$ is the fundamental solution to the Cauchy problem for $\partial_t - L^{(\alpha)}$, in particular,

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(y-x) f(y) dy - f(x) \right| \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (\text{B.14})$$

In case **B** we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - tb(y) - x) f(y) dy - f(x) \right| \\
& \leq C \int_{\mathbb{R}^d} \left| g_t^{(\alpha)}(y - tb(y) - x) - g_t^{(\alpha)}(y - tb(x) - x) \right| dy \\
& \quad + \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - x - tb(x)) f(y) dy - f(x) \right| \\
& =: J_1(t, x) + J_2(t, x).
\end{aligned} \tag{B.15}$$

Note that by (3.7)

$$\begin{aligned}
& \left| g_t^{(\alpha)}(y - tb(y) - x) - g_t^{(\alpha)}(y - tb(x) - x) \right| \\
& \leq t \int_0^1 \left| \frac{(b(y) - b(x))}{t^{1/\alpha}} \frac{1}{t^{d/\alpha}} (\nabla g^{(\alpha)}) \left(\frac{y - x - tb(x) - st(b(y) - b(x))}{t^{1/\alpha}} \right) \right| ds \\
& \leq Ct \int_0^1 \frac{|y - x|^\gamma}{t^{1/\alpha}} \frac{1}{t^{d/\alpha}} G^{(\alpha+1)} \left(\frac{y - x - tb(x) - st(b(y) - b(x))}{t^{1/\alpha}} \right) ds \\
& \leq Ct \frac{|y - x|^\gamma}{t^{1/\alpha}} \frac{1}{t^{d/\alpha}} G^{(\alpha+1)} \left(\frac{y - x - tb(x)}{t^{1/\alpha}} \right),
\end{aligned}$$

where in the last line we used that the estimate (B.4) also holds true for $\frac{1}{t^{d/\alpha}} G^{(\alpha+1)}(\frac{\cdot}{t^{1/\alpha}})$ instead of $K_t(\cdot)$. Using the triangle inequality, (3.4) and (3.6), we derive

$$\begin{aligned}
& \left| g_t^{(\alpha)}(y - tb(y) - x) - g_t^{(\alpha)}(y - tb(x) - x) \right| \\
& \leq C(t^{1+\gamma-1/\alpha} + t^{1-1/\alpha+\gamma/\alpha}) g_t^{(\alpha)}(y - x - tb(x)).
\end{aligned}$$

Since in case **B** we assumed that $\alpha > (1 + \gamma)^{-1}$, we have $\alpha > 1 - \gamma$. Thus,

$$\sup_x J_1(t, x) \leq C(t^{1+\gamma-1/\alpha} + t^{1-1/\alpha+\gamma/\alpha}) \rightarrow 0, \quad t \rightarrow 0.$$

For $J_2(t, x)$ we have under the additional assumption that $f \in C_\infty^1(\mathbb{R}^d)$

$$J_2(t, x) = \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(z) (f(z + x + tb(x)) - f(x)) dy \right| \leq Ct \rightarrow 0, \quad t \rightarrow 0,$$

uniformly in x . The general case $f \in C_\infty(\mathbb{R}^d)$ follows by the approximation argument. This completes the proof in case **B**.

In case **C** the argument is similar. We split

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(\theta_t(y) - x) f(y) dy - f(x) \right| \\
& \leq C \int_{\mathbb{R}^d} \left| g_t^{(\alpha)}(\theta_t(y) - x) - g_t^{(\alpha)}(y - \chi_t(x)) \right| dy \\
& \quad + \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(y - \chi_t(x)) f(y) dy - f(x) \right| \\
& =: J_1(t, x) + J_2(t, x).
\end{aligned} \tag{B.16}$$

Using (3.7), (3.6) and (B.11), we get

$$\begin{aligned} & |g_t^{(\alpha)}(\theta_t(y) - x) - g_t^{(\alpha)}(y - \chi_t(x))| \\ & \leq t \int_0^1 \left| \frac{b(\chi_{qt}(\theta_t(y))) - b(\chi_{qt}(x))}{t^{1/\alpha}} \frac{1}{t^{d/\alpha}} (\nabla g^{(\alpha)}) \left(\frac{\chi_{qt}(\theta_t(y)) - \chi_{qt}(x)}{t^{1/\alpha}} \right) \right| dq \\ & \leq Ct g_t^{(\alpha)}(y - \chi_t(x)), \end{aligned}$$

which implies that $\sup_x J_1(t, x) \rightarrow 0$ as $t \rightarrow 0$.

For $J_2(t, x)$ we have by the same argument as in case B

$$J_2(t, x) = \left| \int_{\mathbb{R}^d} g_t^{(\alpha)}(z) (f(z + \chi_t(x)) - f(x)) dy \right| \rightarrow 0, \quad t \rightarrow 0,$$

uniformly in x . This finishes the proof in case C. □

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