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THE HEAT KERNEL ON AN ASYMPTOTICALLY CONIC MANIFOLD

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We investigate the long-time structure of the heat kernel on a Riemannian manifold M that is asymptotically conic near infinity. Using geometric microlocal analysis and building on results of Guillarmou and Hassell, we give a complete description of the asymptotic structure of the heat kernel in all spatial and temporal regimes. We apply this structure to define and investigate a renormalized zeta function and determinant of the Laplacian on M.

1. Introduction

We study the heat kernel on asymptotically conic manifolds. Asymptotically conic manifolds should be thought of as those complete manifolds which are approximately conic near infinity. More specifically:

Definition [Guillarmou and Hassell 2008]. Let (M, g) be a complete Riemannian manifold without boundary of dimension n, and let $\overline{M}$ be the usual radial compactification of M. Let (N, h_0) be a closed Riemannian manifold of dimension $n - 1$. We say that (M, g) is asymptotically conic with cross-section (N, h_0) if in a neighborhood of $\partial \overline{M}$, $\overline{M}$ is isometric to $[0, \delta)_x \times N_y$ with the metric

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}. \tag{1}$$

Here x is a smooth function on $\overline{M}$ with $x = 0$ and $dx \neq 0$ on $\partial \overline{M}$ (we call this a *boundary defining function* for $\partial \overline{M}$) and a smooth family of metrics $h(x)$ on N with $h(0) = h_0$. Throughout, we let z be a global coordinate on M, writing $z = (x, y)$ in a neighborhood of the boundary of $\overline{M}$.

In particular, Euclidean space $\mathbb{R}^n$ is asymptotically conic with cross-section S^{n-1}; we may choose $x = r^{-1}$. Any complete manifold which is exactly Euclidean or conic near infinity is, of course, also asymptotically conic. The condition (1) may be weakened by replacing $h(x)$ with any symmetric 2-tensor $h'(x, y)$ which restricts to a metric $h_0(x)$ on the boundary at $x = 0$; an observation of Melrose and Wunsch [2004] shows that these conditions are in fact equivalent.

Asymptotically conic manifolds are a relatively well-behaved class of manifolds, and as such the theory of the heat equation is relatively advanced. In particular, it is easy to see from (1) that all sectional curvatures of (M, g) approach zero as x goes to zero, and thus that (M, g) has bounded sectional curvature. For complete manifolds of bounded sectional curvature, a classical theorem of Cheng, Li, and Yau gives the following Gaussian upper bound for the heat kernel:

Theorem 1 [Cheng et al. 1981]. *There are nonzero constants C_1 and C_2 such that the heat kernel on M,
denoted $H^M(t, z, z')$, satisfies*

$$H^M(t, z, z') \leq \frac{C_1}{t^{n/2}} e^{-|z-z'|^2/C_2 t}. \tag{2}$$

However, for many applications to spectral theory, one needs finer information about the structure of
the heat kernel. The example we have in mind is the definition of the zeta function. Recall that if M is
compact, the zeta function is defined for $\Re s > n/2$ by

$$\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{Tr} H^M(t) - 1) \, t^{s-1} \, dt. \tag{3}$$

The zeta function has a well-known meromorphic continuation to all of $\mathbb{C}$ with a regular value at $s = 0$;
the key is that the trace of the heat kernel has a short-time asymptotic expansion, which, along with the
long-time exponential decay, enables us to write down an explicit meromorphic continuation [Rosenberg
1997]. The *determinant of the Laplacian* is then given by $\exp(-\zeta_M'(0))$; the determinant plays a key role
in many problems in spectral theory, including the isospectral compactness results of Osgood, Phillips,
and Sarnak [1988b; 1988a; 1989].

We would like to define such a zeta function and determinant when M is asymptotically conic, with
an eye towards applying these concepts to the spectral and scattering theory of asymptotically conic
manifolds. There are several obstacles. First, the heat kernel is no longer trace class, so $\operatorname{Tr} H^M(t)$ does
not make sense. Instead, we define the *renormalized heat trace* $^R\operatorname{Tr} H^M(t)$ to be the finite part at $\delta = 0$
of the divergent asymptotic expansion in δ of

$$\int_{x \geq \delta} H^M(t, z, z) \, dz. \tag{4}$$

Details, including the existence of this divergent asymptotic expansion, may be found in Section 3. We
then formally define the *renormalized zeta function*:

$$^R\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty {}^R\operatorname{Tr} H^M(t) \, t^{s-1} \, dt. \tag{5}$$

However, to make sense of this definition and obtain a meromorphic continuation, we still need to
understand the behavior of the renormalized trace — and hence of the heat kernel itself — as $t \to 0$ and
$t \to \infty$. In particular, we need asymptotics in both the short and long time regimes.

The short-time behavior of the heat kernel on an asymptotically conic manifold is relatively well-
understood. Short-time heat kernels may be analyzed using techniques from semiclassical analysis. In
this approach, the goal is to develop a "semiclassical functional calculus" containing the heat kernel,
modeled on standard semiclassical techniques as developed, for example, in [Dimassi and Sjöstrand
1999]. The key functional calculus for this purpose, at least in the asymptotically Euclidean setting, is the
Weyl calculus of Hörmander [1979].

An alternate approach, and one more suited to analysis of the renormalized trace and determinant, is
to use geometric microlocal analysis to first construct the heat kernel and then analyze its fine structure.
The techniques of geometric microlocal analysis were first developed by Melrose and Mendoza [1983]

to study elliptic PDE on manifolds with asymptotically cylindrical ends. They have been extended by many other mathematicians and play a key role in the modern analysis of linear PDE on singular and non-compact spaces. In particular, Melrose [1994] discusses some aspects of spectral and scattering theory on asymptotically conic manifolds. Albin [2007] uses these methods to investigate the short-time heat kernel on a variety of complete spaces, including asymptotically conic manifolds. His work can be used to obtain the fine structure that we need for the short-time heat kernel.

The long-time problem is trickier: in the asymptotically conic setting, we no longer have exponential decay of the heat kernel as $t \to \infty$. Indeed, from the structure of the Euclidean heat kernel and (2), we expect that the leading-order behavior of $H^M(t, z, z')$ as $t \to \infty$ will be $Ct^{-n/2}$, and the leading-order behavior of the renormalized heat trace may be even worse. This lack of decay means that the zeta function may not be well-defined a priori for any s. We may split (5) into two integrals by breaking it up at $t = 1$, but there is no obvious reason for the integral from $t = 1$ to ∞ to have a meromorphic continuation to all of $\mathbb{C}$. In order to obtain such a meromorphic continuation, we need an asymptotic expansion for the heat kernel as $t \to \infty$. Moreover, we must understand how this expansion interacts with the heat trace renormalization.

1.1. *Main results.* We solve this problem by using the methods of geometric microlocal analysis to obtain a complete description of the asymptotic structure of the heat kernel on M in all spatial and temporal regimes. The key concepts, including blow-ups and polyhomogeneous conormal functions, were originally introduced by Melrose [1993; 1996], and a good introduction may be found in [Grieser 2001]. In Section 2, we discuss these concepts briefly and then use them to define a new blown-up manifold with corners which we call $M^2_{w,sc}$. The space $M^2_{w,sc}$ was originally defined by Guillarmou and Hassell [2008], and we use their labeling of the boundary hypersurfaces; see Section 2 for the definitions. Our main theorem is the following:

Theorem 2. *Let M be asymptotically conic. For any $n \geq 2$, and for any fixed time $T > 0$, the heat kernel on M is polyhomogeneous conormal on $M^2_{w,sc}$ for $t > T$, where $w = t^{-1/2}$. The leading orders at the boundary hypersurfaces are at least 0 at sc and n at each of bf_0, rb_0, lb_0, and zf, with infinite-order decay at lb, rb, and bf.*

This theorem gives a complete description of the asymptotic structure of the heat kernel for long time; previously, only estimates such as Theorem 1 were known. The analogous structure for the short-time heat kernel is well-understood (see Section 2 for the definition of M^2_{sc}):

Theorem 3. *For $t < 1$, the heat kernel $H^M(t, z, z')$ is polyhomogeneous conormal on*

$$\left[M^2_{sc}(z, z') \times [0, 1]_{\sqrt{t}} \, ; \, \{\sqrt{t} = 0, z = z'\} \right].$$

Moreover, there is infinite-order decay at all faces except the scattering front face sc *and the face F obtained by the final blow-up.*

Theorem 3 follows immediately from the work of Albin [2007]; however, the precise statement above does not appear in the literature. Therefore, in the Appendix, we give a simple proof using the machinery
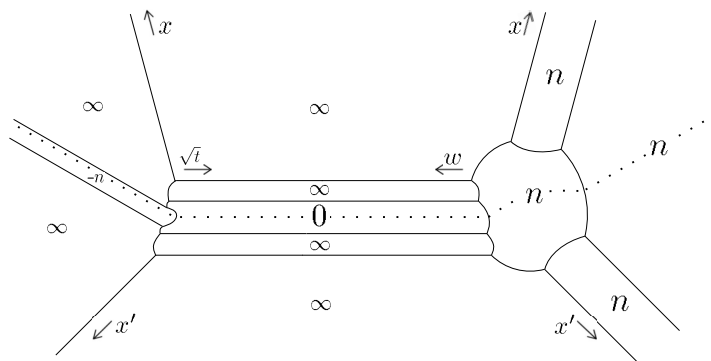
Figure 1. Asymptotic structure of the heat kernel on M.

developed in [Albin 2007]. Combining Theorem 2 with Theorem 3 gives a complete geometric-microlocal description of the structure of the heat kernel. This structure is illustrated in Figure 1; the short-time structure is the left-hand side of the diagram, and the long-time structure is the right-hand side. We also indicate the leading order of the heat kernel at each of the boundary hypersurfaces, in terms of $\sqrt{t}$ at $t = 0$, $w = t^{-1/2}$ at $t = \infty$, and x or x' respectively at all the finite-time boundaries.

As an example of how polyhomogeneous structure may be used to read off the behavior in all asymptotic regimes, let a be a parameter, and fix (y, x', y'); consider the heat kernel $H^Z(a^2, a^{-1}, y, x', y')$ as a approaches infinity. In the compactified space in Figure 1, as a approaches infinity, the arguments approach a point in the center of the face lb_0, where the leading order of the heat kernel is n. Since $w = a^{-1}$, and w is a boundary defining function for lb_0, we conclude that $H^Z(a^2, a^{-1}, y, x', y')$ has a polyhomogeneous asymptotic expansion in a^{-1} as $a \to \infty$, with leading term $C_n a^{-n}$. A similar analysis may be performed in any asymptotic regime.

As an application, Theorems 2 and 3 give us precisely the polyhomogeneous structure we need to define and investigate the renormalized zeta function on M:

Theorem 4. *Let M be asymptotically conic. The renormalized zeta function, defined formally by (5), is well-defined and has a meromorphic continuation to all of $\mathbb{C}$.*

We may then define the *renormalized determinant* of the positive Laplacian Δ_M on M by

$$\log {}^R\!\det \Delta_M = -{}^R\zeta'_M(0),$$

where ${}^R\zeta'_M(0)$ is the coefficient of s in the Laurent series for ${}^R\zeta_M(s)$ around $s = 0$.

In a companion paper [Sher 2012a], we use Theorems 2, 3, and 4 to analyze the behavior of the determinant of the Laplacian on a family of manifolds degenerating to a manifold with conical singularities. We expect that this work will have applications to spectral theory and to index theory on singular spaces, including the study of the Cheeger–Müller theorem on manifolds with conical singularities. The key theorem from [Sher 2012a] is as follows: let Ω_0 be a manifold with an exact conic singularity (with arbitrary base) and let Z be a manifold conic near infinity with the same base. For each $\epsilon > 0$, we define

a smooth manifold Ω_ϵ replacing the tip of Ω_0 with an ϵ-scaled copy of Z; as $\epsilon \to 0$, the manifolds Ω_ϵ converge to Ω_0 in the Gromov–Hausdorff sense. Then it is proven in [Sher 2012a] that

Theorem 5. *As $\epsilon \to 0$,*

$$\log \det \Delta_{\Omega_\epsilon} = -2 \log \epsilon \, ({}^R\zeta_Z(0)) + \log \det \Delta_{\Omega_0} + \log \, {}^R\!\det \Delta_Z + o(1).$$

1.2. *Outline of the proofs.* The usual geometric-microlocal approach to the fine structure of the heat kernel is a direct parametrix construction, which involves the construction of an initial approximation to the heat kernel and then the removal of the error via a Neumann series argument. This is the method adopted in [Albin 2007]. However, parametrix constructions are not well-suited for analysis of the long-time heat kernel; the problem is global rather than local. In order to obtain the asymptotic structure of the heat kernel at long time, we instead take an indirect approach. Recall that the functional calculus shows that the heat kernel and the resolvent are related by

$$H^M(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\Delta_M + \lambda)^{-1} \, d\lambda, \tag{6}$$

where Γ is a contour around the spectrum. Guillarmou and Hassell [2008; 2009] have analyzed the asymptotic structure of the resolvent $(\Delta_M + \lambda)^{-1}$ at low energy, again giving a complete description in all regimes. They have shown:

Theorem 6 [Guillarmou and Hassell 2008]. *Suppose that M is an asymptotically conic manifold of dimension $n \geq 3$. Then the Schwartz kernel of $(\Delta_M + e^{i\theta}k^2)^{-1}$ is polyhomogeneous conormal on $M^2_{k,\mathrm{sc}}$ for each $\theta \in (-\pi, \pi)$, with a conormal singularity at the spatial diagonal and all coefficients smoothly depending on θ. It decays to infinite order at the faces lb, rb, and bf, with leading orders at sc, bf$_0$, rb$_0$, lb$_0$ and zf given by 0, $n-2$, $n-2$, $n-2$, and 0 respectively.*

Note that Guillarmou and Hassell require $n \geq 3$. In Section 4, we adapt their methods to extend Theorem 6 to the two-dimensional case:

Theorem 7. Theorem 6 *also holds when $n = 2$; all the leading orders are the same, except that we have logarithmic growth instead of order 0 at* zf.

In Section 2, we use geometric microlocal analysis, in particular Melrose's pushforward theorem, to prove Theorem 2. The key is to push the structure of Theorems 6 and 7 through the contour integral (6). To state the main technical theorem, we must first compactify $M^2_{k,\mathrm{sc}}$ to $\overline{M}^2_{k,\mathrm{sc}}$ by introducing a new boundary face at $k = \infty$, with boundary defining function k^{-1}. In a neighborhood of the new face, which we call tf, $\overline{M}^2_{k,\mathrm{sc}}$ is $M^2_{\mathrm{sc}} \times [0, 1)_{k^{-1}}$. The main technical theorem is this:

Theorem 8. *Let M be an asymptotically conic manifold, and let E be a vector bundle over M. Let $A : C^\infty(E) \to C^{-\infty}(E)$ be a pseudodifferential operator with the following properties*:

(a) $\sigma(A) \subset [0, \infty]$.

(b) (*Low-energy resolvent behavior*) *If k is bounded above, the Schwartz kernel of the resolvent $(A + e^{i\theta}k^2)^{-1}$ is polyhomogeneous conormal on $M^2_{k,\mathrm{sc}}$ for each $\theta \in (-\pi, \pi)$, with a conormal*

singularity at the spatial diagonal and all coefficients smoothly depending on θ. Moreover, it decays to infinite order at the faces lb, rb, *and* bf, *with index sets at* sc, bf_0, rb_0, lb_0, *and* zf *given by* R_{sc}, R_{bf_0}, R_{rb_0}, R_{lb_0}, *and* R_{zf} *respectively.*

(c) (*High-energy resolvent behavior*) *For each* $θ ∈ (−π, π)$ *and for k bounded below, the Schwarz kernel of* $(A + e^{iθ}k^2)^{-1}$ *is phg conformal on* $\overline{M}^2_{k,sc}$, *with infinite-order decay at* lb, rb, *and* bf, *index set* R_{sc} *at* sc, *and index set* R_{tf} *at* tf.

Then for t greater than any fixed $T > 0$, *the kernel of* e^{-tA} *is polyhomogeneous conformal on* $M^2_{w,sc}$, *where* $w = t^{-1/2}$. *It decays to infinite order at* lb, rb, *and* bf, *and has index sets at* sc, bf_0, rb_0, lb_0, *and* zf *which are subsets of* R_{sc}, $R_{bf_0} + 2$, $R_{rb_0} + 2$, $R_{lb_0} + 2$, *and* $R_{zf} + 2$ *respectively.*

Once we have proven Theorem 8, Theorem 2 is an almost immediate consequence, though there is a slight twist involving the leading orders.

In Section 3, we use Theorem 2 and some additional geometric microlocal techniques to analyze the renormalized heat trace and prove Theorem 4. We also analyze the renormalized zeta function and determinant in the special case where M is exactly conic (or Euclidean) outside a compact set. In Section 4, we extend the methods of [Guillarmou and Hassell 2008] to prove Theorem 7. Finally, in the Appendix, we use the framework in [Albin 2007] to prove Theorem 3.

2. From resolvent to heat kernel

The goal of this section is to prove Theorems 8 and 2.

2.1. *Preliminaries.* We first give a brief summary of the key relevant concepts in geometric microlocal analysis; again, a self-contained introduction may be found in [Grieser 2001]. A *manifold with corners* of dimension n is a topological space which is locally modeled on $\mathbb{R}^k_+ × \mathbb{R}^{n-k}$ for some k; a simple example is the n-dimensional unit cube. *Blow-up* is a way of creating new manifolds with corners from old ones, and is used to resolve certain geometric singularities. The idea is to formally introduce polar coordinates around a submanifold of a manifold with corners, in order to distinguish between directions of approach to that submanifold. For example, consider the origin as a submanifold of $\mathbb{R}^2_+$. To blow up the origin, we introduce polar coordinates $(r, θ)$, which corresponds to replacing the point $(0, 0)$ with a quarter-circle, which corresponds to the inward-pointing spherical normal bundle of $(0, 0) ⊂ \mathbb{R}^2_+$. See [Melrose 1993; 1996; Grieser 2001], or the appendix of [Sher 2012b] for a more detailed explanation and more general examples of blow-ups.

By Taylor's theorem, smooth functions on a manifold with corners are precisely those functions which have Taylor expansions at each boundary hypersurface and joint Taylor expansions at every corner. *Polyhomogeneous conformal distributions*, which we abbreviate as phg or phg conformal, are a generalization of smooth functions. In particular, if we let x be a boundary defining function, we allow terms of the form $x^s (\log x)^p$ for any $x ∈ \mathbb{C}$ and any $p ∈ \mathbb{N}_0$ to appear in the asymptotic expansions at the boundary and in the joint expansions at the corners. The *index set* of a phg conformal distribution u at a particular boundary hypersurface H is simply the set of (s, p) which appear in the asymptotic expansion of u at H.

Polyhomogeneous conormal functions are well-behaved under addition and multiplication, but also under more complicated operations, namely pullback and pushforward. To discuss these, we first need to discuss properties of a map $f : W \to Z$ between manifolds with corners. Roughly, we say that f is a *b-map* if it is smooth up to the boundary and product-type near the boundary in terms of the local coordinate models (see [Grieser 2001] for a precise definition). If additionally f does not map any boundary hypersurface of W into a corner of Z, and f is also a fibration over the interior of every boundary hypersurface, we call f a *b-fibration*. Two results of Melrose will be critical in the analysis to follow:

Proposition 9 (Melrose's [1992] pullback and pushforward theorems). *Let $f : W \to Z$ be a smooth map of manifolds with corners.*

(a) *If f is a b-map and u is phg conormal on Z, then f^*u is phg conormal on W. Moreover, the index sets of f^*u may be computed explicitly from those of u and the geometry of the map f.*

(b) *If f is also a b-fibration, v is phg conormal on W, and f_*v is well-defined (the pushforward is integration along the fibers, which may not converge), then f_*v is phg conormal on Z, and again the index sets may be computed explicitly.*

Finally, we need to consider distributions which have pseudodifferential-type *conormal singularities* at submanifolds in the interior of a manifold with corners.

Definition [Grieser 2001]. Let $y = (x_1, \ldots, x_k)$ and $z = (x_{k+1}, \ldots, x_n)$, and let N be the set $\{z = 0\}$ in $\mathbb{R}^n_{y,z}$. A distribution u on $\mathbb{R}^n$ has a conormal singularity at N of order m if it can be written

$$u(y, z) = \int_{\mathbb{R}^{n-k}} e^{iz \cdot \xi} a(y, \xi) \, d\xi,$$

where a is a classical symbol; that is, a has asymptotics as $|\xi| \to \infty$

$$a(y, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}\left(y, \frac{\xi}{|\xi|}\right) |\xi|^{m-j},$$

with each coefficient a_{m-j} smooth in y and $\xi/|\xi|$.

This definition may be extended, by using the local coordinate models, to define distributions with a conormal singularity at any *p-submanifold* of a manifold with corners; a p-submanifold is a subset which, in each local coordinate chart, may be identified with a coordinate submanifold. Variants of the pullback and pushforward theorems also hold for polyhomogeneous conormal distributions with interior conormal singularities [Melrose 1996; Epstein et al. 1991].

2.2. The space $M^2_{k,\mathrm{sc}}$. We now introduce the space $M^2_{k,\mathrm{sc}}$, which appears in [Guillarmou and Hassell 2008; 2009; Guillarmou et al. 2012] and was first proposed in an unpublished note of Melrose and Sá Barreto. To construct $M^2_{k,\mathrm{sc}}$, we begin with the space $M^2_k = [0, \infty)_k \times \overline{M} \times \overline{M}$; coordinates on this space near $[0, \infty)_k \times \partial \overline{M} \times \partial \overline{M}$ are (k, x, y, x', y'). There are three boundary hypersurfaces: $\{k = 0\}$, which we call zf, $\{x = 0\}$, which we call lb, and $\{x' = 0\}$, which we call rb.
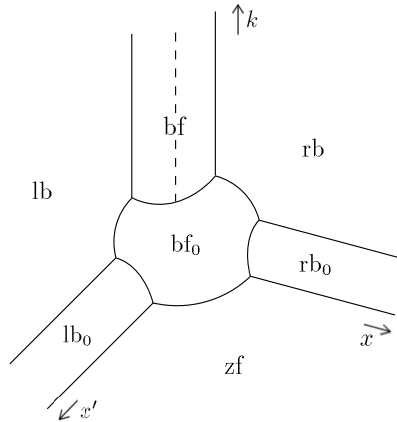
Figure 2. The space $M_{k,b}^2$.

First we blow up the corner $\{x = 0, x' = 0, k = 0\}$, which corresponds to the introduction of polar coordinates near that corner; we call the front face of this blow-up bf_0. We then blow up three codimension-2 submanifolds: we blow up $\{x = 0, x' = 0\}$ and call the new face bf, we blow up $\{x = 0, k = 0\}$ and call the resulting face lb_0, and we blow up $\{x' = 0, k = 0\}$ and call the resulting face rb_0. The resulting manifold with corners, which we call $M_{k,b}^2$ (as in [Guillarmou et al. 2012]), is shown in Figure 2, with y and y' suppressed. Using the definition of a "b-stretched product" from [Melrose and Singer 2008], we can identify this manifold near $x = 0$, $x' = 0$ with $X_b^3(x, x', k) \times N_y \times N_{y'}$. We use these b-stretched products X_b^n from [Melrose and Singer 2008] throughout the arguments.

Finally, consider the intersection of the closure of the interior spatial diagonal with the face bf. In coordinates near the boundary, this is $\{x/x' = 1, y = y', x/k = 0\}$, and it is marked with a dotted line in Figure 2. We blow this up to create a new boundary hypersurface, which we call sc (for "scattering"). The resulting space is $M_{k,sc}^2$, and it has eight boundary hypersurfaces, illustrated in Figure 3. The spatial diagonal $D_{k,sc}$ is defined to be the closure in $M_{k,sc}^2$ of the interior spatial diagonal; its intersection with the boundary is marked with a dotted line in Figure 3.
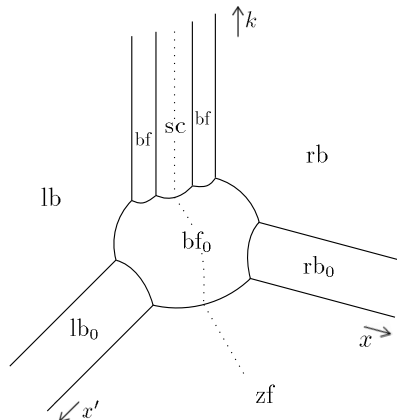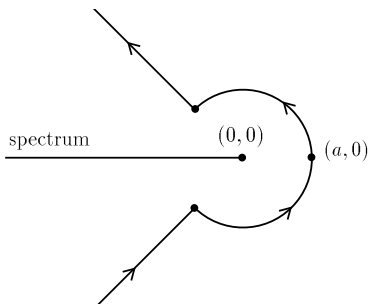

Figure 3. The space $M_{k,sc}^2$.

Figure 4. Contour of integration Γ_a.

We now describe some useful coordinate systems on $M^2_{k,\mathrm{sc}}$. Near the intersection of zf, rb$_0$, and bf$_0$, we use the coordinates

$$\left(x, \sigma = \frac{x'}{x}, y, y', \kappa' = \frac{k}{x}\right).$$

In these coordinates, x is a boundary defining function (bdf) for bf$_0$, σ is a bdf for rb$_0$, and κ' is a bdf for zf. Similarly, near the intersection of zf, lb$_0$, and bf$_0$, we use the coordinates

$$\left(x', \sigma' = \frac{x}{x'}, y, y', \kappa = \frac{k}{x}\right).$$

Coordinates near sc are slightly more complicated; before the final blow-up, good coordinates are $(x', \sigma', y, y', x/k)$. After the blow-up, we use the coordinates

$$\left(X = k\left(\frac{1}{x'} - \frac{1}{x}\right), Y = k\frac{y - y'}{x}, \lambda = \frac{x}{k}, y, k\right).$$

These are valid in a neighborhood of the intersection of $D_{k,\mathrm{sc}}$ with sc and bf$_0$; however, they are not good coordinates as we approach bf.

In addition to the b-stretched products of [Melrose and Singer 2008] and the space $M^2_{k,\mathrm{sc}}$, we also define the *scattering double space $M^2_{\mathrm{sc}}(z, z')$*, originally described in [Melrose 1994]. It is a blown-up version of $\overline{M} \times \overline{M}$; the first blow-up is of $\{x = x' = 0\}$, and the second blow-up is of the boundary fiber diagonal $\{x' = 0, x/x' = 1, y = y'\}$. Notice that each cross-section of $M^2_{k,\mathrm{sc}}$ corresponding to a fixed $k > 0$ is a copy of M^2_{sc}.

2.3. *Proof of Theorem 8.* Let A be an operator satisfying hypotheses (a)–(c) of Theorem 8, and let $R(\lambda, z, z')$ be the Schwartz kernel of $(A+\lambda)^{-1}$. The spectrum of A is $[0, \infty]$, so $R(\lambda, z, z')$ is holomorphic outside the non-positive real axis. Fix $\varphi \in (\pi/2, \pi)$. For any $a > 0$, let Γ_a be the path in $\mathbb{C}$ consisting of two half-rays along $\theta = -\varphi$ and $\theta = \varphi$, connected by the portion of the circle of radius a from $\theta = -\varphi$ to $\theta = \varphi$, and traversed counterclockwise. Moreover, let $\Gamma_{a,1}$ be the portion of Γ_a along the circle of radius a, and let $\Gamma_{a,2}$ be the remainder; that is, the two half-rays. These contours are illustrated in Figure 4.

Let $F(w, z, z')$ be the heat kernel at time $t = w^{-2}$. Then, by the functional calculus, we have

$$F(w, z, z') = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda/w^2} R(\lambda, z, z')\, d\lambda. \tag{7}$$

We let $a = w^2$ and $\Gamma = \Gamma_{w^2}$, and then consider the integral (7) over $\Gamma_{w^2,1}$ and $\Gamma_{w^2,2}$ separately.

On $\Gamma_{w^2,1}$, $\lambda = w^2 e^{i\theta}$, so $d\lambda = w^2 d\theta$, and we have

$$\frac{w^2}{2\pi i} \int_{-\varphi}^{\varphi} e^{e^{i\theta}} R(\theta, w, z, z') \, d\theta. \tag{8}$$

By condition (b), for each θ, the integrand in (8) is phg conormal on $M_{w,\mathrm{sc}}^2$ with a conormal singularity at $\Delta_{w,\mathrm{sc}}$, and the dependence of all coefficients on $\theta \in [-\varphi, \varphi]$ is smooth. Therefore, the integral (8) is phg conormal on $M_{w,\mathrm{sc}}^2$, with a possible conormal singularity at $D_{w,\mathrm{sc}}$. The index sets of (8) on $M_{w,\mathrm{sc}}^2$ are those of w^2 plus those of $R(w, z, z')$. The function w^2 is smooth and has order 2 as a function at zf, bf$_0$, rb$_0$, and lb$_0$ and order 0 everywhere else, so we add 2 to the index sets of the resolvent at those faces. This procedure gives precisely the index sets claimed in Theorem 8.

It remains to consider the integral over $\Gamma_{w^2,2}$. $\Gamma_{w^2,2}$ consists of two half-rays; we consider only the half-ray corresponding to $\theta = \varphi$, as the other is analogous. Since θ is fixed, we suppress θ in the notation; the integral runs from $r = w^2$ to $r = \infty$. After changing variables from r to $s = \sqrt{r}$, and dropping the overall factor of $(2\pi i)^{-1}$ (which does not affect polyhomogeneity), the $\Gamma_{w^2,2}$ portion of (7) becomes

$$\int_{w}^{\infty} 2s e^{(\cos\varphi)s^2/w^2} e^{i(\sin\varphi)s^2/w^2} R(s^2, z, z') \, ds. \tag{9}$$

First consider the behavior of the integrand as $s \to \infty$. Since $\cos\varphi < 0$ and w is bounded above, the term $\exp((\cos\varphi)s^2/w^2)$, and hence the entire integrand of (9), will decay to infinite order at $s = \infty$. There will still be a conormal singularity at the spatial diagonal, but the coefficients decay to infinite order at $s = \infty$.

In order to analyze (9), we break $R(s^2, z, z')$ into pieces. First we separate out the conormal singularity in a neighborhood of $D_{s,\mathrm{sc}}$ and analyze (9) using explicit local coordinates. Then we deal with the remainder, which is smooth on $M_{s,\mathrm{sc}}^2$, by breaking it into two pieces and applying the pushforward theorem.

2.4. The conormal singularity. Using a partition of unity, we let $R = R_s + R_c$, where R_c is supported in a neighborhood of $D_{s,\mathrm{sc}}$ and R_s is supported away from $D_{s,\mathrm{sc}}$, as in Figure 5. In a neighborhood of sc $\cap$ bf$_0$, we use the coordinates

$$\left(X = s \left(\frac{1}{x} - \frac{1}{x'} \right), Y = s \frac{y - y'}{x}, y, \mu = \frac{x}{s}, s \right).$$

In these coordinates, R has a conormal singularity at $\{X = Y = 0\}$. On the other hand, in a neighborhood of bf$_0 \cap$ zf $\cap D_{s,\mathrm{sc}}$, we use a slight modification of the coordinates in the previous subsection:

$$\left(\hat{X} = \left(1 - \frac{x}{x'} \right), \hat{Y} = y - y', y, \frac{s}{x}, x \right).$$

Since we use different coordinates in different regimes, write $R_c = R_1 + R_2 + R_3$ by using a smooth partition of unity near $x/s = 1$; R_1 is supported near the boundary but away from sc (say $s/x < 2$), R_2
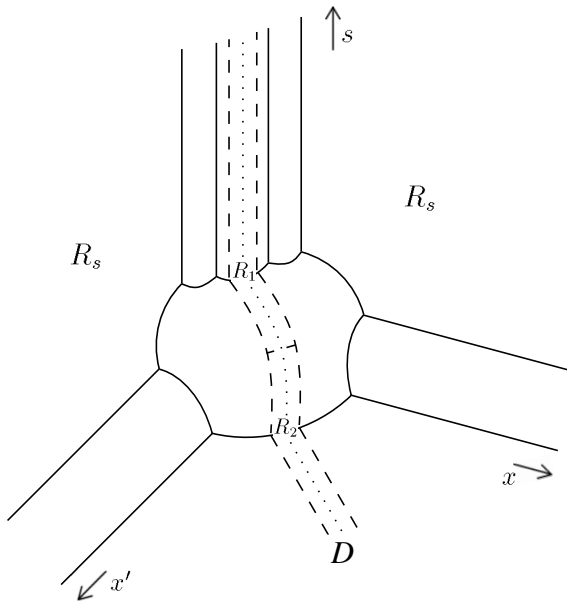
Figure 5. Decomposition of R.

near the boundary but away from zf (say $x/s < 2$), and R_3 in the interior. The decomposition at the boundary is illustrated in Figure 5.

First look at R_1. Using the explicit symbolic form of a conormal singularity, we may write

$$R_1 \sim \int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\xi_1, \xi_2)} \sum_{j=0}^{\infty} a_j \left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|} \right) |\xi|^{2-j} \, d\xi. \tag{10}$$

This is an asymptotic sum, modulo smooth functions on $M_{s,\mathrm{sc}}^2$; we pick a particular representative which is supported in a small neighborhood of $D_{s,\mathrm{sc}}$, and absorb the remainder into R_s. The coefficients a_j are phg conormal in x and s/x with index sets independent of j; they are also smooth in y and $\xi/|\xi|$. We plug (10) into (9) and then interchange the convergent s-integral with the asymptotic sum and the oscillatory integral over $\mathbb{R}^n$. The result is

$$\int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\xi_1, \xi_2)} \sum_{j=0}^{\infty} \int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} a_j \left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|} \right) ds \, |\xi|^{2-j} \, d\xi. \tag{11}$$

By Melrose's pullback theorem [1996], the pullback of each a_j to $X_b^3(s, x, w) \times N_y \times S_{\xi/|\xi|}^{n-1}$ via projection is also phg conormal with index sets independent of j. As a result, the integrand in

$$\int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} a_j \left(\frac{s}{x}, x, y, \frac{\xi}{|\xi|} \right) ds \tag{12}$$

is phg conormal on $X_b^3(s, w, x) \times N_y \times S_{\xi/|\xi|}^{n-1}$, with a cutoff singularity at $s/w = 1$. Moreover, it has infinite-order decay at $s = \infty$, independent of $w < 1$ and $x < 1$, and hence integration in s is well-defined. Integration in s is a b-fibration from $X_b^3(s, w, x)$ to $X_b^2(w, x)$, by [Melrose and Singer 2008,

Proposition 4.4], and is hence also a b-fibration when we take the direct product with $N_y \times S^{n-1}_{\xi/|\xi|}$. Moreover, integration in s is transverse to the cutoff singularity at $s/w = 1$. Therefore, by the pushforward theorem with conormal singularities (from the appendix of [Epstein et al. 1991]), (12) is phg conormal on $X^2_b(w, x) \times N_y \times S^{n-1}_{\xi/|\xi|}$, with index sets independent of j.

Since $w < s < 2x$ on the support of R_1, (12) has a phg conormal expansion in $(w/x, x)$. Therefore, the integral (11), in the coordinates $(\hat{X}, \hat{Y}, y, w/x, x)$, corresponding to the R_1 piece of (9), is phg conormal in $(w/x, x)$ with index sets independent of j, and smoothly dependent on y, with an interior conormal singularity at $\hat{X} = \hat{Y} = 0$. Thus (11) is phg conormal on $M^2_{w,\text{sc}}$ with a conormal singularity at the diagonal.

We now consider R_2; the analysis is similar. Write

$$R_2 \sim \int_{\mathbb{R}^n} e^{i(X,Y)\cdot(\xi_1,\xi_2)} \sum_{j=0}^{\infty} b_j\left(\frac{x}{s}, s, y, \frac{\xi}{|\xi|}\right) |\xi|^{2-j} \, d\xi, \tag{13}$$

where the b_j are phg conormal in x/s and s with index sets independent of j, and also smooth in y and $\xi/|\xi|$.

It is helpful to consider the regimes $w > x/2$ and $w < 2x$ separately. First assume that $w > x/2$, and let $\bar{X} = w(1/x - 1/x')$, $\bar{Y} = w(y - y')/x$, and $\bar{\lambda} = x/w$. We expect a conormal singularity at $\bar{X} = \bar{Y} = 0$ in this regime. Noting that $(X, Y) = (s/w)(\bar{X}, \bar{Y})$, we change variables in (13) and let $\zeta = (s/w)\xi$. The result is

$$R_2 \sim \int_{\mathbb{R}^n} e^{i(\bar{X},\bar{Y})\cdot(\zeta_1,\zeta_2)} \sum_{j=0}^{\infty} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right)\left(\frac{s}{w}\right)^{j-2-n} |\zeta|^{2-j} \, d\zeta. \tag{14}$$

As before, plug (14) into (9) and interchange the sums and convergent integrals: the part of (9) coming from R_2 is

$$\int_{\mathbb{R}^n} e^{i(\bar{X},\bar{Y})\cdot(\zeta_1,\zeta_2)} \sum_{j=0}^{\infty} \int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right)\left(\frac{s}{w}\right)^{j-2-n} ds \, |\zeta|^{2-j} \, d\zeta. \tag{15}$$

Consider the coefficients

$$\int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right)\left(\frac{s}{w}\right)^{j-2-n} ds. \tag{16}$$

If we can show that the coefficients (16) are phg conormal in $(x/w, w)$ with respect to some index sets independent of j, with smooth dependence on y and $\zeta/|\zeta|$, then (15) is phg conormal on $M^2_{w,\text{sc}}$ with a conormal singularity at the diagonal when $w > x/2$. To show this phg conormality, note that again the integrands in (16) are each phg conormal on $X^3_b(s, x, w) \times N_y \times S^{n-1}_{\zeta/|\zeta|}$. Moreover, the index sets are independent of j, as $s/w > 1$, and there is always infinite-order decay at $s = \infty$, independent of w. As before, we use the pushforward theorem to integrate in s, and we conclude that the coefficients (16) are phg conormal on $X^2_b(x, w) \times N_y \times S^{n-1}_{\zeta/|\zeta|}$ with respect to index sets independent of j. Since $w > x/2$, this yields expansions in x/w and w, which is precisely what we need.

On the other hand, suppose that $w < 2x$. Then we expect a conormal singularity at $\hat{X} = \hat{Y} = 0$. Since $(X, Y) = (s/x)(\hat{X}, \hat{Y})$, we change variables in (13), letting $\zeta' = (s/x)\xi$. Following the exact same

procedure as in the $w > x/2$ case, we see that the part of (9) coming from R_2, when $w < 2x$, is

$$\int_{\mathbb{R}^n} e^{i(\hat{X}, \hat{Y}) \cdot (\zeta_1', \zeta_2')} \sum_{j=0}^{\infty} \int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta'}{|\zeta'|}\right) \left(\frac{s}{x}\right)^{j-2-n} ds\, |\zeta'|^{2-j}\, d\zeta'. \tag{17}$$

The coefficients can be rewritten as

$$\left(\frac{w}{x}\right)^{j-2-n} \int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} b_j\left(\frac{x}{s}, s, y, \frac{\zeta}{|\zeta|}\right) \left(\frac{s}{w}\right)^{j-2-n} ds. \tag{18}$$

These are just $(w/x)^{j-2-n}$ times (16), and $(w/x)^{j-2-n}$ is phg conormal on $X_b^2(w, x)$, so (18) is also phg conormal on $X_b^2(w, x)$ for each j. Since we are only considering $w < 2x$, the orders only improve as j increases. In particular, all the coefficients are phg conormal on $X_b^2(w, x)$ with respect to subsets of the index set of the $j = 0$ coefficient. This is again sufficient to prove that (17) is phg conormal on $M_{w,\text{sc}}^2$ with a conormal singularity at the diagonal.

Finally, consider the interior term R_3; it is the simplest of the lot, since z and z' are in a compact subset of M. We let η be a dual variable to $z - z'$ and write

$$R_3 \sim \int_{\mathbb{R}^n} e^{i(z-z') \cdot \eta} \sum_{j=0}^{\infty} c_j\left(s, z, \frac{\eta}{|\eta|}\right) |\eta|^{2-j}\, d\eta. \tag{19}$$

Here the c_j are phg conormal at $s = 0$ and $s = \infty$, with index sets independent of j at $s = 0$ and $s = \infty$. Following the same procedure as in the previous two cases, simplified since $z - z'$ and η are independent of s, we conclude that the part of (9) coming from R_3 is

$$\int_{\mathbb{R}^n} e^{i(z-z') \cdot \eta} \sum_{j=0}^{\infty} \int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} c_j\left(s, z, \frac{\eta}{|\eta|}\right) ds\, |\eta|^{2-j}\, d\eta. \tag{20}$$

Analyzing the coefficients, we see that the integrand in each is phg conormal on $X_b^2(s, w)$, with index sets independent of j, z, and $\eta/|\eta|$, and with infinite-order decay at $s = \infty$. By the pushforward theorem, each coefficient is phg conormal at $w = 0$, with index sets independent of j, z, and $\eta/|\eta|$. Moreover, (20) has compact support in (z, z'). Therefore, (20), and hence the part of (9) corresponding to R_c, is phg conormal on $M_{w,\text{sc}}^2$ with a conormal singularity at the diagonal.

Technically, we need to compute the index sets of the coefficients of the conormal singularity at each boundary face of $M_{w,\text{sc}}^2$. This may be done directly via the pushforward theorem, but it is easier to apply the analysis we will develop in the next section. Note that a_j and b_j may be viewed as phg conormal functions on the diagonal $D_{s,\text{sc}} \subset M_{s,\text{sc}}^2$, with fixed index sets R_{sc}, R_{bf_0}, and R_{zf} at the boundary hypersurfaces sc, bf$_0$, and zf. Observe that they can be extended smoothly to functions defined in a neighborhood of $D_{s,\text{sc}}$ which are themselves phg conormal on $M_{s,\text{sc}}^2$ with the given index sets; call these extensions d_j. Then the coefficients of the conormal singularity of the integral (9) are the restrictions to the diagonal of

$$\int_w^{\infty} 2s e^{-(s/w)^2 e^{i\varphi}} d_j(s, x, y, x', y')\, ds.$$

Applying Lemma 10 to each d_j, these coefficients are all phg conormal on $M^2_{w,\mathrm{sc}}$, with index sets obtained by adding 2 at the faces bf_0, rb_0, lb_0, and zf. Therefore the restrictions to the diagonal are all phg conormal on $D_{w,\mathrm{sc}}$, with leading orders matching those in Theorem 8, as expected. This completes the analysis of the conormal singularity.

2.5. *Finishing the proof.* It remains to consider the integral

$$\int_w^\infty 2s e^{-(s/w)^2 e^{i\varphi}} R_s(s^2, z, z') \, ds, \tag{21}$$

where $R_s(s^2, z, z')$ is phg conormal on $\overline{M}^2_{s,\mathrm{sc}}$ and smooth across the diagonal. We claim:

Lemma 10. *Let $T(s, z, z')$ be any function which is phg conormal on $\overline{M}^2_{s,\mathrm{sc}}$, smooth in the interior, and decaying to infinite order at* lb, rb, *and* bf. *Then*

$$\int_w^\infty 2s e^{-(s/w)^2 e^{i\varphi}} T(s, z, z') \, ds \tag{22}$$

is phg conormal on $M^2_{w,\mathrm{sc}}$ for w bounded above. Moreover, if the index sets of T at the various boundary hypersurfaces are T_{sc}, T_{bf_0}, T_{zf}, T_{rb_0}, T_{lb_0}, and T_{tf}, then the index sets of (22) are

$$T_{\mathrm{sc}}, \quad T_{\mathrm{bf}_0} + 2, \quad T_{\mathrm{zf}} + 2, \quad T_{\mathrm{rb}_0} + 2, \quad T_{\mathrm{lb}_0} + 2.$$

We defer the proof for the moment. Applying Lemma 10 to $T(s) = R_s(s^2)$, we conclude that (21) is phg conormal on $M^2_{w,\mathrm{sc}}$ with index sets precisely as in Theorem 8. Combining this with our analysis of R_c, we have now shown that $F(w, z, z')$ is phg conormal on $M^2_{w,\mathrm{sc}}$ possibly with a conormal singularity at the spatial diagonal, and with leading orders as specified in Theorem 8. However, $F(w, z, z')$ is a heat kernel, so it has no conormal singularity at the diagonal. This completes the proof of Theorem 8.

Finally, to prove Theorem 2, we apply Theorem 8. Condition (a) is true since the Laplacian is essentially self-adjoint and non-negative. Condition (b) follows from Theorems 6 and 7. Condition (c) is a well-known consequence of the semiclassical scattering calculus. The scattering calculus was first introduced by Melrose [1994] and the semiclassical version was developed by Vasy, Wunsch, and Zworski among others [Vasy and Zworski 2000; Wunsch and Zworski 2000]. The exact statement we need, along with a summary of the semiclassical scattering calculus, may be found in [Hassell and Wunsch 2008, Section 10]; $\hbar$ in the semiclassical calculus corresponds to k^{-1} in our context. Applying Theorem 8 gives us the polyhomogeneity we claim, and once we plug in the leading orders from [Guillarmou and Hassell 2008] and the Appendix, we see that the heat kernel has leading orders of 0 at sc and n at each of bf_0, rb_0, and lb_0.

Unfortunately, Theorem 8 does not by itself give us the claimed order-n behavior at zf; instead, we only see quadratic decay at zf when $n \geq 3$ and decay of the form $w^2 \log w$ when $n = 2$. However, by the estimate of Cheng–Li–Yau (Theorem 1), the heat kernel is uniformly bounded for large time by $Ct^{-n/2} = Cw^n$. Thus the leading order of the heat kernel at zf must actually be at least n, which completes the proof of Theorem 2.
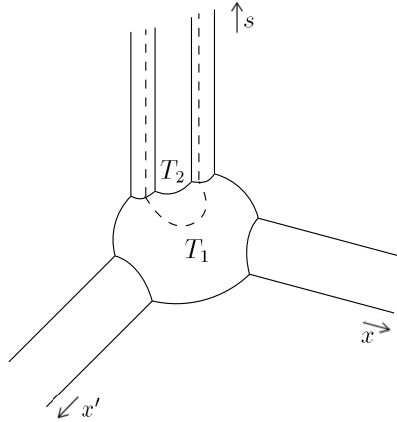
Figure 6. Decomposition of T.

Note that the lack of sharpness in the order calculation of Theorem 8 reflects the fact that our real-analytic approach does not take into account the complex-analytic structure of the resolvent; there is cancellation between the top and bottom parts of the integral that our approach cannot see. In particular, we could instead move the contour Γ towards the spectrum and represent the heat kernel as an integral with respect to the spectral measure. Guillarmou, Hassell, and Sikora demonstrate in [Guillarmou et al. 2012] that there is cancellation between the top and bottom parts of the contour in the spectral measure. In particular, the spectral measure at zf vanishes to order $n - 1$ by Theorem 1.2 of their paper; integrating $e^{-\lambda t}$ against this spectral measure, we obtain an alternative proof of the fact that the heat kernel vanishes to order n at zf.

2.6. *Proof of Lemma 10.* We now prove Lemma 10; the proof involves extensive use of Melrose's pullback and pushforward theorems. First, write $T(s, z, z')$ as $T_1 + T_2$, where T_1 is supported away from sc and T_2 is supported in a neighborhood of sc. This partition is illustrated in Figure 6. Then decompose (22) into two integrals, corresponding to T_1 and T_2.

Consider the first integral:

$$\int_0^\infty \chi(\{s \geq w\}) \, 2s e^{-(s/w)^2 e^{i\varphi}} T_1(s, z, z') \, \frac{ds}{s}. \tag{23}$$

Notice that $T_1(s, z, z')$ is phg conormal on $M^2_{s,\mathrm{sc}}$ but supported away from sc, so it is in fact phg conormal on the blown-down space $M^2_{s,b}(z, z')$ (see Figure 2). The rest of the terms in the integrand are phg conormal on $X^2_b(s, w)$, with a cutoff singularity (which is an example of a conormal singularity) at $s = w$. We now define a space $M^2_{s,w,b}(z, z')$ as follows: start with $[0, T)_w \times [0, \infty]_s \times M_z \times M_{z'}$. Then blow up, in order,

- The submanifold where all four of (x, x', s, w) are zero;
- The four now-disjoint submanifolds where exactly three of (x, x', s, w) are zero;
- The six now-disjoint submanifolds where exactly two of (x, x', s, w) are zero.

This construction mimics the construction of the b-stretched product $X_b^4(x, x', s, w)$, and in fact $M_{s,w,b}^2(z, z')$ is precisely $X_b^4(x, x', s, w) \times N_y \times N_{y'}$ in a neighborhood of $\{x = x' = s = w = 0\}$. By the same arguments as for the b-stretched products in [Melrose and Singer 2008], the projection-induced maps from $M_{s,w,b}^2(z, z')$ to $M_{w,b}^2(z, z')$ (isomorphic to $X_b^3(x, x', w) \times N_y \times N_{y'}$ near $\{x = x' = w = 0\}$) and to $X_b^2(s, w)$ are well-defined b-fibrations. Therefore, by the pullback theorem, the integrand of (23) is phg conormal on $M_{s,w,b}^w(z, z')$, with a conormal singularity at $s = w$. Since the fibers of the projection map to $M_{w,b}^2(z, z')$ are transverse to the singularity at $s = w$, and the integrand has order ∞ at $s = \infty$, the pushforward theorem implies that (23) itself is phg conormal on $M_{w,b}^2(z, z')$. Since $M_{w,b}^2(z, z')$ is a blow-down of $M_{w,sc}^2(z, z')$, we conclude that (23) is phg conormal on $M_{w,sc}^2(z, z')$ as desired.

For T_2, we may use $(z, z') = (x, y, x', y')$ since T_2 is supported in a small neighborhood of sc. We have

$$\int_0^\infty \chi(\{s \geq w\}) \, 2s e^{-(s/w)^2 e^{i\varphi}} T_2(s, x, y, x', y') \frac{ds}{s}. \tag{24}$$

Let $\bar{\sigma} = (x/x' - 1, y - y')$; $\bar{\sigma}$ is an n-dimensional coordinate, and $M_{s,sc}^2$ is created from $X_b^3(s, x, x') \times N_y \times N_{y'}$ by blowing up $\{\bar{\sigma} = x/s = 0\}$. In particular, $T_2(s, x, y, x', y')$, having compact support in x/x', is phg conormal on

$$\left[X_b^2(s, x) \times N_y \, ; \{\bar{\sigma} = x/s = 0\} \right].$$

This space is the subset of $M_{w,sc}^2$ with $1/2 < x/x' < 2$, so label its boundary hypersurfaces bf, sc, bf_0, and zf. In this labeling, T_2 is supported away from zf, decays to infinite order at bf, and has leading orders t_{sc} at sc and t_{bf_0} at bf_0.

We analyze the integrand in (24) as a function on the space

$$S = \left(X_b^3(s, w, x) \cap \{s \geq w\} \right) \times B(\bar{\sigma}) \times N_y.$$

Here B is the unit ball in $\mathbb{R}^n$. A diagram of S is given in Figure 7, with $\bar{\sigma}$ and y suppressed; we label the boundary hypersurfaces A–E.
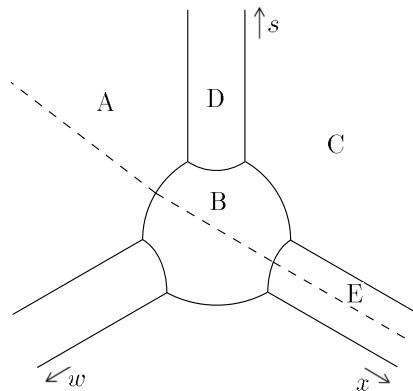

Figure 7. The space S.

We now define an iterated blow-up of S. Let P_1 be the p-submanifold of S given by $A \cap \{\bar{\sigma} = 0\}$. Blowing up P_1 creates a new space $S_1 = [S ; P_1]$; call the front face of this blow-up F. Now let P_2 be the p-submanifold of S_1 given by the closure of the lift of $D° \cap \{\bar{\sigma} = 0\}$. Then let

$$S_2 = [S_1 ; P_2] = [[S ; P_1] ; P_2],$$

and let G be the new front face. The following two propositions allow us to analyze (24); their proofs are deferred for the moment.

Proposition 11. *The map*

$$\pi_w : S_2 \cap \{s \geq x\} \to \left[\left(X_b^2(s, x) \cap \{s \geq x\} \right) \times B_1(\bar{\sigma}) \times N_y ; \{\bar{\sigma} = x/s = 0\} \right],$$

given in the interior of S_2 by projection off the variable w and extending continuously to the boundary, is a b-map.

Proposition 12. *The map*

$$\pi_s : S_1 \to \left[X_b^2(w, x) \times B_1(\bar{\sigma}) \times N_y ; \{\bar{\sigma} = x/w = 0\} \right],$$

given in the interior of S_1 by projection off the variable s and extending continuously to the boundary, is a b-fibration. Moreover, if we let ρ_H be a bdf for each hypersurface H, we have

$$(\pi_s)^*(\rho_{zf}) = \rho_C \rho_E, \quad (\pi_s)^*(\rho_{bf_0}) = \rho_B \rho_D, \quad (\pi_s)^*(\rho_{sc}) = \rho_F, \quad (\pi_s)^*(\rho_{bf}) = \rho_A. \quad (25)$$

Since T_2 is supported in $\{s \geq x\}$ and its support does not intersect the lift of $\{s = x\}$, Proposition 11 and the pullback theorem imply that the pullback of T_2 is phg conormal on S_2. Moreover, the remainder of the integrand in (24) is phg conormal on $X_b^2(s, w) \cap \{s > w\}$, so pulling back first to $X_b^3(s, w, x) \cap \{s > w\}$, and then to S_2, we see that it is phg conormal on S_2 as well. Therefore, the entire integrand in (24) is phg conormal on $S_2 = [S_1 ; P_2]$. However, the factor of e^{-s^2/w^2}, and hence the integrand, vanishes to infinite order at the front face G; consequently the integrand in (24) is actually phg conormal on S_1. By the pushforward theorem from [Epstein et al. 1991], since π_s is a b-fibration transverse to the conormal singularity at $s = w$, the pushforward (24) is phg conormal on the target space $[X_b^2(w, x) \times B_1(\bar{\sigma}) \times N_y ; \{\bar{\sigma} = x/w = 0\}]$. From Figure 8, we see that this space is a subset of $M_{w,sc}^2$; we have therefore shown that (24) is phg conormal on $M_{w,sc}^2$. This completes the proof of the polyhomogeneity statement in Lemma 10, modulo the proofs of Propositions 11 and 12.

It remains to check the index sets claimed in Lemma 10. However, this calculation is a straightforward application of the pullback and pushforward theorems (explicit descriptions of the pullback and pushforward index sets may be found in [Grieser 2001]). A computation of the leading orders may be found in [Sher 2012b] and computing the index sets themselves is no harder.

2.7. *Propositions 11 and 12.* Finally, we prove Propositions 11 and 12. These propositions are proved in [Sher 2012b] using explicit local coordinates, but here we instead give a simpler proof based on the machinery developed by Hassell, Mazzeo, and Melrose [Hassell et al. 1995].
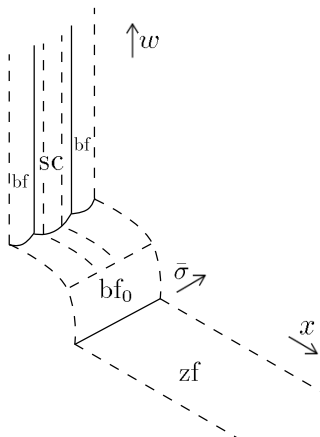
Figure 8. $[X_b^2(w, x) ; \{\bar{\sigma} = x/w = 0\}]$.

Observe first that there are projection-induced maps from S to both $X_b^2(s, x) \times B(\bar{\sigma}) \times N_y$ and $X_b^2(w, x) \times B(\bar{\sigma}) \times N_y$. We call these maps $\tilde{\pi}_w$ and $\tilde{\pi}_s$ respectively. It is easy to see directly that both of these maps are in fact b-fibrations; see also the analysis of b-stretched products in [Melrose and Singer 2008]. Moreover, it may be checked by hand [ibid.] that each entry of the "exponent matrix" associated to each of these maps is either 0 or 1; see [Grieser 2001] or [Mazzeo 1991] for a discussion of exponent matrices.

To prove Proposition 11, consider the p-submanifold $\{\bar{\sigma} = x/s = 0\}$ of the target space of $\tilde{\pi}_w$. Its lift under $\tilde{\pi}_w$ is a union of two p-submanifolds of S: $A \cap \{\bar{\sigma} = 0\}$ and $D \cap \{\bar{\sigma} = 0\}$. S_2 is precisely the space we obtain from S by blowing up those two p-submanifolds (first A, then D). We may therefore apply Lemma 10 from [Hassell et al. 1995, Section 2] to conclude that the lift of $\tilde{\pi}_w$ to a map from S_2 to $[X_b^2(s, x) \times B(\bar{\sigma}) \times N_y ; \{\bar{\sigma} = x/s = 0\}]$ is a b-fibration; but this lift is precisely π_w. Since a b-fibration is certainly a b-map, this completes the proof of Proposition 11.

Proposition 12 is proved in exactly the same way: the lift of $\{\bar{\sigma} = x/w = 0\}$ to S under $\tilde{\pi}_s$ is just $A \cap \{\bar{\sigma} = 0\}$, which is precisely P_1. An identical application of the lemma just cited allows us to conclude that π_s is a b-fibration. The computation of the pullbacks of boundary defining functions is not hard and may be done directly using local coordinates; the details may be found in [Sher 2012b].

3. Renormalized heat trace and zeta function

In this section, we define the renormalized heat trace, zeta function, and determinant on an asymptotically conic manifold M. These definitions ultimately allow us, in [Sher 2012a], to state and prove Theorem 5. The first step is to define the *renormalized trace*. This definition is inspired by Melrose's b-heat trace, which is a renormalized heat trace for manifolds with asymptotically cylindrical ends. Albin [2007] also defined renormalized heat traces in the asymptotically hyperbolic setting; later, Albin, Aldana, and Rochon [Albin et al. 2013] defined and investigated a renormalized determinant of the Laplacian on asymptotically hyperbolic surfaces.

3.1. *The renormalized heat trace.* Pick any cutoff function $\chi_1(r)$ on $\mathbb{R}_+$ which is supported on $\{r \leq 2\}$ and equal to 1 on $\{r \leq 1/2\}$. Assume that either

(a) $\chi_1(r)$ is a non-increasing smooth function of r (smooth cutoff), or

(b) $\chi_1(r)$ is precisely the characteristic function of $[0, 1]$ (sharp cutoff).

Then for any $\delta < 1/2$, let $\chi_{1,\delta}$ be a function on M, equal to $\chi_1(r\delta)$ for $r \geq 1$ and equal to 1 inside $\{r = 1\}$. Consider the integral

$$\int_M \chi_{1,\delta}(z) H^M(t, z, z) \, dz. \tag{26}$$

Theorem 13. *Let χ_1 be either the smooth or the sharp cutoff. The integral* (26) *has a polyhomogeneous expansion in δ for each fixed t. Moreover, the finite part at $\delta = 0$, which we denote $P(t)$, has polyhomogeneous expansions in t at $t = 0$ and t^{-1} at $t = \infty$.*

This theorem allows us to define the renormalized heat trace on an asymptotically conic manifold. Roughly, this corresponds to integrating the heat kernel on the diagonal over regions where $r \leq \delta^{-1}$, and renormalizing by subtracting the divergent parts at $\delta = 0$. Renormalization in this fashion is often called Hadamard renormalization (for details, see [Albin 2007; Albin et al. 2013]).

Definition. Let $\chi_1(r)$ be the sharp cutoff. The renormalized heat trace, denoted $^R\mathrm{Tr}\, H^M(t)$, is the finite part at $\delta = 0$ of (26).

Proof of Theorem 13. The key ingredient is the following observation on the structure of the heat kernel on the diagonal near the boundary, which is a consequence of the structure theorem we have proven for the heat kernel. The asymptotic structure of $H^M(t, x, y, x, y)$ reflected in this proposition is illustrated in Figure 9.

Proposition 14. (a) *For t bounded above, $H^M(t, x, y, x, y)$ is phg conormal in $(\sqrt{t}, x)$, with smooth dependence on y.*

(b) *For t bounded below, let $w = t^{-1/2}$; then $H^M(t, x, y, x, y)$ is phg conormal as a function of w and x on $X_b^2(w, x)$, again with smooth dependence on y.*

Proof. Proposition 14 is an immediate consequence of restricting to the spatial diagonal D in Theorem 2; since D is a p-submanifold of the space on which H^M is polyhomogeneous, the restriction of H^M to D
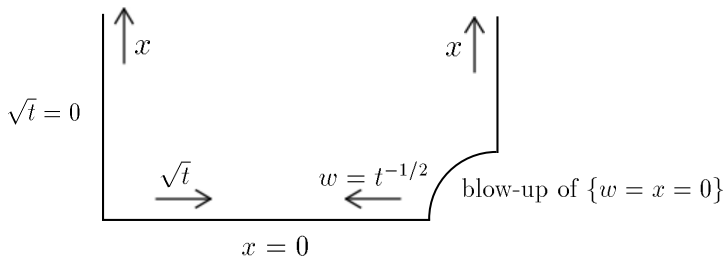

Figure 9. Asymptotic structure of $H^M(t, x, y, x, y)$.

is also polyhomogeneous. Comparing D with Figure 9, we see that D is precisely the space described in Proposition 14. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To prove Theorem 13, we analyze (26), which may be rewritten as

$$\int_N \int_0^1 \chi_1(\delta/x) H^M(t, x, y, x, y) x^{-n-1} \, dx \, dy + \int_{x \geq 1} H^M(t, z, z) \, dz. \tag{27}$$

First analyze the second term; the region $\{x \geq 1\}$ is bounded away from spatial infinity. Therefore, by Theorem 2, $H^M(t, z, z)$ has polyhomogeneous expansions in t at $t = 0$ and $t^{-1/2}$, hence t^{-1}, at $t = \infty$, and these expansions are uniform in z with smooth coefficients. Integrating in z results in a function of t which is phg conormal at $t = 0$ and $t = \infty$; this function contributes only to the finite part $P(t)$ at $\delta = 0$ and satisfies the polyhomogeneity claimed in Theorem 13.

It remains to analyze the first term in (27). We consider the small-t and large-t regimes separately, analyzing the integrand

$$\chi_1(\delta/x) H^M(t, x, y, x, y) x^{-n-1} \tag{28}$$

in each regime as a function of (t, x, δ). In each case, $\chi_1(\delta/x)$ is phg conormal on $X_b^2(x, \delta)$; if χ_1 is the sharp cutoff, there is also a cutoff singularity, which is a type of conormal singularity, at $\delta/x = 1$.

For small t, $H^M(t, x, y, x, y)$ is phg conormal in $(\sqrt{t}, x)$, so (28) is phg conormal on $\mathbb{R}_+(\sqrt{t}) \times X_b^2(x, \delta)$, possibly with a conormal singularity at $\delta/x = 1$. The projection map π_x is a b-fibration from this space onto the first quadrant in $(\sqrt{t}, \delta)$ and is transverse to $\delta/x = 1$; moreover, the integral in x is well-defined, as the integrand is supported away from the $x = 0, \delta > 0$ face. By the pushforward theorem from [Epstein et al. 1991], the first term of (27) is phg conormal in $(\sqrt{t}, \delta)$ for bounded t.

On the other hand, for large t, H^M is phg conormal on $X_b^2(w, x)$ and χ_1 is phg conormal on $X_b^2(x, \delta)$. Since the maps from $X_b^3(w, x, \delta)$ to each of these spaces are b-maps (also b-fibrations), the integrand is phg conormal on $X_b^3(w, x, \delta)$ by the pullback theorem; there may again be a conormal singularity at $\delta/x = 1$. Integration in x is pushforward by a b-fibration onto $X_b^2(w, \delta)$. Again, the integrand is supported in $\{x > \delta\}$, and the fibration is transverse to $\delta/x = 1$, so we apply the pushforward theorem from [Epstein et al. 1991] to conclude that the first term in (27) is phg conormal on $X_b^2(w, \delta)$ for bounded w. Combining these results, we have shown that (27) is phg conormal on the space in Figure 10.

In particular, for any fixed t, (27) has a polyhomogeneous expansion as $\delta \to 0$. Moreover, $P(t)$ is simply the coefficient of the t^0 term at the $(0 < t < \infty, \delta = 0)$ face. By the definition of phg conormality
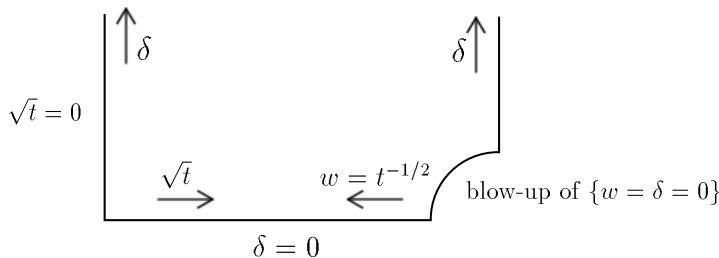

Figure 10. Asymptotic structure of (27).

(also see the discussion surrounding [Mazzeo 1991, Lemma A.4]), $P(t)$ therefore has polyhomogeneous conormal expansions at $t = 0$ and $t = \infty$. This completes the proof of Theorem 13. $\qquad\square$

Using Theorem 13, we now define the meromorphic continuation of the renormalized zeta function:

$$^R\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty {}^R\mathrm{Tr}\, H^M(t)\, t^{s-1}\, dt. \tag{29}$$

We break up the integral (29) at $t = 1$, and consider first the short-time piece,

$$\frac{1}{\Gamma(s)} \int_0^1 {}^R\mathrm{Tr}\, H^M(t)\, t^{s-1}\, dt. \tag{30}$$

By the phg conormality of $^R\mathrm{Tr}\, H^M(t)$, we can write, for any $N > 0$,

$$^R\mathrm{Tr}\, H^M(t) = \sum_{i=0}^{k_N} a_i t^{z_i} (\log t)^{p_i} + \mathbb{O}(t^N).$$

Plug this expansion into (30). The $\mathbb{O}(t^N)$ contribution is well-defined and meromorphic whenever $\Re s > -N$, and the continuations of the other terms are integrals of the form

$$\frac{a_i}{\Gamma(s)} \int_0^1 t^{z_i+s-1} (\log t)^{p_i}\, dt.$$

These integrals may be evaluated directly, and give explicit meromorphic functions of s, each with finitely many poles. Therefore, (30), though initially defined only when $\Re s > -z_0$, has a meromorphic extension to all of $\mathbb{C}$.

On the other hand, the long-time piece is

$$\frac{1}{\Gamma(s)} \int_1^\infty {}^R\mathrm{Tr}\, H^M(t)\, t^{s-1}\, dt. \tag{31}$$

Writing $u = 1/t$ and substituting, this becomes

$$\frac{1}{\Gamma(-(-s))} \int_0^1 {}^R\mathrm{Tr}\, H^M(1/u)\, u^{(-s)-1}\, du.$$

We have a phg conormal expansion for $^R\mathrm{Tr}\, H^M(1/u)$ as $u \to 0$; say the leading order term is of the form $u^{z_\infty} (\log u)^p$. Proceeding exactly as in the analysis of (30), we conclude that (31), though initially defined only when $\Re(-s) > -z_\infty$, has a meromorphic continuation to all of $\mathbb{C}$.

This allows us to define the renormalized zeta function and determinant on any asymptotically conic manifold M.

Definition. The renormalized zeta function on M, $^R\zeta_M(s)$, is given by the meromorphic continuation of (29).

Depending on the orders, there may be no s in $\mathbb{C}$ for which (29) is defined; however, once we split the integral at $t = 1$, both pieces are defined in half-planes and continue meromorphically to all of $\mathbb{C}$.

Definition. The *renormalized determinant* of the Laplacian on M is $e^{-R\zeta'_M(0)}$, where $^R\zeta'_M(0)$ is the coefficient of s in the Laurent series for $^R\zeta_M(s)$ at $s = 0$.

3.2. *Manifolds conic near infinity.* We now specialize to the case of manifolds which are precisely conic outside a compact set. In particular, let Z be any asymptotically conic manifold without boundary which is isometric to a cone outside a compact set. Without loss of generality, assume that Z is isometric to a cone when $r \geq 1/2$. We examine the asymptotic expansion of $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z)\, dz$ as $\delta \to 0$; the finite part is precisely the renormalized heat trace. However, for applications, such as the study of conic degeneration in [Sher 2012a], we are also interested in identifying the divergent terms in the expansion. The fact that Z is conic near infinity allows us to identify those terms:

Theorem 15. *Let Z be conic near infinity as above, and let χ_1 be the sharp cutoff. We have the following asymptotic expansion for $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z)\, dz$ as $\delta \to 0$:*

$$\int_Z \chi_{1,\delta}(z) H^Z(t, z, z)\, dz = \sum_{k=0}^{n-1} f_k(t)\delta^{k-n} + f_{\log}(t) \log \delta + {}^R\mathrm{Tr}\, H^Z(t) + R(\delta, t). \tag{32}$$

Here $R(\delta, t)$ goes to zero as δ goes to zero for each fixed t. Moreover, if we let $u_k(1, y)$ be the coefficient of $t^{(k-n)/2}$ in the short-time heat expansion on C_N at the point $(1, y)$, then

$$f_k(t) = \frac{t^{(k-n)/2}}{k-n} \int_N u_k(1, y)\, dy \quad \text{and} \quad f_{\log}(t) = - \int_N u_n(1, y)\, dy.$$

Proof of Theorem 15. Note first that $\int_Z \chi_{1,\delta}(z) H^Z(t, z, z)$ does in fact have a polyhomogeneous expansion in δ, by Theorem 13, so it is just a matter of identifying the terms. The proof involves a comparison of the heat kernels on Z and on C_N; Z and C_N are identical near infinity, which allows us to formulate and prove the following lemma:

Lemma 16. *Let C_N be the infinite cone over N. Then $\left| H^{C_N}(t, z, z) - H^Z(t, z, z) \right|$, defined whenever $r \geq 1$, decays to infinite order in $|z|$ as $|z|$ goes to infinity.*

Proof. Let $\hat{Z} = Z \cap \{r \geq 1\}$. It is a complete manifold with boundary at $r = 1$, and is a subset of both C_N and Z. On $\hat{Z}$, $H^{C_N}(t, z, z') - H^Z(t, z, z')$ is a solution of the heat equation for each z', with initial data equal to zero and boundary data at $r = 1$ given by $H^{C_N}(t, 1, y, z') - H^Z(t, 1, y, z')$. Fix any $T > 0$. We claim that for all $t < T$, $y \in N$, and z' with $|z'| > 2$, there is a constant K so that the absolute value of the boundary data is less than K.

We show this for $H^{C_N}(t, 1, y, z')$ and for $H^Z(t, 1, y, z')$ separately. For C_N, by scaling and noting that $|z'| < 2$,

$$H^{C_N}(t, 1, y, z') = \frac{1}{|z'|^n} H^{C_N}\left(\frac{t}{|z'|^2}, \frac{1}{|z'|}, y, \frac{z'}{|z'|}\right) < H^{C_N}\left(\frac{t}{|z'|^2}, \frac{1}{|z'|}, y, 1, y'\right).$$

For each fixed y', the heat kernel with point source at $(1, y')$ is continuous for $r < 1/2$ (i.e., the tip of the cone), and hence is bounded for $r < 1/2$ and for $t/|z'|^2 < t < T$ by some universal constant K. Since y'

varies only over a compact set, the proof is complete. As for $H^Z(t, 1, y, z')$, consider the region

$$W = \left\{ (t, 1, y, x', y') \,\middle|\, t < T, x' < \tfrac{1}{2}, y \in N, y' \in N \right\}$$

as a subset of (t, x, y, x', y') space. The kernel H^Z has infinite-order decay at each boundary hypersurface of the space in Figure 1 with which W has nontrivial intersection. We conclude that H^Z is bounded on W, so there is a constant K so that $H^Z(t, 1, y, z') < K$ for all $t < T$, all $y \in N$, and all z' with $|z'| > 2$.

Since we have an upper bound for the boundary data, we can construct a supersolution and apply the parabolic maximum principle. Let $g(t, r)$ be the solution of the heat equation on $\hat{Z}$ with zero initial condition and boundary data at $r = 1$ equal to K for all t. By the maximum principle, we see that $\left| H^{C_N}(t, z, z', t) - H^Z(t, z, z') \right| < g(t, |z'|)$ uniformly for $|z'| > 2$ and $t < T$. We claim that $g(t, r)$ decays to infinite order in r, uniformly in t for $t < T$. This can be seen either from Bessel function expansions or by constructing a further supersolution $\hat{g}(t, r)$ modeled on the heat kernel on $\mathbb{R}^n$. In particular, we can use

$$\hat{g}(t, r) = \frac{K/\alpha}{(4\pi)^{n/2}} \sum_{k=-1}^{T_0} \frac{1}{(t-k)^{n/2}} e^{-r^2/4(t-k)} \chi_{\{t>k\}},$$

where $\alpha = (8\pi)^{n/2} e^{1/4}$ and T_0 is the greatest integer less than or equal to T. This supersolution has the uniform exponential decay property we want, so a final application of the parabolic maximum principle finishes the proof of the lemma. $\square$

Corollary 17. *For any fixed t and any $\chi_{1,\delta}$ (either a sharp cutoff or a smooth cutoff),*

$$\left| \int_{C_N} \chi_{1,\delta}(z) H^{C_N}(t, z, z)\, dz - \int_Z \chi_{1,\delta}(z) H^Z(t, z, z)\, dz \right| \tag{33}$$

converges as $\delta \to 0$.

It now suffices to show that

$$\int_{C_N} \chi_{1,\delta}(z) H^{C_N}(t, z, z)\, dz$$

has a divergent asymptotic expansion of the form claimed in Theorem 15, as (33) converges as $\delta \to 0$ and hence contributes only to the finite part of the expansion. (Recall that χ is the sharp cutoff.)

Lemma 18. *Fix t. The divergent terms in the expansion of $\int_{|z| \le 1/\delta} H^{C_N}(t, z, z)$ as $\delta \to 0$ are given by*

$$\sum_{k=0}^{n-1} \frac{t^{(k-n)/2}}{k-n} \delta^{k-n} \int_N u_k(1, y)\, dy - C \log \delta,$$

where C is equal to $\int_N u_n(1, y)\, dy$.

Proof. The integral is, modulo a term independent of δ,

$$\int_1^{1/\delta} \int_N H^{C_N}(t, r, y, r, y)\, r^{n-1}\, dy\, dr.$$

By the conformal homogeneity of C_N, $H^{C_N}(t, r, y, r, y) = r^{-n}H^{C_N}(t/r^2, 1, y, 1, y)$. So the integral becomes

$$\int_1^{1/\delta}\int_N H^{C_N}\left(\frac{t}{r^2}, 1, y, 1, y\right)\frac{1}{r}\,dy\,dr.$$

Now let $s = t/r^2$ and switch to an integral in s; we get

$$\frac{1}{2}\int_{\delta^2 t}^t\int_N H^{C_N}(s, 1, y, 1, y)\frac{1}{s}\,dy\,ds. \tag{34}$$

From short-time heat asymptotics, we know that

$$\int_N H^{C_N}(s, 1, y, 1, y)\,dy = \sum_{k=0}^{n-1}s^{(k-n)/2}\int_N u_k(1, y)\,dy + \int_N u_n(1, y)\,dy + \mathbb{O}(s^{1/2}). \tag{35}$$

Here $u_k(1, y)$ are the heat coefficients on the cone C_N at the point $(1, y)$. We plug (35) into (34) and get

$$\sum_{k=0}^{n-1}\frac{t^{(k-n)/2}}{k-n}\delta^{k-n}\int_N u_k(1, y)\,dy - \log\delta\int_N u_n(1, y)\,dy + g(\delta, t),$$

where $g(\delta, t)$ is finite as $\delta \to 0$. This is what we wanted to prove.   □

Combining this lemma with the preceding corollary and the definition of the renormalized heat trace completes the proof of Theorem 15.   □

Finally, it is also useful to investigate the analogous divergent expansion when a smooth cutoff, rather than a sharp cutoff, is used.

Lemma 19. *Let $X_1(r)$ be as in condition* (a) *of* Section 3.1: *smooth and non-increasing, supported in $r \le 2$ and 1 when $r \le 1/2$. Then*

$$\int_Z X_{1,\delta}(z)H^Z(t, z, z)\,dz = \sum_{k=0}^{n-1}l_k f_k(t)\delta^{k-n} + f_{\log}(t)\log\delta + {}^R\mathrm{Tr}\,H^Z(t) + l_{\log}f_{\log}(t) + \tilde{R}(\delta, t), \tag{36}$$

where $l_k = -\int_{1/2}^2 X_1'(r)r^{k-n}\,dr$ and $l_{\log} = -\int_{1/2}^2 X_1'(r)\log r\,dr$, and $\tilde{R}(\delta, t)$ goes to zero as δ goes to zero for every fixed t.

Proof. Let $\xi(r)$ be any function which is equal to a constant a for $r \le 1/2$ and supported in $\{r \le 2\}$. For any $\delta < 1/2$, we may define a function $\xi_\delta(z)$ on Z by letting $\xi_\delta(z)$ be equal to $\xi \delta r$ for $r = |z| \ge 1$ and a for $\{r \le 1\}$. Then consider the integral

$$\int_Z \xi_\delta(z)H^Z(t, z, z)\,dz, \tag{37}$$

and examine its behavior as $\delta \to 0$.

When $\xi(r)$ is the characteristic function of $[0, 1]$, we have the expansion (32). By replacing δ with δ/b for any $b \in [1/2, 2]$, we can compute the $\delta \to 0$ expansion of (37) for $\xi(r)$ equal to the characteristic

function of $[0, b]$. By linearity, we see that the expansion of (37) for $\xi(r) = (\Delta h)\chi_{[a,b]}$ is

$$\sum_{k=0}^{n-1} (\Delta h) f_k(t) \left(b^{k-n} - a^{k-n}\right) \delta^{k-n} + (\Delta h) f_{\log}(t)\left(\log b - \log a\right) + (\Delta h)\left(R(\delta/b, t) - R(\delta/a, t)\right).$$

Now let $\xi(r) = \chi_1(r) - \chi_{\{r \leq 1\}}$; this is the difference between the sharp and smooth cutoffs. Since the expansions of (37) are linear in ξ, we can approximate by step functions and then integrate by summing over thin horizontal rectangles. Assume for simplicity that $\chi_1(r) = 1$ for all $r \leq 1$ (in the general case, there are some negative signs, but we get the same answer). The thickness of the rectangle at height h is Δh. The length of the rectangle is $\chi_1^{-1}(h) - 1$. Putting all of this together, the expansion of (37) with respect to $\xi(r)$ is

$$\sum_{k=0}^{n-1} f_k(t) \int_0^1 \left((\chi_1^{-1}(h))^{k-n} - 1\right) dh\, \delta^{k-n} + f_{\log}(t) \int_0^1 \log \chi_1^{-1}(h)\, dh + \tilde{R}(\delta, t),$$

where $\tilde{R}(\delta, t)$ is the contribution from the remainder terms.

Finally, perform the change of variables $u = \chi_1^{-1}(h)$, then add the expansion for $\chi_{\{r \leq 1\}}$; we obtain precisely the expansion claimed in the statement of the lemma. This finishes the proof, as long as we can control the remainder term $\tilde{R}(\delta, t)$. Indeed, for each fixed t, we claim that $\tilde{R}(\delta, t)$ goes to zero as δ goes to zero; define a new function $S(\delta, t)$ by letting

$$S(\delta, t) = \sup_{1/2 \leq \gamma \leq 2} \left|R(\delta/\gamma, t)\right|.$$

When $\xi(r) = (\Delta h)\chi_{[a,b]}$, the remainder is bounded in absolute value by $(\Delta h)S(\delta, t)$. So the integral from $h = 0$ to 1 is bounded by $S(\delta, t)$, which goes to zero as $\delta \to 0$; this shows boundedness of the remainder term and finishes the proof of the lemma. □

It is worth examining the dependence on the zeta function on the choice of cutoff χ_1; we used a sharp cutoff to define it, but we could use a smooth cutoff instead. In this case, the finite part of the divergent δ-expansion changes from ${}^R\mathrm{Tr}\, H^Z(t)$ to ${}^R\mathrm{Tr}\, H^Z(t) + l_{\log} f_{\log}(t)$. But $f_{\log}$ and $l_{\log}$ are constants. So the renormalized heat trace only depends on the choice of cutoff function by the addition of a constant, independent of t. However, it can be easily shown by breaking up the integral at $t = 1$ that the meromorphic continuation of $\int_0^\infty C t^k t^{s-1}\, dt$ is identically zero for any constant C. We have shown:

Proposition 20. *Let Z be conic near infinity. The renormalized zeta function and determinant of the Laplacian on Z are independent of the choice of cutoff function $\chi_{1,\delta}$.*

We have now shown the existence of a renormalized zeta function and determinant of the Laplacian on any asymptotically conic manifold M; moreover, when M is conic near infinity, we have computed the divergent terms in the expansion which leads to those renormalizations.

4. The low-energy resolvent in two dimensions

In this section, we extend the techniques used by Guillarmou and Hassell in [2008] to prove Theorem 7. In particular, we construct the low-energy resolvent on an asymptotically conic surface. The resolvent is

$$R(\theta, k, z, z') = (\Delta_M + e^{i\theta}k^2)^{-1}(z, z').$$

For simplicity, we set $\theta = 0$, so that R is a function of (k, z, z'). At the end of the section, we return to discuss allowing arbitrary $\theta \in [-\varphi, \varphi]$ and showing smoothness in θ; however, this is not difficult.

4.1. Strategy. Our goal is to construct the Schwartz kernel of the resolvent, $R(k, z, z')$, as a distribution on $M^2_{k,\mathrm{sc}}$. To do this, as in [Guillarmou and Hassell 2008], we will first construct a parametrix $G(k)$ so that $(\Delta_M + k^2)G(k) = \mathrm{Id} + E(k)$, where $E(k)$ is an error term. $G(k)$ will be a family (in k) of pseudodifferential operators on M whose Schwartz kernel is polyhomogeneous conormal on $M^2_{w,\mathrm{sc}}$ with an interior conormal singularity at the spatial diagonal. By examining the leading order behavior of the equation $(\Delta_M + k^2)G(k) = \mathrm{Id}$ at each boundary hypersurface of $M^2_{w,\mathrm{sc}}$, we obtain a model problem at each hypersurface. The leading order of the parametrix $G(k)$ at each hypersurface should solve the model problem. We first choose solutions of the model problem at each hypersurface, and then check that they are consistent; that is, that they may be glued together to obtain a parametrix $G(k)$. Finally, we analyze the error $E(k)$ and show that it can be removed via a Neumann series argument.

In order to define the appropriate space of pseudodifferential operators, we use certain density conventions, all the same as in [Guillarmou and Hassell 2008]. We consider $P = \Delta_M$ as an operator on *scattering half-densities* by writing

$$P\big(f(x, y)\,|x^{-n-1}\,dx\,dy|^{1/2}\big) = (\Delta_M f)(x, y)\,|x^{-n-1}\,dx\,dy|^{1/2}.$$

As in [Guillarmou and Hassell 2008], we expect a transition between scattering behavior for $k > 0$ and b-behavior at $k = 0$, which leads us to define the conformally related b-metric $g_b = x^2 g$. The space (M, g_b) is asymptotically cylindrical. We then define $P_b = x^{-n/2-1}P x^{n/2-1}$ with respect to scattering half-densities. However, we want to consider P_b acting with respect to *b-half densities* $|x^{-1}\,dx\,dy|^{1/2}$. After this shift, the relationship between P_b acting on b-half densities and P acting on scattering half-densities is $P = x P_b x$.

Let $\tilde{\Omega}_b^{1/2}$ be the bundle of half-densities on $M^2_{k,\mathrm{sc}}$ which is spanned by sections of the form

$$\left| f(k, x, y, x', y') \frac{dg_b\,dg_b'\,dk}{k} \right|^{1/2}.$$

Let ν be a smooth nonvanishing section of this bundle. Since it involves the b-metric g_b, $\tilde{\Omega}_b^{1/2}$ is not the natural bundle near sc. In particular, the kernel of the identity operator on M has leading order $-n/2$ at sc with respect to $\tilde{\Omega}_b^{1/2}$ [Guillarmou and Hassell 2008]. We can now define spaces of pseudodifferential operators, precisely as in [Guillarmou and Hassell 2008]:

Definition. Let ρ_{sc} be a boundary defining function for sc. The space $\Psi_k^{m,\mathscr{E}}(M; \tilde{\Omega}_b^{1/2})$ is the space of half-density kernels $K = K_1 + K_2$ on $M^2_{k,\mathrm{sc}}$ satisfying:

(1) $\rho_{\mathrm{sc}}^{n/2} K_1$ is supported near $\Delta_{k,\mathrm{sc}}$, and has an interior conormal singularity of order m at $\Delta_{k,\mathrm{sc}}$, with coefficients whose behavior at the boundary is specified by $\mathscr{E}$;

(2) $\rho_{\mathrm{sc}}^{n/2} K_2$ is polyhomogeneous conormal on $M_{k,\mathrm{sc}}^2$ with index family $\mathscr{E}$, and moreover decays to infinite order at bf, lb, and rb.

The factor of $\rho_{\mathrm{sc}}^{n/2}$ corrects for the use of b-half densities near sc. Using this definition, we can compute as in [Guillarmou and Hassell 2008] that

$$(P + k^2) \in \Psi_k^{2,\mathscr{E}}(M; \tilde{\Omega}_b^{1/2}),$$

with index sets 0 at sc, 2 at bf_0, 0 at zf, 2 at lb_0, and 2 at rb_0.

As proven in [Guillarmou and Hassell 2008], these spaces satisfy a composition rule:

Proposition 21. *Suppose that $A \in \Psi_k^{m,\mathscr{E}}$ and $B \in \Psi_k^{m',\mathscr{F}}$. Then $A \circ B$ is well defined and an element of $\Psi_k^{m+m',\mathscr{G}}$, where*

$$\mathscr{G}_{\mathrm{sc}} = \mathscr{E}_{\mathrm{sc}} + \mathscr{F}_{\mathrm{sc}}, \quad \mathscr{G}_{\mathrm{zf}} = (\mathscr{E}_{\mathrm{zf}} + \mathscr{F}_{\mathrm{zf}}) \,\overline{\cup}\, (\mathscr{E}_{\mathrm{rb}_0} + \mathscr{F}_{\mathrm{lb}_0}), \quad \mathscr{G}_{\mathrm{bf}_0} = (\mathscr{E}_{\mathrm{bf}_0} + \mathscr{F}_{\mathrm{bf}_0}) \,\overline{\cup}\, (\mathscr{E}_{\mathrm{lb}_0} + \mathscr{F}_{\mathrm{rb}_0}),$$

$$\mathscr{G}_{\mathrm{lb}_0} = (\mathscr{E}_{\mathrm{bf}_0} + \mathscr{F}_{\mathrm{lb}_0}) \,\overline{\cup}\, (\mathscr{E}_{\mathrm{lb}_0} + \mathscr{F}_{\mathrm{zf}}), \quad \mathscr{G}_{\mathrm{rb}_0} = (\mathscr{E}_{\mathrm{rb}_0} + \mathscr{F}_{\mathrm{bf}_0}) \,\overline{\cup}\, (\mathscr{E}_{\mathrm{zf}} + \mathscr{F}_{\mathrm{rb}_0}).$$

We therefore expect our parametrix G to be in $\Psi_k^{-2,\mathscr{G}}$ for some index family $\mathscr{G}$. To gain more information, we need to start analyzing the model problems.

4.2. *The two-dimensional problem.* We begin our analysis of the model problems at the face zf. In order to identify the leading-order part of the equation at a boundary hypersurface, we need to pick a coordinate to use as a boundary defining function in the interior of that face. For all the faces in the lift of $\{k = 0\}$, we use k, which is the easiest choice, since it commutes with P. Since $P + k^2 = x P_b x + k^2$, the leading order part of the operator at zf, which we call the *normal operator*, is $x P_b x$, and the model problem is $(x P_b x) G_{\mathrm{zf}}^0 = \mathrm{Id}$. We therefore expect that G_{zf}^0 will be $(xx')^{-1}$ times some right inverse for P_b.

In order to invert P_b, we use the b-calculus of [Melrose 1993], identifying zf, near bf_0, with the b-double space $X_b^2(x, x') \times N_y \times N_{y'}$. The corner $\mathrm{zf} \cap \mathrm{bf}_0$ corresponds to the front face ff in the b-double space. An easy calculation, following [Guillarmou and Hassell 2008], shows that

$$P_b = -(x\partial_x)^2 + \left(\frac{n}{2} - 1\right)^2 + \Delta_N + W,$$

where W is a lower-order term; that is, W vanishes as a b-differential operator at $x = 0$. In fact, P_b is an elliptic b-differential operator, and hence may be inverted by following the procedure of Melrose, which is described in [Melrose 1993] and [Mazzeo 1991].

The first step in this procedure is to consider the indicial operator, which is the leading order part of P_b at the front face ff. Using the coordinates $(\sigma' = x/x', x', y, y')$, this is

$$I_{\mathrm{ff}}(P_b) = -(\sigma'\partial_{\sigma'})^2 + \left(\frac{n}{2} - 1\right)^2 + \Delta_N.$$

With this terminology, the key theorem is as follows:

Theorem 22 [Melrose 1993]. *P_b is Fredholm as an operator between $x^\delta H_b^2$ and $x^\delta L_b^2$ if and only if δ is not an* indicial root *of $I_{\text{ff}}(P_b)$. (See* [Melrose 1993] *or* [Mazzeo 1991] *for definitions of $x^\delta H_b^2$ and $x^\delta L_b^2$).*

In our setting, as in [Guillarmou and Hassell 2008], the indicial roots are precisely

$$\pm \nu_i = \pm \sqrt{\left(\frac{n}{2} - 1\right)^2 + \lambda_i}\,; \quad \lambda_i \in \sigma(\Delta_N).$$

When $n > 2$, 0 is not an indicial root, and Guillarmou and Hassell show that P_b is not only Fredholm but invertible for $\delta = 0$, and then set G_{zf}^0 to be $(xx')^{-1}$ times that inverse. However, in our case, $n = 2$, so $\nu_i = \sqrt{\lambda_i}$, and 0 is an indicial root. So P_b is not even Fredholm from H_b^2 to L_b^2. This is precisely why the $n = 2$ case is not considered by Guillarmou and Hassell.

4.3. *An example: Euclidean space.* In order to gain some intuition for the behavior of the resolvent near zf in the $n = 2$ setting, we examine the simplest case, which is $M = \mathbb{R}^2$. The resolvent on $\mathbb{R}^2$, acting on scattering half-densities $|dz| = |x^{-3}\, dx\, dy|$, is

$$-\frac{1}{2\pi} H_0\big(k\,|z - z'|\big) = -\frac{1}{2\pi} H_0\left(k \left|\frac{y}{x} - \frac{y'}{x'}\right|\right),$$

where H_0 is the Hankel function of order zero. From the asymptotics of the Hankel function, we know that $H_0(r)$ decays exponentially as $r \to \infty$, and for small r,

$$H_0(r) \sim -\log r + \log 2 - \gamma + \mathbb{O}(r).$$

Using these asymptotics, one can show that the resolvent on $\mathbb{R}^2$ is phg conormal on $M_{k,\text{sc}}^2$. We are most interested in the leading order behavior near zf. In a neighborhood of zf, we have $k\,|z - z'| < 1$, so this leading order behavior is controlled by the small-r asymptotics

$$H_0\big(k\,|z - z'|\big) \sim -\log\big(k\,|z - z'|\big) + \log 2 - \gamma = -\log k - \log|z - z'| + \log 2 - \gamma.$$

Some observations on these asymptotics:

- As we approach zf, the resolvent increases logarithmically. This is a major difference from the $n \geq 3$ case studied in [Guillarmou and Hassell 2008], in which the resolvent is continuous down to zf. On the other hand, the resolvent is continuous down to bf_0, lb_0, and rb_0.

- The function $-(1/2\pi)\log|z - z'|$ is the Green's function for the Laplacian on $\mathbb{R}^2$.

These observations suggest that in two dimensions, we will have a logarithmic term at zf in addition to a zero-order term. We write these terms as $G_{\text{zf}}^{0,1} \log k$ and G_{zf}^0 respectively. From the Euclidean-space example, we expect that, on scattering half-densities,

$$G_{\text{zf}}^{0,1} \log k + G_{\text{zf}}^0 = -C \log k + F(z, z'),$$

where $F(z, z')$ is a right inverse for the operator P. Moreover, since $-C \log k$ has logarithmic growth at bf_0, lb_0, and rb_0 but the resolvent on $\mathbb{R}^2$ does not, we expect that $F(z, z')$ will have logarithmic growth at those faces, with the right coefficient to cancel the logarithmic growth coming from $-C \log k$.

4.4. *Construction of the initial parametrix.* We now construct our parametrix $G(k)$ by specifying its leading order behavior at each boundary hypersurface and then checking that the models are consistent.

4.4.1. *The diagonal,* sc, *and* bf$_0$. The resolvent has an interior conormal singularity at the diagonal $\{z, z'\}$. The symbol of $P + k^2$ is $|\eta|^2 + k^2$, where η is the dual variable of $z - z'$. One can compute that $|\eta|^2 + k^2$ is elliptic on $M_{k,\mathrm{sc}}^2$, with leading orders 0 at sc, 2 at bf$_0$, and 0 at zf. As in [Guillarmou and Hassell 2008], we let the symbol of $G(k)$ be the inverse of $|\eta|^2 + k^2$, in the sense of operator composition. This determines the diagonal symbol of $G(k)$ up to symbols of order $-\infty$, and hence determines $G(k)$ up to operators with smooth Schwartz kernels in the interior of $M_{w,\mathrm{sc}}^2$.

At sc, the analysis is identical to that of Guillarmou and Hassell, so we omit some of the details. The key point is that sc can be described as a fiber bundle with $\mathbb{R}^n$ fibers, parametrized by $y' \in N$ and k. The normal operator of $P + k^2$ is $\Delta_{\mathbb{R}^n} + k^2$, which has a well-defined inverse for $k > 0$. In each fiber, we let

$$G_{\mathrm{sc}}^0 = \left(\Delta_{\mathbb{R}^2} + k^2\right)^{-1}.$$

At bf$_0$, we again follow [Guillarmou and Hassell 2008] exactly. We use the coordinates $(\kappa = k/x, \kappa' = k/x', y, y')$, with k a bdf for bf$_0$. Note that these are only good coordinates on the interior of bf$_0$ — for example, they become degenerate near zf. We then view the interior of bf$_0$ as $\mathbb{R}_+(\kappa) \times \mathbb{R}_+(\kappa') \times N_y \times N_{y'}'$. The normal operator at bf$_0$ is

$$I_{\mathrm{bf}_0}\left(k^{-2}(P + k^2)\right) = \kappa^{-1}\left(-(\kappa \partial_\kappa)^2 + \Delta_N + \kappa^2\right)\kappa^{-1}.$$

Letting $P_{\mathrm{bf}_0} = -(\kappa \partial_\kappa)^2 + \Delta_N + \kappa^2$, the model problem is $(\kappa P_{\mathrm{bf}_0} \kappa) G_{\mathrm{bf}_0}^{-2} = \delta\kappa - \kappa'\delta - y'$. To solve it, we separate variables and invert P_{bf_0}. For each eigenvalue λ_j of Δ_N, write $\nu_j = \sqrt{\lambda_j}$ (these are the indicial roots). Let $E_{\nu_j} \subset L^2(N)$ be the corresponding eigenspace of Δ_N, and let $\Pi_{E_{\nu_j}}$ be projection in $L^2(N)$ onto E_{ν_j}. Then the inverse of P_{bf_0} is

$$Q_{\mathrm{bf}_0} = \sum_{j=0}^{\infty} \Pi_{E_{\nu_j}}\left(I_{\nu_j}(\kappa)K_{\nu_j}(\kappa')\chi_{\{\kappa' > \kappa\}} + I_{\nu_j}(\kappa')K_{\nu_j}(\kappa)\chi_{\{\kappa' < \kappa\}}\right).$$

The only difference between our setting and Guillarmou and Hassell's is that we have $\nu_0 = 0$ as opposed to $\nu_0 > 0$. We then set

$$G_{\mathrm{bf}_0}^{-2} = (\kappa\kappa')Q_{\mathrm{bf}_0}.$$

We need to check consistency between G_{sc}^0 and $G_{\mathrm{bf}_0}^{-2}$; that is, we need to show that they agree to leading order in a neighborhood of sc $\cap$ bf$_0$. This proof is the same as in [Guillarmou and Hassell 2008]; the model problems and formal expressions for G_{sc}^0 and $G_{\mathrm{bf}_0}^{-2}$ are identical. We do have $\nu_0 = 0$, and $K_0(r)$ has different small-r asymptotics from $K_{\nu_j}(r)$ for $\nu_j > 0$; this will be reflected in the asymptotics of $G_{\mathrm{bf}_0}^{-2}$ near zf. However, since κ and κ' both approach infinity near sc, only the large-r asymptotics are relevant for this consistency check, and the large-r asymptotics of $I_\nu(r)$ and $K_\nu(r)$ are no different when $\nu = 0$.

Technically, we also need to check consistency between the diagonal symbol and the models at bf$_0$ and sc. However, this is also the same as in [Guillarmou and Hassell 2008]; the models at bf$_0$ and sc themselves satisfy elliptic pseudodifferential equations given by the leading order part of $(P + k^2)G(k) = \mathrm{Id}$ at those

faces. As a result, their symbols at the diagonal are determined up to symbols of order $-\infty$, and agree up to order $-\infty$ with the inverse of $|\xi|^2 + k^2$.

4.4.2. *The leading order term at* zf. At zf, the model problem with respect to b-half-densities is

$$(x P_b x)\big(G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0\big) = \mathrm{Id},$$

which translates to

$$(x P_b x) G_{\mathrm{zf}}^0 = \mathrm{Id}, \quad (x P_b x) G_{\mathrm{zf}}^{0,1} = 0.$$

Translating our observations in the $M = \mathbb{R}^2$ case to b-half-densities, we expect

$$G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0 = (xx')^{-1}(-C \log k + F(z, z')),$$

where $F(z, z')$ is a right inverse for P_b. We need to pick the correct right inverse; in particular, if we have one right inverse, we may add any function of z' to obtain another right inverse. The correct choice should have logarithmic singularities at all faces and should be consistent with our choice of $G_{\mathrm{bf}_0}^{-2}$. To check consistency, we need to show that $k^0\big(G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0\big)$ and $k^{-2} G_{\mathrm{bf}_0}^{-2}$ agree to leading order at ff $= \mathrm{bf}_0 \cap \mathrm{zf}$, which is the same as checking if $(xx')\big(G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0\big)$ and $Q_{\mathrm{bf}_0} = (\kappa\kappa')^{-1} G_{\mathrm{bf}_0}^{-2}$ agree to leading order there.

First examine the leading order part of Q_{bf_0} at zf. When $x < x'$, we use the coordinates (s, κ, x', y, y'), and we have, where $V = \mathrm{Vol}(N)$,

$$Q_{\mathrm{bf}_0} = V^{-1} I_0(\kappa \sigma') K_0(\kappa) + \sum_{j=1}^{\infty} \Pi_{E_j} I_{\nu_j}(\kappa \sigma') K_{\nu_j}(\kappa). \tag{38}$$

The boundary defining function for zf is κ, so we need to examine the small-κ asymptotics. For $\nu > 0$ we know by standard asymptotics of Bessel functions in [Watson 1944] that

$$I_0(r) \sim 1, \quad K_0(r) \sim -\log r + \log 2 - \gamma, \quad I_\nu(r) \sim \frac{1}{\Gamma(\nu + 1)}\left(\frac{r}{2}\right)^\nu, \quad K_\nu(r) \sim \frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu}.$$

Here γ is the Euler–Mascheroni constant. Plugging these asymptotics into (38) shows that the leading order term in κ is

$$V^{-1}\big(-\log \kappa + \log 2 - \gamma\big) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{(\sigma')^{\nu_j}}{2\nu_j}.$$

On the other hand, when $x > x'$, we use the coordinates $(\sigma, \kappa', x, y, y')$ and perform the same sort of calculations to obtain that the leading order term in κ is

$$V^{-1}\big(-\log \kappa' + \log 2 - \gamma\big) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{\sigma^{\nu_j}}{2\nu_j}.$$

The $x < x'$ and $x > x'$ cases may be combined; we see that the leading order term of Q_{bf_0} at zf is

$$V^{-1}\big(-\log k + \log 2 - \gamma + \log x' + \mathcal{X}_{\sigma' < 1} \log \sigma'\big) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{e^{-\nu_j |\log \sigma'|}}{2\nu_j}. \tag{39}$$

We see immediately that we must have $C = V^{-1}$, and hence we set

$$G_{\text{zf}}^{0,1} = -V^{-1}(xx')^{-1}.$$

We then need to construct G_{zf}^0 so that $(xx')G_{\text{zf}}^0$ has leading order at bf_0 given by

$$V^{-1}\left(\log 2 - \gamma + \log x' + \chi_{\sigma'<1}\log\sigma'\right) + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{e^{-\nu_j|\log\sigma'|}}{2\nu_j}. \tag{40}$$

To construct G_{zf}^0, we must first find the correct right inverse for P_b. Fix δ with $0 < \delta < \nu_1$; then P_b is Fredholm from $x^{-\delta}H_b^2$ to $x^{-\delta}L_b^2$ by Theorem 22. Following the usual b-calculus construction in [Melrose 1993] and [Mazzeo 1991], we obtain a generalized inverse $Q_b^{-\delta}$. We claim:

Lemma 23. *P_b is surjective onto $x^{-\delta}L_b^2$.*

The lemma implies that $Q_b^{-\delta}$ is an exact right inverse for P_b.

Proof. By taking adjoints, the lemma is equivalent to the statement that P_b is injective on $x^{\delta}L_b^2$. Suppose that $u|dg_b|^{1/2}$ is in $x^{\delta}L_b^2$ and satisfies $P_b u = 0$. By regularity of solutions to b-elliptic equations, u is phg conormal on M near $x = 0$; since $u \in x^{\delta}L_b^2$, it decays to at least order δ at $x = 0$. On the other hand, since $P = xP_b x$ and $|dg_b|^{1/2} = x|dg|^{1/2}$, $u|dg|^{1/2}$ is in the kernel of $P = \Delta_M$, and hence $\Delta_M u = 0$. By the maximum principle, $u = 0$, which completes the proof of the lemma. $\qquad\square$

The correct right inverse will be a slight modification of $Q_b^{-\delta}$. In order to check consistency, we need to understand the structure of $Q_b^{-\delta}$ near the front face $\text{ff} = \text{bf}_0 \cap \text{zf}$. This structure is described in detail in [Melrose 1993] and [Mazzeo 1991]. In particular, the leading order of $Q_b^{-\delta}$ at ff is precisely the indicial operator $I_{\text{ff}}(Q_b^{-\delta})$, which satisfies the equation

$$\left(-(\sigma'\partial_{\sigma'})^2 + \Delta_N\right)I_{\text{ff}}(Q_b) = \delta(\sigma' = 1, y = y'). \tag{41}$$

Moreover, from [Melrose 1993] and [Mazzeo 1991], $I_{\text{ff}}(Q_b^{-\delta})$ has polyhomogeneous expansions at $\sigma' = 0$ and $\sigma' = \infty$, with leading order terms at worst $(\sigma')^{-\delta}$ at each end; that is, a small amount of growth is allowed at $\sigma' = 0$, and a small amount of decay is required at $\sigma' = \infty$.

We now separate variables and solve (41) directly. For each $j \geq 1$, $(\sigma')^{\pm\nu_j}$ span the kernel of $-(\sigma'\partial_{\sigma'})^2 + \nu_j^2$. Therefore, the solutions corresponding to E_{ν_j} are combinations of $(\sigma')^{\nu_j}$ and $(\sigma')^{-\nu_j}$ away from $\sigma' = 1$. By the requirements at $\sigma' = 0$ and $\sigma' = \infty$, our solution is a multiple of $(\sigma')^{-\nu_j}$ for $\sigma' > 1$ and of $(\sigma')^{\nu_j}$ for $\sigma' < 1$. Using the matching conditions at $\sigma' = 1$ arising from the delta function singularity, the solution on the eigenspace E_{ν_j} is

$$\Pi_{E_{\nu_j}} \frac{e^{-\nu_j|\log\sigma'|}}{2\nu_j}.$$

We have to consider $\nu_0 = 0$ separately; the kernel of $-(\sigma'\partial_{\sigma'})^2$ is spanned by 1 and $\log\sigma'$. Because we require decay at $\sigma' = \infty$, the solution for $\sigma' > 1$ must be zero. Then the matching conditions at $\sigma' = 1$ imply that the solution is $\log\sigma'$ for $\sigma' < 1$. Since projection onto E_0 is simply V^{-1}, the zero-eigenspace

solution is $V \chi_{\{\sigma' < 1\}} \log \sigma'$. Therefore the leading order part of $Q_b^{-\delta}$ at ff is

$$I_{\mathrm{ff}}(Q_b^{-\delta}) = V^{-1} \chi_{\{\sigma' < 1\}} \log \sigma' + \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} \frac{e^{-\nu_j |\log \sigma'|}}{2\nu_j}. \qquad (42)$$

Now compare (42) with (39). Let $\chi(z')$ be a smooth cutoff function on M, equal to 1 when $x' \in [0, 1]$ and 0 whenever $x' \geq 2$. We see immediately that if we let

$$G_{\mathrm{zf}}^0 = (xx')^{-1} \big(Q_b^{-\delta} + V^{-1} \chi(z') \log x' + V^{-1} (\log 2 - \gamma) \big),$$

then $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ and $G_{\mathrm{bf}_0}^{-2}$ are consistent. Additionally, G_{zf}^0 solves the model problem $(x P_b x) G_{\mathrm{zf}}^0 = \mathrm{Id}$ at zf; the key is that any function of z' is independent of (x, y) and hence is in the kernel of P_b. Similarly, $G_{\mathrm{zf}}^{0,1} \log k = -(V x x')^{-1} \log k$ is in the kernel of $x P_b x$ and hence solves the model problem. Moreover, the diagonal symbol is consistent with $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ for the same reason that it is consistent with G_{sc}^0 and $G_{\mathrm{bf}_0}^{-2}$.

4.4.3. *The model terms at* rb_0. Finally, we need to specify the leading-order behavior of the parametrix at rb_0; in fact, we need to specify some lower-order terms as well. We use the coordinates (x, y, κ', y'); the $\kappa' = 0$ face is $\mathrm{rb}_0 \cap \mathrm{zf}$ and the $x = 0$ face is $\mathrm{rb}_0 \cap \mathrm{bf}_0$. There will be a term $G_{\mathrm{rb}_0}^{\nu_j - 1}$ for each ν_j in $[0, 1)$. The model problem near this face, with k as a boundary defining function, is $(x P_b x) u = 0$, so we need $P_b(x G_{\mathrm{rb}_0}^{-\nu_j}) = 0$ for each $\nu_j \in [0, 1)$.

First we focus on the model of order -1. We let

$$G_{\mathrm{rb}_0}^{-1} = V^{-1} x^{-1} \kappa' K_0(\kappa'),$$

and claim that this is consistent with $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ and $G_{\mathrm{bf}_0}^{-2}$.

To check consistency with $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$, we need to show that the leading order of $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ agrees with the leading order of $k^{-1} G_{\mathrm{rb}_0}^{-1}$ at $\mathrm{zf} \cap \mathrm{rb}_0$. Recall that at rb_0, which corresponds to $s = \infty$, $Q_b^{-\delta}$ decays to a positive order. So $(xx')^{-1} Q_b^{-\delta}$ has leading order greater than -1 at rb_0; therefore, the leading order part of $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ at rb_0 is precisely

$$(xx')^{-1} \big(V^{-1}(-\log \kappa' + \log 2 - \gamma) \big).$$

But by Bessel function asymptotics,

$$k^{-1} G_{\mathrm{rb}_0}^{-1} = V^{-1}(xx')^{-1} K_0(\kappa') \sim (xx')^{-1} V^{-1} \big(-\log \kappa' + \log 2 - \gamma \big).$$

Therefore $G_{\mathrm{zf}}^{0,1} \log k + G_{\mathrm{zf}}^0$ and $G_{\mathrm{rb}_0}^{-1}$ are consistent.

We must also check consistency of $G_{\mathrm{rb}_0}^{-1}$ with $G_{\mathrm{bf}_0}^{-2}$. Near rb_0,

$$k^{-2} G_{\mathrm{bf}_0}^{-2} = V^{-1}(xx')^{-1} I_0(\kappa) K_0(\kappa') + (xx')^{-1} \sum_{j=1}^{\infty} \Pi_{E_{\nu_j}} I_{\nu_j}(\kappa) K_{\nu_j}(\kappa').$$

We are only interested in the order -1 part of this term. Since x' and κ both vanish to first order at rb_0, all the $j > 0$ terms have leading order $-1 + \nu_j$ at rb_0. Since $I_0(0) = 1$, the order -1 part of $k^{-2} G_{\mathrm{bf}_0}^{-2}$ at

rb_0 is precisely

$$(xx')^{-1}V^{-1}K_0(\kappa') = k^{-1}\big(V^{-1}x^{-1}\kappa'K_0(x')\big) = k^{-1}G_{\mathrm{bf}_0}^{-1}.$$

We conclude that $G_{\mathrm{rb}_0}^{-1}$ is consistent with the models at zf and bf_0.

We also need to specify some lower order terms at bf_0; for this we precisely follow [Guillarmou and Hassell 2008, Section 4]. At zf, they need to match with the asymptotics of $Q_b^{-\delta}$, and at bf_0, they need to match with the higher order Bessel functions. Both of these involve only the nonzero indicial roots, so the terms and arguments are identical to [Guillarmou and Hassell 2008]. In particular, for any $0 < \nu_j < 1$ in the indicial set, we let

$$G_{\mathrm{rb}_0}^{\nu_j-1} = x^{-1}\frac{\kappa'K_{\nu_j}(\kappa')}{\Gamma(\nu_j)2^{\nu_j-1}}\, v_{\nu_j}(z, y'),$$

where $v_{\nu_j}(z, y')$ is in the kernel of P_b with asymptotic

$$v_{\nu_j}(x, y, y') = (2\nu_j)^{-1}\Pi_{E_j}x^{-\nu_j} + \mathbb{O}(x^{-\nu_j-1}\log x).$$

The function v_{ν_j} is there to match with the asymptotics of Q_b at rb_0, as in [Guillarmou and Hassell 2008, Section 4]. In fact, these models are consistent with our models at bf_0 and at zf by precisely the same argument as in [Guillarmou and Hassell 2008]; we will not repeat it here.

4.5. *The final parametrix and resolvent.* We have now constructed models at sc, bf_0, zf, and rb_0 which are consistent with each other and also with the diagonal symbol. Moreover, all the models decay to infinite order as we approach lb, rb, or bf. Therefore, we specify our parametrix $G(k)$ to be any pseudodifferential operator in $\Psi_k^{-2,\mathscr{E}}$ with kernel having the specified diagonal symbol and specified leading-order terms at sc, bf_0, zf, and rb_0. The consistency we checked guarantees that such an operator exists. The behavior of the kernel of $G(k)$ at lb_0 may be freely chosen as long as the leading-order term is order -1 and it matches with our models at zf and bf_0; a term of order -1 will, however, be required.

Now let $E(k) = (P+k^2)G(k) - \mathrm{Id}$. Since $G(k)$ has diagonal symbol equal to the inverse of the symbol of $P+k^2$, the Schwartz kernel of $E(k)$ is smooth on the interior of $M_{w,\mathrm{sc}}^2$. Moreover, since $P+k^2$ is a differential operator, the Schwartz kernel of $E(k)$ is phg conormal on $M_{w,\mathrm{sc}}^2$.

- At lb, rb, and bf, the Schwartz kernel of $G(k)$ vanishes to infinite order along with all derivatives, so the same is true of $E(k)$.

- At sc, G_{sc}^0 solves the model problem, so $E(k)$ has positive leading order at sc.

- At bf_0, $G(k)$ has order -2, but $G_{\mathrm{bf}_0}^{-2}$ solves the model problem, and moreover $P+k^2$ vanishes to second order. Therefore $E(k)$ has positive leading order at bf_0.

- At zf, G_{zf}^0 and $G_{\mathrm{zf}}^{0,1}$ solve the model problem, so $E(k)$ has positive leading order.

- At lb_0, $G(k)$ has order -1. The variables k and x both vanish at lb_0, so $k^2G(k)$ has order 1 and $xG(k)$ has order 0. Since P_b is a b-differential operator, $P_b(xG(k))$ also has order 0, and hence $(P+k^2)G(k) = (xP_bx+k^2)G(k)$ has order 1. Since Id is supported away from lb_0, $E(k)$ decays to at least order 1 at lb_0.

- At rb_0, $G(k)$ has order -1, but all the terms G^{ν_j-1} for $\nu_j \in [0, 1)$ solve the model problem. Therefore, the error $E(k)$ has leading order at worst 0.

To summarize, if $\mathscr{E}$ is the index set for $E(k)$, we have shown:

$$\mathscr{E}_{sc} > 0, \quad \mathscr{E}_{zf} > 0, \quad \mathscr{E}_{bf_0} > 0, \quad \mathscr{E}_{lb_0} \geq 1, \quad \mathscr{E}_{rb_0} \geq 0, \quad \mathscr{E}_{lb} = \mathscr{E}_{rb} = \mathscr{E}_{bf} = \varnothing.$$

Now we iterate away the error. By Proposition 21, $E(k)^2$ vanishes to positive order at all faces of $M^2_{k,sc}$; suppose that the order of vanishing at each face is greater than $\epsilon > 0$. Again applying Proposition 21, we see that for each $N \in \mathbb{N}$, the order of vanishing of $E(k)^{2N}$ and $E(k)^{2N+1}$ at each face of $M^2_{k,sc}$ is greater than $N\delta$. Therefore the Neumann series

$$\left(\mathrm{Id} + E(k)\right)^{-1} = \sum_{i=0}^{\infty} E(k)^i$$

may be summed asymptotically, and the sum defines an element of $\Psi_k^{-\infty,\hat{\mathscr{E}}}$ for some index family $\hat{\mathscr{E}}$.

Finally, let $R(k) = G(k)(\mathrm{Id} + E(k))^{-1}$; we see that $(P + k^2)R(k) = \mathrm{Id}$. Since $P + k^2$ is invertible for all positive k, its only right inverse is the resolvent. We conclude that $R(k)$ is in fact the resolvent, and it is an element of $\Psi_k^{-\infty,\mathscr{R}}(M, \tilde{\Omega}_b^{1/2})$ for some index family $\mathscr{R}$.

In order to prove Theorem 7, we need to perform this construction for any angle $\theta \in (-\pi, \pi)$, not just for $\theta = 0$. However, as claimed in [Guillarmou and Hassell 2008], the construction is essentially unchanged. Indeed, we just use $e^{i\theta/2}k$ as our boundary defining function for the $k = 0$ faces instead of k, and correspondingly change the model at sc from $(\Delta_{\mathbb{R}^2} + k^2)^{-1}$ to $(\Delta_{\mathbb{R}^2} + e^{i\theta}k^2)^{-1}$. The construction is then precisely analogous to the $\theta = 0$ case; by construction, the index sets are independent of θ. Moreover, by the continuity of the resolvent outside the spectrum (also by construction), all the dependence on θ is smooth. This completes the proof.

4.6. *Leading orders of the resolvent.*

Since $R(k) = G(k) - G(k)E(k) + G(k)E(k)^2 - \cdots$, we can obtain some information about the leading orders of $R(k)$ at each face. For $n \geq 3$, it is shown in [Guillarmou and Hassell 2008] that the leading orders of $R(k)$ are the same as those of $G(k)$; we claim that the same is true when $n = 2$.

When $n = 2$, $G(k)$ has leading orders -1 at lb_0 and rb_0, order 0 at sc, order -2 at bf_0, and logarithmic growth at zf. $E(k)$ has non-negative leading orders at all faces, and it is easy to use Proposition 21 to show that the leading orders of $G(k)E(k)$ are no worse than those of $G(k)$. Similarly, it may be shown that the leading orders of $G(k)E(k)^l$ are no worse than those of $G(k)$. Since $G(k)$ is fixed and $\mathrm{Id} - E(k) + E(k)^2 - \cdots$ is asymptotically summable, the series $G(k) - G(k)E(k) + \cdots$ is also asymptotically summable; therefore, the leading orders of $R(k)$ are no worse than those of $G(k)$. The leading order terms themselves may be affected by $G(k)E(k)$, but the orders are not.

To summarize, when $n = 2$, the leading orders of the exact resolvent $R(k)$ are no worse than 0 at sc, -2 at bf_0, logarithmic at zf, and -1 at lb_0 and rb_0. When $n \geq 3$, the orders are at worst 0 at sc, -2 at bf_0, 0 at zf, and $n/2 - 2$ at lb_0 and rb_0, as in [Guillarmou and Hassell 2008]. However, these orders are for the resolvent acting on b-half-densities, rather than the more natural scattering half-densities. Switching to

scattering half-densities requires an order shift, adding $n/2$ at each of $\{x = 0\}$ and $\{x' = 0\}$. So we need to add $n/2$ to the orders at lb_0 and rb_0, and n at bf_0. The order at sc remains unchanged, because the extra factor of $\rho_{sc}^{n/2}$ in the definition of the calculus already incorporates the shift. So: viewing the resolvent as a scattering half-density $|dg\,dg'|^{1/2}$ acting on scattering half-densities for each k, or equivalently as a function acting on functions on M by integration against dg, it has leading orders given by

- 0 at sc and $n - 2$ at bf_0, rb_0, and lb_0;
- r_{zf} at zf, where $r_{zf} = 0$ if $n \geq 3$ and $r_{zf} = (0, 1)$ (that is, leading order behavior of $\log \rho_{zf}$) if $n = 2$.

Appendix: Construction of the short-time heat kernel

Albin has created a framework for the construction of the heat kernel on an asymptotically conic manifold; essentially all of the hard work involved in this construction has already been done [2007]. To complete the construction and prove Theorem 3, all we need to do is create an initial parametrix for the heat kernel. This construction is the content of this short appendix and is based on [Albin 2007, Section 5], in which the heat kernel on an edge manifold is constructed.

The space in Theorem 3, which we call S_{heat}, is obtained by taking the manifold $M_{sc}^2 \times [0, T)_t$ and then blowing up the $t = 0$ diagonal. We call the scattering face of $M_{sc}^2 \times [0, T)_t$ sf and call the front face at $t = 0$ ff. The heat operator is precisely $\partial_t + \Delta_M$. Our goal is to create a parametrix which, to first order, solves the normal equations at sf and ff.

First analyze the situation at sf. As first discussed in [Melrose 1994] and elaborated upon in [Guillarmou and Hassell 2008], sf has a Euclidean structure, parametrized by $y \in N$ and $t \in [0, T)$. By the same analysis as in the latter paper, the normal operator at sf is precisely $\partial_t + \Delta_{\mathbb{R}^n}$. We then simply let the model at sf be the Euclidean heat kernel, $H^{\mathbb{R}^n}$. This is analogous to the construction in [Albin 2007, Section 5] for the edge setting, in which the model is the heat kernel on hyperbolic space times the heat kernel on the fiber.

The analysis at ff is also standard, since ff corresponds to the short-time regime on the interior of the manifold, where the heat kernel asymptotics are local. We know that $d(z, z')/\sqrt{t}$ is a good coordinate along ff, zero at the spatial diagonal and increasing to infinity as we approach the original $t = 0$ face. We therefore let the leading-order model at ff be

$$H_{ff} = \frac{1}{(4\pi t)^{n/2}} e^{-d(z,z')^2/4t}.$$

The choice of model at ff is again based on the Euclidean heat kernel, and is precisely the same as the choice of model in the edge setting [Albin 2007].

Each model vanishes to infinite order as we approach all boundary hypersurfaces other than sf and ff. Moreover, the models are consistent, as the leading orders of each are precisely the Euclidean heat kernel at $sf \cap ff$. We may therefore pick a pseudodifferential operator whose Schwartz kernel agrees with our models to leading order at sf and ff, and decays to infinite order at all other boundary faces. Albin [2007, Section 4] proves a composition rule for time-dependent pseudodifferential operators whose kernels are polyhomogeneous conormal on S_{heat} — our setting is the "scattering" setting, which is included

in his analysis. We then use this composition rule and an iteration argument, precisely as in [Albin 2007, Section 5], to construct the heat kernel as a polyhomogeneous conormal distribution on S_{heat}. This completes the proof of Theorem 3.

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