

Simple CLE in doubly connected domains

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Abstract. Loop Ensemble (CLE_κ) in doubly connected domains: annuli, the punctured disc, and the punctured plane. We restrict attention to CLE_κ for which the loops are simple, i.e. $\kappa \in (8/3, 4]$. In *Ann. of Math. (2)* **176** (2012) 1827–1917, simple CLE in the unit disc is introduced and constructed as the collection of outer boundaries of outermost clusters of the Brownian loop soup. For simple CLE in the unit disc, any fixed interior point is almost surely surrounded by some loop of CLE. The gasket of the collection of loops in CLE, i.e. the set of points that are not surrounded by any loop, almost surely has Lebesgue measure zero. In the current paper, simple CLE in an annulus is constructed similarly: it is the collection of outer boundaries of outermost clusters of the Brownian loop soup conditioned on the event that there is no cluster disconnecting the two components of the boundary of the annulus. Simple CLE in the punctured disc can be viewed as simple CLE in the unit disc conditioned on the event that the origin is in the gasket. Simple CLE in the punctured plane can be viewed as simple CLE in the whole plane conditioned on the event that both the origin and infinity are in the gasket. We construct and study these three kinds of CLE's, along with the corresponding exploration processes.

Résumé. Nous étudions l'ensemble des boucles conformes (CLE_κ) dans des domaines connexes du type : anneau, disque percé et plan percé. Nous considérons les cas CLE_κ pour lesquels les boucles sont simples, i.e. $\kappa \in (8/3, 4]$. Dans *Ann. of Math. (2)* **176** (2012) 1827–1917, l'ensemble CLE dans le disque unité est introduit et construit comme la collection de frontières extérieures des amas les plus excentrés de la soupe de boucles Browniennes. Dans le cas du disque unité, n'importe quel point intérieur est presque sûrement entouré par une boucle du CLE. L'ensemble des points qui ne sont entourés par aucune boucle a une mesure de Lebesgue nulle presque sûrement. Dans notre article, le CLE dans un anneau est construit de façon similaire : il s'agit de la collection de frontières extérieures des amas de la soupe de boucles Browniennes conditionnés sur l'évènement qu'il n'existe pas d'amas séparant les deux composantes de la frontière de l'anneau. Dans le cas du disque percé, le CLE correspond au conditionnement par le fait que l'origine n'est pas entourée par une boucle. Dans le cas du plan percé, le conditionnement est tel que l'origine et l'infini ne sont pas entourés. Nous construisons et étudions ces trois types de CLE et les processus d'exploration correspondants.

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1. Introduction

Schramm Loewner Evolution (SLE) curves were introduced by Oded Schramm [8] as candidates for the scaling limit of various interfaces in discrete statistical physics models. For each $\kappa \geq 0$, SLE_κ is a random curve in a simply connected domain (which is non-empty and is not the whole plane) connecting one boundary point to another boundary point that satisfies certain conformal symmetry and so-called domain Markov property [8]. Since their introduction, SLE_κ have been proved to be the scaling limits of many discrete models. For example, SLE_3 has been proved to be the scaling limit of the interface in critical Ising model [1,2]; SLE_4 has been proved to be the scaling limit of a level line of the discrete Gaussian Free Field [9,10].

When one studies the scaling limit of the collection of all interfaces in a discrete statistical physics models (as opposed to a single interface), one is led to the notion of Conformal Loop Ensemble (CLE). For each $\kappa \in (8/3, 8]$, one can define CLE_κ in the unit disc which is a random countable collection of loops that are contained in the unit disc. Only for $\kappa \in (8/3, 4]$, the loops are simple and disjoint. We occasionally use the term “simple CLE” to refer to a non-nested disjoint Conformal Loop Ensemble CLE_κ for $\kappa \in (8/3, 4]$, and we will focus exclusively on these CLE’s for $\kappa \in (8/3, 4]$. In [11,13], simple CLE in the unit disc is defined and studied. The Brownian loop soup is a random collection of the Brownian loops which are Brownian paths start and end at the same point (see Section 2.2). In [13], simple CLE in the unit disc is constructed from Brownian loop soup and the authors prove that CLE_κ for $\kappa \in (8/3, 4]$ is the only one-parameter family of collections of loops that satisfies conformal invariance and the domain Markov property (as we will define in Section 2.3.1), and each loop of which looks locally like an SLE_κ . Now CLE_3 is conjectured to be the scaling limit of the collection of interfaces in the critical Ising model; CLE_4 has been proved to be the collection of level lines of Gaussian Free Field. (The details have not all written, but a reasonably detailed proof appears in Jason Miller’s lecture slides [6].) Later in [3], the nested CLE in Riemann sphere is defined and studied. Most of the effort in [3] is devoted to showing that the nested CLE in whole plane is invariant under inversion $z \mapsto 1/z$.

Given a collection of CLE loops in the unit disc, it is natural to ask what is the “distance” between a loop, say the loop containing the origin denoted by $\gamma(0)$, and the boundary $\partial\mathbb{U}$, or what is the “distance” between two loops. Since CLE is conformal invariant, such a distance should also be conformal invariant, and should depend on the collection of the loops between $\gamma(0)$ and the boundary. It turns out that the collection of loops between $\gamma(0)$ and $\partial\mathbb{U}$ is a collection of loops in the annulus. Therefore, to find such a distance between loops, we need to understand the properties of CLE in the annulus. This is the motivation for this paper.

We construct CLE in the annulus as the collection of the outer boundaries of outermost clusters of Brownian loop soup in the annulus conditioned on the event that there is no cluster disconnecting the two components of the boundary of the annulus. Our main results about CLE in the annulus can be summarized as follows:

- CLE in the annulus satisfies an annulus version of conformal invariance and the domain Markov property (detailed description in Section 3).
- CLE in the annulus and CLE in the unit disc are related in the following way: for a CLE in the unit disc, fix the loop containing a particular interior point. Then, given this loop, the conditional law of the collection of loops between this particular loop and the boundary of the domain has the same law as CLE in the annulus.

Consider CLE in the annulus with inradius $r \in (0, 1)$ and outradius 1. We show that, as r goes to zero, CLE in the annulus converges, and the limit object can be viewed as CLE in the unit disc “conditioned” on the event that the origin is in the gasket. This is a variant of CLE in which the origin plays a special role. We call it CLE in the punctured disc. This version of CLE has the nice properties as we would expect:

- CLE in the punctured disc satisfies conformal invariance and the domain Markov property (see Section 4).
- The law of the set of loops that are “near to the boundary of the disc” is approximately the same for CLE in the unit disc and CLE in the punctured disc (in a sense we will make precise in Proposition 4.5).

In the construction of CLE in the punctured disc, we let the inradius of the annulus go to zero. We can also let the outradius go to infinity at the same time as the inradius goes to zero: Consider CLE in the annulus with inradius r and outradius $1/r$. When r goes to zero, CLE in the annulus also converges, and we call the limit object CLE in the punctured plane. For CLE in the punctured plane, there is no loop separating the origin from infinity, and we define the gasket to be the set of points that are not separated by any loop from infinity (or equivalently, not separated by any loop from the origin). For CLE in punctured plane, the invariance under inversion $z \mapsto 1/z$ is true by construction (which is not trivially true for nested CLE in whole plane [3]).

We use the name “CLE in doubly connected regions” to indicate the above three versions of CLE: CLE in the annulus, CLE in the punctured disc, and CLE in the punctured plane.

In [13], the authors describe an exploration procedure to discover the loops in CLE progressively. The conformal invariance and domain Markov property of CLE make this exploration procedure easy to control. In our paper, we use the same procedure to explore the loops in CLE in the punctured disc. We will give a precise quantitative relation between the continuous exploration process of CLE in the punctured disc and the continuous exploration process of CLE in the unit disc. The authors are in the process of carrying out a program to define a conformal invariant distance

on CLE_4 and other CLE loop configurations which includes [12,15–17]. The continuous exploration process is an important ingredient in describing the “distance” between loops, and this quantitative relation between exploration processes would shed lights on the asymptotic of the “distance.”

Outline. In Section 2, we give preliminaries about CLE in the unit disc and other tools. We construct and study CLE in the annulus in Section 3, CLE in the punctured disc in Section 4, and CLE in the punctured plane in Section 5.

2. Preliminaries

In this paper, we denote the disc, circle and annulus as follows: for $0 < r < R$, $x \in \mathbb{C}$,

$$\begin{aligned} B(x, r) &= \{z \in \mathbb{C} : |z - x| < r\}, & \mathbb{D} &= B(0, 1), \\ C(x, r) &= \{z \in \mathbb{C} : |z - x| = r\}, & C_r &= C(0, r), \\ \mathbb{A}(r, R) &= \{z \in \mathbb{C} : r < |z| < R\}, & \mathbb{A}_r &= \mathbb{A}(r, 1). \end{aligned}$$

Denote the punctured disc and punctured plane in the following way

$$\mathbb{D}^\dagger = \mathbb{D} \setminus \{0\}, \quad \mathbb{C}^\dagger = \mathbb{C} \setminus \{0\}.$$

Throughout the paper, we fix the following constants:

$$\kappa \in \left(\frac{8}{3}, 4\right], \quad \beta = \frac{8}{\kappa} - 1, \quad \alpha = \frac{(8 - \kappa)(3\kappa - 8)}{32\kappa}, \quad c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}. \quad (2.1)$$

For general positive functions f and g , we write $f \lesssim g$ if f/g is bounded from above by some universal constant; $f \gtrsim g$ if $g \lesssim f$; and $f \asymp g$ if $f \lesssim g$ and $f \gtrsim g$.

2.1. Conformal radius and conformal modulus

In this section, we are interested in two kinds of domains: non-trivial simply connected domains and annuli.

A non-trivial simply connected domain D is a non-empty open subset of \mathbb{C} , which is not all of \mathbb{C} , such that both D and its complement in the Riemann sphere are connected. From the Riemann mapping theorem, we know that, for any non-trivial simply connected domain D and an interior point $z \in D$, there exists a unique conformal map Φ from D onto the unit disc \mathbb{D} such that $\Phi(z) = 0$ and $\Phi'(z) > 0$. We define the *conformal radius* of D seen from z as

$$\text{CR}(D; z) = 1/\Phi'(z).$$

We write $\text{CR}(D) = \text{CR}(D; z)$ if $z = 0$.

Consider a closed subset K of \mathbb{D} such that $\mathbb{D} \setminus K$ is simply connected and $0 \in \mathbb{D} \setminus K$. There exists a unique conformal map Φ_K from $\mathbb{D} \setminus K$ onto \mathbb{D} normalized at the origin: $\Phi_K(0) = 0$, and $\Phi'_K(0) > 0$. In fact $\Phi'_K(0) \geq 1$, and

$$\text{CR}(\mathbb{D} \setminus K) = 1/\Phi'_K(0) \leq 1.$$

The Schwarz lemma and the Koebe one quarter theorem imply that

$$d \leq \text{CR}(\mathbb{D} \setminus K) \leq 4d, \quad (2.2)$$

where $d = \text{dist}(0, K)$ is the Euclidean distance from the origin to K .

Define the *capacity* of K in \mathbb{D} seen from the origin as

$$\text{cap}(K) = -\log \text{CR}(\mathbb{D} \setminus K) \geq 0.$$

By convention, if $0 \in K$, we set $\text{CR}(\mathbb{D} \setminus K) = 0$ and $\text{cap}(K) = \infty$. When K is small, i.e. the diameter $R(K)$ of K is less than $1/2$, we have that¹

$$\text{cap}(K) \lesssim R(K)^2.$$

An annular domain A is a connected open subset of \mathbb{C} such that its complement in the Riemann sphere has two connected components and both of them contain more than one point. Then there exists a unique constant $r \in (0, 1)$ such that A can be conformally mapped onto the standard annulus \mathbb{A}_r . We define the *conformal modulus* of A , denoted as $\text{CM}(A)$, to be this unique constant r . Note that two annuli with different conformal radii cannot be conformally mapped onto each other.

The following lemma describes the relation between the conformal radius of a non-trivial simply connected domain and the conformal modulus of an annulus.

Lemma 2.1. *Suppose K is a closed subset of \mathbb{D} such that $\mathbb{D} \setminus K$ is simply connected and $0 \in \mathbb{D} \setminus K$. Clearly $\mathbb{A}_r \setminus K$ is an annulus for r small enough. We have that*

$$\frac{\text{CM}(\mathbb{A}_r \setminus K)}{\text{CM}(\mathbb{A}_r)} \rightarrow \text{CR}(\mathbb{D} \setminus K)^{-1}, \quad \text{as } r \rightarrow 0.$$

Proof. Suppose Φ is the conformal map from $\mathbb{D} \setminus K$ onto \mathbb{D} normalized at the origin: $\Phi(0) = 0$, and $\Phi'(0) > 0$. Note that $\Phi(\mathbb{A}_r \setminus K)$ equals $\mathbb{D} \setminus \Phi(r\overline{\mathbb{D}})$, the inner hole of which is asymptotically a disk of radius $\Phi'(0)r$ as $r \rightarrow 0$. Thus we have that

$$\lim_{r \rightarrow 0} \frac{\text{CM}(\mathbb{A}_r \setminus K)}{\text{CM}(\mathbb{A}_r)} = \lim_{r \rightarrow 0} \frac{1}{r} \text{CM}(\Phi(\mathbb{A}_r \setminus K)) = \Phi'(0).$$

This implies the conclusion. □

2.2. Brownian loop soup

We now briefly recall some results from [5]. It is well known that Brownian motion in \mathbb{C} is conformal invariant. Let us now define, for all $t \geq 0$, the law $\mu_t(z, z)$ of the two-dimensional Brownian bridge of time length t that starts and ends at z and define

$$\mu^{\text{loop}} = \int_{\mathbb{C}} \int_0^\infty d^2z \frac{dt}{t} \mu_t(z, z),$$

where d^2z is the Lebesgue measure in \mathbb{C} . We stress that μ^{loop} is a measure on *unrooted* loops modulo time-reparameterization (see [5]). And μ^{loop} inherits a striking conformal invariance property. Namely, if for any subset $D \subset \mathbb{C}$, one defines the Brownian loop measure μ_D^{loop} in D as the restriction of μ^{loop} to the set of loops contained in D , then it is shown in [5] that:

- For two domains $D' \subset D$, $\mu_{D'}^{\text{loop}}$ restricted to the loops contained in D' is the same as $\mu_{D'}^{\text{loop}}$ (this is a trivial consequence of the definition of these measures).
- For two connected domains D_1, D_2 , suppose Φ is a conformal map from D_1 onto D_2 , then the image of $\mu_{D_1}^{\text{loop}}$ under Φ has the same law as $\mu_{D_2}^{\text{loop}}$ (this non-trivial fact is inherited from the conformal invariance of planar Brownian motion).

Suppose D is a domain and V_1, V_2 are two subsets of D . We denote by

$$\Lambda(V_1, V_2; D)$$

the measure of the set of Brownian loops in a domain D that intersect both V_1 and V_2 .

¹We may assume K is contained in $B(1, R(K))$. Then $\text{cap}(K) \leq \text{cap}(B(1, R(K)) \cap \mathbb{D}) \asymp R(K)^2$.

Proposition 2.2 ([4], Lemma 3.1, Equation (22)). *Suppose $0 < r < 1, R \geq 2$. Then we have that*

$$\Lambda(C_1, C_R; \mathbb{C} \setminus \mathbb{D}_r) = 2 \int_r^1 s^{-1} \rho(R/s) ds,$$

where the function ρ satisfies the following estimate: there exists a universal constant $C < \infty$ such that, for $u \geq 2$

$$\left| \rho(u) - \frac{1}{2 \log u} \right| \leq \frac{C}{u \log u}.$$

For a fixed domain $D \subset \mathbb{C}$ and a constant $c > 0$, a *Brownian loop-soup with intensity c in D* is a Poisson point process with intensity $c \mu_D^{\text{loop}}$. From the properties of Brownian loop measure, we have the following: Fix a domain D and a constant $c > 0$, and suppose D' is a subset of D . Suppose \mathcal{L} is a Brownian loop-soup in D , let \mathcal{L}_1 be the collection of loops in \mathcal{L} that are totally contained in D' and let $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Then \mathcal{L}_1 has the same law as the Brownian loop soup in D' , and \mathcal{L}_1 and \mathcal{L}_2 are independent.

2.3. CLE in the unit disc

2.3.1. Definition and properties

A simple loop in the plane is the image of the unit circle under a continuous injective map. The Jordan Theorem says that a simple loop L separates the plane into two connected components that we call its interior $\text{int}(L)$ (the bounded one) and its exterior (the unbounded one). We will use the σ -field Σ generated by all the events of the type $\{O \subset \text{int}(L)\}$ where O spans the set of open sets in the plane. Consider (at most countable) collections $\Gamma = (L_j, j \in J)$ of non-nested disjoint simple loops that are locally finite, i.e., for each $\varepsilon > 0$, only finitely many loops L_j have a diameter greater than ε . The space of collections of locally finite, non-nested, disjoint simple loops is equipped with the σ -field generated by the sets $\{\Gamma : \#\Gamma \cap A = k\}$ where $A \in \Sigma$ and $k \geq 0$. Therefore, to characterize the law of Γ , we only need to characterize the laws of macroscopic loops in Γ . In other words, if we characterize the law of all loops in Γ with diameter greater than ε for each $\varepsilon > 0$, then the law of Γ is determined.

Let us now briefly recall some features of CLE for $\kappa \in (8/3, 4)$ – we refer to [13] for details (and the proofs) of these statements. A CLE in \mathbb{D} is a collection Γ of non-nested disjoint simple loops $(\gamma_j, j \in J)$ in \mathbb{D} that possesses a particular conformal restriction property. In fact, this property, which we will now recall, characterizes these CLE's:

- (Conformal Invariance) For any Möbius transformation Φ of \mathbb{D} onto itself, the laws of Γ and $\Phi(\Gamma)$ are the same. This makes it possible to define, for any non-trivial simply connected domain D (that can therefore be viewed as the conformal image of \mathbb{D} via some map $\tilde{\Phi}$), the law of CLE in D as the distribution of $\tilde{\Phi}(\Gamma)$ (because this distribution does not depend on the actual choice of conformal map $\tilde{\Phi}$ from \mathbb{D} onto D).
- (Domain Markov Property) For any non-trivial simply connected domain $D \subset \mathbb{D}$, define the set $D^* = D^*(D, \Gamma)$ obtained by removing from D all the loops (and their interiors) of Γ that do not entirely lie in D . Then, conditionally on D^* , and for each connected component U of D^* , the law of those loops of Γ that do stay in U is exactly that of a CLE in U .

As we mentioned in Section 1, the loops in a given CLE are SLE_κ type loops for some value of $\kappa \in (8/3, 4]$ (and they look locally like SLE_κ curves). In fact for each such value of κ , there exists exactly one CLE distribution that has SLE_κ type loops.

As explained in [13], a construction of these particular families of loops can be given in terms of outer boundaries of outermost clusters of the Brownian loops in a Brownian loop soup with intensity $c \in (0, 1]$ which is a function in κ given by:

$$c = c(\kappa) = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}.$$

Throughout the paper, we will denote the law of CLE in a non-trivial simply connected domain D by μ_D^\sharp .

2.3.2. Exploration of CLE in the unit disc

In [13], the authors introduce a discrete exploration process of CLE loop configuration. The conformal invariance and the domain Markov property make the discrete exploration much easier to control. Consider a CLE in the unit disc, draw a small disc $B(x, \varepsilon)$ and let γ^ε be the loop that intersects $B(x, \varepsilon)$ with largest radius. Define the quantity

$$u(\varepsilon) = \mathbb{P}[\gamma^\varepsilon \text{ contains the origin}]. \tag{2.3}$$

In fact, $u(\varepsilon) = \varepsilon^{\beta+o(1)}$ as ε goes to zero where $\beta = 8/\kappa - 1$.

Proposition 2.3 ([13, Section 4]). *The law of γ^ε normalized by $1/u(\varepsilon)$ converges towards a bubble measure, denoted as $\nu_{\mathbb{D};x}^{\text{bub}}$ which we call SLE bubble measure in \mathbb{D} rooted at x . This $\nu_{\mathbb{D};x}^{\text{bub}}$ is an infinite σ -finite measure, and we have:*

- (1) $\nu_{\mathbb{D};x}^{\text{bub}}[\gamma \text{ contains the origin}] = 1$;
- (2) For r small enough, $\nu_{\mathbb{D};x}^{\text{bub}}[R(\gamma) \geq r] \asymp r^{-\beta}$ where $R(\gamma)$ is the smallest radius r such that γ is contained in $B(x, r)$.

Because of the conformal invariance and the domain Markov property, we can repeat the “small semi-disc exploration” until we discover the loop containing the origin: Suppose we have a CLE loop configuration in the unit disc \mathbb{D} . We draw a small semi-disc of radius ε whose center is uniformly chosen on the unit circle. The loops that intersect this small semi-disc are the loops we discovered. If we do not discover the loop containing the origin, we refer to the connected component of the remaining domain that contains the origin as the *to-be-explored domain*. Let f_1^ε be the conformal map from the to-be-explored domain onto the unit disc normalized at the origin. We also define γ_1^ε as the loop we discovered with largest radius. Because of the conformal invariance and the domain Markov property of CLE, the image of the loops in the to-be-explored domain under the conformal map f_1^ε has the same law as simple CLE in the unit disc. Thus we can repeat the same procedure for the image of the loops under f_1^ε . We draw a small semi-disc of radius ε whose center is uniformly chosen on the unit circle. The loops that intersect the small semi-disc are the loops we discovered at the second step. If we do not discover the loop containing the origin, define the conformal map f_2^ε from the to-be-explored domain onto the unit disc normalized at the origin. The image of the loops in the to-be-explored domain under f_2^ε has the same law as CLE in the unit disc, etc. At some finite step N , we discover the loop containing the origin, we define γ_N^ε as the loop containing the origin discovered at this step and stop the exploration. We summarize the properties and notations in this discrete exploration below.

- Before N , all steps of discrete exploration are i.i.d.
- The number of the step N , when we discover the loop containing the origin, has the geometric distribution:

$$\mathbb{P}[N > n] = \mathbb{P}[\gamma^\varepsilon \text{ does not contain the origin}]^n = (1 - u(\varepsilon))^n.$$

- Define the conformal map

$$\Phi^\varepsilon = f_{N-1}^\varepsilon \circ \dots \circ f_2^\varepsilon \circ f_1^\varepsilon.$$

As ε goes to zero, the discrete exploration will converge to a Poisson point process of bubbles with intensity measure given by

$$\nu_{\mathbb{D}}^{\text{bub}} = \int_{\partial\mathbb{D}} dx \nu_{\mathbb{D};x}^{\text{bub}},$$

where dx is Lebesgue length measure on $\partial\mathbb{D}$. See [13] for details.

Now we can reconstruct CLE loops from the Poisson point process of SLE bubbles. Let $(\gamma_t, t \geq 0)$ be a Poisson point process with intensity $\nu_{\mathbb{D}}^{\text{bub}}$. Namely, let $((\gamma_j, t_j), j \in J)$ be a Poisson point process with intensity $\nu_{\mathbb{D}}^{\text{bub}} \times [0, \infty)$, and then arrange the bubble according to the time t_j , i.e. denote γ_t as the bubble γ_j if $t = t_j$, and γ_t is empty set if there is no t_j that equals t . Clearly, there are only countably many bubbles in $(\gamma_t, t \geq 0)$ that are not empty set. Define

$$\tau = \inf\{t : \gamma_t \text{ contains the origin}\}.$$

For each $t < \tau$, the bubble γ_t does not contain the origin. Define f_t to be the conformal map from the connected component of $\mathbb{D} \setminus \gamma_t$ containing the origin onto the unit disc and normalized at the origin: $f_t(0) = 0, f_t'(0) > 0$. For this Poisson point process, we have the following properties [13, Section 4, Section 7]:

- The time τ has the exponential law: $\mathbb{P}[\tau > t] = e^{-t}$.
- For $r > 0$ small, let $t_1(r), t_2(r), \dots, t_j(r)$ be the times t before τ at which the bubble γ_t has radius greater than r . Define $\Psi^r = f_{t_j(r)} \circ \dots \circ f_{t_1(r)}$. Then Ψ^r almost surely converges towards some conformal map Ψ in the Carathéodory topology seen from the origin as r goes to zero. And Ψ can be interpreted as $\Psi = \circ_{t < \tau} f_t$.
- Generally, for each $t \leq \tau$, we can define $\Psi_t = \circ_{s < t} f_s$. Then

$$(L_t := \Psi_t^{-1}(\gamma_t), 0 \leq t \leq \tau)$$

is a collection of loops in the unit disc and L_τ is a loop containing the origin.

The relation between this Poisson point process of bubbles and the discrete exploration process we described above is given via the following result.

Proposition 2.4 ([13, Section 4]). Φ^ε converges in distribution to Ψ in the Carathéodory topology seen from the origin. And L_τ has the same law as the loop of CLE in \mathbb{D} containing the origin.

Write

$$D_t = \Psi_t^{-1}(\mathbb{D}), \quad t \leq \tau.$$

We call the sequence of domains $(D_t, t \leq \tau)$ the continuous exploration process of CLE in \mathbb{D} (targeted at the origin).

3. CLE in the annulus

3.1. Definition and properties of CLE in the annulus

In this section, we will construct CLE loop configuration in the annulus \mathbb{A}_r for $r \in (0, 1)$. We want to use the same idea of constructing CLE in \mathbb{D} from the Brownian loop soup. Suppose $\mathcal{L}(\mathbb{A}_r)$ is a Brownian loop soup in \mathbb{A}_r . Note that $\mathcal{L}(\mathbb{A}_r)$ can have clusters that disconnect the inner boundary C_r from the outer boundary C_1 and this is the case we will not address in the current paper. We will consider the loop-soup conditioned on the event $E(\mathcal{L}(\mathbb{A}_r))$ that there is no cluster of $\mathcal{L}(\mathbb{A}_r)$ that disconnects the inner boundary from the outer boundary.

On the event $E(\mathcal{L}(\mathbb{A}_r))$, let $\Gamma(\mathbb{A}_r)$ be the collection of the outer boundaries of outermost clusters of $\mathcal{L}(\mathbb{A}_r)$. Clearly, $\Gamma(\mathbb{A}_r)$ is a collection of disjoint simple loops in \mathbb{A}_r . We define CLE in the annulus \mathbb{A}_r as the law of $\Gamma(\mathbb{A}_r)$ conditioned on the event $E(\mathcal{L}(\mathbb{A}_r))$. Since the event $E(\mathcal{L}(\mathbb{A}_r))$ has positive probability, the above CLE in the annulus is well-defined.

For any annulus A , suppose its conformal modulus is r and φ is a conformal map from \mathbb{A}_r onto A . Then CLE in the annulus A can be defined as the image of CLE in the annulus \mathbb{A}_r under the map φ . And we denote the law of CLE in the annulus A as $\mu^\sharp(A)$.

We denote $p(\mathbb{A}_r)$ as the probability of the event $E(\mathcal{L}(\mathbb{A}_r))$. Clearly, $p(\mathbb{A}_r)$ only depends on r , so we may also denote it by $p(r)$. The following lemma summarizes the asymptotic behavior of $p(r)$ as r goes to zero. Recall the relation in Equation (2.1).

Proposition 3.1 ([7, Lemma 7, Corollary 8]). Suppose $p(r)$ is the probability of the event $E(\mathcal{L}(\mathbb{A}_r))$. Then p is nondecreasing and there exists a universal constant $C < \infty$ such that, for $0 < r, r' < 1$,

$$\frac{1}{C} p(r)p(r'/C) \leq p(rr') \leq p(r)p(r'). \tag{3.1}$$

Furthermore, we have that

- There exists a constant $C \geq 1$ such that, for r small enough,

$$r^\alpha \leq p(r) \leq Cr^\alpha. \tag{3.2}$$

- For any constant $\lambda \in (0, 1)$, we have

$$\lim_{r \rightarrow 0} \frac{p(\lambda r)}{p(r)} = \lambda^\alpha. \tag{3.3}$$

Proof. Equation (3.1) is proved in [7, Lemma 7] and Equation (3.2) is proved in [7, Corollary 8], and we will give a short proof of Equation (3.3). Suppose $\mathcal{L}_{\lambda r}$ is a Brownian loop soup in the annulus $\mathbb{A}(\lambda, 1/r)$ and let \mathcal{L}_r be the collection of loops in $\mathcal{L}_{\lambda r}$ that are contained in $\mathbb{A}(1, 1/r)$. Denote by $E(\mathcal{L}_{\lambda r})$ (resp. $E(\mathcal{L}_r)$) the event that there is no cluster in $\mathcal{L}_{\lambda r}$ (resp. \mathcal{L}_r) that disconnects the origin from infinity. Note that

$$\frac{p(\lambda r)}{p(r)} = \frac{p(\mathbb{A}(\lambda, 1/r))}{p(\mathbb{A}(1, 1/r))} = \mathbb{P}[E(\mathcal{L}_{\lambda r})|E(\mathcal{L}_r)],$$

thus the limit of $p(\lambda r)/p(r)$ exists as $r \rightarrow 0$. We denote this limit by $f(\lambda)$. Then clearly, for any $\lambda, \lambda' \in (0, 1)$, we have that

$$f(\lambda\lambda') = f(\lambda)f(\lambda').$$

This implies that there exists some constant $\alpha' > 0$ such that $f(\lambda) = \lambda^{\alpha'}$ for all $\lambda \in (0, 1)$. From Equation (3.2), we know that $\alpha' = \alpha$. □

By the conformal invariance of the Brownian loop soup, we have the following:

Proposition 3.2. *The CLE in the annulus $\mathbb{A}(r, \frac{1}{r})$ is invariant under $z \mapsto 1/z$.*

The following is the annulus version of the domain Markov property:

Proposition 3.3. *Suppose that Γ is a CLE in the annulus \mathbb{A}_r , and that D is an open subset of \mathbb{A}_r . Let D^* be the set obtained by removing from D all the loops (and their interiors) in Γ that are not totally contained in D .*

- (1) *If D is simply connected, then each connected component of D^* is also simply connected; and given D^* , for each connected component U of D^* , the conditional law of the loops in Γ that stay in U is the same as CLE in U .*
- (2) *If D is an annulus, then the connected components of D^* can be simply connected or annular; and given D^* , for each connected component U of D^* , the conditional law of the loops in Γ that stay in U is the same as CLE in U .*

Proof. We only prove the case when both D and U are annuli. Other cases can be proved similarly. Let $U_n \subset U$ be an approximation of U whose boundary is a simple path in the lattice $2^{-n}\mathbb{Z}^2$ (see Figure 1). Suppose F is any bounded function on loop configurations that only depends on macroscopic loops (i.e. the loops with diameter greater than 4×2^{-n}). Then, for any deterministic set V_n such that the probability of $\{U_n = V_n\}$ is strictly positive, we only need to show that, when Γ is a CLE in the annulus \mathbb{A}_r , and $\Gamma|_{V_n}$ is the collection of loops of Γ that are contained in V_n , we have that

$$\mu_{\mathbb{A}_r}^\sharp[F(\Gamma|_{V_n})|U_n = V_n] = \mu_{V_n}^\sharp[F].$$

Suppose \mathcal{L} is a Brownian loop soup in \mathbb{A}_r . Let $E(\mathcal{L})$ be the event that no cluster of \mathcal{L} that disconnects C_r from C_1 , and let $\Gamma(\mathcal{L})$ be the collection of outer boundaries of outermost clusters of \mathcal{L} . Then we have that

$$\begin{aligned} & \mu_{\mathbb{A}_r}^\sharp[F(\Gamma|_{V_n})1_{U_n=V_n}] \\ &= \mathbb{E}[F(\Gamma(\mathcal{L})|_{V_n})1_{U_n=V_n}1_{E(\mathcal{L})}]/p(r) \\ &= \mathbb{E}[F(\Gamma(\mathcal{L})|_{V_n})1_{U_n=V_n}1_{E_1}1_{E_2}]/p(r), \end{aligned}$$

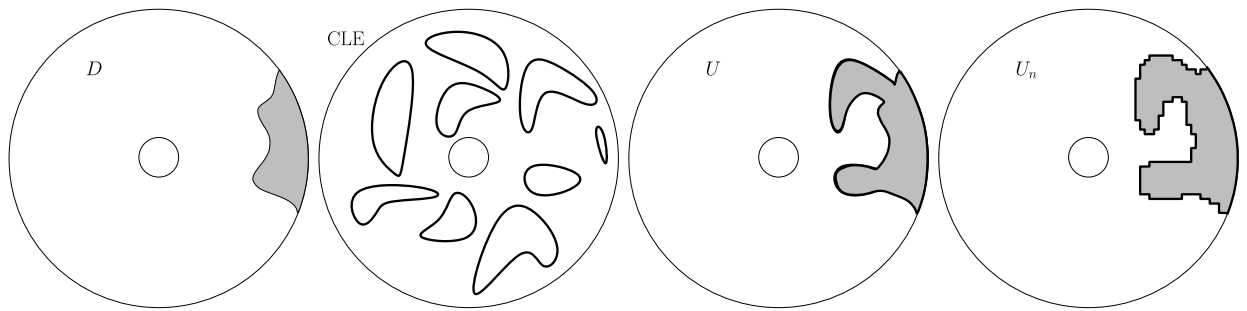


Fig. 1. The first panel indicates the annular region D . The second panel indicates a CLE loop configuration in the annulus \mathbb{A}_r . The third panel indicates one annular connected component U of D^* . The last panel indicates the approximation U_n of U .

where the events E_1, E_2 are defined in the following way: Consider the loops in $\Gamma(\mathcal{L})$ that are contained in V_n , the event E_1 is that no loop disconnects C_r from C_1 . Consider the loops in $\Gamma(\mathcal{L})$ that are not totally contained in V_n , the event E_2 is that no loop disconnects C_r from C_1 . Note that the event E_1 is measurable with respect to $\mathcal{L}|_{V_n}$, which are the loops of \mathcal{L} that are contained in V_n . The event E_2 is measurable with respect to the event $\{U_n = V_n\}$ which is independent of $\mathcal{L}|_{V_n}$. Thus we have

$$\begin{aligned} & \mathbb{E}[F(\Gamma(\mathcal{L})|_{V_n})1_{U_n=V_n}1_{E_1}1_{E_2}]/p(r) \\ &= \mathbb{E}[\mu_{V_n}^\sharp[F]p(V_n)1_{U_n=V_n}1_{E_2}]/p(r) \\ &= \mu_{V_n}^\sharp[F]p(V_n)\mathbb{P}[U_n = V_n, E_2]/p(r). \end{aligned}$$

Thus

$$\begin{aligned} & \mu_{\mathbb{A}_r}^\sharp[F(\Gamma|_{V_n})|U_n = V_n] \\ &= \frac{\mu_{\mathbb{A}_r}^\sharp[F(\Gamma|_{V_n})1_{U_n=V_n}]}{\mu_{\mathbb{A}_r}^\sharp[1_{U_n=V_n}]} \\ &= \frac{\mu_{V_n}^\sharp[F]p(V_n)\mathbb{P}[U_n = V_n, E_2]/p(r)}{p(V_n)\mathbb{P}[U_n = V_n, E_2]/p(r)} \\ &= \mu_{V_n}^\sharp[F]. \end{aligned}$$

□

Propositions 3.4 and 3.5 describe two ways to find CLE in annuli from CLE in simply connected domains.

Proposition 3.4. *Suppose Γ is a CLE in \mathbb{D} and $D \subset \mathbb{D}$ is an annulus. Let D^* be the set obtained by removing from D all the loops (and their interiors) in Γ that are not totally contained in D . Note that the connected components of D^* can be simply connected or annular. Then given D^* , for each connected component U of D^* , the conditional law of the loops in Γ that stay in U is the same as CLE in U .*

Proof. We only prove the case when U is an annulus. Suppose \mathcal{L} is a Brownian loop soup in \mathbb{D} , and let Γ be the collection of the outer boundaries of outermost clusters of \mathcal{L} . Then Γ has the law of simple CLE in \mathbb{D} . Suppose $U_n \subset U$ is the approximation of U whose boundary is a simple path in the lattice $2^{-n}\mathbb{Z}^2$ (see Figure 2). Suppose F is any bounded function on loop configurations that only depends on macroscopic loops (i.e. the loops with diameter greater than 4×2^{-n}). Then, for any deterministic annular set V_n such that the probability of $\{U_n = V_n\}$ is strictly positive, we only need to show that

$$\mathbb{E}[F(\Gamma|_{V_n})|U_n = V_n] = \mu_{V_n}^\sharp[F], \tag{3.4}$$

where $\Gamma|_{V_n}$ is the collection of loops of Γ that are contained in V_n .

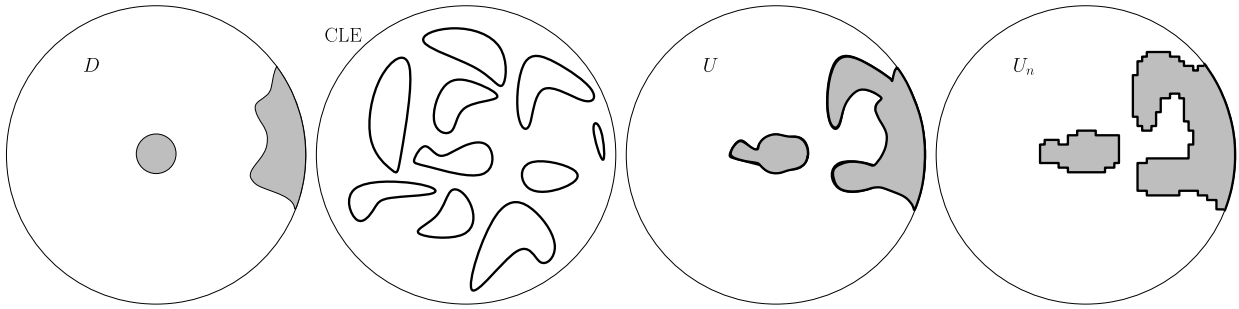


Fig. 2. The first panel indicates the annulus region D . The second panel indicates a simple CLE loop configuration. The third panel indicates one annulus connected component U of D^* . The last panel indicates the approximation U_n of U .

Denote by $\mathcal{L}|_{V_n}$ the collection of loops in \mathcal{L} that are totally contained in V_n ; and denote by $E(\mathcal{L}|_{V_n})$ the event that there is no loop cluster in $\mathcal{L}|_{V_n}$ disconnecting the two components of the boundary of V_n . Then we can see that, given $E(\mathcal{L}|_{V_n})$, the event $[U_n = V_n]$ is conditionally independent of $\mathcal{L}|_{V_n}$. Thus we have that

$$\mathbb{E}[F(\Gamma|_{V_n})1_{U_n=V_n}] = \mu_{V_n}^\# [F]p(V_n)\mathbb{P}[U_n = V_n|E(\mathcal{L}|_{V_n})].$$

This implies Equation (3.4). □

Proposition 3.5. *Suppose Γ is a CLE in \mathbb{D} and $\gamma(0)$ is the loop in Γ that contains the origin. Let D^* be the subset of \mathbb{D} obtained by removing from \mathbb{D} the loop $\gamma(0)$ and its interior. Then given D^* , the conditional law of the loops in Γ that stay in D^* is the same as CLE in the annulus D^* .*

Proof. The conclusion can be derived by setting $D = \mathbb{A}_r$ in Proposition 3.4 and then letting r go to zero. □

3.2. The SLE bubble measure in the annulus

Suppose Γ is a simple CLE in \mathbb{D} . Recall the discrete exploration of Γ : we fix $x \in \partial\mathbb{D}$, and explore the loops in Γ that intersect $B(x, \varepsilon)$, suppose γ^ε is the loop we discovered with largest radius. The probability of the event that γ^ε surrounds the origin is $u(\varepsilon) = \varepsilon^{\beta+o(1)}$ and the law of γ^ε normalized by $1/u(\varepsilon)$ converges to the bubble measure $\nu_{\mathbb{D};x}^{\text{bub}}$ (recall Proposition 2.3).

We use a similar idea to define the bubble measure of CLE in the annulus. Fix $r \in (0, 1)$ and suppose Γ_r is a CLE in the annulus \mathbb{A}_r . We fix $x \in \partial\mathbb{D}$ and explore the loops in Γ_r that intersect $B(x, \varepsilon)$; suppose γ_r^ε is the loop we discovered with largest radius. Then we have the following conclusion which is a counterpart of Proposition 2.3 for CLE in the annulus (recall the definitions of $u(\varepsilon)$ in Equation (2.3) and the constant c in Equation (2.1)):

Proposition 3.6. *The law of γ_r^ε normalized by $1/u(\varepsilon)$ converges to a bubble measure in \mathbb{A}_r , denoted as $\nu_{\mathbb{A}_r;x}^{\text{bub}}$ which we call SLE bubble measure in \mathbb{A}_r rooted at x . Furthermore, the Radon–Nikodym derivative between $\nu_{\mathbb{A}_r;x}^{\text{bub}}$ and $\nu_{\mathbb{D};x}^{\text{bub}}$ is given by*

$$\frac{\nu_{\mathbb{A}_r;x}^{\text{bub}}}{\nu_{\mathbb{D};x}^{\text{bub}}}[d\gamma] = 1_{\{\gamma \subset \mathbb{A}_r\} \cap E(\gamma)} \frac{p(\mathbb{A}_r \setminus \gamma)}{p(\mathbb{A}_r)} \exp(c\Lambda(r\mathbb{D}, \gamma; \mathbb{D})),$$

where $E(\gamma)$ is the event that γ does not surround the origin and $\mathbb{A}_r \setminus \gamma$ indicates the subset of \mathbb{A}_r obtained by removing γ and its interior from \mathbb{A}_r .

Proof. Suppose \mathcal{L} is a Brownian loop soup in \mathbb{D} , and let Γ be the collection of the outer boundaries of outermost clusters of \mathcal{L} . Consider the loops in Γ that intersect $B(x, \varepsilon)$, let γ^ε be the loop with largest radius. Suppose \mathcal{L}_1 is the collection of loops in \mathcal{L} that are totally contained in \mathbb{A}_r . On the event $E(\mathcal{L}_1)$, let Γ_1 be the collection of the outer

boundaries of outermost clusters of \mathcal{L}_1 . Consider the loops in Γ_1 that intersect $B(x, \varepsilon)$, let γ_1^ε be the loop with largest radius. Denote $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Note that \mathcal{L}_1 and \mathcal{L}_2 are independent. Then, for any integrable test function F , we have that

$$\begin{aligned} & \mu_{\mathbb{A}_r}^\# [F(\gamma_r^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_r^\varepsilon; \mathbb{D}))] \\ &= \mathbb{E}[F(\gamma_1^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_1^\varepsilon; \mathbb{D})) 1_{E(\mathcal{L}_1)}] / p(r) \\ &= \mathbb{E}[F(\gamma_1^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_1^\varepsilon; \mathbb{D})) 1_{E_1^1} 1_{E_1^2}] / p(r), \end{aligned}$$

where the events E_1^1, E_1^2 are defined in the following way: Consider the loops in Γ_1 that intersect $B(x, \varepsilon)$, the event E_1^1 is that no loop disconnects C_r from C_1 ; consider the loops in Γ_1 that are totally contained in $\mathbb{A}_r \setminus B(x, \varepsilon)$, the event E_1^2 is that no loop disconnects C_r from C_1 . Note that, given the loops in Γ_1 that intersect $B(x, \varepsilon)$ and the event E_1^1 , the event E_1^2 has probability $p(D_{1,\varepsilon}^*)$ where $D_{1,\varepsilon}^*$ is the set obtained by removing from \mathbb{A}_r all loops (with their interiors) in Γ_1 that intersect $B(x, \varepsilon)$. We also know that the quantity $\exp(-c\Lambda(r\mathbb{D}, \gamma_1^\varepsilon; \mathbb{D}))$ is the probability of the event that no loop in \mathcal{L}_2 that intersects γ_1^ε , which is equivalent to the event that $\{\gamma^\varepsilon = \gamma_1^\varepsilon\}$. Thus we have

$$\begin{aligned} & \mathbb{E}[F(\gamma_1^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_1^\varepsilon; \mathbb{D})) 1_{E_1^1} 1_{E_1^2}] / p(r) \\ &= \mathbb{E}[F(\gamma_1^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_1^\varepsilon; \mathbb{D})) 1_{E_1^1} p(D_{1,\varepsilon}^*)] / p(r) \\ &= \mathbb{E}[F(\gamma_1^\varepsilon) 1_{\{\gamma^\varepsilon = \gamma_1^\varepsilon\}} 1_{E_1^1} p(D_{1,\varepsilon}^*)] / p(r). \end{aligned}$$

Note that, when ε is very small, $D_{1,\varepsilon}^*$ is very close to the set $\mathbb{A}_r \setminus \gamma_1^\varepsilon$. We have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)p(r)} \mathbb{E}[F(\gamma_1^\varepsilon) 1_{\{\gamma^\varepsilon = \gamma_1^\varepsilon\}} 1_{E_1^1} p(D_{1,\varepsilon}^*)] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)p(r)} \mathbb{E}[F(\gamma^\varepsilon) 1_{\{\gamma^\varepsilon = \gamma_1^\varepsilon\}} 1_{E^1} p(\mathbb{A}_r \setminus \gamma^\varepsilon)] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)p(r)} \mathbb{E}[F(\gamma^\varepsilon) 1_{\{\gamma^\varepsilon \subset \mathbb{A}_r\}} 1_{E^1} p(\mathbb{A}_r \setminus \gamma^\varepsilon)] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)p(r)} \mathbb{E}[F(\gamma^\varepsilon) 1_{\{\gamma^\varepsilon \subset \mathbb{A}_r\}} 1_{E(\gamma^\varepsilon)} p(\mathbb{A}_r \setminus \gamma^\varepsilon)], \end{aligned}$$

where the events E^1 and $E(\gamma^\varepsilon)$ are defined in the following way: consider the loops in Γ that intersect $B(x, \varepsilon)$, the event E^1 is that no loop disconnects C_r from C_1 ; the event $E(\gamma^\varepsilon)$ is that γ^ε does not disconnect C_r from C_1 .

Combining all these relations, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)} \mu_{\mathbb{A}_r}^\# [F(\gamma_r^\varepsilon) \exp(-c\Lambda(r\mathbb{D}, \gamma_r^\varepsilon; \mathbb{D}))] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{u(\varepsilon)p(r)} \mathbb{E}[F(\gamma^\varepsilon) 1_{\{\gamma^\varepsilon \subset \mathbb{A}_r\}} 1_{E(\gamma^\varepsilon)} p(\mathbb{A}_r \setminus \gamma^\varepsilon)] \\ &= \nu_{\mathbb{D};x}^{\text{bub}} \left[F(\gamma) 1_{\{\gamma \subset \mathbb{A}_r\}} 1_{E(\gamma)} \frac{p(\mathbb{A}_r \setminus \gamma)}{p(r)} \right]. \end{aligned} \quad \square$$

4. CLE in the punctured disc

4.1. Construction of CLE in the punctured disc

We are going to define CLE in the punctured disc. Roughly speaking, it is the limit of CLE in the annulus \mathbb{A}_r as the inradius r goes to zero. There is another natural way to define CLE in the punctured disc: the limit of CLE in

the disc conditioned on the event that the loop containing the origin has diameter at most ε as ε goes to zero. From Proposition 3.5, we could check that the two limiting procedures would give the same limit. Thus, we also refer to CLE in the punctured disc as CLE in the unit disc conditioned that the origin is in the gasket.

Lemma 4.1. *There exists a universal constant $C < \infty$ such that the following is true. For any $\delta \in (0, 1)$, $0 < r' < r < \delta^2$, and any subset $D \subset \mathbb{A}_\delta$, suppose Γ_r (resp. $\Gamma_{r'}$) is a CLE in the annulus \mathbb{A}_r (resp. $\mathbb{A}_{r'}$), and D_r^* (resp. $D_{r'}^*$) is the set obtained by removing from D all loops (and their interiors) of Γ_r (resp. $\Gamma_{r'}$) that are not totally contained in D . Then there exists a coupling between Γ_r and $\Gamma_{r'}$ such that the probability of the event $\{D_r^* = D_{r'}^*\}$ is at least*

$$1 - C \frac{\log(1/\delta)}{\log(1/r)}.$$

Furthermore, on the event $\{D_r^* = D_{r'}^*\}$, the collection of loops of Γ_r restricted to D_r^* is the same as the collection of loops of $\Gamma_{r'}$ restricted to $D_{r'}^*$.

Proof. Suppose \mathcal{L} is a Brownian loop soup in $\mathbb{A}_{r'}$. Denote by \mathcal{L}_1 the collection of loops of \mathcal{L} that are totally contained in \mathbb{A}_r , and write $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Note that \mathcal{L}_1 and \mathcal{L}_2 are independent. On the event $E(\mathcal{L})$, define Γ (resp. Γ_1) to be the collection of outer boundaries of outermost clusters of \mathcal{L} (resp. \mathcal{L}_1). Note that, conditioned on $E(\mathcal{L})$, the collection Γ (resp. Γ_1) has the same law as CLE in the annulus $\mathbb{A}_{r'}$ (resp. \mathbb{A}_r). Let D^* (resp. D_1^*) be the set obtained by removing from D all loops (and their interiors) of Γ (resp. Γ_1) that are not totally contained in D . Clearly

$$D^* \subset D_1^* \subset \mathbb{A}_\delta.$$

Here is a simple observation: on the event $E(\mathcal{L})$, if there is no loop of \mathcal{L}_2 that intersects D_1^* , then we have $D^* = D_1^*$. Define $S(\mathcal{L}_2, \mathbb{A}_\delta)$ as the event that there exists loop of \mathcal{L}_2 intersecting \mathbb{A}_δ . Thus we have

$$\begin{aligned} & \mathbb{P}[D^* \neq D_1^*, E(\mathcal{L})] / p(r') \\ & \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta), E(\mathcal{L})] / p(r') \\ & \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta), E_1, E_2] / p(r'), \end{aligned}$$

where the events E_1 and E_2 are defined in the following way: the event E_1 is that no loop of Γ_1 that disconnects C_r from C_1 ; consider the loops of \mathcal{L} that are totally contained in the annulus $\mathbb{A}(r', r)$, the event E_2 is that there is no cluster that disconnects $C_{r'}$ from C_r . Clearly, the events $E_1, E_2, S(\mathcal{L}_2, \mathbb{A}_\delta)$ are independent, and the probability of E_1 (resp. E_2) is $p(r)$ (resp. $p(r'/r)$). Thus we have

$$\begin{aligned} & \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta), E_1, E_2] / p(r') \\ & \leq \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)] p(r) p(r'/r) / p(r') \\ & \lesssim \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)], \end{aligned}$$

where the constant in \lesssim can be decided from Proposition 3.1 and is universal. To complete the proof, we only need to show that

$$\mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)] \lesssim \frac{\log(1/\delta)}{\log(1/r)}.$$

Note that the event $S(\mathcal{L}_2, \mathbb{A}_\delta)$ is the same as the event that there exists a loop in \mathcal{L} intersecting both C_r and C_δ . The latter event has the probability

$$1 - \exp(-c\Lambda(C_r, C_\delta; \mathbb{A}_{r'})).$$

From Proposition 2.2, we have that

$$\begin{aligned}
 & \mathbb{P}[S(\mathcal{L}_2, \mathbb{A}_\delta)] \\
 &= 1 - \exp(-c\Lambda(C_r, C_\delta; \mathbb{A}_{r'})) \\
 &\lesssim \Lambda(C_r, C_\delta; \mathbb{A}_{r'}) \\
 &= \Lambda(C_r, C_\delta; \mathbb{C} \setminus r'\mathbb{D}) - \Lambda(C_r, C_1; \mathbb{C} \setminus r'\mathbb{D}) \\
 &= 2 \int_{r'}^r \frac{1}{s} \left(\rho\left(\frac{\delta}{s}\right) - \rho\left(\frac{1}{s}\right) \right) ds \\
 &\lesssim \int_{r'}^r \frac{1}{s} \frac{\log(1/\delta)}{(\log \frac{1}{s})^2} ds \\
 &\lesssim \frac{\log(1/\delta)}{\log(1/r)}. \quad \square
 \end{aligned}$$

Theorem 4.2. *There exists a unique measure on collections of disjoint simple loops in the punctured disc, which we call CLE in the punctured disc or CLE in \mathbb{D} conditioned on the event that the origin is in the gasket, to which CLE in the annulus \mathbb{A}_r converge in the following sense. There exists a universal constant $C < \infty$ such that for any $\delta > 0$, any subset $D \subset \mathbb{A}_\delta$, suppose Γ^\dagger is a CLE in the punctured disc and Γ_r is a CLE in the annulus \mathbb{A}_r , and $D^{\dagger,*}$ (resp. D_r^*) is the set obtained by removing from D all loops of Γ^\dagger (resp. Γ_r) that are not totally contained in D , then Γ^\dagger and Γ_r can be coupled so that the probability of the event $\{D^{\dagger,*} = D_r^*\}$ is at least*

$$1 - C \frac{\log(1/\delta)}{\log(1/r)}.$$

Furthermore, on the event $\{D^{\dagger,*} = D_r^*\}$, the collection of loops of Γ^\dagger restricted to $D^{\dagger,*}$ is the same as the collection of loops of Γ_r restricted to D_r^* .

Proof. Define r_k to be the sequence of positive values so that:

$$\log \log \frac{1}{r_k} = k.$$

Note that $r_k \rightarrow 0$ as $k \rightarrow \infty$. For $k \geq 1$, suppose Γ_k is a CLE in the annulus \mathbb{A}_{r_k} and D_k^* is the set obtained by removing from D all loops of Γ_k that are not totally contained in D . From Lemma 4.1, Γ_k and Γ_{k+1} can be coupled so that the probability of $\{D_k^* \neq D_{k+1}^*\}$ is at most

$$C e^{-k} \log(1/\delta),$$

and on the event $\{D_k^* = D_{k+1}^*\}$, the collection of loops of Γ_k restricted to D_k^* is the same as the collection of loops of Γ_{k+1} restricted to D_{k+1}^* . Suppose that, for each $k \geq 1$, Γ_k and Γ_{k+1} are coupled in this way. Then with probability 1, for all but finitely many couplings, we have that $D_k^* = D_{k+1}^*$. Suppose that this is true for all $k \geq l$, and define, for $k \geq l$,

$$D^{\dagger,*} = D_k^*,$$

and define Γ^\dagger restricted to $D^{\dagger,*}$ to be the collection of loops of Γ_k restricted to D_k^* . Then, for any $k_0 \geq 1$, the probability of $\{D^{\dagger,*} \neq D_{k_0}^*\}$ is at most

$$\sum_{k \geq k_0} C e^{-k} \log(1/\delta) \lesssim e^{-k_0} \log(1/\delta).$$

For any $r > 0$, suppose $r_{k_0} \leq r \leq r_{k_0-1}$. Then CLE in the annulus \mathbb{A}_r , denoted by Γ_r , can be coupled with Γ_{k_0} so that the probability of $\{D_r^* \neq D_{k_0}^*\}$ is at most

$$C \frac{\log(1/\delta)}{\log(1/r)},$$

where D_r^* is the set obtained by removing from D all loops of Γ_r that are not totally contained in D . And on the event $\{D_r^* = D_{k_0}^*\}$, the collection of loops of Γ_r restricted to D_r^* is the same as the collection of loops of Γ_{k_0} restricted to $D_{k_0}^*$. Therefore, the probability of the event $\{D_r^{\dagger,*} \neq D_r^*\}$ is at most

$$C \log(1/\delta)e^{-k_0} + C \frac{\log(1/\delta)}{\log(1/r)} \lesssim \frac{\log(1/\delta)}{\log(1/r)}.$$

This completes the proof. □

4.2. Properties of CLE in the punctured disc

Clearly, CLE in the punctured disc is invariant under rotation. Thus, it is possible to define CLE in any non-trivial simply connected domain D with a singular point $z \in D$ via conformal image, and we call it CLE in D conditioned on the event that z is in the gasket. Propositions 4.3 and 4.4 describe the domain Markov properties of CLE in the punctured disc.

Proposition 4.3. *Suppose Γ^\dagger is a CLE in the punctured disc. For any subset $D \subset \mathbb{D}$ such that 0 is an interior point of $\mathbb{D} \setminus D$ and that D is either simply connected or an annulus, let $D^{\dagger,*}$ be the set obtained by removing from D all loops of Γ^\dagger that are not totally contained in D . Then, given $D^{\dagger,*}$, for each connected component U of $D^{\dagger,*}$, the conditional law of the loops in Γ^\dagger that stay in U is the same as CLE in U .*

Proof. The conclusion is a direct consequence of the construction of CLE in the punctured disc in Theorem 4.2 and the domain Markov property of CLE in the annulus in Proposition 3.3. □

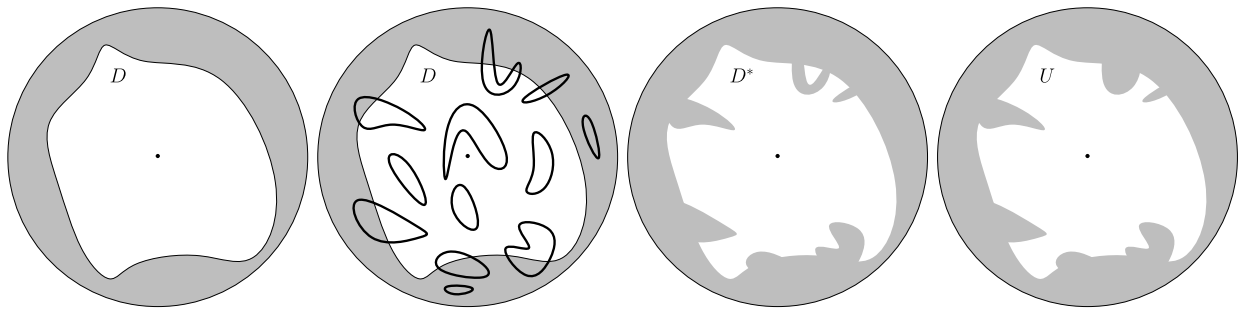
Proposition 4.4. *Suppose Γ^\dagger is a CLE in the punctured disc. For any simply connected domain $D \subset \mathbb{D}$ such that $0 \in D$, let $D^{\dagger,*}$ be the set obtained by removing from D all loops of Γ^\dagger that are not totally contained in D . Suppose U is the connected component of $D^{\dagger,*}$ that contains the origin. Then, given $D^{\dagger,*}$, the conditional law of loops in Γ^\dagger that stay in U is the same as CLE in U conditioned on the event that the origin is in the gasket.*

Proof. For $r > 0$ small, denote $D_r = D \cap \mathbb{A}_r$, and denote by D_r^* the set obtained by removing from D_r all loops of Γ^\dagger that are not totally contained in D_r . Note that, when r is small, it is unlikely that Γ^\dagger has a loop intersecting both $\mathbb{D} \setminus D$ and $r\mathbb{D}$. Suppose there is no such loop and let U_r be the connected component of D_r^* that is contained in U (see Figure 3). From Proposition 4.3, we know that, given U_r , the collection of loops of Γ^\dagger restricted to U_r has the same law as CLE in the annulus. To complete the proof, we only need to point out that, almost surely, $\text{CM}(U_r) \rightarrow 0$ as $r \rightarrow 0$. □

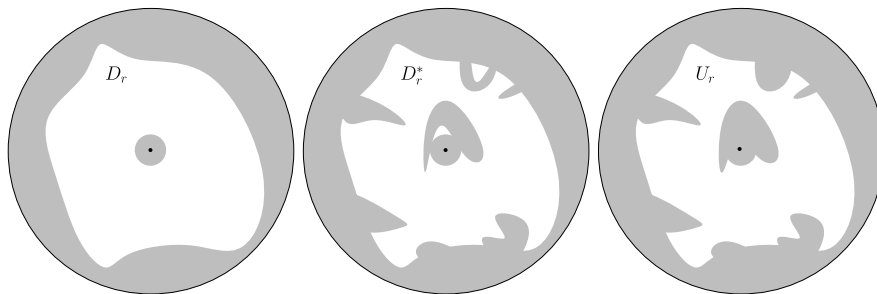
We will describe the relation between CLE in the disc and CLE in the punctured disc. Roughly speaking, if the loops we are interested are far from the singular point, then these loops in the punctured disc are close to those in the disc. We switch from the unit disc to the upper-half plane so that it is easier to describe “the loops are far away from the singular point.”

Proposition 4.5. *Denote $D = \mathbb{D} \cap \mathbb{H}$, and let $y > 0$ be large. Suppose Γ_y^\dagger is a CLE in \mathbb{H} conditioned on the event that iy is in the gasket and Γ is a CLE in \mathbb{H} , and denote $D_y^{\dagger,*}$ (resp. D^*) as the subset of D obtained by removing from D all loops of Γ_y^\dagger (resp. Γ) that are not totally contained in D . Then there exists a universal constant $C < \infty$ such that Γ_y^\dagger and Γ can be coupled so that the probability of the event $\{D_y^{\dagger,*} = D^*\}$ is at least*

$$1 - \frac{C}{\log y}.$$



(a) The first panel indicates the domain D who contains the origin. The second panel indicates a sample of CLE in the punctured disc. The third panel indicates the corresponding set D^* . The last panel indicates the connected component U .



(b) The first panel indicates the set $D_r = D \cap \mathbb{A}_r$. The second panel indicates the corresponding D_r^* . The last panel indicates the connected component U_r .

Fig. 3. The domain Markov property of CLE in the punctured disc.

Furthermore, on the event $\{D_y^{\dagger,*} = D^*\}$, the collection of loops of Γ_y^\dagger restricted to $D_y^{\dagger,*}$ is the same as the collection of loops of Γ restricted to D^* .

Proof. Suppose Γ is a simple CLE in \mathbb{H} and $\gamma(iy)$ is the loop in Γ that contains the point iy . In this proof we write $\gamma(iy)$ to refer to the union of the loop and its interior. We fix a constant $\eta > 1/\beta$, and set

$$R = \frac{y}{(\log y)^\eta}.$$

From Proposition 3.5, we know that, given $\gamma(iy)$, the collection of loops in Γ restricted to $\mathbb{H} \setminus \gamma(iy)$, denoted by Γ_1 , has the same law as CLE in the annulus. Given $\gamma(iy)$ and on the event that $\gamma(iy) \cap C_R = \emptyset$, we have $\{D^* = D_1^*\}$ where D^* (resp. D_1^*) is the set obtained by removing from D all loops of Γ (resp. Γ_1) that are not totally contained in D .

With the similar idea in the proof of Lemma 4.1, Γ_1 can be coupled with CLE in the annulus $\mathbb{H} \setminus B(iy, 1)$, denoted by Γ_2 , so that the probability of $\{D_1^* \neq D_2^*\}$ is at most

$$C\Lambda(\gamma(iy), C_1; \mathbb{H} \setminus B(iy, 1)),$$

where D_2^* is the set obtained by removing from D all loops of Γ_2 that are not totally contained in D . On the event $\{\gamma(iy) \cap C_R = \emptyset\}$, this quantity is less than

$$C\Lambda(C_R, C_1; \mathbb{H}).$$

By [4, Lemma 4.5], we have that

$$\Lambda(C_R, C_1; \mathbb{H}) \lesssim \frac{1}{\log R}.$$

From Theorem 4.2, Γ_2 can be coupled with Γ_y^\dagger in $\mathbb{H} \setminus \{iy\}$ so that, the probability of $\{D_2^* \neq D_y^{\dagger,*}\}$ is at most

$$\frac{C}{\log y}.$$

Combining all these, we conclude that Γ and Γ_y^\dagger can be coupled so that the probability of $\{D^* \neq D_y^{\dagger,*}\}$ is less than

$$C \left(\left(\frac{R}{y} \right)^{-\beta+o(1)} + \frac{1}{\log y} + \frac{1}{\log R} \right) \lesssim \frac{1}{\log y}.$$

This completes the proof. □

4.3. Exploration of CLE in the punctured disc

We will explore CLE in the punctured disc in a way similar to the discrete exploration of simple CLE described in Section 2.3.2. Suppose Γ^\dagger is CLE in the punctured disc. We explore the loops of Γ^\dagger that intersect $B(x, \varepsilon)$ for some $x \in \partial\mathbb{D}$. Let $\gamma^{\dagger,\varepsilon}$ be the loop of Γ^\dagger with the largest radius. Then we have the following conclusion which is a counterpart of Proposition 2.3, recall the definition of $u(\varepsilon)$ in Equation (2.3):

Proposition 4.6. *The law of $\gamma^{\dagger,\varepsilon}$ normalized by $1/u(\varepsilon)$ converges to a measure, denoted by $\nu_{\mathbb{D}^\dagger;x}^{\text{bub}}$ which we call the SLE bubble measure in \mathbb{D}^\dagger rooted at x . Furthermore, the Radon–Nikodym derivative of $\nu_{\mathbb{D}^\dagger;x}^{\text{bub}}$ with respect to the SLE bubble measure $\nu_{\mathbb{D};x}^{\text{bub}}$ in \mathbb{D} rooted at x is given by*

$$\frac{\nu_{\mathbb{D}^\dagger;x}^{\text{bub}}}{\nu_{\mathbb{D};x}^{\text{bub}}}[d\gamma] = 1_{E(\gamma)} \text{CR}(\mathbb{D} \setminus \gamma)^{-\alpha},$$

where $E(\gamma)$ is the event that γ does not surround the origin and $\mathbb{D} \setminus \gamma$ indicates the subset of \mathbb{D} obtained by removing from \mathbb{D} the bubble γ and its interior.

Proof. Combination of Proposition 3.6 and Lemma 2.1 implies the conclusion. □

Suppose γ is a bubble in \mathbb{D} rooted at x , and recall that $R(\gamma)$ is the smallest r for which γ is contained in $B(x, r)$. Recall the constants defined in Equation (2.1); we have the following quantitative results for $\nu_{\mathbb{D};x}^{\text{bub}}$ and $\nu_{\mathbb{D}^\dagger;x}^{\text{bub}}$:

Lemma 4.7.

$$\int 1_{E(\gamma) \cap [R(\gamma) \geq 1/2]} \text{CR}(\mathbb{D} \setminus \gamma)^{-\eta} \nu_{\mathbb{D};x}^{\text{bub}}[d\gamma] < \infty \quad \text{as long as } \eta < 1 - \kappa/8.$$

In particular, this implies that

$$\nu_{\mathbb{D}^\dagger;x}^{\text{bub}}[R(\gamma) \geq 1/2] < \infty.$$

Proof. Conditioned on $\{R(\gamma) > 1/2\} \cap E(\gamma)$, we can parameterize the bubble γ clockwise by the capacity seen from the origin starting from the root and ending at the root: $(\gamma(t), 0 \leq t \leq T)$. Suppose S is the first time that γ exits the ball $B(x, 1/2)$. Then we know that, given $\gamma[0, S]$, the future part of the curve $\gamma[S, T]$ has the same law as a chordal SLE in $\mathbb{D} \setminus \gamma[0, S]$ from $\gamma(S)$ to x . Thus we only need to show that the integral is finite when we replace the curve by a chordal SLE curve.

Precisely, suppose $\gamma = (\gamma_t, t \geq 0)$ is a chordal SLE in the upper-half plane \mathbb{H} from 0 to ∞ (parameterized by the half-plane capacity). We only need to show that

$$\mathbb{E}[\text{CR}(\mathbb{H} \setminus \gamma; i)^{-\eta}] < \infty, \tag{4.1}$$

where $\text{CR}(\mathbb{H} \setminus \gamma; i)$ is the conformal radius of $\mathbb{H} \setminus \gamma$ in \mathbb{H} seen from i . This is true for chordal SLE, see [14, Equation (6.9)]. \square

Lemma 4.8.

$$\int R(\gamma)^\eta v_{\mathbb{D}^\dagger; x}^{\text{bub}}[d\gamma] < \infty \quad \text{as long as } \eta > \beta.$$

Note that $\beta \in [1, 2)$, thus we have

$$\int R(\gamma)^2 v_{\mathbb{D}^\dagger; x}^{\text{bub}}[d\gamma] < \infty.$$

Proof. We divide the integral into two parts:

$$\int 1_{[R(\gamma) \leq 1/2]} R(\gamma)^\eta v_{\mathbb{D}^\dagger; x}^{\text{bub}}[d\gamma] \quad \text{and} \quad \int 1_{[R(\gamma) \geq 1/2]} R(\gamma)^\eta v_{\mathbb{D}^\dagger; x}^{\text{bub}}[d\gamma].$$

The first part is finite due to Propositions 4.6 and 2.3. The second part is finite by Lemma 4.7. \square

Now we can describe the exploration process of CLE in the punctured disc. Most of the proofs are similar to the proofs in [13] for simple CLE's. To be self-contained, we rewrite the proofs in the current setting.

Suppose $(\gamma_t^\dagger, t \geq 0)$ is a Poisson point process with intensity

$$v_{\mathbb{D}^\dagger}^{\text{bub}} = \int_{\partial\mathbb{D}} dx v_{\mathbb{D}^\dagger; x}^{\text{bub}}.$$

For any time t , let f_t^\dagger be the conformal map from $\mathbb{D} \setminus \gamma_t^\dagger$ onto \mathbb{D} normalized at the origin. For any fixed $T > 0$ and $r > 0$, let $t_1(r) < \dots < t_j(r)$ be the times t before T at which $R(\gamma_t^\dagger)$ is greater than r . Define

$$\Psi_T^{\dagger, r} = f_{t_1(r)}^\dagger \circ \dots \circ f_{t_j(r)}^\dagger.$$

Then we have the following:

Lemma 4.9. $\Psi_T^{\dagger, r}$ converges almost surely in the Carathéodory topology seen from the origin towards some conformal map, denoted as Ψ_T^\dagger , as r goes to zero.

Proof. Lemma 4.8 guarantees that

$$\begin{aligned} & \mathbb{E} \left[\sum_{t < T} \text{cap}(\gamma_t^\dagger) 1_{R(\gamma_t^\dagger) \leq 1/2} \right] \\ &= T v_{\mathbb{D}^\dagger}^{\text{bub}} [\text{cap}(\gamma^\dagger) 1_{\{R(\gamma^\dagger) \leq 1/2\}}] \\ &\lesssim T v_{\mathbb{D}^\dagger}^{\text{bub}} [R(\gamma^\dagger)^2 1_{\{R(\gamma^\dagger) \leq 1/2\}}] < \infty. \end{aligned}$$

Since there are only finitely many times t before T at which $R(\gamma_t^\dagger) \geq 1/2$, we have that, almost surely,

$$\sum_{t < T} \text{cap}(\gamma_t^\dagger) < \infty,$$

and this implies the convergence in the Carathéodory topology (see [13, Stability of Loewner chains]). \square

Define $(D_t^\dagger = (\Psi_t^\dagger)^{-1}(\mathbb{D}), t \geq 0)$. This is a decreasing sequence of simply connected domains containing the origin, and we call it *the continuous exploration process of CLE in the punctured disc*. Write, for $t > 0$, $L_t^\dagger = (\Psi_t^\dagger)^{-1}(\gamma_t^\dagger)$. It is clear that

$$D_t^\dagger = \bigcap_{s \leq t} D_s^\dagger, \quad D_{t+}^\dagger := \bigcup_{s > t} D_s^\dagger = D_t^\dagger \setminus L_t^\dagger.$$

Suppose $(\gamma_t, t \geq 0)$ is a Poisson point process with intensity $\nu_{\mathbb{D}}^{\text{bub}}$, and $(D_t, t \leq \tau)$ is the continuous exploration process of simple CLE in \mathbb{D} defined from the process $(\gamma_t, t \geq 0)$ as described in Section 2.3.2. Define, for $\eta > 0$,

$$\theta(\eta) := \int (e^{\eta \text{cap}(\gamma)} - 1) 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma], \quad (4.2)$$

where $E(\gamma)$ is the event that γ does not surround the origin, and we first assume that $\theta(\alpha)$ is a positive finite constant. Then the relation between the process $(D_t^\dagger, t \geq 0)$ and the process $(D_t, t \leq \tau)$ can be described using the following proposition.

Proposition 4.10. *For any $t > 0$, the law of $(\gamma_s^\dagger, s < t)$ is the same as the law of $(\gamma_s, s < t)$ conditioned on $\{\tau \geq t\}$ and weighted by M_t where*

$$M_t = \exp\left(\alpha \sum_{s < t} \text{cap}(\gamma_s) - \theta(\alpha)t\right). \quad (4.3)$$

In particular, for any $t > 0$, the law of D_t^\dagger is the same as the law of D_t conditioned on $\{\tau \geq t\}$ and weighted by

$$\text{CR}(D_t)^{-\alpha} e^{-\theta(\alpha)t}.$$

Proof. We first note that the process $(\gamma_s, s < t)$ conditioned on $\{\tau \geq t\}$ has the same law as a Poisson point process with intensity $1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}$ restricted to the time interval $[0, t)$. Suppose $(\hat{\gamma}_s, s \geq 0)$ is a Poisson point process with intensity $1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}$, and define

$$\hat{M}_t = \exp\left(\alpha \sum_{s < t} \text{cap}(\hat{\gamma}_s) - \theta(\alpha)t\right).$$

We only need to show that, for any function f on bubbles that makes every integral finite, we have

$$\mathbb{E}\left[\exp\left(-\sum_{s < t} f(\hat{\gamma}_s)\right) \hat{M}_t\right] = \exp\left(-t \int (1 - e^{-f(\gamma)}) e^{\alpha \text{cap}(\gamma)} 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma]\right).$$

This can be obtained by direct calculation:

$$\begin{aligned} & -\log \mathbb{E}\left[\exp\left(-\sum_{s < t} f(\hat{\gamma}_s)\right) \hat{M}_t\right] \\ &= -\log \mathbb{E}\left[\exp\left(-\sum_{s < t} (f(\hat{\gamma}_s) - \alpha \text{cap}(\hat{\gamma}_s))\right)\right] + \theta(\alpha)t \\ &= t \int (1 - e^{-f(\gamma) + \alpha \text{cap}(\gamma)}) 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] + \theta(\alpha)t \\ &= t \int (1 - e^{-f(\gamma)}) e^{\alpha \text{cap}(\gamma)} 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma]. \end{aligned}$$

□

The fact that $\theta(\alpha)$ is a positive finite constant is guaranteed by the following lemma.

Lemma 4.11. *The quantity $\theta(\eta)$, which is defined in Equation (4.2), is finite as long as $\eta < 1 - \kappa/8$.*

The quantity

$$\int (e^{\eta \text{cap}(\gamma)} - 1) \nu_{\mathbb{D}^\dagger}^{\text{bub}}[d\gamma] \tag{4.4}$$

is finite as long as $\eta < 2/\kappa - \kappa/32$.

Proof. The integral in $\theta(\eta)$ may explode when $R(\gamma)$ is small or when γ is close to the origin. We will control the two parts separately.

For the first part, we have

$$\begin{aligned} & \int (e^{\eta \text{cap}(\gamma)} - 1) 1_{\{R(\gamma) \leq 1/2\}} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] \\ & \lesssim \int \text{cap}(\gamma) 1_{\{R(\gamma) \leq 1/2\}} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] \\ & \lesssim \int R(\gamma)^2 1_{\{R(\gamma) \leq 1/2\}} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] < \infty. \end{aligned}$$

For the second part, by Lemma 4.7, we get

$$\begin{aligned} & \int (e^{\eta \text{cap}(\gamma)} - 1) 1_{\{R(\gamma) \geq 1/2\}} 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] \\ & \leq \int \text{CR}(\mathbb{D} \setminus \gamma)^{-\eta} 1_{\{R(\gamma) \geq 1/2\}} 1_{E(\gamma)} \nu_{\mathbb{D}}^{\text{bub}}[d\gamma] < \infty. \end{aligned}$$

The quantity in Equation (4.4) is finite as long as $\eta + \alpha < 1 - \kappa/8$. □

We conclude this section by explaining that the sequence of loops $(L_t^\dagger, t \geq 0)$ obtained from the sequence of bubbles $(\gamma_t^\dagger, t \geq 0)$ (the Poisson point process with intensity $\nu^{\text{bub}}(\mathbb{D}^\dagger)$) does correspond to the loops in CLE in the punctured disc. Namely, we first remove from \mathbb{D} all loops L_t^\dagger (with their interiors) for $t \geq 0$, and then, in each connected component, sample independent simple CLE's. We will argue that the collection of these loops from simple CLE together with the sequence $(L_t^\dagger, t \geq 0)$ has the same law as the collection of loops in CLE in the punctured disc. The idea is very similar to the one used in [13, Section 7] to show that the loops obtained from the bubbles have the same law as the loops in CLE.

Suppose Γ^\dagger is a CLE in the punctured disc. Fix a point $z \in \mathbb{D}^\dagger$. Let $L^\dagger(z)$ be the loop in Γ^\dagger that contains z . Following is the discrete exploration of Γ^\dagger to discover $L^\dagger(z)$. Fix $\varepsilon > 0$ small and $\delta > \varepsilon$ small. Sample $x_1 \in \partial\mathbb{D}$ uniformly chosen from the circle. The loops of Γ^\dagger that intersect $B(x_1, \varepsilon)$ are the loops we discovered. Call the connected component of the remaining domain that contains the origin the to-be-explored domain and let $f_1^{\dagger, \varepsilon}$ be the conformal map from the to-be-explored domain onto the unit disc normalized at the origin. Let $\gamma_1^{\dagger, \varepsilon}$ be the discovered loop with largest radius. The image of the loops in the to-be-explored domain under $f_1^{\dagger, \varepsilon}$ has the same law as CLE in the unit disc conditioned on the event that the origin is in the gasket. Thus we can repeat the same procedure, define $f_2^{\dagger, \varepsilon}, \gamma_2^{\dagger, \varepsilon}$ etc. For $k \geq 1$, define

$$\Phi_k^{\dagger, \varepsilon} = f_k^{\dagger, \varepsilon} \circ \dots \circ f_1^{\dagger, \varepsilon}.$$

We also need to keep track of the point z : let $z_k = \Phi_k^{\dagger, \varepsilon}(z)$, and let K be the largest k such that $z \in (\Phi_k^{\dagger, \varepsilon})^{-1}(\mathbb{D})$. Define another auxiliary stopping time $K' \leq K$ as the first step k at which either $|z_k| \geq 1 - \delta$ or $k = K$. If $K' < K$, this means that the point z is conformally far from the origin and is likely to be cut off in the discrete exploration.

We first address the case that z is discovered at step $K + 1$. Note that $\Phi_K^{\dagger, \varepsilon}$ will converge in distribution towards some conformal map Ψ_S^\dagger (with similar explanation as for Proposition 2.4) obtained from the Poisson point process

$(\gamma_r^\dagger, t \geq 0)$. This implies that $L^\dagger(z)$ has the same law as $(\Psi_S^\dagger)^{-1}(\gamma_S^\dagger)$ as we expected. Now we will deal with the case that z is cut off from the origin: we stop the discrete exploration at step $K' - 1$. At step K' , instead of discovering the loops intersecting the ball of radius ε , we discover the loops intersecting the circle centered at $z_{K'-1}$ with radius $\sqrt{\delta}$. After this step, we continue the discrete exploration (of size ε) by targeting at the image of the point z , in the same way for the discrete exploration of simple CLE targeted at the image of the point z . We will discover the point z at some step. Let ε and δ go to zero in proper way, we can also prove the conclusion for $L^\dagger(z)$ in this case.

We also need to show the conclusion for the joint law of $(L^\dagger(z_1), \dots, L^\dagger(z_k))$ where $z_1, \dots, z_k \in \mathbb{D}^\dagger$. The argument is almost the same as above and we leave it to the interested reader.

5. CLE in the punctured plane

In this section we discuss CLE in the punctured plane. The following lemma is analogous to Lemma 4.1, and we delay its proof to the end of this section.

Lemma 5.1. *There exists a universal constant $C < \infty$ such that the following is true. Let $\delta \in (0, 1)$, $0 < r' < r < \delta^2$, and let $D \subset \mathbb{A}(\delta, 1/\delta)$. Suppose Γ_r (resp. $\Gamma_{r'}$) is a CLE in the annulus $\mathbb{A}(r, 1/r)$ (resp. $\mathbb{A}(r', 1/r')$) and D_r^* (resp. $D_{r'}^*$) is the set obtained by removing from D all loops (and their interiors) of Γ_r (resp. $\Gamma_{r'}$) that are not totally contained in D . Then there exists a coupling between Γ_r and $\Gamma_{r'}$ such that the probability of the event $\{D_r^* = D_{r'}^*\}$ is at least*

$$1 - C \frac{\log(1/\delta)}{\log(1/r)}.$$

Furthermore, on the event $\{D_r^* = D_{r'}^*\}$, the collection of loops of Γ_r restricted to D_r^* is the same as the collection of loops of $\Gamma_{r'}$ restricted to $D_{r'}^*$.

Theorem 5.2. *There exists a unique measure on collections of disjoint simple loops in the punctured plane, which we call CLE in the punctured plane, or CLE in \mathbb{C} conditioned on the event that both the origin and infinity are in the gasket, to which CLE in the annulus $\mathbb{A}(r, 1/r)$ converges in the following sense. There exists a universal constant $C < \infty$ such that for any $\delta > 0$ and any subset $D \subset \mathbb{A}(\delta, 1/\delta)$, if Γ^\dagger is a CLE in the punctured plane and Γ_r is a CLE in the annulus $\mathbb{A}(r, 1/r)$, and $D^{\dagger,*}$ (resp. D_r^*) is the set obtained by removing from D all loops of Γ^\dagger (resp. Γ_r) that are not totally contained in D , then Γ^\dagger and Γ_r can be coupled so that the probability of the event $\{D^{\dagger,*} = D_r^*\}$ is at least*

$$1 - C \frac{\log(1/\delta)}{\log(1/r)}.$$

Furthermore, on the event $\{D^{\dagger,*} = D_r^*\}$, the collection of loops of Γ^\dagger restricted to $D^{\dagger,*}$ is the same as the collection of loops of Γ_r restricted to D_r^* .

It is clear that CLE in the punctured plane can also be viewed as the limit of CLE in $R\mathbb{D}$ conditioned on the event that the origin is in the gasket as $R \rightarrow \infty$ or the limit of CLE in $\mathbb{C} \setminus r\mathbb{D}$ conditioned on the event that infinity is in the gasket as $r \rightarrow 0$.

Proposition 5.3. *CLE in the punctured plane satisfies the conformal invariance:*

- (1) CLE in the punctured plane is invariant under the conformal map: $z \mapsto \lambda z$, for any $\lambda \in \mathbb{C}$.
- (2) CLE in the punctured plane is invariant under the conformal map: $z \mapsto 1/z$.

Note that, Lemma 5.1 and Theorem 5.2 are the counterparts of Lemma 4.1 and Theorem 4.2. The proof of Theorem 5.2 from Lemma 5.1 is almost the same as the proof of Theorem 4.2 from Lemma 4.1. The conformal invariance of CLE in the punctured plane in Proposition 5.3 is then a direct consequence of the construction in Theorem 5.2. The rest of this subsection is devoted to the proof of Lemma 5.1, since the proof of Lemma 4.1 does not work directly here, we need some extra effort to complete the proof of Lemma 5.1.

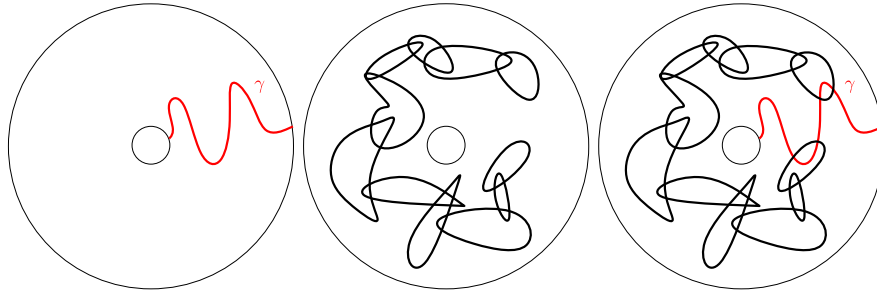


Fig. 4. The first panel indicates a continuous path γ in the annulus connecting the two pieces of the boundary. The second panel indicates a Brownian loop soup in the annulus and $E(\mathcal{L})$ holds. The third panel indicates that $E_\gamma(\mathcal{L})$ does not hold even though $E(\mathcal{L})$ holds.

Proof of Lemma 5.1. We first introduce a quantity $q(r)$ for $r > 0$ small: Let \mathcal{L} be a Brownian loop soup in \mathbb{A}_r , define $E(\mathcal{L})$ as the event that there is no cluster of \mathcal{L} disconnecting C_r from C_1 . Suppose γ is a continuous path in \mathbb{A}_r connecting C_r to C_1 . Define $E_\gamma(\mathcal{L})$ to be the event that there is no cluster of $\mathcal{L} \cup \{\gamma\}$ disconnecting C_r from C_1 . See Figure 4. Clearly, $E_\gamma(\mathcal{L}) \subset E(\mathcal{L})$. Define

$$q(r) = \sup_{\gamma} \mathbb{P}[E_\gamma(\mathcal{L})|E(\mathcal{L})],$$

where the sup is taken over all possible continuous paths γ in \mathbb{A}_r that connect C_r to C_1 . We can see that $q(r) \rightarrow 0$ as r goes to zero.

Take $r, r' > 0$ small. Let \mathcal{L} be a Brownian loop soup in $\mathbb{A}_{rr'}$. Suppose \mathcal{L}_1 (resp. \mathcal{L}_2) is the collection of loops of \mathcal{L} that are contained in $\mathbb{A}(rr', r)$ (resp. \mathbb{A}_r). Let γ be any continuous path in $\mathbb{A}_{rr'}$ connecting $C_{rr'}$ to C_1 . Suppose γ_1 (resp. γ_2) is part of γ that is a continuous path in $\mathbb{A}(rr', r)$ (resp. \mathbb{A}_r) connecting $C_{rr'}$ to C_r (resp. connecting C_r to C_1). Then we have that

$$\begin{aligned} &\mathbb{P}[E_\gamma(\mathcal{L})|E(\mathcal{L})] \\ &= \mathbb{P}[E_\gamma(\mathcal{L})]/p(rr') \\ &\leq \mathbb{P}[E_{\gamma_1}(\mathcal{L}_1), E_{\gamma_2}(\mathcal{L}_2)]/p(rr') \\ &= \mathbb{P}[E_{\gamma_1}(\mathcal{L}_1)|E(\mathcal{L}_1)] \times \mathbb{P}[E_{\gamma_2}(\mathcal{L}_2)|E(\mathcal{L}_2)] \times p(r')p(r)/p(rr') \\ &\lesssim q(r)q(r'). \end{aligned}$$

Thus, there exists universal constant C so that

$$q(rr') \leq Cq(r)q(r').$$

Together with the fact that $q(r) \rightarrow 0$ as r goes to zero, we have that there exists some constant $\tilde{\alpha} > 0$ such that, for $r > 0$ small,

$$q(r) \leq r^{\tilde{\alpha}}.$$

Now we are ready to complete the proof. Suppose \mathcal{L} is a Brownian loop soup in $\mathbb{A}(r', 1/r')$. Let \mathcal{L}_1 be the collection of loops of \mathcal{L} that are contained in $\mathbb{A}(r, 1/r)$. On the event $E(\mathcal{L})$, let Γ (resp. Γ_1) be the collection of the outer boundaries of outermost clusters of \mathcal{L} (resp. \mathcal{L}_1). Let D^* (resp. D_1^*) be the set obtained by removing from D all loops of Γ (resp. Γ_1) that are not totally contained in D . Note that, if $\{D^* \neq D_1^*\}$, there must exist a loop in \mathcal{L} intersecting both C_r and C_δ or intersecting both $C_{1/r}$ and $C_{1/\delta}$. Define $S(\mathcal{L}, C_r, C_\delta)$ to be the event that there exists a loop of \mathcal{L} that intersects both C_r and C_δ . Then we have that

$$\begin{aligned} &\mathbb{P}[D^* \neq D_1^*, E(\mathcal{L})]/p((r')^2) \\ &\leq 2\mathbb{P}[S(\mathcal{L}, C_r, C_\delta), E(\mathcal{L})]/p((r')^2). \end{aligned}$$

We divide the loops in \mathcal{L} into independent collections: Let \mathcal{L}_2 be the loops in \mathcal{L} that are contained in $\mathbb{A}(r', r)$, \mathcal{L}_3 be the loops in \mathcal{L} that are contained in $\mathbb{A}(r, \delta)$, \mathcal{L}_4 be the loops in \mathcal{L} that are contained in $\mathbb{A}(\delta, 1/r')$, and \mathcal{L}_5 be the collection of loops in \mathcal{L} that intersect both C_r and C_δ . Clearly, $\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5$ are independent and the event $S(\mathcal{L}, C_r, C_\delta)$ is the same as $\{\mathcal{L}_5 \neq \emptyset\}$. Define E_2 (resp. E_3, E_4) to be the event that there is no cluster of \mathcal{L}_2 (resp. $\mathcal{L}_3, \mathcal{L}_4$) disconnecting $C_{r'}$ from C_r (resp. disconnecting C_r from C_δ , disconnecting C_δ from $C_{1/r'}$). Then E_2, E_3, E_4 are independent, and their probabilities are $p(r'/r), p(r/\delta), p(\delta r')$ respectively. Thus

$$\begin{aligned} & \mathbb{P}[S(\mathcal{L}, C_r, C_\delta), E(\mathcal{L})]/p((r')^2) \\ &= \mathbb{P}[\mathcal{L}_5 \neq \emptyset, E(\mathcal{L}), E_2, E_3, E_4]/p((r')^2) \\ &= \mathbb{P}[\mathcal{L}_5 \neq \emptyset, E(\mathcal{L})|E_2, E_3, E_4] \times p(r'/r)p(r/\delta)p(\delta r')/p((r')^2) \\ &\lesssim \mathbb{P}[\mathcal{L}_5 \neq \emptyset, E(\mathcal{L})|E_2, E_3, E_4] \\ &\leq \mathbb{P}[\mathcal{L}_5 \neq \emptyset, E(\mathcal{L}_3 \cup \mathcal{L}_5)|E_2, E_3, E_4] \\ &= \mathbb{P}[\mathcal{L}_5 \neq \emptyset, E(\mathcal{L}_3 \cup \mathcal{L}_5)|E_3] \\ &\leq q(r/\delta), \end{aligned}$$

where $E(\mathcal{L}_3 \cup \mathcal{L}_5)$ is the event that there is no cluster of $\mathcal{L}_3 \cup \mathcal{L}_5$ disconnecting C_r from C_δ . This implies the conclusion. \square

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