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Excited random walk with periodic cookies

Gady Kozma[1,3], Tal Orenshtein[1] and Igor Shinkar[2]

Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel.
E-mail: gady.kozma@weizmann.ac.il; tal.orenshtein@weizmann.ac.il; igor.shinkar@weizmann.ac.il

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Abstract. In this paper we consider an excited random walk (ERW) on $\mathbb{Z}$ in identically piled periodic environment. This is a discrete time process on $\mathbb{Z}$ defined by parameters $(p_1, \dots, p_M) \in [0,1]^M$ for some positive integer M, where the walker upon the ith visit to $z \in \mathbb{Z}$ moves to $z+1$ with probability $p_{i \pmod M}$, and moves to $z-1$ with probability $1 - p_{i \pmod M}$. We give an explicit formula in terms of the parameters $(p_1, \dots, p_M)$ which determines whether the walk is recurrent, transient to the left, or transient to the right. In particular, in the case that $\frac{1}{M}\sum_{i=1}^{M} p_i = \frac{1}{2}$ all behaviors are possible, and may depend on the order of the p_i. Our framework allows us to reprove some known results on ERW and branching processes with migration with no additional effort.

Résumé. Dans ce papier, nous considérons une marche aléatoire excitée (MAE) sur $\mathbb{Z}$ en environnement empilé de manière identique et périodique. Il s'agit d'un processus à temps discret sur $\mathbb{Z}$ défini par des paramètres $(p_1, \dots, p_M) \in [0,1]^M$ pour un certain entier strictement positif M, où le marcheur, après la $i$ième visite au site $z \in \mathbb{Z}$ se déplace soit en $z+1$ avec probabilité $p_{i \pmod M}$, soit en $z-1$ avec probabilité $1 - p_{i \pmod M}$. Nous donnons une formule explicite en fonction des paramètres $(p_1, \dots, p_M)$ qui détermine si la marche est récurrente, transiente vers la gauche, ou transiente vers la droite. En particulier, dans le cas où $\frac{1}{M}\sum_{i=1}^{M} p_i = \frac{1}{2}$, tous les comportements sont possibles et peuvent dépendre de l'ordre des p_i. Notre approche permet de retrouver directement certains résultats connus sur les MAE et les processus de branchement sans immigration.

MSC: 60K35; 60J85

Keywords: Excited random walk; Cookie walk; Recurrence; Transience; Bessel process; Lyapunov function; Branching process with migration

1. Introduction

Excited Random Walk (ERW), also known as the Cookie Walk, was introduced by Benjamini and Wilson [3] as a non-Markovian local perturbation of simple random walk on $\mathbb{Z}^d$, $d \geq 1$. In this model we have a stack of cookies placed on each vertex of the lattice, and each cookie encodes a probability distribution on the next step of the walker (also known as the cookie monster). In each round the walker eats the top cookie in the stack in her current position, and makes a random step according to the probability distribution encoded by this cookie. In their paper Benjamini and Wilson [3] showed that by placing a single biased cookie in each vertex the walk is recurrent in $\mathbb{Z}$ and is transient in $\mathbb{Z}^d$ for all $d \geq 2$. The case $d = 1$ has been later generalized by Zerner [15] by placing more biased cookies in each vertex of the lattice. There has been a lot of work done on this model, in both deterministic and random cookie environments. For more background see the recent survey of Kosygina and Zerner [10] and the references therein.

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In this paper we shall discuss only the case $d = 1$. ERW is a discrete time stochastic process $X = (X_n)_{n\geq 0}$ on the integer lattice $\mathbb{Z}$. The process X in the cookie environment $\omega \in [0,1]^{\mathbb{Z}\times\mathbb{N}}$ is initiated at some $X_0 = x \in \mathbb{Z}$. If at time n the walker is in position y, and this is her jth visit to y, then she moves to $y+1$ with probability $\omega(y, j)$, and moves to $y-1$ with probability $1-\omega(y, j)$.

In this paper we shall assume that the initial position x of the process is 0, and that the stacks in the cookie environment ω are *identically piled*, that is $\omega(x,i) = \omega(0,i)$ for all $x \in \mathbb{Z}$ and $i \in \mathbb{N}$. For a vector $p \in [0,1]^{\mathbb{N}}$ we shall write $\omega(p)$ to denote the identically piled cookie environment ω where $\omega(x,i) = p_i$ for all $x \in \mathbb{Z}$ and $i \in \mathbb{N}$. In this case the ERW mechanism has a simple form, namely $\mathbb{P}[X_0 = 0] = 1$ and

$$\mathbb{P}\big[X_{n+1} = X_n + 1 | L_n(X_n) = j\big] = 1 - \mathbb{P}\big[X_{n+1} = X_n - 1 | L_n(X_n) = j\big] = p_j,$$

where $L_n(x) = \#\{k \leq n\colon\ X_k = x\}$ is the number of visits to $x \in \mathbb{Z}$ in n steps of the walk. With a minor abuse of notation we say that p is a cookie environment, when actually referring to the identically piled cookie environment $\omega(p)$. Next, we introduce some definitions for cookie environments.

Definition 1.1. *Let $p \in [0,1]^{\mathbb{N}}$ be a cookie environment.*

- *The environment p is called* elliptic *if $p \in (0,1)^{\mathbb{N}}$.*
- *The environment p is called* non-degenerate *if $\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty}(1-p_i) = \infty$.*
- *The environment p is called* positive *if $p \in [\frac{1}{2}, 1]^{\mathbb{N}}$.*
- *The environment p is called* bounded *if there is some $M \in \mathbb{N}$ such that $p_i = \frac{1}{2}$ for all $i > M$.*
- *The environment p is called* periodic *if for some $M \in \mathbb{N}$ it holds that $p_i = p_{i+M}$ for all $i \in \mathbb{N}$. We denote such an environment by $\omega(p_1, \dots, p_M)$.*

1.1. *Our results*

Consider an ERW in an elliptic periodic environment. That is, the environment is defined by parameters $(p_1, \dots, p_M) \in (0,1)^M$ for some $M \in \mathbb{N}$, where the walker upon the ith visit to $z \in \mathbb{Z}$ moves to $z+1$ with probability $p_{i \ (\mathrm{mod}\ M)}$, and moves to $z-1$ with probability $1 - p_{i \ (\mathrm{mod}\ M)}$ (where we identify p_0 with p_M). Our main result is the following.

Theorem 1.2 (Transience criterion for periodic environments). *Let $(p_1, \dots, p_M) \in (0,1)^M$ for some $M \in \mathbb{N}$, and let $\bar{p} = \frac{1}{M}\sum_{i=1}^{M} p_i$ be the average of the p_i's. Let $X = (X_n)_{n\geq 0}$ be an ERW in the periodic environment $\omega(p_1, \dots, p_M)$.*

(1) *If $\bar{p} > \frac{1}{2}$, then $\mathbb{P}$-a.s. $X_n \to +\infty$ as $n \to \infty$.*
(2) *If $\bar{p} < \frac{1}{2}$ then $\mathbb{P}$-a.s. $X_n \to -\infty$ as $n \to \infty$.*
(3) *Suppose that $\bar{p} = \frac{1}{2}$, and let*

$$\theta(p_1, \dots, p_M) = \frac{\sum_{i=1}^{M} \delta_i (1-p_i)}{4\sum_{j=1}^{M} p_j(1-p_j)}, \tag{1}$$

where $\delta_i = \sum_{j=1}^{i}(2p_j - 1)$.

- *If $\theta(p_1, \dots, p_M) > 1$, then $\mathbb{P}$-a.s. $X_n \to +\infty$.*
- *If $\theta(1-p_1, \dots, 1-p_M) > 1$, then $\mathbb{P}$-a.s. $X_n \to -\infty$.*
- *If both $\theta(p_1, \dots, p_M) \leq 1$ and $\theta(1-p_1, \dots, 1-p_M) \leq 1$, then $\mathbb{P}$-a.s. $X_n = 0$ infinitely often.*

It is interesting to compare Theorem 1.2 to results about the case of bounded environment. Recall [9] that in the case of bounded environment the only value that matters is the *total drift*, i.e., the sum $\sum_i (2p_i - 1)$. If the total drift is bigger than 1, then the walk is transient to the right, if it is smaller than -1 then the walk is transient to the left, and if it is in $[-1, 1]$ then the walk is recurrent. For other phase transitions in the total drift see [11]. Comparing to the bounded case, the cases $\bar{p} > \frac{1}{2}$ and $\bar{p} < \frac{1}{2}$ (which corresponds to total drift infinite and negative infinite, respectively) are not surprising. For the case $\bar{p} = \frac{1}{2}$ one could have naively conjectured that it corresponds to total drift 0 and hence should be recurrent. We see that this is not necessarily the case, and further, that the question of recurrence depends also on the *order* of the p_i, a phenomenon which has no obvious analog with a bounded number of cookies.

A less naive but still wrong conjecture would be "what matters is the *average* total drift." For example if we have 10 positive cookies followed by 10 negative cookies the "total drift after n cookies" fluctuates as n changes between a large positive number and 0, so maybe the average should be compared to 1. This turns out to be wrong on two accounts. First one should not take a simple average but a weighted average. And even then, this only explains the numerator in the definition of θ. The denominator has a different origin, which we will explain after a necessary tour of the Kesten–Kozlov–Spitzer approach.

The approach of Kesten, Kozlov, and Spitzer [7] (which may also be referred to Harris [6]) for processes on $\mathbb{Z}$ is to examine the number of times the edge $(n-1,n)$ was crossed up to a certain event – denote it by Z_n – and study it as a process in n. In the original application, random walk in random environment, Z_n behaved like a branching process with different branching rules in different times. The approach was first applied to excited random walk by Basdevant and Singh [2], who studied the case of finitely many cookies, and in that case Z_n turned out to be a branching process *with immigration*. In our case, however, the branching process terminology is not as useful, and it is best to think about the Kesten–Kozlov–Spitzer process as just some Markov process on $\mathbb{N}_0=\{0,1,2,\dots\}$, the set of non-negative integers.

We will use a variation of this approach due to Kosygina and Zerner [9] in which there are two processes, Z_n^+ and Z_n^-. For a given cookie environment ω define a corresponding Markov chain $Z^+=(Z_n^+)_{n\ge 0}$ on $\mathbb{N}_0$ starting at $Z_0^+=1$, where Z_n^+ counts the number of right crossings of the directed edge $(n-1,n)$ of the ERW before hitting -1 for the first time. Then, ERW on ω is transient to the right if and only if Z^+ does not return to zero with positive probability. Similarly, for the Markov chain $Z^-=(Z_n^-)_{n\ge 0}$ on $\mathbb{N}_0$ starting at $Z_0^-=1$, where Z_n^- counts the number of left crossings of the directed edge $(-n+1,-n)$ of the ERW before hitting 1. Then, ERW on ω is transient to the left if and only if Z^- does not return to zero with positive probability. Theorem 1.2 follows from an analysis of these two Markov chains, together with zero–one laws for right/left transience of ERW in identically piled environments. We shall discuss more about Z^+ and Z^-, and their correspondence with ERW in Section 2.

Thus, the question of recurrence/transience of ERW reduces to a question about Markov chains on $\mathbb{N}_0$. Our next step will be to formulate a criterion for transience of Markov chains on $\mathbb{N}_0$ which is suitable for the kind of Markov chains we will encounter.

Let Z be an irreducible discrete time Markov chain on $\mathbb{N}_0$, and let U be its step distribution. That is, for all $n\ge 0$ the distribution of Z_{n+1} conditioned on Z_n is defined as $\mathbb{P}[Z_{n+1}=y\mid Z_n=x]=\mathbb{P}[U(x)=y]$. Assume that the limit $\lim_{x\to\infty}\frac{\mathbb{E}[U(x)]}{x}$ exists, and denote it by μ. Furthermore, assume that U is concentrated around its expectation. That is, there is a constant $c>0$ such that for all $x\in\mathbb{N}_0$ sufficiently large and for all $\varepsilon>0$ it holds that

$$\mathbb{P}\left[\left|\frac{U(x)}{x}-\mu\right|>\varepsilon\right]\le 2\exp\bigl(-c\varepsilon' x\bigr)$$

for some ε' that depends on ε. (See the statement of Theorem 1.3 for the precise assumption.) We now define some quantities associated with U. For each $x\in\mathbb{N}_0$ let

Drift: $\rho(x)=\mathbb{E}[U(x)-\mu x]$.
Diffusion: $\nu(x)=\frac{\mathbb{E}[(U(x)-\mu x)^2]}{x}$.
Ratio: $\theta(x)=\frac{2\rho(x)}{\nu(x)}$.

Note that since Z is irreducible the random variable $U(x)$ is not constant. Therefore, $\nu(x)>0$ for all $x\in\mathbb{N}_0$, and so $\theta(x)$ is well defined. We prove the following theorem.

Theorem 1.3 (Transience criterion for Markov chains on $\mathbb{N}_0$). *Let Z be an irreducible discrete time Markov chain on $\mathbb{N}_0$ as above, and let U be its step distribution. Assume that the limit $\mu=\lim_{x\to\infty}\frac{\mathbb{E}[U(x)]}{x}$ exists, and furthermore that U is concentrated in the sense that there is a constant $c>0$ such that for all $x\in\mathbb{N}_0$ sufficiently large and for all $\varepsilon>0$ it holds that*

$$\mathbb{P}\left[\left|\frac{U(x)}{x}-\mu\right|>\varepsilon\right]\le 2\exp\left(-\frac{c\varepsilon^2}{1+\mu+\varepsilon}x\right).$$

Suppose that $\mu \neq 1$.

- *If* $\mu > 1$, *then* $\mathbb{P}[Z_n > 0 \text{ for all } n] > 0$.
- *If* $\mu < 1$, *then* $\mathbb{P}[Z_n = 0 \text{ for some } n] = 1$.

Suppose that $\mu = 1$.

- *If* $\theta(x) < 1 + \frac{1}{\ln(x)} - \alpha(x) \cdot x^{-1/2}$ *for all sufficiently large* $x \in \mathbb{N}_0$, *where* $\alpha(x)$ *is such that* $\alpha(x)\nu(x) \to +\infty$ *as* $x \to \infty$, *then* $\mathbb{P}[Z_n = 0 \text{ for some } n] = 1$.
- *If* $\theta(x) > 1 + \frac{2}{\ln(x)} + \alpha(x) \cdot x^{-1/2}$ *for all sufficiently large* $x \in \mathbb{N}_0$, *where* $\alpha(x)$ *is such that* $\alpha(x)\nu(x) \to +\infty$ *as* $x \to \infty$, *then* $\mathbb{P}[Z_n > 0 \text{ for all } n] > 0$.

Remark 1.4. *Note that Theorem 1.3 does not cover the cases where* $\theta(x)$ *is between* $1 + \frac{1}{\ln(x)}$ *and* $1 + \frac{2}{\ln(x)}$, *and so it is not applicable for all Markov chains that satisfy* $\theta = \lim_{x\to\infty} \theta(x) = 1$. *Nevertheless, it will be enough for Theorem 1.2 since in the periodic case we have* $|\theta(x) - \theta| \leq C \log^4(x)/\sqrt{x}$, *as well as to reprove known results for bounded cookie environments, positive cookie environments, and branching processes with migration, where in all cases* $|\theta(x) - \theta| = O(x^{-1/2})$.

Remark 1.5. *If* $\liminf_{x\to\infty} \nu(x) > 0$, *then in the case* $\mu = 1$ *in Theorem 1.3 it is enough to assume* $\alpha(x) \to \infty$ *instead of* $\alpha(x)\nu(x) \to \infty$. *In particular, in the applications to Theorem 1.3 in this paper there exists a limit* $\lim_{x\to\infty} \nu(x) = \nu > 0$.

Remark 1.6. *The denominator* $(1 + \mu + \varepsilon)$ *in the concentration assumption is somewhat non-standard. Note however, that for small values of* ε *this is equivalent to the standard assumption* $\mathbb{P}[|\frac{U(x)}{x} - \mu| > \varepsilon] \leq 2\exp(-c\varepsilon^2 x)$, *while for large values of* ε *this is essentially equivalent to* $\mathbb{P}[|\frac{U(x)}{x} - \mu| > \varepsilon] \leq 2\exp(-c\varepsilon x)$.

Theorem 1.3 is quite easy to understand intuitively even in the case that $\mu = 1$. Assume $\theta(x)$ converges to some θ. Then Z is a discrete version of a *Bessel process* in dimension $\theta + 1$ (this connection between excited random walk and Bessel processes has already been noted in [8]).

Similar results were proved by Lamperti [12] and Menshikov, Asymonth and Iasnogorodski [14] in slightly different settings. Fortunately, the concentration assumptions in Theorem 1.3 are sufficient for our purposes, and we prove Theorem 1.3 following the same ideas as in [12] and [14] by using the classic approach of Lyapunov functions. As this is standard, the proof will be given in the Appendix.

We are now in a position to explain Theorem 1.2. We will show below that the θ given by (1) is exactly the θ of Theorem 1.3 when applied to the process Z_n. In fact, the numerator of (1) is 2ρ and the denominator is ν. Theorem 1.3 can also be used to give short proofs of existing results such as in the case of bounded environments studied in [9]. In this case the quantity ρ is exactly the total drift $\sum_i (2p_i - 1)$, and ν is equal to 2. Thus, the appearance of the parameter ν (the denominator in (1)) is another phenomenon of infinite environments, which has no analog in bounded environments. See Section 4 for details.

In order to apply Theorem 1.3 in the case of periodic environment $\omega(p)$ considered in Theorem 1.2 we define the corresponding Markov chain $Z^+ = (Z_n^+)_{n\geq 0}$ as explained above with step distribution U_p. We then do the following.

(1) Formulate the measure of the corresponding step distribution U_p in terms of p.
(2) Prove concentration bounds for U_p.
(3) Calculate the parameters μ and θ as a function of p.
(4) Prove that when $\lim_{x\to\infty} \theta(x) = 1$, the convergence of the ratio $\theta(x)$ is sufficiently fast (that is, faster than $\frac{1}{\ln(x)}$).

1.2. *Structure of the paper*

In Section 2 we define the correspondence between ERW and the Markov chain Z^+ on $\mathbb{N}_0$, and prove some properties of the step distribution U_p defined by the environment p. Theorem 1.2 is proven in Section 3. The proofs in this section include the calculations of μ and θ, and are rather technical. In Section 4 we reprove some existing results on ERW for the case of positive environments [15] and for the case of bounded environments [9], and a result on branching processes with migration. We conclude the paper with some open problems in Section 5. We prove Theorem 1.3 in the Appendix.

1.3. *Basic notations*

Throughout the paper we distinguish between $\mathbb{N} = \{1, 2, \ldots\}$, the set of strictly positive integers and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the set of non-negative integers. For a positive integer $M \in \mathbb{N}$ we denote $[M] = \{1, 2, \ldots, M\}$. For a vector $v = (v_1, v_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, and for $j \in \mathbb{N}$ denote by $s^j(v) = (v_j, v_{j+1}, \ldots) \in \mathbb{R}^{\mathbb{N}}$, that is, $s^j(v)$ is the shift of v by $j - 1$ to the left. Similarly, for a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we denote by $s^j(v)$ its cyclic rotation by $j - 1$ to the left, i.e., $s^j(v) = (v_j, \ldots, v_n, v_1, \ldots, v_{j-1})$.

2. Associating ERW with a Markov chain on $\mathbb{N}_0$

In this section we are setting up the tools needed in order to prove Theorem 1.2. The main idea behind the proof of Theorem 1.2 is to study a different process, which is Markovian, unlike the original ERW. This connection between the Markov chain Z^+ and right transience is well-known, and may be found, for example, in Section 2 of [9]. We describe the correspondence between the two processes here for the reader's convenience.

Definition 2.1. *Fix an elliptic and non-degenerate cookie environment $p = (p_i)_{i\in\mathbb{N}}$. For each $x \geq 0$ define a random variable $U_p(x)$ in the following way. For each $x > 0$ let*

$$U_p(x) = \inf\left\{k \in \mathbb{N}\colon \sum_{i=1}^{k}(1 - B_i) = x\right\} - x,$$

where $B_i \sim B(p_i)$ are independent Bernoulli random variables. In words, $U_p(x)$ counts the number of "successes" in a sequence of Bernoulli trials with probabilities $p_1, p_2, \ldots$ until reaching x "failures."

Finally, define a Markov chain $Z^+ = (Z_n^+)_{n\geq 0}$ on $\mathbb{N}_0$ where $Z_0^+ = 1$, and U_p is its step distribution. That is,

$$Z_0^+ = 1 \quad \textit{and} \quad Z_{n+1}^+ \sim U_p\bigl(Z_n^+\bigr). \tag{2}$$

To ensure that Z be irreducible, set $U_p(0) = 1$.

The basic observation due to Kosygina and Zerner [9] is that if X is an ERW in $\omega(p)$, then on the event $T_{-1} < \infty$ the sequence Z_n^+ has the same distribution as the number of right crossings of the directed edges $(n - 1, n)$ by X until T_{-1}, where $T_{-1} = \inf\{t \geq 0\colon\ X_t = -1\}$ is the hitting time of -1 by X. Moreover, on $T_{-1} = \infty$ the process Z_n^+ stochastically dominates the corresponding number of left crossings. Therefore, we have $Z_n^+ > 0$ for all $n \in \mathbb{N}$ if and only if $T_{-1} = \infty$. Since in this paper we are only interested in environments that are identically piled and elliptic the range of the walk in such environments is a.s. infinite, and hence we have $\mathbb{P}[T_{-1} = \infty] > 0$ if and only if $\mathbb{P}[X_n \to \infty] > 0$. (To see why the range of the walk has to be infinite, note first that if it is non-degenerate then its range is infinite a.s. by the Borel–Cantelli Lemma. Otherwise, since the environment is assumed to be elliptic and identically piled, if $\sum_i p_i < \infty$ it is transient to the left, and if $\sum_i (1 - p_i) < \infty$ it is transient to the right, again using the Borel–Cantelli Lemma.) Therefore, $\mathbb{P}[Z_n^+ > 0 \text{ for all } n] > 0$ if and only if $\mathbb{P}[X_n \to \infty] > 0$. The reader is referred to Section 2 of [1] for a complete argument using the so-called arrow environments.

Analogously, we define the view of ERW "to the left" and associate it with the following Markovian process Z^-. Let q be the cookie environment defined by $q_i = 1 - p_i$ for all $i \in \mathbb{N}$. Define $U_q(x) = \inf\{k \in \mathbb{N}\colon \sum_{i=1}^{k}(1 - B_i') = x\} - x$, where $B_i' \sim B(q_i)$ are independent Bernoulli random variables, and let $Z^- = (Z_n^-)_{n\geq 0}$ be a Markov chain on $\mathbb{N}_0$ defined as

$$Z_0^- = 1 \quad \text{and} \quad Z_{n+1}^- \sim U_q\bigl(Z_n^-\bigr). \tag{3}$$

Symmetrically to Z^+, we have $\mathbb{P}[Z_n^- > 0 \text{ for all } n] > 0$ if and only if $\mathbb{P}[X_n \to -\infty] > 0$.

We will use the following result of Amir et al. from [1] that asserts a zero–one law for directional transience of X.

Theorem 2.2. *Let p be an elliptic cookie environment, and let X be an ERW in $\omega(p)$. Then $\mathbb{P}[X_n \to +\infty], \mathbb{P}[X_n \to -\infty] \in \{0, 1\}$.*

This implies that in order to prove that X is right transient a.s. it is enough to show that $\mathbb{P}[X_n \to +\infty] > 0$. (An analogous equivalence holds also for left transience.) By the discussion above we get the following corollary from Theorem 2.2.

Theorem 2.3. *Let p be an elliptic and non-degenerate cookie environment, and let X be an ERW in $\omega(p)$. Then, the following holds.*

- $\mathbb{P}[Z_n^+ > 0$ *for all* $n] > 0$ *if and only if* $\mathbb{P}[X_n \to +\infty] = 1$.
- $\mathbb{P}[Z_n^- > 0$ *for all* $n] > 0$ *if and only if* $\mathbb{P}[X_n \to -\infty] = 1$.
- $\mathbb{P}[Z_n^+ = 0$ *for some* $n] = \mathbb{P}[Z_n^- = 0$ *for some* $n] = 1$ *if and only if* $\mathbb{P}[X_n = 0$ *i.o.*$] = 1$.

Therefore, in order to prove Theorem 1.2 we need to understand when the Markov chains Z^+ and Z^- have a positive probability to keep above 0 for all $n \geq 0$.

2.1. *Studying the step distribution U_p*

In order to understand the Markov chain Z^+ we analyze its step distribution U_p. Recall that by Theorem 1.3 we need to understand the relevant parameters of U_p, namely μ, $\rho(x)$ and $\nu(x)$. In addition, in order to apply Theorem 1.3 we need to show that $U_p(x)$ is concentrated around μx in the appropriate sense.

We start by computing the expectation parameter μ explicitly, and by showing that $U_p(x)$ is concentrated around its expectation. In order to do so, it will be convenient to define the random variables

$$F_n = \sum_{i=1}^{n} B_i', \tag{4}$$

where $B_i' = 1 - B_i \sim B(q_i)$ are independent Bernoulli random variables with $q_i = 1 - p_i$ for all $i \in \mathbb{N}$. Note that by definition of F_n we have

$$\{F_n < x\} = \{U_p(x) > n - x\} \quad \text{and} \quad \{F_{n-1} \geq x\} = \{U_p(x) < n - x\}. \tag{5}$$

For each $n \in \mathbb{N}$ define

$$\bar{p}_n = \frac{1}{n} \sum_{i=1}^{n} p_i. \tag{6}$$

We claim that for any environment p such that for some real numbers K and $\bar{p}$ it holds that $|\bar{p}_n - \bar{p}| \leq \frac{K}{n}$ for all n, we have $\mu = \frac{\bar{p}}{1-\bar{p}}$, and $\frac{U_p(x)}{x}$ is concentrated around μ. This is proven in the following proposition.

Proposition 2.4 (Concentration bound for U_p). *Let p be a cookie environment. Suppose it satisfies the assumptions as above, namely, the limit $\bar{p} = \lim_{n\to\infty} \bar{p}_n \in (0, 1)$ exists and there is some $K \in \mathbb{R}$ such that $|\bar{p}_n - \bar{p}| \leq \frac{K}{n}$ for all $n \in \mathbb{N}$. Then, for all $\varepsilon > 0$ it holds that*

$$\mathbb{P}\left[\left|\frac{U_p(x)}{x} - \mu\right| > \varepsilon\right] \leq 2\exp\left(-\frac{c\varepsilon^2}{1 + \mu + \varepsilon} x\right),$$

where $\mu = \frac{\bar{p}}{1-\bar{p}}$, and $c > 0$ is some constant that depends only on p.

Note that the bound is interesting only for $\varepsilon > \frac{C}{\sqrt{x}}$ for some $C > 0$ sufficiently large, and so we shall assume that this is indeed the case.

Proof of Proposition 2.4. We rely on the correspondence between $U_p(x)$ and F_n stated in (5), and use the concentration of F_n in order to prove the proposition. By (5) we have

$$\mathbb{P}\left[\left|\frac{U_p(x)}{x}-\mu\right|>\varepsilon\right]\le \mathbb{P}[F_{\lceil(1+\mu+\varepsilon)x\rceil}<x]+\mathbb{P}[F_{\lfloor(1+\mu-\varepsilon)x\rfloor}\ge x].$$

We shall bound each of the two terms using Hoeffding's inequality.

For the first term, define $n=\lceil(1+\mu+\varepsilon)x\rceil$ and note that

$$\begin{aligned}\mathbb{E}[F_n] &= \sum_{i=1}^n q_i = n(1-\bar{p}_n) \overset{(*)}{=} n(1-\bar{p})+\mathrm{O}(1)\\ &\overset{(**)}{=} x\left(1+\frac{\bar{p}}{1-\bar{p}}+\varepsilon\right)(1-\bar{p})+\mathrm{O}(1)=x+x\varepsilon(1-\bar{p})+\mathrm{O}(1),\end{aligned} \tag{7}$$

where in $(*)$ we used the assumption on $|\bar{p}_n-\bar{p}|$, and in $(**)$ the definitions of n and μ. Since F_n is a sum of n independent bounded summands we get from Hoeffding's inequality that

$$\mathbb{P}[F_n<x]\overset{(7)}{\le}\mathbb{P}\big[F_n-\mathbb{E}[F_n]<-c'x\varepsilon\big]\le\exp\big(-c(\varepsilon x)^2/n\big)\le\exp\left(-\frac{c\varepsilon^2}{1+\mu+\varepsilon}x\right)$$

for some constants $c,c'>0$ that depends only on p. The bound for $\mathbb{P}[F_{\lfloor(1+\mu-\varepsilon)x\rfloor}\ge x]$ is similar, and the proposition is proved. □

2.2. *The centralized second moment of $U_p(x)$*

The following theorem gives an explicit formula for the second moment of $U_p(x)-x$. Recall the definition of $\bar{p}_n$ in (6).

Lemma 2.5. *Let p be a cookie environment, and let U_p be the step distribution of the corresponding Markov chain Z^+. Suppose that the limit $\lim_{n\to\infty}\bar{p}_n$ exists and equals to $\frac{1}{2}$. For each $n\in\mathbb{N}$ define*

$$A_n=\frac{1}{n}\sum_{i=1}^n p_i(1-p_i). \tag{8}$$

Suppose that the limit $\lim_{n\to\infty}A_n$ also exists and is strictly positive. Denote this limit by A. Assume further that there is some K such that $|\bar{p}_n-\frac{1}{2}|\le\frac{K}{n}$ and $|A_n-A|\le\frac{K}{n}$ for all $n\in\mathbb{N}$. Then, the limit $\lim_{x\to\infty}\frac{1}{x}\mathbb{E}[(U_p(x)-x)^2]$ exists and is equal to

$$\lim_{x\to\infty}\frac{1}{x}\mathbb{E}\big[\big(U_p(x)-x\big)^2\big]=8A.$$

Moreover, the rate of convergence is bounded by $C\cdot\log^4(x)/\sqrt{x}$, that is, for all $x\in\mathbb{N}_0$ sufficiently large it holds that

$$\frac{1}{x}\cdot\mathbb{E}\big[\big(U_p(x)-x\big)^2\big]=8A+\mathrm{O}\left(\frac{\log^4(x)}{\sqrt{x}}\right),$$

where the constant implicit in the $\mathrm{O}(\cdot)$ notation depends only on p.

Let us sketch the argument before starting the proof proper. We write $\mathbb{E}[(U_p(x)-x)^2]=\sum(2t+1)\mathbb{P}[|U_p(x)-x|>t]$. We then rewrite each term in the sum in the language of F_n using (5). But F_n is just a sum of independent variables, so we can estimate it using the Berry–Esseen theorem. This gives a sum over Φ, the Gaussian cumulative distribution function over a (small perturbation of a) linear progression. We approximate the sum with an integral and the integral may be calculated explicitly.

Thus the proof is quite simple in principle, but there are multiple approximation steps each of which has to be done carefully, and the details will fill the rest of this section.

Proof of Lemma 2.5. We start by writing the expression of $\mathbb{E}[(U_p(x)-x)^2]$ as a sum.

$$\mathbb{E}\big[\big(U_p(x)-x\big)^2\big]=\sum_{t=0}^{\infty}(2t+1)\cdot\mathbb{P}\big[\big|U_p(x)-x\big|>t\big].$$

Note that

$$\sum_{t=0}^{\infty}\mathbb{P}\big[\big|U_p(x)-x\big|>t\big]=\mathbb{E}\big[\big|U_p(x)-x\big|\big],$$

which by Proposition 2.4 is $O(\sqrt{x})$. Therefore, in order to prove the lemma it is enough to show that

$$\left|\frac{1}{x}\sum_{t=0}^{\infty}t\cdot\mathbb{P}\big[\big|U_p(x)-x\big|>t\big]-4A\right|=O\left(\frac{\log^4(x)}{\sqrt{x}}\right).$$

Recall the random variables F_n are defined in (4) as $F_n=\sum_{i=1}^{n}B_i'$, where $B_i'=1-B_i\sim B(q_i)$ are independent Bernoulli random variables, and $q_i=1-p_i$ for all $i\in\mathbb{N}$. Then, using the connection between F and U_p from (5) it is enough to prove that

$$\left|\frac{1}{x}\left(\sum_{t=0}^{\infty}t\cdot\big(\mathbb{P}[F_{2x+t}<x]+\mathbb{P}[F_{2x-t-1}\ge x]\big)\right)-4A\right|=O\left(\frac{\log^4(x)}{\sqrt{x}}\right). \tag{9}$$

We now divide the sum into two parts, the "head" and the "tail." For $x,a\in\mathbb{N}_0$ define

$$H_x(a)=\sum_{t=0}^{\lfloor a\sqrt{x}\rfloor}t\big(\mathbb{P}[F_{2x+t}<x]+\mathbb{P}[F_{2x-t-1}\ge x]\big), \tag{10}$$

$$T_x(a)=\sum_{t=\lfloor a\sqrt{x}\rfloor+1}^{\infty}t\big(\mathbb{P}[F_{2x+t}<x]+\mathbb{P}[F_{2x-t-1}\ge x]\big). \tag{11}$$

We shall take $a=a(x)$ that grows to infinity with x sufficiently slow. The following three claims prove Lemma 2.5.

Claim 2.6. *Let $A,a>0$ and $x\in\mathbb{N}$ be such that $a\le\sqrt{x}$. Then, for $H_x(a)$ as in (10) the following holds.*

$$\left|\frac{1}{x}\cdot H_x(a)-\frac{1}{x}\cdot\sum_{t=0}^{\lfloor a\sqrt{x}\rfloor}2t\cdot\Phi\left(\frac{-t}{\sqrt{8Ax}}\right)\right|\le\frac{Ca^4}{\sqrt{x}}.$$

Here Φ is the cumulative distribution function of the normal distribution. Estimating the sum in Claim 2.6 is a standard exercise in approximating sums by integrals: let us formulate it as a claim.

Claim 2.7. *Let $A>0$ be a constant, and let $a=a(x)$. Then*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{t=0}^{\lfloor a\sqrt{x}\rfloor}2t\cdot\Phi\left(\frac{-t}{\sqrt{8Ax}}\right)=4A.$$

Furthermore, the rate of convergence in at most $O(\frac{a}{\sqrt{x}})$, that is

$$\frac{1}{x}\sum_{t=0}^{\lfloor a\sqrt{x}\rfloor}2t\cdot\Phi\left(\frac{-t}{\sqrt{8Ax}}\right)=4A+O\left(\frac{a}{\sqrt{x}}+\exp(-ca)\right).$$

Finally, for the tail we have the following estimate.

Claim 2.8. *Let $A > 0$ be a constant, and let $a = a(x)$. Then, for all $x \in \mathbb{N}_0$ sufficiently large it holds that*

$$\frac{1}{x} \cdot T_x(a) \le C \exp(-ca).$$

In all three claims the constants c, C and the constants implicit in the $\mathrm{O}(\cdot)$ notation depend on p but are independent of a and x.

The lemma follows by letting $a = K \ln(x)$ for some K sufficiently large and applying the claims. By Claim 2.8 we get that

$$\frac{1}{x} \cdot T_x(a) \ll \frac{1}{\sqrt{x}}.$$

By combining Claims 2.6 and 2.7 we get that

$$\frac{1}{x} \cdot H_x(a) = 4A + \mathrm{O}\left(\frac{\log^4(x)}{\sqrt{x}}\right).$$

This proves (9), which, in turn, concludes the proof of Lemma 2.5. □

Proof of Claim 2.8. By Proposition 2.4 the random variable $U_p(x)$ is concentrated, and hence

$$\begin{aligned} T_x(a) &= \sum_{t=\lfloor a\sqrt{x}\rfloor+1}^{\infty} t \cdot \mathbb{P}\big[\big|U_p(x) - x\big| > t\big] \\ &\le \sum_{i=a}^{\infty} \sum_{t=\lfloor i\sqrt{x}\rfloor+1}^{\lfloor (i+1)\sqrt{x}\rfloor} t \cdot \exp\left(-\frac{ct^2}{2x+t}\right). \end{aligned}$$

The inner sum has at most $\sqrt{x}$ terms, each upper bounded by $4i\sqrt{x} \cdot \exp(-\frac{ci^2x}{2x+i\sqrt{x}}) \le 4i\sqrt{x} \cdot \exp(-c'i)$. Therefore

$$\begin{aligned} T_x(a) &\le \sum_{i=a}^{\infty} \sqrt{x} \cdot 4i\sqrt{x} \cdot \exp(-c'i) \\ &\le 4x \sum_{i=a}^{\infty} i \cdot \exp(-c'i) \le Cx \exp(-c'a). \end{aligned}$$

□

Proof of Claim 2.6. Recall the definition of F_n in (4), and denote $\sigma_i^2 = \mathbb{E}[(B_i' - q_i)^2] = p_i q_i$ and $\rho_i = \mathbb{E}[|B_i - q_i|^3]$ for all $i \in \mathbb{N}$. By the Berry–Esseen theorem [4,5] there exists an absolute constant C_0 so that for all $\alpha \in \mathbb{R}$ it holds that

$$\left|\mathbb{P}\left[\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right] - \Phi(\alpha)\right| \le C_0 \cdot \left(\sum_{i=1}^{n} \sigma_i^2\right)^{-3/2} \cdot \left(\sum_{i=1}^{n} \rho_i\right),$$

where A_n is defined in (8), and Φ is the cumulative distribution function of the normal variable $\mathcal{N}(0,1)$. In our case we have $\rho_i \le 1$ for all $i \in \mathbb{N}$, and hence

$$\left(\sum_{i=1}^{n} \sigma_i^2\right)^{-3/2} \cdot \left(\sum_{i=1}^{n} \rho_i\right) \le (nA_n)^{-3/2} \cdot n \le Cn^{-1/2}.$$

Therefore, for all $\alpha \in \mathbb{R}$ we have

$$\mathbb{P}\left[\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right] = \Phi(\alpha) + \mathrm{O}\big(n^{-1/2}\big), \tag{12}$$

where the constant implicit in the O notation depends only on p_i's.

We next observe that we may replace A_n with A and $\bar{q}_n$ with $\frac{1}{2}$ in (12). This is summarized in the following claim.

Claim 2.9. *For all $n \in \mathbb{N}$ and for all $\alpha \in \mathbb{R}$ such that $|\alpha| \le \sqrt{\frac{n}{2A}}$ it holds that*

$$\mathbb{P}\left[\frac{F_n - n/2}{\sqrt{nA}} \le \alpha\right] = \Phi(\alpha) + \mathrm{O}\big(n^{-1/2}\big),$$

where the constant implicit in the O *notation depends only on p_i's.*

Proof. We show below that

$$\left|\mathbb{P}\left[\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right] - \mathbb{P}\left[\frac{F_n - n/2}{\sqrt{nA}} \le \alpha\right]\right| \le Cn^{-1/2}.$$

This, together with (12) will imply the claim.

$$\begin{aligned}
&\left|\mathbb{P}\left[\frac{F_n - \bar{q}_n n}{\sqrt{nA_n}} \le \alpha\right] - \mathbb{P}\left[\frac{F_n - n/2}{\sqrt{nA}} \le \alpha\right]\right| \\
&\quad = \left|\mathbb{P}\left[\frac{F_n - \bar{q}_n n}{\sqrt{nA_n}} \le \alpha\right] - \mathbb{P}\left[\frac{F_n - \bar{q}_n n}{\sqrt{nA_n}} \le \sqrt{\frac{A}{A_n}} \cdot \alpha + \frac{n(1/2 - \bar{q}_n)}{\sqrt{nA_n}}\right]\right| \\
&\quad \overset{(12)}{\le} \left|\Phi(\alpha) - \Phi\left(\sqrt{\frac{A}{A_n}} \cdot \alpha + \frac{n(1/2 - \bar{q}_n)}{\sqrt{nA_n}}\right)\right| + Cn^{-1/2}.
\end{aligned}$$

Using the fact that the function Φ is $\frac{1}{\sqrt{2\pi}}$-Lipschitz together with the assumptions that $|A_n - A| < \frac{K}{n}$ and $|n(\frac{1}{2} - \bar{q}_n)| \le K$ the difference is bounded by

$$\begin{aligned}
&\left|\mathbb{P}\left[\frac{F_n - \bar{q}_n n}{\sqrt{nA_n}} \le \alpha\right] - \mathbb{P}\left[\frac{F_n - n/2}{\sqrt{nA}} \le \alpha\right]\right| \\
&\quad \le \frac{1}{\sqrt{2\pi}}\left(\left|\sqrt{\frac{A}{A_n}} - 1\right| \cdot \alpha + \frac{K}{\sqrt{nA_n}}\right) + Cn^{-1/2} \\
&\quad \le \frac{1}{\sqrt{2\pi}}\left(\frac{C'K}{nA} \cdot \alpha + \frac{K}{\sqrt{nA - K}}\right) + Cn^{-1/2} \le C''n^{-1/2},
\end{aligned}$$

for some constants $C, C', C'' > 0$. □

We now return to the proof of Claim 2.6. Recall the definition of $H_x(a)$ in (10),

$$H_x(a) = \sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} t\big(\mathbb{P}[F_{2x+t} < x] + \mathbb{P}[F_{2x-t-1} \ge x]\big).$$

Let us rewrite $H_x(a)$ as follows

$$\begin{aligned}
H_x(a) = \sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} t \cdot \Bigg(&\mathbb{P}\left[\frac{F_{2x+t} - (2x+t)/2}{\sqrt{(2x+t)A}} < \frac{-t}{2\sqrt{(2x+t)A}}\right] \\
&+ \mathbb{P}\left[\frac{F_{2x-t-1} - (2x-t-1)/2}{\sqrt{(2x-t-1)A}} \ge \frac{t+1}{2\sqrt{(2x-t-1)A}}\right]\Bigg).
\end{aligned}$$

By Claim 2.9 we have

$$H_x(a) = \sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} t \cdot \left(\Phi\left(\frac{-t}{2\sqrt{(2x+t)A}}\right) + 1 - \Phi\left(\frac{t+1}{2\sqrt{(2x-t-1)A}}\right) + \mathrm{O}\left(x^{-1/2}\right)\right),$$

where the last term $\mathrm{O}(x^{-1/2})$ is the error term from Claim 2.9 applied twice with $n = 2x + t$ and with $n = 2x - t - 1$ for each $t \geq 0$. Therefore, by the assumption that $a \leq \sqrt{x}$, for x sufficiently large the error term $\mathrm{O}(n^{-1/2})$ is bounded by $\mathrm{O}(x^{-1/2})$. Since Φ is $\frac{1}{\sqrt{2\pi}}$-Lipschitz on $\mathbb{R}$, it follows that for $t \geq 0$ we have

$$\left|\Phi\left(\frac{-t}{2\sqrt{(2x+t)A}}\right) - \Phi\left(\frac{-t}{\sqrt{8xA}}\right)\right| \leq \frac{1}{\sqrt{2\pi}}\left|\frac{-t}{2\sqrt{(2x+t)A}} - \frac{-t}{2\sqrt{2xA}}\right| \leq \frac{Ct^2}{x^{3/2}}.$$

An analogous calculation gives

$$\left|\Phi\left(\frac{t+1}{2\sqrt{(2x-t-1)A}}\right) - \Phi\left(\frac{t}{\sqrt{8xA}}\right)\right| \leq \frac{Ct^2}{x^{3/2}}.$$

Therefore,

$$\left|H_x(a) - \sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} t \cdot \left(\Phi\left(\frac{-t}{\sqrt{8Ax}}\right) + 1 - \Phi\left(\frac{t}{\sqrt{8Ax}}\right)\right)\right|$$

$$\leq C \sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} t \cdot \left(\frac{t^2}{x^{3/2}} + x^{-1/2}\right) \leq Ca^4 x^{1/2}.$$

The claim follows from the fact that $\Phi(\alpha) = 1 - \Phi(-\alpha)$ for all $\alpha \in \mathbb{R}$. □

Proof of Claim 2.7. Since the function $f(t) = 2t \cdot \Phi(\frac{-t}{\sqrt{8Ax}})$ is Lipschitz on $\mathbb{R}$ (with a constant independent of x), it follows that

$$\left|\sum_{t=0}^{\lfloor a\sqrt{x}\rfloor} f(t) - \int_0^{a\sqrt{x}} f(t)\,\mathrm{d}t\right| \leq Ca\sqrt{x}.$$

To estimate the integral, do a linear change of variable and get

$$\int_0^{a\sqrt{x}} 2t \cdot \Phi\left(\frac{-t}{\sqrt{8Ax}}\right)\mathrm{d}t = 16Ax \int_{-\sqrt{a/8A}}^{0} u \cdot \Phi(u)\,\mathrm{d}u.$$

Write

$$\int_{-\sqrt{a/8A}}^{0} = \int_{-\infty}^{0} - \int_{-\infty}^{-\sqrt{a/8A}}.$$

The first integral can be show to be equal to $\frac{1}{4}$ with a simple integration by parts. The second can be bounded by $C\exp(-ca)$ since $\Phi(x) \leq C\exp(-cx^2)$. This completes the proof of Claim 2.7 and hence also of that of Lemma 2.5. □

3. Application: The periodic case

In this section we prove Theorem 1.2. Recall that $\bar{p}$ is the average of the p_i, and that θ given in (1) is defined as

$$\theta(p_1, \ldots, p_M) = \frac{\sum_{i=1}^{M} \delta_i(1-p_i)}{4\sum_{j=1}^{M} p_j(1-p_j)}, \tag{13}$$

where $\delta_i = \sum_{j=1}^{i}(2p_j - 1)$. We need to prove that if $\bar{p} \neq \frac{1}{2}$ then the transience of Z depends on whether $\bar{p}$ is smaller or larger than $\frac{1}{2}$, and if $\bar{p} = \frac{1}{2}$ then its transience depends on θ.

In order to prove the theorem let us fix $p = (p_1, \dots, p_M) \in (0,1)^M$, and let $\bar{p} = \frac{1}{M}\sum_{i=1}^{M} p_i$. Let U_p be the step distribution of the Markov chain Z^+ defined by the environment $\omega(p_1, \dots, p_M)$. We wish to apply Theorem 1.3 in order to prove Theorem 1.2. Recall the parameters of U_p considered in Theorem 1.3.

$$\mu = \lim_{x\to\infty} \frac{\mathbb{E}(U_p(x))}{x}, \qquad \rho(x) = \mathbb{E}\big[U_p(x) - \mu x\big],$$
$$\nu(x) = \frac{\mathbb{E}[(U_p(x) - \mu x)^2]}{x}, \qquad \theta(x) = \frac{2\rho(x)}{\nu(x)}.$$

Let ρ, ν, and θ be the corresponding limits whenever they exist. The following proposition supplies the ingredients required for the proof of Theorem 1.2.

Proposition 3.1. *Let $M \in \mathbb{N}$, and let $p = (p_1, \dots, p_M) \in (0,1)^M$ be a periodic cookie environment. Let U_p be the step distribution of the corresponding Markov chain Z^+. Then*

(1) *We have $\mu = \frac{\bar{p}}{1-\bar{p}}$. In particular, $\bar{p} > \frac{1}{2}$ if and only if $\mu > 1$, and $\bar{p} < \frac{1}{2}$ if and only if $\mu < 1$.*
(2) *Suppose that $\bar{p} = \frac{1}{2}$. Then*
   (a) *Let*

$$\rho = \frac{2}{M}\sum_{i=1}^{M}(1 - p_i) \cdot \sum_{j=1}^{j}(2p_i - 1).$$

   Then $|\rho(x) - \rho| \le \exp(-cx)$, where c depends on p but not on x.
   (b) *Let*

$$\nu = 8 \cdot \frac{1}{M}\sum_{i=1}^{M} p_i(1 - p_i).$$

   Then $|\nu(x) - \nu| \le C\log^4(x)/\sqrt{x}$, where C depends on p but not on x.
   (c) *Let*

$$\theta = \frac{2\rho}{\nu}.$$

   Then $|\theta(x) - \theta| \le C\log^4(x)/\sqrt{x}$, where C depends on p but not on x.

Theorem 1.2 is now a simple corollary of Theorem 1.3, Proposition 3.1 and Theorem 2.3. Here are the details:

Proof of Theorem 1.2 given Proposition 3.1. Let $p = (p_1, \dots, p_M) \in (0,1)^M$ be a periodic environment. We shall consider the Markov chains Z^+ and Z^- defined by $\omega(p)$, and the corresponding step distributions U_p and U_q, where $q = (q_i = 1 - p_i)_{i\in\mathbb{N}}$. Recall that by Proposition 2.4 the step distributions $U_p(x)$ and $U_q(x)$ are concentrated, as required in the conditions of Theorem 1.3.

Suppose first that $\bar{p} > \frac{1}{2}$ and consider the step distribution $U_p(x)$ that corresponds to the Markov chain Z^+ defined by $\omega(p)$. Then, by the first item of Proposition 3.1 we have $\mu > 1$, and thus by Theorem 1.3 it holds that Z_n^+ is transient. Therefore, by Theorem 2.3 the ERW in $\omega(p)$ is right transient a.s.

Analogously, if $\bar{p} < \frac{1}{2}$, then if we consider the Markov chain Z^- defined by $\omega(p)$ we get that $\mu < 1$, and thus by Theorem 1.3 it holds that Z_n^- is transient. Therefore, by Theorem 2.3 the ERW in $\omega(p)$ is left transient a.s.

Suppose now that $\bar{p} = \frac{1}{2}$, which corresponds to $\mu = 1$ for both Z^+ and Z^-. Suppose first that $\theta(p_1, \dots, p_M) > 1$. Then, by Theorem 1.3 we have Z_n^+ is transient, and thus, by Theorem 2.3 the ERW in $\omega(p)$ is right transient a.s.

Analogously, if $\bar{p} = \frac{1}{2}$ and $\theta(1 - p_1, \dots, 1 - p_M) > 1$. Then, by Theorem 1.3 we have Z_n^- is transient, and thus, by Theorem 2.3 the ERW in $\omega(p)$ is left transient a.s.

Finally, if both $\theta(p_1, \dots, p_M) \le 1$ and $\theta(1 - p_1, \dots, 1 - p_M) \le 1$, then by Theorem 1.3 we have $\mathbb{P}[Z_n^+ = 0 \text{ for some } n] = 1$ and $\mathbb{P}[Z_n^- = 0 \text{ for some } n] = 1$. Therefore, by Theorem 2.3 the ERW in $\omega(p)$ is recurrent a.s. □

We now turn to the proof of Proposition 3.1. Let U_p be the step distribution of Z^+ defined by the periodic environment $\omega(p_1, \dots, p_M)$. Recall (Definition 2.1 on page 1027) that $U_p(x)$ is the number of successes in a sequence of Bernoulli trials with periodic parameters prior to x failures. Suppose we had already counted how many successes we had up to the first i failures and we wish to proceed to $i + 1$. Because the cookies p_i are periodic, we do not need to remember our exact "position" in the pile of cookies, but only its value modulo M. These values form a *Markov chain* with M states, with i being the time. Thus, we arrived at a description of U_p in terms of two sequence: the Markov chain of the values modulo M (which we will denote by R_i) and the number of failures at the ith step (which we will denote by g_i). Here is a more formal description.

Definition 3.2. *For a periodic cookie environment $p \in [0, 1]^{\mathbb{N}}$, and for $j \in [M]$ let $U^{(j)} = U_{s^j(p)}(1)$ be the number of successes in a sequence of Bernoulli trials with probabilities $p_j, p_{j+1}, \dots$ until the first failure.*[4] *Define two sequences $(R_i \in [M])_{i\ge 0}$ and $(g_i \in \mathbb{N}_0)_{i\ge 0}$ as follows. We start with $R_0 = 1$ and g_0 distributed as $U^{(1)}$. Inductively, for each $i \in \mathbb{N}$ define $R_i = R_{i-1} + g_{i-1} + 1 \pmod M$, and define g_i to be distributed as $U^{(R_i)}$. Other than the dependency on R_i, the random variable g_i is independent of all previous $\{g_j, R_j\colon j < i\}$. Informally speaking, R_i represents the location $\pmod M$ of the next available cookie after the ith failure, and g_i represents the number of successes between the ith and the $(i + 1)$st failure.*

We show below that $\{R_i\colon i \ge 0\}$ is a Markov chain, and that $U_p(x) = \sum_{i=0}^{x-1} g_i$.

Claim 3.3. *Let $(R_i)_{i\ge 0}$ and $(g_i)_{i\ge 0}$ be as above. Then*

(1) *$\sum_{i=0}^{x-1} g_i$ is distributed according to $U_p(x)$.*
(2) *$(R_i)_{i\ge 0}$ is a Markov chain on $[M]$ with transition matrix $P = (P_{j,k})_{j,k\in[M]}$ given by*

$$P_{j,k} = \mathbb{P}[R_i = k | R_{i-1} = j] = \frac{a_{j,k}(1 - p_{k-1})}{1 - p_1 \cdot p_2 \cdots p_M},$$

where

$$a_{j,k} := \begin{cases} 1 & j = k - 1, \\ p_j p_{j+1} \cdots p_{k-2} & \text{otherwise.} \end{cases}$$

In particular, since $p_i \in (0, 1)$ for all $i \in [M]$, the Markov chain $(R_i)_{i\ge 0}$ is irreducible and aperiodic, and, therefore, has a unique stationary distribution $\pi = (\pi_1, \dots, \pi_M)$.

Here and below expressions such as $p_j \cdots p_{k-2}$ should be read "cyclically" i.e. $p_j \cdots p_{k-2}$ if $j \le k - 2$ and $p_j \cdots p_M p_1 \cdots p_{k-2}$ otherwise. The product always contains between 1 and M terms.

Proof of Claim 3.3. Recall that $U_p(x) = \inf\{k \in \mathbb{N}\colon \sum_{i=1}^{k}(1 - B_i) = x\} - x$, where $B_i = B(p_i)$ are independent Bernoulli random variables. Note that g_0 counts the number of successes until the first failure. Hence, the $(g_0 + 1)$st Bernoulli trial is a failure, and g_1 starts counting successes until the next failure, starting from p_j, where $j = R_0 + g_0 + 1$. The process continues until reaching x failures, and $\sum_{i=0}^{x-1} g_i$ counts the number of successes until then.

For the second item, the fact that $(R_i)_{i\ge 0}$ is a Markov chain follows from the definition of R, since the next step R_i depends only on R_{i-1}, as g_{i-1} is defined by independent Bernoulli trials.

[4] Recall that $s^j(p) = (p_j, p_{j+1}, \dots, p_{j-1})$ is the left shift by $j - 1$ of the environment p.

Finally we show the formula for $P_{j,k}$. We write the event "the first failure is when $i \equiv k-1 \pmod M$" as a sum of the probabilities that the first failure is at $k-1+tM$ for some $t \in \{0,1,\dots\}$ (the case $t=0$ is irrelevant if $j > k-1$). We get

$$\begin{aligned} P_{j,k} &= \sum_t p_j \cdot p_{j+1} \cdots p_{k-2+tM} \cdot (1 - p_{k-1+tM}) \\ &= (1 - p_{k-1}) \sum_{t=0}^{\infty} a_{j,k} \cdot (p_1 \cdots p_M)^t \\ &= \frac{a_{j,k} \cdot (1 - p_{k-1})}{1 - p_1 \cdots p_M}. \end{aligned}$$

□

3.1. *Calculating* μ

We are now ready to prove the first item of Proposition 3.1.

Lemma 3.4 (Calculating μ). *Let $(p_1,\dots,p_M) \in (0,1)^M$, and let U_p be the corresponding step distribution in the environment $\omega(p_1,\dots,p_M)$. Let $\mu = \lim_{x\to\infty} \frac{\mathbb{E}[U_p(x)]}{x}$. Then $\mu = \frac{\bar{p}}{1-\bar{p}}$, where $\bar{p} = \frac{1}{M}\sum_{i=1}^M p_i$.*

Proof. Recall Definition 3.2 on page 1035, where $U^{(j)}$, R_j and g_j are defined. Define an M-dimensional vector $E = (\mathbb{E}[U^{(1)}],\dots,\mathbb{E}[U^{(M)}]) \in \mathbb{R}^M$. We claim that

$$\mu = \langle \pi, E \rangle = \sum_{j=1}^M \pi_j \cdot \mathbb{E}\big[U^{(j)}\big]. \tag{14}$$

Indeed, by definition of μ we have

$$\mu = \lim_{x\to\infty} \frac{\mathbb{E}[U_p(x)]}{x} = \lim_{x\to\infty} \frac{\sum_{i=1}^x \mathbb{E}[g_i]}{x} = \lim_{x\to\infty} \frac{\sum_{i=1}^x \mathbb{E}[\mathbb{E}[g_i|R_i]]}{x}.$$

Now, since $(R_i)_{i\ge 0}$ is an irreducible and aperiodic Markov chain, it converges to a unique stationary distribution π, and therefore as i grows to infinity the expectation $\mathbb{E}[\mathbb{E}[g_i|R_i]]$ converges to $\sum_{j=1}^M \pi_j \cdot \mathbb{E}[g_i|R_i = j]$, which is equal to $\sum_{j=1}^M \pi_j \cdot \mathbb{E}[U^{(j)}] = \langle \pi, E\rangle$. The following two claims provide the calculations of π and E.

Claim 3.5. *The unique stationary distribution π of the Markov chain $(R_i)_{i\ge 0}$ is given by*

$$\pi_j = \frac{1 - p_{j-1}}{\sum_{k=1}^M (1 - p_k)}, \quad j = 1,\dots,M,$$

where we identify p_0 with p_M.

Claim 3.6. *For each $j \in [M]$ the expectation $\mathbb{E}[U^{(j)}]$ is equal to*

$$\mathbb{E}\big[U^{(j)}\big] = \frac{\sum_{k=j}^{j-1} p_j \cdots p_k}{1 - p_1 \cdots p_M},$$

where the product $p_j \cdots p_k$ is cyclic for $j > k$.

The calculation of $\mu = \langle \pi, E\rangle$ is a straightforward application of the claims (the sum over k in the formula for $\mathbb{E}[U^{(j)}]$ cancels telescopically after multiplication with the terms $1 - p_{j-1}$ in π and summing over j). We omit the tedious details. □

Proof of Claim 3.5. We show that $(\pi P)_\ell = \pi_\ell$ for all $\ell \in [M]$, where the matrix $P = (P_{j,k})_{j,k\in[M]}$ with

$$P_{j,k} = \frac{a_{j,k}(1-p_{k-1})}{1-p_1p_2\cdots p_M}$$

is given by Claim 3.3. Computing $(\pi P)_\ell$ we have

$$\begin{aligned}(\pi P)_\ell &= \sum_{j=1}^{M} \pi_j P_{j,\ell} \\ &= \sum_{j=1}^{M} \frac{1-p_{j-1}}{\sum_{k=1}^{M}(1-p_k)} \cdot \frac{a_{j,\ell}(1-p_{\ell-1})}{1-p_1p_2\cdots p_M} \\ &= \frac{1-p_{\ell-1}}{\sum_{k=1}^{M}(1-p_k)} \cdot \frac{\sum_{j=1}^{M}(1-p_{j-1})\cdot a_{j,\ell}}{1-p_1p_2\cdots p_M}.\end{aligned}$$

Recalling the definition of $a_{j,\ell}$ we see that the sum in the numerator of the second term cancels telescopically, leaving $1-p_1p_2\cdots p_M$. Therefore $(\pi P)_\ell = \pi_\ell$ for all $\ell \in [M]$, and the claim follows. □

Proof of Claim 3.6. By symmetry it is enough to calculate $\mathbb{E}[U^{(1)}]$. For convenience write $a_j = p_1\cdots p_j$ for $j\in[M]$, $a_0 = 1$.

$$\begin{aligned}\mathbb{E}\big[U^{(1)}\big] &= \sum_{\ell=0}^{\infty} \ell\cdot \mathbb{P}\big[U^{(1)}=\ell\big] \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{M-1}(kM+j)\cdot\mathbb{P}\big[U^{(1)}=kM+j\big] \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{M-1}(kM+j)\cdot(a_M)^k\cdot a_j\cdot(1-p_{j+1}) \\ &= M\cdot\left(\sum_{k=0}^{\infty}k\cdot(a_M)^k\cdot\sum_{j=0}^{M-1}a_j\cdot(1-p_{j+1})\right)+\left(\sum_{k=0}^{\infty}(a_M)^k\cdot\sum_{j=0}^{M-1}j\cdot a_j\cdot(1-p_{j+1})\right) \\ &= M\cdot\left(\sum_{k=0}^{\infty}k(a_M)^k\right)\cdot\left(\sum_{j=0}^{M-1}a_j(1-p_{j+1})\right)+\left(\sum_{k=0}^{\infty}(a_M)^k\right)\cdot\left(\sum_{j=0}^{M-1}ja_j(1-p_{j+1})\right) \\ &= M\cdot\frac{a_M}{(1-a_M)^2}\cdot(1-a_M)+\frac{1}{1-a_M}\cdot\left(\sum_{j=1}^{M}a_j - Ma_M\right) \\ &= \frac{1}{1-a_M}\sum_{j=1}^{M}a_j,\end{aligned}$$

as required. □

3.2. *Calculating* ρ

In this section we compute ρ in the case $\bar{p}=\frac{1}{2}$. Recall that by Lemma 3.4 this implies that $\mu=1$.

Lemma 3.7. *Let* $p=(p_1,\dots,p_M)\in(0,1)^M$ *be a periodic environment with* $\bar{p}=\frac{1}{2}$. *Let*

$$\rho=\frac{2}{M}\sum_{i=1}^{M}(1-p_i)\cdot\sum_{j=1}^{i}(2p_j-1).$$

Then $\lim_{x\to\infty}\rho(x)=\rho$. *Furthermore for all* $x\in\mathbb{N}_0$ *we have* $|\rho(x)-\rho|\le\exp(-Cx)$ *for some constant* C *that depends on* p, *but not on* x.

Proof. We first prove that the limit $\lim_{x\to\infty}\rho(x)$ exists. Using the notations $U^{(j)}$, R_j and g_j (see Definition 3.2) we have

$$\begin{aligned}
\rho(x) &= \left(\sum_{i=0}^{x-1}\mathbb{E}[g_i]\right)-\mu x\\
&= \sum_{i=0}^{x-1}\bigl(\mathbb{E}\bigl[\mathbb{E}\bigl[U^{(R_i)}|R_i\bigr]\bigr]-\mu\bigr)\\
&\overset{(14)}{=} \sum_{i=0}^{x-1}\sum_{j=1}^{M}\mathbb{P}[R_i=j]\cdot\mathbb{E}\bigl[U^{(j)}\bigr]-\pi_j\cdot\mathbb{E}\bigl[U^{(j)}\bigr]\\
&= \sum_{i=0}^{x-1}\sum_{j=1}^{M}\bigl(\mathbb{P}[R_i=j]-\pi_j\bigr)\cdot\mathbb{E}\bigl[U^{(j)}\bigr]\\
&= \sum_{j=1}^{M}\mathbb{E}\bigl[U^{(j)}\bigr]\cdot\sum_{i=1}^{x}\bigl(\mathbb{P}[R_i=j]-\pi_j\bigr).
\end{aligned}$$

Since $(R_i)_{i\ge 0}$ is irreducible and aperiodic, it converges exponentially fast to the stationary distribution, that is, there is some $c\in\mathbb{R}$ and some $\alpha\in(0,1)$ such that $|\mathbb{P}[R_i=j]-\pi_j|\le c\cdot\alpha^i$ for all $i\in\mathbb{N}$ and for all $j\in[M]$ (see, e.g., Theorem 4.9 in [13]). It now follows that $\rho(x)$ converges, and if we denote its limit by ρ, then $|\rho(x)-\rho|\le\exp(-Cx)$ for some constant C that does not depend on x.

Next, we turn to computing the limit ρ explicitly. For every $j=1,\dots,M$, define $\rho^{(j)}$ to be the value of ρ which corresponds to the environment $s^j(p)=(p_j,p_{j+1},\dots,p_{j-1})$. We are interested in $\rho^{(1)}$, and our approach will be to find M independent linear relations between the variables $\rho^{(j)}$. We will also need the notations $\rho^{(j)}(x)$ and $U^{(j)}(x)$ which are $\rho(x)$ and $U(x)$ with respect to the environment $s^j(p)$.

Step 1. We first extract $M-1$ relations between the $\rho^{(j)}$ as follows. Since $U^{(j)}$ counts successes, examine the very first cookie and divide according to whether is was a success or failure. We get the following equality

$$\mathbb{E}\bigl[U^{(j)}(x)\bigr]=p_j\cdot\bigl(1+\mathbb{E}\bigl[U^{(j+1)}(x)\bigr]\bigr)+(1-p_j)\cdot\mathbb{E}\bigl[U^{(j+1)}(x-1)\bigr].$$

Subtracting μx from both sides of the equality we get

$$\begin{aligned}
\rho^{(j)}(x)&=p_j\cdot\bigl(1+\mathbb{E}\bigl[U^{(j+1)}(x)\bigr]-\mu x\bigr)+(1-p_j)\cdot\bigl(\mathbb{E}\bigl[U^{(j+1)}(x-1)\bigr]-\mu x\bigr)\\
&=p_j\cdot\bigl(1+\rho^{(j+1)}(x)\bigr)+(1-p_j)\cdot\bigl(\rho^{(j+1)}(x-1)-\mu\bigr).
\end{aligned}$$

Taking $x \to \infty$ we get

$$\rho^{(j)} = \rho^{(j+1)} + p_j - (1 - p_j) \cdot \mu. \tag{15}$$

Recall that we assume that $\bar{p} = \frac{1}{2}$. Therefore, by Lemma 3.4 if follows that $\mu = 1$. Hence, (15) gives us the constraints $\rho^{(j+1)} = \rho^{(j)} + 1 - 2p_j$. Summing from 1 to $j - 1$ we obtain

$$\rho^{(j)} = \rho^{(1)} + \sum_{k=1}^{j-1}(1 - 2p_k) \quad \text{for all } j \in [M]. \tag{16}$$

These are our first $M - 1$ relations.

Step 2. The remaining relation will be extracted from the stationarity of π. If we start with $j \in [M]$ distributed according to π, and then wait until the first failure we get again j distributed like π. This means that we can write

$$\sum_{j=1}^{M} \pi_j \mathbb{E}\big[U^{(j)}(x)\big] = x \sum_{j=1}^{M} \pi_j \mathbb{E}\big[U^{(j)}(1)\big]. \tag{17}$$

For $\rho(x)$ this gives

$$\begin{aligned} \sum_{j=1}^{M} \pi_j \cdot \rho^{(j)}(x) &= \sum_{j=1}^{M} \pi_j \cdot \big(\mathbb{E}\big[U^{(j)}(x)\big] - \mu x\big) \\ &\overset{(17)}{=} x \sum_{j=1}^{M} \pi_j \big(\mathbb{E}\big[U^{(j)}(1)\big] - \mu\big) \\ &\overset{(14)}{=} 0. \end{aligned}$$

Passing to the limit as x goes to infinity we get

$$\sum_{j=1}^{M} \pi_j \cdot \rho^{(j)} = 0.$$

By Claim 3.5 we have $\pi_j = (1 - p_{j-1}) / \sum_{k=1}^{M}(1 - p_k)$. Plugging this in the equation above, and simplifying it we get

$$\sum_{j=1}^{M}(1 - p_{j-1}) \cdot \rho^{(j)} = 0.$$

Substituting ρ_j with its values in (16) we get

$$\sum_{j=1}^{M}(1 - p_{j-1}) \cdot \left(\rho^{(1)} - \sum_{k=1}^{j-1}(2p_k - 1)\right) = 0.$$

Isolating the variable $\rho = \rho^{(1)}$ we finally obtain the desired formula.

$$\rho \cdot \sum_{j=1}^{M}(1 - p_j) = \sum_{j=1}^{M}\left[(1 - p_{j-1}) \cdot \sum_{k=1}^{j-1}(2p_k - 1)\right].$$

By the assumption $\bar{p}=\frac{1}{2}$ we have $\sum_{j=1}^{M}(1-p_j)=\frac{M}{2}$ and $\sum_{j=1}^{M}(2p_j-1)=0$. Hence,

$$\rho=\frac{2}{M}\sum_{j=1}^{M}\left[(1-p_j)\cdot\sum_{k=1}^{j}(2p_k-1)\right].$$

This completes the proof of Lemma 3.7. □

We finally prove Proposition 3.1.

Proof of Proposition 3.1. Lemma 3.4 proves item (1) of the proposition. Item (2a) is proven in Lemma 3.7. In order to prove item (2b) note first that $|\bar{p}_n-\frac{1}{2}|\le\frac{M}{n}$ and $A_n=\frac{1}{n}\sum_{i=1}^{n}p_i(1-p_i)\to\frac{1}{M}\sum_{i=1}^{M}p_i(1-p_i)=:A$ as $n\to\infty$. Moreover, $A>0$ as $p_i\in(0,1)$ and $|A_n-A|<\frac{M/4}{n}$. Therefore, Item (2b) is a direct application of Lemma 2.5. Item (2c) now follows from items (2a) and (2b) using the fact that $A_n\to\frac{1}{M}\sum_{i=1}^{M}p_i(1-p_i)=\nu/8>0$ and the triangle inequality. □

3.3. *A concrete example of a periodic environment*

In this section we provide a concrete example of a periodic environment. Let $M\in\mathbb{N}$ be an even integer, and let $p\in(0,1)$ be a parameter. Define a periodic environment $\omega(p,M)$ with first $M/2$ cookies being p, and the last $M/2$ cookies being $1-p$. The average of the cookies in a period is equal to $\frac{1}{2}$ and hence $\mu=1$. By Lemma 3.7 we have

$$\rho=\frac{2}{M}\sum_{i=1}^{M}(1-p_i)\cdot\sum_{j=1}^{i}(2p_j-1).$$

A tedious calculation gives

$$\rho=(2p-1)\frac{M}{4}-\frac{(2p-1)^2}{2}.$$

By Lemma 2.5 we have

$$\nu=\frac{8}{M}\sum_{i=1}^{M}p_i(1-p_i)=8p(1-p),$$

and hence,

$$\theta=\frac{2\rho}{\nu}=\frac{(M/2-(2p-1))\cdot(2p-1)}{8p(1-p)}.$$

Therefore, by Theorem 1.2 we have the following corollary.

Corollary 3.8. *Let $p\in(\frac{1}{2},1)$, and let M be an even positive integer. Define a periodic environment $\omega(p,M)$ with first $M/2$ cookies having probabilities p, and the last $M/2$ cookies having probabilities $1-p$. Then, ERW in $\omega(p,M)$ is right transient if and only if $M>\frac{8p-8p^2+2}{2p-1}$, and is recurrent otherwise.*

In particular for $M=2$ ERW in the periodic environment $\omega(p,1-p)$ is a.s. recurrent for all $p\in(0,1)$.

4. More applications: Reproving known results

In this section we show how to use Theorem 1.3 in order to reprove transience criterion for several known cases of ERW in identically piled environments. We shall assume that the discussed environments p are always non-degenerate. In the case that the environment is degenerate, then we must have that either $p_i\to 0$ or $p_i\to 1$, which clearly imply transience. For example, assume that $p_i\to 1$. Then, transience can be proven by coupling the Kesten–Kozlov–Spitzer process Z^+ with the corresponding process in bounded environment p' defined by $p'_i:=p_i$ for all $i \le M$ and $p_i' = \frac{1}{2}$ for all $i > M$ for M sufficiently large to make sure that $p_i > \frac{1}{2}$ for all $i > M$ and also $\sum_{i=1}^{M}(2p_i - 1) > 1$ (we will explain this coupling in detail below, in the proof of Claim 4.6). Transience in such environment follows from Theorem 4.1.

4.1. *ERW in bounded environments*

In this section we reprove the following theorem of Kosygina and Zerner from [9] (the original proof applies in the more general setting of random environments).

Theorem 4.1 (Kosygina–Zerner [9]). *Let $p = (p_i)_{i\in\mathbb{N}}$ be an elliptic bounded cookie environment. That is, $p_i \in (0,1)$ for all $i \in \mathbb{N}$, and there is some $M \in \mathbb{N}$ such that $p_i = \frac{1}{2}$ for all $i > M$. Let*

$$\delta = \sum_{i=1}^{M}(2p_i - 1).$$

Let $X = (X_n)_{n\ge 0}$ be an ERW in $\omega(p)$. Then

(1) *If $\delta > 1$ then $X_n \to +\infty$ a.s.*
(2) *If $\delta < -1$ then $X_n \to -\infty$ a.s.*
(3) *If $-1 \le \delta \le 1$, then $X_n = 0$ i.o. a.s.*

Proof. Consider the step distribution U_p of the Markov chain Z^+ defined by the environment p. We start the proof by computing the expectation $\mathbb{E}[U_p(x)]$ for all $x > M$. Let L be the number of failures in the first M Bernoulli trials. Then

$$\mathbb{E}\big[U_p(x)\big] = M - \mathbb{E}[L] + \mathbb{E}\left[\mathbb{E}\left[NB\left(x - L, \frac{1}{2}\right)\Big| L\right]\right], \tag{18}$$

where $NB(x - L, \frac{1}{2})$ is the negative binomial distribution. Indeed, the last term is $\mathbb{E}[\mathbb{E}[NB(x - L, \frac{1}{2})|L]]$ due to the assumption that there are at most M biased cookies. Thus, after M Bernoulli trials the rest are just $B(p_i = \frac{1}{2})$ for all $i > M$, and we count the number of successes in unbiased Bernoulli trials until reaching additional $x - L$ failures.

By definition $\mathbb{E}[L]$ is equal to

$$\mathbb{E}[L] = \sum_{i=1}^{M}(1 - p_i) = M - \sum_{i=1}^{M} p_i = \frac{M}{2} - \frac{\delta}{2}.$$

The last term in (18) is equal to

$$\mathbb{E}\left[\mathbb{E}\left[NB\left(x - L, \frac{1}{2}\right)\Big| L\right]\right] = \mathbb{E}[x - L] = x - \mathbb{E}[L].$$

Therefore, for $x > M$ we have

$$\mathbb{E}\big[U_p(x)\big] = x + M - 2\mathbb{E}[L] = x + \delta. \tag{19}$$

That is, in the setting of Theorem 1.3 the parameters μ and $\rho(x)$ for $U_p(x)$ are

$$\mu = 1, \tag{20}$$

$$\rho(x) = \delta \quad \text{for all } x > M. \tag{21}$$

In order to compute $\nu(x)$ we assume again that $x > M$ and compute $\mathbb{E}[(U_p(x) - x)^2]$. Note that for $x > M$ we can write $U_p(x) = U_p(M) + NB(x - M, \frac{1}{2})$, where the two summands are independent. Therefore, if we let $c =$

$\mathbb{E}[(U_p(M)-M)^2]<\infty$, then

$$\mathbb{E}\big[\big(U_p(x)-x\big)^2\big]=\mathbb{E}\bigg[\bigg(\big(U_p(M)-M\big)+\bigg(NB\bigg(x-M,\frac{1}{2}\bigg)-(x-M)\bigg)\bigg)^2\bigg]$$
$$=\mathbb{E}\big[\big(U_p(M)-M\big)^2\big]+\mathbb{E}\bigg[\bigg(NB\bigg(x-M,\frac{1}{2}\bigg)-(x-M)\bigg)^2\bigg]$$
$$=c+2(x-M),$$

where the second equality is by independence of $U_p(M)$ and $NB(x-M,\frac{1}{2})$. This gives us that

$$\nu(x)=\frac{\mathbb{E}[(U_p(x)-x)^2]}{x}=2+\frac{c-2M}{x}=2+\mathrm{O}\bigg(\frac{1}{x}\bigg). \tag{22}$$

Using (21) and (22) we get that for all $x>M$ it holds that

$$\theta(x)=\frac{2\rho(x)}{\nu(x)}=\delta+\mathrm{O}\bigg(\frac{1}{x}\bigg). \tag{23}$$

Next we apply Theorem 1.3 on Z^+. Recall that by Proposition 2.4 the step distributions $U_p(x)$ is concentrated, as required in the conditions of Theorem 1.3. By applying Theorem 1.3 we conclude that the Markov chain Z^+ that corresponds to ERW in $\omega(p)$ is transient if and only if $\delta>1$. Therefore, by Theorem 2.3 ERW in $\omega(p)$ is right transient a.s. if and only if $\delta>1$.

Analogously, the Markov chain Z^- that corresponds to ERW in $\omega(p)$ is transient if and only if $\delta<-1$, and hence ERW in $\omega(p)$ is left transient a.s. if and only if $\delta<-1$.

Lastly, if $\delta\in[-1,1]$, then both Z^+ and Z^- are a.s. recurrent, and thus ERW in $\omega(p)$ visits the origin i.o. a.s. □

4.2. *ERW in positive environments*

In this section we assume that our cookie environments p are positive, that is $p_i\geq\frac{1}{2}$ for all $i\in\mathbb{N}$, and reprove the following theorem of Zerner [15] (the original proof applies in the more general setting of random environments).

Theorem 4.2 (Zerner [15]). *Let $p=(p_i)_{i\in\mathbb{N}}$ be an elliptic and positive cookie environment, and let*

$$\delta=\sum_{i=1}^{\infty}(2p_i-1).$$

Let $X=(X_n)_{n\geq 0}$ be an ERW in $\omega(p)$. Then

(1) *If $\delta>1$ then $X_n\to+\infty$ a.s.*
(2) *Otherwise $X_n=0$ i.o. a.s.*

Proof. Note first that if $\delta=\infty$, then the walk is right transient. This can be shown by coupling the Kesten–Kozlov–Spitzer process Z^+ with a corresponding process in bounded environment as explained in Claim 4.6. Actually this coupling can be done for all $\delta>1$ and it is left to prove the recurrence part. However, we prefer here to show how to deduce it from Theorem 2.3. Suppose now that $\delta<\infty$. We prove the theorem by considering the step distribution $U_p(x)$ of the corresponding Markov chain Z^+, and computing the corresponding parameters μ and θ.

Lemma 4.3. *Let p be a positive and elliptic cookie environment. Suppose that $\delta=\sum_{i=1}^{\infty}(2p_i-1)<\infty$. Let U_p be the step distribution of the corresponding Markov chain Z^+. Then $\lim_{x\to\infty}\rho(x)=\lim_{x\to\infty}(\mathbb{E}[U_p(x)]-x)=\delta$. Furthermore, $\rho(x)\leq\delta$ for all $x\geq 0$.*

Lemma 4.4. *Let p be a positive and elliptic cookie environment. Suppose that $\delta<\infty$. Let U_p be the step distribution of the corresponding Markov chain Z^+. Then $\nu(x)=\frac{1}{x}\mathbb{E}[(U_p(x)-x)^2]\to 2$. Furthermore, for all $x\in\mathbb{N}_0$ sufficiently large we have $|\nu(x)-2|\leq C\log^4(x)/\sqrt{x}$ for some constant $C\in\mathbb{R}$ that depends only on p.*

The following corollary is immediate from Lemmas 4.3 and 4.4.

Corollary 4.5. *Let p be a positive and elliptic cookie environment. Suppose that $\delta < \infty$. Let U_p be the step distribution of the corresponding Markov chain Z^+. Then*

(1) $\mu = \lim_{x\to\infty} \frac{\mathbb{E}[U_p(x)]}{x} = 1$.
(2) $\lim_{x\to\infty} \theta(x) = \lim_{x\to\infty} \frac{2\rho(x)}{\nu(x)} = \delta$.
(3) *For all $x \in \mathbb{N}_0$ sufficiently large we have $\theta(x) \le \delta + C \cdot \log^4(x)/\sqrt{x}$ for some constant $C \in \mathbb{R}$ that depends only on p.*

Theorem 4.2 follows by applying Theorem 1.3 with the parameters given in Corollary 4.5, together with Theorem 2.3. Consider the step distribution U_p of the Markov chain Z^+ defined by p, and recall that by Proposition 2.4 we have concentration of $\frac{U_p(x)}{x}$ around μ as required in the conditions of Theorem 1.3. By the first item of Corollary 4.5 we have that $\mu = 1$.

If $\delta > 1$, then by the second item of Corollary 4.5 we have $\lim_{x\to\infty} \theta(x) = \delta > 1$, and thus, by Theorem 1.3 Z^+ is transient a.s. Therefore, by Theorem 2.3 ERW in $\omega(p)$ is right transient a.s.

Suppose now that $\delta \le 1$. Then, by the second and the third items of Corollary 4.5 we have $\theta(x) \le 1 + O(\log^4(x)/\sqrt{x})$ for all $x \in \mathbb{N}_0$ sufficiently large, and thus by Theorem 1.3 Z^+ is recurrent a.s. Therefore, by Theorem 2.3 ERW in $\omega(p)$ is not right transient a.s. In order to see that ERW in $\omega(p)$ cannot be left transient either we can couple the Markov chain Z^- with the one defined by a simple random walk on $\mathbb{Z}$. Therefore, if $\delta \le 1$, then ERW on $\omega(p)$ returns to the origin i.o. a.s. □

We now turn to prove Lemmas 4.3 and 4.4.

4.2.1. *Proof of Lemma* 4.3
We start with the first part of the lemma.

Claim 4.6. *Let p be a positive and elliptic cookie environment, and let $\delta < \infty$. Then $\lim_{x\to\infty} \rho(x) = \lim_{x\to\infty}(\mathbb{E}[U_p(x)] - x) = \delta$.*

Proof. Fix $\varepsilon > 0$ sufficiently small. We claim that there is some $M \in \mathbb{N}$ large enough so that $|\mathbb{E}[U_p(x) - x] - \delta| \le \varepsilon$ for all $x \ge M$.

Let M be sufficiently large so that $\sum_{i=M}^{\infty}(2p_i - 1) < \frac{\varepsilon}{2}$. For $x > M$ define a bounded environment by p' by letting $p'_i = p_i$ for $i < x$ and $p'_i = \frac{1}{2}$ for all $i \ge x$. That is, p' is obtained from p by "forgetting" all its cookies above level x. Then $\sum_{i=1}^{\infty} |p_i - p'_i| = \frac{1}{2}\sum_{i=x}^{\infty}(2p_i - 1) < \frac{\varepsilon}{4}$. Since p' is a bounded environment, by (19) we have

$$\left|\mathbb{E}\left[U_{p'}(x)\right] - x - \delta\right| \le \left|\mathbb{E}\left[U_{p'}(x)\right] - x - \sum_{i=1}^{x}(2p_i - 1)\right| + \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

and so, it is left to prove that

$$\left|\mathbb{E}\left[U_p(x)\right] - \mathbb{E}\left[U_{p'}(x)\right]\right| < \frac{\varepsilon}{2}. \tag{24}$$

We prove (24) by coupling the two processes in the natural way. For each $i \in \mathbb{N}$ let $Y_i \sim U[0,1]$ be i.i.d. uniform random variables. Define $U_p(x) = \inf\{k \in \mathbb{N}\colon \sum_{i=1}^{k} \mathbf{1}_{[Y_i > p_i]} = x\} - x$, and analogously let $U_{p'}(x) = \inf\{k \in \mathbb{N}\colon \sum_{i=1}^{k} \mathbf{1}_{[Y_i > p'_i]} = x\} - x$. Clearly both $U_p(x)$ and $U_{p'}(x)$ have the correct distribution. In addition we have $U_p(x) \ge U_{p'}(x)$. Let $T = U_{p'}(x) + x$ be the time when $U_{p'}(x)$ reaches x failures, and let $K = x - \sum_{i=1}^{T} \mathbf{1}_{[Y_i > p_i]}$, be the number of failures of $U_p(x)$ after time T. Then

$$\left|U_p(x) - U_{p'}(x)\right| \sim U_{s^{T+1}(p)}(K),$$

where $s^{T+1}(p) = (p_{T+1}, p_{T+2}, \ldots)$ is the left shift of the cookie environment p. Taking the expectation on both sides, we get

$$\mathbb{E}\big[\big|U_p(x) - U_{p'}(x)\big|\big] = \mathbb{E}\big[U_{s^{T+1}(p)}(K)\big] \le \alpha \cdot \mathbb{E}[K],$$

where $\alpha = \sup_{k\ge M}\{\mathbb{E}[U_{s^k(p)}(1)]\}$. We show below that $\mathbb{E}[K] \le \frac{\varepsilon}{4}$ and $\alpha < 2$, which is clearly enough in order to prove (24), since $T \ge x > M$,

In order to see that $\mathbb{E}[K] \le \frac{\varepsilon}{4}$ note that $K \le \sum_{i=1}^{\infty} \mathbf{1}_{[p'_i < Y_i \le p_i]}$. Therefore, taking the expectation we get

$$\mathbb{E}[K] \le \sum_{i=1}^{\infty} \big|p_i - p'_i\big| < \frac{\varepsilon}{4}.$$

In order to prove that $\alpha < 2$ note that in every environment p we have

$$\mathbb{E}\big[U_p(1)\big] = \sum_{n=1}^{\infty} \mathbb{P}\big[U_p(1) \ge n\big] = \sum_{n=1}^{\infty}\prod_{i=1}^{n} p_i. \tag{25}$$

In particular, if for some $\gamma < 1$ it holds that $p_i < \gamma$ for all $i \ge k$, then

$$\mathbb{E}\big[U_{s^k(p)}(1)\big] \le \sum_{n=1}^{\infty} \gamma^n = \frac{\gamma}{1-\gamma}. \tag{26}$$

Recall that M is sufficiently large so that $\sum_{i=M}^{\infty}(2p_i - 1) < \frac{\varepsilon}{2}$, and in particular $p_i < \frac{1}{2} + \frac{\varepsilon}{4}$ for all $i \ge M$. Therefore, it follows that $\alpha < \frac{1/2+\varepsilon/4}{1/2-\varepsilon/4} < 2$ for all $\varepsilon < 2/3$. This completes the proof of Claim 4.6. □

Next, we prove the "furthermore" part of Lemma 4.3.

Claim 4.7. *Let p be a positive and elliptic cookie environment, and let $\delta < \infty$. Then $\rho(x) \le \delta$ for all $x \ge 0$.*

Proof. By Claim 4.6, $\lim_{x\to\infty} \rho(x) = \delta$ and so the claim will follow once we show that $\rho(x)$ is monotonically increasing in x. Note that for $p = (p_1, p_2, \ldots)$, we have that $\mathbb{E}[U_p(1)]$ is monotonically increasing in each p_i. Indeed, this can be seen either from the explicit formula (25), or using the natural coupling specified in the proof of Claim 4.6. (Actually, for every x it holds that $\mathbb{E}[U_p(x)]$ is monotonically increasing in each p_i, but we do not use that.) By comparing to the constant $1/2$ environment we observe that for a positive environment p it holds that $\mathbb{E}[U_p(1) - 1] \ge 0$. Therefore, $\rho(x+1) = \mathbb{E}[U_p(x+1) - (x+1)] = \mathbb{E}[U_p(x) - x + U_{p'}(1) - 1)] = \mathbb{E}[U_p(x) - x] + E[U_{p'}(1) + 1)] \ge \mathbb{E}[U_p(x) - x] = \rho(x)$, where p' is some random (but a.s. finite) shift of p and hence also positive, and the inequality follows from the last observation. □

4.2.2. *Proof of Lemma 4.4*

The lemma is an immediate consequence of Lemma 2.5. Indeed, since $\sum_{i=1}^{\infty}(2p_i - 1) < \infty$ it follows that $|\bar{p}_n - \frac{1}{2}| = |\frac{1}{n}\sum_{i=1}^{n} p_i - \frac{1}{2}| = \frac{1}{2n}\sum_{i=1}^{n}(2p_i - 1) \le \frac{\delta}{2n}$. Therefore, by Lemma 2.5 the limit of $\frac{\mathbb{E}[(U_p(x)-x)^2]}{x}$ as x tends to infinity exists, and is equal to

$$\lim_{x\to\infty} \frac{1}{x}\mathbb{E}\big[\big(U_p(x) - x\big)^2\big] = 8A,$$

where

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} p_i(1-p_i) = \frac{1}{4}.$$

By the "moreover" part of Lemma 2.5 it follows that the rate of convergence is bounded by $C \cdot \log^4(x)/\sqrt{x}$, that is, for all $x \in \mathbb{N}_0$ sufficiently large it holds that

$$\left| \frac{1}{x} \cdot \mathbb{E}\big[\big(U_p(x) - x\big)^2\big] - 2 \right| = \mathrm{O}\left(\frac{\log^4(x)}{\sqrt{x}} \right),$$

where the constant implicit in the $\mathrm{O}(\cdot)$ notation depends only on p. This completes the proof of Lemma 4.4.

4.3. *Branching process with migration*

As a corollary from Theorem 1.3 we obtain the following result on branching processes with migration. In order to define a branching process with migration let ξ and η be two random variables, where the support of ξ is $\mathbb{N}_0$ and $\eta \in \mathbb{Z}$. Suppose that both ξ and η have an exponential tail. That is, there is some $\alpha > 0$ and t_0 such that $\mathbb{P}[\xi > t] \le \exp(-\alpha t)$ and $\mathbb{P}[|\eta| > t] \le \exp(-\alpha t)$ for all $t > t_0$.

Let $\mu = \mathbb{E}[\xi]$, $\rho = \mathbb{E}[\eta]$, $\nu = \mathrm{Var}[\xi]$, and let $\theta = \frac{2\rho}{\nu}$. Note that by the assumption on ξ and η all these quantities are finite. For $i, n, m \in \mathbb{N}$ let $\xi_i^{(n)}$ and $\eta_{(m)}$ be independent random variables so that $\xi_i^{(n)} \sim \xi$ and $\eta_{(m)} \sim \eta$.

A branching process with migration is a random sequence $Z = (Z_n)_{n \ge 0}$ defined by setting $Z_0 = 1$, and for each $n \ge 0$ the random variable Z_{n+1} conditioned on Z_n is distributed as

$$Z_{n+1} = \begin{cases} \max\{\sum_{i=1}^{Z_n} \xi_i^{(n)} + \eta_{(n+1)}, 0\} & \text{if } Z_n > 0, \\ 0 & \text{if } Z_n = 0. \end{cases}$$

The random variable ξ is the *offspring distribution*, and η is the *migration distribution*.

We say that the process Z *survives* if $Z_n > 0$ for all n. Otherwise we say that Z dies out. The following theorem gives necessary and sufficient conditions for survival of Z.

Theorem 4.8. *Consider the branching process with migration $Z = (Z_n)_{n \ge 0}$ as above. Then*

- *If $\mu > 1$, then Z survives with positive probability.*
- *If $\mu < 1$ then Z a.s. dies out.*
- *Assume $\mu = 1$, then Z dies out a.s. if and only if $\theta = \frac{2\rho}{\nu} \le 1$.*

Proof. Note that the process $Z = (Z_n)_{n \ge 0}$ is a Markov chain on $\mathbb{N}_0$ with the step distribution

$$U(x) = \begin{cases} \max\{\sum_{i=1}^{x} \xi_i^{(1)} + \eta, 0\} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Our Theorem 1.3 is formulated for irreducible chains, but we can simply change $U(0)$ to be, say, 1, and replace "dies out" with "reaches 0" and we are back in the irreducible case. We now apply Theorem 1.3 to the process Z. Note that:

(1) The sum $\sum_{i=1}^{x} \xi_i^{(1)} + \eta$ is concentrated around its mean, which follows from Hoeffding's type inequality for random variables with exponential tails. In particular, $\frac{U(x)}{x}$ is concentrated around μ.
(2) This shows that the effect of taking the maximum with zero is negligible. Indeed, for large values of x we have

$$\begin{aligned} \mathbb{P}\big[U(x) = 0\big] &\le \mathbb{P}\left[\sum_{i=1}^{x} \xi_i^{(1)} > \mu x/2 \text{ and } \eta < -\mu x/2 \right] + \mathbb{P}\left[\sum_{i=1}^{x} \xi_i^{(1)} < \mu x/2 \right] \\ &\le \mathbb{P}[\eta < -\mu x/2] + \mathbb{P}\left[\sum_{i=1}^{x} \xi_i^{(1)} < \mu x/2 \right] \\ &\le \exp(-cx), \end{aligned}$$

for some constant c which depends on ξ and η but not on x. Therefore $|\mathbb{E}[U(x)] - \mu x + \rho| \le \exp(-c'x)$ for some constant $c' > 0$ that depends on ξ and η but not on x.

(3) By independence of $\xi_i^{(n)}$'s and $\eta_{(m)}$ we have

$$\begin{aligned}\mathbb{E}\big[\big(U(x)-\mu x\big)^2\big] &= \mathbb{E}\Bigg[\sum_{i=1}^{x}\big(\xi_i^{(1)}-\mu\big)+\eta^2\Bigg]+\mathrm{O}\big(\mathrm{e}^{-cx}\big)\\ &= \sum_{i=1}^{x}\mathbb{E}\big[(\xi-\mu)^2\big]+\mathbb{E}\big[\eta^2\big]+\mathrm{O}\big(\mathrm{e}^{-cx}\big)\\ &= \nu x+\mathbb{E}\big[\eta^2\big]+\mathrm{O}\big(\mathrm{e}^{-cx}\big),\end{aligned}$$

and hence $\frac{\mathbb{E}[U(x)-x]^2}{x}=\nu+\frac{\mathbb{E}[\eta^2]}{x}+\mathrm{O}(\mathrm{e}^{-cx})$.

Therefore, by applying Theorem 1.3 we get the desired conclusion. □

5. Open problems

(1) For ERW with periodic environments, compute the speed in terms of the period.
(2) Find an identically piled (uniformly) elliptic cookie environments so that $\mu=\theta=1$ and the walk is right transient. Note that by Theorem 1.3 it is enough to find an environment so that $\theta(x)-1$ is eventually larger than $\frac{2}{\ln(x)}+\alpha(x)\cdot x^{-1/2}$ for some $\alpha(x)$ such that $\alpha(x)\nu(x)\to+\infty$.

Appendix: Survival of irreducible Markov chains on $\mathbb{N}_0$

In this appendix we prove our criterion for transience of Markov chains on $\mathbb{N}_0$ stated in Theorem 1.3. Recall that we denote by $Z=(Z_n)_{n\geq 0}$ an irreducible discrete time Markov chain on $\mathbb{N}_0$ starting at $Z_0=1$, and that we denote by $U=(U(x))_{x\geq 0}$ its step distribution. Recall also the asymptotic mean μ, the drift $\rho(x)$, the diffusion constant $\nu(x)$ and the ratio $\theta(x)$ defined just before Theorem 1.3 (page 1025). The proof of Theorem 1.3 relies on the classical approach of *Lyapunov functions*. Theorems 2.1 and 2.2 of Lamperti [12] will serve as a convenient reference. The following theorem is an immediate corollary of them.

Theorem A.1. *Let Z be an irreducible discrete time Markov chain on $\mathbb{N}_0$, with step distribution $U=(U(x))_{x\geq 0}$. That is $\mathbb{P}[Z_{n+1}=y|Z_n=x]=\mathbb{P}[U(x)=y]$ for all $n\geq 0$. Then*

(1) *Z is recurrent whenever there is some function $V:\mathbb{N}_0\to(0,\infty)$ such that $\lim_{x\to\infty}V(x)=\infty$ and $\mathbb{E}[V(U(x))]\leq V(x)$ for all sufficiently large values of x.*
(2) *Z is transient whenever there is some function $V:\mathbb{N}_0\to(0,\infty)$ such that $\lim_{x\to\infty}V(x)=0$ and $\mathbb{E}[V(U(x))]\leq V(x)$ for all sufficiently large values of x.*

A function V satisfying one of the two possibilities in Theorem A.1 is called *Lyapunov function* for the Markov chain defined by U.

We start our proof with the two simple cases of $\mu<1$ and $\mu>1$.

The case $\mu<1$: We apply Theorem A.1 on U with Lyapunov function $V(x)=x$. We claim that for all x sufficiently large it holds that $\mathbb{E}[U(x)]\leq x$. Indeed, $\mathbb{E}[U(x)]-x\leq(\mu-1)x+\mathrm{o}(x)<0$ for all sufficiently large x since $\mu<1$. We are done since $V(x)\to\infty$ as $x\to\infty$.

The case $\mu>1$: Define $V(x):=\frac{1}{x+1}$. We claim that for all sufficiently large x we have $\mathbb{E}[\frac{1}{U(x)+1}]\leq\frac{1}{x+1}$. Indeed, using first order Taylor expansion applied to the function $f(U)=\frac{1}{U+1}$ around x we have

$$\mathbb{E}\bigg[\frac{1}{U(x)+1}\bigg]=\frac{1}{x+1}-\mathbb{E}\bigg[\frac{1}{(1+\xi)^2}\cdot\big(U(x)-x\big)\bigg]$$

for some ξ lying between x and $U(x)$. By the concentration of U for x sufficiently large we have $\frac{1+\mu}{2} \leq \frac{U(x)}{x} \leq 2\mu$ with high probability, in which case the expression in the expectation is $\frac{U(x)-x}{(1+\xi)^2} \geq \frac{c}{x}$ for some constant c that depends only on μ. Note that either way the expression in the expectation is at least $-x$. Therefore, if we denote $p_\mu = \mathbb{P}[\frac{U(x)}{x} \in [\frac{1+\mu}{2}, 2\mu]]$, then

$$\mathbb{E}\left[\frac{1}{U(x)+1}\right] \leq \frac{1}{x+1} - \left(p_\mu \cdot \frac{c}{x} + (1-p_\mu)\cdot(-x)\right) \leq \frac{1}{x+1},$$

where the last inequality follows from the concentration of $U(x)$, which implies that p_μ is exponentially close to 1. This completes the proof of the case $\mu > 1$.

The case $\mu = 1$: The proof for the case $\mu = 1$ uses again Theorem A.1 with an appropriate Lyapunov function. For the recurrence case the function we will use is $V(x) = \ln\ln(x) \to \infty$, and for the transience we will use $V(x) = \ln^{-1}(x) \to 0$. In both cases we use Taylor expansion of V around x to prove that $V(U(x))$ satisfies the super-martingale property, namely, that $\mathbb{E}[V(U(x))] \leq V(x)$ for all x sufficiently large.

The case $\theta(x) - 1 \ll \frac{1}{\ln(x)}$: This case is summarized in the following claim.

Claim A.2. *Suppose that* $\theta(x) < 1 + \frac{1}{\ln(x)} - \alpha(x)\cdot x^{-1/2}$ *for all sufficiently large* $x \in \mathbb{N}_0$, *where* $\alpha(x)$ *is such that* $\alpha(x)\nu(x) \to +\infty$. *Then* $\mathbb{P}[Z_n = 0 \text{ for some } n] = 1$.

Proof. We define our Lyapunov function to be $V(x) = \ln\ln(x)$.[5] We claim that for all x sufficiently large it holds that $\mathbb{E}[\ln\ln U(x)] \leq \ln\ln x$, which by Theorem A.1 implies the claim.

We state the first three derivatives of V, which hold for all sufficiently large values of x.

$$V'(x) = \frac{1}{x\ln(x)},$$
$$V^{(2)}(x) = -\frac{1}{x^2\ln(x)} - \frac{1}{x^2\ln^2(x)},$$
$$V^{(3)}(x) = \frac{2}{x^3\ln(x)} + \frac{3}{x^3\ln^2(x)} + \frac{2}{x^3\ln^3(x)}.$$

Using 3rd order Taylor expansion of V around x with Cauchy remainder we have for all large enough x

$$\mathbb{E}\big[\ln\ln(U)\big] = \ln\ln(x) + V'(x)\cdot\mathbb{E}[U-x] + \frac{1}{2!}V^{(2)}(x)\cdot\mathbb{E}\big[(U-x)^2\big] + \frac{1}{3!}\mathbb{E}\big[V^{(3)}(\xi)\cdot(U-x)^3\big],$$

where ξ is some random value between x and U. It is easy to see that the exponential concentration of U implies that the remainder is $\mathrm{O}(x^{-3/2}\ln^{-1}(x))$ (regardless of how V is defined for small values of x). Inserting the definitions of $\rho(x)$ and $\nu(x)$ we have

$$\mathbb{E}\big[\ln\ln(U)\big] = \ln\ln(x) + \frac{\rho(x)}{x\ln(x)} - \frac{x\nu(x)}{2x^2\ln(x)} - \frac{x\nu(x)}{2x^2\ln^2(x)} + \mathrm{O}\left(\frac{x^{-3/2}}{\ln x}\right).$$

Multiplying by $\frac{2x\ln(x)}{\nu(x)}$, and recalling that $\theta(x) = \frac{2\rho(x)}{\nu(x)}$ we see that it is enough to show that

$$\theta(x) \leq 1 + \frac{1}{\ln(x)} + \mathrm{O}\big(x^{-1/2}\big).$$

Since our assumption of θ was that $\theta(x) < 1 + \frac{1}{\ln(x)} - \alpha(x)x^{-1/2}$, and $\alpha(x)\nu(x) \to +\infty$, the required inequality holds for all x sufficiently large, and therefore for such values of x we have $\mathbb{E}[\ln\ln U(x)] \leq \ln\ln x$, as required. □

[5]Note that $V(x)$ is not defined properly for $x \leq e$. We overcome this by defining V in a range slightly larger that $[0, e]$ arbitrarily, while making sure that V is smooth and positive.

The case $\theta(x) - 1 \gg \frac{2}{\ln(x)}$: This case is summarized in the following claim.

Claim A.3. *Suppose that* $\theta(x) > 1 + \frac{2}{\ln(x)} + \alpha(x) \cdot x^{-1/2}$ *for all sufficiently large* $x \in \mathbb{N}_0$, *where* $\alpha(x)$ *is such that* $\alpha(x)\nu(x) \to +\infty$. *Then* $\mathbb{P}[Z_n > 0$ *for all* $n] > 0$.

Proof. We define our Lyapunov function to be $V(x) = \ln^{-1}(x)$.[6] We claim that for all x sufficiently large it holds that $\mathbb{E}[\ln^{-1}(U(x))] \le \ln^{-1}(x)$, which by Theorem A.1 implies the claim.

We state the first three derivatives of V, which hold for all sufficiently large values of x.

$$V'(x) = -\frac{1}{x\ln^2(x)},$$
$$V^{(2)}(x) = \frac{1}{x^2\ln^2(x)} + \frac{2}{x^2\ln^3(x)},$$
$$V^{(3)}(x) = -\frac{2}{x^3\ln^2(x)} - \frac{6}{x^3\ln^3(x)} - \frac{6}{x^3\ln^4(x)}.$$

Using 3rd order Taylor expansion of V around x with Cauchy remainder we have

$$\mathbb{E}\big[\ln^{-1}(U)\big] = \ln^{-1}(x) + V'(x) \cdot \mathbb{E}[U - x] + \frac{1}{2!}V^{(2)}(x) \cdot \mathbb{E}\big[(U-x)^2\big] + \frac{1}{3!}\mathbb{E}\big[V^{(3)}(\xi) \cdot (U-x)^3\big]$$

for some random ξ between x and U. As before the exponential concentration of U gives that the error is $\mathrm{O}(x^{-3/2}\ln^{-2}(x))$. By the definition of $\rho(x)$ and $\nu(x)$ we have

$$\mathbb{E}\big[\ln^{-1}(U)\big] = \ln^{-1}(x) - \frac{\rho(x)}{x\ln^2(x)} + \frac{x\nu(x)}{2x^2\ln^2(x)} + \frac{x\nu(x)}{x^2\ln^3(x)} + \mathrm{O}\left(\frac{x^{-3/2}}{\ln^2(x)}\right).$$

Therefore, in order to prove that $\mathbb{E}[\ln^{-1}(U(x))] \le \ln^{-1}(x)$ it is enough to show that

$$\frac{\rho(x)}{x\ln^2(x)} \ge \frac{\nu(x)}{2x\ln^2(x)} + \frac{\nu(x)}{x\ln^3(x)} + \mathrm{O}\left(\frac{x^{-3/2}}{\ln^2(x)}\right).$$

Multiplying both sides of the inequality by $\frac{2x\ln^2(x)}{\nu(x)}$, and substituting $\theta(x) = \frac{2\rho(x)}{\nu(x)}$ this is equivalent to showing that

$$\theta(x) \ge 1 + \frac{2}{\ln(x)} + \mathrm{O}\big(x^{-1/2}\big).$$

Therefore, if $\theta(x) > 1 + \frac{2}{\ln(x)} + \alpha(x)x^{-1/2}$ for some $\alpha(x)$ such that $\alpha(x)\nu(x) \to +\infty$, then the above inequality holds for all large enough x. The claim, and hence also Theorem 1.3, follow. □

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[6]Just like in the previous case $V(x)$ is not defined in $x = 1$, and it is not positive for $x < 1$. Again, we overcome this by defining V in the interval $[0, 2]$ arbitrarily, while making sure that V is smooth and positive.

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