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How big is the minimum of a branching random walk?

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Abstract. Let $\mathbb{M}_n$ be the minimal position in the nth generation, of a real-valued branching random walk in the boundary case. As $n \to \infty$, $\mathbb{M}_n - \frac{3}{2}\log n$ is tight (see (*Ann. Probab.* **37** (2009) 1044–1079, *Ann. Probab.* **41** (2013) 1362–1426, *Ann. Probab.* **37** (2009) 615–653)). We establish here a law of iterated logarithm for the upper limits of $\mathbb{M}_n$: upon the system's non-extinction, $\limsup_{n\to\infty} \frac{1}{\log\log\log n}(\mathbb{M}_n - \frac{3}{2}\log n) = 1$ almost surely. We also study the problem of moderate deviations of $\mathbb{M}_n$: $\mathbb{P}(\mathbb{M}_n - \frac{3}{2}\log n > \lambda)$ for $\lambda \to \infty$ and $\lambda = o(\log n)$. This problem is closely related to the small deviations of a class of Mandelbrot's cascades.

Résumé. Soit $\mathbb{M}_n$ la position minimale à la n^{ieme} génération, d'une marche aléatoire branchante réelle dans le cas frontière. Quand $n \to \infty$, $\mathbb{M}_n - \frac{3}{2}\log n$ est tendue (voir (*Ann. Probab.* **37** (2009) 1044–1079, *Ann. Probab.* **41** (2013) 1362–1426, *Ann. Probab.* **37** (2009) 615–653)). Nous établissons une loi du logarithme itéré pour décrire les limites supérieures de $\mathbb{M}_n$: sur l'événement de la survie du système, $\limsup_{n\to\infty} \frac{1}{\log\log\log n}(\mathbb{M}_n - \frac{3}{2}\log n) = 1$ presque sûrement. Nous étudions également les déviations modérées de $\mathbb{M}_n$: $\mathbb{P}(\mathbb{M}_n - \frac{3}{2}\log n > \lambda)$ pour $\lambda \to \infty$ et $\lambda = o(\log n)$. Ce problème est directement lié aux petites déviations d'une classe des cascades de Mandelbrot.

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1. Introduction

Let $\{V(u), u \in \mathbb{T}\}$ be a discrete-time branching random walk (BRW) on the real line $\mathbb{R}$ driven by a point process Θ. At generation 0, there is a single particle at the origin from which we generate a point process Θ on $\mathbb{R}$. The particles in Θ together with their positions in $\mathbb{R}$ constitute the first generation of the BRW. From the position of each particle at the first generation, we generate an independent copy of Θ. The collection of all particles together with their positions gives the second generation of the BRW, and so on. The genealogy of all particles forms a Galton–Watson tree $\mathbb{T}$ (whose root is denoted by $\varnothing$). For any particle $u \in \mathbb{T}$, we denote by $V(u)$ its position in $\mathbb{R}$ and $|u|$ its generation in $\mathbb{T}$. The whole system may die out or survive forever.

Plainly $\Theta = \sum_{|u|=1} \delta_{\{V(u)\}}$. Let $\nu = \Theta(\mathbb{R})$. Throughout this paper and unless stated otherwise, we shall assume that the BRW is in the boundary case, i.e.

$$\mathbb{E}[\nu] \in (1, \infty], \qquad \mathbb{E}\Bigg[\sum_{|u|=1} \mathrm{e}^{-V(u)}\Bigg] = 1, \qquad \mathbb{E}\Bigg[\sum_{|u|=1} V(u)\mathrm{e}^{-V(u)}\Bigg] = 0. \tag{1.1}$$

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Notice that under (1.1), it is possible that $\mathbb{P}(\nu = \infty) > 0$. See Jaffuel [22] for detailed discussions on how to reduce a general branching random walk to the boundary case.

Denote by $\mathbb{M}_n := \min_{|u|=n} V(u)$ the minimum of the branching random walk in the nth generation (with convention: $\inf_\emptyset \equiv \infty$). Hammersley [18], Kingman [23] and Biggins [6] established the law of large numbers for $\mathbb{M}_n$ (for any general branching random walk), whereas the second order limits have attracted many recent attentions, see [1,2,8,21] and the references therein. In particular, Aïdékon [2] proved the convergence in law of $\mathbb{M}_n - \frac{3}{2}\log n$ under (1.1) and some mild conditions.

On the almost sure limits of $\mathbb{M}_n$, it was shown in [21] that there is the following phenomena of fluctuation at the logarithmic scale. Assume (1.1). If there exists some $\delta > 0$ such that $\mathbb{E}[\nu^{1+\delta}] < \infty$ and $\mathbb{E}[\int_{\mathbb{R}} (e^{\delta x} + e^{-(1+\delta)x})\Theta(dx)] < \infty$, then

$$\limsup_{n\to\infty} \frac{\mathbb{M}_n}{\log n} = \frac{3}{2} \quad \text{and} \quad \liminf_{n\to\infty} \frac{\mathbb{M}_n}{\log n} = \frac{1}{2}, \quad \mathbb{P}^*\text{-a.s.},$$

where here and in the sequel,

$$\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot | \mathcal{S}),$$

and $\mathcal{S} := \{\mathbb{T} \text{ is not finite}\}$ denotes the event that the whole system survives.

It turns out that much more can be said on the lower limits $\frac{1}{2}\log n$ of $\mathbb{M}_n$: Under (1.1) and the following integrability condition

$$\sigma^2 := \mathbb{E}\Big[\sum_{|u|=1} \big(V(u)\big)^2 e^{-V(u)}\Big] < \infty, \qquad \mathbb{E}\big[\zeta\big((\log\zeta)^+\big)^2 + \widetilde{\zeta}(\log\widetilde{\zeta})^+\big] < \infty, \tag{1.2}$$

with $\zeta := \sum_{|u|=1} e^{-V(u)}$, $\widetilde{\zeta} := \sum_{|u|=1} (V(u))^+ e^{-V(u)}$ and $x^+ := \max(0, x)$, Aïdékon and Shi [4] proved that

$$\liminf_{n\to\infty} \Big(\mathbb{M}_n - \frac{1}{2}\log n\Big) = -\infty, \quad \mathbb{P}^*\text{-a.s.}$$

Furthermore, by following Aïdékon and Shi's [4] methods, we established ([20]) an integral test to describe the lower limits of $\mathbb{M}_n - \frac{1}{2}\log n$. As a consequence, we have that under (1.1) and (1.2),

$$\liminf_{n\to\infty} \frac{1}{\log\log n}\Big(\mathbb{M}_n - \frac{1}{2}\log n\Big) = -1, \quad \mathbb{P}^*\text{-a.s.} \tag{1.3}$$

In this paper, we wish to investigate how big $\mathbb{M}_n - \frac{3}{2}\log n$ can be. The following law of iterated logarithm (LIL) describes the upper limits of $\mathbb{M}_n$:

Theorem 1.1. *Assume* (1.1), (1.2) *and that* $\mathbb{E}[\sum_{|u|=1} (V(u)^+)^3 e^{-V(u)}] < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{1}{\log\log\log n}\Big(\mathbb{M}_n - \frac{3}{2}\log n\Big) = 1, \quad \mathbb{P}^*\text{-}a.s. \tag{1.4}$$

The integrability of $\sum_{|u|=1} (V(u)^+)^3 e^{-V(u)}$ will be used only in the proof of Lemma 4.2, see Remark 4.3, Section 4. Usually, to establish such LIL, the first step would be the study of the moderate deviations:

$$\mathbb{P}^*\Big(\mathbb{M}_n - \frac{3}{2}\log n > \lambda\Big), \quad \text{when } \lambda = o(\log n) \text{ and } \lambda, n \to \infty.$$

Denote by $p_j = \mathbb{P}(\nu = j)$, $j \ge 0$, the offspring distribution of the Galton–Watson tree $\mathbb{T}$. Concerning the small deviations of the size of $\mathbb{T}$, there exist two cases: either $p_0 + p_1 > 0$ (namely the Schröder case) or $p_0 = p_1 = 0$ (namely the Böttcher case), see e.g. Fleischmann and Wachtel [15,16] and the references therein. Basically in the

Schröder case, the tree $\mathbb{T}$ may grow linearly whereas it always grows exponentially in the Böttcher case. For the branching random walk, we shall prove that the moderate deviations of $\mathbb{M}_n$ decay exponentially fast or double-exponentially fast depending on the growth rate of $\mathbb{T}$.

Let $q := \mathbb{P}(\mathbb{T} \text{ is finite}) = \mathbb{P}(\mathcal{S}^c) \in [0, 1)$ be the extinction probability. We introduce two separate cases:

(The Schröder case) *If the following hypotheses hold:*

$$\mathbb{E}\left[1_{(\nu\geq 1)} q^{\nu-1} \sum_{|u|=1} e^{\gamma V(u)}\right] = 1, \quad \textit{for some constant } \gamma > 0, \tag{1.5}$$

and

$$\mathbb{E}\left[\sum_{|u|=1} e^{aV(u)}\right] < \infty, \quad \textit{for some } a > \gamma. \tag{1.6}$$

(The Böttcher case) *If the following hypotheses hold:*

$$p_0 = p_1 = 0, \tag{1.7}$$

$$\sup_{|u|=1} V(u) \leq K, \quad \textit{for some constant } K > 0. \tag{1.8}$$

Remark 1.2.

(i) *When a.s. $\nu \geq 1$ in the Schröder case, the condition* (1.5) *just amounts to*

$$\mathbb{E}\left[1_{(\nu=1)} \sum_{|u|=1} e^{\gamma V(u)}\right] = 1, \quad \textit{if } q = 0. \tag{1.9}$$

(ii) *Under* (1.1), *the condition* (1.6) *or* (1.8) *implies that $\mathbb{E}[\nu] < \infty$. The technical conditions* (1.6) *and* (1.8) *are made to avoid too large jumps of Θ in the moderate deviations.*

(iii) *In the Böttcher case, we can define a parameter $\beta > 0$ by*

$$\beta := \sup\left\{a > 0 \colon \mathbb{P}\left(\sum_{|u|=1} e^{-aV(u)} \geq 1\right) = 1\right\}. \tag{1.10}$$

Note that $\beta < 1$ if we assume (1.1).

The parameters γ and β will naturally appear in the small deviations of a class of Mandelbrot's cascades. Under (1.1) and (1.2), the so-called derivative martingale (with convention: $\sum_\emptyset := 0$)

$$D_n := \sum_{|u|=n} V(u) e^{-V(u)}, \quad n \geq 0,$$

converges almost surely to some limit D_∞ which is $\mathbb{P}^*$-a.s. positive (see e.g. Biggins and Kyprianou [7] and Aïdékon [2]). The non-negative random variable D_∞ satisfies the following equation in law (Mandelbrot's cascade):

$$D_\infty \overset{\text{law}}{=} \sum_{|u|=1} e^{-V(u)} D_\infty^{(u)}, \tag{1.11}$$

where conditioned on $\{V(u), |u| = 1\}$, $(D_\infty^{(u)})_{|u|=1}$ are independent copies of D_∞. The moderate deviations of $\mathbb{M}_n$ will be naturally related to the small deviations of D_∞ which were already studied in the literature, see e.g. Liu [25,26] and the references therein.

We shall work under a more general setting in order that Theorem 1.3 could also be applied to the non-degenerated case of Mandelbrot's cascades. Instead of (1.1), we assume that there exists some constant $\chi \in (0, 1]$ such that

$$\mathbb{E}\Big[\sum_{|u|=1} \mathrm{e}^{-\chi V(u)}\Big] \le 1, \quad \text{and} \quad \mathbb{E}[\nu] \in (1, \infty], \tag{1.12}$$

where as before, $\nu := \sum_{|u|=1} 1$.

The condition (1.12) ensures that there exists a non-trivial non-negative solution Z to the following equation:

$$Z \overset{\text{law}}{=} \sum_{|u|=1} \mathrm{e}^{-V(u)} Z^{(u)}, \tag{1.13}$$

where conditioned on $\{V(u), |u| = 1\}$, $(Z^{(u)})_{|u|=1}$ are independent copies of Z, see Liu [26], Proposition 1.1.

Denote by $f(x) \asymp g(x)$ [resp.: $f(x) \sim g(x)$] as $x \to x_0$ if $0 < \liminf_{x\to x_0} f(x)/g(x) \le \limsup_{x\to x_0} f(x)/g(x) < \infty$ [resp.: $\lim_{x\to x_0} f(x)/g(x) = 1$]. The following result may arise an interest in Mandelbrot's cascades.

Theorem 1.3. *Assume* (1.12). *Let $Z \ge 0$ be a non-trivial solution of* (1.13).

(The Schröder case) *Assume* (1.5) *and* (1.6). *Then*

$$\mathbb{P}(0 < Z < \varepsilon) \asymp \varepsilon^{\gamma}, \quad \text{as } \varepsilon \to 0, \tag{1.14}$$

and $\mathbb{E}[\mathrm{e}^{-tZ} 1_{(Z>0)}] \asymp t^{-\gamma}$ *as* $t \to \infty$.

(The Böttcher case) *Assume* (1.7), (1.8) *and that* $\sum_{|u|=1} \mathrm{e}^{-\chi V(u)} \not\equiv 1$. *Then*

$$\mathbb{E}\big[\mathrm{e}^{-tZ}\big] = \mathrm{e}^{-t^{\beta+\mathrm{o}(1)}}, \quad t \to \infty, \tag{1.15}$$

and $\mathbb{P}(Z < \varepsilon) = \mathrm{e}^{-\varepsilon^{-\beta/(1-\beta)+\mathrm{o}(1)}}$, *as* $\varepsilon \to 0$, *with* β *defined in* (1.10).

Obviously we can apply Theorem 1.3 to $Z := D_\infty$ with $\chi = 1$. In the Böttcher case, the two conditions (1.12) and $\sum_{|u|=1} \mathrm{e}^{-\chi V(u)} \not\equiv 1$ imply that $\beta < \chi$, hence $\beta < 1$; moreover, $\operatorname{ess\,inf} \sum_{|u|=1} \mathrm{e}^{-\beta V(u)} = 1$.

Let us mention that (1.14) confirms a prediction in Liu [26] who already proved that if $q = 0$, then for any $a > 0$, $\mathbb{E}[Z^{-a}] < \infty$ if and only if $a < \gamma$. When all $V(u)$, $|u| = 1$, are equal to some random variable, (1.15) is in agreement with Liu [25], Theorem 6.1. If furthermore, all $V(u)$ are equal to some constant, then (1.14) and (1.15) give some rough estimates on the limiting law of Galton–Watson processes, see Fleischmann and Wachtel [15,16] for the precise estimates. We refer to [5] for further studies of the conditioned Galton–Watson tree itself. For instance, we could seek the asymptotic behaviors of the BRW conditioned on $\{0 < D_\infty < \varepsilon\}$, as $\varepsilon \to 0$, but this problem exceeds the scope of the present paper.

Our moderate deviations result on $\mathbb{M}_n$ reads as follows:

Theorem 1.4. *Assume* (1.1), (1.2). *Let* $\lambda, n \to \infty$ *and* $\lambda = \mathrm{o}(\log n)$.

(The Schröder case) *Assume* (1.5) *and that* (1.6) *hold for all* $a > 0$. *Then*

$$\mathbb{P}^*\Big(\mathbb{M}_n > \frac{3}{2}\log n + \lambda\Big) = \mathrm{e}^{-(\gamma+\mathrm{o}(1))\lambda}. \tag{1.16}$$

(The Böttcher case) *Assume* (1.7) *and* (1.8). *Then*

$$\mathbb{P}\Big(\mathbb{M}_n > \frac{3}{2}\log n + \lambda\Big) = \exp\big(-\mathrm{e}^{(\beta+\mathrm{o}(1))\lambda}\big). \tag{1.17}$$

The same estimates hold if we replace $\mathbb{M}_n$ *by* $\max_{n\le k\le 2n} \mathbb{M}_k$.

We refer to Aïdékon [2], Proposition 4.1, for the precise estimate on $\mathbb{P}(\mathbb{M}_n < \frac{3}{2}\log n - \lambda)$ as $\lambda \le \frac{3}{2}\log n$ and $\lambda \to \infty$.

Comparing Theorem 1.1 and Theorem 1.4, we remark that the almost sure behaviors of $\mathbb{M}_n$ are not related to the moderate deviations of $\mathbb{M}_n$. This can be explained as follows: Define for all $\lambda \ge 0$ and $u \in \mathbb{T}$,

$$\tau_\lambda(u) := \inf\bigl\{1 \le i \le |u|:\ V(u_i) > \lambda\bigr\} \quad \text{(with convention } \inf_\emptyset = \infty), \tag{1.18}$$

where here and in the sequel, $\{u_0 = \emptyset, u_1, \dots, u_{|u|} := u\}$ denotes the shortest path from $\emptyset$ to u such that $|u_i| = i$ for all $0 \le i \le |u|$. We introduce the stopping lines:

$$\pounds_\lambda := \bigl\{u \in \mathbb{T}:\ \tau_\lambda(u) = |u|\bigr\}, \quad \lambda \ge 0. \tag{1.19}$$

Roughly speaking, the almost sure limits of $\mathbb{M}_n$ ($\limsup$ of $\mathbb{M}_n$) are determined by those of $\#\pounds_\lambda$, whereas the moderate deviations of $\mathbb{M}_n$ are by the small deviations of $\#\pounds_\lambda$. By Nerman [30], $\mathbb{P}^*$-almost surely, $\#\pounds_\lambda$ is of order $\mathrm{e}^{(1+o(1))\lambda}$; however, to make $\#\pounds_\lambda$ to be as small as possible (and conditioned on $\{\#\pounds_\lambda > 0\}$), in the Schröder case, $\pounds_\lambda$ will be essentially a singleton or a set of few points with exponential costs (see Lemma 5.3), which is no longer possible in the Böttcher case. To relate $\#\pounds_\lambda$ to D_∞, we shall use the martingale (D_n) at the stopping line $\pounds_\lambda$:

$$D_{\pounds_\lambda} := \sum_{u \in \pounds_\lambda} V(u)\mathrm{e}^{-V(u)}, \tag{1.20}$$

which, as shown in Biggins and Kyprianou [7], converges almost surely to D_∞ as $\lambda \to \infty$. For $u \in \pounds_\lambda$, $V(u) \approx \lambda$, hence $D_{\pounds_\lambda} \approx \lambda \mathrm{e}^{-\lambda}\#\pounds_\lambda$. Then the problem of small values of $\#\pounds_\lambda$ will be reduced to that of $D_{\pounds_\lambda}$ and D_∞ as $\lambda \to \infty$. The hypothesis (1.6) and (1.8) are made to control the possible overshoots.

The rest of the paper is organized as follows: In Section 2, we collect some facts on a one-dimensional random walk and on the branching random walk. In Section 3, we study the cascade equation (1.13) and prove Theorem 1.3. In Section 4, we first prove some uniform tightness of $\mathbb{M}_n - \frac{3}{2}\log n$ (Lemma 4.5) and then Theorem 1.1. Finally, in Section 5, we prove Theorem 1.4 in two separate subsections on the Schröder case and on the Böttcher case.

Throughout the paper, we adopt the usual conventions that $\sum_\emptyset := 0$, $\sup_\emptyset := 0$, $\prod_\emptyset := 1$, $\inf_\emptyset := \infty$; we also denote by $(c_i, 1 \le i \le 15)$ some positive constants, and by C, C' and C'' (eventually with a subscript) some unimportant positive constants whose values can vary from one paragraph to another one.

2. Preliminaries

2.1. *Estimates on a centered real-valued random walk*

We collect here some estimates on a real-valued random walk $\{S_k, k \ge 0\}$, under $\mathbb{P}$, centered and with finite variance $\sigma^2 > 0$. Write $\mathbb{P}_x$ and $\mathbb{E}_x$ when $S_0 = x$. Let $\underline{S}_n := \min_{0 \le i \le n} S_i$, $\forall n \ge 0$. The renewal function $R(x)$ related to the random walk S is defined as follows:

$$R(x) := \sum_{k=0}^{\infty} \mathbb{P}(S_k \ge -x, S_k < \underline{S}_{k-1}), \quad x \ge 0, \tag{2.1}$$

and $R(x) = 0$ if $x < 0$. Moreover (see Feller [14], p. 612),

$$\lim_{x \to \infty} \frac{R(x)}{x} = c_1 > 0. \tag{2.2}$$

Lemma 2.1. *Let S be a centered random walk with finite and positive variance. There exists some constant $c_2 > 0$ such that for any $b \ge a \ge 0, x \ge 0, n \ge 1$,*

$$\mathbb{P}_x\bigl(S_n \in [a, b], \underline{S}_n \ge 0\bigr) \le c_2(1 + x)(1 + b - a)(1 + b)n^{-3/2}. \tag{2.3}$$

For any fixed $0 < r < 1$, *there exists some* $c_3 \equiv c_{3,r} > 0$ *such that for all* $b \geq a \geq 0$, $x, y \geq 0$ *and* $n \geq 1$,

$$\mathbb{P}_x\Big(S_n \in [y+a, y+b], \underline{S}_n \geq 0, \min_{rn \leq i < n} S_i \geq y\Big) \leq c_3(1+x)(1+b-a)(1+b)n^{-3/2}, \tag{2.4}$$

$$\mathbb{P}_x\Big(\underline{S}_n \geq 0, \min_{rn \leq i < n} S_i > y, S_n \leq y\Big) \leq c_3(1+x)n^{-3/2}. \tag{2.5}$$

For any $a > 0$, *if* $\mathbb{E}[S_1^2 e^{aS_1}] < \infty$, *then there exists some* $C_a > 0$ *such that for any* $b \geq 0$,

$$\mathbb{P}(S_{\tau_b} - b > x) \leq C_a e^{-ax}, \quad \forall x \geq 0, \tag{2.6}$$

where $\tau_b := \inf\{j \geq 0 \colon S_j > b\}$.

Proof. See Aïdékon and Shi [4] for (2.3) and (2.4). To get (2.6), note that $\mathbb{E}[S_1^2 e^{aS_1}] < \infty$ if and only if $\mathbb{E}[(S_1^+)^2 e^{aS_1^+}] < \infty$. By Doney ([11], p. 250), this condition ensures that $\mathbb{E}[S_{\tau_0} e^{aS_{\tau_0}}] < \infty$. Then in view of Chang ([9], Proposition 4.2), we have that uniformly on $b > 0$, $\mathbb{E}[e^{a(S_{\tau_b} - b)}] \leq C_a$ for some constant $C_a > 0$, which implies (2.6) by Chebychev's inequality.

It remains to check (2.5). Let $f(x) := \mathbb{P}(S_1 \leq -x)$, $x \geq 0$. It follows from the Markov property at $n-1$ that the probability in LHS of (2.5) equals

$$\begin{aligned}
&\mathbb{E}_x\big[1_{(\underline{S}_{n-1} \geq 0, \min_{rn \leq i < n} S_i > y)} f(S_{n-1} - y)\big] \\
&\quad \leq \sum_{j=0}^{\infty} f(j)\mathbb{P}_x\Big(\underline{S}_{n-1} \geq 0, \min_{rn \leq i \leq n-1} S_i > y, y+j < S_{n-1} \leq y+j+1\Big) \\
&\quad \leq C(1+x)n^{-3/2}\sum_{j=0}^{\infty} f(j)(2+j) \quad \text{(by (2.4))} \\
&\quad \leq C'(1+x)n^{3/2},
\end{aligned}$$

yielding (2.5). □

2.2. *Change of measures for the branching random walk*

In this subsection, we recall some change of measure formulas in the branching random walk, for the details we refer to [4,7,10,19,28,31] and the references therein.

At first let us fix some notations: For $|u| = n$, we write as before $\{u_0 := \varnothing, u_1, \ldots, u_{n-1}, u_n = u\}$ the path from the root $\varnothing$ to u such that $|u_i| = i$ for any $0 \leq i \leq n$. Define $\overline{V}(u) := \max_{1 \leq i \leq n} V(u_i)$ and $\underline{V}(u) := \min_{1 \leq i \leq n} V(u_i)$. For any $u, v \in \mathbb{T}$, we use the partial order $u < v$ if u is an ancestor of v and $u \leq v$ if $u < v$ or $u = v$. We also denote by $\overleftarrow{u}$ the parent of u and by $\nu(u)$ the number of children of u. Define $\mho(u) := \{v \colon \overleftarrow{v} = \overleftarrow{u}, v \neq u\}$ the set (eventually empty) of brothers of u for any $u \neq \varnothing$. For any $u \in \mathbb{T}$, we denote by $\mathbb{T}_u := \{v \in \mathbb{T} \colon u \leq v\}$ the subtree of $\mathbb{T}$ rooted at u.

Under (1.1), there exists a centered real-valued random walk $\{S_n, n \geq 0\}$ such that for any $n \geq 1$ and any measurable $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}\bigg[\sum_{|u|=n} e^{-V(u)} f\big(V(u_1), \ldots, V(u_n)\big)\bigg] = \mathbb{E}\big(f(S_1, \ldots, S_n)\big), \tag{2.7}$$

which is often referred as the "many-to-one" formula. Moreover under (1.2), $\mathrm{Var}(S_1) = \sigma^2 = \mathbb{E}[\sum_{|u|=1}(V(u))^2 \times e^{-V(u)}] \in (0, \infty)$. We shall use the notation

$$\tau_0 := \inf\{j \geq 1 \colon S_j > 0\}. \tag{2.8}$$

Denote by $(\mathcal{F}_n, n \ge 0)$ the natural filtration of the branching random walk. Under (1.1), the process $W_n := \sum_{|u|=n} e^{-V(u)}$, $n \ge 1$, is a $(\mathbb{P}, (\mathcal{F}_n))$-martingale. It is well-known (see [4,7,10,19,28,31]) that on some enlarged probability space (more precisely on the space of marked trees enlarged by an infinite ray $(\mathrm{w}_n, n \ge 0)$, called spine), we may construct a probability $\mathbb{Q}$ such that the following statements (i), (ii) and (iii) hold:

(i) For all $n \ge 1$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_n} = W_n, \quad \text{and} \quad \mathbb{Q}(\mathrm{w}_n = u | \mathcal{F}_n) = \frac{1}{W_n} e^{-V(u)}, \quad \forall |u| = n.$$

(ii) Under $\mathbb{Q}$, the process $\{V(\mathrm{w}_n), n \ge 0\}$ along the spine $(\mathrm{w}_n)_{n \ge 0}$, is distributed as the random walk $(S_n, n \ge 0)$ under $\mathbb{P}$. Moreover, $(\sum_{u \in \mho(\mathrm{w}_k)} \delta_{\{\Delta V(u)\}}, \Delta V(\mathrm{w}_k))_{k \ge 1}$ are i.i.d. under $\mathbb{Q}$, where $\Delta V(u) := V(\overleftarrow{u}) - V(u)$ for any $u \ne \emptyset$.

(iii) Let $\mathcal{G}_n := \sigma\{u, V(u)\colon \overleftarrow{u} \in \{\mathrm{w}_k, 0 \le k < n\}\}$, $n \ge 0$. Then $\mathcal{G}_\infty$ is the σ-algebra generated by the spine. Under $\mathbb{Q}$ and conditioned on $\mathcal{G}_\infty$, for all $u \notin \{\mathrm{w}_k, k \ge 0\}$ but $\overleftarrow{u} \in \{\mathrm{w}_k, k \ge 0\}$ the induced branching random walk $(V(uv), |v| \ge 0)$ are independent and are distributed as $\mathbb{P}_{V(u)}$, where $\{uv, |v| \ge 0)$ is the subtree $\mathbb{T}_u$.

We mention that the above change of measure still holds for the stopping line $£_\lambda$ (see e.g. [3], Proposition 3, for the detailed statement): i.e. replace $|u| = n$ by $u \in £_\lambda$, $\mathcal{F}_n$ by $\mathcal{F}_{£_\lambda}$ the σ-filed generated by the BRW up to $£_\lambda$, and W_n by

$$W_{£_\lambda} := \sum_{u \in £_\lambda} e^{-V(u)}. \tag{2.9}$$

For brevity, we shall write $\mathbb{Q}[X]$ for the expectation of some random variable X under the probability $\mathbb{Q}$.

3. Proof of Theorem 1.3

The following result is due to Liu [26]:

Lemma 3.1 (Liu [26]). *Assuming* (1.5), (1.6) *and* (1.12). *Let $Z \ge 0$ be a non-trivial solution of* (1.13). *For any $0 < \varepsilon < \gamma$, there exists some positive constant $c_4 = c_4(\varepsilon)$ such that*

$$\mathbb{E}\big[e^{-tZ} 1_{(Z>0)}\big] \le c_4 t^{-\gamma+\varepsilon}, \quad \forall t \ge 1. \tag{3.1}$$

Proof. At first we remark that

$$\mathbb{P}(Z = 0) = q. \tag{3.2}$$

In fact, we easily deduce from (1.13) that the probability $\mathbb{P}(Z = 0)$ is a solution of $x = \mathbb{E}[x^\nu]$ which only has two solutions q and 1 for $x \in [0, 1]$. This gives (3.2).

In the case $q = 0$, namely $Z > 0$ a.s., γ is defined through (1.9), it is easy to check that $\mathbb{P}(\sum_{|u|=1} e^{-V(u)} \ne 1) > 0$, then (3.1) follows exactly from Liu [26], Theorem 2.4, after a standard Tauberian argument (see Lemma 4.4 in [25]). We only need to check that the case $q > 0$ can be reduced to the case $q = 0$.

For brevity, let us denote by $\{A_i, 1 \le i \le \nu\}$ the family $\{e^{-V(u)}, |u| = 1\}$ [the order of A_i is arbitrary]. Then Z satisfies the equation in law

$$Z \stackrel{\text{law}}{=} \sum_{i=1}^{\nu} A_i Z_i, \tag{3.3}$$

with $(Z_i, i \ge 1)$ independent copies of Z, and independent of $(A_i)_{1 \le i \le \nu}$. Let $\{\xi, \xi_i, i \ge 1\}$ be a family of i.i.d. Bernoulli random variables, independent of everything else, with common law $\mathbb{P}(\xi = 0) = q = 1 - \mathbb{P}(\xi = 1)$. Let

$\widehat{Z}$ be a random variable distributed as Z conditioned on $\{Z>0\}$. Since $\mathbb{P}(Z>0)=1-q$, we have that $Z\stackrel{\text{law}}{=}\xi\widehat{Z}$. Then we deduce from (3.3) that

$$\widehat{Z}\stackrel{\text{law}}{=}\sum_{i=1}^{\nu}A_i\xi_i\widehat{Z}_i\quad\text{conditioned on }\left\{\sum_{i=1}^{\nu}\xi_i>0\right\},$$

where $(\widehat{Z}_i, i\ge 1)$ are i.i.d. copies of $\widehat{Z}$, and $(\nu, A_i, 1\le i\le\nu)$ and $(\xi_i, i\ge 1)$ are three independent families of random variables. Let $\{\widehat{A}_i, 1\le i\le\widehat{\nu}\}$ be a family of random variables such that for any non-negative measurable function f,

$$\mathbb{E}\big[\mathrm{e}^{-\sum_{i=1}^{\widehat{\nu}}f(\widehat{A}_i)}\big]=\mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{\nu}\xi_i f(A_i)}\,\middle|\,\sum_{i=1}^{\nu}\xi_i>0\right].\tag{3.4}$$

In other words, $\sum_{i=1}^{\widehat{\nu}}\delta_{\{\widehat{A}_i\}}$ has the same law as the point process $\sum_{1\le i\le\nu,\xi_i\ne 0}\delta_{\{A_i\}}$ conditioning the latter does not vanish everywhere. Elementary calculations show that $\mathbb{P}(\sum_{i=1}^{\nu}\xi_i>0)=1-\mathbb{E}[q^{\nu}]=1-q$ and for any non-negative measurable function f,

$$\mathbb{E}\left[\sum_{i=1}^{\widehat{\nu}}f(\widehat{A}_i)\right]=\mathbb{E}\left[\sum_{i=1}^{\nu}\xi_i f(A_i)\,\middle|\,\sum_{i=1}^{\nu}\xi_i>0\right]=\frac{1}{1-q}\mathbb{E}\left[\sum_{i=1}^{\nu}\xi_i f(A_i)\right]=\mathbb{E}\left[\sum_{i=1}^{\nu}f(A_i)\right].\tag{3.5}$$

In particular, $\mathbb{E}[\sum_{i=1}^{\widehat{\nu}}\widehat{A}_i^{\chi}]=\mathbb{E}[\sum_{i=1}^{\nu}A_i^{\chi}]\le 1$ and $\mathbb{E}[\widehat{\nu}]=\mathbb{E}[\nu]\in(1,\infty]$. Moreover, we deduce from (3.4) that $\widehat{\nu}$ is distributed as $\sum_{i=1}^{\nu}\xi_i$ conditioned on $\{\sum_{i=1}^{\nu}\xi_i>0\}$, hence $\widehat{\nu}\ge 1$ a.s. It is easy (e.g. by using the Laplace transform) to see that

$$\widehat{Z}\stackrel{\text{law}}{=}\sum_{i=1}^{\widehat{\nu}}\widehat{A}_i\widehat{Z}_i.$$

Therefore we can apply the case $q=0$ of (3.1) to $\widehat{Z}$ once we have determined the corresponding parameter γ (as in (1.9)) for $\widehat{Z}$. To this end, let $t_\xi=\inf\{1\le i\le\nu\colon\ \xi_i=1\}$. Then $\widehat{A}_1=A_{t_\xi}$ if $t_\xi<\infty$. We have

$$\begin{aligned}\mathbb{E}\big[(\widehat{A}_1)^{-\gamma}1_{(\widehat{\nu}=1)}\big]&=\mathbb{E}\left[A_{t_\xi}^{-\gamma}1_{(\sum_{i=1}^{\nu}\xi_i=1)}\,\middle|\,\sum_{i=1}^{\nu}\xi_i>0\right]\\&=\frac{1}{1-q}\mathbb{E}\left[1_{(\nu\ge 1)}\sum_{k=1}^{\nu}A_k^{-\gamma}1_{(\xi_k=1,\xi_i=0,\forall i\ne k,1\le i\le\nu)}\right]\\&=\mathbb{E}\left[1_{(\nu\ge 1)}q^{\nu-1}\sum_{k=1}^{\nu}A_k^{-\gamma}\right]=\mathbb{E}\left[1_{(\nu\ge 1)}q^{\nu-1}\sum_{|u|=1}\mathrm{e}^{\gamma V(u)}\right]=1,\end{aligned}$$

by (1.5). Therefore $\mathbb{E}[\mathrm{e}^{-t\widehat{Z}}]=\mathrm{O}(t^{-\gamma+\varepsilon})$ as $t\to\infty$. The lemma follows from the fact that $\mathbb{P}(0<Z<x)=(1-q)\mathbb{P}(\widehat{Z}<x)$ for any $x>0$. $\square$

3.1. *Proof of Theorem 1.3: The Schröder case*

As shown in the proof of Lemma 3.1, we can assume $q=0$ (hence we assume (1.9)) in this proof without any loss of generality. Let $\Phi(t):=\mathbb{E}[\mathrm{e}^{-tZ}]$ for $t\ge 0$. We are going to prove that

$$\Phi(t)\asymp t^{-\gamma},\quad t\to\infty.\tag{3.6}$$

To this end, we have by (3.3) that

$$\Phi(t)=\mathbb{E}\left[\prod_{i=1}^{\nu}\Phi(tA_i)\right],\quad t\ge 0. \tag{3.7}$$

Note also that the condition (1.9) can be rewritten as $\mathbb{E}[1_{(\nu=1)}A_1^{-\gamma}]=1$. Define $g(t):=t^\gamma\Phi(t)$ for all $t\ge 0$. Then for any $t>0$,

$$g(t)=t^\gamma\Phi(t)\ge t^\gamma\mathbb{E}\big[1_{(\nu=1)}\Phi(tA_1)\big]=\mathbb{E}\big[1_{(\nu=1)}A_1^{-\gamma}g(tA_1)\big]=\mathbb{E}\big[g(t\widetilde{A}_1)\big], \tag{3.8}$$

where $\widetilde{A}_1$ denotes a (positive) random variable whose law is determined by $\mathbb{E}[f(\widetilde{A}_1)]:=\mathbb{E}[1_{(\nu=1)}A_1^{-\gamma}f(A_1)]$ for any measurable bounded function f. In particular, $\mathbb{E}[\log\widetilde{A}_1]=\mathbb{E}[1_{(\nu=1)}A_1^{-\gamma}\log A_1]$.

Define $f(t):=\mathbb{E}[1_{(\nu=1)}\sum_{|u|=1}\mathrm{e}^{tV(u)}]\equiv\mathbb{E}[1_{(\nu=1)}A_1^{-t}]$ which is finite for $t\in[-\chi,\gamma]$, in particular $f(-\chi)<1$ and $f(0)<1=f(\gamma)$. By the assumption of integrability in Theorem 1.3, $\mathbb{E}[1_{(\nu=1)}A_1^{-\gamma}(-\log A_1)^+]<\infty$ which implies that $f'(\gamma-)$ exists and equals $-\mathbb{E}[1_{(\nu=1)}A_1^{-\gamma}\log A_1]$. By convexity, $f'(\gamma-)\ge\frac{f(\gamma)-f(0)}{\gamma}>0$. Hence

$$\mathbb{E}[\log\widetilde{A}_1]=-f'(\gamma-)<0. \tag{3.9}$$

Let $(\widetilde{A}_i)_{i\ge 2}$ be a sequence of i.i.d. copies of $\widetilde{A}_1$ and define $X_j:=-\sum_{i=1}^{j}\log\widetilde{A}_i$ for all $j\ge 1$. Let $r>1$ and put

$$\alpha_r:=\inf\{j\ge 1:\ X_j>\log r\}, \tag{3.10}$$

which is a.s. finite thanks to (3.9). Going back to (3.8), we get that

$$\begin{aligned}g(r)&\ge\mathbb{E}\big[g(r\widetilde{A}_1)1_{(r\widetilde{A}_1<1)}\big]+\mathbb{E}\big[g(r\widetilde{A}_1)1_{(r\widetilde{A}_1\ge 1)}\big]\\&\ge\mathbb{E}\big[g(r\widetilde{A}_1)1_{(r\widetilde{A}_1<1)}\big]+\mathbb{E}\big[g(r\widetilde{A}_1\widetilde{A}_2)1_{(r\widetilde{A}_1\ge 1)}\big],\end{aligned}$$

where to get the last inequality, we have applied (3.8) with t replaced by $r\widetilde{A}_1$ and $\widetilde{A}_1$ replaced by $\widetilde{A}_2$. Then we obtain that

$$\begin{aligned}g(r)&\ge\mathbb{E}\big[g(r\widetilde{A}_1)1_{(r\widetilde{A}_1<1)}\big]+\mathbb{E}\big[g(r\widetilde{A}_1\widetilde{A}_2)1_{(r\widetilde{A}_1\ge 1,r\widetilde{A}_1\widetilde{A}_2<1)}\big]+\mathbb{E}\big[g(r\widetilde{A}_1\widetilde{A}_2)1_{(r\widetilde{A}_1\ge 1,r\widetilde{A}_1\widetilde{A}_2\ge 1)}\big]\\&=\mathbb{E}\left[g\left(r\prod_{i=1}^{\alpha_r}\widetilde{A}_i\right)1_{(\alpha_r\le 2)}\right]+\mathbb{E}\big[g(r\widetilde{A}_1\widetilde{A}_2)1_{(\alpha_r>2)}\big].\end{aligned}$$

By induction, we get that for any $n\ge 1$,

$$g(r)\ge\mathbb{E}\left[g\left(r\prod_{i=1}^{\alpha_r}\widetilde{A}_i\right)1_{(\alpha_r\le n)}\right]+\mathbb{E}\left[g\left(r\prod_{i=1}^{n}\widetilde{A}_i\right)1_{(\alpha_r>n)}\right]\ge\mathbb{E}\left[g\left(r\prod_{i=1}^{\alpha_r}\widetilde{A}_i\right)1_{(\alpha_r\le n)}\right].$$

Since $\alpha_r<\infty$ a.s., we let $n\to\infty$ and deduce from the monotone convergence theorem that

$$g(r)\ge\mathbb{E}\left[g\left(r\prod_{i=1}^{\alpha_r}\widetilde{A}_i\right)\right]=\mathbb{E}\big[g\big(\mathrm{e}^{-\mathcal{R}_r}\big)\big],$$

where $\mathcal{R}_r:=X_{\alpha_r}-\log r>0$ denotes the overshoot of the random walk (X_j) at the level $\log r$. Note that for any $0<t\le 1$, $g(t)=t^\gamma\Phi(t)\ge\Phi(1)t^\gamma$, hence

$$g(r)\ge\Phi(1)\mathbb{E}\big[\mathrm{e}^{-\gamma\mathcal{R}_r}\big],\quad\forall r>1. \tag{3.11}$$

By the assumption (1.6), $\mathbb{E}[((-\log \widetilde{A}_1)^+)^2] = \mathbb{E}[1_{(\nu=1)} \sum_{|u|=1} (V(u)^+)^2 \mathrm{e}^{\gamma V(u)}] < \infty$, then by Lorden [27], Theorem 1, $\sup_{r\geq 1} \mathbb{E}[\mathcal{R}_r] < \infty$. Consequently for some positive constant C,

$$g(r) \geq \Phi(1)\mathrm{e}^{-\gamma \mathbb{E}[\mathcal{R}_r]} \geq C > 0, \quad \forall r > 1.$$

Hence

$$\Phi(r) \geq C r^{-\gamma}, \quad \forall r > 1, \tag{3.12}$$

which implies the lower bound in (3.6).

To prove the upper bound in (3.6), let $a > \gamma$ be as in (1.6) such that $\mathbb{E}[\sum_{i=1}^{\nu} A_i^{-a}] \equiv \mathbb{E}[\sum_{|u|=1} \mathrm{e}^{aV(u)}] < \infty$. Choose (and then fix) $0 < \varepsilon < \frac{1}{2}\min(a-\gamma, \gamma)$ small and $b := \frac{\gamma+\varepsilon}{2} < \gamma$. By Lemma 3.1, $\Phi(t) \leq c_4 t^{-b}$ for all $t \geq 1$ (with $c_4 \geq 1$). Since $\Phi(t) \leq 1$ for all $0 < t < 1$, we obtain immediately that

$$g(t) \leq c_4 t^{\gamma - b}, \quad \forall t > 0. \tag{3.13}$$

By (3.7) and using again the notation $\widetilde{A}_i$, $i \geq 1$, we get that for any $t > 0$,

$$\begin{aligned} g(t) &\leq t^{\gamma}\mathbb{E}\big[\Phi(tA_1)1_{(\nu=1)}\big] + t^{\gamma}\mathbb{E}\big[1_{(\nu\geq 2)}\Phi(tA_1)\Phi(tA_2)\big] \\ &= \mathbb{E}\big[g(t\widetilde{A}_1)\big] + t^{-\gamma}\mathbb{E}\big[1_{(\nu\geq 2)} g(tA_1) g(tA_2) A_1^{-\gamma} A_2^{-\gamma}\big] \\ &\leq \mathbb{E}\big[g(t\widetilde{A}_1)\big] + c_4^2 t^{\gamma - 2b}\mathbb{E}\big[1_{(\nu\geq 2)} A_1^{-b} A_2^{-b}\big] \quad \text{(by (3.13))} \\ &=: \mathbb{E}\big[g(t\widetilde{A}_1)\big] + C_\varepsilon t^{-\varepsilon}, \end{aligned} \tag{3.14}$$

with $C_\varepsilon := c_4^2 \mathbb{E}[1_{(\nu\geq 2)} A_1^{-b} A_2^{-b}] \leq c_4^2 \mathbb{E}[\sum_{i=1}^{\nu} A_i^{-2b}]$ by Cauchy–Schwarz' inequality. Then $C_\varepsilon < \infty$ by the assumption (1.6) and the choice that $b < a/2$.

Let $r > 1$. As before, we shall iterate (3.14) up to the stopping time α_r (cf. (3.10)). We have that

$$\begin{aligned} g(r) &\leq C_\varepsilon r^{-\varepsilon} + \mathbb{E}\big[g(r\widetilde{A}_1)1_{(\alpha_r=1)}\big] + \mathbb{E}\big[1_{(\alpha_r>1)}\big(C_\varepsilon (r\widetilde{A}_1)^{-\varepsilon} + g(r\widetilde{A}_1\widetilde{A}_2)\big)\big] \\ &= C_\varepsilon r^{-\varepsilon} + C_\varepsilon \mathbb{E}\big[(r\widetilde{A}_1)^{-\varepsilon} 1_{(\alpha_r>1)}\big] + \mathbb{E}\left[g\left(r\prod_{i=1}^{2\wedge\alpha_r} \widetilde{A}_i\right)\right]. \end{aligned}$$

By induction, we get that for any $n \geq 2$,

$$\begin{aligned} g(r) &\leq C_\varepsilon r^{-\varepsilon} + C_\varepsilon \sum_{k=1}^{n-1} \mathbb{E}\left[1_{(\alpha_r>k)}\left(r\prod_{i=1}^{k}\widetilde{A}_i\right)^{-\varepsilon}\right] + \mathbb{E}\left[g\left(r\prod_{i=1}^{n\wedge\alpha_r}\widetilde{A}_i\right)\right] \\ &= C_\varepsilon r^{-\varepsilon} + C_\varepsilon \mathbb{E}\left[\sum_{k=1}^{n\wedge\alpha_r - 1} \mathrm{e}^{\varepsilon(X_k - \log r)}\right] + \mathbb{E}\big[g\big(r\mathrm{e}^{-X_{n\wedge\alpha_r}}\big)\big], \end{aligned} \tag{3.15}$$

by using the random walk $X_j \equiv -\sum_{i=1}^{j} \log \widetilde{A}_i$, $j \geq 1$. The random walk (X_j) has positive drift and $\mathbb{E}[X_1^2] = \mathbb{E}[1_{(\nu=1)} \sum_{|u|=1} (V(u))^2 \mathrm{e}^{\gamma V(u)}] < \infty$ by the assumption (1.6), then by Lemma 5 in [3],

$$\mathbb{E}\left[\sum_{k=1}^{\alpha_r - 1} \mathrm{e}^{\varepsilon(X_k - \log r)}\right] \leq C'_\varepsilon < \infty,$$

for some constant C'_ε independent of r. On the other hand, $g(r\mathrm{e}^{-X_{\alpha_r}}) \leq 1$ (since $r\mathrm{e}^{-X_{\alpha_r}} \leq 1$), then we obtain that for all $r > 1$, $n \geq 2$,

$$g(r) \leq C_\varepsilon + C'_\varepsilon + 1 + \mathbb{E}\big[g\big(r\mathrm{e}^{-X_n}\big)1_{(n<\alpha_r)}\big] \leq C''_\varepsilon + c_4 r^{\varepsilon}\mathbb{E}\big[\mathrm{e}^{-\varepsilon X_n} 1_{(n<\alpha_r)}\big], \tag{3.16}$$

where in the last inequality we have used the facts that $t := r\mathrm{e}^{-X_n} \ge 1$ on $\{n < \alpha_r\}$ and that $g(t) \le c_4 t^{\varepsilon}$ for any $t \ge 1$ by Lemma 3.1.

Remark that $\mathbb{E}[\mathrm{e}^{-\varepsilon X_1}] = \mathbb{E}[(\widetilde{A}_1)^{\varepsilon}] = \mathbb{E}[1_{(\nu=1)}(A_1)^{-\gamma+\varepsilon}] < 1$ by convexity. Then $\mathbb{E}[\mathrm{e}^{-\varepsilon X_n}] \to 0$ as $n \to \infty$, which in view of (3.16) yield that for any $r > 1$ (ε being fixed), $g(r) \le C''_{\varepsilon}$, i.e.

$$\Phi(r) \le C''_{\varepsilon} r^{-\gamma}, \quad \forall r > 1.$$

This and (3.12) imply (3.6): $\Phi(r) \asymp r^{-\gamma}$ for all $r \ge 1$. The small deviation in (1.14) follows from a standard Tauberian argument (see e.g. [25], Lemma 4.4).

3.2. *Proof of Theorem* 1.3: *The Böttcher case*

The proof of (1.15) goes in the same spirit as that of (3.6). Let $h(t) := -\log \mathbb{E}[\mathrm{e}^{-tZ}]$, $t \ge 0$. Note that h is an increasing, concave function and vanishing at zero. Using the notations introduced in (3.3), we get that

$$\mathrm{e}^{-h(t)} = \mathbb{E}\big[\mathrm{e}^{-\sum_{i=1}^{\nu} h(tA_i)}\big], \quad \forall t \ge 0.$$

On an enlarged probability space, we may find a random variable ξ such that

$$\mathbb{P}(\xi = i | \mathcal{A}) = \frac{A_i^{\beta}}{\sum_{j=1}^{\nu} A_j^{\beta}}, \quad 1 \le i \le \nu,$$

where $\mathcal{A} := \sigma\{A_i, 1 \le i \le \nu, \nu\}$. Then $\sum_{i=1}^{\nu} h(tA_i) = (\sum_{i=1}^{\nu} A_i^{\beta})\mathbb{E}[\frac{h(tA_\xi)}{A_\xi^{\beta}} | \mathcal{A}]$, and by Jensen's inequality, we have that for any $t \ge 0$,

$$\mathrm{e}^{-\sum_{i=1}^{\nu} h(tA_i)} \le \mathbb{E}\left[\exp\left(-\left(\sum_{i=1}^{\nu} A_i^{\beta}\right)\frac{h(tA_\xi)}{A_\xi^{\beta}}\right)\Big|\mathcal{A}\right].$$

Write for brevity

$$B := A_\xi, \qquad \eta := \frac{1}{A_\xi^{\beta}}\left(\sum_{i=1}^{\nu} A_i^{\beta}\right) > 1, \quad \text{a.s.}$$

[$\eta > 1$ because $\nu \ge 2$ a.s.] Then for any $t \ge 0$, we have

$$\mathrm{e}^{-h(t)} \le \mathbb{E}\big[\mathrm{e}^{-\eta h(tB)}\big]. \tag{3.17}$$

We shall iterate the inequality (3.17) up to some random times: Let $(\eta_i, B_i)_{i\ge 1}$ be an i.i.d. copies of (η, B). Let $r > 1$ and define

$$\Upsilon_r := \inf\left\{i \ge 1: \prod_{j=1}^{i} B_j \le \frac{1}{r}\right\}.$$

Observe that

$$\mathbb{E}[\log B] = \mathbb{E}\left[\frac{\sum_{i=1}^{\nu} A_i^{\beta} \log A_i}{\sum_{i=1}^{\nu} A_i^{\beta}}\right] = -\mathbb{E}\left[\frac{\sum_{|u|=1} \mathrm{e}^{-\beta V(u)} V(u)}{\sum_{|u|=1} \mathrm{e}^{-\beta V(u)}}\right] = \psi'(\beta),$$

where $\psi(x) := \mathbb{E}[\log \sum_{|u|=1} \mathrm{e}^{-xV(u)}]$ for $0 \le x \le \chi$. Note that ψ is convex on $[0, \chi]$, $\psi(\chi) < \log \mathbb{E}[\sum_{|u|=1} \mathrm{e}^{-\chi V(u)}] \le 0$, and $\psi(\beta) \ge 0$ since $\sum_{|u|=1} \mathrm{e}^{-\beta V(u)} \ge 1$ by the definition of β. By convexity, $\psi'(\beta) \le$

$\frac{\psi(\chi)-\psi(\beta)}{\chi-\beta} < 0$. Then $\mathbb{E}[\log B] < 0$ which implies that $\Upsilon_r < \infty$, a.s. By (3.17), we see that for

$$\begin{aligned} e^{-h(r)} &\le \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(rB_1 \le 1)}\big] + \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(rB_1>1)}\big] \\ &= \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(\Upsilon_r=1)}\big] + \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(rB_1>1)}\big]. \end{aligned}$$

Applying (3.17) to $t = rB_1$, we get that

$$e^{-\eta_1 h(rB_1)} \le \big(\mathbb{E}\big[e^{-\eta_2 h(rB_1B_2)} \big| \sigma\{\eta_1, B_1\}\big]\big)^{\eta_1} \le \mathbb{E}\big[e^{-\eta_1\eta_2 h(rB_1B_2)} \big| \sigma\{\eta_1, B_1\}\big],$$

by Jensen's inequality, since $\eta_1 > 1$. It follows that $\mathbb{E}[e^{-\eta_1 h(rB_1)} 1_{(rB_1>1)}] \le \mathbb{E}[1_{(rB_1>1)} e^{-\eta_1\eta_2 h(rB_1B_2)}]$, hence

$$\begin{aligned} e^{-h(r)} &\le \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(\Upsilon_r=1)}\big] + \mathbb{E}\big[1_{(rB_1>1)} e^{-\eta_1\eta_2 h(rB_1B_2)}\big] \\ &= \mathbb{E}\big[e^{-\eta_1 h(rB_1)} 1_{(\Upsilon_r=1)}\big] + \mathbb{E}\big[e^{-\eta_1\eta_2 h(rB_1B_2)} 1_{(\Upsilon_r=2)}\big] + \mathbb{E}\big[1_{(rB_1B_2>1)} e^{-\eta_1\eta_2 h(rB_1B_2)}\big]. \end{aligned}$$

Again applying (3.17) to $t = rB_1B_2$ and using Jensen's inequality (since $\eta_1\eta_2 > 1$), we get that $\mathbb{E}[1_{(rB_1B_2>1)} \times e^{-\eta_1\eta_2 h(rB_1B_2)}] \le \mathbb{E}[1_{(rB_1B_2>1)} e^{-\eta_1\eta_2\eta_3 h(rB_1B_2B_3)}]$, and so on. We get that for any $n \ge 1$,

$$\begin{aligned} e^{-h(r)} &\le \mathbb{E}\big[e^{-(\prod_{i=1}^{\Upsilon_r}\eta_i) h(r\prod_{i=1}^{\Upsilon_r} B_i)} 1_{(\Upsilon_r \le n)}\big] + \mathbb{E}\big[e^{-(\prod_{i=1}^{n}\eta_i) h(r\prod_{i=1}^{n} B_i)} 1_{(\Upsilon_r > n)}\big] \\ &=: A_{(3.18)} + C_{(3.18)}. \end{aligned} \tag{3.18}$$

By (1.8), $B \ge e^{-K}$ a.s., then $\frac{1}{r} \ge \prod_{i=1}^{\Upsilon_r} B_i > \frac{1}{r} e^{-K}$. Notice that by (1.10) the definition of β, $\sum_{i=1}^{\nu} A_i^{\beta} \ge 1$ a.s. Then $\eta \ge B^{-\beta}$ and $\prod_{i=1}^{\Upsilon_r} \eta_i \ge r^{\beta}$. It follows that for any n,

$$A_{(3.18)} \le e^{-r^{\beta} h(e^{-K})}.$$

To deal with $C_{(3.18)}$, we remark that on $\{\Upsilon_r > n\}$, $r\prod_{i=1}^{n} B_i \ge 1$. It follows that

$$C_{(3.18)} \le \mathbb{E}\big[e^{-h(1)\prod_{i=1}^{n}\eta_i}\big].$$

Since $\eta_i > 1$ a.s., $\prod_{i=1}^{n} \eta_i \uparrow \infty$ as $n \to \infty$, then by the monotone convergence theorem $\limsup_{n\to\infty} C_{(3.18)} = 0$. Letting $n \to \infty$ in (3.18), we obtain that

$$\mathbb{E}\big[e^{-rZ}\big] \equiv e^{-h(r)} \le e^{-h(e^{-K}) r^{\beta}}, \quad \forall r > 1, \tag{3.19}$$

which is stronger than the upper bound in (1.15).

To prove the lower bound, recalling that $\operatorname{ess\,inf} \sum_{i=1}^{\nu} A_i^{\beta} = 1$ and $A_i \ge e^{-K}$, we deduce that for any small $\varepsilon > 0$, there are some integer $2 \le k \le \operatorname{ess\,sup} \nu$, and some real numbers $a_1, \ldots, a_k \in (0, 1)$ such that $\sum_{i=1}^{k} a_i^{\beta} \ge 1$ and $\sum_{i=1}^{k} a_i^{\beta+\varepsilon} < 1$ and $p := \mathbb{P}(A_i \le a_i, \forall 1 \le i \le k, \nu = k) > 0$. Therefore

$$e^{-h(t)} = \mathbb{E}\big[e^{-\sum_{i=1}^{\nu} h(tA_i)}\big] \ge p e^{-\sum_{i=1}^{k} h(ta_i)}, \quad t \ge 0.$$

Let $b := \log(1/p) > 0$ and define a random variable $Y \in \{a_1, \ldots, a_k\}$ such that for any measurable and non-negative function f, $\mathbb{E}[f(Y)] = \frac{1}{k}\sum_{i=1}^{k} f(a_i)$. Therefore,

$$h(t) \le b + k\mathbb{E}\big[h(tY)\big], \quad \forall t \ge 0. \tag{3.20}$$

As in the proof of the upper bound, we shall iterate the above inequality up to some random times: Let $(Y_j)_{j\ge 1}$ be an i.i.d. copies of Y. For $r > 1$, we define

$$\theta := \theta_r := \inf\left\{ j \ge 1: \prod_{i=1}^{j} Y_i \le \frac{1}{r} \right\}.$$

Since $Y \le \max_{1\le i\le k} a_i < 1$, θ is a bounded random variable. Going back to (3.20), we get that

$$\begin{aligned} h(r) &\le b + k\mathbb{E}\big[h(rY_1)1_{(rY_1\le 1)}\big] + k\mathbb{E}\big[h(rY_1)1_{(rY_1>1)}\big] \\ &\le b + k\mathbb{E}\big[h(rY_1)1_{(\theta=1)}\big] + k\mathbb{E}\big[1_{(rY_1>1)}\big(b + kh(rY_1Y_2)\big)\big] \\ &= b + k\mathbb{E}\big[h(rY_1)1_{(\theta=1)}\big] + bk\mathbb{P}(\theta > 1) + k^2\mathbb{E}\big[1_{(rY_1>1)}h(rY_1Y_2)\big]. \end{aligned}$$

By induction, we get that for any $n \ge 1$,

$$\begin{aligned} h(r) \;&\le\; b\sum_{j=0}^{n} k^j\mathbb{P}(\theta > j) + \mathbb{E}\left[k^{\theta\wedge n}h\left(r\prod_{i=1}^{\theta\wedge n} Y_i\right)\right] \\ &=: A_{(3.21)} + C_{(3.21)}. \end{aligned} \tag{3.21}$$

Elementary computations yield that

$$A_{(3.21)} = \frac{b}{k-1}\mathbb{E}\big[k^{\theta\wedge(n+1)} - 1\big] \le \frac{b}{k-1}\mathbb{E}\big[k^{\theta}\big].$$

Recalling θ is bounded hence $\mathbb{E}[k^\theta] < \infty$. For $C_{(3.21)}$, we use the fact that $Y_i \le \max_{1\le j\le k} a_j =: \overline{a} < 1$. Remark that $r\prod_{i=1}^{n} Y_i \le 1$. Then

$$\begin{aligned} C_{(3.21)} :&= \mathbb{E}\left[k^\theta h\left(r\prod_{i=1}^{\theta} Y_i\right)1_{(\theta\le n)}\right] + \mathbb{E}\left[k^n h\left(r\prod_{i=1}^{n} Y_i\right)1_{(\theta>n)}\right] \\ &\le\; h(1)\mathbb{E}\big[k^\theta\big] + h\big(r\overline{a}^n\big)\mathbb{E}\big[k^n 1_{(\theta>n)}\big] \\ &\le\; h(1)\mathbb{E}\big[k^\theta\big] + h\big(r\overline{a}^n\big)\mathbb{E}\big[k^\theta\big]. \end{aligned}$$

Since $r\overline{a}^n \to 0$ as $n \to \infty$, we get that [recalling that θ depends on r]

$$h(r) \le \left(h(1) + \frac{b}{k-1}\right)\mathbb{E}\big[k^\theta\big], \quad \forall r > 1. \tag{3.22}$$

To estimate $\mathbb{E}[k^\theta]$, let us find $\lambda > 0$ such that $\mathbb{E}[Y^\lambda] = \frac{1}{k}$. By the law of Y, this is equivalent to $\sum_{i=1}^{k} a_i^\lambda = 1$.

By the choice of (a_i), we have $\beta \le \lambda < \beta + \varepsilon$. Then the process $n \to k^n\prod_{i=1}^{n} Y_i^\lambda$ is a martingale (moreover uniformly integrable on $[0, \theta]$). Hence the optional stopping theorem implies that

$$1 = \mathbb{E}\left[k^\theta\prod_{i=1}^{\theta} Y_i^\lambda\right] \ge \mathbb{E}\big[k^\theta\big]r^{-\lambda}\min_{1\le i\le k} a_i^\lambda,$$

since $\prod_{i=1}^{\theta} Y_i \ge \frac{1}{r}\min_{1\le i\le k} a_i$. This and (3.22) give that

$$h(r) \le \left(h(1) + \frac{b}{k-1}\right)\max_{1\le i\le k} a_i^{-\lambda}r^\lambda, \quad \forall r > 1,$$

yielding the lower bound in (1.15) since $\lambda < \beta + \varepsilon$. This completes the proof of (1.15). Finally, by using the elementary inequalities: for any $\varepsilon, t > 0$, $\mathrm{e}^{-\varepsilon t}\mathbb{P}(Z < \varepsilon) \le \mathbb{E}[\mathrm{e}^{-tZ}] \le \mathbb{P}(Z < \varepsilon) + \mathrm{e}^{-\varepsilon t}$, we immediately deduce from (1.15) that $\mathbb{P}(Z < \varepsilon) = \mathrm{e}^{-\varepsilon^{-\beta/(1-\beta)+o(1)}}$ as $\varepsilon \to 0$.

4. Proof of Theorem 1.1

Let us give some preliminary estimates on the branching random walk:

Lemma 4.1. *Assume* (1.1) *and* (1.2). *There exists some constants* $c_5, c_6 > 0$ *such that for* $n \ge 1$,

$$\mathbb{P}\Big(\min_{|u|=n} \overline{V}(u) < c_5 n^{1/3}\Big) \le c_6 \mathrm{e}^{-c_5 n^{1/3}}, \tag{4.1}$$

where we recall that for any $|u| = n$, $\overline{V}(u) := \max_{1\le i\le n} V(u_i)$. *Consequently, for any* $0 < \lambda \le c_5 n^{1/3}$, *we have*

$$\mathbb{P}\Big(\max_{u\in \pounds_\lambda} |u| > n\Big) \le c_6 \mathrm{e}^{-c_5 n^{1/3}}. \tag{4.2}$$

We mention that under an extra integrability condition, i.e. $\exists\delta > 0$ such that $\mathbb{E}[\nu^{1+\delta}] < \infty$, $n^{-1/3}\min_{|u|=n} \overline{V}(u) \to (\frac{3\pi^2\sigma^2}{2})^{1/3}$, $\mathbb{P}^*$-a.s. (see [13] and [12]) and the probability term in (4.1) is equal to $\mathrm{e}^{(c_5-(3\pi^2\sigma^2/2)^{1/3}+o(1))n^{1/3}}$ for any $0 < c_5 < (\frac{3\pi^2\sigma^2}{2})^{1/3}$ (see [13], Proposition 2.3). Here, we only assume (1.1) and (1.2), and we do not seek the precise upper bound in (4.1).

Proof of Lemma 4.1. We shall use the following fact (see Shi [31]):

$$\mathbb{P}\Big(\inf_{u\in\mathbb{T}} V(u) < -\lambda\Big) \le \mathrm{e}^{-\lambda}, \quad \forall\lambda \ge 0. \tag{4.3}$$

Consider $0 < c < (\frac{\pi^2\sigma^2}{8})^{1/3}$. Then

$$\begin{aligned}
\mathbb{P}\Big(\min_{|u|=n} \overline{V}(u) < cn^{1/3}, \inf_{u\in\mathbb{T}} V(u) \ge -cn^{1/3}\Big) &\le \mathbb{E}\Big[\sum_{|u|=n} 1_{(\max_{1\le i\le n}|V(u_i)|\le cn^{1/3})}\Big] \\
&= \mathbb{E}\big[\mathrm{e}^{S_n} 1_{(\max_{1\le i\le n}|S_i|\le cn^{1/3})}\big] \quad \text{(by (2.7))} \\
&\le \mathrm{e}^{cn^{1/3}} \mathbb{P}\Big(\max_{1\le i\le n}|S_i| \le cn^{1/3}\Big) \\
&= \mathrm{e}^{cn^{1/3}} \mathrm{e}^{-(\pi^2\sigma^2/(8c^2)+o(1))n^{1/3}},
\end{aligned}$$

where the last equality follows from Mogulskii [29]. This and (4.3) easily yield the lemma by choosing a sufficiently small constant c. □

Recall (1.19). Define for $a \in (0, \infty]$ and $\lambda > 0$,

$$\pounds_\lambda^{(a)} := \big\{u \in \pounds_\lambda\colon\ V(u) \le \lambda + a\big\}. \tag{4.4}$$

In particular, $\pounds_\lambda^{(\infty)} = \pounds_\lambda$. Recall (1.20). Since the function $x \to x\mathrm{e}^{-x}$ is decreasing for $x \ge 1$, then for any $\lambda > 1$, $D_{\pounds_\lambda} \le \lambda \mathrm{e}^{-\lambda}\#\pounds_\lambda$, which implies that

$$\liminf_{\lambda\to\infty} \lambda\mathrm{e}^{-\lambda}\#\pounds_\lambda \ge D_\infty > 0 \quad \text{a.s. on } \mathcal{S}. \tag{4.5}$$

If $\nu = \infty$ [which is allowed under (1.1) and (1.2)], then $\#\pounds_\lambda = \infty$ hence (4.5) cannot be strengthened into a true limit. We present a similar result for $\pounds_\lambda^{(a)}$:

Lemma 4.2. *Assume* (1.1), (1.2) *and that* $\mathbb{E}[\sum_{|u|=1}(V(u)^+)^3\mathrm{e}^{-V(u)}] < \infty$. *There exists some* $a_0 > 0$ *such that for all large* $a \ge a_0$, *almost surely on the set of non-extinction* $\mathcal{S}$,

$$0 < \liminf_{\lambda\to\infty} \lambda\mathrm{e}^{-\lambda}\#\pounds_\lambda^{(a)} \le \limsup_{\lambda\to\infty} \lambda\mathrm{e}^{-\lambda}\#\pounds_\lambda^{(a)} < \infty.$$

Proof. We only deal with the case when the distribution of Θ is non-lattice, in this case, the limit exists. The lattice case can be treated in a similar way, by applying Gatzouras ([17], Theorem 5.2), a lattice version of Nerman's [30] result, but the cyclic phenomenon could prevent from the existence of limit. In the non-lattice case, we are going to prove that for any $a > 0$, almost surely on the set of non-extinction $\mathcal{S}$,

$$\lim_{\lambda\to\infty} \lambda e^{-\lambda} \# \pounds_\lambda^{(a)} = c_7(a) D_\infty, \tag{4.6}$$

where $c_7(a) := \frac{1}{\mathbb{E}[S_{\tau_0}]}\mathbb{E}[e^{\min(a,S_{\tau_0})} - 1]$, and $S.$ and τ_0 are defined by (2.7) and (2.8) respectively. Obviously, $c_7(a) > 0$ for all large a.

To get (4.6), we consider a new point process $\widehat{\Theta} := \sum_{u\in\pounds_0} \delta_{\{V(u)\}}$ on $(0,\infty)$. Generate a branching random walk $(\widehat{V}(u), u \in \widehat{\mathbb{T}})$ from the point process $\widehat{\Theta}$, in the same way as $(V(u), u \in \mathbb{T})$ do from Θ. Remark that $\mathcal{S} = \{\sup_{u\in\mathbb{T}} V(u) = \infty\} = \{\widehat{\mathbb{T}} \text{ is infinite}\}$, and

$$\#\pounds_\lambda^{(a)} = \sum_{u\in\widehat{\mathbb{T}}} \phi_u\bigl(\lambda - \widehat{V}(u)\bigr), \qquad \sum_{u\in\pounds_\lambda} e^{-V(u)+\lambda} = \sum_{u\in\widehat{\mathbb{T}}} \psi_u\bigl(\lambda - \widehat{V}(u)\bigr),$$

where

$$\phi_u(y) := 1_{(y\ge 0)} \sum_{v:\overleftarrow{v}=u} 1_{(y<\widehat{V}(v)-\widehat{V}(u)\le y+a)}, \qquad \psi_u(y) := 1_{(y\ge 0)} \sum_{v:\overleftarrow{v}=u} e^{y-(\widehat{V}(v)-\widehat{V}(u))} 1_{(\widehat{V}(v)-\widehat{V}(u)>y)}.$$

Applying Theorem 6.3 in Nerman [30] (with $\alpha = 1$ there) gives that almost surely on $\mathcal{S}$,

$$\frac{\sum_{u\in\widehat{\mathbb{T}}} \phi_u(\lambda - \widehat{V}(u))}{\sum_{u\in\widehat{\mathbb{T}}} \psi_u(\lambda - \widehat{V}(u))} \to \frac{\mathbb{E}[\sum_{|u|=1,u\in\widehat{\mathbb{T}}}(e^{-(\widehat{V}(u)-a)^+} - e^{-\widehat{V}(u)})]}{\mathbb{E}[\sum_{|u|=1,u\in\widehat{\mathbb{T}}} \widehat{V}(u)e^{-\widehat{V}(u)}]}.$$

Remark that $\mathbb{E}[\sum_{|u|=1,u\in\widehat{\mathbb{T}}}(e^{-(\widehat{V}(u)-a)^+} - e^{-\widehat{V}(u)})] = \mathbb{E}[\sum_{u\in\pounds_0}(e^{-(V(u)-a)^+} - e^{-V(u)})] = \mathbb{E}[e^{\min(a,S_{\tau_0})} - 1]$ and $\mathbb{E}[\sum_{|u|=1,u\in\widehat{\mathbb{T}}} \widehat{V}(u)e^{-\widehat{V}(u)}] = \mathbb{E}[\sum_{u\in\pounds_0} V(u)e^{-V(u)}] = \mathbb{E}[S_{\tau_0}]$. Hence on $\mathcal{S}$, a.s.,

$$\frac{\#\pounds_\lambda^{(a)}}{\sum_{u\in\pounds_\lambda} e^{-V(u)+\lambda}} \to c_7(a). \tag{4.7}$$

On the other hand, almost surely,

$$D_{\pounds_\lambda} = \lambda e^{-\lambda}\left(\sum_{u\in\pounds_\lambda} e^{-V(u)+\lambda} + \frac{1}{\lambda}\eta_\lambda\right) \to D_\infty, \quad \lambda\to\infty, \tag{4.8}$$

where $\eta_\lambda := \sum_{u\in\pounds_\lambda}(V(u) - \lambda)e^{-V(u)+\lambda}$. By the many-to-one formula and the assumption, $\mathbb{E}[(S_1^+)^3] = \mathbb{E}[\sum_{|u|=1}(V(u)^+)^3 e^{-V(u)}] < \infty$. Then by Doney [11], $\mathbb{E}[S_{\tau_0}^2] < \infty$.

Note that $\eta_\lambda = \sum_{u\in\widehat{\mathbb{T}}} \widetilde{\psi}_u(\lambda - \widehat{V}(u))$ with $\widetilde{\psi}_u(y) := 1_{(y\ge 0)} \sum_{v:\overleftarrow{v}=u} e^{y-(\widehat{V}(v)-\widehat{V}(u))}(\widehat{V}(v) - \widehat{V}(u) - y) \times 1_{(\widehat{V}(v)-\widehat{V}(u)>y)}$. In the same manner we get that almost surely on $\mathcal{S}$,

$$\lim_{\lambda\to\infty} \frac{\eta_\lambda}{\sum_{u\in\pounds_\lambda} e^{-V(u)+\lambda}} = c_8, \tag{4.9}$$

with $c_8 := \frac{1}{2}\frac{\mathbb{E}[S_{\tau_0}^2]}{\mathbb{E}[S_{\tau_0}]} > 0$. It follows that a.s. on $\mathcal{S}$, $\sum_{u\in\pounds_\lambda} e^{-V(u)+\lambda} \sim \frac{1}{\lambda}e^\lambda D_{\pounds_\lambda} \sim \frac{1}{\lambda}e^\lambda D_\infty$ as $\lambda\to\infty$. This combined with (4.7) and (4.8) yield (4.6), as desired. $\square$

Remark 4.3. *The condition $\mathbb{E}[\sum_{|u|=1}(V(u)^+)^3 e^{-V(u)}] < \infty$ was used in the above proof of Lemma 4.2 only to obtain (4.9) which controls the contribution of η_λ in $D_{\pounds_\lambda}$. We do not know how to relax this condition.*

We consider now some deviations on the minimum $\mathbb{M}_n$. If the distribution of Θ is non-lattice, Aïdékon (Proposition 4.1, [2]) proved that for any $A>0$ and for all large n,λ such that $A\le\lambda\le\frac{3}{2}\log n-A$,

$$\mathbb{P}\biggl(\mathbb{M}_n<\frac{3}{2}\log n-\lambda\biggr)=\bigl(c_9+o_A(1)\bigr)\lambda e^{-\lambda},$$

with c_9 some positive constant and $o_A(1)\to 0$ as $A\to\infty$ uniformly on n,λ. We shall need in the proof of Theorem 1.1 an estimate which holds uniformly on λ.

Lemma 4.4. *Assume* (1.1) *and* (1.2). *There is some constant $c_{10}>0$ such that*

$$\mathbb{P}\biggl(\mathbb{M}_n<\frac{3}{2}\log n-\lambda\biggr)\le c_{10}(1+\lambda)e^{-\lambda},\quad \forall n\ge 1,\lambda\ge 0.$$

Proof. We are going to prove that there exists some constant $C>0$ such that for any $n\ge1,\lambda\ge0,\alpha>0$,

$$\mathbb{P}\biggl(\mathbb{M}_n\le\frac{3}{2}\log n-\lambda,\min_{|u|\le n}V(u)\ge-\alpha\biggr)\le C(1+\alpha)e^{-\lambda}\biggl(1+\frac{(1+(\alpha+(3/2)\log n-\lambda)^+)^5}{n^{1/2}}\biggr).\tag{4.10}$$

Then by taking $\alpha=\lambda$ in (4.10) and (4.3), we get the Lemma.

To prove (4.10), we write for brevity $b:=\frac{3}{2}\log n-\lambda-1$. Note that we can assume $b+1>-\alpha$. otherwise there is nothing to prove in (4.10). For those $|u|=n$ such that $V(u)<b+1$, either $\min_{n/2\le j\le n}V(u_j)>b$, or $\min_{n/2\le j\le n}V(u_j)\le b$; for the latter case, we shall consider the first $j\ge\frac{n}{2}$ such that $V(u_j)\le b$. Then

$$\mathbb{P}\biggl(\mathbb{M}_n\le\frac{3}{2}\log n-\lambda,\min_{|u|\le n}V(u)\ge-\alpha\biggr)\le\mathbb{P}(E_{(4.11)})+\mathbb{P}(F_{(4.11)}),\tag{4.11}$$

with

$$E_{(4.11)}:=\Bigl\{\exists|u|=n\colon V(u)\le b+1,\underline{V}(u)\ge-\alpha,\min_{n/2\le j\le n}V(u_j)>b\Bigr\},$$

$$F_{(4.11)}:=\bigcup_{n/2\le j\le n}\Bigl\{\exists|u|=n\colon V(u)\le b+1,\min_{n/2\le i<j}V(u_i)>b,V(u_j)\le b,\underline{V}(u)\ge-\alpha\Bigr\},$$

where as before, $\underline{V}(u):=\min_{1\le i\le n}V(u_i)$ for any $|u|=n$. We estimate $\mathbb{P}(E_{(4.11)})$ as follows:

$$\begin{aligned}\mathbb{P}(E_{(4.11)})&\le\mathbb{E}\biggl[\sum_{|u|=n}1_{(V(u)\le b+1,\underline{V}(u)\ge-\alpha,\min_{n/2\le j\le n}V(u_j)>b)}\biggr]\\&=\mathbb{E}\bigl[e^{S_n}1_{(S_n\le b+1,\underline{S}_n\ge-\alpha,\min_{n/2\le j\le n}S_j>b)}\bigr]\quad\text{(by (2.7))}\\&\le e^b c_3(1+\alpha)n^{-3/2}\quad\text{(by (2.4))}\\&\le c_3(1+\alpha)e^{-\lambda}.\end{aligned}$$

To deal with $\mathbb{P}(F_{(4.11)})$, we consider $v=u_j$ and use the notation $|u|_v=n-j$ and $V_v(u):=V(u)-V(v)$ for $|u|=n$ and $v<u$. Then

$$\begin{aligned}&\mathbb{P}(F_{(4.11)})\\&\quad\le\sum_{n/2\le j\le n}\mathbb{E}\biggl[\sum_{|v|=j}1_{(\underline{V}(v)\ge-\alpha,\min_{n/2\le i<j}V(v_i)>b,V(v)\le b)}\sum_{|u|_v=n-j}1_{(V_v(u)\le b+1-V(v),\min_{j\le i\le n}V_v(u_i)\ge-\alpha-V(v))}\biggr]\\&\quad=\sum_{n/2\le j\le n}\mathbb{E}\biggl[\sum_{|v|=j}1_{(\underline{V}(v)\ge-\alpha,\min_{n/2\le i<j}V(v_i)>b,V(v)\le b)}\phi\bigl(V(v),n-j\bigr)\biggr]&(4.12)\\&\quad=:A_{(4.13)}+B_{(4.13)},&(4.13)\end{aligned}$$

where $A_{(4.13)}$ denotes the sum $\sum_{n/2\le j\le 3n/4}$ and $B_{(4.13)}$ the sum $\sum_{3n/4<j\le n}$ in (4.12), and

$$\begin{aligned}\phi(x,n-j) &:= \mathbb{E}\Big[\sum_{|u|_v=n-j} 1_{(V_v(u)\le b+1-V(v),\min_{j\le i\le n} V_v(u_i)\ge -\alpha-V(v))}\Big| V(v)=x\Big] \\ &= \mathbb{E}\big[e^{S_{n-j}} 1_{(S_{n-j}\le b+1-x,\underline{S}_{n-j}\ge -\alpha-x)}\big].\end{aligned}$$

Obviously, $\phi(x,n-j)\le e^{b+1-x}$. It also follows from (2.3) that

$$\phi(x,n-j)\le c_2(1+\alpha+x)(1+\alpha+b)^2(n-j+1)^{-3/2}e^{b-x}. \tag{4.14}$$

By the estimate that $\phi(x,n-j)\le e^{b+1-x}$, we get that

$$\begin{aligned}B_{(4.13)} &\le \sum_{3n/4\le j\le n}\mathbb{E}\Big[\sum_{|v|=j} 1_{(\underline{V}(v)\ge -\alpha,\min_{n/2\le i<j} V(v_i)>b, V(v)\le b)} e^{b+1-V(v)}\Big] \\ &= e^{b+1}\sum_{3n/4\le j\le n}\mathbb{P}\Big(\underline{S}_j\ge -\alpha,\ \min_{n/2\le i<j} S_i>b,\ S_j\le b\Big) \\ &\le c_3 e^b(1+\alpha)n^{-3/2} \quad (\text{by } (2.5)) \\ &\le c_3(1+\alpha)e^{-\lambda},\end{aligned}$$

since $b=\frac{3}{2}\log n-\lambda-1$. By using (4.14) and the many-to-one formula (2.7), we obtain that

$$\begin{aligned}A_{(4.13)} &\le C'\sum_{n/2\le j\le 3n/4}(1+b+\alpha)^2 n^{-3/2}e^b\mathbb{E}\big[(1+\alpha+S_j)1_{(\underline{S}_j\ge -\alpha,\min_{n/2\le i<j} S_i>b,S_j\le b)}\big] \\ &\le C'(1+b+\alpha)^3e^{-\lambda}\sum_{n/2\le j\le 3n/4}\mathbb{P}(\underline{S}_j\ge -\alpha,\ S_j\le b) \\ &\le C''(1+\alpha)(1+b+\alpha)^5e^{-\lambda}\sum_{n/2\le j\le 3n/4} j^{-3/2} \quad (\text{by } (2.3)) \\ &\le C(1+\alpha)\frac{(1+\alpha+(3/2)\log n-\lambda)^5}{n^{1/2}}e^{-\lambda},\end{aligned}$$

yielding (4.10) and completing the proof of the lemma. □

The tightness of $(\mathbb{M}_n-\frac{3}{2}\log n)_{n\ge 1}$ under (1.1) and (1.2) was implicitly contained in Aïdékon ([2]) (see also [8], and see [1] for exponential decay under some additional assumptions): Assume (1.1) and (1.2). We have[2]

$$\limsup_{\lambda\to\infty}\limsup_{n\to\infty}\mathbb{P}^*\Big(\mathbb{M}_n-\frac{3}{2}\log n\ge\lambda\Big)=0, \tag{4.15}$$

where as before, $\mathbb{P}^*(\cdot):=\mathbb{P}(\cdot|\mathcal{S})$. We need some tightness uniformly on n:

Lemma 4.5. *Assume* (1.1) *and* (1.2). *For any fixed $a>1$, we have*

$$\limsup_{n\to\infty}\mathbb{P}^*\Big(\max_{n\le k\le an}\mathbb{M}_k\ge\frac{3}{2}\log n+x\Big)\to 0,\quad \textit{as } x\to\infty.$$

---

[2] In fact, by Lemma 3.6 in [2] and using the fact that $\mathbb{M}_n$ is stochastically smaller than M_n^{kill}, we obtain that $\sup_{n\ge 3}\mathbb{P}(\mathbb{M}_n\ge\frac{3}{2}\log n)\le e^{-C}$ for some (small) constant $C>0$. For any $k\ge 1$, denote by $Z_k:=\sum_{|u|=k}1$ the number of individuals at generation k. By the triangular inequality and the branching property at k, we get that for any $n\ge k+3$, $\mathbb{P}(\mathbb{M}_n\ge\frac{3}{2}\log n+\lambda,\mathcal{S})\le\mathbb{P}(\exists|u|=k\colon\ V(u)>\lambda)+\mathbb{E}[1_{(Z_k>0)}e^{-CZ_k}]$. Letting $\lambda\to\infty$ and then $k\to\infty$, we get (4.15). The left tail $\limsup_{n\to\infty}\mathbb{P}(\mathbb{M}_n-\frac{3}{2}\log n<-\lambda)$, as $\lambda\to\infty$, follows from [2], see also Lemma 4.4.

Proof. Obviously, it is enough to prove the Lemma for $a = 2$. By Lemma 4.4, there exists some $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ and for all $n \le k \le 3n$,

$$\mathbb{P}\left(\mathbb{M}_{4n-k} \ge \frac{3}{2}\log n - \lambda\right) \ge \exp\bigl(-2c_{10}\lambda e^{-\lambda}\bigr). \tag{4.16}$$

Let $x \ge 2\lambda_0$ and $n \gg x$. Define

$$\kappa_x \equiv \kappa_x(n) := \inf\left\{k \ge n\colon\ \mathbb{M}_k \ge \frac{3}{2}\log n + x\right\} \quad \bigl(\inf_{\emptyset} = \infty\bigr).$$

Let $n \le k \le 3n$. Denote by $\mathcal{S}_k$ the event that the Galton–Watson tree $\mathbb{T}$ survives up to the generation k. Then $\mathcal{S}_k$ is non-increasing on k. On the set $\{\kappa_x = k\} \cap \mathcal{S}_k$, $V(u) > \frac{3}{2}\log n + x$ for any $|u| = k$. Let $0 < y < x - \lambda_0$. It follows from the branching property that on $\{\kappa_x = k\} \cap \mathcal{S}_k$,

$$\begin{aligned}
\mathbb{P}\left(\mathbb{M}_{4n} > \frac{3}{2}\log n + y \,\middle|\, \mathcal{F}_k\right) &= \prod_{|u|=k} \mathbb{P}\left(\mathbb{M}_{4n-k} \ge \frac{3}{2}\log n - \lambda\right)\bigg|_{\lambda = V(u)-y} \\
&\ge \exp\left(-2c_{10}\sum_{|u|=k}\bigl(V(u) - y\bigr)e^{-(V(u)-y)}\right) \\
&\ge \exp\bigl(-2c_{10}e^{y} D_k\bigr),
\end{aligned}$$

where we have used (4.16) to get the above first inequality.

Therefore for any $\varepsilon > 0$ and $n \le k \le 3n$,

$$\begin{aligned}
\mathbb{P}\left(\mathbb{M}_{4n} > \frac{3}{2}\log n + y, \mathcal{S}_k, \kappa_x = k\right) &\ge \mathbb{E}\bigl[e^{-2c_{10}e^{y}D_k} 1_{(\mathcal{S}_k \cap \{\kappa_x = k\})}\bigr] \\
&\ge e^{-\varepsilon}\mathbb{P}\bigl(A_{(4.18)} \cap \{\kappa_x = k\}\bigr),
\end{aligned} \tag{4.17}$$

where

$$A_{(4.18)} := \mathcal{S} \cap \left\{\sup_{j \ge 0} D_j \le \frac{\varepsilon}{2c_{10}} e^{-y}\right\}. \tag{4.18}$$

Since $\mathcal{S}_k \subset \mathcal{S}_n$ for $k \ge n$, (4.17) still holds if we replace $\mathcal{S}_k$ by $\mathcal{S}_n$ in the LHS. Taking the sum over $n \le k \le 3n$ for (4.17) (with $\mathcal{S}_k$ replaced by $\mathcal{S}_n$), we get that for any $\varepsilon > 0$, $0 < y < x - \lambda_0$ and all $n \ge n_0$,

$$\begin{aligned}
\mathbb{P}\left(A_{(4.18)} \cap \left\{\max_{n \le k \le 3n} \mathbb{M}_k \ge \frac{3}{2}\log n + x\right\}\right) &\le e^{\varepsilon}\mathbb{P}\left(\mathbb{M}_{4n} > \frac{3}{2}\log n + y, \mathcal{S}_n\right) \\
&\le e^{\varepsilon}\mathbb{P}\left(\mathbb{M}_{4n} > \frac{3}{2}\log n + y, \mathcal{S}\right) + e^{\varepsilon}\mathbb{P}\bigl(\mathcal{S}^c \cap \mathcal{S}_n\bigr) \\
&\le \varepsilon + e^{\varepsilon}\mathbb{P}\bigl(\mathcal{S}^c \cap \mathcal{S}_n\bigr),
\end{aligned} \tag{4.19}$$

by using (4.15) if we choose a sufficiently large constant $y = y(\varepsilon)$ only depending on ε. Since $\lim_{n\to\infty}\mathbb{P}(\mathcal{S}^c \cap \mathcal{S}_n) = 0$, then for $x > y(\varepsilon) + \lambda_0$ and all large $n \ge n_1(\varepsilon)$,

$$\mathbb{P}\left(A_{(4.18)} \cap \left\{\max_{n \le k \le 3n} \mathbb{M}_k \ge \frac{3}{2}\log n + x\right\}\right) \le 2\varepsilon. \tag{4.20}$$

Note the factor $3n$ in the above estimate and we fix our choice of $y \equiv y(\varepsilon)$ in $A_{(4.18)}$.

Now, we shall get rid of the term $A_{(4.18)}$ in (4.20). Let $z \in (y, x - \lambda_0)$. Recalling the definition of $\pounds_z$ in (1.19). Define

$$A_{(4.21)} := \left\{ \exists u \in \pounds_z \colon |u| \le x, V(u) \le x, \sup_{j \ge 0} D_j^{(u)} \le \frac{\varepsilon}{2c_{10}} \mathrm{e}^{-y}, \mathcal{S}^{(u)} \right\}, \tag{4.21}$$

where $(D_j^{(u)}, j \ge 0), \mathbb{M}_\cdot^{(u)}, \mathcal{S}^{(u)}$ are defined from the subtree $\mathbb{T}_u$ in the same way as $(D_j, j \ge 0), \mathbb{M}_\cdot, \mathcal{S}$ do from $\mathbb{T}$. Let $n > 2x$. The event $\{\max_{n \le k \le 2n} \mathbb{M}_k \ge \frac{3}{2} \log n + 2x, \mathcal{S}\}$ implies that for some $n \le k \le 2n$, for any $|v| = k$, $V(v) \ge \frac{3}{2} \log n + 2x$. If $A_{(4.21)} \ne \emptyset$, then we take an arbitrary $u \in A_{(4.21)}$ and get that $\mathbb{M}_{k-|u|}^{(u)} \ge \frac{3}{2} \log n + 2x - V(u) \ge \frac{3}{2} \log n + x$. It follows that for $z \in (y, x - \lambda_0)$ and for all large $n \ge n_2(x, \varepsilon)$,

$$\begin{aligned} \mathbb{P}\left(\max_{n \le k \le 2n} \mathbb{M}_k \ge \frac{3}{2} \log n + 2x, \mathcal{S}, A_{(4.21)} \ne \emptyset \right) &\le \max_{1 \le j \le x} \mathbb{P}\left(A_{(4.18)} \cap \left\{ \max_{n-j \le k \le 2n-j} \mathbb{M}_k \ge \frac{3}{2} \log n + x \right\} \right) \\ &\le 2\varepsilon, \end{aligned} \tag{4.22}$$

by applying (4.20) to $n - j$. On the other hand,

$$\begin{aligned} &\mathbb{P}(A_{(4.21)} = \emptyset, \mathcal{S}) \\ &\quad \le \mathbb{P}\left(\forall u \in \pounds_z, \sup_{j \ge 0} D_j^{(u)} \ge \frac{\varepsilon}{2c_{10}} \mathrm{e}^{-y} \text{ or } \left(\mathcal{S}^{(u)}\right)^c, \pounds_z \ne \emptyset \right) + \mathbb{P}\left(\max_{u \in \pounds_z} \max\big(V(u), |u|\big) \ge x \right) \\ &\quad = \mathbb{E}\left[\mathrm{e}^{-p(\varepsilon, y) \# \pounds_z} 1_{(\# \pounds_z > 0)} \right] + \mathbb{P}\left(\max_{u \in \pounds_z} \max\big(V(u), |u|\big) \ge x \right), \end{aligned} \tag{4.23}$$

where the last equality is due to the branching property at $\pounds_z$, and $p(\varepsilon, y) > 0$ is defined by $\mathrm{e}^{-p(\varepsilon, y)} := \mathbb{P}(\sup_{j \ge 0} D_j \ge \frac{\varepsilon}{2c_{10}} \mathrm{e}^{-y} \text{ or } \mathcal{S}^c)$.

Assembling (4.22) and (4.23) give that for any $z > y$,

$$\begin{aligned} C_{(4.24)} &:= \limsup_{x \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\max_{n \le k \le 2n} \mathbb{M}_k \ge \frac{3}{2} \log n + 2x, \mathcal{S} \right) \\ &\le \mathbb{E}\left[\mathrm{e}^{-p(\varepsilon, y) \# \pounds_z} 1_{(\# \pounds_z > 0)} \right] + 2\varepsilon. \end{aligned} \tag{4.24}$$

Notice that $\{\# \pounds_z > 0\}$ is non-increasing on z and its limit as $z \to \infty$ equals $\mathcal{S}$. Then $\mathbb{P}(\{\# \pounds_z > 0\} \cap \mathcal{S}^c) \to 0$ as $z \to \infty$. On $\mathcal{S}$, we have from (4.5) that $\pounds_z \to \infty$ as $z \to \infty$ almost surely, hence $\mathbb{E}[\mathrm{e}^{-p(\varepsilon, y) \# \pounds_z} 1_{(\# \pounds_z > 0)}] \le \mathbb{E}[\mathrm{e}^{-p(\varepsilon, y) \# \pounds_z} 1_{\mathcal{S}}] + \mathbb{P}(\{\# \pounds_z > 0\} \cap \mathcal{S}^c) \to 0$ as $z \to \infty$. Then letting $z \to \infty$, we see that $C_{(4.24)} \le 2\varepsilon$. This proves the Lemma since ε can be arbitrarily small. □

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. *Proof of the lower bound in Theorem* 1.1. Consider large integer j. Let $n_j := 2^j$ and $\lambda_j := a \log \log \log n_j$ with some constant $0 < a < 1$. Fix $\alpha > 0$ and put

$$A_j := \left\{ \mathbb{M}_{n_j} > \frac{3}{2} \log n_j + \lambda_j \right\}.$$

Recall that if the system dies out at generation n_j, then by definition $\mathbb{M}_{n_j} = \infty$. Define $\mathbb{M}_\cdot^{(u)}$ from the subtree $\mathbb{T}_u$ in the same way as $\mathbb{M}_\cdot$ does from $\mathbb{T}$. Then $A_j = \{\forall |u| = n_{j-1}, \mathbb{M}_{n_j - n_{j-1}}^{(u)} \ge \frac{3}{2} \log n_j + \lambda_j - V(u)\}$, which by the branching property at n_{j-1} implies that

$$\mathbb{P}(A_j | \mathcal{F}_{n_{j-1}}) = \prod_{|u| = n_{j-1}} \mathbb{P}\left(\mathbb{M}_{n_j - n_{j-1}} \ge \frac{3}{2} \log n_j + \lambda_j - x \right) \Big|_{x = V(u)},$$

with the usual convention: $\prod_{\emptyset} := 1$. By the lower limits of $\mathbb{M}_{n_{j-1}}$ (cf. (1.3)), a.s. for all large j, $\mathbb{M}_{n_{j-1}} \geq \frac{1}{3} \log n_{j-1} \sim \frac{\log 2}{3} j$, hence $x \equiv V(u) \gg \lambda_j$ since $\lambda_j \sim a \log\log j$. Applying Lemma 4.4 gives that on $\{\mathbb{M}_{n_{j-1}} \geq \frac{1}{3} \log n_{j-1}\}$, for some constant $C > 0$, for all $|u| = n_{j-1}$,

$$\mathbb{P}\biggl(\mathbb{M}_{n_j - n_{j-1}} < \frac{3}{2} \log n_j + \lambda_j - x\biggr)\bigg|_{x=V(u)} \leq C V(u) \mathrm{e}^{-(V(u)-\lambda_j)}.$$

It follows that

$$\begin{aligned}
\mathbb{P}(A_j | \mathcal{F}_{n_{j-1}}) &\geq 1_{(\mathbb{M}_{n_{j-1}} \geq (1/3) \log n_{j-1})} \prod_{|u|=n_{j-1}} \bigl(1 - C V(u) \mathrm{e}^{-(V(u)-\lambda_j)}\bigr) \\
&\geq 1_{(\mathbb{M}_{n_{j-1}} \geq (1/3) \log n_{j-1})} \exp\biggl(-2C \sum_{|u|=n_{j-1}} V(u) \mathrm{e}^{-(V(u)-\lambda_j)}\biggr) \\
&= 1_{(\mathbb{M}_{n_{j-1}} \geq (1/3) \log n_{j-1})} \exp\bigl(-2C \mathrm{e}^{\lambda_j} D_{n_{j-1}}\bigr).
\end{aligned}$$

Since $D_{n_{j-1}} \to D_\infty$, a.s., and $\mathrm{e}^{\lambda_j} \sim (\log j)^a$ with $a < 1$, we get that almost surely,

$$\sum_j \mathbb{P}(A_j | \mathcal{F}_{n_{j-1}}) = \infty,$$

which according to Lévy's conditional form of Borel–Cantelli's lemma ([24], Corollary 68), implies that $\mathbb{P}(A_i, \text{i.o.}) = 1$. Then

$$\limsup_{n\to\infty} \frac{1}{\log\log\log n} \biggl(\mathbb{M}_n - \frac{3}{2} \log n\biggr) \geq a, \qquad \text{a.s.}$$

The lower bound follows by letting $a \to 1$.

Proof of the upper bound in Theorem 1.1. Let $\delta > 0$ be small. Recall (4.4). Let $a \geq a_0$ be as in Lemma 4.2 such that a.s. on $\mathcal{S}$, $\#\pounds_\lambda^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda}$ for all large λ. Let $b > 0$ such that $\mathrm{e}^{-b} > q \equiv \mathbb{P}(\mathcal{S}^c)$. By Lemma 4.5, there exists some constant $x_0 > 0$ such that

$$\mathbb{P}\Bigl(\max_{n \leq k \leq 4n} \mathbb{M}_k > \frac{3}{2} \log n + x_0\Bigr) \leq \mathrm{e}^{-b}, \quad \forall n \geq n_0.$$

Let $x_1 := x_0 + a$. Consider large integer j and define $n_j := 2^j$, $\lambda_j := (1 + 2\delta) \log\log\log n_j$. Define

$$B_j := \Bigl\{\max_{n_j < k \leq n_{j+1}} \mathbb{M}_k > \frac{3}{2} \log n_j + \lambda_j + x_1\Bigr\} \cap \mathcal{S}.$$

Then,

$$\begin{aligned}
&\mathbb{P}\Bigl(B_j, \#\pounds_{\lambda_j}^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda_j}, \max_{u \in \pounds_{\lambda_j}^{(a)}} |u| \leq n_{j-1}\Bigr) \\
&\quad \leq \mathbb{P}\Bigl(\forall u \in \pounds_{\lambda_j}^{(a)}\colon \max_{n_{j-1} \leq k \leq n_{j+1}} \mathbb{M}_k^{(u)} > \frac{3}{2} \log n_j + x_0, \#\pounds_{\lambda_j}^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda_j}\Bigr) \\
&\quad \leq \exp\bigl(-b \mathrm{e}^{(1-\delta)\lambda_j}\bigr),
\end{aligned}$$

whose sum on j converges [δ being small]. On the other hand, by (4.2), $\mathbb{P}(\max_{u \in \pounds_{\lambda_j}^{(a)}} |u| > n_{j-1}) \leq c_6 \mathrm{e}^{-c_5 n_{j-1}^{1/3}}$ whose sum again converges. Therefore, $\sum_j \mathbb{P}(B_j, \#\pounds_{\lambda_j}^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda_j}) < \infty$. By Borel–Cantelli's lemma, almost surely, for all large j, the event $\{B_j, \#\pounds_{\lambda_j}^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda_j}\}$ does not hold; but we have chosen a such that on $\mathcal{S}$, $\#\pounds_{\lambda_j}^{(a)} \geq \mathrm{e}^{(1-\delta)\lambda_j}$ for all large j. Hence a.s. on $\mathcal{S}$, for all large j, $\max_{n_j<k\le n_{j+1}} \mathbb{M}_k \le \frac{3}{2}\log n_j + \lambda_j + x_1$, from which we get that a.s. on $\mathcal{S}$,

$$\limsup_{n\to\infty} \frac{1}{\log\log\log n}\left(\mathbb{M}_n - \frac{3}{2}\log n\right) \le 1+2\delta,$$

yielding the upper bound as $\delta > 0$ can be arbitrarily small. $\square$

5. Proof of Theorem 1.4

5.1. *The Böttcher case*: *Proof of* (1.17)

Recall (1.19) for the stopping line $\pounds_\lambda$.

Lemma 5.1 (The Böttcher case). *Under the same assumptions as in Theorem* 1.4, *for any constant* $a > 0$, *we have*

$$\mathbb{E}\big[e^{-a\#\pounds_\lambda}\big] = e^{-e^{(\beta+o(1))\lambda}}, \quad \lambda\to\infty. \tag{5.1}$$

Proof. Let us check at first the lower bound in (5.1). Observe that $\mathbb{P}$-almost surely,

$$D_\infty = \sum_{u\in\pounds_\lambda} e^{-V(u)} D_\infty(u), \tag{5.2}$$

where conditioned on $\{V(u), u\in\pounds_\lambda\}$, $D_\infty(u)$ are independent copies of D_∞. Take K_0 large enough such that $\mathbb{E}[e^{-K_0 D_\infty}] \le e^{-a}$, that is possible because $D_\infty > 0$, $\mathbb{P}$-a.s. Let $x = K_0 e^{\lambda+K}$, where $K = \operatorname{ess\,sup}\max_{|u|=1} V(u) < \infty$ is as in (1.8). Therefore

$$\mathbb{E}\big[e^{-xD_\infty}\big] = \mathbb{E}\bigg[\prod_{u\in\pounds_\lambda} \mathbb{E}\big[e^{-xe^{-y}D_\infty}\big]\Big|_{y=V(u)\le\lambda+K}\bigg] \le \mathbb{E}\bigg[\prod_{u\in\pounds_\lambda} e^{-a}\bigg] = \mathbb{E}\big[e^{-a\#\pounds_\lambda}\big].$$

Hence $\mathbb{E}[e^{-a\#\pounds_\lambda}] \ge \mathbb{E}[e^{-xD_\infty}] = e^{-x^{\beta+o(1)}} = e^{-e^{(\beta+o(1))\lambda}}$ gives the lower bound of (5.1).

For the upper bound of (5.1), we use again (5.2) to see that $D_\infty \le e^{-\lambda}\sum_{u\in\pounds_\lambda} D_\infty(u)$. Take a constant $b > 0$ such that $\mathbb{E}[e^{-bD_\infty}] \ge e^{-a}$. It follows that

$$\mathbb{E}\big[e^{-be^\lambda D_\infty}\big] \ge \mathbb{E}\big[e^{-b\sum_{u\in\pounds_\lambda} D_\infty(u)}\big] \ge \mathbb{E}\big[e^{-a\#\pounds_\lambda}\big],$$

since conditioned on $\pounds_\lambda$, $(D_\infty(u))_{u\in\pounds_\lambda}$ are i.i.d. copies of D_∞. Then (1.15) implies the upper bound of (5.1). $\square$

Proof of (1.17). By Lemmas 4.4 and 4.5, we can choose two positive constants c_{11} and c_{12} such that for any $n\ge 1$,

$$\min_{n/2\le j\le n} \mathbb{P}\left(\mathbb{M}_j \ge \frac{3}{2}\log n - c_{11}\right) \ge e^{-c_{12}}, \tag{5.3}$$

$$\mathbb{P}\left(\max_{n/2\le j\le 3n} \mathbb{M}_j \ge \frac{3}{2}\log n + c_{11}\right) \le e^{-c_{12}}. \tag{5.4}$$

For any $u\in\mathbb{T}$, define as before $\mathbb{M}_j^{(u)} := \min_{v\in\mathbb{T}_u, |v|=|u|+j}(V(v)-V(u))$ for any $j\ge 0$. It follows that

$$\begin{aligned}\mathbb{P}\left(\mathbb{M}_n > \frac{3}{2}\log n + \lambda - c_{11}\right) &\ge \mathbb{P}\left(\forall u\in\pounds_\lambda, |u|\le\frac{n}{2}, M_{n-|u|}^{(u)} \ge \frac{3}{2}\log n - c_{11}\right)\\ &\ge \mathbb{E}\big[e^{-c_{12}\#\pounds_\lambda} 1_{(\max_{u\in\pounds_\lambda}|u|\le n/2)}\big]\end{aligned}$$

$$
\begin{aligned}
&\ge \mathbb{E}\big[\mathrm{e}^{-c_{12}\#\pounds_\lambda}\big] - \mathbb{P}\Big(\max_{u\in\pounds_\lambda}|u| > \frac{n}{2}\Big)\\
&\ge \mathrm{e}^{-\mathrm{e}^{(\beta+\mathrm{o}(1))\lambda}} - c_6\mathrm{e}^{-c_5 n^{1/3}},
\end{aligned}
$$

by Lemma 5.1 and (4.2). The lower bound in (1.17) follows from the assumption that $\lambda = \mathrm{o}(\log n)$.

To get the upper bound in (1.17), we use the hypothesis (1.8) and obtain that

$$
\begin{aligned}
&\mathbb{P}\Big(\max_{n\le k\le 2n}\mathbb{M}_k > \frac{3}{2}\log n + \lambda + c_{11} + K\Big)\\
&\quad\le \mathbb{P}\Big(\forall u\in\pounds_\lambda, \max_{u\in\pounds_\lambda}|u| \le \frac{n}{2}, \max_{n\le k\le 2n}\mathbb{M}^{(u)}_{k-|u|} \ge \frac{3}{2}\log n + c_{11}\Big) + \mathbb{P}\Big(\max_{u\in\pounds_\lambda}|u| > \frac{n}{2}\Big)\\
&\quad\le \mathbb{E}\big[\mathrm{e}^{-c_{12}\#\pounds_\lambda}\big] + c_6\mathrm{e}^{-c_5 n^{1/3}},
\end{aligned}
$$

by (5.4) and (4.2). The upper bound follows from Lemma 5.1. $\square$

5.2. *The Schröder case*: *Proof of* (1.16)

In the case $q := \mathbb{P}(\mathcal{S}^c) > 0$, we need to estimate the probability that the extinction happens after $\pounds_\lambda$:

Lemma 5.2. *Assume* (1.1), (1.2) *and* (1.5). *Then for any* $\lambda > 0$,

$$\mathbb{P}\big(\{\pounds_\lambda \ne \emptyset\}\cap\mathcal{S}^c\big) = \mathbb{E}\big[q^{\#\pounds_\lambda}1_{(\#\pounds_\lambda>0)}\big] \le q\mathrm{e}^{-\gamma\lambda}.$$

Proof. The above equality is an immediate consequence of the branching property at the optional line $\pounds_\lambda$ (cf. [7]).

To show the above inequality, we recall that $\nu(u)$, for any $u\in\mathbb{T}$, denotes the number of children of u. Write $u < \pounds_\lambda$ if there exists some particle $v\in\pounds_\lambda$ such that $u < v$ [i.e. u is an ancestor of v]. Then for the tree up to $\pounds_\lambda$, the following equality holds: almost surely,

$$\#\pounds_\lambda = 1 + \sum_{\emptyset\le u<\pounds_\lambda}\big(\nu(u)-1\big). \tag{5.5}$$

Recall (1.5). Define a process

$$X_n := \sum_{|u|=n}\prod_{i=0}^{n-1}\big(q^{\nu(u_i)-1}1_{(\nu(u_i)\ge 1)}\big)\mathrm{e}^{\gamma V(u)}, \quad n\ge 1,$$

where as before, u_i denotes the ancestor of u at ith generation. It is straightforward to check, by using the branching property, that $(X_n)_{n\ge1}$ is a (non-negative) martingale with mean 1. Define

$$X_{\pounds_\lambda} := \sum_{u\in\pounds_\lambda}\prod_{i=0}^{|u|-1}\big(q^{\nu(u_i)-1}1_{(\nu(u_i)\ge 1)}\big)\mathrm{e}^{\gamma V(u)}, \quad \lambda > 0.$$

According to Biggins and Kyprianou ([7], Lemma 14.1), $\mathbb{E}[X_{\pounds_\lambda}]$ equals $\mathbb{E}[X_1]$ times some probability term, hence $\mathbb{E}[X_{\pounds_\lambda}] \le \mathbb{E}[X_1] = 1$.

Notice that for any $u\in\pounds_\lambda$, $\nu(u_i)\ge 1$ for all $i<|u|$ and $\prod_{i=0}^{|u|-1}(q^{\nu(u_i)-1}1_{(\nu(u_i)\ge1)}) = q^{\sum_{0\le i<|u|}(\nu(u_i)-1)} \ge q^{\#\pounds_\lambda-1}$ by (5.5) [recalling $q<1$]. Then $X_{\pounds_\lambda} \ge q^{\#\pounds_\lambda-1}\mathrm{e}^{\gamma\lambda}$ on $\{\#\pounds_\lambda>0\}$. The Lemma follows from $\mathbb{E}[X_{\pounds_\lambda}]\le 1$. $\square$

Lemma 5.3. *Assume* (1.1), (1.2), (1.5) *and* (1.6). *For any* $\delta>0$, *there exist an integer* $m_\delta\ge 1$ *and a constant* $\lambda_0(\delta)>0$ *such that for all* $\lambda\ge\lambda_0(\delta)$,

$$\mathbb{P}(0<\#\pounds_\lambda\le m_\delta) \ge \mathrm{e}^{-(\gamma+\delta)\lambda}.$$

Proof. We discuss the case $q=0$ and the case $q>0$ separately.

(i) *First case:* $q=0$. We shall prove that

$$\mathbb{P}(\#\pounds_\lambda = 1) \geq \mathrm{e}^{-(\gamma+\mathrm{o}(1))\lambda}, \tag{5.6}$$

where as usual o(1) denotes a quantity which goes to 0 as $\lambda\to\infty$. To this end, we have by the change of measure (see Section 2.2 and (2.9)) that

$$\mathbb{P}(\#\pounds_\lambda = 1) = \mathbb{Q}\left[\frac{1}{W_{\pounds_\lambda}} 1_{(\#\pounds_\lambda=1)}\right] = \mathbb{Q}\left[\mathrm{e}^{V(\mathrm{w}_{\tau_\lambda(\mathrm{w})})} 1_{(\#\pounds_\lambda=1)}\right] \geq \mathrm{e}^{\lambda}\mathbb{Q}(\#\pounds_\lambda = 1). \tag{5.7}$$

Notice that under $\mathbb{Q}$, $\{\#\pounds_\lambda = 1\}$ means that $\pounds_\lambda = \{\mathrm{w}_{\tau_\lambda(\mathrm{w})}\}$. Recall that $\nu(u)$ denotes the number of children of $u\in\mathbb{T}$. Then $\mathbb{Q}(\#\pounds_\lambda = 1|\mathcal{G}_\infty) = 1_{(0\leq k<\tau_\lambda(\mathrm{w}),\nu(\mathrm{w}_k)=1)}$ and thus

$$\mathbb{P}(\#\pounds_\lambda = 1) \geq \mathrm{e}^{\lambda}\mathbb{Q}\big(0\leq k<\tau_\lambda(\mathrm{w}), \nu(\mathrm{w}_k)=1\big). \tag{5.8}$$

Recall (1.9) for γ. We claim that

$$\mathbb{Q}\big(0\leq k<\tau_\lambda(\mathrm{w}), \nu(\mathrm{w}_k)=1\big) = \mathrm{e}^{-(1+\gamma+\mathrm{o}(1))\lambda}. \tag{5.9}$$

To get (5.9), we use the fact (cf. Section 2.2) that $(\sum_{u\in\mho(\mathrm{w}_k)}\delta_{\{\Delta V(u)\}}, \Delta V(\mathrm{w}_k))_{k\geq1}$ are i.i.d. under $\mathbb{Q}$, where $\Delta V(u) := V(u) - V(\overleftarrow{u})$ for any $u\neq\emptyset\equiv\mathrm{w}_0$. Notice that $\nu(\mathrm{w}_{k-1}) = 1+\#\mho(\mathrm{w}_k)$.

Let us check that the process

$$U_n := \mathrm{e}^{(1+\gamma)V(\mathrm{w}_n)} 1_{(\forall 1\leq k\leq n, \nu(\mathrm{w}_{k-1})=1)}, \quad n\geq 1,$$

is a $\mathbb{Q}$-martingale of mean 1. In fact, U_n is a product of n i.i.d. variables, then it is enough to check that $\mathbb{Q}[U_1]=1$. But $\mathbb{Q}[U_1] = \mathbb{Q}[\mathrm{e}^{(1+\gamma)V(\mathrm{w}_1)}1_{(\nu(\mathrm{w}_0)=1)}] = \mathbb{E}[\sum_{|u|=1}\mathrm{e}^{\gamma V(u)}1_{(\nu=1)}] = 1$, as claimed. By the optional stopping theorem and the Fatou lemma, we get that $\mathbb{Q}[U_{\tau_\lambda(\mathrm{w})}]\leq 1$, which implies the upper bound in (5.9) since $V(\tau_\lambda(\mathrm{w}))>\lambda$ [under $\mathbb{Q}$, $\tau_\lambda(\mathrm{w})$ is a.s. finite].

To get the lower bound in (5.9), let $\varepsilon>0$ be small. Fix some large constant C whose value will be determined later. Let us find some γ_C such that the process

$$U_n^{(C)} := \mathrm{e}^{(1+\gamma_C)V(\mathrm{w}_n)} 1_{(\forall 1\leq k\leq n, \nu(\mathrm{w}_{k-1})=1, \Delta V(\mathrm{w}_k)\leq C)}, \quad n\geq 1,$$

is a $\mathbb{Q}$-martingale with mean 1. As for U_n, the constant γ_C[3] is determined by

$$1 = \mathbb{E}\left[\sum_{|u|=1}\mathrm{e}^{\gamma_C V(u)}1_{(\nu=1, V(u)\leq C)}\right],$$

where for $|u|=1$, $\Delta V(u) = V(u)$. Plainly $\gamma_C\to\gamma$ as $C\to\infty$. Choose C sufficiently large such that $\gamma_C\leq\gamma+\varepsilon$. Since $(U_k^{(K)}, k\leq\tau_\lambda(\mathrm{w}))$ is uniformly bounded by $\mathrm{e}^{(1+\gamma_C)(\lambda+C)}$. By the optional stopping theorem, we obtain that

$$1 = \mathbb{Q}\big[U^{(C)}_{\tau_\lambda(\mathrm{w})}\big] \leq \mathrm{e}^{(1+\gamma_C)(\lambda+C)}\mathbb{Q}\big(\forall 1\leq k\leq n, \nu(\mathrm{w}_{k-1})=1\big),$$

finishing the proof of (5.9) as ε can be arbitrarily small. The lemma (in the case $q=0$) follows from (5.9) and (5.8).

[3]For the existence of such constant, we used the integrability assumption (1.6): the convex function $f: b\to\mathbb{E}[\sum_{|u|=1}\mathrm{e}^{bV(u)}1_{(\nu=1)}]$ has a derivative $f'(\gamma)\geq\frac{f(\gamma)-f(0)}{\gamma}>0$ hence f is increasing at γ. Then $f(a)>f(\gamma)=1$. Take C_0 large enough such that $\mathbb{E}[\sum_{|u|=1}\mathrm{e}^{aV(u)}1_{(\nu=1, V(u)\leq C_0)}]>1$, then such γ_C exists for all $C\geq C_0$. We shall use the existences of similar constants later without further explanations.

(ii) *Second (and last) case:* $q > 0$. We can not repeat the same proof as before, for instance $p_1 \equiv \mathbb{P}(\nu = 1)$ may vanish.

Again by the change of measure we have that for any integer $m \geq 1$,

$$\mathbb{P}(0 < \#\pounds_\lambda \leq m) = \mathbb{Q}\left[\frac{1}{W_{\pounds_\lambda}} 1_{(\#\pounds_\lambda \leq m)}\right] \geq \frac{1}{m} \mathrm{e}^{\lambda} \mathbb{Q}(\#\pounds_\lambda \leq m), \tag{5.10}$$

where we used the facts that $W_{\pounds_\lambda} = \sum_{u \in \pounds_\lambda} \mathrm{e}^{-V(u)} \leq m\mathrm{e}^{-\lambda}$ on $\{\#\pounds_\lambda \leq m\}$ and under $\mathbb{Q}$, $\pounds_\lambda$ contains at least the singleton $\{\mathrm{w}_{\tau_\lambda(\mathrm{w})}\}$. Define for any $x > 0$,

$$q(x) := \mathbb{P}\Big(\sup_{v \in \mathbb{T}} V(v) \leq x\Big) = \mathbb{P}(\pounds_x = \emptyset),$$

with the usual convention that $\sup_\emptyset = 0$. Plainly, $\lim_{x \to \infty} q(x) = \mathbb{P}(\sup_{v \in \mathbb{T}} V(v) < \infty) = \mathbb{P}(\mathcal{S}^c) = q$. For any small $\varepsilon > 0$, there exists some $x_0 = x_0(\varepsilon) > 0$ such that $q(x) \geq q - \varepsilon$ for all $x \geq x_0$.

Let $\delta > 0$ be small. Before bounding below $\mathbb{Q}(\#\pounds_\lambda \leq m)$ with some $m = m_\delta$, we first choose some constants. Let α be large and ε be small whose values will be determined later. Recall that $\mho(\mathrm{w}_k)$ denotes the set of brothers of w_k. Let us choose a constant $\gamma_{\alpha,\varepsilon}$ such that

$$U_n^{(\alpha,\varepsilon)} := \mathrm{e}^{(1+\gamma_{\alpha,\varepsilon})V(\mathrm{w}_n)} (q - \varepsilon)^{\sum_{0 \leq k < n}(\nu(\mathrm{w}_k)-1)} 1_{(\forall k < n, \forall u \in \mho(\mathrm{w}_k), \Delta V(u) \leq \alpha)}, \quad n \geq 1, \tag{5.11}$$

is a $\mathbb{Q}$-martingale with mean 1. As before, such $\gamma_{\alpha,\varepsilon}$ is determined by the following equalities

$$1 = \mathbb{Q}\big[\mathrm{e}^{(1+\gamma_{\alpha,\varepsilon})V(\mathrm{w}_1)} (q - \varepsilon)^{\nu(\mathrm{w}_0)-1} 1_{(\max_{|u|=1, u \neq \mathrm{w}_1} V(u) \leq \alpha)}\big] = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{\gamma_{\alpha,\varepsilon} V(u)} (q - \varepsilon)^{\nu - 1} 1_{(\max_{|v|=1, v \neq u} V(v) \leq \alpha)}\right].$$

The existence of $\gamma_{a,\varepsilon}$ follows from (1.5) and (1.6). Clearly $\gamma_{\alpha,\varepsilon} \to \gamma$ as $\alpha \to \infty$ and $\varepsilon \to 0$. Fix now $\alpha \equiv \alpha(\delta) > 0$ (large enough) and $\varepsilon \equiv \varepsilon(\delta) > 0$ (small enough) such that $\gamma_{\alpha,\varepsilon} < \gamma + \delta$. Choose a constant $x_0 \equiv x_0(\delta) > 0$ such that $q(x) \geq q - \varepsilon$ for all $x \geq x_0$.

On the other hand, we remark that (1.1) and (1.5) imply that

$$\mathbb{P}\Big(1 \leq \nu < \infty, \max_{|u|=1} V(u) > 0\Big) > 0. \tag{5.12}$$

In fact,

$$\begin{aligned}\mathbb{E}\left[1_{(1 \leq \nu < \infty)} q^{\nu-1} \sum_{|u|=1} \mathrm{e}^{\gamma V(u)} 1_{(V(u) > 0)}\right] &= 1 - \mathbb{E}\left[1_{(1 \leq \nu < \infty)} q^{\nu-1} \sum_{|u|=1} \mathrm{e}^{\gamma V(u)} 1_{(V(u) \leq 0)}\right] \\ &> 1 - \mathbb{E}\big[1_{(1 \leq \nu < \infty)} q^{\nu-1} \nu\big] > 0,\end{aligned}$$

hence (5.12) holds. It follows that there are some integer $n_* \geq 1$ and some positive constants c_* and b_* such that

$$b_* \leq \mathbb{E}\left[1_{(\nu \leq n_*)} \sum_{|u|=1} \mathrm{e}^{-V(u)} 1_{(V(u) \geq c_*)}\right] = \mathbb{Q}\big(\nu(\mathrm{w}_0) \leq n_*, V(\mathrm{w}_1) \geq c_*\big), \tag{5.13}$$

where the last equality follows from the change of measure formula (Section 2.2 (i), $\mathrm{w}_0 = \emptyset$).

Choose (and fix) a constant $L \geq \alpha + x_0$ such that $\frac{L}{c_*}$ is an integer. Define $m_\delta := (n_*)^{L/c_*}$. Recall (1.18) for the definition of $\tau_\lambda(u)$. For any $\lambda > 2L$, we consider the following events

$$A_1 := \big\{\forall k < \tau_{\lambda-L}(\mathrm{w}), \forall u \in \mho(\mathrm{w}_k), \Delta V(u) \leq \alpha, \pounds_\lambda^{(u)} = \emptyset\big\},$$

$$A_2 := \left\{\forall \tau_{\lambda-L}(\mathrm{w}) \leq k < \tau_{\lambda-L}(\mathrm{w}) + \frac{L}{c_*}, \forall u \in \mho(\mathrm{w}_k), \nu(u) = 0, \nu(\mathrm{w}_{k-1}) \leq n_*, \Delta V(\mathrm{w}_k) \geq c_*\right\},$$

where $\pounds_\lambda^{(u)} := \mathbb{T}_u \cap \pounds_\lambda$ and $\nu(u)$ denotes the number of children of u.

Observe that on $A_1 \cap A_2$, $\tau_\lambda(\mathrm{w}) \le \tau_{\lambda-L}(\mathrm{w}) + \frac{L}{c_*}$, and $\#\pounds_\lambda \le (n_*)^{L/c_*} \equiv m_\delta$. Since $q > 0$, $p_0 \equiv \mathbb{P}(\nu = 0) > 0$, it follows from the spinal decomposition (Section 2.2 (iii)) that

$$\begin{aligned}
\mathbb{Q}(\#\pounds_\lambda \le m_\delta) &\ge \mathbb{Q}(A_1 \cap A_2) \\
&= \mathbb{Q}\left[B_1 \prod_{k=\tau_{\lambda-L}(\mathrm{w})}^{\tau_{\lambda-L}(\mathrm{w})+L/c_*-1} \prod_{u\in\mho(\mathrm{w}_k)} p_0 \times 1_{(\nu(\mathrm{w}_{k-1})\le n_*, \Delta V(\mathrm{w}_k)\ge c_*)}\right] \\
&\ge p_0^{m_\delta} \mathbb{Q}\left[B_1 \prod_{k=\tau_{\lambda-L}(\mathrm{w})}^{\tau_{\lambda-L}(\mathrm{w})+L/c_*-1} 1_{(\nu(\mathrm{w}_{k-1})\le n_*, \Delta V(\mathrm{w}_k)\ge c_*)}\right],
\end{aligned} \tag{5.14}$$

where

$$B_1 := \prod_{k<\tau_{\lambda-L}(\mathrm{w})} \prod_{u\in\mho(\mathrm{w}_k)} q\big(\lambda - V(u)\big) 1_{(\Delta V(u)\le\alpha)} \ge \prod_{k<\tau_{\lambda-L}(\mathrm{w})} (q-\varepsilon)^{\nu(\mathrm{w}_{k-1})-1} 1_{(\max_{u\in\mho(\mathrm{w}_k)} \Delta V(u)\le\alpha)} =: B_2,$$

by using the fact that for any $u \in \mho(\mathrm{w}_k)$ with $k < \tau_{\lambda-L}(\mathrm{w})$, $V(u) \le \lambda - L + \alpha \le \lambda - x_0$, and $q(\lambda - V(u)) \ge q(x_0) \ge q - \varepsilon$.

Recall that under $\mathbb{Q}$, $(\sum_{u\in\mho(\mathrm{w}_k)} \delta_{\{\Delta V(u)\}}, \Delta V(\mathrm{w}_k))_{k\ge1}$ are i.i.d.; then the strong Markov property implies that under $\mathbb{Q}$ and conditioned on $\mathcal{G}_{\tau_{\lambda-L}(\mathrm{w})}$, $(\nu(\mathrm{w}_{k-1}), \Delta V(\mathrm{w}_k))_{k\ge\tau_{\lambda-L}(\mathrm{w})}$ are i.i.d., of common law that of $(\nu(\mathrm{w}_0), V(\mathrm{w}_1))$. Therefore,

$$\mathbb{Q}(\#\pounds_\lambda \le m_\delta) \ge p_0^{m_\delta} \mathbb{Q}[B_2] \mathbb{Q}\big(\nu(\mathrm{w}_0) \le n_*, V(\mathrm{w}_1) \ge c_*\big)^{L/c_*} \ge p_0^{m_\delta} b_*^{L/c_*} \mathbb{Q}[B_2]. \tag{5.15}$$

It remains to estimate $\mathbb{Q}[B_2]$. Going back to (5.11) and applying the optional stopping theorem at $\tau_{\lambda-L}$ for $U^{(\alpha,\varepsilon)}$ (which remains bounded up to $\tau_{\lambda-L}$), we get that

$$\mathbb{Q}[B_2] = \mathbb{Q}\big[(q-\varepsilon)^{\sum_{0\le k<\tau_{\lambda-L}(\mathrm{w})}(\nu(\mathrm{w}_k)-1)} 1_{(\forall k<n, \forall u\in\mho(\mathrm{w}_k), \Delta V(u)\le\alpha)}\big] \ge \mathrm{e}^{-(1+\gamma_{\alpha,\varepsilon})(\lambda-L+\alpha)}.$$

In view of (5.10) and (5.15), this implies that

$$\mathbb{P}(0 < \#\pounds_\lambda \le m_\delta) \ge \frac{1}{m_\delta} p_0^{m_\delta} b_*^{L/c_*} \mathrm{e}^{L-\alpha} \mathrm{e}^{-\gamma_{\alpha,\varepsilon}(\lambda-L+\alpha)}.$$

Then we have proved the Lemma in the case $q > 0$ [by choosing a sufficiently large $\lambda_0(\delta)$]. □

Lemma 5.4 (The Schröder case). *Under the same assumptions as in Theorem 1.4, for any constant $a > 0$, we have*

$$\mathbb{E}\big[\mathrm{e}^{-a\#\pounds_\lambda} 1_{(\#\pounds_\lambda>0)}\big] = \mathrm{e}^{-(\gamma+\mathrm{o}(1))\lambda}, \quad \lambda \to \infty. \tag{5.16}$$

Proof. From Lemma 5.3, the lower bound of (5.16) follows immediately. We also mention that in the cases when $q = 0$ or $q > 0$ but $0 < a < \log(1/q)$, we can give a proof of the lower bound of (5.16) in the same way as that of (5.1).

For the upper bound, we proceed in the same way as in the proof of Lemma 5.1, but by paying attention to the possibility of extinction of the system. Take $b > 0$ such that $\mathbb{E}[\mathrm{e}^{-bD_\infty}] \ge \mathrm{e}^{-a}$. By (5.2), $\mathrm{e}^\lambda D_\infty \le \sum_{u\in\pounds_\lambda} D_\infty(u)$, then

$$\begin{aligned}
\mathbb{E}\big[\mathrm{e}^{-b\mathrm{e}^\lambda D_\infty} 1_{(D_\infty>0)}\big] &\ge \mathbb{E}\big[\mathrm{e}^{-b\sum_{u\in\pounds_\lambda} D_\infty(u)} 1_{(D_\infty>0)}\big] \\
&\ge \mathbb{E}\big[\mathrm{e}^{-b\sum_{u\in\pounds_\lambda} D_\infty(u)} 1_{(\#\pounds_\lambda>0)}\big] - \mathbb{P}\big(\{\#\pounds_\lambda > 0\} \cap \mathcal{S}^c\big) \\
&\ge \mathbb{E}\big[\mathrm{e}^{-a\#\pounds_\lambda} 1_{(\#\pounds_\lambda>0)}\big] - \mathbb{P}\big(\{\#\pounds_\lambda > 0\} \cap \mathcal{S}^c\big).
\end{aligned}$$

By (1.14), $\mathbb{E}[\mathrm{e}^{-b\mathrm{e}^\lambda D_\infty} 1_{(D_\infty>0)}] \le C\mathrm{e}^{-\gamma\lambda}$, which together with Lemma 5.2 yield the upper bound in (5.16). □

We now are ready to give the proof of (1.16):

Proof of (1.16). Let us prove at first the lower bound in (1.16). By Lemma 4.4, there are $c_{13} > 0$ (large enough) and $c_{14} > 0$ (small enough) such that $\min_{n/2 \le k \le n} \mathbb{P}(\mathbb{M}_k \ge \frac{3}{2} \log n - c_{13}, \mathcal{S}) \ge c_{14}$ for all $n \ge 1$.

Let $\delta > 0$ be small and let $m_\delta \ge 1$ and $\lambda_0(\delta) > 0$ be as in Lemma 5.3. Let $\lambda \ge \lambda_0(\delta)$. Remark that

$$\mathbb{P}\left(\mathbb{M}_n > \frac{3}{2}\log n + \lambda - c_{13}, \mathcal{S}\right) \ge \mathbb{P}\left(0 < \#£_\lambda \le m_\delta, \forall u \in £_\lambda, \mathbb{M}^{(u)}_{n-|u|} > \frac{3}{2}\log n - c_{13}, |u| \le \frac{n}{2}, \mathcal{S}^{(u)}\right),$$

where as before, $S^{(u)} = \{\mathbb{T}_u \text{ survives}\}$ and $\mathbb{M}^{(u)}_j := \min_{v \in \mathbb{T}_u, |v|=|u|+j}(V(v) - V(u))$ for any $j \ge 0$. It follows that

$$\begin{aligned}\mathbb{P}\left(\mathbb{M}_n > \frac{3}{2}\log n + \lambda - c_{13}, \mathcal{S}\right) &\ge (c_{14})^{m_\delta}\mathbb{P}\left(0 < \#£_\lambda \le m_\delta, \max_{u\in £_\lambda}|u| \le \frac{n}{2}\right)\\ &\ge (c_{14})^{m_\delta}\left(\mathbb{P}(0 < \#£_\lambda \le m_\delta) - \mathbb{P}\left(\max_{u\in £_\lambda}|u| > \frac{n}{2}\right)\right)\\ &\ge (c_{14})^{m_\delta}\big(\mathrm{e}^{-(\gamma+\delta)\lambda} - c_6\mathrm{e}^{-c_5 n^{1/3}}\big),\end{aligned}$$

by Lemma 5.3 and (4.2). The lower bound of (1.16) follows.

We prove now the upper bound in (1.16). By assumption (1.6) holds for any $a > 0$, hence S_1 has all exponential moments. It follows from (2.6) that for any $a > 0$, there exists some $C_a > 0$ such that

$$\mathbb{P}(S_{\tau_\lambda} - \lambda \ge x) \le C_a \mathrm{e}^{-ax}, \quad \forall x \ge 0. \tag{5.17}$$

Let $\delta > 0$ be small and $a > (1+\gamma)/\delta + 1$. Then

$$\mathbb{P}\Big(\max_{u\in £_\lambda} V(u) > (1+\delta)\lambda\Big) \le \mathbb{E}\left[\sum_{u\in £_\lambda} 1_{(V(u)>(1+\delta)\lambda)}\right] = \mathbb{E}\big[\mathrm{e}^{S_{\tau_\lambda}} 1_{(S_{\tau_\lambda} \ge (1+\delta)\lambda)}\big] = \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big), \tag{5.18}$$

where the last equality follows easily from (5.17). Define

$$A_{(5.19)} := \left\{\max_{u\in £_\lambda} V(u) \le (1+\delta)\lambda, \max_{u\in £_\lambda}|u| \le \frac{n}{2}\right\}. \tag{5.19}$$

Then by (4.2), for all large $n \ge n_0$ and $0 < \lambda = \mathrm{o}(\log n)$,

$$\begin{aligned}\mathbb{P}\big(A^c_{(5.19)}\big) &\le \mathbb{P}\Big(\max_{u\in £_\lambda} V(u) > (1+\delta)\lambda\Big) + \mathbb{P}\Big(\max_{u\in £_\lambda}|u| > \frac{n}{2}\Big)\\ &\le \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big) + c_6\mathrm{e}^{-c_5 n^{1/3}} = \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big).\end{aligned}$$

On $\mathcal{S} \cap \{\mathbb{M}_n > \frac{3}{2}\log n + (1+2\delta)\lambda\}$, $£_\lambda \ne \emptyset$. Consider λ such that $\delta\lambda < \log n$. Therefore,

$$\begin{aligned}&\mathbb{P}\Big(\max_{n\le k\le 2n} \mathbb{M}_k > \frac{3}{2}\log n + (1+2\delta)\lambda, \mathcal{S}\Big)\\ &\quad\le \mathbb{P}\Big(\max_{n\le k\le 2n} \mathbb{M}_k > \frac{3}{2}\log n + (1+2\delta)\lambda, A_{(5.19)}, £_\lambda \ne \emptyset\Big) + \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big)\\ &\quad\le \mathbb{P}\Big(\forall u \in £_\lambda, \max_{n/2\le j\le 2n} \mathbb{M}^{(u)}_j > \frac{3}{2}\log n + \delta\lambda, £_\lambda \ne \emptyset\Big) + \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big)\\ &\quad=: B_{(5.20)} + \mathrm{o}\big(\mathrm{e}^{-\gamma\lambda}\big),\end{aligned} \tag{5.20}$$

where $\mathbb{M}_k^{(u)} := \max_{v\in\mathbb{T}_u,|v|=|u|+k}(V(v)-V(u))$. Conditioning on $\mathcal{F}_{£_\lambda}$, $\mathbb{M}^{(u)}_{\cdot}$ are i.i.d. copies of $\mathbb{M}_{\cdot}$. By Lemma 4.5 (with $a=4$), there exist some $c_{15}>0$ and λ_0 such that (δ being fixed) for all large $n\ge n_0(\lambda_0)$,

$$\mathbb{P}\Bigl(\max_{n/2\le k\le 2n}\mathbb{M}_k\ge\frac{3}{2}\log n+\delta\lambda_0\Bigr)\le\mathbb{P}\bigl(\mathcal{S}^c\bigr)+\mathbb{P}^*\Bigl(\max_{n/2\le k\le 2n}\mathbb{M}_k\ge\frac{3}{2}\log n+\delta\lambda_0\Bigr)\le \mathrm{e}^{-c_{15}}.$$

Then by conditioning on $\mathcal{F}_{£_\lambda}$, we get that

$$B_{(5.20)}\le\mathbb{E}\bigl[\mathrm{e}^{-c_{15}\#£_\lambda}1_{(£_\lambda\ne\emptyset)}\bigr]=\mathrm{e}^{-(\gamma+o(1))\lambda},$$

by Lemma 5.4. This and (5.20) prove the upper bound in (1.16) since δ can be arbitrarily small. $\square$

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