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SHARP POLYNOMIAL DECAY RATES FOR THE DAMPED WAVE EQUATION ON THE TORUS

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We address the decay rates of the energy for the damped wave equation when the damping coefficient b does not satisfy the geometric control condition (GCC). First, we give a link with the controllability of the associated Schrödinger equation. We prove in an abstract setting that the observability of the Schrödinger equation implies that the solutions of the damped wave equation decay at least like $1/\sqrt{t}$ (which is a stronger rate than the general logarithmic one predicted by the Lebeau theorem).

Second, we focus on the 2-dimensional torus. We prove that the best decay one can expect is $1/t$, as soon as the damping region does not satisfy GCC. Conversely, for smooth damping coefficients b vanishing flatly enough, we show that the semigroup decays at least like $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$. The proof relies on a second microlocalization around trapped directions, and resolvent estimates.

In the case where the damping coefficient is a characteristic function of a strip (hence discontinuous), Stéphane Nonnenmacher computes in an appendix part of the spectrum of the associated damped wave operator, proving that the semigroup cannot decay faster than $1/t^{2/3}$. In particular, our study emphasizes that the decay rate highly depends on the way b vanishes.

CONTENTS

MSC2010: primary 35A21, 35B35, 35L05, 35P20, 35S05; secondary 35B37, 93C20.
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Part I. The damped wave equation

1. Decay of energy: a survey of existing results

Let (M, g) be a smooth compact connected Riemannian d-dimensional manifold, with or without boundary ∂M. We denote by Δ the (nonpositive) Laplace–Beltrami operator on M for the metric g. Given a bounded nonnegative function, $b \in L^\infty(M)$, $b(x) \geq 0$ on M, we want to understand the asymptotic behavior as $t \to +\infty$ of the solution u of the problem

$$\begin{cases} \partial_t^2 u - \Delta u + b(x)\,\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial M \ (\text{if } \partial M \neq \varnothing), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1-1}$$

The energy of a solution is defined by

$$E(u, t) = \tfrac{1}{2}(\|\nabla u(t)\|^2_{L^2(M)} + \|\partial_t u(t)\|^2_{L^2(M)}). \tag{1-2}$$

Multiplying (1-1) by $\partial_t u$ and integrating on M yields the dissipation identity

$$\frac{d}{dt}E(u, t) = -\int_M b|\partial_t u|^2\,dx,$$

which, as b is nonnegative, implies a decay of the energy. As soon as $b \geq C > 0$ on a nonempty open subset of M, the decay is strict and $E(u, t) \to 0$ as $t \to +\infty$. The question is then to know at what rate the energy goes to zero.

The first interesting issue concerns uniform stabilization: under which condition does there exist a function $F(t)$, $F(t) \to 0$, such that

$$E(u, t) \leq F(t)E(u, 0)\ ? \tag{1-3}$$

The answer was given by Rauch and Taylor [1974] in the case $\partial M = \varnothing$ and by Bardos, Lebeau and Rauch [Bardos et al. 1992] in the general case (see also [Burq and Gérard 1997] for the necessity of this condition): assuming that $b \in \mathscr{C}^0(\overline{M})$, uniform stabilization occurs if and only if the set $\{b > 0\}$ satisfies the geometric control condition (GCC). Recall that a set $\omega \subset M$ is said to satisfy GCC if there exists $L_0 > 0$ such that every geodesic γ (resp. generalized geodesic in the case $\partial M \neq \varnothing$) of M with length larger than L_0 satisfies $\gamma \cap \omega \neq \varnothing$. When (1-3) is satisfied, one can take $F(t) = Ce^{-\kappa t}$ (for some constants

$C, \kappa > 0$) in (1-3), and the energy decays exponentially. Finally, Lebeau [1996] gives the explicit value of the best exponential decay rate κ in terms of the spectral abscissa of the generator of the semigroup and the mean value of the function b along the rays of geometrical optics.

In the case where $\{b > 0\}$ does not satisfy GCC, i.e., in the presence of "trapped rays" that do not meet $\{b > 0\}$, what can be said about the decay rate of the energy? As soon as $b \geq C > 0$ on a nonempty open subset of M, Lebeau [1996] shows that the energy of smoother initial data (satisfying the boundary condition if $\partial M \neq \varnothing$) goes at least logarithmically to zero:

$$E(u,t) \leq C(f(t))^2\big(\|u_0\|^2_{H^2(M)} + \|u_1\|^2_{H^1(M)}\big) \quad \text{for all } t > 0, \tag{1-4}$$

with $f(t) = 1/\log(2+t)$ (see also [Burq 1998]). Note that here, $(f(t))^2$ characterizes the decay of the energy, whereas $f(t)$ is that of the associated semigroup. Moreover, the author constructed a series of explicit examples of geometries for which this rate is optimal, including for instance the case where $M = \mathbb{S}^2$ is the two-dimensional sphere and $\{b > 0\} \cap N_\varepsilon = \varnothing$, where N_ε is a neighborhood of an equator of $\mathbb{S}^2$. This result is generalized in [Lebeau and Robbiano 1997] for a wave equation damped on a (small) part of the boundary. In this paper, the authors also make the following comment about the result they obtain:

> Notons toutefois qu'une étude plus approfondie de la localisation spectrale et des taux de décroissance de l'énergie pour des données régulières doit faire intervenir la dynamique globale du flot géodésique généralisé sur M. Les théorèmes 1 et 2 [de cet article] ne fournissent donc que les bornes a priori qu'on peut obtenir sans aucune hypothèse sur la dynamique, en n'utilisant que les inégalités de Carleman qui traduisent «l'effet tunnel».

In all examples where the optimal decay rate is logarithmic, the trapped ray is a stable trajectory from the point of view of the dynamics of the geodesic flow. This means basically that an important amount of the energy can stay concentrated, for a long time, in a neighborhood of the trapped ray, i.e., away from the damping region.

If the trapped trajectories are less stable, or unstable, one can expect to obtain an intermediate decay rate between exponential and logarithmic. We shall say that the energy decays at rate $f(t)$ if (1-4) is satisfied (more generally, see Definition 2.2 below in the abstract setting). This problem has already been addressed and, in some particular geometries, several different behaviors have been exhibited. Two main directions have been investigated.

On the one hand, Liu and Rao [2005] considered the case where M is a square and the set $\{b > 0\}$ contains a vertical strip. In this situation, the trapped trajectories consist of a family of parallel vertical geodesics; these are unstable, in the sense that nearby geodesics diverge at a linear rate. They proved that the energy decays at rate $(\log(t)/t)^{1/2}$ (i.e., that (1-4) is satisfied with $f(t) = (\log(t)/t)^{1/2}$). This was extended by Burq and Hitrik [2007] (see also [Nishiyama 2009]) to the case of partially rectangular two-dimensional domains, if the set $\{b > 0\}$ contains a neighborhood of the nonrectangular part. Phung [2007] proved a decay at rate $t^{-\delta}$ for some (unspecified) $\delta > 0$ in a three-dimensional domain having two parallel faces. In all these situations, the only obstruction to GCC is due to a "cylinder of periodic orbits".

The geometry is flat and the instabilities of the geodesic flow around the trapped rays are relatively weak (geodesics diverge at a linear rate).

In [Burq and Hitrik 2007], the authors argue that the optimal decay in their geometry should be of the form $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$. They provide conditions on the damping coefficient $b(x)$ under which one can obtain such decay rates, and wonder whether this is true in general. Our main theorem (Theorem 2.6) extends these results to more general damping functions b on the two-dimensional torus.

On the other hand, Christianson [2007] proved that the energy decays at rate $e^{-C\sqrt{t}}$ for some $C > 0$, in the case where the trapped set is a hyperbolic closed geodesic. Schenck [2011] proved an energy decay at rate e^{-Ct} on manifolds with negative sectional curvature, if the trapped set is "small enough" in terms of topological pressure (for instance, a small neighborhood of a closed geodesic), and if the damping is "large enough" (that is, starting from a damping function b, βb will work for any $\beta > 0$ sufficiently large). In these two papers, the geodesic flow near the trapped set enjoys strong instability properties: the flow on the trapped set is uniformly hyperbolic, and in particular all trajectories are exponentially unstable.

These cases confirm the idea that the decay rate of the energy strongly depends on the stability of trapped trajectories.

One may now want to compare these geometric situations to situations where the Schrödinger group is observable (or, equivalently, controllable), i.e., for which there exist $C > 0$ and $T > 0$ such that, for all $u_0 \in L^2(M)$, we have

$$\|u_0\|^2_{L^2(M)} \le C \int_0^T \|\sqrt{b}\, e^{-it\Delta} u_0\|^2_{L^2(M)}\, dt. \tag{1-5}$$

The conditions under which this property holds are also known to be related to stability of the geodesic flow. In particular, [Bardos et al. 1992], [Liu and Rao 2005], [Burq and Hitrik 2007; Nishiyama 2009] and [Christianson 2007; Schenck 2011] can be seen as counterparts for damped wave equations of the articles [Lebeau 1992], [Haraux 1989a; Jaffard 1990], [Burq and Zworski 2004] and [Anantharaman and Rivière 2012], respectively, in the context of observation of the Schrödinger group.

Our main results are twofold. First, we clarify (in an abstract setting) the link between the observability (or the controllability) of the Schrödinger equation and polynomial decay for the damped wave equation. This follows the spirit of [Haraux 1989b; Miller 2005], exploring the links between the different equations and their control properties, such as observability, controllability, and stabilization. More precisely, we prove that the controllability of the Schrödinger equation implies a polynomial decay at rate $1/\sqrt{t}$ for the damped wave equation (Theorem 2.3).

Second, we study precisely the damped wave equation on the flat torus $\mathbb{T}^2$ in case GCC fails. We give the following a priori lower bound on the decay rate, revisiting the argument of [Burq and Hitrik 2007]: (1-1) is not stable at a better rate than $1/t$, provided that GCC is not satisfied. In this situation, the Schrödinger group is known to be controllable (see [Jaffard 1990; Komornik 1992] and the more recent [Anantharaman and Macià 2010; Burq and Zworski 2012]). Thus, one cannot hope to have a decay better than polynomial in our previous result, i.e., under the mere assumption that the Schrödinger flow is observable.

The remainder of the paper is devoted to studying the gap between the a priori lower and upper bounds given respectively by $1/t$ and $1/\sqrt{t}$ on flat tori. For some *smooth* nonvanishing damping coefficient $b(x)$, we prove that the energy decays at rate $1/t^{1-\varepsilon}$ for all $\varepsilon > 0$. This result holds without making any dynamical assumption on the damping coefficient, but only on the order of vanishing of b. It generalizes a result of [Burq and Hitrik 2007], which holds in the case where b is invariant in one direction. Our analysis is, again, inspired by the recent microlocal approach proposed in [Anantharaman and Macià 2010] and [Burq and Zworski 2012] for the observability of the Schrödinger group. More precisely, we follow here several ideas and tools introduced in [Macià 2010] and [Anantharaman and Macià 2010].

In the situation where b is a characteristic function of a vertical strip of the torus (hence discontinuous), Stéphane Nonnenmacher proves in Appendix B that the decay rate cannot be better than $1/t^{2/3}$. This is done by explicitly computing the high frequency eigenvalues of the damped wave operator which are closest to the imaginary axis; see, for instance, the figures in [Asch and Lebeau 2003; Anantharaman and Léautaud 2012]. That the decay rate $1/t$ is not achieved in this situation was observed in the numerical computations from this last paper.

In contrast to the control problem for the Schrödinger equation on the torus, this result shows that the stabilization of the wave equation is not only sensitive to the global properties of the geodesic flow, but also to the rate at which the damping function vanishes.

2. Main results of the paper

Our first result can be stated in a general abstract setting that we now introduce. We come back to the case of the torus afterwards.

2A. *The damped wave equation in an abstract setting.* Let H and Y be two Hilbert spaces (resp. the state space and the observation/control space) with norms $\|\cdot\|_H$ and $\|\cdot\|_Y$, and associated inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Y$.

We denote by $A\colon D(A) \subset H \to H$ a *nonnegative* selfadjoint operator with compact resolvent, and by $B \in \mathscr{L}(Y; H)$ a control operator. We recall that $B^* \in \mathscr{L}(H; Y)$ is defined by $(B^*h, y)_Y = (h, By)_H$ for all $h \in H$ and $y \in Y$.

Definition 2.1. We say that the system

$$\partial_t u + iAu = 0, \quad y = B^* u, \tag{2-1}$$

is observable in time T if there exists a constant $K_T > 0$ such that, for all solution of (2-1), we have

$$\|u(0)\|_H^2 \le K_T \int_0^T \|y(t)\|_Y^2 \, dt.$$

We recall that the observability of (2-1) in time T is equivalent to the exact controllability in time T of the adjoint problem

$$\partial_t u + iAu = Bf, \quad u(0) = u_0, \tag{2-2}$$

(see, for instance, [Lebeau 1992] or [Ramdani et al. 2005]). More precisely, given $T > 0$, the exact controllability in time T is the ability of finding for any $u_0, u_1 \in H$ a control function $f \in L^2(0, T; Y)$ so that the solution of (2-2) satisfies $u(T) = u_1$.

We equip $\mathcal{H} = D(A^{\frac{1}{2}}) \times H$ with the graph norm

$$\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|(A + \mathrm{Id})^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2,$$

and define the seminorm

$$|(u_0, u_1)|_{\mathcal{H}}^2 = \|A^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2.$$

Of course, if A is coercive on H, $|\cdot|_{\mathcal{H}}$ is a norm on $\mathcal{H}$ equivalent to $\|\cdot\|_{\mathcal{H}}$.

We also introduce in this abstract setting the damped wave equation on the space $\mathcal{H}$

$$\begin{cases} \partial_t^2 u + Au + BB^*\, \partial_t u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}, \end{cases} \tag{2-3}$$

which can be recast on $\mathcal{H}$ as a first order system

$$\begin{cases} \partial_t U = \mathcal{A} U, \\ U|_{t=0} = {}^t(u_0, u_1), \end{cases} \qquad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -A & -BB^* \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}}). \tag{2-4}$$

The compact injections $D(A) \hookrightarrow D(A^{\frac{1}{2}}) \hookrightarrow H$ imply that $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ compactly, and that the operator $\mathcal{A}$ has a compact resolvent.

We define the energy of solutions of (2-3) by

$$E(u, t) = \tfrac{1}{2}\big(\|A^{\frac{1}{2}} u\|_H^2 + \|\partial_t u\|_H^2\big) = \tfrac{1}{2}|(u, \partial_t u)|_{\mathcal{H}^2}^2.$$

Definition 2.2. Let f be a function such that $f(t) \to 0$ when $t \to +\infty$. We say that system (2-3) is stable at rate $f(t)$ if there exists a constant $C > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A})$, we have

$$E(u, t)^{\frac{1}{2}} \le C f(t) |\mathcal{A}(u_0, u_1)|_{\mathcal{H}} \quad \text{for all } t > 0.$$

If it is the case, for all $k > 0$, there exists a constant $C_k > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}^k)$, we have (see, for instance, [Batty and Duyckaerts 2008, page 767])

$$E(u, t)^{\frac{1}{2}} \le C_k (f(t))^k \|\mathcal{A}^k(u_0, u_1)\|_{\mathcal{H}} \quad \text{for all } t > 0.$$

Theorem 2.3. *Suppose that there exists $T > 0$ such that system (2-1) is observable in time T. Then system (2-3) is stable at rate $1/\sqrt{t}$.*

Note that the gain of the $\log(t)^{\frac{1}{2}}$ with respect to [Liu and Rao 2005; Burq and Hitrik 2007] is not essential in our work. It is due to the optimal characterization of polynomially decaying semigroups obtained in [Borichev and Tomilov 2010].

This theorem may be compared with the works (both presented in a similar abstract setting) [Haraux 1989b], proving that the controllability of wave-type equations in some time is equivalent to uniform stabilization of (2-3), and [Miller 2005], showing that the controllability of wave-type equations in some time implies the controllability of Schrödinger-type equations in any time.

The link between this abstract setting and that of problem (1-1) is as follows: $H = Y = L^2(M)$; $A = -\Delta$ with $D(A) = H^2(M)$ if $\partial M = \varnothing$ and $H^2(M) \cap H_0^1(M)$ otherwise; B is the multiplication in $L^2(M)$ by the bounded function $\sqrt{b}$.

As a first application of Theorem 2.3 we obtain a different proof of the polynomial decay results for wave equations of [Liu and Rao 2005] and [Burq and Hitrik 2007] as consequences of the associated control results for the Schrödinger equation of [Haraux 1989a] and [Burq and Zworski 2004], respectively. Moreover, Theorem 2.3 also provides several new stability results for system (1-1) in particular geometric situations; namely, in all following situations, the Schrödinger group is proved to be observable, and Theorem 2.3 gives the polynomial stability at rate $1/\sqrt{t}$ for (1-1):

- For any nonvanishing $b(x) \geq 0$ in the 2-dimensional square (resp. torus), as a consequence of [Jaffard 1990] (resp. [Macià 2010; Burq and Zworski 2012]); for any nonvanishing $b(x) \geq 0$ in the d-dimensional rectangle (resp. d-dimensional torus) as a consequence of [Komornik 1992] (resp. [Anantharaman and Macià 2010]).
- If M is the Bunimovich stadium and $b(x) > 0$ on the neighborhood of one half-disc and on one point of the opposite side, as a consequence of [Burq and Zworski 2004].
- If M is a d-dimensional manifold of constant negative curvature and the set of trapped trajectories (as a subset of S^*M, see [Anantharaman and Rivière 2012, Theorem 2.5] for a precise definition) has Hausdorff dimension lower than d, as a consequence of [Anantharaman and Rivière 2012].

Moreover, Lebeau [1996, Théorème 1(ii)] gives several 2-dimensional examples for which the decay rate $1/\log(2+t)$ is optimal. For all these geometrical situations, Theorem 2.3 implies that the Schrödinger group is not observable.

The proof of Theorem 2.3 relies on the following characterization of polynomial decay for system (2-3). For $z \in \mathbb{C}$, we define on H the operator

$$P(z) = A + z^2 \operatorname{Id} + zBB^*, \quad \text{with domain } D(P(z)) = D(A). \tag{2-5}$$

Proposition 2.4. *Suppose that*

$$\text{for any eigenvector } \varphi \text{ of } A, \text{ we have } B^*\varphi \neq 0. \tag{2-6}$$

Then, for all $\alpha > 0$, the following five assertions are equivalent:

The system (2-3) is stable at rate $1/t^\alpha$. (2-7)

There exist $C > 0$ and $s_0 \geq 0$ such that $\|(is\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|s|^{\frac{1}{\alpha}}$ for all $s \in \mathbb{R}$, $|s| \geq s_0$. (2-8)

There exist $C > 0$ and $s_0 \geq 0$ such that for all $z \in \mathbb{C}$ satisfying $|z| \geq s_0$ and $|\operatorname{Re}(z)| \leq \dfrac{1}{C|\operatorname{Im}(z)|^{\frac{1}{\alpha}}}$,

we have $\|(z\operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\operatorname{Im}(z)|^{\frac{1}{\alpha}}$. (2-9)

There exist $C > 0$ and $s_0 \geq 0$ such that $\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq C|s|^{\frac{1}{\alpha}-1}$ for all $s \in \mathbb{R}$, $|s| \geq s_0$. (2-10)

There exist $C > 0$ and $s_0 \geq 0$ such that for all $s \in \mathbb{R}$, $|s| \geq s_0$ and $u \in D(A)$,

$$\text{we have } \|u\|_H^2 \leq C\big(|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2 + |s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2\big). \tag{2-11}$$

Theorem 2.3 and Proposition 2.4 are proved in Part II, as consequences of the characterization of polynomial decay for general semigroups in terms of resolvent estimates given in [Borichev and Tomilov 2010], providing the equivalence between (2-7) and (2-8). See also [Batty and Duyckaerts 2008] for general decay rates in Banach spaces. Note that the proof of a decay rate is reduced to the proof of a resolvent estimate on the imaginary axes. By the way, this estimate implies the existence of a "spectral gap" between the spectrum of $\mathcal{A}$ and the imaginary axis, given by (2-9).

Note finally that the estimates (2-8), (2-10) and (2-11) can be equivalently restricted to $s > 0$, since $P(-is)\bar{u} = \overline{P(is)u}$ for $s \in \mathbb{R}$.

2B. *Decay rates for the damped wave equation on the torus.* The main results of this article deal with the decay rate for problem (1-1) on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$. In this setting, as well as in the abstract setting, we shall write $P(z) = -\Delta + z^2 + zb(x)$.

First, we give an a priori lower bound for the decay rate of the damped wave equation, on $\mathbb{T}^2$, when GCC is "strongly violated", that is, assuming that $\operatorname{supp}(b)$ does not satisfy GCC (instead of $\{b > 0\}$). This theorem is proved by constructing explicit *quasimodes* for the operator $P(is)$.

Theorem 2.5. *Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0,\ \tau \in \mathbb{R}\} = \varnothing.$$

Then there exist two constants $C > 0$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\|P(in\kappa_0)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq C. \tag{2-12}$$

As a consequence of Proposition 2.4, polynomial stabilization at rate $1/t^{1+\varepsilon}$ for $\varepsilon > 0$ is not possible if there is a strongly trapped ray (i.e., that does not intersect $\operatorname{supp}(b)$). More precisely, in such geometry, Theorem 2.5 combined with Lemma 4.6 and [Batty and Duyckaerts 2008, Proposition 1.3] shows that $m_1(t) \geq C/(1+t)$, for some $C > 0$ (with the notation of [Batty and Duyckaerts 2008], where $m_1(t)$ denotes the best decay rate).

The main goal of this paper is to explore the gap between the a priori upper bound $1/\sqrt{t}$ for the decay rate, given by Theorem 2.3, and the a priori lower bound $1/t$ of Theorem 2.5. Our results are twofold (somehow in two opposite directions) and concern either the case of smooth damping functions b, or the case $b = \mathbb{1}_U$, with $U \subset \mathbb{T}^2$.

2B1. *The case of smooth damping coefficients.* Our main result deals with the case of smooth damping coefficients. Without any geometric assumption, but with an additional hypothesis on the order of vanishing of the damping function b, we prove a weak converse of Theorem 2.5.

Theorem 2.6. *Let $M = \mathbb{T}^2$ with the standard flat metric. There exists $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ satisfying the following property: Suppose that $b \in W^{k_0,\infty}(\mathbb{T}^2)$ is a nonnegative, nonvanishing function on $\mathbb{T}^2$ and that* *there exist* $\varepsilon \in (0, \varepsilon_0)$ *and* $C_\varepsilon > 0$ *such that*

$$|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x) \quad \text{for } x \in \mathbb{T}^2. \tag{2-13}$$

Then there exist $C > 0$ *and* $s_0 \geq 0$ *such that for all* $s \in \mathbb{R}$, $|s| \geq s_0$, *we have*

$$\|P(is)^{-1}\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq C|s|^\delta, \quad \text{with } \delta = 4\varepsilon. \tag{2-14}$$

As a consequence of Proposition 2.4, in this situation, the damped wave equation (1-1) is stable at rate $1/t^{1/(1+\delta)}$.

Remark 2.7. Following carefully the steps of the proof, one sees that $\varepsilon_0 = \frac{1}{29}$ works, but the proof is not optimized with respect to this parameter, and it is likely that it could be much improved.

The regularity assumption $b \in W^{k_0,\infty}(\mathbb{T}^2)$ is required since we make use of symbolic calculus in the proof of Lemma 7.1 (and only at this point of the paper). We only use the two following properties: (i) that the commutator of b with some Fourier multipliers is given by the usual principal term plus a lower order perturbation; (ii) the sharp Gårding inequality for a symbol depending on ∇b. It seems that (in 2 space dimensions) $k_0 = 8$ suffices in these two different applications of symbolic calculus (see [Sjöstrand 1995, Proposition 5.1] for a Gårding inequality with this regularity or [Lerner 2010, pp. 117–118] for a related discussion).

One of the main difficulties in understanding the decay rates is that there exists no general monotonicity property of the type "$b_1(x) \leq b_2(x)$ for all $x \Rightarrow$ the decay rate associated to the damping b_2 is larger (or smaller) than the decay rate associated to the damping b_1." This makes a significant difference with observability or controllability problems of the type (1-5).

Assumption (2-13) is only a local assumption in a neighborhood of $\partial\{b > 0\}$ (even if it is stated here globally on $\mathbb{T}^2$). Far from this set, i.e., on each compact set $\{b \geq b_0\}$ for $b_0 > 0$, the constant C_ε can be chosen uniformly, depending only on b_0, and not on ε. Hence, ε somehow quantifies the vanishing rate of the damping function b.

An interesting situation is when the smooth function b vanishes like e^{-1/x^α} in smooth local coordinates, for some $\alpha > 0$. In this case, assumption (2-13) is satisfied for any $\varepsilon > 0$, and the associated damped wave equation (1-1) is stable at rate $1/t^{1-\delta}$ for any $\delta > 0$. This shows that the lower bound given by Theorem 2.5 is sharp, in the sense that one cannot improve upon the exponent of t. This phenomenon had already been remarked by Burq and Hitrik [2007] in the case where b is invariant in one direction.

An example of a smooth function not satisfying assumption (2-13) is a function vanishing like $\sin(1/x)^2 e^{-1/x}$. We do not have any idea of the decay rate achieved in this case (except for the a priori upper and lower bounds $1/\sqrt{t}$ and $1/t$).

Theorem 2.6 generalizes the result of [Burq and Hitrik 2007], which only holds if b is assumed to be invariant in one direction. Moreover, our condition (2-13) is weaker than the assumption (3.2) of Burq and Hitrik. Actually their proof only uses the condition $|b'| \leq C_\varepsilon b^{1-\varepsilon}$ and $|b''| \leq C_\varepsilon b^{1-2\varepsilon}$ for some $\varepsilon < \frac{1}{4}$ (which is similar to ours), to obtain the same decay at rate $t^{-1/(1+4\varepsilon)}$.

The proof of Theorem 2.6 occupies Part III and is sketched in its introduction. It is based on ideas and tools developed in [Macià 2010; Anantharaman and Macià 2010] and especially the notion of *two-microlocal* semiclassical measures. One of the key technical points appears in Section 12: we have to construct, for each trapped direction, a cutoff function invariant in that direction and adapted to the damping coefficient b. We do not know how to adapt this technical construction to tori of higher dimension $d > 2$; hence we do not know whether Theorem 2.6 holds in higher dimension (although we have no reason to suspect it should not hold). Only in the particular case where b is invariant in $d-1$ directions can our methods (or those of [Burq and Hitrik 2007]) be applied to prove the analogue of Theorem 2.6.

Note that if GCC is satisfied, one has (on a general compact manifold M) for some $C > 1$ and all $|s| \geq s_0$ the estimate

$$\|P(is)^{-1}\|_{\mathscr{L}(L^2(M))} \leq C|s|^{-1} \tag{2-15}$$

instead of (2-14). Estimate (2-15) is in turn equivalent to uniform stabilization (see [Huang 1985] together with Lemma 4.6).

Remark 2.8. As a consequence of Theorem 2.6 on the torus, we can deduce that the decay rate $t^{-1/(1+\delta)}$ also holds for (1-1) if $M = (0,\pi)^2$ is the square, one considers Dirichlet or Neumann boundary conditions, and the damping function b is smooth, vanishes near ∂M and satisfies assumption (2-13). First, we extend the function b as an even (with respect to both variables) smooth function on the larger square $(-\pi,\pi)^2$, and using the injection $\imath : (-\pi,\pi)^2 \to \mathbb{T}^2$, as a smooth function on $\mathbb{T}^2$, still satisfying (2-13). Moreover, $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(0,\pi)^2$ can be identified as the closed subspace of odd (resp. even) functions of $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(-\pi,\pi)^2$. Using again the injection $\imath$, it can also be identified with a closed subspace of $H^2(\mathbb{T}^2)$. The estimate

$$\|u\|_{L^2(\mathbb{T}^2)} \leq C|s|^{\delta}\|P(is)u\|_{L^2(\mathbb{T}^2)} \quad \text{for all } u \in H^2(\mathbb{T}^2)$$

is thus also true on the square $(0,\pi)^2$ for Dirichlet or Neumann boundary conditions. In particular, this strongly improves the results of [Liu and Rao 2005].

The lower bound of Theorem 2.5 can be similarly extended to the case of a square with Dirichlet or Neumann boundary conditions, implying that the rate $1/t$ is optimal if GCC is strongly violated.

2B2. *The case of discontinuous damping functions.* Appendix B (by Stéphane Nonnenmacher) deals with the case where b is the characteristic function of a vertical strip, i.e., $b = \tilde{B}\mathbb{1}_{U\times\mathbb{T}}$, for some $\tilde{B} > 0$ and $U \subset \mathbb{T}$, U a nonempty open interval with $\overline{U} \neq \mathbb{T}$. Due to the invariance of b in one direction, the spectrum of the damped wave operator $\mathscr{A}$ splits into countably many "branches" of eigenvalues. This structure of the spectrum is illustrated in the numerics of [Asch and Lebeau 2003; Anantharaman and Léautaud 2012].

The branch closest to the imaginary axis is explicitly computed; it contains a sequence of eigenvalues $(z_i)_{i\in\mathbb{N}}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \leq C_0/(\operatorname{Im} z_i)^{\frac{3}{2}}$. This result is in agreement with the numerical tests given in [Anantharaman and Léautaud 2012].

As a consequence, for any $\varepsilon > 0$ and $C > 0$, the strip $\{|\mathrm{Re}\, z| \leq C|\mathrm{Im}(z)|^{-\frac{3}{2}+\varepsilon}\}$ contains infinitely many poles of the resolvent $(z\,\mathrm{Id} - \mathcal{A})^{-1}$, so item (2-9) in Proposition 2.4 implies the following obstruction to the stability of this damped system:

Corollary 2.9. *For any* $\varepsilon > 0$, *the damped wave equation* (1-1) *on* $\mathbb{T}^2$ *with the damping function* (B-1) *cannot be stable at the rate* $1/t^{\frac{2}{3}+\varepsilon}$.

The same result holds on the square with Dirichlet or Neumann boundary conditions.

More precisely, in this situation, Lemma 4.6 and [Batty and Duyckaerts 2008, Proposition 1.3] yield that $m_1(t) \geq C/(1+t)^{\frac{2}{3}}$, for some $C > 0$ (with the notation of that reference, where $m_1(t)$ denotes the best decay rate).

This corollary shows in particular that the regularity conditions in Theorem 2.6 cannot be completely disposed of if one wants a stability at the rate $1/t^{1-\varepsilon}$ for small ε.

2C. ***Some related open questions.*** The various results obtained in this article lead to several open questions.

(1) In the case where b is the characteristic function of a vertical strip, our analysis shows that the best decay rate lies somewhere between $1/t^{\frac{1}{2}}$ and $1/t^{\frac{2}{3}}$, but the "true" decay rate is not yet clear.

(2) It would also be interesting to investigate the spectrum and the decay rates for damping functions b invariant in one direction, but having a less singular behavior than a characteristic function. In particular, is it possible to give a precise link between the vanishing rate of b and the decay rate?

(3) In the general setting of Section 2A (as well as in the case of the damped wave equation on the torus), is the a priori upper bound $1/t^{\frac{1}{2}}$ for the decay rate optimal?

(4) For smooth damping functions vanishing like e^{-1/x^α}, Theorem 2.6 yields stability at rate $1/t^{1-\delta}$ for all $\delta > 0$. Is the decay rate $1/t$ reached in this situation? Can one find a damping function b such that the decay rate is exactly $1/t$?

(5) The lower bound of Theorem 2.5 is still valid in higher-dimensional tori. Is there an analogue of Theorem 2.6 (i.e., for general "smooth" damping functions) for $\mathbb{T}^d$, with $d \geq 3$?

Part II. Resolvent estimates and stabilization in the abstract setting

3. Proof of Theorem 2.3 assuming Proposition 2.4

To prove Theorem 2.3, we express the observability condition as a resolvent estimate (also known as the Hautus test), as introduced by Burq and Zworski [2004], and further developed by Miller [2005] and Ramdani, Takahashi, Tenenbaum and Tucsnak [Ramdani et al. 2005]. For a survey of this notion, we refer to the book [Tucsnak and Weiss 2009, Section 6.6].

In particular [Miller 2005, Theorem 5.1] (or [Tucsnak and Weiss 2009, Theorem 6.6.1]) yields that system (2-1) is observable in some time $T > 0$ if and only if there exists a constant $C > 0$ such that we have

$$\|u\|_H^2 \leq C\big(\|(A - \lambda\,\mathrm{Id})u\|_H^2 + \|B^*u\|_Y^2\big) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } u \in D(A).$$

As a first consequence, assumption (2-6) is satisfied and Proposition 2.4 applies in this context. Moreover, recalling that $P(z)$ is defined in (2-5), we have, for all $s \in \mathbb{R}$ and $u \in D(A)$,

$$\begin{aligned}\|u\|_H^2 &\le C\big(\|(A - s^2\,\mathrm{Id} + isBB^* - isBB^*)u\|_H^2 + \|B^*u\|_Y^2\big)\\ &\le C\big(\|P(is)u\|_H^2 + s^2\|BB^*u\|_H^2 + \|B^*u\|_Y^2\big)\end{aligned} \tag{3-1}$$

Since $B \in \mathscr{L}(Y; H)$, we obtain for $s \ge 1$ and for some $C > 0$,

$$\|u\|_H^2 \le C\big(\|P(is)u\|_H^2 + s^2\|B^*u\|_Y^2\big) \le C\big(s^2\|P(is)u\|_H^2 + s^2\|B^*u\|_Y^2\big).$$

Proposition 2.4 then yields the polynomial stability at rate $1/\sqrt{t}$ for (2-3). This concludes the proof of Theorem 2.3. □

4. Proof of Proposition 2.4

Our proof relies strongly on the characterization of polynomially stable semigroups given in [Borichev and Tomilov 2010, Theorem 2.4], which can be reformulated as follows.

Theorem 4.1 (Borichev and Tomilov). *Let $(e^{t\dot{\mathscr{A}}})_{t\ge 0}$ be a bounded $\mathscr{C}^0$-semigroup on a Hilbert space $\dot{\mathscr{H}}$, generated by $\dot{\mathscr{A}}$. Suppose that $i\mathbb{R} \cap \mathrm{Sp}(\dot{\mathscr{A}}) = \varnothing$. Then, the following conditions are equivalent*:

$$\|e^{t\dot{\mathscr{A}}}\dot{\mathscr{A}}^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})} = \mathscr{O}(t^{-\alpha}) \quad \text{as } t \to +\infty, \tag{4-1}$$

$$\|(is\,\mathrm{Id} - \dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})} = \mathscr{O}(|s|^{\frac{1}{\alpha}}) \quad \text{as } s \to \infty. \tag{4-2}$$

Let us first describe some spectral properties of the operator $\mathscr{A}$ defined in (2-4).

Lemma 4.2. *The spectrum of $\mathscr{A}$ contains only isolated eigenvalues, and we have*

$$\mathrm{Sp}(\mathscr{A}) \subset \big((-\tfrac{1}{2}\|B^*\|^2_{\mathscr{L}(H;Y)}, 0) + i\mathbb{R}\big) \cup ([-\|B^*\|^2_{\mathscr{L}(H;Y)}, 0] + 0i),$$

with $\ker(\mathscr{A}) = \ker(A) \times \{0\}$.

Moreover, the operator $P(z)$ is an isomorphism from $D(A)$ onto H if and only if $z \notin \mathrm{Sp}(\mathscr{A})$. If this is satisfied, we have

$$(z\,\mathrm{Id} - \mathscr{A})^{-1} = \begin{pmatrix} P(z)^{-1}(BB^* + z\,\mathrm{Id}) & P(z)^{-1} \\ P(z)^{-1}(zBB^* + z^2\,\mathrm{Id}) - \mathrm{Id} & zP(z)^{-1} \end{pmatrix}. \tag{4-3}$$

The localization properties for the spectrum of $\mathscr{A}$ stated in the first part of this lemma are illustrated, for instance, in [Asch and Lebeau 2003] or [Anantharaman and Léautaud 2012].

This lemma leads us to introduce the spectral projector of $\mathscr{A}$ on $\ker(\mathscr{A})$, given by

$$\Pi_0 = \frac{1}{2i\pi}\int_\gamma (z\,\mathrm{Id} - \mathscr{A})^{-1}\,dz \in \mathscr{L}(\mathscr{H}),$$

where γ denotes a positively oriented circle centered on 0 with a radius so small that 0 is the single eigenvalue of $\mathscr{A}$ in the interior of γ. We set $\dot{\mathscr{H}} = (\mathrm{Id} - \Pi_0)\mathscr{H}$ and equip this space with the norm

$$\|(u_0, u_1)\|^2_{\dot{\mathscr{H}}} := |(u_0, u_1)|^2_{\mathscr{H}} = \|A^{\frac{1}{2}}u_0\|_H^2 + \|u_1\|_H^2,$$

and associated inner product. This is indeed a norm on $\dot{\mathcal{H}}$ since $\|(u_0, u_1)\|_{\dot{\mathcal{H}}} = 0$ is equivalent to $(u_0, u_1) \in \ker(A) \times \{0\} = \Pi_0 \mathcal{H}$.

We also set $\dot{\mathcal{A}} = \mathcal{A}|_{\dot{\mathcal{H}}}$ with domain $D(\dot{\mathcal{A}}) = D(\mathcal{A}) \cap \dot{\mathcal{H}}$. A first remark is that $\mathrm{Sp}(\dot{\mathcal{A}}) = \mathrm{Sp}(\mathcal{A}) \setminus \{0\}$, so that $\mathrm{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} = \varnothing$.

The remainder of the proof consists in applying Theorem 4.1 to the operator $\dot{\mathcal{A}}$ in $\dot{\mathcal{H}}$. We first check the assumptions of Theorem 4.1 and describe the solutions of the evolution problem (2-4) (or equivalently (2-3)).

Lemma 4.3. *The operator $\dot{\mathcal{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathcal{H}}$, denoted $(e^{t\dot{\mathcal{A}}})_{t\geq 0}$. Moreover, for all initial data $U_0 \in \mathcal{H}$, problem (2-4) (or equivalently (2-3)) has a unique solution $U \in \mathscr{C}^0(\mathbb{R}^+; \mathcal{H})$, issued from U_0, that can be decomposed as*

$$U(t) = e^{t\dot{\mathcal{A}}}(\mathrm{Id} - \Pi_0)U_0 + \Pi_0 U_0 \quad \textit{for all } t \geq 0. \tag{4-4}$$

As a consequence, we can apply Theorem 4.1 to the semigroup generated by $\dot{\mathcal{A}}$. The proof of Proposition 2.4 will be achieved when the following lemmata are proved.

Lemma 4.4. *Conditions (2-7) and (4-1) are equivalent.*

Lemma 4.5. *Conditions (2-10) and (2-11) are equivalent. Conditions (2-8) and (2-9) are equivalent.*

Lemma 4.6. *There exist $C > 1$ and $s_0 > 0$ such that for $s \in \mathbb{R}$, $|s| \geq s_0$, we have*

$$\|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} - \frac{C}{|s|} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} + \frac{C}{|s|}, \tag{4-5}$$

and

$$C^{-1}|s|\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C(1 + |s|\|P(is)^{-1}\|_{\mathcal{L}(H)}). \tag{4-6}$$

In particular this implies that (4-2), (2-8) and (2-10) are equivalent.

The proof of Lemma 4.6 is more or less classical and we follow [Lebeau 1996; Burq and Hitrik 2007].

Proof of Lemma 4.2. Since $\mathcal{A}$ has compact resolvent, its spectrum contains only isolated eigenvalues. Suppose that $z \in \mathrm{Sp}(\mathcal{A})$; then, for some $(u_0, u_1) \in D(\mathcal{A}) \setminus \{0\}$, we have

$$u_1 = z u_0, \qquad -Au_0 - BB^* u_1 = z u_1,$$

and in particular

$$Au_0 + z^2 u_0 + zBB^* u_0 = 0, \tag{4-7}$$

with $u_0 \in D(A) \setminus \{0\}$.

Suppose that $z \in i\mathbb{R}$; then, this yields $Au_0 - \mathrm{Im}(z)^2 u_0 + i\,\mathrm{Im}(z)BB^* u_0 = 0$. Following [Lebeau 1996], taking the inner product of this equation with u_0 yields $i\,\mathrm{Im}(z)\|B^* u_0\|_Y^2 = 0$. Hence, either $\mathrm{Im}(z) = 0$ or $B^* u_0 = 0$. In the first case, $Au_0 = 0$, i.e., $u_0 \in \ker(A)$, and $u_1 = 0$. This yields $\ker(\mathcal{A}) \subset \ker(A) \times \{0\}$ (and the other inclusion is clear). In the second case, u_0 is an eigenvector of A associated to the eigenvalue $\mathrm{Im}(z)^2$ and satisfies $B^* u_0 = 0$, which is absurd, according to assumption (2-6). Thus, $\mathrm{Sp}(\mathcal{A}) \cap i\mathbb{R} \subset \{0\}$.

Now, for a general eigenvalue $z \in \mathbb{C}$, taking the inner product of (4-7) with u_0 yields

$$\begin{aligned} (Au_0, u_0)_H + (\mathrm{Re}(z)^2 - \mathrm{Im}(z)^2)\|u_0\|_H^2 + \mathrm{Re}(z)\|B^*u_0\|_Y^2 = 0, \\ 2\,\mathrm{Re}(z)\,\mathrm{Im}(z)\|u_0\|_H^2 + \mathrm{Im}(z)\|B^*u_0\|_Y^2 = 0. \end{aligned} \tag{4-8}$$

If $\mathrm{Im}(z) \neq 0$, then the second equation of (4-8) together with $\mathrm{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} \subset \{0\}$ gives

$$0 > \mathrm{Re}(z) = -\frac{1}{2}\frac{\|B^*u_0\|_Y^2}{\|u_0\|_H^2} \geq -\frac{1}{2}\|B^*\|_{\mathcal{L}(H;Y)}^2.$$

If $\mathrm{Im}(z) = 0$, then the first equation of (4-8) together with $(\dot{A}u_0, u_0)_H \geq 0$ gives

$$-\mathrm{Re}(z)\|B^*u_0\|_Y^2 \geq \mathrm{Re}(z)^2\|u_0\|_H^2,$$

which yields

$$0 \geq \mathrm{Re}(z) \geq -\|B^*\|_{\mathcal{L}(H;Y)}^2.$$

Following [Lebeau 1996], we now give the link between $P(z)^{-1}$ and $(z\,\mathrm{Id} - \mathcal{A})^{-1}$ for $z \notin \mathrm{Sp}(\mathcal{A})$. Taking $F = (f_0, f_1) \in \mathcal{H}$, and $U = (u_0, u_1)$, we have

$$F = (z\,\mathrm{Id} - \mathcal{A})U \iff \begin{cases} u_1 = zu_0 - f_0, \\ P(z)u_0 = f_1 + (BB^* + z\,\mathrm{Id})f_0. \end{cases} \tag{4-9}$$

As a consequence, we obtain that $P(z) : D(A) \to H$ is invertible if and only if $(z\,\mathrm{Id} - \mathcal{A}) : D(\mathcal{A}) \to \mathcal{H}$ is invertible, i.e., if and only if $z \notin \mathrm{Sp}(\mathcal{A})$. Moreover, for such values of z, the condition on the right-hand side of (4-9) is equivalent to

$$u_0 = P(z)^{-1}f_1 + P(z)^{-1}(BB^* + z\,\mathrm{Id})f_0 \quad \text{and} \quad u_1 = zP(z)^{-1}f_1 + zP(z)^{-1}(BB^* + z\,\mathrm{Id})f_0 - f_0,$$

which can be rewritten as (4-3). This concludes the proof of Lemma 4.2. □

Proof of Lemma 4.3. Let us check that $\dot{\mathcal{A}}$ is a maximal dissipative operator on $\dot{\mathcal{H}}$ [Pazy 1983]. First, it is dissipative since, for $U = (u_0, u_1) \in D(\dot{\mathcal{A}})$,

$$(\dot{\mathcal{A}}U, U)_{\dot{\mathcal{H}}} = (A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_0)_H - (Au_0, u_1)_H - (BB^*u_1, u_1)_H = -\|B^*u_1\|_Y^2 \leq 0.$$

Next, the fact that $\mathcal{A} - \mathrm{Id}$ is onto is a consequence of Lemma 4.2. Hence, for all $F \in \dot{\mathcal{H}} \subset \mathcal{H}$, there exists $U \in D(\mathcal{A})$ such that $(\mathcal{A} - \mathrm{Id})U = F$. Applying $(\mathrm{Id} - \Pi_0)$ to this identity yields $(\dot{\mathcal{A}} - \mathrm{Id})(\mathrm{Id} - \Pi_0)U = F$, so $\dot{\mathcal{A}} - \mathrm{Id} : D(\dot{\mathcal{A}}) \to \dot{\mathcal{H}}$ is onto. According to the Lumer–Phillips theorem (see, for instance, [Pazy 1983, Chapter 1, Theorem 4.3]) $\dot{\mathcal{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathcal{H}}$. Then, formula (4-4) directly comes from the linearity of (2-4) (or equivalently (2-3)) together with the decomposition of the initial condition $U_0 = (I - \Pi_0)U_0 + \Pi_0 U_0$. □

Proof of Lemma 4.4. Condition (4-1) is equivalent to the existence of $C > 0$ such that for all $t > 0$, and $\dot{U}_0 \in \dot{\mathcal{H}}$, we have

$$\|e^{t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-1}\dot{U}_0\|_{\dot{\mathcal{H}}} \leq \frac{C}{t^\alpha}\|\dot{U}_0\|_{\dot{\mathcal{H}}}.$$

This can be rephrased as

$$\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}} \leq \frac{C}{t^\alpha}\|\dot{\mathcal{A}}\dot{U}_0\|_{\dot{\mathcal{H}}}, \tag{4-10}$$

for all $t > 0$, and $\dot{U}_0 \in D(\dot{\mathcal{A}})$. Now, take any $U_0 = (u_0, u_1) \in D(\mathcal{A})$, and associated projection $\dot{U}_0 = (\mathrm{Id} - \Pi_0)U_0 \in D(\dot{\mathcal{A}})$. According to (4-4), we have

$$E(u,t) = \tfrac{1}{2}\big(\|A^{\frac{1}{2}}u(t)\|_H^2 + \|\partial_t u(t)\|_H^2\big) = \tfrac{1}{2}|e^{t\dot{\mathcal{A}}}\dot{U}_0 + \Pi_0 U_0|_{\mathcal{H}}^2 = \tfrac{1}{2}\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}}^2,$$

and

$$|\mathcal{A}U_0|_{\mathcal{H}} = |\dot{\mathcal{A}}\dot{U}_0 + \mathcal{A}\Pi_0 U_0|_{\mathcal{H}} = \|\dot{\mathcal{A}}\dot{U}_0\|_{\dot{\mathcal{H}}}.$$

This shows that (4-10) is equivalent to (2-7), and concludes the proof of Lemma 4.4. □

Proof of Lemma 4.5. First, (2-10) clearly implies (2-11). To prove the converse, for $u \in D(\mathcal{A})$, we have

$$(P(is)u, u)_H = \big((A - s^2\,\mathrm{Id})u, u\big)_H + is\|B^*u\|_Y^2.$$

Taking the imaginary part of this identity gives $s\|B^*u\|_Y^2 = \mathrm{Im}(P(is)u,u)_H$, so that, using the Young inequality, we obtain for all $\varepsilon > 0$,

$$|s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2 = |s|^{\frac{1}{\alpha}-1}|\mathrm{Im}(P(is)u,u)_H| \le \frac{|s|^{\frac{2}{\alpha}-2}}{4\varepsilon}\|P(is)u\|_H^2 + \varepsilon\|u\|_H^2.$$

Plugging this into (2-11) and taking ε sufficiently small, we obtain that for some $C > 0$ and $s_0 \ge 0$, for any $s \in \mathbb{R}$, $|s| \ge s_0$, we have

$$\|u\|_H^2 \le C|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2,$$

which yields (2-10). Hence (2-10) and (2-11) are equivalent.

Second, Condition (2-9) clearly implies (2-8) and it only remains to prove the converse. For $z \in \mathbb{C}$, we write $r = \mathrm{Re}(z)$ and $s = \mathrm{Im}(z)$. We have the identity

$$((r+is)\,\mathrm{Id} - \mathcal{A})^{-1} = (is\,\mathrm{Id} - \mathcal{A})^{-1}\big(\mathrm{Id} + r(is\,\mathrm{Id} - \mathcal{A})^{-1}\big)^{-1}. \tag{4-11}$$

Hence, assuming

$$\|r(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le \tfrac{1}{2}, \tag{4-12}$$

this gives

$$\big\|\big(\mathrm{Id} + r(is\,\mathrm{Id} - \mathcal{A})^{-1}\big)^{-1}\big\|_{\mathcal{L}(\mathcal{H})} = \Big\|\sum_{k=0}^{\infty}(-1)^k\big(r(is\,\mathrm{Id} - \mathcal{A})^{-1}\big)^k\Big\|_{\mathcal{L}(\mathcal{H})} \le 2.$$

As a consequence of (4-11) and (2-8), we then obtain

$$\|((r+is)\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le 2\|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le 2C|s|^{\frac{1}{\alpha}},$$

for all $s \ge s_0$, under condition (4-12). Finally, (2-8) also yields

$$\|r(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le |r|C|s|^{\frac{1}{\alpha}},$$

so that condition (4-12) is realized as soon as $|r| \le 1/(2C|s|^{\frac{1}{\alpha}})$. This proves (2-9) and concludes the proof of Lemma 4.5. □

Proof of Lemma 4.6. To prove (4-5), we first remark that the norms $\|\cdot\|_{\dot{\mathcal{H}}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent on $\dot{\mathcal{H}}$, so that the norms $\|\cdot\|_{\mathcal{L}(\dot{\mathcal{H}})}$ and $\|\cdot\|_{\mathcal{L}(\mathcal{H})}$ are equivalent on $\mathcal{L}(\dot{\mathcal{H}})$. Next, we have

$$(is\,\mathrm{Id}-\dot{\mathcal{A}})^{-1}(\mathrm{Id}-\Pi_0)=(is\,\mathrm{Id}-\mathcal{A})^{-1}(\mathrm{Id}-\Pi_0)$$

and

$$\begin{aligned}\|(is\,\mathrm{Id}-\dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\mathcal{H})}&=\|(is\,\mathrm{Id}-\dot{\mathcal{A}})^{-1}(\mathrm{Id}-\Pi_0)\|_{\mathcal{L}(\mathcal{H})}=\|(is\,\mathrm{Id}-\mathcal{A})^{-1}(\mathrm{Id}-\Pi_0)\|_{\mathcal{L}(\mathcal{H})}\\&\leq\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}+\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})},\end{aligned}$$

together with

$$\begin{aligned}\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}&=\|(is\,\mathrm{Id}-\dot{\mathcal{A}})^{-1}(\mathrm{Id}-\Pi_0)+(is\,\mathrm{Id}-\mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})}\\&\leq\|(is\,\mathrm{Id}-\dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\mathcal{H})}+\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})}.\end{aligned}$$

Moreover, for $|s|\geq 1$, we have

$$\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})}=\|(is)^{-1}\Pi_0\|_{\mathcal{L}(\mathcal{H})}=\frac{1}{|s|}\|\Pi_0\|_{\mathcal{L}(\mathcal{H})}=\frac{C}{|s|},$$

which concludes the proof of (4-5).

Let us now prove (4-6). For concision, we set $H_1=D(A^{\frac12})$ endowed with the graph norm $\|u\|_{H_1}=\|(A+\mathrm{Id})^{\frac12}u\|_H$ and denote by $H_{-1}=D(A^{\frac12})'$ its dual space. The operator A can be uniquely extended as an operator $\mathcal{L}(H_1;H_{-1})$, still denoted A for simplicity. With this notation, the space H_{-1} can be equipped with the natural norm $\|u\|_{H_{-1}}=\|(A+\mathrm{Id})^{-\frac12}u\|_H$.

As a consequence of formula (4-3), and using the fact that $\mathrm{Sp}(\mathcal{A})\cap i\mathbb{R}\subset\{0\}$, there exist constants $C>1$ and $s_0>0$ such that for all $s\in\mathbb{R}$, $|s|\geq s_0$, we have

$$C^{-1}M(s)\leq\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\leq CM(s),\tag{4-13}$$

with

$$\begin{aligned}M(s)=\big(&\|P(is)^{-1}(BB^*+is\,\mathrm{Id})\|_{\mathcal{L}(H_1)}+\|P(is)^{-1}\|_{\mathcal{L}(H;H_1)}\\&+\|P(is)^{-1}(isBB^*-s^2\,\mathrm{Id})-\mathrm{Id}\,\|_{\mathcal{L}(H_1;H)}+\|sP(is)^{-1}\|_{\mathcal{L}(H)}\big).\end{aligned}\tag{4-14}$$

On the one hand, this directly yields

$$|s|\|P(is)^{-1}\|_{\mathcal{L}(H)}\leq C\|(is\,\mathrm{Id}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})},$$

for $s\in\mathbb{R}$, $|s|\geq s_0$. This proves that (4-2) implies (2-10).

On the other hand, we have to estimate each term of (4-14). First, using $Au=P(is)u+s^2u-isBB^*u$, we have

$$\begin{aligned}\|u\|_{H_1}^2&=\|A^{\frac12}u\|_H^2+\|u\|_H^2=(P(is)u+s^2u-isBB^*u,u)_H+\|u\|_H^2\\&=\mathrm{Re}(P(is)u,u)_H+(s^2+1)\|u\|_H^2\leq C\big(\|P(is)u\|_H^2+(s^2+1)\|u\|_H^2\big)\\&\leq C\big(1+(s^2+1)\|P(is)^{-1}\|_{\mathcal{L}(H)}^2\big)\|P(is)u\|_H^2,\end{aligned}$$

so that

$$\|P(is)^{-1}\|_{\mathscr{L}(H;H_1)} \le C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-15}$$

Second, the same computation for $(P(is)^{-1})^* = (A - s^2\,\mathrm{Id} - isBB^*)^{-1}$ (the adjoint of $P(is)^{-1}$ in the space H) in place of $P(is)^{-1}$ leads to

$$(P(is)^{-1})^* \in \mathscr{L}(H;\,H_1),$$

together with the estimate

$$\|(P(is)^{-1})^*\|_{\mathscr{L}(H;H_1)} \le C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big).$$

By transposition, we have ${}^t(P(is)^{-1})^* \in \mathscr{L}(H_{-1};\,H)$, together with the estimate

$$\|{}^t(P(is)^{-1})^*\|_{\mathscr{L}(H_{-1};H)} \le \|(P(is)^{-1})^*\|_{\mathscr{L}(H;H_1)} \le C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-16}$$

Moreover, ${}^t(P(is)^{-1})^*$ is defined, for every $u \in H$, $v \in H_{-1}$, by

$$\big({}^t(P(is)^{-1})^* v, u\big)_H = \langle v, (P(is)^{-1})^* u\rangle_{H_{-1},H_1} = \big((A+\mathrm{Id})^{-\frac12} v, (A+\mathrm{Id})^{\frac12}(P(is)^{-1})^* u\big)_H.$$

In particular, taking $v \in H$ gives

$$\big({}^t(P(is)^{-1})^* v, u\big)_H = \big(P(is)^{-1} v, u\big)_H\,,$$

which implies that the restriction of the operator ${}^t(P(is)^{-1})^*$ to H coincides with $P(is)^{-1}$. For simplicity, we will denote $P(is)^{-1}$ for ${}^t(P(is)^{-1})^*$.

Equation (4-16) can thus be rewritten

$$\|P(is)^{-1}\|_{\mathscr{L}(H_{-1};H)} \le C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-17}$$

Then, we have $P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id} = P(is)^{-1}A$, so that

$$\begin{aligned}\|P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id}\,\|_{\mathscr{L}(H_1;H)} &= \|P(is)^{-1}A\|_{\mathscr{L}(H_1;H)} \le \|P(is)^{-1}\|_{\mathscr{L}(H_{-1};H)}\|A\|_{\mathscr{L}(H_1;H_{-1})}\\ &\le \big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big)\,. \end{aligned}\tag{4-18}$$

Third, for $|s| \ge 1$ we write

$$P(is)^{-1}(BB^* + is\,\mathrm{Id}) = \frac{i}{s}(P(is)^{-1}A - \mathrm{Id}), \tag{4-19}$$

and it remains to estimate the term $\|P(is)^{-1}A\|_{\mathscr{L}(H_1)}$ in (4-14). For $f \in H_1$, we set $u = P(is)^{-1}Af$. We have $u \in H_1$, together with

$$(A - s^2\,\mathrm{Id} + isBB^*)u = Af.$$

Taking the real part of the inner product of this identity with u, we find

$$\|A^{\frac12}u\|_H^2 - s^2\|u\|_H^2 = \mathrm{Re}(Af, u)_H \le \|Af\|_{H_{-1}}\|u\|_{H_1} \le C\|f\|_{H_1}\|u\|_{H_1},$$

since $A \in \mathscr{L}(H_1, H_{-1})$. Hence

$$\|u\|_{H_1}^2 \le C(1+s^2)\|u\|_H^2 + C\|f\|_{H_1}^2.$$

Using (4-17), this gives

$$\begin{aligned}\|u\|_{H_1}^2 &\le C(1+s^2)\|P(is)^{-1}A\|^2_{\mathscr{L}(H_1;H)}\|f\|^2_{H_1} + C\|f\|^2_{H_1}\\ &\le C(1+s^2)\|P(is)^{-1}\|^2_{\mathscr{L}(H_{-1};H)}\|f\|^2_{H_1} + C\|f\|^2_{H_1}\\ &\le C(1+s^2)\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big)^2\|f\|^2_{H_1},\end{aligned}$$

and finally $\|P(is)^{-1}A\|_{\mathscr{L}(H_1)} \le C(1+|s|)\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big)$. Coming back to (4-19), we have, for $|s| \ge 1$,

$$\|P(is)^{-1}(BB^* + is\,\mathrm{Id})\|_{\mathscr{L}(H_1)} \le C\big(1+|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-20}$$

Finally, combining (4-15), (4-18) and (4-20), together with (4-13)–(4-14), we obtain for $|s| \ge 1$,

$$\|(is\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le C\big(1+|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\big).$$

This concludes the proof of Lemma 4.6. □

Part III. Proof of Theorem 2.6: smooth damping coefficients on the torus

To prove Theorem 2.6, we argue by contradiction, assuming that estimate (2-10) does not hold (which provides a sequence of "quasimodes" defined in Section 5). The proof of Theorem 2.6 then relies on the study of semiclassical measures (a semiclassical version of microlocal defect measures) associated to quasimodes. This standard technique originates in the work of Lebeau [1996], but the novelty here is that we introduce a second microlocalization which allows us to study different scales of concentration around periodic orbits.

Sections 5 and 6 are preliminaries: Section 5 deals with the notion of semiclassical measures in a general setting, while Section 6 specializes to the torus case. Lemmata 6.1 and 6.4 reduce everything to understanding the semiclassical measure μ restricted to frequencies of rational slopes.

From Section 7 on, a frequency of rational slope is fixed; it is parametrized by a submodule Λ of $\mathbb{Z}^2$ of rank 1 (rather than by the slope). More precisely, we study the restriction $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$. The main outcome of this section is the technical Proposition 7.3: it says that a quasimode which is small in the support of b must also be small in a whole strip of direction $\Lambda^\perp$.

The core of the proof occupies Sections 8–10. Section 8 introduces tools of second microlocal calculus. The idea is to study in a finer way the rate of concentration of our quasimodes on $\mathbb{T}^2 \times (\Lambda^\perp \setminus \{0\})$. Section 9 is inspired by [Anantharaman and Macià 2010]: the two-microlocal defect measures gain some additional structure, which depends on the rate of concentration. The final argument is in Section 10, showing that the semiclassical measure μ must vanish everywhere, thus obtaining a contradiction since it was by construction a probability measure.

The last two sections are devoted to more technical lemmata.

5. The invariant semiclassical measure μ

5A. ***Quasimodes.*** To prove Theorem 2.6, we shall instead prove estimate (2-10) with $\alpha = 1/(1+\delta)$ (which, according to Proposition 2.4, is equivalent to the statement of Theorem 2.6). Let us first recast (2-10) with $\alpha = 1/(1+\delta)$ in the semiclassical setting: taking $h = s^{-1}$, we are left to prove that there exist $C > 1$ and $h_0 > 0$ such that for all $h \le h_0$, for all $u \in H^2(\mathbb{T}^2)$, we have

$$\|u\|_{L^2(\mathbb{T}^2)} \le Ch^{-\delta}\|P(i/h)u\|_{L^2(\mathbb{T}^2)}, \tag{5-1}$$

where $P(z)$ is defined in (2-5).

We prove this inequality by contradiction, using the notion of semiclassical measures. The idea of developing such a strategy for proving energy estimates, together with the associated technology, originates from Lebeau [1996].

We assume that (5-1) is not satisfied, and will obtain a contradiction at the end of Section 10. Hence, for all $n \in \mathbb{N}$, there exists $0 < h_n \le 1/n$ and $u_n \in H^2(\mathbb{T}^2)$ such that

$$\|u_n\|_{L^2(\mathbb{T}^2)} > \frac{n}{h_n^\delta}\|P(i/h_n)u_n\|_{L^2(\mathbb{T}^2)}.$$

Setting $v_n = u_n/\|u_n\|_{L^2(\mathbb{T}^2)}$ and

$$P_b^{h_n} = -h_n^2\Delta - 1 + ih_n b(x) = h_n^2 P(i/h_n),$$

we then have, as $n \to \infty$,

$$h_n \to 0^+, \quad \|v_n\|_{L^2(\mathbb{T}^2)} = 1, \quad h_n^{-2-\delta}\|P_b^{h_n} v_n\|_{L^2(\mathbb{T}^2)} \to 0.$$

Our goal is now to associate to the sequence (u_n, h_n) a semiclassical measure on the cotangent bundle μ on $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ (where $(\mathbb{R}^2)^*$ is the dual space of $\mathbb{R}^2$). To obtain a contradiction, we shall prove both that $\mu(T^*\mathbb{T}^2) = 1$, and that $\mu = 0$ on $T^*\mathbb{T}^2$.

From now on, we drop the subscript n of the sequences above, and write h in place of h_n and v_h in place of v_n. We study sequences (h, v_h) such that $h \to 0^+$ and

$$\begin{cases} \|v_h\|_{L^2(\mathbb{T}^2)} = 1, \\ \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}) \quad \text{as } h \to 0^+. \end{cases} \tag{5-2}$$

We call such sequences “sequences of $o(h^{2+\delta})$-quasimodes,” or simply “quasimodes of order $2+\delta$.” In particular, this last equation also yields the key information

$$(bv_h, v_h)_{L^2(\mathbb{T}^2)} = h^{-1}\operatorname{Im}(P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}) \quad \text{as } h \to 0^+.$$

In the following, it will be convenient to identify $(\mathbb{R}^2)^*$ and $\mathbb{R}^2$ through the usual inner product. In particular, the cotangent bundle $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ will be identified with $\mathbb{T}^2 \times \mathbb{R}^2$.

5B. ***Semiclassical measures.*** We denote by $\overline{T^*\mathbb{T}^2}$ the compactification of $T^*\mathbb{T}^2$ obtained by adding a point at infinity to each fiber (i.e., the set $\mathbb{T}^2 \times (\mathbb{R}^2 \cup \{\infty\})$). A neighborhood of $(x, \infty) \in \overline{T^*\mathbb{T}^2}$ is a set

$U \times (\{\infty\} \cup \mathbb{R}^2 \setminus K)$, where U is a neighborhood of x in $\mathbb{T}^2$ and K a compact set in $\mathbb{R}^2$. Endowed with this topology, the set $\overline{T^*\mathbb{T}^2}$ is compact.

We denote by $S^0(T^*\mathbb{T}^2)$, S^0 for short, the space of functions $a(x, \xi)$ that satisfy the following properties:

(1) $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$.

(2) There exists a compact set $K \subset \mathbb{R}^2$ and a constant $k_0 \in \mathbb{C}$ such that $a(x, \xi) = k_0$ for all $\xi \in \mathbb{R}^2 \setminus K$.

Note that we have in particular $\mathscr{C}_c^\infty(T^*\mathbb{T}^2) \subset S^0(T^*\mathbb{T}^2)$.

To a symbol $a \in S^0(T^*\mathbb{T}^2)$, we associate its semiclassical Weyl quantization $\mathrm{Op}_h(a)$ by formula (A-1), which according to the Calderón–Vaillancourt theorem (see Appendix A) defines a uniformly bounded operator on $L^2(\mathbb{T}^2)$.

From the sequence (v_h, h) (see, for instance, [Gérard and Leichtnam 1993]), we can define (using again the Calderón–Vaillancourt theorem) the associated Wigner distribution $V^h \in (S^0)'$ by

$$\langle V^h, a\rangle_{(S^0)', S^0} = (\mathrm{Op}_h(a) v_h, v_h)_{L^2(\mathbb{T}^2)} \quad \text{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5-3}$$

Decomposing v_h and a in Fourier series,

$$\hat{v}_h(k) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} v_h(x)\,dx, \quad \hat{a}(h,k,\xi) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} a(h,x,\xi)\,dx,$$

the expression (5-3) can be more explicitly rewritten as

$$\langle V^h, a\rangle_{(S^0)', S^0} = \frac{1}{2\pi}\sum_{k,j\in\mathbb{Z}^2} \hat{a}\Big(h, j-k, \frac{h}{2}(k+j)\Big)\hat{v}_h(k)\overline{\hat{v}_h(j)}.$$

Proposition 5.1. *The family (V^h) is bounded in $(S^0)'$. Hence, there exists a subsequence of the sequence (h, v_h) and an element $\mu \in (S^0)'$ such that $V^h \rightharpoonup \mu$ weakly in $(S^0)'$, that is,*

$$(\mathrm{Op}_h(a) v_h, v_h)_{L^2(\mathbb{T}^2)} \to \langle \mu, a\rangle_{(S^0)', S^0} \quad \textit{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5-4}$$

In addition, $\langle \mu, a\rangle_{(S^0)', S^0}$ is nonnegative if a is; in other words, μ may be identified with a nonnegative Radon measure on $\overline{T^*\mathbb{T}^2}$.

Notation: in what follows, we shall denote by $\mathcal{M}^+(\overline{T^*\mathbb{T}^2})$ the set of nonnegative Radon measures on $\overline{T^*\mathbb{T}^2}$.

Proof. The proof is an adaptation from the original proof of Gérard [1991] (see also [Gérard and Leichtnam 1993] in the semiclassical setting).

The fact that the Wigner distributions V^h are uniformly bounded in $(S^0)'$ follows from the Calderón–Vaillancourt theorem (see Appendix A), and from the boundedness of (v_h) in $L^2(\mathbb{T}^2)$.

The sharp Gårding inequality (see for instance [Sjöstrand 1995, Proposition 5.1] or [Lerner 2010, Section 2.5.2]) gives the existence of $C > 0$ such that, for all $a \geq 0$ and $h > 0$,

$$\big(\mathrm{Op}_h(a) v_h, v_h\big)_{L^2(\mathbb{T}^2)} \geq -Ch\|v_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the distribution μ is nonnegative (and hence is a measure). □

5C. ***Properties of μ for zeroth and first order quasimodes.*** To simplify the notation, we set

$$P_b^h = P_0^h + ihb(x), \quad \text{with } P_0^h = -h^2\Delta - 1 = \mathrm{Op}_h(|\xi|^2 - 1).$$

The geodesic flow on the torus $\phi_\tau : T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ for $\tau \in \mathbb{R}$ is the flow generated by the Hamiltonian vector field associated to the symbol $\frac{1}{2}(|\xi|^2 - 1)$, i.e., by the vector field $\xi \cdot \partial_x$ on $T^*\mathbb{T}^2$. Explicitly, we have

$$\phi_\tau(x, \xi) = (x + \tau\xi, \xi), \quad \tau \in \mathbb{R},\ (x, \xi) \in T^*\mathbb{T}^2.$$

Note that ϕ_τ preserves the ξ-component, and in particular every energy layer $\{|\xi|^2 = C > 0\} \subset T^*\mathbb{T}^2$.

Now, we describe the first properties of the measure μ implied by (5-2).

We recall that for $\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$, $(\phi_\tau)_*\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$ is defined by $\langle (\phi_\tau)_*\nu, a\rangle = \langle \nu, a \circ \phi_\tau\rangle$ for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$. In particular, $(\phi_\tau)_*\nu$ is a measure if ν is. We shall say that ν is an *invariant measure* if it is invariant by the geodesic flow, i.e., $(\phi_\tau)_*\nu = \nu$ for all $\tau \in \mathbb{R}$.

Proposition 5.2. *Let μ be as in Proposition 5.1 with v_h satisfying (5-2). We have*

(1) $\mathrm{supp}(\mu) \subset \{|\xi|^2 = 1\}$ (*hence is compact in $T^*\mathbb{T}^2$*),

(2) $\mu(T^*\mathbb{T}^2) = 1$,

(3) *μ is invariant by the geodesic flow, i.e., $(\phi_\tau)_*\mu = \mu$,*

(4) $\langle \mu, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2), \mathscr{C}^0(T^*\mathbb{T}^2)} = 0$, *where $\mathcal{M}_c(T^*\mathbb{T}^2)$ denotes the space of compactly supported measures on $T^*\mathbb{T}^2$.*

In other words, μ is an invariant probability measure on $T^*\mathbb{T}^2$ vanishing on $\{b > 0\}$.

These are standard arguments that we reproduce here for the reader's comfort. In particular, we recover all information required to prove the Bardos–Lebeau–Rauch–Taylor uniform stabilization theorem under GCC. The proof of this proposition only uses that μ is a measure associated to a $o(h)$-quasimode, and not the full information in (5-2) (which is the key point to prove Theorem 2.6).

Proof. First, we take $\chi \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ depending only on the ξ variable, such that $\chi \geq 0$, $\chi(\xi) = 0$ for $|\xi| \leq 2$, and $\chi(\xi) = 1$ for $|\xi| \geq 3$. Hence, $\chi(\xi)/(|\xi|^2 - 1) \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ and we have the exact composition formula

$$\mathrm{Op}_h(\chi) = \mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h,$$

since both operators are Fourier multipliers. Moreover, $\mathrm{Op}_h(\chi(\xi)/(|\xi|^2 - 1))$ is a bounded operator on $L^2(\mathbb{T}^2)$. As a consequence, we have

$$\langle V^h, \chi\rangle_{(S^0)', S^0} \to \langle \mu, \chi\rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})},$$

together with

$$\begin{aligned}
\langle V^h, \chi\rangle_{(S^0)', S^0} &= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\
&= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_b^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} - ih\left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) b v_h, v_h\right)_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

Since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(1)$ and $\|v_h\|_{L^2(\mathbb{T}^2)} = 1$, both terms in this expression vanish in the limit $h \to 0^+$. This implies $\langle \mu, \chi \rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}),\mathscr{C}^0(\overline{T^*\mathbb{T}^2})} = 0$. Since this holds for all χ as above, we have $\operatorname{supp}(\mu) \subset \{|\xi|^2 = 1\}$, which proves item (1).

In particular, this implies $\mu(\overline{T^*\mathbb{T}^2} \setminus T^*\mathbb{T}^2) = 0$. Now, item (2) is a direct consequence of item (1) and $1 = \|v_h\|^2_{L^2(\mathbb{T}^2)} \to \langle \mu, 1 \rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}),\mathscr{C}^0(\overline{T^*\mathbb{T}^2})}$. Item (4) is a direct consequence of $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$.

Finally, for $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, we recall that

$$[P_0^h, \operatorname{Op}_h(a)] = \frac{h}{i} \operatorname{Op}_h(\{|\xi|^2 - 1, a\}) = \frac{2h}{i} \operatorname{Op}_h(\xi \cdot \partial_x a)$$

is a consequence of the Weyl quantization (any other quantization would have left an error term of order $\mathbb{O}(h^2)$). Hence, (5-3) yields

$$\langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)} \to \langle \mu, \xi \cdot \partial_x a \rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)}, \tag{5-5}$$

together with

$$\begin{aligned}
\langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)} &= \frac{i}{2h} ([P_0^h, \operatorname{Op}_h(a)] v_h, v_h)_{L^2(\mathbb{T}^2)} \\
&= \frac{i}{2h} (\operatorname{Op}_h(a) v_h, P_0^h v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h} (\operatorname{Op}_h(a) P_0^h v_h, v_h)_{L^2(\mathbb{T}^2)} \\
&= \frac{i}{2h} (\operatorname{Op}_h(a) v_h, P_b^h v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h} (\operatorname{Op}_h(a) P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} \\
&\quad - \frac{1}{2} (\operatorname{Op}_h(a) v_h, b v_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2} (\operatorname{Op}_h(a) b v_h, v_h)_{L^2(\mathbb{T}^2)}.
\end{aligned} \tag{5-6}$$

In this expression, we have $(1/h)(\operatorname{Op}_h(a) v_h, P_b^h v_h)_{L^2(\mathbb{T}^2)} \to 0$ and $(1/h)(\operatorname{Op}_h(a) P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} \to 0$ since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h)$. Moreover, the last two terms can be estimated by

$$|(\operatorname{Op}_h(a) b v_h, v_h)_{L^2(\mathbb{T}^2)}| \le \|\sqrt{b} v_h\|_{L^2(\mathbb{T}^2)} \|\sqrt{b} \operatorname{Op}_h(a) v_h\|_{L^2(\mathbb{T}^2)} = o(1), \tag{5-7}$$

since $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$. This yields $\langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)} \to 0$, so that, using (5-5),

$$\langle \mu, \xi \cdot \partial_x a \rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = 0 \quad \text{for all } a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2).$$

Replacing a by $a \circ \phi_\tau$ and integrating with respect to the parameter τ gives $(\phi_\tau)_* \mu = \mu$, which concludes the proof of item (3). □

6. Geometry on the torus and decomposition of invariant measures

The results of Section 5 were valid on arbitrary manifolds. We now turn to specific properties of the geodesic flow on the torus (and related facts of Fourier analysis). In Lemma 6.1 we use the partition of the cotangent bundle into resonant and nonresonant vectors to decompose any invariant measure according to the long-time behavior of geodesics.

6A. ***Resonant and nonresonant vectors on the torus.*** In this section, we collect several facts concerning the geometry of $T^*\mathbb{T}^2$ and its resonant subspaces. Most of the setting and the notation comes from [Anantharaman and Macià 2010, Section 2].

We shall say that a submodule $\Lambda \subset \mathbb{Z}^2$ is primitive if $\langle\Lambda\rangle \cap \mathbb{Z}^2 = \Lambda$, where $\langle\Lambda\rangle$ denotes the linear subspace of $\mathbb{R}^2$ spanned by Λ. The family of all primitive submodules will be denoted by $\mathcal{P}$.

Let us denote by $\Omega_j \subset \mathbb{R}^2$, for $j = 0, 1, 2$, the following sets:

$$\Omega_j := \{\xi \in \mathbb{R}^2 \text{ such that } \operatorname{rk}(\Lambda_\xi) = 2 - j\}, \quad \text{with } \Lambda_\xi := \{k \in \mathbb{Z}^2 \text{ such that } \xi \cdot k = 0\} = \xi^\perp \cap \mathbb{Z}^2.$$

The set $\Omega_0 \cup \Omega_1$ is referred to as the set of resonant directions, whereas $\Omega_2 = \mathbb{R}^2 \setminus (\Omega_0 \cup \Omega_1)$ is referred to as the set of nonresonant vectors.

Note that the sets Ω_j form a partition of $\mathbb{R}^2$, and that we have

- $\Omega_0 = \{0\}$ (resonance of order 0);
- $\xi \in \Omega_1$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is periodic (resonances of order 1);
- $\xi \in \Omega_2$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is dense in $\mathbb{T}^2$.

On the Fourier analysis side, we will use the following facts. For $\Lambda \in \mathcal{P}$ let us define

$$\Lambda^\perp := \{\xi \in \mathbb{R}^2 \text{ such that } \xi \cdot k = 0 \text{ for all } k \in \Lambda\}.$$

For a function f on $\mathbb{T}^2$ with Fourier coefficients $(\hat{f}(k))_{k\in\mathbb{Z}^2}$, and $\Lambda \in \mathcal{P}$, we shall say that f has only Fourier modes in Λ if $\hat{f}(k) = 0$ for $k \notin \Lambda$. This means that f is constant in the direction $\Lambda^\perp$, or equivalently, that $\sigma \cdot \partial_x f = 0$ for all $\sigma \in \Lambda^\perp$. This is a trivial property if $\operatorname{rk}\Lambda = 2$, but means that f is constant if $\operatorname{rk}\Lambda = 0$ and that f is constant along the 1-dimensional tori

$$\mathbb{T}_{\Lambda^\perp} := \Lambda^\perp/(2\pi\mathbb{Z}^2 \cap \Lambda^\perp)$$

if $\operatorname{rk}\Lambda = 1$.

We shall use the following notation: $L^p_\Lambda(\mathbb{T}^2)$ will stand for the subspace of $L^p(\mathbb{T}^2)$ consisting of functions having only Fourier modes in Λ. For a function $f \in L^2(\mathbb{T}^2)$ (resp. a symbol $a \in S^0(T^*\mathbb{T}^2)$), we denote by $\langle f\rangle_\Lambda$ its orthogonal projection on $L^2_\Lambda(\mathbb{T}^2)$, i.e., the average of f along $\Lambda^\perp$:

$$\langle f\rangle_\Lambda(x) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{f}(k) \quad \left(\text{resp. } \langle a\rangle_\Lambda(x,\xi) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{a}(k,\xi)\right).$$

If $\operatorname{rk}(\Lambda) = 1$ and v is a vector in $\Lambda^\perp \setminus \{0\}$, we also have

$$\langle f\rangle_\Lambda(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x+tv)\,dt. \tag{6-1}$$

In particular, note that $\langle f\rangle_\Lambda$ (resp. $\langle a\rangle_\Lambda$) is nonnegative if f (resp. a) is, and that $\langle f\rangle_\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $\langle a\rangle_\Lambda \in S^0(T^*\mathbb{T}^2)$) if $f \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $a \in S^0(T^*\mathbb{T}^2)$).

Finally, given $f \in L^\infty_\Lambda(\mathbb{T}^2)$, we denote by m_f the bounded operator on $L^2_\Lambda(\mathbb{T}^2)$, consisting in the multiplication by f.

6B. ***Decomposition of invariant measures.*** We denote by $\mathcal{M}^+(T^*\mathbb{T}^2)$ the set of finite, nonnegative measures on $T^*\mathbb{T}^2$. With the definitions above, we have the following decomposition lemmata, proved in [Macià 2010] or [Anantharaman and Macià 2010, Section 2]. These properties are given for general measures $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Of course, they apply in particular to the measure μ defined by Proposition 5.1.

Lemma 6.1. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Then μ decomposes as a sum of nonnegative measures*

$$\mu = \mu|_{\mathbb{T}^2\times\{0\}} + \mu|_{\mathbb{T}^2\times\Omega_2} + \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}. \tag{6-2}$$

This decomposition simply comes from partitioning $\mathbb{R}^2$ into the disjoint, countable union of $\{0\}$, Ω_2 and the sets $\Lambda^\perp\setminus\{0\}$, which for $\mathrm{rk}(\Lambda)=1$ are punctured lines of rational slopes. For such Λ, note that $\xi\in\Lambda^\perp\setminus\{0\}$ implies $\Lambda_\xi=\Lambda$.

Given $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$, we define its Fourier coefficients by the complex measures on $\mathbb{R}^2$:

$$\hat{\mu}(k,\cdot) := \int_{\mathbb{T}^2} \frac{e^{-ik\cdot x}}{2\pi}\mu(dx,\cdot), \qquad k\in\mathbb{Z}.$$

One has, in the sense of distributions, the Fourier inversion formula

$$\mu(x,\xi) = \sum_{k\in\mathbb{Z}^2} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi).$$

Lemma 6.2. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ and $\Lambda\in\mathcal{P}$. Then the distribution*

$$\langle\mu\rangle_\Lambda(x,\xi) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi)$$

is in $\mathcal{M}^+(T^*\mathbb{T}^2)$ and satisfies, for all $a\in\mathscr{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\langle\langle\mu\rangle_\Lambda, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle\mu, \langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)}.$$

Lemma 6.3. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ be an* invariant *measure. Then, for all $\Lambda\in\mathcal{P}$, the measure $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$ is also a nonnegative invariant measure and*

$$\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})} = \langle\mu\rangle_\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}.$$

Let us now come back to the measure μ given by Proposition 5.1, which satisfies all properties listed in Proposition 5.2. In particular, this measure vanishes on the nonempty open subset of $\mathbb{T}^2$ given by $\{b>0\}$ (see item (4) in Proposition 5.2). As a consequence of Proposition 5.2 and of the three lemmata above, this yields the following lemma.

Lemma 6.4. *We have $\mu = \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$.*

As a consequence of Proposition 5.2, we have indeed that the measure μ is supported in $\{|\xi|=1\}$, which implies $\mu|_{\mathbb{T}^2\times\{0\}}=0$. In addition, Lemma 6.3 applied with $\Lambda=\{0\}$ implies that $\mu|_{\mathbb{T}^2\times\Omega_2}$ is constant in x, and thus vanishes everywhere since it vanishes on $\{b>0\}$.

Remark 6.5. Since the measure μ is supported in $\{|\xi| = 1\}$ (Proposition 5.2(1)), we have

$$\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$$

(which simplifies the notation).

As a consequence of these lemmata and the last remark, the study of the measure μ is now reduced to that of all nonnegative invariant measures $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ with $\operatorname{rk}(\Lambda) = 1$.

The aim of the next sections is to prove that the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ vanishes identically, for each periodic direction $\Lambda^\perp$.

6C. ***Adapted coordinates for resonant directions of order* 1.** For each $\Lambda \in \mathcal{P}$, we define

$$\begin{aligned}\Lambda^\perp &:= \{\xi \in \mathbb{R}^2 \text{ such that } \xi \cdot k = 0 \text{ for all } k \in \Lambda\},\\ \mathbb{T}_\Lambda &:= \langle\Lambda\rangle/2\pi\Lambda,\\ \mathbb{T}_{\Lambda^\perp} &:= \Lambda^\perp/(2\pi\mathbb{Z}^2 \cap \Lambda^\perp).\end{aligned}$$

Note that if $\operatorname{rk}(\Lambda) = 1$, $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda^\perp}$ are two submanifolds of $\mathbb{T}^2$ diffeomorphic to one-dimensional tori. Their cotangent bundles admit the global trivializations $T^*\mathbb{T}_\Lambda = \mathbb{T}_\Lambda \times \langle\Lambda\rangle$ and $T^*\mathbb{T}_{\Lambda^\perp} = \mathbb{T}_{\Lambda^\perp} \times \Lambda^\perp$.

To study the measure $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$ for $\Lambda \in \mathcal{P}$, $\operatorname{rk}(\Lambda) = 1$, we need to work in adapted coordinates.

We define the linear isomorphism

$$\chi_\Lambda : \Lambda^\perp \times \langle\Lambda\rangle \to \mathbb{R}^2$$

by $(s, y) \mapsto s + y$, and denote by $\tilde{\chi}_\Lambda : T^*\Lambda^\perp \times T^*\langle\Lambda\rangle \to T^*\mathbb{R}^2$ its extension to the cotangent bundle. This map can be defined as follows: for $(s, \sigma) \in T^*\Lambda^\perp = \Lambda^\perp \times (\Lambda^\perp)^*$ and $(y, \eta) \in T^*\langle\Lambda\rangle = \langle\Lambda\rangle \times \langle\Lambda\rangle^*$, we can extend σ to a covector of $\mathbb{R}^2$ vanishing on $\langle\Lambda\rangle$ and η to a covector of $\mathbb{R}^2$ vanishing on $\Lambda^\perp$. Remember that we identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ through the usual inner product; thus we can also see σ as an element of $\Lambda^\perp$ and η as an element of $\langle\Lambda\rangle$. Then we have

$$\tilde{\chi}_\Lambda(s, \sigma, y, \eta) = (s + y, \sigma + \eta) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*.$$

Conversely, any $\xi \in (\mathbb{R}^2)^*$ can be decomposed into $\xi = \sigma + \eta$, where $\sigma \in \Lambda^\perp$ and $\eta \in \langle\Lambda\rangle$. We denote by P_Λ the orthogonal projection of $\mathbb{R}^2$ onto $\langle\Lambda\rangle$, that is,

$$P_\Lambda \xi = \eta. \tag{6-3}$$

Next, the map χ_Λ goes to the quotient, giving a smooth Riemannian covering of $\mathbb{T}^2$:

$$\pi_\Lambda : \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda \to \mathbb{T}^2, \quad (s, y) \mapsto s + y.$$

We shall denote by $\tilde{\pi}_\Lambda$ its extension to cotangent bundles:

$$\tilde{\pi}_\Lambda : T^*\mathbb{T}_{\Lambda^\perp} \times T^*\mathbb{T}_\Lambda \to T^*\mathbb{T}^2.$$

As the map π_Λ is not an injection (because the torus $\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda$ contains several copies of $\mathbb{T}^2$), we introduce its degree p_Λ, which is also equal to $\operatorname{Vol}(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)/\operatorname{Vol}(\mathbb{T}^2)$.

Then, the map

$$T_\Lambda u := \frac{1}{\sqrt{p_\Lambda}} u \circ \chi_\Lambda$$

defines a linear isomorphism $L^2_{\mathrm{loc}}(\mathbb{R}^2) \to L^2_{\mathrm{loc}}(\Lambda^\perp \times \langle\Lambda\rangle)$. Note that because of the factor $1/\sqrt{p_\Lambda}$, T_Λ maps $L^2(\mathbb{T}^2)$ isometrically into a subspace of $L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$. Moreover, T_Λ maps $L^2_\Lambda(\mathbb{T}^2)$ into $L^2(\mathbb{T}_\Lambda) \subset L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$, since the nonvanishing Fourier modes of $u \in L^2_\Lambda(\mathbb{T}^2)$ correspond only to frequencies $k \in \Lambda$. This reads

$$T_\Lambda u(s, y) = \frac{1}{\sqrt{p_\Lambda}} u(y) \quad \text{for } (s, y) \in \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda. \tag{6-4}$$

Since $\tilde{\chi}_\Lambda$ is linear, we have

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h(a \circ \tilde{\chi}_\Lambda) T_\Lambda, \tag{6-5}$$

for any $a \in \mathscr{C}^\infty(T^*\mathbb{R}^2)$, where on the left Op_h is the Weyl quantization on $\mathbb{R}^2$ (A-1), and on the right Op_h is the Weyl quantization on $\Lambda^\perp \times \langle\Lambda\rangle$. Next, we denote by $\operatorname{Op}_h^{\Lambda^\perp}$ and $\operatorname{Op}_h^\Lambda$ the Weyl quantization operators defined on smooth test functions on $T^*\Lambda^\perp \times T^*\langle\Lambda\rangle$ and acting only on the variables in $T^*\Lambda^\perp$ and $T^*\langle\Lambda\rangle$, respectively, leaving the other frozen. For any $a \in \mathscr{C}_c^\infty(T^*\Lambda^\perp \times T^*\langle\Lambda\rangle)$, we have

$$\operatorname{Op}_h(a) = \operatorname{Op}_h^{\Lambda^\perp} \circ \operatorname{Op}_h^\Lambda(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a). \tag{6-6}$$

Now, if the symbol $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ has only Fourier modes in Λ, we remark, in view of (6-4), that $a \circ \tilde{\pi}_\Lambda$ does not depend on $s \in \mathbb{T}_{\Lambda^\perp}$. Therefore, we sometimes write $a \circ \tilde{\pi}_\Lambda(\sigma, y, \eta)$ for $a \circ \tilde{\pi}_\Lambda(s, \sigma, y, \eta)$, and (6-5) and (6-6) give

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a \circ \tilde{\pi}_\Lambda) T_\Lambda = \operatorname{Op}_h^\Lambda\big(a \circ \tilde{\pi}_\Lambda(hD_s, \cdot, \cdot)\big) T_\Lambda. \tag{6-7}$$

Note that for every $\sigma \in \Lambda^\perp$, the operator $\operatorname{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(\sigma, \cdot, \cdot))$ maps $L^2(\mathbb{T}_\Lambda)$ into itself. More precisely, it maps the subspace $T_\Lambda(L^2_\Lambda(\mathbb{T}^2))$ into itself.

7. Change of quasimode and construction of an invariant cutoff function

In this section, we first construct from the quasimode v_h another quasimode w_h that will be easier to handle when studying the measure $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$. Indeed, w_h is basically a microlocalization of v_h in the direction $\Lambda^\perp$ at a precise concentration rate.

Moreover, we introduce a cutoff function $\chi_h^\Lambda(x) = \chi_h^\Lambda(y, s)$, well adapted to the damping coefficient b and to the invariance of the measure $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$ in the direction $\Lambda^\perp$ (this cutoff function plays the role of the function $\chi(b/h)$ used in [Burq and Hitrik 2007] in the case where b is itself invariant in the direction $\Lambda^\perp$). Its construction is a key point in the proof of Theorem 2.6.

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighborhood of the origin. With P_Λ defined in (6-3), we first set

$$w_h := \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h,$$

which implicitly depends on $\alpha \in (0, 1)$. The following lemma implies that, for δ and α sufficiently small, w_h is also a $o(h^{2+\delta})$-quasimode for P_b^h.

Lemma 7.1. *For any $\alpha > 0$ such that*

$$2\alpha + \delta \le 1 \quad \text{and} \quad 3\alpha + 2\delta < 1, \tag{7-1}$$

we have

$$\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

As a consequence of this lemma, the semiclassical measures associated to w_h satisfy in particular the conclusions of Proposition 5.2. Moreover, the following proposition implies that the sequence w_h contains all the information in the direction $\Lambda^\perp$.

Proposition 7.2. *Suppose that $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta})$ and $0 < \alpha < (1+\delta)/2$. For any $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, we have*

$$\langle \mu|_{\mathbb{T}^2 \times \Lambda^\perp}, a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} = \lim_{h \to 0} (\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)}.$$

Note that under condition (7-1), both assumptions of Proposition 7.2 are satisfied since in particular $\alpha < \frac{1}{3}$.

Next, we state the desired properties of the cutoff function χ_h^Λ. The proof of its existence is a crucial point in the proof of Theorem 2.6.

Proposition 7.3. *For $\delta = 4\varepsilon$ and $\varepsilon < \frac{1}{29}$, there exists α satisfying (7-1), such that for any constant $c_0 > 0$, there exists a cutoff function $\chi_h^\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ valued in $[0, 1]$, such that*

(1) $\chi_h^\Lambda = \chi_h^\Lambda(y)$ *does not depend on the variable s (i.e., χ_h^Λ is $\Lambda^\perp$-invariant),*

(2) $\|(1 - \chi_h^\Lambda) w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,

(3) $b \le c_0 h$ *on* $\mathrm{supp}(\chi_h^\Lambda)$,

(4) $\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,

(5) $\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$.

Note that the function χ_h^Λ implicitly depends on the constant c_0, which will be taken arbitrarily small in Section 9.

In the particular case where the damping function b is invariant in one direction, this proposition is not needed. In this case, one can take as in [Burq and Hitrik 2007] $\chi_h^\Lambda = \chi(b/(c_0 h))$. In the d-dimensional torus, this cutoff function works as well if b is invariant in $d-1$ directions, and an analogue of Theorem 2.6 can be stated in this setting. Unfortunately, our construction of the function χ_h^Λ (see the proof of Proposition 7.3 in Section 12) strongly relies on the fact that all trapped directions are periodic, and fails in higher dimensions.

We give here a proof of Lemma 7.1. Because of their technicality, we postpone the proofs of Propositions 7.2 and 7.3 to Sections 11 and 12, respectively.

Proof of Lemma 7.1. First, we develop

$$P_b^h w_h = P_b^h \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h = \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) P_b^h v_h + ih\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] v_h, \tag{7-2}$$

since P_0^h and $\operatorname{Op}_h(\chi(|P_\Lambda\xi|/h^\alpha))$ are both Fourier multipliers. We know that

$$\Big\|\operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) P_b^h v_h\Big\|_{L^2(\mathbb{T}^2)} \le \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

It only remains to study the operator

$$\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] = ih^{1-\alpha} \operatorname{Op}_h\Big(\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) + \mathcal{O}_{\mathcal{L}(L^2)}(h^{2(1-\alpha)}) \tag{7-3}$$

according to the symbolic calculus.

Moreover, using the pointwise inequality[1] $|\nabla b(x)|^2 \le 2|b|_{W^{2,\infty}} b(x)$ (holding for any nonnegative $W^{2,\infty}$ function b), we have, for some $C > 0$,

$$Cb - \Big|\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big|^2 \ge 0 \quad \text{on } \mathbb{T}^2 \times \mathbb{R}^2.$$

The sharp Gårding inequality applied to this nonnegative symbol then yields

$$\Big(\operatorname{Op}_h\Big(Cb - \Big|\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big|^2\Big) v_h, v_h\Big)_{L^2(\mathbb{T}^2)} \ge -Ch^{1-\alpha},$$

and hence

$$\Big\|\operatorname{Op}_h\Big(\partial_y b\ \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h\Big\|^2_{L^2(\mathbb{T}^2)} \le C(bv_h, v_h)_{L^2(\mathbb{T}^2)} + \mathcal{O}(h^{1-\alpha}).$$

Combining this estimate together with (7-3) gives

$$\Big\|ih\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] v_h\Big\|_{L^2(\mathbb{T}^2)} = o(h^{2-\alpha+\frac{1+\delta}{2}}) + \mathcal{O}(h^{\frac{5-3\alpha}{2}}).$$

Coming back to the expression of $P_b^h w_h$ given in (7-2), this concludes the proof of Lemma 7.1. □

8. Second microlocalization of μ on a resonant affine subspace by ν^Λ and ρ_Λ

We want to analyze precisely the structure of the restriction $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$, using the full information contained in $o(h^{2+\delta})$-quasimodes like v_h and w_h.

From now on, we want to take advantage of the family w_h of $o(h^{2+\delta})$-quasimodes constructed in Section 7, which are microlocalized in the direction $\Lambda^\perp$. Hence, we define the Wigner distribution $W^h \in \mathcal{D}'(T^*\mathbb{T}^2)$ associated to the functions w_h and the scale h, by

$$\langle W^h, a\rangle_{(S^0)', S^0} = (\operatorname{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} \quad \text{for all } a \in S^0(T^*\mathbb{T}^2).$$

[1] To prove this inequality, we denote by Hf the Hessian of f, take $v \in \mathbb{R}^2$ and write the Taylor formula

$$b(x+tv) = b(x) + tv\cdot\nabla b(x) + \int_0^t (t-s) v\cdot Hf(x+sv) v\, ds.$$

Taking $t > 0$ and using that $b(x+tv) \ge 0$, we obtain $-v\cdot\nabla b(x) \le \frac{1}{t} b(x) + \frac{t|v|^2}{2}\|Hf\|_{L^\infty}$ for all $(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2$ and $t > 0$. The conclusion follows when optimizing in t.

According to Proposition 7.2, we recover

$$\langle W^h, a\rangle_{(S^0)', S^0} \to \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}^0_c(T^*\mathbb{T}^2)}$$

in the limit $h \to 0$, for any $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$ (and α satisfying (7-1)).

To provide a precise study of $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$, we shall introduce as in [Macià 2010; Anantharaman and Macià 2010] two-microlocal semiclassical measures, describing at a finer level the concentration of the sequence v_h on the resonant subspace

$$\Lambda^\perp = \{\xi \in \mathbb{R}^2 \text{ such that } P_\Lambda \xi = 0\},$$

where P_Λ is defined in (6-3). These objects were introduced in the local Euclidean case in [Nier 1996; Fermanian-Kammerer 2000a; 2000b]. A specific concentration scale may also be chosen in the in the two-microlocal variable, giving rise to the two-scales semiclassical measures studied in [Miller 1996; 1997; Fermanian-Kammerer and Gérard 2002].

We first have to describe the adapted symbol class (inspired by [Fermanian-Kammerer 2000a] and used in [Anantharaman and Macià 2010]). According to Lemma 6.3 (see also Remark 6.5), it suffices to test the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ with functions constant in the direction $\Lambda^\perp$ (or equivalently, having only x-Fourier modes in Λ, in the sense of the following definition).

Definition 8.1. Given $\Lambda \in \mathcal{P}$, we shall say that $a \in S^1_\Lambda$ if $a = a(x, \xi, \eta) \in \mathscr{C}^\infty(T^*\mathbb{T}^2 \times \langle\Lambda\rangle)$ and

(1) there exists a compact set $K_a \subset T^*\mathbb{T}^2$ such that, for all $\eta \in \langle\Lambda\rangle$, the function $(x, \xi) \mapsto a(x, \xi, \eta)$ is compactly supported in K_a;

(2) a is homogeneous of order zero at infinity in the variable $\eta \in \langle\Lambda\rangle$; i.e., if we denote by $\mathbb{S}_\Lambda := \mathbb{S}^1 \cap \langle\Lambda\rangle$ the unit sphere in $\langle\Lambda\rangle$, there exists $R_0 > 0$ (depending on a) and $a_{\text{hom}} \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that

$$a(x, \xi, \eta) = a_{\text{hom}}\Big(x, \xi, \frac{\eta}{|\eta|}\Big) \quad \text{for } |\eta| \geq R_0 \text{ and } (x, \xi) \in T^*\mathbb{T}^2;$$

for $\eta \neq 0$, we will also use the notation $a(x, \xi, \infty\eta) := a_{\text{hom}}(x, \xi, \eta/|\eta|)$.

(3) a has only x-Fourier modes in Λ, that is,

$$a(x, \xi, \eta) = \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi} \hat{a}(k, \xi, \eta).$$

This last assumption is equivalent to saying that $\sigma \cdot \partial_x a = 0$ for any $\sigma \in \Lambda^\perp$. We denote by $S^{1\,\prime}_\Lambda$ the topological dual space of S^1_Λ.

Let $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighborhood of the origin. Let $R > 0$. The previous remark allows us to define, for $a \in S^1_\Lambda$ the two following elements of $S^{1\,\prime}_\Lambda$:

$$\langle W^{h,\Lambda}_R, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} := \Big\langle W^h, \Big(1 - \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big) a\Big(x, \xi, \frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)}, \tag{8-1}$$

$$\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} := \Big\langle W^h, \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big) a\Big(x, \xi, \frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)}. \tag{8-2}$$

In particular, for any $R > 0$ and $a \in S^1_\Lambda$, we have

$$\Big\langle W^h, a\Big(x, \xi, \frac{P_\Lambda \xi}{h}\Big)\Big\rangle_{\mathcal{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)} = \langle W^{h,\Lambda}_R, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} + \langle W^h_{R,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda}. \tag{8-3}$$

The next two propositions are the analogues of [Fermanian-Kammerer 2000a] in our context. They state the existence of two-microlocal semiclassical measures, as the limit objects of $W^{h,\Lambda}_R$ and $W^h_{R,\Lambda}$.

Proposition 8.2. *There exists a subsequence (h, w_h) and a nonnegative measure $\nu^\Lambda \in \mathcal{M}^+(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that, for all $a \in S^1_\Lambda$, we have*

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W^{h,\Lambda}_R, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} = \Big\langle \nu^\Lambda, a_{\mathrm{hom}}\Big(x, \xi, \frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

To define the limit of the distributions $W^h_{R,\Lambda}$, we need first to introduce operator spaces and operator-valued measures, following [Gérard 1991]. Given a Hilbert space H (in the following, we shall use $H = L^2(\mathbb{T}_\Lambda)$), we denote respectively by $\mathcal{K}(H)$, $\mathcal{L}^1(H)$ the spaces of compact and trace class operators on H. We recall that they are both two-sided ideals of the ring $\mathcal{L}(H)$ of bounded operators on H. We refer for instance to [Reed and Simon 1980, Chapter VI.6] for a description of the space $\mathcal{L}^1(H)$ and its basic properties. Given a Polish space T (in the following, we shall use $T = T^*\mathbb{T}_{\Lambda^\perp}$), we denote by $\mathcal{M}^+(T; \mathcal{L}^1(H))$ the space of nonnegative measures on T, taking values in $\mathcal{L}^1(H)$. More precisely, we have $\rho \in \mathcal{M}^+(T; \mathcal{L}^1(H))$ if ρ is a bounded linear form on $\mathscr{C}^0_c(T)$ such that, for every nonnegative function $a \in \mathscr{C}^0_c(T)$, $\langle \rho, a\rangle_{\mathcal{M}(T),\mathscr{C}^0_c(T)} \in \mathcal{L}^1(H)$ is a nonnegative hermitian operator. As a consequence of [Reed and Simon 1980, Theorem VI.26], these measures can be identified in a natural way to nonnegative linear functionals on $\mathscr{C}^0_c(T; \mathcal{K}(H))$.

Proposition 8.3. *There exists a subsequence (h, w_h) and a nonnegative measure*

$$\rho_\Lambda \in \mathcal{M}^+\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1(L^2(\mathbb{T}_\Lambda))\big),$$

such that, for all $K \in \mathscr{C}^\infty_c\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{K}(L^2(\mathbb{T}_\Lambda))\big)$, we have

$$\lim_{h\to 0}(K(s, hD_s)T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = \operatorname{tr}\Big\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} K(s,\sigma)\rho_\Lambda(ds, d\sigma)\Big\}. \tag{8-4}$$

Moreover (for the same subsequence), for all $a \in S^1_\Lambda$, we have

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} = \operatorname{tr}\Big\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Op}^\Lambda_1\big(a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta)\big)\rho_\Lambda(ds, d\sigma)\Big\}. \tag{8-5}$$

In the left-hand side of (8-4), the inner product actually means

$$(K(s, hD_s)T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp} L^2(\mathbb{T}_\Lambda))} = \int_{s\in\mathbb{T}_{\Lambda^\perp},\, s'\in\Lambda^\perp,\, \sigma\in\Lambda^\perp} e^{\frac{i}{h}(s-s')\cdot\sigma}\Big(K\Big(\frac{s+s'}{2}, \sigma\Big)T_\Lambda w_h(s', y), T_\Lambda w_h(s, y)\Big)_{L^2_y(\mathbb{T}_\Lambda)} ds\, ds'\, d\sigma.$$

In the expression (8-5), remark that for each $\sigma \in \Lambda^\perp$, the operator $\mathrm{Op}^\Lambda_1\big(a(\tilde{\pi}_\Lambda(\sigma, y, 0), \eta)\big)$ is in $\mathcal{L}(L^2(\mathbb{T}_\Lambda))$. Hence, its product with the operator $\rho_\Lambda(ds, d\sigma)$ defines a trace-class operator.

Before proving Propositions 8.2 and 8.3, we explain how to reconstruct the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ from the two-microlocal measures ν^Λ and ρ_Λ. This reduces the study of the measure μ to that of all two-microlocal measures ν^Λ and ρ_Λ, for $\Lambda \in \mathcal{P}$.

We denote by $\mathcal{M}_c^+(T)$ the set of compactly supported measures on T, and by $\langle\cdot,\cdot\rangle_{\mathcal{M}_c(T),\mathscr{C}^0(T)}$ the associated duality bracket.

Proposition 8.4. *For all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ having only x-Fourier modes in Λ (i.e., for all $a \in S^1_\Lambda$ independent of the third variable $\eta \in \langle\Lambda\rangle$), we have*

$$\langle\mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle\nu^\Lambda, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\}, \tag{8-6}$$

and

$$\begin{aligned}\langle\mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} \\ = \langle\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\},\end{aligned} \tag{8-7}$$

where for $\sigma \in \Lambda^\perp$, $m_{a\circ\tilde{\pi}_\Lambda}(\sigma)$ denotes the multiplication in $L^2(\mathbb{T}_\Lambda)$ by the function $y \mapsto a\circ\tilde{\pi}_\Lambda(\sigma, y)$.

Moreover, we have $\nu^\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ and $\rho_\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(L^2(\mathbb{T}_\Lambda)))$ (i.e., both measures are compactly supported).

Formula (8-7) follows immediately from (8-6) by restriction. By the definition of the measure ρ_Λ, we see that it is already supported on $\mathbb{T}^2\times\Lambda^\perp$ (see expression (8-2)).

The end of this section is devoted to the proofs of the three propositions, inspired by [Fermanian-Kammerer 2000a; Anantharaman and Macià 2010].

Proof of Proposition 8.2. The Calderón–Vaillancourt theorem implies that the operators

$$\operatorname{Op}_h\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right) = \operatorname{Op}_1\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{R}\right)\right)a\left(x,h\xi,P_\Lambda\xi\right)\right)$$

are uniformly bounded as $h\to 0$ and $R\to+\infty$. It follows that the family $W_R^{h,\Lambda}$ is bounded in $S^{1\,\prime}_\Lambda$, and thus there exists a subsequence (h, w_h) and a distribution $\tilde{\mu}^\Lambda$ such that

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W_R^{h,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda,S^1_\Lambda} = \langle\tilde{\mu}^\Lambda, a(x,\xi,\eta)\rangle_{S^{1\,\prime}_\Lambda,S^1_\Lambda}.$$

Because of the support properties of the function χ, we notice that $\langle\tilde{\mu}^\Lambda, a\rangle_{S^{1\,\prime}_\Lambda,S^1_\Lambda} = 0$ as soon as the support of a is compact in the variable η. Hence, there exists a distribution $\nu^\Lambda \in \mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ such that

$$\langle\tilde{\mu}^\Lambda, a(x,\xi,\eta)\rangle_{S^{1\,\prime}_\Lambda,S^1_\Lambda} = \left\langle\nu^\Lambda, a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Next, suppose that $a > 0$ (and that $\sqrt{1-\chi}$ is smooth). Then, using [Anantharaman and Macià 2010, Corollary 35], and setting

$$b^R(x,\xi) = \left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right)^{\frac{1}{2}},$$

there exists $C > 0$ such that for all $h \leq h_0$ and $R \geq 1$, we have

$$\left\| \mathrm{Op}_h\Big(\Big(1-\chi\Big(\frac{|P_\Lambda \xi|}{Rh}\Big)\Big)a\Big(x,\xi,\frac{P_\Lambda \xi}{h}\Big)\Big) - \mathrm{Op}_h(b^R)^2 \right\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq \frac{C}{R}.$$

As a consequence, we have,

$$\langle W_R^{h,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} \geq \| \mathrm{Op}_h(b^R) w_h\|^2_{L^2(\mathbb{T}^2)} - \frac{C}{R}\|w_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the limit $\langle \nu^\Lambda, a_{\mathrm{hom}}(x,\xi,\frac{\eta}{|\eta|})\rangle_{\mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}$ is nonnegative. The distribution ν^Λ is nonnegative, and is hence a measure. This concludes the proof of Proposition 8.2. □

Proof of Proposition 8.3. First, the proof of the existence of a subsequence (h, w_h) and the measure ρ_Λ satisfying (8-4) is the analogue of Proposition 5.1 in the context of operator valued measures, viewing the sequence w_h as a bounded sequence of $L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$. It follows the lines of this result, after the adaptation of the symbolic calculus to operator-valued symbols (or more precisely, of [Gérard 1991] in the semiclassical setting).

Second, using the definition (8-2) together with (6-7), we have

$$\begin{aligned}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} &= \Big(\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)}\\ &= \Big(\mathrm{Op}_h^{\Lambda^\perp}\circ \mathrm{Op}_h^{\Lambda}\Big(\chi\Big(\frac{|\eta|}{Rh}\Big)a\Big(\tilde{\pi}_\Lambda(\sigma,y,\eta),\frac{\eta}{h}\Big)\Big)T_\Lambda w_h, T_\Lambda w_h\Big)_{L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)}.\end{aligned}$$

Hence, setting

$$a^h_{R,\Lambda}(\sigma,y,\eta) = \chi\Big(\frac{|\eta|}{R}\Big)a(\tilde{\pi}_\Lambda(\sigma,y,h\eta),\eta),$$

we obtain

$$\langle W^h_{R,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} = \big(\mathrm{Op}_h^{\Lambda^\perp}\circ \mathrm{Op}_1^{\Lambda}(a^h_{R,\Lambda}(\sigma,y,\eta))T_\Lambda w_h, T_\Lambda w_h\big)_{L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)}.$$

We also notice that $\mathrm{Op}_1^\Lambda(a^h_{R,\Lambda}) \in \mathscr{K}(L^2(\mathbb{T}_\Lambda))$, for any $\sigma\in\Lambda^\perp$, since $a^h_{R,\Lambda}$ has compact support with respect to η. Moreover, for any $R > 0$ fixed and $a \in S^1_\Lambda$, the Calderón–Vaillancourt theorem yields

$$\mathrm{Op}_1^\Lambda(a^h_{R,\Lambda}) = \mathrm{Op}_1^\Lambda(a^0_{R,\Lambda}) + hB,$$

for some $B \in \mathscr{L}(L^2(\mathbb{T}_\Lambda))$, uniformly bounded with respect to h. Using (8-4), this implies that for any $R > 0$ fixed and $a \in S^1_\Lambda$, we have

$$\lim_{h\to 0}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\prime}_\Lambda, S^1_\Lambda} = \mathrm{tr}\Big\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Op}_1^\Lambda(a^0_{R,\Lambda})\rho_\Lambda(ds,d\sigma)\Big\}.$$

Moreover, we have

$$\lim_{R\to+\infty}\mathrm{Op}_1^\Lambda(a^0_{R,\Lambda}) = \mathrm{Op}_1^\Lambda(a^0_{\infty,\Lambda}) = \mathrm{Op}_1^\Lambda\big(a(\tilde{\pi}_\Lambda(\sigma,y,0),\eta)\big),$$

in the strong topology of $\mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp};\mathscr{L}(L^2(\mathbb{T}_\Lambda))\big)$. This proves (8-5) and concludes the proof of Proposition 8.3. □

Proof of Proposition 8.4. Taking $a \in S^1_\Lambda$ independent of the third variable $\eta \in \langle \Lambda \rangle$ gives

$$\langle W^h, a(x,\xi)\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)} \to \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathscr{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)},$$

together with

$$\langle W^{h,\Lambda}_R, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} \to \langle \nu^\Lambda, a\rangle_{\mathscr{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)},$$

(according to Proposition 8.2) and

$$\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} \to \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Op}^\Lambda_1\big(a(\tilde{\pi}_\Lambda(\sigma,y,0))\big)\rho_\Lambda(ds,d\sigma)\right\} = \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\}$$

(according to Proposition 8.3). Now, using the last three equations together with (8-3) directly gives (8-7).

As both terms in the right hand-side of (8-7) are nonnegative measures and the left-hand side is a compactly supported nonnegative measure, this implies that ν^Λ and ρ_Λ are both compactly supported. □

9. Propagation laws for the two-microlocal measures ν^Λ and ρ_Λ

In this section, we study the propagation properties of ν^Λ and ρ_Λ defined in Propositions 8.2 and 8.3, respectively. The key point here is the use of the cutoff function introduced in Proposition 7.3.

We will use repeatedly this fact, which follows from item (2) in Proposition 7.3: if A is a bounded operator on $L^2(\mathbb{T}^2)$, we have

$$(Aw_h, w_h)_{L^2(\mathbb{T}^2)} = (A\chi^\Lambda_h w_h, \chi^\Lambda_h w_h)_{L^2(\mathbb{T}^2)} + \|A\|_{\mathscr{L}(L^2)}\, o(1). \tag{9-1}$$

To simplify the notation, we shall write $A_{c_0,h}$ for $\chi^\Lambda_h A \chi^\Lambda_h$.

9A. *Propagation of ν^Λ*. We define for $(x,\xi,\eta) \in T^*\mathbb{T}^2 \times \langle\Lambda\rangle$ and $\tau \in \mathbb{R}$ the flows

$$\phi^0_\tau(x,\xi,\eta) := (x+\tau\xi, \xi, \eta),$$

generated by the vector field $\xi\cdot\partial_x$ and, for $\eta \neq 0$,

$$\phi^1_\tau(x,\xi,\eta) := \left(x+\tau\frac{\eta}{|\eta|}, \xi, \eta\right)$$

generated by the vector field $(\eta/|\eta|)\cdot\partial_x$. With these definitions, we have the following propagation laws for the two-microlocal measure ν^Λ.

Proposition 9.1. *The measure ν^Λ is ϕ^0_τ- and ϕ^1_τ-invariant, that is,*

$$(\phi^0_\tau)_*\nu^\Lambda = \nu^\Lambda \quad \textit{and} \quad (\phi^1_\tau)_*\nu^\Lambda = \nu^\Lambda \quad \textit{for every } \tau \in \mathbb{R}.$$

The key result here is the additional "transverse propagation law" given by the flow ϕ^1_τ. The measure ν^Λ not only propagates along the geodesic flow ϕ^0_τ, but also along directions transverse to $\Lambda^\perp$.

Proof. Fix $a \in S^1_\Lambda$. The computation done in (5-6) is still valid replacing a by

$$\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right),$$

since it only uses the fact that $\mathrm{Op}_h((1-\chi(|P_\Lambda\xi|/Rh))a(x,\xi,P_\Lambda\xi/h))$ is bounded and that $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)}=o(h)$ and $(bw_h,w_h)_{L^2(\mathbb{T}^2)}=o(1)$. This yields

$$\lim_{h\to 0}\langle W_R^{h,\Lambda},\xi\cdot\partial_x a\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}=\lim_{h\to 0}\Big\langle W^h,\xi\cdot\partial_x\Big\{\Big(1-\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\}\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)}=0,$$

and hence, in the limit $R\to+\infty$, we obtain

$$\Big\langle \nu^\Lambda,\xi\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0.$$

Replacing a_{hom} by $a_{\mathrm{hom}}\circ\phi_\tau^0$ and integrating with respect to the parameter τ gives $(\phi_\tau^0)_*\nu^\Lambda=\nu^\Lambda$, which concludes the first part of the proof.

Second, to prove the ϕ_τ^1-invariance of ν^Λ we compute

$$\Big\langle \nu^\Lambda,\frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=\lim_{R\to\infty}\lim_{h\to 0}\Big\langle W_R^{h,\Lambda},\frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}. \tag{9-2}$$

Setting

$$a^R(x,\xi,\eta)=\frac{1}{|\eta|}\Big(1-\chi\Big(\frac{|\eta|}{R}\Big)\Big)a(x,\xi,\eta)$$

and

$$A^R:=\mathrm{Op}_h\Big(a^R\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big) \tag{9-3}$$

we have the relation

$$\Big\langle W_R^{h,\Lambda},\frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}=-\frac{i}{2}([\Delta_\Lambda,A^R]w_h,w_h)_{L^2(\mathbb{T}^2)},$$

where $\Delta_\Lambda=\partial_y^2$ is the laplacian in the direction Λ.

Lemma 9.2. *For any given $c_0>0$ and $R>0$, we have*

$$([\Delta_\Lambda,A^R]w_h,w_h)_{L^2(\mathbb{T}^2)}=([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}+o(1).$$

We postpone the proof of Lemma 9.2 and first indicate how it allows us to prove Proposition 9.1. We now know that

$$\Big\langle \nu^\Lambda,\frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=\lim_{R\to\infty}\lim_{h\to 0}-\frac{i}{2}([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}.$$

Recall that $a\in S_\Lambda^1$ implies that a has only x-Fourier modes in Λ, i.e., $P_\Lambda\xi\cdot\partial_x a=\xi\cdot\partial_x a$. We have also assumed in this section that b has only x-Fourier modes in Λ. As a consequence, we have

$$\begin{aligned}-\frac{i}{2}([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}&=-\frac{i}{2}([\Delta,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}\\&=\frac{i}{2h^2}([P_0^h,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{9-4}$$

Developing the last expression of (9-4), we obtain

$$\frac{i}{2h^2}([P_0^h, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)} = \frac{i}{2h^2}(A^R_{c_0,h}w_h, P_b^h w_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}(A^R_{c_0,h}P_b^h w_h, w_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^R_{c_0,h}w_h, bw_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^R_{c_0,h}bw_h, w_h)_{L^2(\mathbb{T}^2)}. \quad (9\text{-}5)$$

Since $A^R_{c_0,h}$ is bounded in $\mathscr{L}(L^2(\mathbb{T}^2))$, its adjoint $A^R_{c_0,h}$ is also bounded so that the first two terms in the last expression vanish in the limit $h \to 0$, using $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^2)$. To estimate the last two terms, we use again the boundedness of A^R and $(A^R)^*$ and write

$$|(A^R_{c_0,h}w_h, bw_h)_{L^2(\mathbb{T}^2)}| \le \|A^R\| \|\chi_h^\Lambda bw_h\|_{L^2(\mathbb{T}^2)} \le 2c_0 h\ \|A^R\|,$$

according to item (3) in Proposition 7.3. It follows that

$$\limsup_{h\to 0}\left|\frac{1}{2h}(A^R_{c_0,h}w_h, bw_h)_{L^2(\mathbb{T}^2)} + \frac{1}{2h}(A^R_{c_0,h}bw_h, w_h)_{L^2(\mathbb{T}^2)}\right| \le 2c_0 \sup \|A^R\|.$$

Coming back to the expression (9-2), we obtain

$$\left|\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot \partial_x a_{\text{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\right| \le 2c_0 \sup \|A^R\|$$

and since c_0 was arbitrary,

$$\left\langle \nu^\Lambda, \frac{\eta}{|\eta|}\cdot \partial_x a_{\text{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0.$$

Replacing a_{hom} by $a_{\text{hom}} \circ \phi_\tau^1$ and integrating with respect to the parameter τ gives $(\phi_\tau^1)_*\nu^\Lambda = \nu^\Lambda$, which concludes the proof of Proposition 9.1. □

Proof of Lemma 9.2. We are going to show that

$$([\Delta_\Lambda, A^R_{c_0,h}]w_h, w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, A^R]_{c_0,h}w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1). \quad (9\text{-}6)$$

Then, using the fact that $[\Delta_\Lambda, A^R]$ is a bounded operator (its symbol is $\left(1-\chi\left(\frac{|\eta|}{R}\right)\right)\frac{\eta}{|\eta|}\cdot\partial_x a(x,\xi,\eta)$) together with (9-1), this is also $([\Delta_\Lambda, A^R]w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1)$.

To prove (9-6), we develop the difference $[\Delta_\Lambda, A^R_{c_0,h}] - [\Delta_\Lambda, A^R]_{c_0,h}$ as

$$[\Delta_\Lambda, A^R_{c_0,h}] - [\Delta_\Lambda, A^R]_{c_0,h} = [\partial_y^2, \chi_h^\Lambda]\, A^R\chi_h^\Lambda + \chi_h^\Lambda A^R[\partial_y^2, \chi_h^\Lambda]. \quad (9\text{-}7)$$

Then, writing

$$[\partial_y^2, \chi_h^\Lambda] = \partial_y^2\chi_h^\Lambda + 2\partial_y\chi_h^\Lambda\partial_y,$$

we have

$$([\partial_y^2, \chi_h^\Lambda]\, A^R\chi_h^\Lambda w_h, w_h)_{L^2(\mathbb{T}^2)} = (A^R\chi_h^\Lambda\, w_h, \partial_y^2\chi_h^\Lambda\, w_h)_{L^2(\mathbb{T}^2)} + (\partial_y \circ A^R\ \chi_h^\Lambda\ w_h, 2\partial_y\chi_h^\Lambda\ w_h)_{L^2(\mathbb{T}^2)}.$$

Recalling that the operator $\partial_y \circ A^R$ is bounded, and using items (4) and (5) in Proposition 7.3, we obtain

$$\left|([\partial_y^2, \chi_h^\Lambda]\, A^R\chi_h^\Lambda w_h, w_h)_{L^2(\mathbb{T}^2)}\right| \le C\|\partial_y^2\chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} + C\|\partial_y\chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1).$$

The last term in (9-7) is handled similarly. This finally implies (9-6), concluding the proof. □

9B. ***Propagation of*** ρ_Λ**.** We denote by $(\omega_\Lambda^j, e_\Lambda^j)_{j\in\mathbb{N}}$ the eigenvalues and associated eigenfunctions of the operator $-\Delta_\Lambda = -\partial_y^2$ forming a Hilbert basis of $L^2(\mathbb{T}_\Lambda)$. We shall use the projector onto low frequencies of $-\Delta_\Lambda$, that is, for any $\omega \in \mathbb{R}_+$, the operator

$$\Pi_\Lambda^\omega := \sum_{\omega_\Lambda^j \leq \omega} (\,\cdot\,, e_\Lambda^j)_{L^2(\mathbb{T}_\Lambda)} e_\Lambda^j,$$

which has finite rank.

We have the following propagation laws for the two-microlocal measure ρ_Λ.

Proposition 9.3. (1) *For any* $K \in \mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ *independent of* s *(i.e.,* $K(s,\sigma) = K(\sigma)$*) and any* $\omega > 0$, *we have*

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\, \rho_\Lambda(ds, d\sigma)\right\} = 0.$$

(2) *Defining*

$$M_\Lambda := \int_{\mathbb{T}_{\Lambda^\perp}\times\Lambda^\perp} \rho_\Lambda(ds, d\sigma) \in \mathscr{L}^1(L^2(\mathbb{T}_\Lambda)),$$

we have

$$[\Delta_\Lambda, M_\Lambda] = 0.$$

Remark that for any $\sigma \in \Lambda^\perp$, the operator

$$[\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega] = \Pi_\Lambda^\omega[\Delta_\Lambda, K(\sigma)]\,\Pi_\Lambda^\omega$$

has finite rank, so the right-hand side of item (1) is well defined. Note that the definition of M_Λ is meaningful since ρ_Λ has a compact support according to Proposition 8.4.

The commutation relations of items (1) and (2) in this proposition correspond to propagation laws at the operator level. They are formulated here in a "derivated form", which, for item (2) for instance, is equivalent to

$$e^{i\tau\Delta_\Lambda} M_\Lambda e^{-i\tau\Delta_\Lambda} = M_\Lambda \quad \text{for all } \tau \in \mathbb{R},$$

in the "integrated form".

Proof of Proposition 9.3. For $K \in \mathscr{C}_c^\infty\big(\Lambda^\perp; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ (in other words $K \in \mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ independent of $s \in \mathbb{T}_{\Lambda^\perp}$), we denote

$$K^\omega(\sigma) := \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega$$

and we note that K^ω is also in $\mathscr{C}_c^\infty\big(\Lambda^\perp; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$. Hence, we have

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\, \rho_\Lambda(ds, d\sigma)\right\} = -\lim_{h\to 0}([-\Delta_\Lambda, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}.$$

To show that this limit vanishes, we proceed as in (9-4), (9-5) and in the subsequent calculation, replacing the operator A^R by $K^\omega(hD_s)$.

With the notation $\Delta_\Lambda = \partial_y^2$ and $\Delta_{\Lambda^\perp} = \partial_s^2$, we first note that

$$([-\Delta_\Lambda, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = ([-\Delta, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))},$$

since $\Delta = \Delta_\Lambda + \Delta_{\Lambda^\perp}$ and since $[\Delta_{\Lambda^\perp}, K^\omega(hD_s)] = 0$. As a matter of fact, $K^\omega(hD_s) = \mathrm{Op}_h^\Lambda(K^\omega(\sigma))$ and $\Delta_{\Lambda^\perp} = -h^{-2}\,\mathrm{Op}_h^\Lambda(|\sigma|^2)$ are both Fourier multipliers.

The following lemma is proved the same way as Lemma 9.2.

Lemma 9.4. *For any given $c_0 > 0$, we have*

$$([\Delta_\Lambda, K^\omega(\sigma)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} + o(1).$$

Here $K^\omega_{c_0,h}(hD_s)$ means $\chi_h^\Lambda K^\omega(hD_s)\chi_h^\Lambda$.

Writing

$$-h^2\Delta = T_\Lambda P_b^h T_\Lambda^* - ihb \circ \pi_\Lambda,$$

we have

$$\begin{aligned}
&([-\Delta, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\\
&= \frac{1}{h^2}(K^\omega_{c_0,h}(hD_s)T_\Lambda w_h, T_\Lambda P_b^h w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} - \frac{1}{h^2}(K^\omega_{c_0,h}(hD_s)T_\Lambda P_b^h w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\\
&\quad + \frac{i}{h}(K^\omega_{c_0,h}(hD_s)T_\Lambda w_h, T_\Lambda(bw_h))_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} + \frac{i}{h}(K^\omega_{c_0,h}(hD_s)T_\Lambda(bw_h), T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}.
\end{aligned}$$

It follows, as in (9-5), that

$$\limsup_{h\to 0}\left|([-\Delta, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\right| \le 2c_0\|K\|$$

and since c_0 was arbitrary, we can conclude that

$$\lim_{h\to 0}([\Delta_\Lambda, K^\omega(\sigma)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} = 0,$$

which concludes the proof of item (1).

Item (1) gives, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$ constant (which is possible since $\rho_\Lambda(ds, d\sigma)$ has compact support),

$$0 = \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, K^\omega]\, \rho_\Lambda(ds, d\sigma)\right\} = \mathrm{tr}\left\{[\Delta_\Lambda, K^\omega] \int_{T^*\mathbb{T}_{\Lambda^\perp}} \rho_\Lambda(ds, d\sigma)\right\} = \mathrm{tr}\{[\Delta_\Lambda, K^\omega]\, M_\Lambda\}.$$

Using that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A \in \mathcal{L}^1$ and $B \in \mathcal{L}$ together with the linearity of the trace (see [Reed and Simon 1980, Theorem VI.25]), we now obtain, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$ and all $\omega > 0$,

$$0 = \mathrm{tr}\{[\Delta_\Lambda, \Pi_\Lambda^\omega K \Pi_\Lambda^\omega]\, M_\Lambda\} = \mathrm{tr}\{K \Pi_\Lambda^\omega [\Delta_\Lambda, M_\Lambda]\, \Pi_\Lambda^\omega\}.$$

Consequently, we have $\Pi_\Lambda^\omega [\Delta_\Lambda, M_\Lambda]\, \Pi_\Lambda^\omega = 0$ for all $\omega > 0$ (see [Reed and Simon 1980, Theorem VI.26]). Letting ω go to $+\infty$, this yields $[\Delta_\Lambda, M_\Lambda] = 0$ and concludes the proof of item (2). □

10. The measures ν^Λ and ρ_Λ vanish identically. End of the proof of Theorem 2.6

In this section, we prove that both measures ν^Λ and ρ_Λ vanish when paired with the function $\langle b\rangle_\Lambda$. Then, we deduce that these two measures vanish identically. In turn, this implies that $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$, and finally that $\mu=0$, which will conclude the proof of Theorem 2.6.

Proposition 10.1. *We have*

$$\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0 \quad \text{and} \quad \operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\}=0.$$

As a consequence, we prove that ρ_Λ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ vanish.

Proposition 10.2. *We have $\rho_\Lambda=0$ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}=0$. Hence $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$.*

This allows us to conclude the proof of Theorem 2.6. Indeed, as a consequence of the decomposition formula of Proposition 8.4, we obtain $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$ for all $\Lambda\in\mathscr{P}$ such that $\operatorname{rk}(\Lambda)=1$. Using the decomposition of the measure μ given in Lemma 6.1 together with Lemma 6.4, this yields $\mu=0$ on $\mathbb{T}^2$. This is in contradiction with $\mu(T^*\mathbb{T}^2)=1$ (Proposition 5.2), and this contradiction proves Theorem 2.6.

Proof of Proposition 10.1. First, (5-2) implies that $(bv_h, v_h)_{L^2(\mathbb{T}^2)}\to 0$, and hence

$$\langle \mu, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0.$$

Then the decomposition given in Lemma 6.1 into a sum of nonnegative measures yields that, for all $\Lambda\in\mathscr{P}$,

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0, \tag{10-1}$$

since b is also nonnegative. Lemmata 6.2, 6.3 and 6.4 (see also Remark 6.5), then give

$$\begin{aligned}\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)} &= \langle \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}\\ &= \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0,\end{aligned} \tag{10-2}$$

where the function $\langle b\rangle_\Lambda$ is also nonnegative. The decomposition formula of Proposition 8.4 into the two-microlocal semiclassical measures then yields

$$\begin{aligned}&\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}\\ &\qquad= \langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}+\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}.\end{aligned}$$

Since the measure $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ is nonnegative, we get $\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\geq 0$. Similarly, $\rho_\Lambda\in\mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp};\mathscr{L}^1(L^2(\mathbb{T}_\Lambda)))$ and the operator $m_{\langle b\rangle_\Lambda}\in\mathscr{L}(L^2(\mathbb{T}_\Lambda))$ is selfadjoint and nonnegative, which gives $\operatorname{tr}\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\}\geq 0$. Using (10-1) and (10-2), this yields

$$\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0$$

and

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}=0.$$

In this expression, the operator $m_{\langle b\rangle_\Lambda}$ does not depend on (s,σ), so

$$0 = \operatorname{tr}\left\{ m_{\langle b\rangle_\Lambda} \int_{T^*\mathbb{T}_{\Lambda^\perp}} \rho_\Lambda(ds,d\sigma)\right\} = \operatorname{tr}\{m_{\langle b\rangle_\Lambda} M_\Lambda\},$$

which concludes the proof of Proposition 10.1. □

Proof of Proposition 10.2. Let us first prove that $\rho_\Lambda = 0$. We recall that the operator M_Λ is a selfadjoint nonnegative trace-class operator. Moreover, Proposition 9.3 implies that the operators M_Λ and Δ_Λ commute. As a consequence, there exists a Hilbert basis $(\tilde{e}^j_\Lambda)_{j\in\mathbb{N}}$ of $L^2(\mathbb{T}_\Lambda)$ in which M_Λ and Δ_Λ are simultaneously diagonal, i.e., such that

$$-\Delta_\Lambda \tilde{e}^j_\Lambda = \omega^j_\Lambda \tilde{e}^j_\Lambda \quad \text{and} \quad M_\Lambda \tilde{e}^j_\Lambda = \gamma^j_\Lambda \tilde{e}^j_\Lambda,$$

where $(\gamma^j_\Lambda)_{j\in\mathbb{N}}$ are the associated eigenvalues of M_Λ. In particular, we have $\gamma^j_\Lambda \geq 0$ for all $j\in\mathbb{N}$ (and $\gamma^j_\Lambda \in \ell^1$). Note that the basis $(\tilde{e}^j_\Lambda)_{j\in\mathbb{N}}$ is not necessarily the same as the basis $(e^j_\Lambda)_{j\in\mathbb{N}}$ introduced in Section 9B.

Using Proposition 10.1, together with the definition of the trace (see, for instance, [Reed and Simon 1980, Theorem VI.18]) we have

$$0 = \operatorname{tr}\{m_{\langle b\rangle_\Lambda} M_\Lambda\} = \sum_{j\in\mathbb{N}} (m_{\langle b\rangle_\Lambda} M_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)} = \sum_{j\in\mathbb{N}} \gamma^j_\Lambda (\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)}.$$

Since all terms in this sum are nonnegative (because both γ^j_Λ and $\langle b\rangle_\Lambda$ are), we deduce that for all $j\in\mathbb{N}$,

$$\gamma^j_\Lambda (\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)} = 0.$$

Suppose that $\gamma^j_\Lambda \neq 0$ for some $j\in\mathbb{N}$. Then, $(\langle b\rangle_\Lambda \tilde{e}^j_\Lambda, \tilde{e}^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)} = 0$ where $\langle b\rangle_\Lambda$ is nonnegative and not identically zero on $\mathbb{T}_\Lambda$. This yields $\tilde{e}^j_\Lambda = 0$ on the nonempty open set $\{\langle b\rangle_\Lambda > 0\}$. Using a unique continuation property for eigenfunctions of the Laplace operator on $\mathbb{T}_\Lambda$, we finally obtain that the eigenfunction $\tilde{e}^j_\Lambda$ vanishes identically on $\mathbb{T}_\Lambda$. This is absurd, and thus we must have $\gamma^j_\Lambda = 0$ for all $j\in\mathbb{N}$, so that $M_\Lambda = 0$. Since $\rho_\Lambda \in \mathcal{M}^+(T^*\mathbb{T}_{\Lambda^\perp}; \mathcal{L}^1(L^2(\mathbb{T}_\Lambda)))$, this directly gives $\rho_\Lambda = 0$.

Next, we prove that $\nu^\Lambda = 0$. This is a consequence of the additional propagation law of ν^Λ with respect to the flow ϕ^1_τ (see Section 9A). Indeed the torus $\mathbb{T}_\Lambda$ has dimension one, $(\phi^1_\tau)_*\nu^\Lambda = \nu^\Lambda$ (according to Proposition 9.1) and, using Proposition 10.1, ν^Λ vanishes on the (nonempty) set $\{\langle b\rangle_\Lambda > 0\} \times \mathbb{R}^2 \times \mathbb{S}_\Lambda$ (with $\{\langle b\rangle_\Lambda > 0\}$ clearly satisfying GCC on $\mathbb{T}_\Lambda$). Hence, $\nu^\Lambda = 0$.

To conclude the proof of Proposition 10.2, it only remains to use the decomposition formula (8-7) which directly yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp} = 0$. □

11. Proof of Proposition 7.2

In this section, we prove Proposition 7.2. For this, we consider two-microlocal semiclassical measures at the scale h^α. The setting is close to that of [Fermanian Kammerer 2005].

We shall see that the concentration rate of the sequence v_h towards the direction $\Lambda^\perp$ is of the form h^α for all $\alpha \leq (1+\delta)/2$.

First, Lemma 6.3 yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \langle\mu\rangle_\Lambda|_{\mathbb{T}^2\times\Lambda^\perp}$ (see also Remark 6.5); that is,

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)},$$

and it suffices to characterize the action of $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ on $\Lambda^\perp$-invariant symbols. Recall that, for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\langle \mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \lim_{h\to 0}(\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)}.$$

As in (8-1) and (8-2), let us define

$$\langle V_R^{h,\Lambda}, a\rangle_{S_\Lambda^{1\,\prime}, S_\Lambda^1} := \Big\langle V^h, \Big(1-\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big) a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)}, \tag{11-1}$$

$$\langle V_{R,\Lambda}^{h}, a\rangle_{S_\Lambda^{1\,\prime}, S_\Lambda^1} := \Big\langle V^h, \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big) a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)}, \tag{11-2}$$

for $a \in S_\Lambda^1$.

We take $R = R(h) = h^{-(1-\alpha)}$ for some $\alpha\in(0,1)$, so that $Rh = h^\alpha$. The proof of Proposition 8.2 applies verbatim and shows the existence of a subsequence (h, v_h) and a nonnegative measure $\nu_\alpha^\Lambda \in \mathcal{M}^+(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ such that, for all $a\in S_\Lambda^1$, we have

$$\lim_{h\to 0}\langle V_{R(h)}^{h,\Lambda}, a\rangle_{S_\Lambda^{1\,\prime}, S_\Lambda^1} = \Big\langle \nu_\alpha^\Lambda, a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Proposition 11.1. *Let $R(h) = h^{-(1-\alpha)}$ with $\alpha \le (1+\delta)/2$. Then*

$$\nu_\alpha^\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda} = 0.$$

The proof of Proposition 11.1 relies on the following propagation result.

Lemma 11.2. *For $\alpha\le(1+\delta)/2$ the measure ν_α^Λ is ϕ_τ^0- and ϕ_τ^1-invariant*:

$$(\phi_\tau^0)_*\nu_\alpha^\Lambda = \nu_\alpha^\Lambda \quad \textit{and} \quad (\phi_\tau^1)_*\nu_\alpha^\Lambda = \nu_\alpha^\Lambda \quad \textit{for every } \tau\in\mathbb{R}.$$

The proof is very similar to that of Proposition 9.1 but does not use assumption (2-13).

Proof. The proof of ϕ_τ^0-invariance is strictly identical to what has been done for Proposition 9.1 and thus we focus on the ϕ_τ^1-invariance. Equation (9-5) still holds with $R(h) = h^{-(1-\alpha)}$, now reading

$$\Big\langle V_{R(h)}^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime}, S_\Lambda^1} = \frac{i}{2h^2}(A^{R(h)}v_h, P_b^h v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}(A^{R(h)}P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^{R(h)}bv_h, v_h)_{L^2(\mathbb{T}^2)},$$

where A^R was defined in (9-3). Using $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$ together with the boundedness of $A^{R(h)}$, it follows that

$$\lim_{h\to 0}\Big\langle V_{R(h)}^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime}, S_\Lambda^1} = \lim_{h\to 0}\Big(-\frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^{R(h)}bv_h, v_h)_{L^2(\mathbb{T}^2)}\Big). \tag{11-3}$$

Recall from (5-2) that $\|\sqrt{b}v_h\|_{L^2(\mathbb{T}^2)} = o(h^{(1+\delta)/2})$. In addition, with $R(h) = h^{-(1-\alpha)}$ we have

$$A^{R(h)} = \mathrm{Op}_1(\tilde{a}_h), \quad \tilde{a}_h(x,\xi) = \frac{1}{|P_\Lambda \xi|}\big(1-\chi(h^{(1-\alpha)}|P_\Lambda\xi|)\big)a(x, h\xi, P_\Lambda\xi),$$

where $a \in S^1_\Lambda$ is homogeneous of order zero in the third variable and P_Λ is defined in (6-3). Since $h^{1-\alpha}|P_\Lambda\xi| \geq 1$ on $\mathrm{supp}(1-\chi)$, the symbol $\tilde{a}_h$ satisfies

$$|\partial_x^{\beta'}\partial_\xi^{\beta}\tilde{a}_h| \leq C_{\beta,\beta'}h^{1-\alpha}h^{|\beta|(1-\alpha)}.$$

Hence, the Calderón–Vaillancourt theorem (see for instance Theorem A.1) yields $\|A^{R(h)}\|_{\mathcal{L}(L^2)} \leq Ch^{1-\alpha}$, which implies

$$\left|\frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)}\right| \leq Ch^{-1}\|A^{R(h)}\|_{\mathcal{L}(L^2)}\|v_h\|_{L^2(\mathbb{T}^2)}\|\sqrt{b}v_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{1+\delta}{2}-\alpha}).$$

Coming back to (11-3), this finally gives

$$\lim_{h\to 0}\Big\langle V^{h,\Lambda}_{R(h)}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = 0,$$

as soon as $\alpha \leq (1+\delta)/2$. ☐

Proof of Proposition 11.1. We have $\langle \nu_\alpha^\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$, since ν_α^Λ is (ϕ^0_τ)-invariant and $\langle \nu_\alpha^\Lambda, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$. Then, the ϕ^1_τ-invariance of ν_α^Λ implies that $\nu_\alpha^\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}$ vanishes. ☐

Proof of Proposition 7.2. Proposition 11.1 implies that

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big)v_h, v_h\Big)_{L^2(\mathbb{T}^2)}$$

for all $\alpha \leq (1+\delta)/2$ and $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. The same holds if we replace χ by χ^2:

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h\Big(\chi^2\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big)v_h, v_h\Big)_{L^2(\mathbb{T}^2)}.$$

Since

$$\mathrm{Op}_h\Big(\chi^2\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big) = \mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big) + \mathcal{O}(h^{1-\alpha}), \tag{11-4}$$

we obtain

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)v_h, \mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)v_h\Big)_{L^2(\mathbb{T}^2)},$$

for all $\alpha \leq (1+\delta)/2$ and $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. ☐

12. Proof of Proposition 7.3: existence of the cutoff function

Given a constant $c_0 > 0$, we define the following subsets of $\mathbb{T}^2$:

$$\mathscr{E}_h = \langle \{b > c_0 h\} \rangle_\Lambda, \quad \mathscr{F}_h = \left\langle \bigcup_{x \in \{b > c_0 h\}} B(x, (c_0 h)^{2\varepsilon}) \right\rangle_\Lambda = \bigcup_{x \in \mathscr{E}_h} B(x, (c_0 h)^{2\varepsilon}), \quad \mathscr{G}_h = \mathscr{F}_h \setminus \mathscr{E}_h,$$

where for $U \subset \mathbb{T}^2$, we denote $\langle U \rangle_\Lambda := \bigcup_{\tau \in \mathbb{R}} \{U + \tau\sigma\}$ for some $\sigma \in \Lambda^\perp \setminus \{0\}$. Remark that $\mathscr{E}_h \subset \mathscr{F}_h$ and that $\mathbb{T}^2 = \mathscr{E}_h \cup \mathscr{G}_h \cup (\mathbb{T}^2 \setminus \mathscr{F}_h)$. Note also that the sets $\mathscr{E}_h, \mathscr{F}_h$ are nonempty for h small enough, and that $\mathscr{G}_h$ is nonempty (for h small enough) as soon as b vanishes somewhere on $\mathbb{T}^2$ (this condition is assumed here, since otherwise GCC is satisfied).

In this section, we construct the cutoff function χ_h^Λ needed to prove the propagation results of Section 9. In particular, this function will be $\Lambda^\perp$-invariant and will satisfy $\chi_h^\Lambda = 0$ on $\mathscr{E}_h$ and $\chi_h^\Lambda = 1$ on $\mathbb{T}^2 \setminus \mathscr{F}_h$.

The proof of Proposition 7.3 relies on three key lemmata. The first key lemma is a precised version of Proposition 5.2 concerning the localization in $T^*\mathbb{T}^2$ of the semiclassical measure μ. It is an intermediate step towards the propagation result stated in Lemma 12.2.

Lemma 12.1. *For any $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighborhood of the origin, for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, and any $\gamma \leq (3+\delta)/2$, we have*

$$(\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} = \left(\mathrm{Op}_h(a) \,\mathrm{Op}_h\left(\chi\left(\frac{|\xi|^2 - 1}{h^\gamma} \right) \right) w_h, w_h \right)_{L^2(\mathbb{T}^2)} + o(h^{\frac{3+\delta}{2} - \gamma}) \| \mathrm{Op}_h(a) \|_{\mathscr{L}(L^2)}. \quad (12\text{-}1)$$

For all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ and all $\tau \in \mathbb{R}$,

$$(\mathrm{Op}_h(a \circ \phi_\tau) w_h, w_h)_{L^2(\mathbb{T}^2)} = (\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}}) \| \mathrm{Op}_h(a \circ \phi_t) \|_{L^\infty(0,\tau; \mathscr{L}(L^2(\mathbb{T}^2)))}.$$

In this statement, we used the notation

$$\| \mathrm{Op}_h(a \circ \phi_t) \|_{L^\infty(0,\tau; \mathscr{L}(L^2(\mathbb{T}^2)))} := \sup_{t \in (0,\tau)} \| \mathrm{Op}_h(a \circ \phi_t) \|_{\mathscr{L}(L^2(\mathbb{T}^2))}.$$

In turn, this lemma implies the following transport property.

Lemma 12.2. *Suppose that the coefficients α, ε satisfy*

$$0 < 3\varepsilon \leq \alpha \quad \textit{and} \quad \alpha + \varepsilon \leq 1. \quad (12\text{-}2)$$

Then, for any time $\tau \in \mathbb{R}$ uniformly bounded with respect to h and any h-family of functions $\psi = \psi_h \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying

$$\| \partial_x^k \psi \|_{L^\infty(\mathbb{T}^2)} \leq C_k h^{-\varepsilon |k|} \quad \textit{for all } k \in \mathbb{N}^2, \quad (12\text{-}3)$$

we have

$$(\psi(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)}$$
$$= (\psi(s+\tau, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi(s-\tau, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + \mathscr{O}(h^{\alpha - 3\varepsilon}) + \mathscr{O}(h^{1-\alpha-\varepsilon}) + o(h^{\frac{1+\delta}{2}}), \quad (12\text{-}4)$$

where the coordinates (s, y) are the ones introduced in Section 6C.

In view of Proposition 7.3, this lemma will allow us to propagate the smallness of the sequence w_h above the set $\{b > c_0 h\}$ to all $\mathscr{E}_h$.

The third key lemma states a property of the damping function b, as a consequence of (2-13).

Lemma 12.3. *For all $\varepsilon \in (0, 1]$, $x \in \mathbb{T}^2$ and all $z \in B(x, \frac{1}{2}b(x)^\varepsilon)$, we have $\frac{1}{2}b(x) \le b(z) \le e^{\frac{1}{2}}b(x)$.*

Assumption (2-13) is used here. We denoted by $B(x, \frac{1}{2}b(x)^\varepsilon)$ the Euclidean ball in $\mathbb{T}^2$ centered at x of radius $\frac{1}{2}b(x)^\varepsilon$. Note that only the left inequality is used in this paper.

With these three lemmata, we are now able to prove Proposition 7.3.

Proof of Proposition 7.3. In the coordinates (s, y) of Section 6C, we can write

$$\mathscr{E}_h = \mathbb{T}_{\Lambda^\perp} \times E_h, \quad \mathscr{F}_h = \mathbb{T}_{\Lambda^\perp} \times F_h, \quad \text{with } E_h \subset F_h \subset \mathbb{T}_\Lambda.$$

Here, F_h is a union of intervals and has uniformly bounded total length. We can hence cover F_h with $C_1 h^{-\varepsilon}$ subsets of length of order $(c_0 h)^\varepsilon/4$, overlapping on intervals of length of order $(c_0 h)^\varepsilon/10$. Associated to this covering, we denote by $(\psi_j)_{j\in\{1,\dots,J\}}$, $J = J(h)$, a smooth partition of unity on E_h, also satisfying

- $\psi_j \in \mathscr{C}_c^\infty(F_h)$;
- $\sum_{j=1}^J \psi_j(y) = 1$ for $y \in E_h$;
- $\|\partial_y^m \psi_j\|_{L^\infty(\mathbb{T}_\Lambda)} \le C_m h^{-\varepsilon m}$ for all $m \in \mathbb{N}$;
- $J = J(h) \le C h^{-\varepsilon}$.

Similarly, we cover $\mathbb{T}_{\Lambda^\perp}$ with $C_2 h^{-\varepsilon}$ subsets of length of order $(c_0 h)^\varepsilon/4$, overlapping on intervals of length of order $(c_0 h)^\varepsilon/10$, and define $(\psi_k)_{k\in\{1,\dots,K\}}$ an associated partition of unity on $\mathbb{T}_{\Lambda^\perp}$ satisfying

- $\psi_k \in \mathscr{C}_c^\infty(\mathbb{T}_{\Lambda^\perp})$;
- $\sum_{k=1}^K \psi_k(s) = 1$ for $s \in \mathbb{T}_{\Lambda^\perp}$;
- $\|\partial_s^m \psi_k\|_{L^\infty(\mathbb{T}_{\Lambda^\perp})} \le C_m h^{-\varepsilon m}$, for all $m \in \mathbb{N}$;
- $K = K(h) \le C h^{-\varepsilon}$;
- for any $k, k_0 \in \{1, \dots, K\}^2$, there exists τ_k satisfying $|\tau_k| \le \text{Length}(\mathbb{T}_{\Lambda^\perp}) \le C$ and $\psi_k(s+\tau_k) = \psi_{k_0}(s)$.

We set

$$\psi_{kj}(s, y) := \psi_k(s)\psi_j(y) \quad \text{and} \quad \chi_h^\Lambda(s, y) = 1 - \sum_{j=1}^J \sum_{k=1}^K \psi_{kj}(s, y) \in \mathscr{C}^\infty(\mathbb{T}^2),$$

which satisfies $\partial_s \chi_h^\Lambda(s, y) = 0$, i.e., χ_h^Λ is $\Lambda^\perp$-invariant, together with

- $\chi_h^\Lambda = 0$ on $\mathscr{E}_h$ and hence $b \le c_0 h$ on $\text{supp}(\chi_h^\Lambda)$;
- $\chi_h^\Lambda = 1$ on $\mathbb{T}^2 \setminus \mathscr{F}_h$;
- $\chi_h^\Lambda \in [0, 1]$ on $\mathscr{G}_h$, with $|\partial_y \chi_h^\Lambda| \le C h^{-\varepsilon}$ and $|\partial_y^2 \chi_h^\Lambda| \le C h^{-2\varepsilon}$.

To conclude the proof of Proposition 7.3, it remains to check item (2) ($\|(1-\chi_h^\Lambda)w_h\|_{L^2(\mathbb{T}^2)} = o(1)$), item (4) ($\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$) and item (5) ($\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$).

Now, let us fix $j_0 \in \{1, \dots, J\}$. Because of the definition of the set $\mathscr{E}_h$, there exists $k_0 \in \{1, \dots, K\}$ and $x_0 \in \{b > c_0 h\}$ such that $\operatorname{supp}(\psi_{k_0 j_0}) \subset B(x_0, (c_0 h)^\varepsilon/2)$. According to Lemma 12.3, we have

$$B\Big(x_0, \frac{(c_0 h)^\varepsilon}{2}\Big) \subset B\Big(x_0, \frac{b(x_0)^\varepsilon}{2}\Big) \subset \Big\{b > \frac{b(x_0)}{2}\Big\} \subset \Big\{b > \frac{c_0 h}{2}\Big\},$$

so that $\operatorname{supp}(\psi_{k_0 j_0}) \subset \{b > c_0 h/2\}$. This yields

$$\frac{c_0 h}{2}(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} \le (b\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}),$$

and hence $(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$. Moreover, for any $k \in \{1, \dots, K\}$, there exists τ_k satisfying $|\tau_k| \le C_2$ with

$$\psi_{k j_0}(s+\tau_k, y) = \psi_{k_0 j_0}(s, y).$$

Hence, using (12-4), we obtain

$$\begin{aligned} o(h^\delta) &= (\psi_{k_0 j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = (\psi_{k j_0}(s+\tau_k, y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &= (\psi_{k j_0}(s+2\tau_k, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi_{k j_0}(s, y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &\qquad + \mathbb{O}(h^{\alpha - 3\varepsilon}) + \mathbb{O}(h^{1-\alpha-\varepsilon}) + o(h^{\frac{1+\delta}{2}}). \end{aligned} \tag{12-5}$$

Since both terms on the right-hand side are nonnegative, this implies $(\psi_{k j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$ as long as

$$\alpha - 3\varepsilon > \delta, \quad 1 - \alpha - \varepsilon > \delta, \quad \text{and} \quad \frac{1+\delta}{2} \ge \delta$$

(which implies (12-2)). From now on we will take $\delta = 4\varepsilon$ (the reason for this choice will become apparent in the following lines). The existence of α satisfying this condition together with (7-1) is equivalent to having $\varepsilon < \frac{1}{29}$.

To conclude the proof of Proposition 7.3, we first compute

$$((1-\chi_h^\Lambda) w_h, w_h)_{L^2(\mathbb{T}^2)} = \sum_{j=1}^{J} \sum_{k=1}^{K} (\psi_{kj} w_h, w_h)_{L^2(\mathbb{T}^2)} = C h^{-2\varepsilon} o(h^\delta) = o(1),$$

since $\delta \ge 2\varepsilon$. This proves item (2). Next, we have by construction $\operatorname{supp}(\partial_y^2 \chi_h^\Lambda) \subset \operatorname{supp}(\partial_y \chi_h^\Lambda) \subset \mathscr{G}_h$, with $\|\partial_y \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathbb{O}(h^{-\varepsilon})$, $\|\partial_y^2 \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathbb{O}(h^{-2\varepsilon})$. Hence, covering $\operatorname{supp}(\partial_y \chi_h^\Lambda))$ by balls of radius $(c_0 h)^\varepsilon$ and using a propagation argument similar to (12-5) shows that we have $\|w_h\|_{L^2(\operatorname{supp}(\partial_y \chi_h^\Lambda))} = o(h^{\frac{\delta}{2}})$. We thus obtain

$$\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-\varepsilon}) = o(1), \quad \|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-2\varepsilon}) = o(1),$$

(since $\delta \ge 4\varepsilon$), which concludes the proof of items (4) and (5), and that of Proposition 7.3. □

To conclude this section, it remains to prove Lemmata 12.2, 12.1 and 12.3. In the following proofs, we shall systematically write η in place of $P_\Lambda \xi$ and σ in place of $(1-P_\Lambda)\xi$ to lighten the notation. Hence, $\xi \in \mathbb{R}^2$ is decomposed as $\xi = \eta + \sigma$, with $\eta \in \langle \Lambda \rangle$ and $\sigma \in \Lambda^\perp$, in accordance to Section 6C.

Proof of Lemma 12.2 from Lemma 12.1. First, given a function $\psi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying (12-3), we have

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} &= (\mathrm{Op}_h(\psi\circ\phi_\tau) w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}}) \|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))} \\ &= \Big(\mathrm{Op}_h(\psi\circ\phi_\tau)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\,\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\Big) w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &\qquad + \big(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2}-\gamma})\big)\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))},\end{aligned}$$

when using Lemma 12.1 together with $\mathrm{Op}_h(\chi(\eta/(2h^\alpha)))w_h = w_h$. Next, the pseudodifferential calculus yields

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} = \Big(\mathrm{Op}_h\Big(\psi\circ\phi_\tau\,\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\chi\Big(\frac{\eta}{2h^\alpha}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} + \mathcal{O}(h^{2-\gamma-\varepsilon}) + \mathcal{O}(h^{1-\alpha-\varepsilon}) \\ + \big(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2}-\gamma})\big)\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))}. \end{aligned} \tag{12-6}$$

A particular feature of the Weyl quantization in the Euclidean setting is that the Egorov theorem provides an exact formula (see, for instance, [Dimassi and Sjöstrand 1999]): $\mathrm{Op}_h(\psi\circ\phi_t) = e^{-ith\frac{\Delta}{2}}\mathrm{Op}_h(\psi)e^{ith\frac{\Delta}{2}}$, so that $\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))} \le C_0$ uniformly with respect to h. Now, remark that the cutoff function $\chi(\eta/(2h^\alpha))\chi((|\xi|^2-1)/h^\gamma)$ can be decomposed (for h small enough) as

$$\chi\Big(\frac{\eta}{2h^\alpha}\Big)\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\big(\tilde{\chi}_\eta^h(\sigma) + \tilde{\chi}_\eta^h(-\sigma)\big)$$

for some nonnegative function $\tilde{\chi}_\eta^h$ such that $(\sigma,\eta)\mapsto \tilde{\chi}_\eta^h(\sigma) \in \mathscr{C}_c^\infty(\mathbb{R}^2)$, such that $\tilde{\chi}_\eta^h(\sigma) = \chi((|\xi|^2-1)/h^\gamma)$ for $\eta \in \operatorname{supp}\chi(\cdot/(2h^\alpha))$ and $\sigma > 0$, and $\tilde{\chi}_\eta^h(\sigma) = 0$ for $\eta \notin \operatorname{supp}\chi(\cdot/(2h^\alpha))$ or $\sigma \le 0$.

Choosing $\gamma = \alpha$, we have in particular

$$|\sigma - 1| \le Ch^\alpha \quad \text{on } \operatorname{supp}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\Big).$$

Next, we recall that $\psi\circ\phi_\tau(s,y,\sigma,\eta) = \psi(s+\tau\sigma, y+\tau\eta)$, and we focus on the first term (corresponding to $\sigma > 0$) in the right-hand side of the identity

$$\chi\Big(\frac{|\xi|^2-1}{h^\alpha}\Big)\chi\Big(\frac{\eta}{2h^\alpha}\Big)\psi\circ\phi_\tau = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\big(\tilde{\chi}_\eta^h(\sigma) + \tilde{\chi}_\eta^h(-\sigma)\big)\psi\circ\phi_\tau. \tag{12-7}$$

We set

$$\zeta_\tau^{(1)}(s,y,\sigma,\eta) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\psi(s+\tau\sigma, y+\tau\eta) \quad \text{and} \quad \zeta_\tau^{(2)}(s,y,\sigma,\eta) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\psi(s+\tau, y),$$

and we want to compare $\mathrm{Op}_h(\zeta_\tau^{(1)})$ and $\mathrm{Op}_h(\zeta_\tau^{(2)})$. For this, let us estimate, for multiindices $\ell, m \in \mathbb{N}^2$,

$$\begin{aligned}&|\partial_{(s,y)}^\ell \partial_{(\sigma,\eta)}^m (\zeta_\tau^{(2)} - \zeta_\tau^{(1)})(s,y,\sigma,\eta)| \\ &\qquad \le C_m \sum_{\nu\le m}\Big|\partial_{(\sigma,\eta)}^{m-\nu}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\Big)\partial_{(s,y)}^\ell\partial_{(\sigma,\eta)}^\nu\big(\psi(s+\tau\sigma, y+\tau\eta) - \psi(s+\tau, y)\big)\Big|.\end{aligned} \tag{12-8}$$

On the one hand, we have

$$\Big|\partial_{(\sigma,\eta)}^{m-\nu}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\Big)\Big| \le C_{m,\nu}h^{-\alpha|m-\nu|}. \tag{12-9}$$

On the other hand, for $|\nu|>0$ we can also write

$$\begin{aligned}\big|\partial_{(s,y)}^{\ell}\partial_{(\sigma,\eta)}^{\nu}\big(\psi(s+\tau\sigma, y+\tau\eta)-\psi(s+\tau, y)\big)\big| &= |\partial_{(s,y)}^{\ell}\partial_{(\sigma,\eta)}^{\nu}\psi(s+\tau\sigma, y+\tau\eta)| \\ &\le C_{\ell,\nu}|\tau|^{|\nu|}h^{-\varepsilon(|\ell|+|\nu|)} \le C_{\ell,\nu}h^{-\varepsilon(|\ell|+|\nu|)},\end{aligned}$$

since $|\tau|\le C$.

Finally, for $|\nu|=0$, we apply the mean value theorem to the function $(\sigma,\eta)\mapsto \partial_{(s,y)}^{\ell}\psi(s+\tau\sigma, y+\tau\eta)$ and write

$$\big|\partial_{(s,y)}^{\ell}\big(\psi(s+\tau\sigma, y+\tau\eta)-\psi(s+\tau, y)\big)\big| \le (|\eta|+|\sigma-1|)\sup_{T^*\mathbb{T}^2}|\nabla_{(\sigma,\eta)}\partial_{(s,y)}^{\ell}(\psi(s+\tau\sigma, y+\tau\eta))|.$$

With (12-3), this yields

$$\begin{aligned}\big|\partial_{(s,y)}^{\ell}\big(\psi(s+\tau\sigma, y+\tau\eta)-\psi(s+\tau, y)\big)\big| &\le (|\eta|+|\sigma-1|)C_\ell h^{-\varepsilon|\ell|}|\tau|h^{-\varepsilon} \\ &\le (|\eta|+|\sigma-1|)C_\ell h^{-\varepsilon(|\ell|+1)},\end{aligned} \tag{12-10}$$

for $|\tau|\le C$.

Using now that $|\eta|\le Ch^\alpha$ and $|\sigma-1|\le Ch^\alpha$ on $\operatorname{supp}(\chi(\eta/(2h^\alpha))\tilde{\chi}_\eta^h(\sigma))$, and combining (12-8), (12-9) and (12-10), we obtain, for all $m\in\mathbb{N}^2$, $\ell\in\mathbb{N}^2$ and $0<h\le h_0$ sufficiently small,

$$\begin{aligned}h^{|m|}|\partial_{(s,y)}^{\ell}\partial_{(\sigma,\eta)}^{m}(\zeta_\tau^{(2)}-\zeta_\tau^{(1)})(s,y,\sigma,\eta)| &\le C_{\ell,m}h^{\alpha-\varepsilon(|\ell|+1)}h^{|m|}h^{-\alpha|m|}+C_{\ell,m}\sum_{0<\nu\le m}h^{|m|}h^{-\varepsilon(|\ell|+|\nu|)}h^{-\alpha|m-\nu|} \\ &\le C_{\ell,m}(h^{(1-\alpha)|m|}h^{\alpha-\varepsilon(|\ell|+1)}+|m|h^{|m|(1-\alpha)}h^{-\varepsilon|\ell|}h^{\alpha-\varepsilon}) \\ &\le C_{\ell,m}h^{\alpha-\varepsilon(|\ell|+1)}.\end{aligned}$$

Using a precised version of the Calderón–Vaillancourt theorem, as presented in Theorem A.1 below (in which only $|\ell|=2$ derivations are needed with respect to x in dimension two), we obtain

$$\mathrm{Op}_h(\zeta_\tau^{(2)})=\mathrm{Op}_h(\zeta_\tau^{(1)})+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-3\varepsilon}).$$

Similarly, we have

$$\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s+\tau\sigma, y+\tau\eta)\Big)=\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s-\tau, y)\Big)+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-3\varepsilon}).$$

Coming back to (12-6) and using (12-7), we finally obtain, for all $|\tau|\le C$,

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} &= \Big(\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\psi(s+\tau, y)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &\quad+\Big(\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s-\tau, y)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &\quad+\mathcal{O}(h^{\alpha-3\varepsilon})+\mathcal{O}(h^{1-\alpha-\varepsilon})+o(h^{\frac{1+\delta}{2}})+o(h^{\frac{3+\delta}{2}-\alpha}).\end{aligned}$$

With the pseudodifferential calculus, this yields (12-4), which concludes the proof of Lemma 12.2. □

Proof of Lemma 12.1. Here, we only have to make more precise some arguments in the proof of Proposition 5.2. Recall that according to Lemma 7.1, w_h satisfies $P_b^h w_h = o(h^{2+\delta})$.

First, we take $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$, such that $\chi = 1$ in a neighborhood of the origin. Hence, $(1-\chi(r))/r \in \mathscr{C}^\infty(\mathbb{R})$ and we have the exact composition formula

$$\mathrm{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big) = \mathrm{Op}_h\Big(\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\frac{h^\gamma}{|\xi|^2-1}\Big)\frac{P_0^h}{h^\gamma},$$

since both operators are Fourier multipliers. Moreover, $\mathrm{Op}_h\big((1-\chi((|\xi|^2-1)/h^\gamma))h^\gamma/(|\xi|^2-1)\big)$ is uniformly bounded as an operator of $\mathscr{L}(L^2(\mathbb{T}^2))$. As a consequence, we have

$$\begin{aligned}\Big(\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} &\\ = \Big(\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\frac{h^\gamma}{|\xi|^2-1}\Big)\frac{P_0^h}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)} &\\ = \Big(A\frac{P_b^h}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)} - \Big(A\frac{ihb}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)},\end{aligned}$$

where $A = \mathrm{Op}_h(a)\,\mathrm{Op}_h\big((1-\chi((|\xi|^2-1)/h^\gamma))h^\gamma/(|\xi|^2-1)\big)$ is bounded on $L^2(\mathbb{T}^2)$. Using $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$, this gives

$$\Big(\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} = o(h^{\frac{3+\delta}{2}-\gamma})\|\mathrm{Op}_h(a)\|_{\mathscr{L}(L^2)},$$

which in turn implies (12-1).

Next, identity (5-6) yields, for all $a \in \mathscr{C}_c^\infty(\mathbb{T}^2)$,

$$\begin{aligned}(\mathrm{Op}_h(\xi\cdot\partial_x a)w_h, w_h)_{L^2(\mathbb{T}^2)} &= \frac{i}{2h}(\mathrm{Op}_h(a)w_h, P_b^h w_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h}(\mathrm{Op}_h(a)P_b^h w_h, w_h)_{L^2(\mathbb{T}^2)}\\ &\quad - \tfrac{1}{2}(\mathrm{Op}_h(a)w_h, bw_h)_{L^2(\mathbb{T}^2)} - \tfrac{1}{2}(\mathrm{Op}_h(a)bw_h, w_h)_{L^2(\mathbb{T}^2)}\\ &= o(h^{1+\delta})\|\mathrm{Op}_h(a)\|_{\mathscr{L}(L^2)} + o(h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(a)\|_{\mathscr{L}(L^2)},\end{aligned}$$

as a consequence of $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$. Applying this identity to $a\circ\phi_t$ in place of a, and integrating on $t \in [0,\tau]$ finally gives

$$(\mathrm{Op}_h(a\circ\phi_\tau)w_h, w_h)_{L^2(\mathbb{T}^2)} = (\mathrm{Op}_h(a)w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(a\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))},$$

which concludes the proof of Lemma 12.1. □

Proof of Lemma 12.3. First, we have $\nabla(b^\varepsilon) = 0$ on $\{b=0\}$ and $\nabla(b^\varepsilon)(x) = \varepsilon b(x)^{\varepsilon-1}\nabla b(x)$ on $\{b>0\}$. Assumption (2-13) then yields $|\nabla(b^\varepsilon)| \le \varepsilon$ uniformly on $\mathbb{T}^2$. The mean value theorem hence gives, for all $z \in B(x, \frac{1}{2}b(x)^\varepsilon)$,

$$b(x)^\varepsilon \le b(z)^\varepsilon + \varepsilon|x-z| \le b(z)^\varepsilon + \frac{\varepsilon}{2}b(x)^\varepsilon.$$

Hence we obtain $b(z) \geq b(x)(1-\varepsilon/2)^{1/\varepsilon}$. On the interval $(0, 1]$, the function $\varepsilon \mapsto (1/\varepsilon)(1-2^{-\varepsilon})$ is decreasing so that for $\varepsilon \in (0, 1]$, we have $(1/\varepsilon)(1-2^{-\varepsilon}) \geq \frac{1}{2}$. This gives $0 < \varepsilon/2 \leq 1-2^{-\varepsilon}$ so that $b(z) \geq b(x)(2^{-\varepsilon})^{1/\varepsilon}$ for $\varepsilon \in (0, 1]$, which concludes the proof of the left inequality.

The right inequality follows from the same arguments. □

Part IV. An *a priori* lower bound for decay rates on the torus

13. Proof of Theorem 2.5

Under the assumption

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \varnothing, \tag{13-1}$$

for some $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, we construct in this section a constant $\kappa_0 > 0$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathscr{O}(1)$-quasimodes in the limit $n \to +\infty$ for the family of operators $P(in\kappa_0)$.

We use the notation introduced in Sections 6A and 8. First, note that, as a consequence of (13-1), ξ_0 is necessarily a rational direction, and the set $\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\}$ is a one-dimensional subtorus of $\mathbb{T}^2$, given by

$$\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \overline{\{x_0 + \tau\xi_0, \tau \in \mathbb{R}\}} = x_0 + \mathbb{T}_{\Lambda_{\xi_0}^{\perp}}, \quad \text{with } \Lambda_{\xi_0} \in \mathscr{P}.$$

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ such that χ has only x-Fourier modes in Λ_{ξ_0}, $\chi = 0$ on a neighborhood of $\overline{\{b > 0\}}$ and $\chi = 1$ on $x_0 + \mathbb{T}_{\Lambda_{\xi_0}^{\perp}}$.

From assumption (13-1), we have $\operatorname{rk}(\Lambda_{\xi_0}) = 1$, so that one can find $k \in \Lambda_{\xi_0}^{\perp} \cap \mathbb{Z}^2 \setminus \{0\}$. Besides, for all $n \in \mathbb{N}$ we have $nk \in \Lambda_{\xi_0}^{\perp} \cap \mathbb{Z}^2 \setminus \{0\}$.

We then define the sequence of *quasimodes* $(\varphi_n)_{n\in\mathbb{N}}$ by

$$\varphi_n(x) = \chi(x)e^{ink\cdot x}, \quad n \in \mathbb{N},\ x \in \mathbb{T}^2.$$

We have $\varphi_n \in \mathscr{C}^\infty(\mathbb{T}^2)$, together with the decoupling

$$\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) = \chi(y)e^{ink\cdot s}, \quad n \in \mathbb{N},\ (s, y) \in \mathbb{T}_{\Lambda_{\xi_0}^{\perp}} \times \mathbb{T}_{\Lambda_{\xi_0}}.$$

This yields

$$\begin{aligned} -\big(T_{\Lambda_{\xi_0}} \Delta T^*_{\Lambda_{\xi_0}}\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) &= -\big(\Delta_{\Lambda_{\xi_0}} + \Delta_{\Lambda_{\xi_0}^{\perp}}\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) \\ &= -e^{ink\cdot s}\Delta_{\Lambda_{\xi_0}}\chi(y) + n^2|k|^2\chi(y)e^{ink\cdot s}. \end{aligned}$$

Moreover, $b\varphi_n = 0$ since their supports are disjoint. Hence, recalling that

$$P(in|k|) = -\Delta - n^2|k|^2 + in|k|b(x),$$

we have

$$(T_{\Lambda_{\xi_0}} P(in|k|) T^*_{\Lambda_{\xi_0}})\varphi_n \circ \pi_{\Lambda_{\xi_0}} = -e^{ink\cdot s}\Delta_{\Lambda_{\xi_0}}\chi(y),$$

and

$$\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)} = \|(T_{\Lambda_{\xi_0}} P(in|k|)T^*_{\Lambda_{\xi_0}})\varphi_n \circ \pi_{\Lambda_{\xi_0}}\|_{L^2(\mathbb{T}_{\Lambda^\perp_{\xi_0}} \times \mathbb{T}_{\Lambda_{\xi_0}})} = C_0 \|\Delta_{\Lambda_{\xi_0}} \chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}.$$

Since we also have $\|\varphi_n\|_{L^2(\mathbb{T}^2)} = \|T_{\Lambda_{\xi_0}}\varphi_n\|_{L^2(\mathbb{T}_{\Lambda^\perp_{\xi_0}} \times \mathbb{T}_{\Lambda_{\xi_0}})} = C_0\|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}$, we obtain, for all $n \in \mathbb{N}$,

$$\|P^{-1}(in|k|)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq \frac{\|\varphi_n\|_{L^2(\mathbb{T}^2)}}{\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)}} = \frac{\|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}}{\|\Delta_{\Lambda_{\xi_0}}\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}} = C > 0,$$

which concludes the proof of Theorem 2.5. □

Appendix A: Pseudodifferential calculus

In the main part of the article, we use the semiclassical Weyl quantization associating to a function a on $T^*\mathbb{R}^2$ an operator $\mathrm{Op}_h(a)$ defined by

$$(\mathrm{Op}_h(a)u)(x) := \frac{1}{(2\pi h)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{i}{h}\xi\cdot(x-y)} a\Big(\frac{x+y}{2}, \xi\Big)u(y)\,dy\,d\xi. \tag{A-1}$$

For smooth functions a with uniformly bounded derivatives, $\mathrm{Op}_h(a)$ defines a continuous operator on $\mathcal{S}(\mathbb{R}^2)$, and also by duality on $\mathcal{S}'(\mathbb{R}^2)$. On a manifold, the quantization Op_h may be defined by working in local coordinates with a partition of unity. On the torus, formula (A-1) still makes sense: taking $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ is equivalent to taking $a \in \mathscr{C}^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, $(2\pi\mathbb{Z})^2$-periodic with respect to the x-variable. Then the operator defined by (A-1) preserves the space of $(2\pi\mathbb{Z})^2$-periodic distributions on $\mathbb{R}^2$, and hence $\mathcal{D}'(\mathbb{T}^2)$.

We sometimes write, with $D := (1/i)\partial$,

$$a(x, hD) = \mathrm{Op}_h(a).$$

We also note that $\mathrm{Op}_1(a)$ is the classical Weyl quantization, and that we have the relation

$$a(x, hD) = \mathrm{Op}_h(a(x,\xi)) = \mathrm{Op}_1(a(x, h\xi)).$$

Theorem A.1. *There exists a constant $C > 0$ such that for any $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ with uniformly bounded derivatives, we have*

$$\|\mathrm{Op}_1(a)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha\in\{0,1\}^2, \beta\in\{0,1\}^2} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

Equivalently, this can be rewritten as

$$\|\mathrm{Op}_h(a)\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha\in\{0,1\}^2, \beta\in\{0,1\}^2} h^{|\beta|}\|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

This precised version of the Calderón–Vaillancourt theorem for the Weyl quantization is needed in Section 12, and proved in [Boulkhemair 1999, Theorem 1.2]. Here in dimension two, this means that only $|\alpha| = 2$ derivations are needed with respect to the space variable x.

Appendix B: Spectrum of $P(z)$ for a piecewise constant damping (by Stéphane Nonnenmacher)

In this appendix we provide an explicit description of some part of the *spectrum* of the damped wave equation (1-1) on $\mathbb{T}^2$, for a damping function proportional to the characteristic function of a vertical strip. We identify the torus $\mathbb{T}^2$ with the square $\{-1/2 \le x < 1/2,\ 0 \le y < 1\}$. We choose some half-width $\sigma \in (0, 1/2)$, and consider a vertical strip of width 2σ. Due to translation symmetry of $\mathbb{T}^2$, we may center this strip on the axis $\{x = 0\}$. Choosing a damping strength $\tilde{B} > 0$, we then get the damping function

$$b(x, y) = b(x) = \begin{cases} 0 & \text{for } |x| \le \sigma, \\ \tilde{B} & \text{for } \sigma < |x| \le 1/2. \end{cases} \tag{B-1}$$

The reason for centering the strip at $x = 0$ is the parity of the problem with respect to that axis, which greatly simplifies the computations.

We are interested in the spectrum of the operator $\mathcal{A}$ generating the evolution equation (1-1), which amounts (see Lemma 4.2) to solving the eigenvalue problem

$$P(z)u = 0 \quad \text{for } P(z) = -\Delta + zb(x) + z^2,\ z \in \mathbb{C},\ u \in L^2(\mathbb{T}^2),\ u \not\equiv 0.$$

This spectrum consists in a discrete set $\{z_j\}$, which is symmetric with respect to the horizontal axis: indeed, any solution (z, u) admits a "sister" solution $(\bar{z}, \bar{u})$. Furthermore, any solution with $\operatorname{Im} z \ne 0$ satisfies

$$\operatorname{Re} z = -\frac{1}{2}\frac{(u, bu)_{L^2(\mathbb{T}^2)}}{\|u\|^2_{L^2(\mathbb{T}^2)}}, \quad \text{and thus } -\tilde{B}/2 \le \operatorname{Re} z \le 0\,. \tag{B-2}$$

We may thus restrict ourselves to the half-strip $\{-\tilde{B}/2 \le \operatorname{Re} z \le 0, \operatorname{Im} z > 0\}$.

Our aim is to find high-frequency eigenvalues ($\operatorname{Im} z \gg 1$) which are as close as possible to the imaginary axis.

Proposition B.1. *There exists $C_0 > 0$ such that the spectrum (B-2) for the damping function (B-1) contains an infinite subsequence $\{z_i\}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \le C_0/(\operatorname{Im} z_i)^{3/2}$.*

The proof of the proposition will actually give an explicit value for C_0, as a function of $\tilde{B}$, σ.

Proof. To study the high-frequency limit $\operatorname{Im} z \to \infty$ we will change variables and take

$$z = i(1/h + \tilde{\zeta}),$$

where $h \in (0, 1]$ will be a small parameter, while $\tilde{\zeta} \in \mathbb{C}$ is assumed to be uniformly bounded when $h \to 0$. The eigenvalue equation then takes the form

$$(-h^2\Delta + ih(1 + h\tilde{\zeta})b)u = \big(1 + 2h\tilde{\zeta}(1 + h\tilde{\zeta}/2)\big)u\,.$$

Having chosen b independent of y, we may naturally Fourier transform along this direction, that is look for solutions of the form $u(x, y) = e^{2i\pi ny}v(x)$, $n \in \mathbb{Z}$. For each n, we now have to solve the 1-dimensional problem

$$\big(-h^2\partial^2/\partial_x^2 + ih(1 + h\tilde{\zeta})b(x)\big)v = \big(1 - (2\pi hn)^2 + 2h\tilde{\zeta}(1 + h\tilde{\zeta}/2)\big)v.$$

Let us call

$$B \stackrel{\text{def}}{=} \tilde{B}(1+h\tilde{\zeta}), \quad \zeta \stackrel{\text{def}}{=} \tilde{\zeta}(1+h\tilde{\zeta}/2).$$

In terms of these parameters, the above equation reads

$$(-h^2\partial^2/\partial_x^2 + ihB\mathbb{1}_{\{\sigma<|x|\le 1/2\}}(x))v = Ev, \quad \text{with } E = 1-(2\pi hn)^2 + 2h\zeta. \tag{B-3}$$

Since we will assume throughout that $\tilde{\zeta} = \mathbb{O}(1)$, we will have in the semiclassical limit

$$B = \tilde{B} + \mathbb{O}(h), \quad \tilde{\zeta} = \zeta(1 - h\zeta/2 + \mathbb{O}(h^2)). \tag{B-4}$$

At leading order we may forget that the variables B, ζ are not independent from one another, and consider (B-3) as a bona fide linear eigenvalue problem.

Since the function $b(x)$ is even, we may separately search for even (resp. odd) solutions $v(x)$. Let us start with the even solutions. Since $b(x)$ is piecewise constant, any even and periodic solution $v(x)$ takes the following form on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (up to a global normalization factor):

$$v(x) = \begin{cases} \cos(kx) & \text{for } |x| \le \sigma, \\ \beta \cos\left(k'(\frac{1}{2} - |x|)\right) & \text{for } \sigma < |x| \le \frac{1}{2}, \end{cases} \tag{B-5}$$

$$k = \frac{E^{1/2}}{h}, \quad k' = \frac{(E-ihB)^{1/2}}{h}. \tag{B-6}$$

We notice that k, k' are defined modulo a change of sign, so we may always assume that $\operatorname{Re} k \ge 0$, $\operatorname{Re} k' \ge 0$. The factor β is obtained by imposing the continuity of v and of its derivative v' at the discontinuity point $x = \sigma$ (we use the notation $\sigma' \stackrel{\text{def}}{=} \frac{1}{2} - \sigma$):

$$\cos(k\sigma) = \beta\cos(k'\sigma'), \quad -k\sin(k\sigma) = \beta k' \sin(k'\sigma').$$

The ratio of these two equations provides the quantization condition for the even solutions:

$$\tan(k\sigma) = -\frac{k'}{k}\tan(k'\sigma'). \tag{B-7}$$

Similarly, any odd eigenfunction takes the form (modulo a global normalization factor)

$$v(x) = \begin{cases} \sin(kx) & \text{for } |x| \le \sigma, \\ \beta \operatorname{sgn}(x)\sin(k'(\frac{1}{2} - |x|)) & \text{for } \sigma < |x| \le \frac{1}{2}, \end{cases} \tag{B-8}$$

so the associated eigenvalues should satisfy the condition

$$\tan(k\sigma) = -\frac{k}{k'}\tan(k'\sigma'). \tag{B-9}$$

We will now study the solutions of the quantization conditions (B-7) and (B-9), taking into account the relations (B-6) between the wavevectors k, k' and the energy E. To describe the full spectrum (which we plan to present in a separate publication), we would need to consider several régimes, depending on the relative scales of E and h. However, since we are only interested here in proving Proposition B.1, we will focus on the régime leading to the smallest possible values of $|\operatorname{Im}\tilde{\zeta}| = |\operatorname{Re} z|$. What characterizes the corresponding eigenmodes $v(x)$? From (B-2) we see that the mass of $v(x)$ in the damped region,

$2\int_\sigma^{1/2} |v(x)|^2\,dx$, should be small compared to its full mass. Intuitively, if such a mode were carrying a large horizontal "momentum" $\mathrm{Re}(hk)$ in the undamped region, it would then strongly penetrate the damped region, because the boundary at $x=\sigma$ is not reflecting. As a result, the mass in the damped region would be of the same order of magnitude as the one in the undamped one. This hand-waving argument explains why we choose to investigate the eigenmodes for which hk is the smallest possible, namely of order $\mathbb{O}(h)$. This implies that $E=(hk)^2=\mathbb{O}(h^2)$, which means that almost all of the energy is carried by the vertical momentum:

$$hn=(2\pi)^{-1}+\mathbb{O}(h).$$

The study of the full spectrum actually confirms that the smallest values of $\mathrm{Im}\,\tilde{\zeta}$ are obtained in this régime.

Equation (B-6) implies that the wavevector k' in the damped region is then much larger than k:

$$k'=\frac{(-ihB+(hk)^2)^{1/2}}{h}=e^{-i\pi/4}(B/h)^{1/2}+\mathbb{O}(h^{1/2}).$$

$\mathrm{Im}\,k'\sigma'\approx-\sigma'(B/2h)^{1/2}$ is negative and large, so that $\tan(k'\sigma')=-i+\mathbb{O}(e^{2\,\mathrm{Im}(k'\sigma')})$, uniformly with respect to $\mathrm{Re}(k'\sigma')$.

Even eigenmodes. In this situation the even quantization condition (B-7) reads

$$\tan(k\sigma)=i\frac{k'}{k}\big(1+\mathbb{O}(e^{-\sigma'(2B/h)^{1/2}})\big). \tag{B-10}$$

Since the right-hand side is large, $k\sigma$ must be close to a pole of the tangent function. Hence, for each integer m in a bounded interval[2] $0\le m\le M$ we look for a solution of the form

$$k_{m+1/2}=\frac{\pi(m+\frac{1}{2})}{\sigma}+\delta k_{m+1/2},\qquad \text{with } |\delta k_{m+1/2}|\ll 1.$$

The quantization condition (B-10) then reads

$$\sigma\,\delta k_{m+1/2}+\mathbb{O}((\delta k_{m+1/2})^2)=i\frac{k_{m+1/2}}{e^{-i\pi/4}(B/h)^{1/2}+\mathbb{O}(h^{1/2})}\big(1+\mathbb{O}(e^{-\sigma'(2B/h)^{1/2}})\big)$$
$$\Longrightarrow\quad k_{m+1/2}=\frac{\pi(m+\frac{1}{2})}{\sigma}\Big(1+h^{1/2}\frac{e^{i3\pi/4}}{\sigma B^{1/2}}+\mathbb{O}(h)\Big).$$

Using (B-3), the corresponding spectral parameter ζ is then given by

$$\begin{aligned}\zeta_{n,m+1/2}&=\frac{(hk_{m+1/2})^2+(2\pi hn)^2-1}{2h}\\&=\frac{(2\pi hn)^2-1}{2h}+\frac{h}{2}\Big(\frac{\pi(m+\frac{1}{2})}{\sigma}\Big)^2+h^{3/2}\Big(\frac{\pi(m+\frac{1}{2})}{\sigma}\Big)^2\frac{e^{i3\pi/4}}{\sigma B^{1/2}}+\mathbb{O}(h^2).\end{aligned}$$

From the assumptions on the quantum numbers n,m, we check that $\zeta_{n,m+1/2}=\mathbb{O}(1)$. We may now go back to the original variables $\tilde{\zeta}$, $\tilde{B}$, using the relations (B-4). The spectral parameter $\tilde{\zeta}$ has an imaginary

[2] Recall that we only need to study values $\mathrm{Re}\,k\ge 0$.

part

$$\operatorname{Im} \tilde{\zeta}_{n,m+1/2} = \operatorname{Im} \zeta_{n,m+1/2}(1 - h \operatorname{Re} \zeta_{n,m+1/2}) + \mathbb{O}(h^2) = h^{3/2} \frac{(\pi(m+\frac{1}{2}))^2}{\sigma^3 (2\tilde{B})^{1/2}} + \mathbb{O}(h^2). \tag{B-11}$$

Returning to the spectral variable z, the above expression gives a string of eigenvalues $\{z_{n,m+1/2}\}$ with $\operatorname{Im} z_{n,m+1/2} = h^{-1} + \mathbb{O}(1)$, $\operatorname{Re} z_{n,m+1/2} = -\operatorname{Im} \tilde{\zeta}_{n,m+1/2}$. These even-parity eigenvalues prove Proposition B.1, and one can take for C_0 any value greater than $(\pi/2)^2/(\sigma^3(2\tilde{B})^{1/2})$. □

We remark that the leading order of $k_{m+1/2}$ corresponds to the even spectrum of the operator $-h^2\partial^2/\partial_x^2$ on the undamped interval $[-\sigma, \sigma]$, with Dirichlet boundary conditions. The eigenmode $v_{n,m+1/2}$ associated with $\tilde{\zeta}_{n,m+1/2}$ is indeed essentially supported on that interval, where it resembles the Dirichlet eigenmode $\cos(x\pi(\frac{1}{2}+m)/\sigma)$. At the boundary of that interval, it takes the value

$$v_{n,m+1/2}(\sigma) = (-1)^{m+1} e^{i3\pi/4} h^{1/2} \frac{\pi(m+\frac{1}{2})}{\sigma \tilde{B}^{1/2}} + \mathbb{O}(h),$$

and decays exponentially fast inside the damping region, with a "penetration length" $(\operatorname{Im} k')^{-1} \approx (2h/\tilde{B})^{1/2}$. From (B-2) we see that the intensity $|v_{n,m+1/2}(\sigma)|^2 \sim C\,h$ penetrating on a distance $\sim h^{1/2}$ exactly accounts for the size $\sim h^{3/2} = hh^{1/2}$ of the $\operatorname{Re} z_{n,m+1/2}$.

We notice that the smallest damping occurs for the state $v_{n,1/2}$ resembling the ground state of the Dirichlet Laplacian.

Odd eigenmodes. For completeness we also investigate the odd-parity eigenmodes with $k = \mathbb{O}(1)$. The computations are very similar as in the even-parity case. The odd quantization condition reads in this régime

$$\tan(k\sigma) = i\frac{k}{k'}\big(1 + \mathbb{O}(e^{-(2B/h)^{1/2}})\big).$$

The right-hand side is then very small, showing σk is close to a zero of the tangent, so we may take $k_m = \pi m/\sigma + \delta k_m$ with $|\delta k_m| \ll 1$ and $0 \le m \le M$. We easily see that the case $m = 0$ does not lead to a solution. For the case $m > 0$ we get

$$\delta k_m = e^{3i\pi/4} h^{1/2} \frac{\pi m}{\sigma^2 B^{1/2}} + \mathbb{O}(h),$$

and thus

$$k_m = \frac{\pi m}{\sigma}\Big(1 + h^{1/2} \frac{e^{3i\pi/4}}{\sigma B^{1/2}} + \mathbb{O}(h)\Big), \qquad 1 \le m \le M.$$

These values k_m approximately sit on the same "line" $\{s(1 + h^{1/2}e^{3i\pi/4}/(\sigma B^{1/2})),\ s \in \mathbb{R}\}$ as the values $k_{m+1/2}$ corresponding to the even eigenmodes, both types of eigenvalues appearing successively. The corresponding energy parameter $\tilde{\zeta}_{n,m}$ satisfies

$$\operatorname{Im} \tilde{\zeta}_{n,m} = h^{3/2} \frac{(\pi m)^2}{\sigma^3(2\tilde{B})^{1/2}} + \mathbb{O}(h^2). \tag{B-12}$$

As in the even parity case, the eigenmodes $v_{n,m}$ are close to the odd eigenmodes $\sin(x\pi m/\sigma)$ of the semiclassical Dirichlet Laplacian on $[-\sigma, \sigma]$, and penetrate on a length $\sim h^{1/2}$ inside the damped region.

The case of the square. If the torus is replaced by the square $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1]$ with Dirichlet boundary conditions, with the same damping function (B-1), the eigenmodes $P(z)$ can as well be factorized into $u(x, y) = \sin(2\pi n y) v(x)$, with $n \in \frac{1}{2}\mathbb{N} \setminus 0$, and $v(x)$ must be an eigenmode of the operator (B-3) vanishing at $x = \pm\frac{1}{2}$. We notice that the odd-parity eigenstates (B-8) satisfy these boundary conditions, so the eigenvalues $z_{n,m}$ (with real parts given by (B-12)) belong to the spectrum of the damped Dirichlet problem.

Similarly, the eigenmodes factorize as $u(x, y) = \cos(2\pi n y) v(x)$, with $n \in \frac{1}{2}\mathbb{N}$, in the case of Neumann boundary conditions. The even-parity states (B-5) satisfy the Neumann boundary conditions at $x = \pm 1/2$, so that the eigenvalues $z_{n,m+1/2}$ described in (B-11) belong to the Neumann spectrum.

As a result, the Dirichlet and Neumann spectra also satisfy Proposition B.1.

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References

[Anantharaman and Léautaud 2012] N. Anantharaman and M. Léautaud, "Some decay properties for the damped wave equation on the torus", exposé no. 6 in *Journées "Équations aux Dérivées Partielles"* (Biarritz, 2012), Cedram, 2012.

[Anantharaman and Macià 2010] N. Anantharaman and F. Macià, "Semiclassical measures for the Schrödinger equation on the torus", 2010. to appear in *J. Europ. Math. Soc.* arXiv 1005.0296

[Anantharaman and Rivière 2012] N. Anantharaman and G. Rivière, "Dispersion and controllability for the Schrödinger equation on negatively curved manifolds", *Anal. PDE* **5**:2 (2012), 313–338. MR 2970709 Zbl 1267.35176

[Asch and Lebeau 2003] M. Asch and G. Lebeau, "The spectrum of the damped wave operator for a bounded domain in $\mathbf{R}^2$", *Experiment. Math.* **12**:2 (2003), 227–241. MR 2016708 Zbl 1061.35064

[Bardos et al. 1992] C. Bardos, G. Lebeau, and J. Rauch, "Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary", *SIAM J. Control Optim.* **30**:5 (1992), 1024–1065. MR 94b:93067 Zbl 0786.93009

[Batty and Duyckaerts 2008] C. J. K. Batty and T. Duyckaerts, "Non-uniform stability for bounded semi-groups on Banach spaces", *J. Evol. Equ.* **8**:4 (2008), 765–780. MR 2009m:47104 Zbl 1185.47043

[Borichev and Tomilov 2010] A. Borichev and Y. Tomilov, "Optimal polynomial decay of functions and operator semigroups", *Math. Ann.* **347**:2 (2010), 455–478. MR 2011c:47091 Zbl 1185.47044

[Boulkhemair 1999] A. Boulkhemair, "L^2 estimates for Weyl quantization", *J. Funct. Anal.* **165**:1 (1999), 173–204. MR 2001f:35454 Zbl 0934.35217

[Burq 1998] N. Burq, "Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel", *Acta Math.* **180**:1 (1998), 1–29. MR 99j:35119 Zbl 0918.35081

[Burq and Gérard 1997] N. Burq and P. Gérard, "Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes", *C. R. Acad. Sci. Paris Sér. I Math.* **325**:7 (1997), 749–752. MR 98j:93052 Zbl 0906.93008

[Burq and Hitrik 2007] N. Burq and M. Hitrik, "Energy decay for damped wave equations on partially rectangular domains", *Math. Res. Lett.* **14**:1 (2007), 35–47. MR 2008a:35180 Zbl 1122.35015

[Burq and Zworski 2004] N. Burq and M. Zworski, "Geometric control in the presence of a black box", *J. Amer. Math. Soc.* **17**:2 (2004), 443–471. MR 2005d:47085 Zbl 1050.35058

[Burq and Zworski 2012] N. Burq and M. Zworski, "Control for Schrödinger operators on tori", *Math. Res. Lett.* **19**:2 (2012), 309–324. MR 2955763

[Christianson 2007] H. Christianson, "Semiclassical non-concentration near hyperbolic orbits", *J. Funct. Anal.* **246**:2 (2007), 145–195. MR 2008k:58058 Zbl 1119.58018

[Dimassi and Sjöstrand 1999] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, London Mathematical Society Lecture Note Series **268**, Cambridge University Press, 1999. MR 2001b:35237 Zbl 0926.35002

[Fermanian-Kammerer 2000a] C. Fermanian-Kammerer, "Mesures semi-classiques 2-microlocales", *C. R. Acad. Sci. Paris Sér. I Math.* **331**:7 (2000), 515–518. MR 2001h:58028 Zbl 0964.35009

[Fermanian Kammerer 2000b] C. Fermanian Kammerer, "Propagation and absorption of concentration effects near shock hypersurfaces for the heat equation", *Asymptot. Anal.* **24**:2 (2000), 107–141. MR 2001i:35128 Zbl 0979.35058

[Fermanian Kammerer 2005] C. Fermanian Kammerer, "Analyse à deux échelles d'une suite bornée de L^2 sur une sous-variété du cotangent", *C. R. Math. Acad. Sci. Paris* **340**:4 (2005), 269–274. MR 2005i:28008 Zbl 1067.35083

[Fermanian-Kammerer and Gérard 2002] C. Fermanian-Kammerer and P. Gérard, "Mesures semi-classiques et croisement de modes", *Bull. Soc. Math. France* **130**:1 (2002), 123–168. MR 2003i:81074 Zbl 0996.35004

[Gérard 1991] P. Gérard, "Microlocal defect measures", *Comm. Partial Differential Equations* **16**:11 (1991), 1761–1794. MR 92k:35027 Zbl 0770.35001

[Gérard and Leichtnam 1993] P. Gérard and É. Leichtnam, "Ergodic properties of eigenfunctions for the Dirichlet problem", *Duke Math. J.* **71**:2 (1993), 559–607. MR 94i:35146 Zbl 0788.35103

[Haraux 1989a] A. Haraux, "Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire", *J. Math. Pures Appl.* (9) **68**:4 (1989), 457–465. MR 91e:93039 Zbl 0685.93039

[Haraux 1989b] A. Haraux, "Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps", *Portugal. Math.* **46**:3 (1989), 245–258. MR 90k:93037 Zbl 0679.93063

[Huang 1985] F. L. Huang, "Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces", *Ann. Differential Equations* **1**:1 (1985), 43–56. MR 87e:34106 Zbl 0593.34048

[Jaffard 1990] S. Jaffard, "Contrôle interne exact des vibrations d'une plaque rectangulaire", *Portugal. Math.* **47**:4 (1990), 423–429. MR 91j:93051 Zbl 0718.49026

[Komornik 1992] V. Komornik, "On the exact internal controllability of a Petrowsky system", *J. Math. Pures Appl.* (9) **71**:4 (1992), 331–342. MR 93j:93019 Zbl 0889.34055

[Lebeau 1992] G. Lebeau, "Contrôle de l'équation de Schrödinger", *J. Math. Pures Appl.* (9) **71**:3 (1992), 267–291. MR 93i:35018 Zbl 0838.35013

[Lebeau 1996] G. Lebeau, "Équation des ondes amorties", pp. 73–109 in *Algebraic and geometric methods in mathematical physics* (Kaciveli, 1993), edited by A. B. de Monvel and V. Marchenko, Math. Phys. Stud. **19**, Kluwer Acad. Publ., Dordrecht, 1996. MR 97i:58173 Zbl 0863.58068

[Lebeau and Robbiano 1997] G. Lebeau and L. Robbiano, "Stabilisation de l'équation des ondes par le bord", *Duke Math. J.* **86**:3 (1997), 465–491. MR 98c:35104 Zbl 0884.58093

[Lerner 2010] N. Lerner, *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications **3**, Birkhäuser, Basel, 2010. MR 2011b:35002 Zbl 1186.47001

[Liu and Rao 2005] Z. Liu and B. Rao, "Characterization of polynomial decay rate for the solution of linear evolution equation", *Z. Angew. Math. Phys.* **56**:4 (2005), 630–644. MR 2006g:34133 Zbl 1100.47036

[Macià 2010] F. Macià, "High-frequency propagation for the Schrödinger equation on the torus", *J. Funct. Anal.* **258**:3 (2010), 933–955. MR 2011b:81143 Zbl 1180.35438

[Miller 1996] L. Miller, *Propagation d'ondes semiclassiques à travers une interface et mesures 2-microlocales*, Ph.D. thesis, École Polytechnique, Palaiseau, 1996.

[Miller 1997] L. Miller, "Short waves through thin interfaces and 2-microlocal measures", pp. exposé no. 12 in *Journées "Équations aux Dérivées Partielles"* (Saint-Jean-de-Monts, 1997), École Polytech., Palaiseau, 1997. MR 1482278 Zbl 01808670

[Miller 2005] L. Miller, "Controllability cost of conservative systems: resolvent condition and transmutation", *J. Funct. Anal.* **218**:2 (2005), 425–444. MR 2005i:93015 Zbl 1122.93011

[Nier 1996] F. Nier, "A semi-classical picture of quantum scattering", *Ann. Sci. École Norm. Sup.* (4) **29**:2 (1996), 149–183. MR 97a:81044 Zbl 0858.35106

[Nishiyama 2009] H. Nishiyama, "Polynomial decay for damped wave equations on partially rectangular domains", *Math. Res. Lett.* **16**:5 (2009), 881–894. MR 2011a:35308 Zbl 1196.35053

[Pazy 1983] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer, New York, 1983. MR 85g:47061 Zbl 0516.47023

[Phung 2007] K. D. Phung, "Polynomial decay rate for the dissipative wave equation", *J. Differential Equations* **240**:1 (2007), 92–124. MR 2008k:35297 Zbl 1130.35017

[Ramdani et al. 2005] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak, "A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator", *J. Funct. Anal.* **226**:1 (2005), 193–229. MR 2006g:93017 Zbl 1140.93395

[Rauch and Taylor 1974] J. Rauch and M. Taylor, "Exponential decay of solutions to hyperbolic equations in bounded domains", *Indiana Univ. Math. J.* **24** (1974), 79–86. MR 50 #13906 Zbl 0281.35012

[Reed and Simon 1980] M. Reed and B. Simon, *Methods of modern mathematical physics, I: Functional analysis*, 2nd ed., Academic Press, New York, 1980. MR 85e:46002 Zbl 0459.46001

[Schenck 2011] E. Schenck, "Exponential stabilization without geometric control", *Math. Res. Lett.* **18**:2 (2011), 379–388. MR 2012c:58050 Zbl 1244.93144

[Sjöstrand 1995] J. Sjöstrand, "Wiener type algebras of pseudodifferential operators", exposé no. 4 in *Séminaire "Équations aux Dérivées Partielles", 1994–1995*, École Polytech., Palaiseau, 1995. MR 96j:47049 Zbl 0880.35145

[Tucsnak and Weiss 2009] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser, Basel, 2009. MR 2010d:93001 Zbl 1188.93002

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