ANALYSIS & PDE

Volume 6 No. 7 2013

MICHAŁ KOWALCZYK, YONG LIU AND FRANK PACARD

THE CLASSIFICATION OF FOUR-END SOLUTIONS TO THE ALLEN–CAHN EQUATION ON THE PLANE

msp

ANALYSIS AND PDE
Vol. 6, No. 7, 2013
dx.doi.org/10.2140/apde.2013.6.1675

THE CLASSIFICATION OF FOUR-END SOLUTIONS TO THE ALLEN–CAHN EQUATION ON THE PLANE

MICHAŁ KOWALCZYK, YONG LIU AND FRANK PACARD

An entire solution of the Allen–Cahn equation $\Delta u = f(u)$, where f is an odd function and has exactly three zeros at ± 1 and 0, for example, $f(u) = u(u^2 - 1)$, is called a $2k$-end solution if its nodal set is asymptotic to $2k$ half lines, and if along each of these half lines the function u looks like the one-dimensional, heteroclinic solution. In this paper we consider the family of four-end solutions whose ends are almost parallel at ∞. We show that this family can be parametrized by the family of solutions of the Toda system. As a result we obtain the uniqueness of four-end solutions with almost parallel ends. Combining this result with the classification of connected components in the moduli space of the four-end solutions, we can classify all such solutions. Thus we show that four-end solutions form, up to rigid motions, a one parameter family. This family contains the saddle solution, for which the angle between the nodal lines is $\pi/2$, as well as solutions for which the angle between the asymptotic half lines of the nodal set is any $\theta \in (0, \pi/2)$.

1. Introduction

Some entire solutions to the Allen–Cahn equation in $\mathbb{R}^2$. This paper deals with the problem of classification of the family of four-end solutions (precise definition will follow) to the Allen–Cahn equation:

$$\Delta u = F'(u) \quad \text{in } \mathbb{R}^2. \tag{1-1}$$

The function F is a smooth double well potential, which means that we assume the following conditions for F: F is even, nonnegative, and has only two zeros at ± 1, $F'(t) \neq 0$, $t \in (0, 1)$. We also suppose $F''(1) \neq 0$, $F''(0) \neq 0$. For convenience, we assume that F is such that $F''(1) = 2$. A standard example is $F(u) = \frac{1}{4}(1 - u^2)^2$.

It is known that (1-1) has a solution whose nodal set is a straight line. This will be called a *planar solution*. It is obtained simply by taking the unique, odd, heteroclinic solution connecting -1 to 1

$$H'' = F'(H), \quad H(\pm\infty) = \pm 1, \quad H(0) = 0, \tag{1-2}$$

and letting $u(x, y) = H(ax + by + c)$ for some constants a, b, c such that $a^2 + b^2 = 1$. We note that if $a > 0$, then $\partial_x u = aH' > 0$. The De Giorgi conjecture says that if u with $|u| < 1$ is a smooth solution of (1-1)

Kowalczyk was partially supported by Chilean research grants Fondecyt 1090103, Fondo Basal CMM-Chile, Project Añillo ACT-125 CAPDE, and a MathAmSud project NAPDE. Liu was partially supported by Chilean research grants Fondecyt 3100011, Fondo Basal CMM-Chile and NSFC grant 11101141. Pacard was partially supported by the ANR-11-IS01-0002 grant and a MathAmSud project NAPDE..
MSC2010: 35J61.
Keywords: Allen–Cahn equation, entire solutions, moduli space, Toda system, four-end solutions.

such that $\partial_{\boldsymbol{e}} u > 0$ for a certain fixed direction $\boldsymbol{e}$, then u must in fact be a planar solution. Indeed, this conjecture holds in $\mathbb{R}^N$, $N \le 8$ (see [Ghoussoub and Gui 1998] when $N = 2$, [Ambrosio and Cabré 2000] when $N = 3$, and [Savin 2009] for $4 \le N \le 8$ under an additional limit condition), while a counterexample can be given when $N \ge 9$ [del Pino et al. 2011]. It is worth mentioning that the De Giorgi conjecture is a direct analogue of the famous Bernstein conjecture in the theory of minimal surfaces.

In order to proceed with the statement of our results, we will define the family of four-end solutions of (1-1), which is a particular example of a more general family of $2k$-end solutions [del Pino et al. 2013]. Intuitively, a four-end solution u is characterized by the fact that its nodal set $N(u)$ is asymptotic at infinity to four half lines, and along each of these half lines it looks locally like the heteroclinic solution. To describe this precisely, we introduce the set Λ_4 of oriented and ordered four affine lines in $\mathbb{R}^2$. Thus Λ_4 consists of 4-tuples $(\lambda_1, \dots, \lambda_4)$ such that each λ_j can be uniquely written as

$$\lambda_j := r_j \boldsymbol{e}_j^{\perp} + \mathbb{R}\boldsymbol{e}_j$$

for some $r_j \in \mathbb{R}$ and some unit vector $\boldsymbol{e}_j = (\cos\theta_j, \sin\theta_j) \in S^1$, which defines the orientation of the line. Here, the symbol $\perp$ refers to the rotation of angle $\pi/2$ in $\mathbb{R}^2$. Observe that the affine lines are oriented, and hence we do not identify the line corresponding to (r_j, θ_j) and the line corresponding to $(-r_j, \theta_j + \pi)$. Additionally we require that these lines are ordered, which means

$$\theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi + \theta_1.$$

For future purposes we denote by

$$\theta_\lambda := \tfrac{1}{2} \min\{\theta_2 - \theta_1, \theta_3 - \theta_2, \theta_4 - \theta_3, 2\pi + \theta_1 - \theta_4\} \tag{1-3}$$

the half of the minimum of the angles between any two consecutive oriented affine lines of $\lambda_1, \dots, \lambda_4$.

Assume that we are given a 4-tuple of oriented affine lines $\lambda = (\lambda_1, \dots, \lambda_4)$. It is easy to check that for all $R > 0$ large enough and for all $j = 1, \dots, 4$, there exists $s_j \in \mathbb{R}$ such that

(i) the point $\boldsymbol{x}_j := r_j \boldsymbol{e}_j^{\perp} + s_j \boldsymbol{e}_j$ belongs to the circle ∂B_R, with $R > 0$;

(ii) the half lines
$$\lambda_j^+ := \boldsymbol{x}_j + \mathbb{R}^+ \boldsymbol{e}_j \tag{1-4}$$
are disjoint and included in $\mathbb{R}^2 \setminus B_R$;

(iii) the minimum of the distance between two distinct half lines λ_i^+ and λ_j^+ is larger than 4.

The set of affine half lines $\lambda_1^+, \dots, \lambda_4^+$ together with the circle ∂B_R induces a decomposition of $\mathbb{R}^2$ into five slightly overlapping connected components

$$\mathbb{R}^2 = \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_4,$$

where

$$\Omega_0 := B_{R+1},$$

and where, for $j = 1, \dots, 4$,

$$\Omega_j := \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > R - 1 \text{ and } \operatorname{dist}(\boldsymbol{x}, \lambda_j^+) < \operatorname{dist}(\boldsymbol{x}, \lambda_i^+) + 2 \text{ for all } i \neq j\}, \tag{1-5}$$

where $\operatorname{dist}(\boldsymbol{x}, \lambda_j^+)$ denotes the distance of $\boldsymbol{x}$ to λ_j^+. Observe that, for all $j = 1, \dots, 4$, the set Ω_j contains the half line λ_j^+.

We consider a smooth partition of unity of $\mathbb{R}^2$ given by the functions $\mathbb{I}_0, \mathbb{I}_1, \dots, \mathbb{I}_4$, which is subordinate to the above decomposition of $\mathbb{R}^2$. Hence

$$\sum_{j=0}^{4} \mathbb{I}_j \equiv 1,$$

and the support of $\mathbb{I}_j$ is included in Ω_j for $j = 0, \dots, 4$. Without loss of generality, we can also assume that $\mathbb{I}_0 \equiv 1$ in

$$\Omega_0' := B_{R-1},$$

and $\mathbb{I}_j \equiv 1$ in

$$\Omega_j' := \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| > R + 1 \text{ and } \operatorname{dist}(\boldsymbol{x}, \lambda_j^+) < \operatorname{dist}(\boldsymbol{x}, \lambda_i^+) - 2 \text{ for all } i \neq j\}$$

for $j = 1, \dots, 4$. Finally, we assume that

$$\|\mathbb{I}_j\|_{\mathscr{C}^2(\mathbb{R}^2)} \leq C.$$

We now take $\lambda = (\lambda_1, \dots, \lambda_4) \in \Lambda_4$ with $\lambda_j^+ = \boldsymbol{x}_j + \mathbb{R}^+ \boldsymbol{e}_j$ and we define

$$u_\lambda(\boldsymbol{x}) := \sum_{j=1}^{4} (-1)^j \mathbb{I}_j(\boldsymbol{x}) H((\boldsymbol{x} - \boldsymbol{x}_j) \cdot \boldsymbol{e}_j^\perp). \tag{1-6}$$

Observe that, by construction, the function u_λ is, away from a compact set, asymptotic to copies of planar solutions whose nodal set is the affine half lines $\lambda_1^+, \dots, \lambda_4^+$. A simple computation shows that u_λ is not far from being a solution of (1-1) in the sense that $\Delta u_\lambda - F'(u_\lambda)$ is a function which decays exponentially to 0 at infinity (this uses the fact that $\theta_\lambda > 0$).

In this paper we are interested in four-end solutions of (1-1), which means that they are asymptotic to a function u_λ for some choice of $\lambda \in \Lambda_4$. More precisely, we have:

Definition 1.1. Let $\mathscr{S}_4$ denote the set of functions u which are defined in $\mathbb{R}^2$ and which satisfy

$$u - u_\lambda \in W^{2,2}(\mathbb{R}^2) \tag{1-7}$$

for some $\lambda \in \Lambda_4$. We also define the decomposition operator $\mathscr{J}$ by

$$\mathscr{J} : \mathscr{S}_4 \to W^{2,2}(\mathbb{R}^2) \times \Lambda_4, \quad u \mapsto (u - u_\lambda, \lambda).$$

The topology on $\mathscr{S}_4$ is the one for which the operator $\mathscr{J}$ is continuous (the target space being endowed with the product topology). We define the set $\mathscr{M}_4$ of four-end solutions of the Allen–Cahn equation to be the set of solutions u of (1-1) which belong to $\mathscr{S}_4$.

The set $\mathcal{M}_4$ is nonempty. Indeed, it is known [Dang et al. 1992] that (1-1) has a saddle solution U, which is bounded and symmetric:

$$U(x, y) = U(x, -y) = U(-x, y).$$

Moreover, the nodal set of U coincides with the lines $y = \pm x$. Along these two lines, U converges exponentially fast to the "heteroclinic" solution. In addition, in [del Pino et al. 2010] it is shown that there exists a small number ε_0 such that, for all $0 < \theta$ with $\tan\theta < \varepsilon_0$, there exists a four-end solution with corresponding angles of the half lines λ_j^+, $j = 1, \dots, 4$ given by

$$\theta_1 = \theta, \quad \theta_2 = \pi - \theta, \quad \theta_3 = \theta + \pi, \quad \theta_4 = 2\pi - \theta.$$

Observe that the fact that θ is small implies that the ends of this solution are almost parallel and their slopes, given by $\pm\varepsilon$, $\varepsilon = \tan\theta$, are small as well. Clearly, by symmetry, it is easy to see that there also exist solutions with almost parallel ends whose angles are given by

$$\theta_1 = \pi/2 - \theta, \quad \theta_2 = \pi/2 + \theta, \quad \theta_3 = -\theta + 3\pi/2, \quad \theta_4 = 3\pi/2 + \theta.$$

In this case we have $\tan\theta_1 > 1/\varepsilon_0$.

Clearly, any four-end solution can be translated and rotated and multiplied by -1, yielding another four-end solution. In fact, from [Gui 2012] we know that any $u \in \mathcal{M}_4$ is (modulo rigid motions and multiplication of a solution by -1) even in its variables, monotonic in x in the set $x > 0$, and monotonic in y in the set $y < 0$:

$$u(x, y) = u(-x, y) = u(x, -y), \quad u_x(x, y) > 0, \quad x > 0, \quad u_y(x, y) > 0, \quad y < 0. \tag{1-8}$$

Thus, when studying four-end solutions, it is natural to consider the set $\mathcal{M}_4^{\text{even}} \subset \mathcal{M}_4$, consisting precisely of functions satisfying (1-8). With each such function u we may associate in a unique way the angle that the asymptotic line of its nodal set in the first quadrant makes with the x-axis. Thus we can define *the angle map*

$$\theta : \mathcal{M}_4^{\text{even}} \to (0, \pi/2), \quad u \mapsto \theta(u). \tag{1-9}$$

In principle the value of the angle map is not enough to identify in a unique way a solution to (1-1) in $\mathcal{M}_4^{\text{even}}$. However, for solutions with almost parallel ends, we have:

Theorem 1.2. *There exists a small number ε_0 such that, for any two solutions $u_1, u_2 \in \mathcal{M}_4^{even}$ satisfying $\tan\theta(u_1) = \tan\theta(u_2) < \varepsilon_0$, we necessarily have $u_1 \equiv u_2$.*

This result, in some sense, gives a classification of the subfamily of the family of four-end solutions which contains solutions with almost parallel ends. It says that this subfamily consists precisely of the solutions constructed in [del Pino et al. 2010]. Let us explain the importance of this statement from the point of view of classification of all four-end solutions. We will appeal to the following theorem.

Theorem 1.3 [Kowalczyk et al. 2012]. *Let M be any connected component of $\mathcal{M}_4^{even}$. Then the angle map $\theta : M \to (0, \pi/2)$ is surjective.*

Consider, for example, the connected component $M_0 \subset \mathcal{M}_4^{\text{even}}$ which contains the saddle solution U. Theorem 1.3 implies that U can be deformed along M_0 to a solution with the value of the angle map arbitrarily close to 0 or to $\pi/2$, thus yielding a solution in the subfamily of the solutions with almost parallel ends. But these solutions are uniquely determined by the value of the angle map, which follows from the uniqueness statement in Theorem 1.2. As a result we obtain the following classification theorem.

Theorem 1.4. *Any solution $u \in \mathcal{M}_4^{even}$ belongs to M_0 and is a continuous deformation of the saddle solution U.*

We observe that, according to the conjecture of De Giorgi, in two dimensions, any solution u with $|u| < 1$ which is monotone in one direction must be one-dimensional and equal to $u(\boldsymbol{x}) = H(\boldsymbol{a} \cdot \boldsymbol{x} + b)$, that is, it is a planar solution. In the language of multiple end solutions, this solution has *two* (heteroclinic, planar) ends. Theorem 1.4, on the other hand, gives the classification of the family of solutions with *four* planar ends. Since the number of ends of a solution to (1-1) must be even, the family of four-end solutions is the natural object to study. In this context, one may wonder if it is possible to classify solutions to (1-1) assuming, for instance, that the nodal sets of u_x and u_y have just one component. This question is beyond the scope of this paper, however, since partial derivatives of four-end solutions satisfy this assumption, it seems reasonable to conjecture that a result similar to Theorem 1.4 should hold in this more general setting. We should mention here that it is, in principle, possible to study the problem of classification of solutions assuming, for example, that their Morse index is 1. This is natural since the Morse index of u and the number of the nodal domains of u_x and u_y are related. We recall here that the heteroclinic is stable, and, from [Dancer 2005], we know that in dimension $N = 2$, stability of a solution implies that it is necessarily a one-dimensional solution (for the related minimality conjecture, see, for example, [Pacard and Wei 2013; Savin 2009]. We expect that in fact the family of four-end solutions should contain all multiple end solutions with Morse index 1. We recall here that the Morse index of the saddle solution is indeed 1 [Schatzman 1995].

Let us now explain the analogy of Theorem 1.4 with some aspects of the theory of minimal surfaces in $\mathbb{R}^3$. In 1834, Scherk discovered an example of a singly periodic, embedded, minimal surface in $\mathbb{R}^3$ which, in a complement of a vertical cylinder, is asymptotic to 4 half-planes with angle $\pi/2$ between them. This surface, after a rigid motion, has two planes of symmetry, say $\{x_2 = 0\}$ and $\{x_1 = 0\}$, and it is periodic, with period 1 in the x_3 direction. If θ is the angle between the asymptotic end of the Scherk surface contained in $\{x_1 > 0, x_2 > 0\}$ and the $\{x_2 = 0\}$ plane, then $\theta = \pi/4$. This is the so-called second Scherk surface and it will be denoted here by $S_{\pi/4}$. Karcher [1988] found Scherk surfaces other than the original example in the sense that the corresponding angle between their asymptotic planes and the $\{x_2 = 0\}$ plane can be any $\theta \in (0, \pi/2)$. The one parameter family $\{S_\theta\}_{\{0<\theta<\pi/2\}}$ of these surfaces is the family of Scherk singly periodic minimal surfaces. Thus, accepting that the saddle solution of the Allen–Cahn equation U corresponds to the Scherk surface $S_{\pi/4}$, Theorem 1.3 can be understood as an analogue of the result of Karcher. We note that, unlike in the case of the Allen–Cahn equation, the Scherk family is given explicitly. For example, it can be represented as the zero level set of the function

$$F_\theta(x_1, x_2, x_3) = \cos^2\theta \cosh\frac{x_1}{\cos\theta} - \sin^2\theta \cosh\frac{x_2}{\cos\theta} - \cos x_3.$$

From this, it follows immediately that the angle map in this context $S_\theta \mapsto \theta$ is a diffeomorphism. A corresponding result for the family $\mathcal{M}_4^{\text{even}}$ is of course more difficult, since no explicit formula is available in this case.

We will further explore the analogy of our result with the theory of minimal surfaces in $\mathbb{R}^3$, now in the context of the classification of the four-end solutions in Theorem 1.4. The corresponding problem can be stated as follows: if S is an embedded, singly periodic, minimal surface with 4 Scherk ends, what can be said about this surface? It is proven by Meeks and Wolf [2007] that S must be one of the Scherk surfaces S_θ described above (a similar result is proven in [Pérez and Traizet 2007] assuming additionally that the genus of S in the quotient $\mathbb{R}^3/\mathbb{Z}$ is 0). The key results to prove this general statement are in fact the counterparts of Theorem 1.2 and Theorem 1.3.

We now sketch the basic elements in the proof of Theorem 1.2. First of all, let us explain the existence result in [del Pino et al. 2010]. The starting point of the construction is the Toda system:

$$\begin{cases} q_1'' = -c_* e^{\sqrt{2}(q_1-q_2)}, \\ q_2'' = c_* e^{\sqrt{2}(q_1-q_2)}, \end{cases} \tag{1-10}$$

for which $q_1 < 0 < q_2$ and $q_1(x) = -q_2(x)$, as well as $q_j(x) = q_j(-x)$, $j = 1, 2$. Here c_* is a fixed constant depending only on F (when $F(u) = \frac{1}{4}(1-u^2)^2$, $c_* = 12\sqrt{2}$), and $\sqrt{2}$ appears because we have assumed $F''(1) = 2$. Any solution of this system is asymptotically linear, namely,

$$q_j(x) = (-1)^j(m|x| + b) + \mathcal{O}(e^{-2\sqrt{2}m|x|}), \quad x \to \infty,$$

where $m > 0$ is the slope of the asymptotic straight line in the first quadrant. On the other hand, given that we only consider solutions whose trajectories are symmetric with respect to the x-axis, the value of the slope m determines the unique solution of (1-10). When the asymptotic lines become parallel, $m \to 0$ or $m \to \infty$. By symmetry, it suffices to consider the case $m \to 0$, and in this paper we will denote small slopes by $m = \varepsilon$ and the corresponding solutions by $q_{\varepsilon,j}$. Note that if by $q_{1,j}$ we denote a solution with $m = 1$, then

$$q_{\varepsilon,j}(x) = q_{1,j}(\varepsilon x) + \frac{(-1)^j}{\sqrt{2}} \ln \frac{1}{\varepsilon}.$$

Then, the existence result in [del Pino et al. 2010] implies that given a small ε, there exists a four-end solution u to (1-1) whose nodal set $N(u)$ is close to the trajectories of the Toda system given by the graphs of $y = q_{\varepsilon,j}(x)$. It turns out that the idea of relating solutions of the Toda system and the four-end solutions of (1-1) [ibid.] is very important. In fact, what we want to achieve is to parametrize the manifold of four-end solutions with almost parallel ends using corresponding solutions of the Toda system as parameters. To do this, in Sections 3–5 we obtain a very precise control of the nodal sets of the four-end solutions. The key observation is that in every quadrant the nodal set $N(u)$ of any four-end solution is a bigraph, and if we assume that the slope of its asymptotic lines is small, it is a graph of a smooth function, both in the lower and in the upper half-plane. We then have

$$N(u) = \{(x, y) \in \mathbb{R}^2 : y = f_{\varepsilon,j}(x),\ j = 1, 2,\ f_{\varepsilon,1}(x) < 0,\ f_{\varepsilon,2}(x) = -f_{\varepsilon,1}(x)\}$$

for any $u \in \mathcal{M}_4^{\text{even}}$, with $\varepsilon = \tan\theta(u)$. Our main result in Section 4 says that, for each ε small,

$$f_{\varepsilon,1}(x) - q_{\varepsilon,1}(x) = C\varepsilon^{\alpha} + \mathbb{O}(\varepsilon^{\alpha} e^{-\varepsilon\beta|x|})$$

with some positive constants α, β. Next, we define (Section 6) a suitable approximate four-end solution based on the solution of the Toda system with slope ε. To explain this, by $\widetilde{\mathcal{N}}_{\varepsilon,1}$ we denote the graph of the function $y = q_{\varepsilon,1}(x)$, which is contained in the lower half-plane. In a suitable neighborhood of the curve $\widetilde{\mathcal{N}}_{\varepsilon,1}$, we introduce Fermi coordinates $\boldsymbol{x} = (x, y) \mapsto (x_1, y_1)$, where y_1 denotes the signed distance to $\widetilde{\mathcal{N}}_{\varepsilon,1}$, and x_1 is the x coordinate of the projection of the point $\boldsymbol{x}$ onto $\widetilde{\mathcal{N}}_{\varepsilon,1}$. With this notation, we write locally the solution u, with $\varepsilon = \tan\theta(u)$ in the form

$$u(\boldsymbol{x}) = H(y_1 - h_{\varepsilon}(x_1)) + \phi.$$

This definition is suitably adjusted to yield a globally defined function. Here the function h_ε is required to satisfy an orthogonality condition. Then it is proven in Section 6 that $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ are small functions of order $\mathbb{O}(\varepsilon^{\alpha})$ in some weighted norms.

Finally, starting on page 1715 we prove the Lipschitz dependence of the solution u on the function h_ε and conclude the proof of Theorem 1.2 using the mapping property of the linearized operator of the Toda equation.

2. Preliminaries

In this section we collect some facts about the Allen–Cahn equation which will be used later on.

Refined asymptotics theorem for four-end solutions. Let $H(x)$ be the heteroclinic solution of the Allen–Cahn equation. Recall that $F''(1) = 2$. Then it is known that we have asymptotically

$$H(x) = 1 - a_F e^{-\sqrt{2}x} + \mathbb{O}(e^{-2\sqrt{2}x}), \quad H'(x) = a_F\sqrt{2}e^{-\sqrt{2}x} + \mathbb{O}(e^{-2\sqrt{2}x}), \quad x \to \infty, \tag{2-1}$$

with similar estimates when $x \to -\infty$, where a_F is a constant depending on F.

We consider the linearized operator

$$L_0\phi = -\phi'' + F''(H)\phi.$$

It is known that the principal eigenvalue of this operator is $\mu_0 = 0$ and the corresponding eigenfunction is H'. In general, the operator L_0 has possibly infinite, discrete spectrum $0 < \mu_1 < \cdots \le \alpha_0^2$, and essential spectrum which is $[\alpha_0^2, \infty)$, $\alpha_0 = \sqrt{F''(1)}$. It may also happen that L_0 has just one eigenvalue, $\mu_0 = 0$ and continuous spectrum, in which case we will set $\mu_1 = \alpha_0^2$.

Next, we recall some facts about the moduli space theory developed in [del Pino et al. 2013]. We will mostly use this theory in the case of four-end solutions. Thus we will restrict the presentation to this situation only. We keep the notations introduced above. Thus we let

$$\lambda = (\lambda_1, \ldots, \lambda_4) \in \Lambda_4,$$

and we write $\lambda_j^+ = \boldsymbol{x}_j + \mathbb{R}^+\boldsymbol{e}_j$ as in (1-4). We denote by $\Omega_0, \ldots, \Omega_4$ the decomposition of $\mathbb{R}^2$ associated to these four affine half lines and $\mathbb{I}_0, \dots, \mathbb{I}_4$ the partition of unity subordinate to this partition. Given $\gamma, \delta \in \mathbb{R}$, we define a *weight function* $\Gamma_{\gamma,\delta}$ by

$$\Gamma_{\gamma,\delta}(\boldsymbol{x}) := \mathbb{I}_0(\boldsymbol{x}) + \sum_{j=1}^{4} \mathbb{I}_j(\boldsymbol{x}) e^{\gamma(\boldsymbol{x}-\boldsymbol{x}_j)\cdot \boldsymbol{e}_j} (\cosh((\boldsymbol{x}-\boldsymbol{x}_j)\cdot \boldsymbol{e}_j^{\perp}))^{\delta}, \tag{2-2}$$

so that, by construction, γ is the rate of decay or blow up along the half lines λ_j^+, and δ is the rate of decay or blow up in the direction orthogonal to λ_j^+.

With this definition in mind, we define the weighted Lebesgue space

$$L^2_{\gamma,\delta}(\mathbb{R}^2) := \Gamma_{\gamma,\delta} L^2(\mathbb{R}^2), \tag{2-3}$$

and the weighted Sobolev space

$$W^{2,2}_{\gamma,\delta}(\mathbb{R}^2) := \Gamma_{\gamma,\delta} W^{2,2}(\mathbb{R}^2). \tag{2-4}$$

Observe that, even though this does not appear in the notation, the partition of unity, the weight function, and the induced weighted spaces all depend on the choice of $\lambda \in \Lambda_4$.

Our first result shows that, if u is a solution of (1-1) which is close to u_λ (in $W^{2,2}$ topology), then $u - u_\lambda$ tends to 0 exponentially fast at infinity.

Proposition 2.1 (refined asymptotics). *Assume that $u \in \mathcal{S}_4$ is a solution of (1-1) and define $\lambda \in \Lambda_4$, so that*

$$u - u_\lambda \in W^{2,2}(\mathbb{R}^2).$$

Then there exist $\delta \in (0, \alpha_0)$, $\alpha_0 = \sqrt{F''(1)}$ and $\gamma > 0$ such that

$$u - u_\lambda \in W^{2,2}_{-\gamma,-\delta}(\mathbb{R}^2). \tag{2-5}$$

More precisely, $\delta > 0$ and $\gamma > 0$ can be chosen so that

$$\gamma \in (0, \sqrt{\mu_1}), \quad \gamma^2 + \delta^2 < \alpha_0^2 \quad \textit{and} \quad \alpha_0 > \delta + \gamma \cot \theta_\lambda, \tag{2-6}$$

where θ_λ is equal to the half of the minimum of the angles between two consecutive oriented affine lines $\lambda_1, \dots, \lambda_4$ (see (1-3)), and μ_1 is the second eigenvalue of the operator L_0 (or $\mu_1 = \alpha_0^2$ if 0 is the only eigenvalue).

We recall here that in this paper for convenience we have assumed $\alpha_0 = \sqrt{F''(1)} = \sqrt{2}$.

It is well known that for any solution of (1-1) the following is true: if by $N(u)$ we denote the nodal set of u and by $\mathrm{d}(N(u), \boldsymbol{x})$ the distance of $\boldsymbol{x}$ to $N(u)$, then

$$|u(\boldsymbol{x})^2 - 1| + |\nabla u(\boldsymbol{x})| + |D^2 u(\boldsymbol{x})| \le C e^{-\beta \mathrm{d}(N(u),\boldsymbol{x})}, \tag{2-7}$$

where $\beta > 0$. This type of estimate is relatively easy to obtain using a comparison argument; see [Berestycki et al. 1997; Kowalczyk et al. 2012]. On the other hand, the estimate (2-5) is nontrivial.

The balancing formulas. We will now briefly describe the balancing formulas for four-end solutions in the form they were introduced in [del Pino et al. 2013]. Assume that u is a solution of (1-1) which is defined in $\mathbb{R}^2$. Assume that X and Y are two vector fields also defined in $\mathbb{R}^2$. In coordinates, we can write

$$X = \sum_j X^j \partial_{x_j}, \quad Y = \sum_j Y^j \partial_{x_j},$$

and, if f is a smooth function, we use the notations

$$X(f) := \sum_j X^j \partial_{x_j} f, \quad \nabla f := \sum_j \partial_{x_j} f \partial_{x_j}, \quad \operatorname{div} X := \sum_i \partial_{x_i} X^i,$$

and

$$\mathrm{d}^* X := \frac{1}{2} \sum_{i,j} (\partial_{x_i} X^j + \partial_{x_j} X^i)\, dx_i \otimes dx_j,$$

so that

$$\mathrm{d}^* X(Y, Y) = \sum_{i,j} \partial_{x_i} X^j Y^i Y^j.$$

We will need the following *balancing formula*, which is proved by direct computation:

$$\operatorname{div}\big(\big(\tfrac{1}{2}|\nabla u|^2 + F(u)\big)X - X(u)\nabla u\big) = \big(\tfrac{1}{2}|\nabla u|^2 + F(u)\big) \operatorname{div} X - \mathrm{d}^* X(\nabla u, \nabla u). \tag{2-8}$$

Translations of $\mathbb{R}^2$ correspond to the constant vector field

$$X := X_0,$$

where X_0 is a fixed vector, while rotations correspond to the vector field

$$X := x\partial_y - y\partial_x.$$

In either case, we have $\operatorname{div} X = 0$ and $\mathrm{d}^* X = 0$. Therefore, we conclude that

$$\operatorname{div}\big(\big(\tfrac{1}{2}|\nabla u|^2 + F(u)\big)X - X(u)\nabla u\big) = 0$$

for these two vector fields. The divergence theorem implies that

$$\int_{\partial\Omega} \big(\big(\tfrac{1}{2}|\nabla u|^2 + F(u)\big)X - X(u)\nabla u\big) \cdot \nu\, ds = 0, \tag{2-9}$$

where ν is the (outward pointing) unit normal vector field to $\partial\Omega$.

To see how this identity is applied let us fix a unit vector $\boldsymbol{e} \in \mathbb{R}^2$ and let $X = \boldsymbol{e}$. For any $s \in \mathbb{R}$ we consider a straight line $L_s = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} = s\boldsymbol{e} + t\boldsymbol{e}^\perp, t \in \mathbb{R}\}$. Then we get

$$\int_{L_s} \big[\tfrac{1}{2}|\nabla u|^2 - |\nabla u \cdot \boldsymbol{e}|^2 + F(u)\big]\, dS = \text{const}$$

for any 4 end solution u of (1-1), as long as the direction of L_s does not coincide with that of any end, that is, $\boldsymbol{e} \neq \boldsymbol{e}_j$, $j = 1, \dots, 4$. In a particular case $\boldsymbol{e} = (0, 1)$ we get a *Hamiltonian identity* [Gui 2008]:

$$\int_{y=s} \big[\tfrac{1}{2}(\partial_x u)^2 - \tfrac{1}{2}(\partial_y u)^2 + F(u)\big]\, dx = \text{const}. \tag{2-10}$$

Summary of the existence result for small angles in [del Pino et al. 2010]. To state the existence result precisely, we assume that we are given an even symmetric solution of the Toda system (1-10) represented by a pair of functions $q_1(t) < 0 < q_2(t)$, where $q_1(t) = -q_2(t)$ as well as $q_1(t) = q_1(-t)$. In addition let us assume that the slope of q_1 at ∞ is -1. Then, asymptotically we have

$$q_j(x) = (-1)^j(|x| + b) + \mathcal{O}(e^{-2\sqrt{2}|x|}), \quad x \to \infty.$$

Given $\varepsilon > 0$, we define the vector valued function $\boldsymbol{q}_\varepsilon$, whose components are given by

$$q_{j,\varepsilon}(x) := q_j(\varepsilon x) + \frac{(-1)^j}{\sqrt{2}} \ln \frac{1}{\varepsilon}. \tag{2-11}$$

It is easy to check that the $q_{j,\varepsilon}$ are again solutions of (1-10).

Observe that, according to the asymptotic description of the functions q_j, the graphs of the functions $q_{j,\varepsilon}$ are asymptotic to oriented half lines with slopes $\pm\varepsilon$ at infinity. In addition, for $\varepsilon > 0$ small enough, these graphs are disjoint and in fact their mutual distance is given by $\sqrt{2}\ln\frac{1}{\varepsilon} + \mathcal{O}(1)$ as ε tends to 0.

It will be convenient to agree that χ^+ (respectively χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $x > 1$ (respectively for $x < -1$) and identically equal to 0 for $x < -1$ (respectively for $x > 1$), and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the four-dimensional space

$$D := \operatorname{Span}\{x \mapsto \chi^{\pm}(x),\ x \mapsto x\chi^{\pm}(x)\}, \tag{2-12}$$

and, for all $\mu \in (0, 1)$ and all $\tau \in \mathbb{R}$, we define the space $\mathscr{C}^{2,\mu}_\tau(\mathbb{R})$ of $\mathscr{C}^{2,\mu}$ functions r which satisfy

$$\|r\|_{\mathscr{C}^{\ell,\mu}_\tau(\mathbb{R})} := \|(\cosh x)^\tau r\|_{\mathscr{C}^{\ell,\mu}(\mathbb{R})} < \infty.$$

Theorem 2.2. *For all $\varepsilon > 0$ sufficiently small, there exists an entire solution u_ε of the Allen–Cahn equation (1-1) whose nodal set is the union of 2 disjoint curves $\widetilde{\Gamma}_{1,\varepsilon}$, $\widetilde{\Gamma}_{2,\varepsilon}$ which are the graphs of the functions*

$$x \mapsto q_{j,\varepsilon}(x) + r_{j,\varepsilon}(\varepsilon x)$$

for some functions $r_{j,\varepsilon} \in \mathscr{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$ satisfying

$$\|r_{j,\varepsilon}\|_{\mathscr{C}^{2,\mu}_\tau(\mathbb{R})\oplus D} \le C\varepsilon^\alpha$$

for some constants $C, \alpha, \tau, \mu > 0$ independent of $\varepsilon > 0$.

In other words, given a solution of the Toda system, we can find a one parameter family of four-end solutions of (1-1) which depend on a small parameter $\varepsilon > 0$. As ε tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions $q_{j,\varepsilon}$.

Going through the proof, one can be more precise about the description of the solution u_ε. If $\Gamma \subset \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ which is the graph over the x-axis of some function, we denote by $\Upsilon(\cdot, \Gamma)$ the signed distance to Γ which is positive in the upper half of $\mathbb{R}^2 \setminus \Gamma$ and is negative in the lower half of $\mathbb{R}^2 \setminus \Gamma$.

Proposition 2.3. *The solution of (1-1) provided by Theorem 2.2 satisfies*

$$\|e^{\varepsilon\hat{\alpha}|\boldsymbol{x}|}(u_\varepsilon - u_\varepsilon^*)\|_{L^\infty(\mathbb{R}^2)} \le C\varepsilon^{\bar{\alpha}}$$

for some constants $C, \bar{\alpha}, \hat{\alpha} > 0$ independent of ε, where $\boldsymbol{x} = (x, y)$ and

$$u_\varepsilon^* = \sum_{j=1}^{2} (-1)^{j+1} H(\Upsilon(\,\cdot\,, \widetilde{\Gamma}_{j,\varepsilon})) - 1, \tag{2-13}$$

in the set

$$V = \{(x, y) : |y| \le C\varepsilon^{-1}\sqrt{1 + x^2}\},$$

with some positive constant C (depending on $\widetilde{\Gamma}_{j,\varepsilon}$), and outside of this set u^* is defined by smoothly interpolating with 1 in the upper half-plane and with -1 in the lower half-plane.

3. The nodal sets of solutions

After a rigid motion, any four-end solution is even symmetric [Gui 2012], and thus we will always consider solutions in $\mathcal{M}_4^{\text{even}}$ which in particular satisfy (1-8). Note that $\mathcal{M}_4^{\text{even}}$ is a one-dimensional manifold, possibly with more than one connected component. For any solution $u \in \mathcal{M}_4^{\text{even}}$, the angle map $\theta(u)$ is defined to be the asymptotic angle at ∞ between the nodal set of u in the first quadrant and the x-axis. By the results proven in [Kowalczyk et al. 2012], the angle map on any connected component of the moduli space $\mathcal{M}_4^{\text{even}}$ of four-end, even solutions is surjective, and in particular it contains solutions whose nodal lines are almost parallel ($\theta(u) \approx 0$ or $\pi/2 - \theta(u) \approx 0$).

By $N(u)$ we will denote in this paper the nodal set of $u \in \mathcal{M}_4^{\text{even}}$. We are interested in solutions whose nodal lines are almost parallel at ∞, and, by symmetry, we can restrict our considerations to the case $\theta(u) \approx 0$. In this case $N(u)$ will consist of two components, one of them is a graph of a smooth function in the lower half-plane and the other one is contained in the upper half-plane.

Basic properties of solutions with almost parallel ends. It is expected that as $\theta(u) \to 0$, the distance between the upper and the lower nodal line of u will tend to infinity. This is the content of Lemma 3.1 below. In the sequel we will denote the first quadrant in $\mathbb{R}^2$ by Q_1.

Lemma 3.1. *Suppose $\{u_n\}_{n=1}^\infty$ is a sequence of four-end solutions such that $\theta(u_n) \to 0$ and $p_n \in N(u_n) \cap \partial Q_1$. Then $|p_n| \to +\infty$, as $n \to +\infty$. Moreover, p_n is point on the y axis for n large.*

Proof. To show that $|p_n| \to \infty$, we suppose by contradiction that $p_n \to p^*$, $|p^*| < \infty$. We know that, up to a subsequence, u_n converges in $\mathscr{C}^2_{\text{loc}}(\mathbb{R}^2)$ to a solution u^* of the Allen–Cahn equation. By similar arguments as in [Kowalczyk et al. 2012, Lemma 5.1], we know that u^* cannot be identically zero. Since $|p^*| < \infty$, u^* cannot be the constant solution 1 or -1. Therefore, by the maximum principle, $u_x^* > 0$, $x > 0$, $u_y^* < 0$, $y > 0$. Then, by [Gui 2008, Theorem 4.4], u^* must be a solution to (1-1), whose nodal set in the first quadrant is asymptotically a straight line with positive slope equal to $\tan\theta^* \neq 0$. It can also be proven using the refined asymptotic theorem (Proposition 2.1), that $u^* \in \mathcal{M}_4^{\text{even}}$. By the Hamiltonian identity,

$$\int_{\mathbb{R}} \left(\frac{1}{2} \left| \frac{\partial u_n(x,0)}{\partial x} \right|^2 + F(u_n(x,0)) \right) dx = 2\boldsymbol{e}_F \sin\theta(u_n) \to 0, \tag{3-1}$$

where $\boldsymbol{e}_F = \int_{\mathbb{R}} (H')^2$. But on the other hand, for any fixed $r > 0$,

$$\int_{-r}^{r} \left(\frac{1}{2} \left| \frac{\partial u_n(x,0)}{\partial x} \right|^2 + F(u_n(x,0)) \right) dx \to \int_{-r}^{r} \left(\frac{1}{2} \left| \frac{\partial u^*(x,0)}{\partial x} \right|^2 + F(u^*(x,0)) \right) dx > \delta > 0.$$

This is a contradiction.

It remains to show that p_n is in the y axis when n is large enough. To this end, we argue by contradiction and assume that p_n is in the x axis for large n. Observe that as p_n goes to infinity, locally around the nodal line, u_n will resemble the heteroclinic solution. Therefore, for any $\varepsilon > 0$, if n is large enough,

$$\int_{\mathbb{R}} \left(\frac{1}{2} \left| \frac{\partial u_n(x,0)}{\partial x} \right|^2 + F(u_n(x,0)) \right) dx > 2\boldsymbol{e}_F - \varepsilon.$$

But on the other hand, by (3-1), the left side is equal to $2\boldsymbol{e}_F \sin\theta(u_n)$, which tends to zero. This is a contradiction. □

We know that when the angle of u_n is small, the nodal set $N(u_n)$ in the upper half-plane is a graph of a smooth function $y = f_n(x)$. For this function, we have the following.

Lemma 3.2. *Suppose $\{u_n\}$ is a sequence of solutions in $\mathcal{M}_4^{even}$ such that $\theta(u_n) \to 0$, as $n \to \infty$. We have*

$$\lim_{n \to +\infty} \|f_n'\|_{\mathscr{C}^0(\mathbb{R})} = 0.$$

Proof. Using the monotonicity of u_n in the upper half-plane and the validity of the De Giorgi conjecture in dimension 2, one can show that, for any $r > 0$,

$$\lim_{n \to +\infty} \|f_n'\|_{\mathscr{C}^0([-r,r])} = 0.$$

Now, we claim that for each $\delta > 0$, there exists $r(\delta) > 0$ such that

$$|f_n'(x) - \tan\theta(u_n)| < \delta \quad \text{for all } x > r(\delta) \text{ and } n \in \mathbb{N}.$$

Indeed, if this were not true, then, using the fact that

$$\lim_{x \to +\infty} f_n'(x) = \tan\theta(u_n) \to 0 \quad \text{as } n \to +\infty,$$

we could find sequences $\{n_k\}$, $\{x_k\}$, $\{y_k\}$, all tending to infinity and $x_k < y_k$, such that

$$\frac{\delta}{4} \le |f_{n_k}'(x)| \le C, \quad x \in [x_k, y_k],$$

and

$$|f_{n_k}'(x_k) - f_{n_k}'(y_k)| = \frac{\delta}{2}. \tag{3-2}$$

Now we consider two lines L_{1,n_k} and L_{2,n_k} with slopes -1 passing though the points $(x_k, f_{n_k}(x_k))$ and $(y_k, f_{n_k}(y_k))$, respectively. Note that since the nodal lines $N(u_{n_k})$ are bigraphs, the lines L_{i,n_k} must be transversal to $N(u_{n_k})$ at their points of intersection.

Next, consider the domain $\Omega_{n_k} \subset Q_1$ bounded by the two axes and the lines L_{i,n_k}, $i = 1, 2$. Let X be the vector field $(0, 1)$. The balancing formula (2-9) tells us

$$\int_{\partial\Omega_{n_k}} \big(\big(\tfrac{1}{2}|\nabla u_{n_k}|^2 + F(u_{n_k})\big)X - X(u_{n_k})\nabla u_{n_k}\big)\cdot \nu\, dS = 0.$$

Note that the integral over the segment $\partial\Omega_{n_k} \cap \{x = 0\}$ is automatically 0 by the choice of the vector field X and the evenness of u_{n_k}.

Following similar arguments as in [Kowalczyk et al. 2012, Lemma 5.2], one can show suitable exponential decay of $|u_n| - 1$ along the x axis, and it follows that, as $k \to +\infty$,

$$\int_{\partial\Omega_{n_k}\cap\{y=0\}} ((\tfrac{1}{2}|\nabla u_{n_k}|^2 + F(u_{n_k}))X - X(u_{n_k})\nabla u_{n_k})\cdot \nu\, dS \to 0. \tag{3-3}$$

Now we estimate the integrals along the segments $\partial\Omega_{n_k} \cap L_{i,n_k}$. For this purpose it is convenient to denote

$$\alpha_{1,n_k} = \arctan f'_{n_k}(x_k), \qquad \alpha_{2,n_k} = \arctan f'_{n_k}(y_k),$$

and

$$\boldsymbol{e}^{\perp}_{1,n_k} = (\sin\alpha_{1,n_k}, -\cos\alpha_{1,n_k}).$$

By the validity of the De Giorgi conjecture in dimension 2, we know that locally around $(x_k, f_{n_k}(x_k))$, as k goes to infinity, the function u_{n_k} converges to

$$H(\boldsymbol{e}^{\perp}_{1,n_k}\cdot(x - x_k, y - f_{n_k}(x_k)).$$

Moreover, by (2-7), on the segment $\partial\Omega_{n_k} \cap L_{1,n_k}$,

$$|u^2_{n_k}(\boldsymbol{x}) - 1| + |\nabla u_{n_k}(\boldsymbol{x})| \le Ce^{-\beta|x_k - x|}, \qquad \boldsymbol{x} = (x, y).$$

Similar results hold around $(y_k, f_{n_k}(y_k))$. Using these facts, after some calculation, we get

$$\int_{\partial\Omega_{n_k}\cap L_{i,n_k}} \big(\big(\tfrac{1}{2}|\nabla u_{n_k}|^2 + F(u_{n_k})\big)X - X(u_{n_k})\nabla u_{n_k}\big)\cdot \nu\, dS = (-1)^{i+1}\sin\alpha_{i,n_k}\boldsymbol{e}_F + o(1),$$

where $o(1)$ is a term that goes to 0 as $k \to +\infty$. Combining all the above estimates, we infer

$$\sin\alpha_{1,n_k} - \sin\alpha_{2,n_k} = o(1),$$

which is a contradiction. □

A refinement of the asymptotic behavior of the nodal set. Let u be a four-end solution with small angle $\theta(u)$. We set $\varepsilon = \tan\theta(u)$ and, for simplicity, use ε as a small parameter. To obtain more precise information about this solution, our first step is to define a good approximate solution and estimate the corresponding error term. As we will see later, this enables us to know more precisely the behavior of the nodal lines.

The nodal set $N(u)$ in the lower half-plane is the graph of a function $y = f(x)$. Strictly speaking the function f depends on u, but we will not indicate this dependence. We have shown that $\|f'\|_{\mathscr{C}^0(\mathbb{R})} \to 0$ as $\theta(u) \to 0$. Recall that by the validity of the De Giorgi conjecture in dimension 2, locally around the nodal line, u behaves like the heteroclinic solution. Using this fact and that $u(x, f(x)) = 0$, it is not difficult to show that $\|f'\|_{\mathscr{C}^1(\mathbb{R})} \to 0$ as $\theta(u) \to 0$. For future reference, we finally observe that, in general, $N(u) \cap Q_1$ is at least a $\mathscr{C}^3(\mathbb{R})$ function and, bootstrapping the above argument, it is not hard to show that $\|f'\|_{\mathscr{C}^2(\mathbb{R})} = o(1)$ as $\theta(u) \to 0$.

To fix attention, we will always work with the solution whose nodal lines have a small slope $\varepsilon = \tan\theta(u)$ at ∞. This means that these lines are asymptotically parallel, as $\varepsilon \to 0$, to the x axis, and one of them is contained in the lower half-plane and the other in the upper half-plane. We know that they are symmetric with respect to the x axis. In the sequel it will be convenient to denote the component of the nodal set $N(u)$ in the lower half-plane by $\mathscr{N}_{\varepsilon,1}$, and the one in the upper half-plane by $\mathscr{N}_{\varepsilon,2}$. Due to the evenness of u, the nodal lines are obviously graphs of some even functions: $\mathscr{N}_{\varepsilon,i} = \{(x, y) \,|\, y = f_{\varepsilon,i}(x)\}$.

To introduce the functional analytic tools used in this paper, we first define the weight functions

$$W_a(\boldsymbol{x}) := (\cosh x)^a, \quad \boldsymbol{x} = (x, y), \quad a \geq 0.$$

For $\ell = 0, 1, 2$, let $\mathscr{C}_a^{\ell,\mu}(\mathbb{R}^2) := W_a^{-1}\mathscr{C}^{\ell,\mu}(\mathbb{R}^2)$, endowed with the weighted norm

$$\|\phi\|_{\mathscr{C}_a^{\ell,\mu}(\mathbb{R}^2)} := \sup_{\boldsymbol{x}\in\mathbb{R}^2} W_a(\boldsymbol{x}) \|\phi\|_{\mathscr{C}^{\ell,\mu}(B(\boldsymbol{x},1))}.$$

Likewise, we let $\overline{W}_a(x) = (\cosh x)^a$ and define the weighted space $\mathscr{C}_a^{\ell,\mu}(\mathbb{R})$ by

$$\|f\|_{\mathscr{C}_a^{\ell,\mu}(\mathbb{R})} := \sup_{x\in\mathbb{R}} \overline{W}_a(x) \|f\|_{\mathscr{C}^{\ell,\mu}((x-1,x+1))}.$$

In what follows we will measure the size of various functions involved in the $\mathscr{C}_a^{2,\mu}(\mathbb{R}^2)$, and in the $\mathscr{C}_a^{2,\mu}(\mathbb{R})$ norms. Mostly we will have $\mu \in (0, 1)$, $a \sim \varepsilon$, or $a = 0$.

Remark 3.3. In this paper, we will frequently estimate the usual $\mathscr{C}^{\ell,\mu}$ norm, as well as the $\mathscr{C}_a^{\ell,\mu}$ norm ($a \sim \varepsilon$) of various functions. In many cases, the argument for the weighted norms and the usual $\mathscr{C}^{\ell,\mu}$ norm is almost identical. Therefore, for notational convenience, the symbol $\mathscr{C}_a^{\ell,\mu}$, with $a = 0$, will just denote the space $\mathscr{C}^{\ell,\mu}$, rather than the space of compactly supported functions.

Let us recall that a four-end solution u is asymptotic to a model solution u_λ defined in the introduction. Using Proposition 2.1, we know that $u - u_\lambda \in W^{2,2}_{-\varepsilon\tau_0,-\delta}(\mathbb{R}^2)$ with some small $\tau_0 > 0$ and $\delta > 0$, which can be chosen independent of the small parameter ε. It follows that

$$u - u_\lambda \in \mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R}^2). \tag{3-4}$$

To see this, we denote by $\boldsymbol{e}$ the asymptotic direction of the end of u in Q_1. Then, by definition of the weight function $\Gamma_{\varepsilon\tau_0,\delta}$ in (2-2), taking R large, we see that when $\delta \geq \varepsilon\tau_0$,

$$\Gamma_{\varepsilon\tau_0,\delta}(\boldsymbol{x}) \sim (\cosh((\boldsymbol{x}-\boldsymbol{x}_{\varepsilon,1})\cdot\boldsymbol{e}))^{\varepsilon\tau_0}(\cosh((\boldsymbol{x}-\boldsymbol{x}_{\varepsilon,1})\cdot\boldsymbol{e}^{\perp}))^{\delta} \geq C(\cosh x)^{\varepsilon\tau_0}, \quad \boldsymbol{x}\in Q_1\setminus B_R.$$

From this, $u-u_\lambda \in \mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R}^2)$ follows immediately. This estimate can be bootstrapped to yield the $\mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R}^2)$ estimate as claimed.

Additionally, using (3-4) and the fact that $u(x, f_{\varepsilon,2}(x)) = 0$, we get that, with some constant $\mathscr{A}_\varepsilon$,

$$H((f_{\varepsilon,2}(x)-\varepsilon x-\mathscr{A}_\varepsilon)\cos(\theta(u))) = \mathbb{O}_{\mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R})}(e^{-\varepsilon\tau_0|x|}), \quad x\to+\infty, \tag{3-5}$$

from which one can show

$$\|f_{\varepsilon,2}-\varepsilon|x|-\mathscr{A}_\varepsilon\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R})} + \|f'_{\varepsilon,2}-\varepsilon\,\mathrm{sign}(x)\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R})} + \|f''_{\varepsilon,2}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R})} < \infty. \tag{3-6}$$

Fermi coordinates near the nodal lines. We will now describe some neighborhoods of the nodal lines $\mathscr{N}_{\varepsilon,i}$, $i=1,2$, where one can define the Fermi coordinates of $\boldsymbol{x}\in\mathbb{R}^2$ as the unique (x_i, y_i) such that

$$\boldsymbol{x} = (x_i, f_{\varepsilon,i}(x_i)) + y_i n_{\varepsilon,i}(x_i), \quad n_{\varepsilon,i}(x) := \frac{(-f'_{\varepsilon,i}(x), 1)}{\sqrt{1+(f'_{\varepsilon,i}(x))^2}}.$$

We will first find a large, expanding neighborhood of $\mathscr{N}_{\varepsilon,i}$ in which the map $\boldsymbol{x}\mapsto(x_i, y_i)$ is a diffeomorphism. Because of symmetry, it suffices to consider a neighborhood of $\mathscr{N}_{\varepsilon,1}$.

We define the (multivalued) projection of a point $\boldsymbol{x}\in\mathbb{R}^2$ onto $\mathscr{N}_{\varepsilon,1}$ to be the set of points that realize the distance between $\boldsymbol{x}$ and $\mathscr{N}_{\varepsilon,1}$:

$$\pi_{\varepsilon,1}(\boldsymbol{x}) := \{(x_1, f_{\varepsilon,1}(x_1)) : \mathrm{dist}(\boldsymbol{x}, (x_1, f_{\varepsilon,1}(x_1))) = \mathrm{dist}(\boldsymbol{x}, \mathscr{N}_{\varepsilon,1})\}.$$

Let $(-\bar{m}_\varepsilon(x_1), \bar{m}_\varepsilon(x_1))$ be the maximal interval where the projection function is single valued:

$$\bar{m}_\varepsilon(x_1) := \sup\{m : \pi_{\varepsilon,1}((x_1, f_{\varepsilon,1}(x_1)) + t n_{\varepsilon,1}(x_1)) = (x_1, f_{\varepsilon,1}(x_1)) \text{ for } |t|\leq m\}.$$

In a certain sense, we can regard the function $\bar{m}_\varepsilon$ as the measure of the size of the maximal neighborhood of $\mathscr{N}_{\varepsilon,1}$ where the Fermi coordinate could be defined. Finally, for technical reasons, for any $x_1\in\mathbb{R}$, let us define

$$m_\varepsilon(x_1) := \min\left\{\frac{1}{\sqrt{|f''_{\varepsilon,1}(x_1)|}}, \bar{m}_\varepsilon(x_1)\right\}.$$

Lemma 3.4. *Let τ be 0 or τ_0. Then there exists a constant C_0 such that*

$$e^{-m_\varepsilon(x)}(\cosh x)^{\varepsilon\tau} \leq C_0\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}. \tag{3-7}$$

Proof. Given $x_1\in\mathbb{R}$, if $m_\varepsilon(x_1) = 1/\sqrt{|f''_{\varepsilon,1}(x_1)|}$, then

$$e^{-m_\varepsilon(x_1)}(\cosh x_1)^{\varepsilon\tau} \leq C|f''_{\varepsilon,1}(x_1)|^2(\cosh x_1)^{\varepsilon\tau} \leq C\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}.$$

Therefore estimate (3-7) holds in this case.

If $m_\varepsilon(x_1) < 1/\sqrt{|f''_{\varepsilon,1}(x_1)|}$ by definition $m_\varepsilon(x_1) = \bar{m}_\varepsilon(x_1)$, and therefore one could find points $\boldsymbol{x}_1 = (x_1, f_{\varepsilon,1}(x_1))$, $\boldsymbol{x}_2 = (x_2, f_{\varepsilon,1}(x_2))$, and $\boldsymbol{x}_0$ with $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \pi_{\varepsilon,1}(\boldsymbol{x}_0)$ and

$$\|\boldsymbol{x}_0 - \boldsymbol{x}_1\| = \|\boldsymbol{x}_0 - \boldsymbol{x}_2\| = m_\varepsilon(x_1).$$

In particular, $\boldsymbol{x}_j$, $j = 1, 2$ lie on the circle S whose center is $\boldsymbol{x}_0$.

We observe that, by the choice of $\boldsymbol{x}_0$, the distance from $\boldsymbol{x}_0$ to $\mathcal{N}_{\varepsilon,1}$ is $m_\varepsilon(x_1)$, and therefore $\mathcal{N}_{\varepsilon,1}$ is tangent with S at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Since $\mathcal{N}_{\varepsilon,1}$ is a graph, it is easy to see that the shorter arc of S between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is the graph of a function $y = g(x)$, $x \in [x_1, x_2]$.

Now an elementary calculation yields

$$\min_{x\in[x_1,x_2]} |g''(x)| \geq \frac{1}{m_\varepsilon(x_1)}.$$

On the other hand,

$$|g'(x_2) - g'(x_1)| = |f'_{\varepsilon,1}(x_2) - f'_{\varepsilon,1}(x_1)|.$$

Therefore, one can find a point $\boldsymbol{x}_3 = (x_3, f_{\varepsilon,1}(x_3)) \in \mathcal{N}_{\varepsilon,1}$, with $x_3 \in [x_1, x_2]$, which satisfies

$$|f''_{\varepsilon,1}(x_3)| \geq \min_{x\in[x_1,x_2]} |g''(x)| \geq \frac{1}{m_\varepsilon(x_1)}. \tag{3-8}$$

Observe that $x_3 \in (x_1 - 2m_\varepsilon(x_1), x_1 + 2m_\varepsilon(x_1))$. Therefore, as ε is small,

$$\begin{aligned} e^{-m_\varepsilon(x_1)}(\cosh x_1)^{\varepsilon\tau} &\leq Ce^{-m_\varepsilon(x_1)}e^{2m_\varepsilon(x_1)\varepsilon\tau}(\cosh x_3)^{\varepsilon\tau} \\ &\leq e^{-(1/2)m_\varepsilon(x_1)}(\cosh x_3)^{\varepsilon\tau}. \end{aligned}$$

Then, using (3-8), we also get the desired estimate:

$$\begin{aligned} e^{-m_\varepsilon(x_1)}(\cosh x_1)^{\varepsilon\tau} &\leq e^{-1/(2|f''_{\varepsilon,1}(x_3)|)}(\cosh x_3)^{\varepsilon\tau} \\ &\leq C\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}. \end{aligned}$$

□

By the above lemma, we know that m_ε satisfies

$$m_\varepsilon(x) \geq \varepsilon\tau \ln\cosh x - \ln(C_0\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}),$$

where τ is either 0 or τ_0, and, in particular, when $\tau = 0$,

$$m_\varepsilon(x) \geq -\ln(C_0\|f''_{\varepsilon,1}\|^2_{\mathscr{C}^0(\mathbb{R})}).$$

Now we set

$$\hat{\mathrm{d}}_\varepsilon(x) = \max\{\varepsilon\tau_0 \ln\cosh x - \ln(C_0\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau_0}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}), -\ln(C_0\|f''_{\varepsilon,1}\|^2_{\mathscr{C}^0(\mathbb{R})})\} - 1.$$

Recall that $\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})} \to 0$ as $\varepsilon \to 0$. Therefore $\hat{\mathrm{d}}_\varepsilon(x)$ is positive. Modifying $\hat{\mathrm{d}}_\varepsilon(x)$ in a neighborhood of the point where it is not smooth, we get a smooth positive function $\mathrm{d}_\varepsilon(x)$ satisfying $\mathrm{d}_\varepsilon(x) \leq \hat{\mathrm{d}}_\varepsilon(x) + \frac{1}{2}$, $\|\mathrm{d}'_\varepsilon\|_{C^1(\mathbb{R})} \leq C$, and a similar estimate as (3-7):

$$e^{-\mathrm{d}_\varepsilon(x)}(\cosh x)^{\varepsilon\tau} \leq C\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}. \tag{3-9}$$

With this choice, the change of variables $\boldsymbol{x} = (x, y) = \boldsymbol{x}_{\varepsilon,1}(x_1, y_1)$ given by

$$(x_1, y_1) \mapsto (x_1, f_{\varepsilon,1}(x_1)) + y_1 n_{\varepsilon,1}(x_1) = (x, y)$$

is a diffeomorphism in the set $\{(x_1, y_1) : |y_1| < \mathrm{d}_\varepsilon(x_1)\}$. Denote the corresponding neighborhood of $\mathcal{N}_{\varepsilon,1}$ by $\mathbb{O}_1$. Note that the transformation $\boldsymbol{x}_{\varepsilon,1}$ is given explicitly by

$$x = x_1 - \frac{f'_{\varepsilon,1}(x_1)}{\sqrt{1 + (f'_{\varepsilon,1}(x_1))^2}} y_1, \quad y = f_{\varepsilon,1}(x_1) + \frac{y_1}{\sqrt{1 + (f'_{\varepsilon,1}(x_1))^2}}. \tag{3-10}$$

Similarly, for the graph of $y = f_{\varepsilon,2}(x) = -f_{\varepsilon,1}(x)$, which is the symmetric image $\mathcal{N}_{\varepsilon,2}$ of $\mathcal{N}_{\varepsilon,1}$ with respect to the x axis in the upper half-plane one can associate a Fermi coordinate $(x_2, y_2) \in \mathbb{R} \times (-\mathrm{d}_\varepsilon, \mathrm{d}_\varepsilon)$, in $\mathbb{O}_2$, which is the symmetric image of $\mathbb{O}_1$ defined above, and y_2 is the signed distance, positive in the upper part of $\mathcal{N}_{\varepsilon,2}$. Also, we use $\boldsymbol{x}_{\varepsilon,2}$ to denote the corresponding diffeomorphism

$$(x_2, y_2) \mapsto (x_2, f_{\varepsilon,2}(x_2)) + y_2 n_{\varepsilon,2}(x_2).$$

Furthermore, for any function $w : \mathbb{O}_i \to \mathbb{R}$, we will define its pullback by $\boldsymbol{x}_{\varepsilon,i}$ by setting $(\boldsymbol{x}^*_{\varepsilon,i} w)(x_i, y_i) = w \circ \boldsymbol{x}_{\varepsilon,i}(x_i, y_i)$.

4. Asymptotic profile of a solution near its nodal line

***An approximate solution of* (1-1).** We will now define an approximate solution to (1-1) which accounts accurately for the asymptotic behavior of the true solution as $\varepsilon \to 0$. We will use the nodal lines $\mathcal{N}_{\varepsilon,i}$ as the point of departure and will base our construction on the neighborhoods $\mathbb{O}_i$, which are expanding as $x \to \infty$.

To be precise, we let η_i be a smooth cutoff function satisfying $\eta_i(\boldsymbol{x}) = 0$, $\boldsymbol{x} \notin \mathbb{O}_i$, and $\eta_i(\boldsymbol{x}) = 1$ for any point $\boldsymbol{x} \in \mathbb{O}_i$ such that $\mathrm{dist}(\boldsymbol{x}, \partial\mathbb{O}_i) > 1$. Moreover, η_i could be chosen in such a way that $\|\eta_i\|_{\mathscr{C}^3(\mathbb{R}^2)} \le C$. We will use (x_i, y_i) to denote the Fermi coordinates associated to $\mathcal{N}_{\varepsilon,i}$, $i = 1, 2$. Finally, we introduce an unknown function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$, which a priori is of class $\mathscr{C}^3$, and we let $H_{\varepsilon,1} : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathscr{C}^3$ function that, outside of $\mathbb{O}_1$, is equal to 1 (above $\mathcal{N}_{\varepsilon,1}$) and -1 (below $\mathcal{N}_{\varepsilon,1}$), and otherwise is given by

$$(\boldsymbol{x}^*_{\varepsilon,1} H_{\varepsilon,1})(x_1, y_1) = (\boldsymbol{x}^*_{\varepsilon,1}\eta_1) H(y_1 - h_\varepsilon(x_1)) + (1 - \boldsymbol{x}^*_{\varepsilon,1}\eta_1) \frac{H(y_1 - h_\varepsilon(x_1))}{|H(y_1 - h_\varepsilon(x_1))|}. \tag{4-1}$$

Furthermore, we define

$$H_{\varepsilon,2}(x, y) = -H_{\varepsilon,1}(x, -y), \quad \bar{u}_\varepsilon = H_{\varepsilon,1} - H_{\varepsilon,2} - 1.$$

The function h_ε is called the modulation function and it will be defined (Lemma 5.1) through the orthogonality condition:

$$\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,i}[(u - \bar{u}_\varepsilon)\rho_{\varepsilon,i} H'_{\varepsilon,i}]\, dy_i = 0 \quad \text{for all } x_i \in \mathbb{R},$$

where

$$(\boldsymbol{x}^*_{\varepsilon,i} H'_{\varepsilon,i})(x_i, y_i) = (\boldsymbol{x}^*_{\varepsilon,i}\eta_i) H'(y_i - (-1)^{i+1} h_\varepsilon(x_i)), \quad i = 1, 2,$$

and the smooth cutoff functions $\rho_{\varepsilon,i}$ are defined by

$$(\boldsymbol{x}^*_{\varepsilon,i}\rho_{\varepsilon,i})(x_i, y_i) = \rho(y_i - (-1)^{i+1}h_\varepsilon(x_i)),$$

where ρ is an even function satisfying

$$\rho(t) = \begin{cases} 1, & |t| \le \min\{\mathrm{d}_\varepsilon(0), f_{\varepsilon,2}(0)\} - 2, \\ 0, & |t| \ge \min\{\mathrm{d}_\varepsilon(0), f_{\varepsilon,2}(0)\} - 1, \\ 0 < \rho < 1, & \text{otherwise.} \end{cases}$$

The proof of existence of the modulation function h_ε will be given later on, but, anticipating it, we observe that due to the exponential decay in x of the functions involved, we have $h_\varepsilon \in \mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})$, and in fact we will show

$$\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\varepsilon^2. \tag{4-2}$$

If we let $\phi = u - \bar{u}_\varepsilon$, we have

$$L_{\bar{u}_\varepsilon}\phi := -\Delta\phi + F''(\bar{u}_\varepsilon)\phi = E(\bar{u}_\varepsilon) - P(\phi),$$

where $E(\bar{u}_\varepsilon) = \Delta\bar{u}_\varepsilon - F'(\bar{u}_\varepsilon)$ and $P(\phi) = F'(\bar{u}_\varepsilon + \phi) - F'(\bar{u}_\varepsilon) - F''(\bar{u}_\varepsilon)\phi$. Our first result is the following.

Proposition 4.1. *Let τ be 0 or τ_0. For all $\mu \in (0, 1)$, the following estimate holds*:

$$\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} + \|\phi\|_{C^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} + \|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\varepsilon^2.$$

The proof of this proposition, which is based on the a priori estimates for the linear operator $L_{\bar{u}_\varepsilon}$ and careful estimates of the error $E(\bar{u}_\varepsilon)$ of the approximation function is postponed for now and will be given in Section 5. However, it is not hard to show that, a priori, we have $\|\phi\|_{\mathscr{C}^0(\mathbb{R}^2)} = o(1)$ as $\varepsilon \to 0$. A proof of this fact is based on the validity of the De Giorgi conjecture in $\mathbb{R}^2$.

Precise asymptotics of the nodal lines. The point of this section is to describe precisely, and in particular uniformly as $\varepsilon \to 0$, estimates for the function $f_{\varepsilon,i}$. Our curve of reference will be given by a solution of the Toda system:

$$\begin{cases} q_1'' = -c_* e^{\sqrt{2}(q_1 - q_2)}, \\ q_2'' = c_* e^{\sqrt{2}(q_1 - q_2)}, \end{cases} \tag{4-3}$$

for which $q_1(x) = -q_2(x)$, as well as $q_j(x) = q_j(-x)$, $j = 1, 2$, and

$$c_* = \frac{a_F \int_{\mathbb{R}} [F''(1) - F''(H(y))] H'(y) e^{\sqrt{2}y}\, dy}{\int_{\mathbb{R}} (H'(y))^2\, dy}.$$

Here a_F is the constant appearing in the asymptotic expansion (2-1) of H. Keep in mind that we have assumed for convenience $F''(1) = 2$.

To find all solutions to (4-3) with the properties described above, we only need to solve

$$q_1'' = -c_* e^{2\sqrt{2}q_1} \tag{4-4}$$

in the class of even functions. It is easy to see that solutions of (4-4) form a one parameter family, and each solution of this family has asymptotically linear behavior. In fact this family can be parametrized by the slope of the asymptotic line. To describe this family precisely, let us consider the unique solution $U_0(x)$ of (4-4), whose slope at ∞ is -1. We have explicitly

$$U_0(x) = \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}}{c_* \cosh^2(\sqrt{2}x)}. \tag{4-5}$$

Asymptotically, as $|x| \to \infty$, we have

$$U_0(x) = -|x| + b_0 + \mathcal{O}(e^{-2\sqrt{2}|x|}),$$

where b_0 is a fixed constant. Then the family of solutions can be written as

$$q_{\varepsilon,1}(x) = U_0(\varepsilon x) - \frac{1}{\sqrt{2}} \ln \frac{1}{\varepsilon}.$$

Thus, given the nodal line $\mathcal{N}_{\varepsilon,1}$ of a solution u, with $\varepsilon = \tan\theta(u)$, by $q_{\varepsilon,1}$ we will denote the solution of (4-4) whose slope at infinity is $-\varepsilon$. Respectively, we set

$$q_{\varepsilon,2} = -q_{\varepsilon,1}.$$

We will denote by $\widetilde{\mathcal{N}}_{\varepsilon,1}$ the curve $y = q_{\varepsilon,1}(x)$ in the lower half-plane and by $\widetilde{\mathcal{N}}_{\varepsilon,2}$ the graph of $y = q_{\varepsilon,2}(\cdot)$. The hope is that the nodal set in the lower half plane of a four-end solution u, with $\varepsilon = \tan\theta(u)$ small, and $\widetilde{\mathcal{N}}_{\varepsilon,1}$ should be close to each other. To quantify this, we state the next result.

Proposition 4.2. *Let u be a four-end solution of (1-1) such that $\varepsilon = \tan\theta(u)$ is small, let $\mathcal{N}_{\varepsilon,1}$ be the nodal line of this solution in the lower half-plane, given as the graph of the function $y = f_{\varepsilon,1}(x)$, and let $h_\varepsilon \in \mathscr{C}^{2,\mu}(\mathbb{R})$ be the modulation function described above. Then there exist $\alpha, \hat{\tau} > 0$ and a constant j_ε, with $|j_\varepsilon| \le C\varepsilon^\alpha$, such that the following estimates hold for the function $\omega_{\varepsilon,1} := f_{\varepsilon,1} + h_\varepsilon + j_\varepsilon - q_{\varepsilon,1}$:*

$$\begin{aligned} \|\omega_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\hat{\tau}}(\mathbb{R})} &\le C\varepsilon^\alpha, \\ \|\omega'_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\hat{\tau}}(\mathbb{R})} &\le C\varepsilon^{1+\alpha}, \\ \|\omega''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\hat{\tau}}(\mathbb{R})} &\le C\varepsilon^{2+\alpha}. \end{aligned} \tag{4-6}$$

This proposition is the main technical tool needed to prove the uniqueness and will be proven in the next section.

5. Proof of Propositions 4.1 and 4.2

We recall that by definition h_ε is required to be such that the following orthogonality condition is satisfied:

$$\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,i}[(u - \bar{u}_\varepsilon)\rho_{\varepsilon,i} H'_{\varepsilon,i}]\, dy_i = 0 \qquad \text{for all } x_i \in \mathbb{R},\ i = 1, 2. \tag{5-1}$$

We will refer to h_ε as the modulation function, and we keep in mind that h_ε is required to be small. Our first objective is to show that the modulation function h_ε indeed exists.

Lemma 5.1. *For each sufficiently small ε there exists a function $h_\varepsilon \in \mathscr{C}^3(\mathbb{R})$ such that (5-1) holds.*

Proof. To find h_ε such that the orthogonality condition (5-1) is satisfied, we first replace the function h_ε in the definition of the functions $H_{\varepsilon,1}$ and $H_{\varepsilon,2}$ by two undetermined, bounded functions $h_{\varepsilon,1}$ and $h_{\varepsilon,2}$. More precisely, given a function $h_{\varepsilon,2}$ in a suitable function space, we have a function $H_{\varepsilon,2}$ which, in the Fermi coordinate (x_2, y_2), is equal to $H(y_2+h_{\varepsilon,2}(x_2))$, at least near $\mathscr{N}_{\varepsilon,2}$. Given this, we want to find the function $h_{\varepsilon,1}$, corresponding to the modulation of the nodal line $\mathscr{N}_{\varepsilon,1}$ such that, for the resulting approximate function $H_{\varepsilon,1}$, the orthogonality condition (5-1) is satisfied for $i=1$. So far the orthogonality condition for $i=2$ still may not hold. However, if it happens that $h_{\varepsilon,2}=h_{\varepsilon,1}$, then, by symmetry, the orthogonality condition is also satisfied for $i=2$ and this will yield the desired modulation function h_ε. To find an $h_{\varepsilon,2}$ such that $h_{\varepsilon,1}=h_{\varepsilon,2}$, we will use a fixed point argument. Now we give more details for this strategy.

Obviously,

$$\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,1}[\bar{u}_\varepsilon \rho_{\varepsilon,1} H'_{\varepsilon,1}]\,dy_1 = -\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,1}[(H_{\varepsilon,2}+1)\rho_{\varepsilon,1} H'_{\varepsilon,1}]\,dy_1.$$

This identity suggests that we should consider the function

$$k_\varepsilon(s,x_1) := \int_{\mathbb{R}} \rho(y_1-s)H'(y_1-s)\boldsymbol{x}^*_{\varepsilon,1}(u+H_{\varepsilon,2}+1)(x_1,y_1)\,dy_1, \quad s, x_1 \in \mathbb{R}.$$

Note that the orthogonality condition (5-1) for $i=1$ is equivalent to $k_\varepsilon(s,x_1)=0$ with $s=h_{\varepsilon,1}(x_1)$. Let us calculate

$$\begin{aligned} -\partial_s k_\varepsilon(s,x_1) &= \int_{\mathbb{R}} [\rho'(y_1-s)H'(y_1-s)+\rho(y_1-s)H''(y_1-s)]\boldsymbol{x}^*_{\varepsilon,1}(u+H_{\varepsilon,2}+1)(x_1,y_1)\,dy_1 \\ &= \underbrace{\int_{\mathbb{R}} [\rho'(y_1-s)H'(y_1-s)+\rho(y_1-s)H''(y_1-s)]H(y_1)\,dy_1}_{l_1} \\ &\quad + \underbrace{\int_{\mathbb{R}} [\rho'(y_1-s)H'(y_1-s)+\rho(y_1-s)H''(y_1-s)]\boldsymbol{x}^*_{\varepsilon,1}(H_{\varepsilon,2}+1)(x_1,y_1)\,dy_1}_{l_2} \\ &\quad + \underbrace{\int_{\mathbb{R}} [\rho'(y_1-s)H'(y_1-s)+\rho(y_1-s)H''(y_1-s)][\boldsymbol{x}^*_{\varepsilon,1}u(x_1,y_1)-H(y_1)]\,dy_1}_{l_3}. \end{aligned}$$

Fix a small constant a. It is easy to see that there exists constant $\delta>0$, independent of ε, such that $l_1>\delta$ for $s\in(-a,a)$. Obviously, the second term l_2 tends to 0 as $\varepsilon\to 0$. Moreover, since u converges locally as $\varepsilon\to 0$ to the heteroclinic solution, we have

$$l_3 \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore $\partial_s k_\varepsilon(s,x_1) > \delta/2$ for $s\in(-a,a)$, and $x_1\in\mathbb{R}$, when ε is small enough.

Next let us write

$$k_\varepsilon(s,x_1)=\underbrace{\int_{\mathbb{R}}\rho(y_1-s)H'(y_1-s)H(y_1)\,dy_1}_{l_4}+\underbrace{\int_{\mathbb{R}}\rho(y_1-s)H'(y_1-s)\boldsymbol{x}^*_{\varepsilon,1}(H_{\varepsilon,2}+1)(x_1,y_1)\,dy_1}_{l_5}$$
$$+\underbrace{\int_{\mathbb{R}}\rho(y_1-s)H'(y_1-s)[\boldsymbol{x}^*_{\varepsilon,1}u(x_1,y_1)-H(y_1)]\,dy_1}_{l_6}.$$

We have

$$l_4(s)=s\int_{\mathbb{R}}\rho(y_1)(H'(y_1))^2\,dy_1+b(s),\quad b(s)\sim s^2, \tag{5-2}$$

while

$$l_5,l_6\to 0,\quad \varepsilon\to 0. \tag{5-3}$$

Hence, taking a smaller if necessary, we may assume $k_\varepsilon(a,x_1)>0$ and $k_\varepsilon(-a,x_1)<0$ for small ε. This together with the monotonicity of k_ε ensures the existence of $h_{\varepsilon,1}$, which fulfills the orthogonality condition (5-1) for $i=1$ and fixed $h_{\varepsilon,2}$.

The above argument implies that, for any $h_{\varepsilon,2}\in\mathscr{C}^0(\mathbb{R})$, $\|h_{\varepsilon,2}\|_{\mathscr{C}^0(\mathbb{R})}<a$, we have a nonlinear map T defined by $h_{\varepsilon,2}\mapsto h_{\varepsilon,1}$. The map T satisfies

$$TB(0,a)\subset B(0,a),\quad B(0,a)=\{h\in\mathscr{C}^0(\mathbb{R}):\|h\|_{\mathscr{C}^0(\mathbb{R})}<a\}.$$

The proof that T is a contraction map is standard and is omitted. At the end we obtain the existence of a fixed point $h_\varepsilon=h_{\varepsilon,1}=h_{\varepsilon,2}$.

One can verify that although $h_{\varepsilon,2}$ is only of class $\mathscr{C}^0$, the function k_ε is of class $\mathscr{C}^1$. Therefore, by the implicit function theorem, h_ε is also of class $\mathscr{C}^1$. It then follows that k_ε is $\mathscr{C}^2$. Therefore the regularity of h_ε can be bootstrapped. This ends the proof. □

Corollary 5.2. *The modulation function h_ε satisfies*

$$\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}(\mathbb{R})}=o(1),\quad \varepsilon\to 0. \tag{5-4}$$

We also have $h_\varepsilon\in\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})$.

Proof. The fact that $\|h_\varepsilon\|_{\mathscr{C}^0(\mathbb{R})}$ tends to 0 as $\varepsilon\to 0$ essentially follows from (5-2) and (5-3). Then the same can be shown for the higher order derivatives. Once the existence of small h_ε is established, one can again use (5-2) and the fact that, a priori, $u\in\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)$ to show that $h_\varepsilon\in\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})$. □

Now let us recall that for a four-end solution with small angle, we have written $u=\bar u_\varepsilon+\phi$. The linearization of the Allen–Cahn equation around $\bar u_\varepsilon$ is $L_{\bar u_\varepsilon}=-\Delta+F''(\bar u_\varepsilon)$. The function ϕ satisfies

$$L_{\bar u_\varepsilon}\phi=\Delta\bar u_\varepsilon-F'(\bar u_\varepsilon)-P(\phi), \tag{5-5}$$

and

$$P(\phi)=F'(\bar u_\varepsilon+\phi)-F'(\bar u_\varepsilon)-F''(\bar u_\varepsilon)\phi\sim\phi^2$$

is a higher order term in ϕ. Note that our definition of $\bar u_\varepsilon$ and the construction of the function h_ε imply that $\phi=u-\bar u_\varepsilon$ satisfies the orthogonality condition (5-1). Our strategy to get suitable estimates for ϕ relies on the a priori estimates for the operator $L_{\bar u_\varepsilon}$, taking into account this orthogonality condition.

To carry out the analysis, we will study the error term $E(\bar{u}_\varepsilon)=\Delta\bar{u}_\varepsilon-F'(\bar{u}_\varepsilon)$. First we consider the projection of $E(\bar{u}_\varepsilon)$ onto the two-dimensional space $K=\operatorname{span}\{H'_{\varepsilon,i}\rho_{\varepsilon,i},\ i=1,2\}$, which we will denote by $E(\bar{u}_\varepsilon)^{\|}$. Explicitly, $E(\bar{u}_\varepsilon)^{\|}=E(\bar{u}_\varepsilon)^{\|}_1+E(\bar{u}_\varepsilon)^{\|}_2$, where $E(\bar{u}_\varepsilon)^{\|}_i$ is equal to 0 outside $\mathbb{O}_i$ and

$$\boldsymbol{x}^*_{\varepsilon,i}E(\bar{u}_\varepsilon)^{\|}_i(x_i,y_i):=c_\varepsilon\boldsymbol{x}^*_{\varepsilon,i}(\rho_{\varepsilon,i}H'_{\varepsilon,i})\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,i}[E(\bar{u}_\varepsilon)\rho_{\varepsilon,i}H'_{\varepsilon,i}]\,dy_i\quad\text{in }\mathbb{O}_i,\ i=1,2.$$

Here

$$c_\varepsilon=\left(\int_{\mathbb{R}}[\boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1}H_{\varepsilon,1})]^2\,dy\right)^{-1}=\left(\int_{\mathbb{R}}(\rho H)^2\,dy\right)^{-1}.$$

Furthermore we set $E(\bar{u}_\varepsilon)^{\perp}=E(\bar{u}_\varepsilon)-E(\bar{u}_\varepsilon)^{\|}$. The main idea in what follows is that the size of the function $f''_{\varepsilon,1}$ is related to $E(\bar{u}_\varepsilon)^{\|}$, while the size of $u-\bar{u}_\varepsilon=\phi$ is controlled by $E(\bar{u}_\varepsilon)^{\perp}$. Of course, both projections of the error $E(\bar{u}_\varepsilon)$ are coupled, in the sense that the dependence on $f_{\varepsilon,1}$ and ϕ appears in both of them, but, as we will see, this coupling is relatively easy to deal with.

As we said, we wish to analyze the error $E(\bar{u}_\varepsilon)$. Observe that

$$\begin{aligned}-F'(H_{\varepsilon,2})-F'(H_{\varepsilon,1}-H_{\varepsilon,2}-1)&=-F'(H_{\varepsilon,2})-F'(H_{\varepsilon,1})+F''(H_{\varepsilon,1})(H_{\varepsilon,2}+1)+\mathbb{O}((H_{\varepsilon,2}+1)^2)\\&=-F'(H_{\varepsilon,1})-[F''(1)-F''(H_{\varepsilon,1})](H_{\varepsilon,2}+1)+\mathbb{O}((H_{\varepsilon,2}+1)^2).\end{aligned}$$

It follows that

$$\begin{aligned}E(\bar{u}_\varepsilon)&=-\Delta(H_{\varepsilon,1}-H_{\varepsilon,2}-1)+F'(H_{\varepsilon,1}-H_{\varepsilon,2}-1)\\&=-\Delta H_{\varepsilon,1}+F'(H_{\varepsilon,1})+\Delta H_{\varepsilon,2}-F'(H_{\varepsilon,2})+[F''(1)-F''(H_{\varepsilon,1})](H_{\varepsilon,2}+1)+\mathbb{O}((H_{\varepsilon,2}+1)^2).\end{aligned}$$

The expression of the Laplace operator in $\mathcal{N}_{\varepsilon,i}$ is

$$\Delta=\frac{1}{A_i}\partial^2_{x_i}+\partial^2_{y_i}+\frac{1}{2}\frac{\partial_{y_i}A_i}{A_i}\partial_{y_i}-\tfrac{1}{2}\frac{\partial_{x_i}A_i}{A_i^2}\partial_{x_i},\tag{5-6}$$

where

$$A_i=1+(f'_{\varepsilon,i}(x_i))^2-2y_i\frac{f''_{\varepsilon,i}(x_i)}{\sqrt{1+(f'_{\varepsilon,i}(x_i))^2}}+y_i^2\frac{(f''_{\varepsilon,i}(x_i))^2}{(1+(f'_{\varepsilon,i}(x_i))^2)^2}.$$

Using these formulas, we can write down the explicit expression of $E(\bar{u}_\varepsilon)$. Because of symmetry, it suffices to carry out the calculation in the lower half plane. The same calculation as that of [del Pino et al. 2010, (5.65)] shows that in the portion of the lower half-plane where both cutoff functions $\eta_{\varepsilon,i}$ equal 1, we have, for $i=1,2$,

$$\begin{aligned}E(\bar{u}_\varepsilon)=\left(\frac{1}{2}\frac{\partial_{y_1}A_1}{A_1}-\frac{h''_\varepsilon(x_1)}{A_1}+\frac{1}{2}\frac{\partial_{x_1}A_1}{A_1^2}h'_\varepsilon(x_1)\right)&H'(y_1-h_\varepsilon(x_1))\\&-\left(\frac{1}{2}\frac{\partial_{y_2}A_2}{A_2}+\frac{h''_\varepsilon(x_2)}{A_2}-\frac{1}{2}\frac{\partial_{x_2}A_2}{A_2^2}h'_\varepsilon(x_2)\right)H'(y_2-h_\varepsilon(x_2))\\&+\left(\frac{(h'_\varepsilon(x_1))^2}{A_1}H''(y_1-h_\varepsilon(x_1))-\frac{(h'_\varepsilon(x_2))^2}{A_2}H''(y_2+h_\varepsilon(x_2))\right)\\&-(F''(1)-F''(H_{\varepsilon,1}))(H_{\varepsilon,2}+1)+\mathbb{O}((H_{\varepsilon,2}+1)^2).\end{aligned}\tag{5-7}$$

Lemma 5.3. *Suppose τ is equal to* 0 *or* τ_0, *and define* $\mathfrak{D}(\boldsymbol{x}) := \operatorname{dist}(\boldsymbol{x}, \mathcal{N}_{\varepsilon,1}) + \operatorname{dist}(\boldsymbol{x}, \mathcal{N}_{\varepsilon,2})$. *Then, for any* $\mu \in (0, 1)$,

$$\|E(\bar{u}_\varepsilon)^\perp\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} = o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} + \|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})}) + \mathbb{O}(\|\exp(-\sqrt{2}\mathfrak{D})\|_{C^0_{\varepsilon\tau}(\mathbb{R}^2)}), \tag{5-8}$$

Proof. First we note that, outside of the set $\mathbb{O}_1 \cup \mathbb{O}_2$, $\bar{u}_\varepsilon$ is equal to 1 or -1, hence the estimate is trivial in this region. Secondly, if $\boldsymbol{x} \in \mathbb{O}_i$ and $\operatorname{dist}(\boldsymbol{x}, \partial\mathbb{O}_i) < 1$, then, using the asymptotic behavior of the heteroclinic solution, it is not difficult to see that

$$\|-\Delta H_{\varepsilon,i} + F'(H_{\varepsilon,i})\|_{\mathscr{C}^{0,\mu}(B(\boldsymbol{x},1))} \le Ce^{-\sqrt{2}\mathrm{d}_\varepsilon(x_i)},$$

where (x_i, y_i) is the Fermi coordinate of $\boldsymbol{x}$. Let (x, y) be the Euclidean coordinate of the point $\boldsymbol{x}$. Then elementary geometry tells us

$$|x_i - x| \le |f'_{\varepsilon,i}(x_i)|\mathrm{d}_\varepsilon(x_i).$$

Therefore, using (3-9), we get

$$\begin{aligned} e^{-\sqrt{2}\mathrm{d}_\varepsilon(x_i)}e^{\varepsilon\tau|x|} &\le e^{-\sqrt{2}\mathrm{d}_\varepsilon(x_i)}e^{\varepsilon\tau|x_i|+\varepsilon\tau|f'_{\varepsilon,i}(x_i)|\mathrm{d}_\varepsilon(x_i)} \\ &\le e^{-\mathrm{d}_\varepsilon(x_i)}e^{\varepsilon\tau|x_i|} \\ &\le C\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})}. \end{aligned}$$

Hence, to prove (5-8), it will suffice to consider the expression (5-7) for $E(\bar{u}_\varepsilon)$.

By (5-7), we get, for instance, the following term in $E(\bar{u}_\varepsilon)^\perp$:

$$T_1 := \frac{\partial_{y_1} A_1}{A_1}\boldsymbol{x}^*_{\varepsilon,1}H'_{\varepsilon,1} - c_\varepsilon \boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1}H'_{\varepsilon,1})\int_{\mathbb{R}} \frac{\partial_{y_1} A_1}{A_1}\rho_{\varepsilon,1}(H'_{\varepsilon,1})^2\,dy_1.$$

Here we have used the fact that $\rho_{\varepsilon,1}H'_{\varepsilon,1}$ is supported in the lower half-plane and $\rho_{\varepsilon,2}H'_{\varepsilon,2}$ is supported in the upper half-plane. Recall that the main order term of A_1 is 1 and

$$\frac{\partial_{y_1} A_1}{A_1} = -2\frac{f''_{\varepsilon,1}(x_1)}{A_1\sqrt{1+(f'_{\varepsilon,1}(x_1))^2}} + 2\frac{y_1(f''_{\varepsilon,1}(x_1))^2}{A_1(1+(f'_{\varepsilon,1}(x_1))^2)^2},$$

whose main order term is, roughly speaking, $-2f''_{\varepsilon,1}$. Substituting this into the expression of T_1 results in

$$T_1 = \frac{\partial_{y_1} A_1}{A_1}H'_{\varepsilon,1} + \frac{2c_\varepsilon\rho_{\varepsilon,1}H'_{\varepsilon,1}f''_{\varepsilon,1}(x_1)}{\sqrt{1+(f'_{\varepsilon,1}(x_1))^2}}\int_{\mathbb{R}}\frac{\rho_{\varepsilon,1}(H'_{\varepsilon,1})^2}{A_1}\,dy_1 - \frac{2c_\varepsilon\rho_{\varepsilon,1}H'_{\varepsilon,1}(f''_{\varepsilon,1}(x_1))^2}{(1+(f'_{\varepsilon,1}(x_1))^2)^2}\int_{\mathbb{R}}\frac{y_1\rho_{\varepsilon,1}(H'_{\varepsilon,1})^2}{A_1}\,dy_1.$$

We notice that although it appears at first that T_1 carries a term of order $\mathbb{O}(\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})})$, there is a cancelation between the first and the second term in T_1. In estimating this term it is important to use the properties of the cut off function $\rho_{\varepsilon,1}$. Note also that although y_1 appears in $\partial_{y_1}A_1/A_1$, it is always multiplied by $f''_{\varepsilon,1}(x_1)$. Since in $\mathbb{O}_1$, $|y_1| \le \mathrm{d}_\varepsilon(x_1)$, we have $|y_1| \le 1/\sqrt{f''_{\varepsilon,1}(x_1)}$. Therefore $y_1 f''_{\varepsilon,1}(x_1)$ is always a small order term.

It is worth mentioning that when we estimate $\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})$ norms we need to take into account the relation between the Fermi coordinate (x_1, y_1) and the Euclidean coordinate (x, y) of a point $\boldsymbol{x} \in \mathbb{O}_1$. Typically, we have

$$|(\cosh x)^{\varepsilon\tau} f''_{\varepsilon,1}(x_1)| \le C e^{\varepsilon\tau|x_1-x|} \| f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})} \le C \exp\{\varepsilon\tau|y_1|\mathbb{O}(\|f'_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})})\}\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})}.$$

Any term of this form is additionally multiplied by $o(1)H'_{\varepsilon,1}$ or $o(1)H''_{\varepsilon,1}$, thus yielding a term of order $o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})})$.

Now, using the fact that $f'_{\varepsilon,1}$ and $f''_{\varepsilon,1}$ are of order $o(1)$ as $\varepsilon \to 0$ and the definition of the cutoff function $\rho_{\varepsilon,1}$, we conclude

$$\|T_1\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} = o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}).$$

Similar estimates hold for the terms involving $h''_\varepsilon(x_1)$. Regarding terms involving $h'_\varepsilon(x_1)$, $h'_\varepsilon(x_2)$, $h''_\varepsilon(x_2)$, we note that they are all multiplied by small order terms. Finally, to estimate the norms of $(H_{\varepsilon,2}+1)H'_{\varepsilon,1}$, we use the fact that

$$(H_{\varepsilon,2}+1)H'_{\varepsilon,1} \sim e^{-\sqrt{2}(|y_1|+|y_2|)}.$$

It follows immediately that

$$\|(H_{\varepsilon,2}+1)H'_{\varepsilon,1}\|_{C^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} \le C\|\exp(-\sqrt{2}\mathfrak{D})\|_{C^0_{\varepsilon\tau}(\mathbb{R}^2)}.$$ □

Observe that there are terms involving h_ε which appear in the right hand side of (5-8). This complicates the situation somewhat. However, since the Fermi coordinates are defined using the nodal line, we have the following.

Lemma 5.4. *Let τ be 0 or τ_0. We have*

$$\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} + C\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R}^2)}. \tag{5-9}$$

Proof. We first recall that if $\boldsymbol{x} \in \mathbb{O}_1$ and $\operatorname{dist}(\boldsymbol{x}, \partial\mathbb{O}_1) > 1$, then

$$(\boldsymbol{x}^*_{\varepsilon,1}u)(x_1, y_1) = H(y_1 - h_\varepsilon(x_1)) - (\boldsymbol{x}^*_{\varepsilon,1}H_{\varepsilon,2})(x_1, y_1) - 1 + (\boldsymbol{x}^*_{\varepsilon,1}\phi)(x_1, y_1). \tag{5-10}$$

Now let us consider any point $\boldsymbol{x}$ on the curve $\mathcal{N}_{\varepsilon,1}$. That is, the Fermi coordinate of $\boldsymbol{x}$ is $(x_1, 0)$. Since the distance of $\boldsymbol{x}$ to $\mathcal{N}_{\varepsilon,2}$ is $\mathfrak{D}(\boldsymbol{x})$, we have

$$|(\boldsymbol{x}^*_{\varepsilon,1}H_{\varepsilon,2})(x_1, 0) + 1| \le C\exp(-\sqrt{2}\mathfrak{D}(\boldsymbol{x})).$$

Then, from $(\boldsymbol{x}^*_{\varepsilon,1}u)(x_1, 0) = 0$ and (5-10), one gets

$$\|h_\varepsilon\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R})} \le C\|\phi\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R}^2)} + C\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^0_{\varepsilon\tau}(\mathbb{R}^2)}.$$

This gives us the $\mathscr{C}^0$ estimate. To estimate the $\mathscr{C}^1$ norm of h_ε, we differentiate the relation (5-10) with respect to x_1 and let $y_1 = 0$ in the resulting equation. Then we find that

$$-H'(-h_\varepsilon(x_1))h'_\varepsilon(x_1) - \frac{\partial}{\partial x_1}(\boldsymbol{x}^*_{\varepsilon,1}H_{\varepsilon,2}) + \frac{\partial}{\partial x_1}(\boldsymbol{x}^*_{\varepsilon,1}\phi) = 0, \tag{5-11}$$

from which the $\mathscr{C}^1_{\varepsilon\tau}$ estimate follows. Similarly, we could differentiate (5-10) twice with respect to x_1 and let $y_1 = 0$ to estimate h''_ε.

Corresponding estimates for the Hölder norm are also straightforward. □

To proceed, we need the following a priori estimate.

Proposition 5.5. *Suppose φ is a solution of the equation*

$$-\Delta\varphi + F''(\bar{u}_\varepsilon)\varphi = f + \sum_{i=1,2} \kappa_{\varepsilon,i}\rho_{\varepsilon,i}H'_{\varepsilon,i} \quad \text{in } \mathbb{R}^2,$$

with some given functions $f \in \mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)$ and $\kappa_{\varepsilon,i} \in \mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})$. Assume furthermore that the function φ satisfies the orthogonality condition:

$$\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,i}(\varphi\rho_{\varepsilon,i}H'_{\varepsilon,i})\,dy_i = 0, \quad i = 1, 2. \tag{5-12}$$

Then we have

$$\|\varphi\|_{C^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} \le C\|f\|_{C^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}, \quad \|\kappa_{\varepsilon,i}\|_{C^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\|f\|_{C^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)},$$

provided ε is small enough.

Sketch of proof. The proof is by contradiction and is essentially the same as that of [del Pino et al. 2010, Proposition 5.1]. First an a priori estimate is proven for a solution of the problem

$$-\Delta\varphi + F''(\bar{u}_\varepsilon)\varphi = f_0 \quad \text{in } \mathbb{R}^2,$$

where φ satisfies the orthogonality condition (5-12). Indeed, using the fact that H', where H is the heteroclinic solution in $\mathbb{R}$, is the only element of the kernel of the corresponding one-dimensional linear operator $d^2/dt^2 + 1 - 3H^2$, one can prove that φ satisfies an estimate of the form claimed. This type of argument can be found, for example, in [del Pino et al. 2011].

Second, we project the equation on the functions of the form $\rho_{\varepsilon,i}H'_{\varepsilon,i}$, $i = 1, 2$, and get the identity

$$\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,i}\{\rho_{\varepsilon,i}H'_{\varepsilon,i}[-\Delta\varphi + F''(\bar{u}_\varepsilon)\varphi]\}\,dy_i - \int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,i}(\rho_{\varepsilon,i}H'_{\varepsilon,i}f)\,dy_i = \kappa_{\varepsilon,i}\int_R (\boldsymbol{x}^*_{\varepsilon,i}\rho_{\varepsilon,i}H'_{\varepsilon,i})^2\,dy_i.$$

After an integration by parts and some calculations, we can use the above identity to prove that the $\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})$ norm of the functions $\kappa_{\varepsilon,i}$ can be controlled by $o(1)\|\varphi\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} + C\|f\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}$. From this and the first step the assertion follows. We omit the details. □

Lemma 5.6. *Let $\phi = u - \bar{u}_\varepsilon$ be the solution of (5-5). The following estimate is true*:

$$\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} \le o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}) + C\|\exp(-\sqrt{2}\mathfrak{D})\|_{C^0_{\varepsilon\tau}(\mathbb{R}^2)}. \tag{5-13}$$

Proof. We will use Proposition 5.5. Thus we write

$$-\Delta\phi + F''(\bar{u}_\varepsilon)\phi = E(\bar{u}_\varepsilon)^\perp - P(\phi) + E(\bar{u}_\varepsilon)^\parallel.$$

Because of Proposition 5.5, to control the size of the function ϕ, it suffices to control the size of $E(\bar{u}_\varepsilon)^\perp$ (which we already do by Lemma 5.3) and the size of $P(\phi)$.

Next we observe that $P(\phi)$ is essentially quadratic in ϕ, and therefore it is not difficult to show

$$\|P(\phi)\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} = o(\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}).$$

Collecting all these estimates, we conclude (5-13). □

The above result indicates that we can control ϕ by $\exp(-\sqrt{2}\mathfrak{D})$ and the second derivative of $f_{\varepsilon,1}$. However, this is not quite enough for our later purpose. Note that for the solution constructed in [del Pino et al. 2010], the corresponding error is, roughly speaking, controlled by $C\varepsilon^2$, and $\|f_{\varepsilon,1}-\varepsilon|x|\|_{\mathscr{C}^0(\mathbb{R})}\sim \ln\frac{1}{\varepsilon}$. For this purpose we first show the following:

Lemma 5.7. *The following estimate holds*:

$$\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}+\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}\le C\|\exp(-\sqrt{2}\mathfrak{D})\|_{C^0_{\varepsilon\tau}(\mathbb{R}^2)}.$$

Proof. Consider the integral $\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}[E(\bar u_\varepsilon)\rho_{\varepsilon,1}H'_{\varepsilon,1}]\,dy_1$. We will show below (Step 1) that on the one hand its $\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})$ norm is controlled by $o(\|\phi\|_{C^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)})$. On the other hand (Step 2) we will show that this integral is related to $f''_{\varepsilon,1}$. The proof will follow by combining this with the previous estimates. (Step 1 can be avoided if we estimate the integral using Proposition 5.5. However, since the computations will be used in the last part of the proof of uniqueness (page 1715), we choose to present them here.)

Step 1. We claim that the relevant norm of the integral $\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}[E(\bar u_\varepsilon)\rho_{\varepsilon,1}H'_{\varepsilon,1}]\,dy_1$ is controlled by $o(\|\phi\|_{C^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)})$.

In fact,

$$\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}[E(\bar u_\varepsilon)\rho_{\varepsilon,1}H'_{\varepsilon,1}]\,dy_1=\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}\{[-\Delta\phi+F''(\bar u_\varepsilon)\phi]\rho_{\varepsilon,1}H'_{\varepsilon,1}\}\,dy_1+\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}[P(\phi)\rho_{\varepsilon,1}H'_{\varepsilon,1}]\,dy_1.$$

To handle the first term appearing in the right side, we write $\Delta_{(x_1,y_1)}=\partial^2_{x_1}+\partial^2_{y_1}$ and

$$T_2:=\underbrace{\int_{\mathbb{R}}[-\Delta_{(x_1,y_1)}\boldsymbol{x}^*_{\varepsilon,1}\phi+F''(H)\boldsymbol{x}^*_{\varepsilon,1}\phi]\boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1}H'_{\varepsilon,1})\,dy_1}_{T_{21}}$$
$$+\underbrace{\int_{\mathbb{R}}[\Delta_{(x_1,y_1)}\boldsymbol{x}^*_{\varepsilon,1}\phi-\boldsymbol{x}^*_{\varepsilon,1}\Delta\phi+\boldsymbol{x}^*_{\varepsilon,1}(F''(\bar u_\varepsilon)\phi)-F''(H)\boldsymbol{x}^*_{\varepsilon,1}\phi]\boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1}H'_{\varepsilon,1})\,dy_1}_{T_{22}}.$$

Since $\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}(\phi\rho_{\varepsilon,1}H'_{\varepsilon,1})\,dy_1=0$, we have $(d^2/dx_1^2)\int_{\mathbb{R}}\boldsymbol{x}^*_{\varepsilon,1}(\phi\rho_{\varepsilon,1}H'_{\varepsilon,1})\,dy_1=0$. Using integration by parts and the fact that $-H''+F'(H)=0$, we find

$$\begin{aligned}T_{21}&=2\int_{\mathbb{R}}\frac{\partial(\boldsymbol{x}^*_{\varepsilon,1}\phi)}{\partial x_1}\frac{\partial(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})}{\partial x_1}\,dy_1+\int_{\mathbb{R}}(\boldsymbol{x}^*_{\varepsilon,1}\phi)\frac{\partial^2(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})}{\partial x_1^2}\,dy_1\\&\qquad-\int_{\mathbb{R}}(\boldsymbol{x}^*_{\varepsilon,1}\phi)\left[\frac{\partial^2(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})}{\partial y_1^2}-F''(H)(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})\right]dy_1\\&=2\int_{\mathbb{R}}\frac{\partial\boldsymbol{x}^*_{\varepsilon,1}\phi}{\partial x_1}\frac{\partial(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})}{\partial x_1}\,dy_1+\int_{\mathbb{R}}(\boldsymbol{x}^*_{\varepsilon,1}\phi)\frac{\partial^2(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}H'_{\varepsilon,1})}{\partial x_1^2}\,dy_1\\&\qquad-\int_{\mathbb{R}}(\boldsymbol{x}^*_{\varepsilon,1}\phi)\left[\frac{\partial^2(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1})}{\partial y_1^2}(\boldsymbol{x}^*_{\varepsilon,1}H'_{\varepsilon,1})+2\frac{\partial(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1})}{\partial y_1}\frac{\partial(\boldsymbol{x}^*_{\varepsilon,1}H'_{\varepsilon,1})}{\partial y_1}\right]dy_1.\end{aligned}$$

Due to the presence of the derivatives of $\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1}$ with respect to x_1, y_1, and also the presence of $H'_{\varepsilon,1}$ in each term, we now obtain that

$$\|T_{21}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} = o(\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}). \tag{5-14}$$

On the other hand,

$$T_{22} = -\int_{\mathbb{R}} \left\{ \left(\frac{1}{A_1} - 1 \right) \partial^2_{x_1} (\boldsymbol{x}^*_{\varepsilon,1}\phi) + \frac{1}{2}\frac{\partial_{y_1} A_1}{A_1} \partial_{y_1} (\boldsymbol{x}^*_{\varepsilon,1}\phi) - \frac{1}{2}\frac{\partial_{x_1} A_1}{A_1^2} \partial_{x_1} (\boldsymbol{x}^*_{\varepsilon,1}\phi) \right\} (\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1} H'_{\varepsilon,1})\, dy_1 \\ + \int_{\mathbb{R}} [\boldsymbol{x}^*_{\varepsilon,1}(F''(\bar{u}_\varepsilon)) - F''(H)] \boldsymbol{x}^*_{\varepsilon,1}(\phi\rho_{\varepsilon,1} H'_{\varepsilon,1})\, dy_1.$$

The desired estimate for T_{22} essentially follows from the fact that $1 - 1/A_1$, $\partial_{y_1} A_1/A_1$, $\partial_{x_1} A_1/A_1^2$, $\boldsymbol{x}^*_{\varepsilon,1}(F''(\bar{u}_\varepsilon)) - F''(H)$ are small terms. Note that we should take into account the relation between the Fermi coordinates and the Euclidean coordinates. For example, let us estimate the Hölder norm of a typical term in T_{22}. First, observe that if $z_1 = (s_1, y_1)$, $z_2 = (s_2, y_1)$ in the Fermi coordinates with respect to $\mathcal{N}_{\varepsilon,1}$, then by the formula (3-10), it is easy to see that

$$|z_1 - z_2| \le C|s_1 - s_2|.$$

Therefore, denoting $(1/A_1 - 1)\partial^2_{x_1}(\boldsymbol{x}^*_{\varepsilon,1}\phi)\boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1}H'_{\varepsilon,1})$ by $\boldsymbol{x}^*_{\varepsilon,1}\mathfrak{G}$, we have

$$\sup_{|s_1-s_2|\le 1} \left| \int_{\mathbb{R}} \frac{\boldsymbol{x}^*_{\varepsilon,1}\mathfrak{G}(s_1, y_1) - \boldsymbol{x}^*_{\varepsilon,1}\mathfrak{G}(s_2, y_1)}{|s_1 - s_2|^\mu}\, dy_1 \right| \le C \sup_{|s_1-s_2|\le 1} \left| \int_{\mathbb{R}} \frac{\mathfrak{G}(z_1) - \mathfrak{G}(z_2)}{|z_1 - z_2|^\mu}\, dy_1 \right| \\ = o(\|\phi\|_{C^{2,\mu}(\mathbb{R})}).$$

Other terms appearing in the definition of T_{22} can be checked similarly. Hence we obtain

$$\|T_{22}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} = o(\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}).$$

This together with (5-14) tells us

$$\|T_{2}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} = o(\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)}).$$

The desired estimate follows from this in a straightforward way.

Step 2. We want to relate the weighted norm of the integral $\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,1}[E(\bar{u}_\varepsilon)\rho_{\varepsilon,1}H'_{\varepsilon,1}]\, dy_1$ to $f''_{\varepsilon,1}$. To do this, we will now check more closely the above integral using the definition of $\bar{u}_\varepsilon$ and the expression of $E(\bar{u}_\varepsilon)$. We see that one term appearing in the integral is

$$\frac{1}{2}\int_{\mathbb{R}} \frac{\partial_{y_1} A_1}{A_1} \boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1} H'^2_{\varepsilon,1})\, dy_1.$$

We will concentrate on this term since the $\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})$ norm of other terms can be estimated by

$$C\|h_\varepsilon\|_{C^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} + C\|e^{-\sqrt{2}\mathfrak{D}}\|_{\mathscr{C}^{0}_{\varepsilon\tau}(\mathbb{R})},$$

as we have seen in the proof of Lemma 5.3. Plugging the formula for A_1 into the above integral, one gets

$$\frac{1}{2}\int_{\mathbb{R}} \frac{\partial_{y_1} A_1}{A_1} \boldsymbol{x}^*_{\varepsilon,1}(\rho_{\varepsilon,1} H'^2_{\varepsilon,1})\,dy_1 = \int_{\mathbb{R}} \frac{1}{A_1}\left(y_1 \frac{(f''_{\varepsilon,1}(x_1))^2}{(1+(f'_{\varepsilon,1}(x_1))^2)^2} - \frac{f''_{\varepsilon,1}(x_1)}{\sqrt{1+(f'_{\varepsilon,1}(x_1))^2}}\right)(\boldsymbol{x}^*_{\varepsilon,1}\rho_{\varepsilon,1} H'^2_{\varepsilon,1})\,dy_1$$
$$= -\frac{1}{c_\varepsilon} f''_{\varepsilon,1}(x_1) + T_4,$$

where T_4 is a function such that

$$\|T_4\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} = o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}).$$

Consequently,

$$\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\Big\| \int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,1}[E(\bar{u}_\varepsilon)\rho_{\varepsilon,1} H'_{\varepsilon,1}\,dy_1\Big\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}$$
$$\le C\|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} + o(\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})}) + C\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^{0}_{\varepsilon\tau}(\mathbb{R})}.$$

This together with (5-9) and (5-13) implies that

$$\|f''_{\varepsilon,1}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^{0}_{\varepsilon\tau}(\mathbb{R})}. \tag{5-15}$$

This combined with Lemma 5.6 yields

$$\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} \le C\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^{0}_{\varepsilon\tau}(\mathbb{R})}.$$ □

To proceed, let us observe that $\|\exp(-\sqrt{2}\mathfrak{D})\|_{\mathscr{C}^0(\mathbb{R}^2)} \le e^{-2\sqrt{2}|f_{\varepsilon,1}(0)|}$. Our next goal is to estimate the quantity $f_{\varepsilon,1}(0)$. To this end, we first need to obtain some exponential decay estimate of ϕ along the y axis away from $\mathscr{N}_{\varepsilon,1}$. Note that, up to now, we have only analyzed the decay behavior of $E(\bar{u}_\varepsilon)$ along the x axis, but actually it also decays exponentially in the direction transversal to the nodal line $\mathscr{N}_{\varepsilon,1}$. The next lemma gives us the necessary information.

Lemma 5.8. *Fix a small constant $\iota_0 > 0$. We have*

$$|\phi(0,y)| \le Ce^{-(2\sqrt{2}-\iota_0)|f_{\varepsilon,1}(0)|} e^{-\iota_0|y-f_{\varepsilon,1}(0)|} \quad \textit{for } y \le 0.$$

Proof. This estimate follows from the maximum principle. We only sketch the proof for $f_{\varepsilon,1}(0) \le y \le 0$, since the case of $y \le f_{\varepsilon,1}(0)$ is similar.

We write the equation satisfied by ϕ as

$$-\Delta\phi + \left(F''(\bar{u}_\varepsilon) + \frac{P(\phi)}{\phi}\right)\phi = E(\bar{u}_\varepsilon). \tag{5-16}$$

Consider the region

$$\Omega := \{(x,y) \in \mathbb{R}^2 \,|\, f_{\varepsilon,1}(0) + r_0 < y < -f_{\varepsilon,1}(0) - r_0\},$$

where r_0 is a fixed large constant satisfying

$$F''(\bar{u}_\varepsilon(\boldsymbol{x})) + \frac{P(\phi)(\boldsymbol{x})}{\phi(\boldsymbol{x})} \ge 1, \quad \boldsymbol{x} \in \Omega.$$

Let $B(x, y) := C_1 e^{2\sqrt{2} f_{\varepsilon,1}(0)} \cosh(\iota_0 y)$. Then

$$-\Delta B + \left(F''(\bar{u}_\varepsilon) + \frac{P(\phi)}{\phi} \right) B \geq (1 - \iota_0^2) B. \tag{5-17}$$

Using (5-16), (5-17), and $\|E(\bar{u}_\varepsilon)\|_{\mathscr{C}^0(\mathbb{R}^2)} + \|\phi\|_{\mathscr{C}^0(\mathbb{R}^2)} \leq Ce^{-2\sqrt{2}|f_{\varepsilon,1}(0)|}$, we find that if the constant C_1 in the definition of B is large enough, $\phi - B < 0$ in $\partial\Omega$ and

$$-\Delta(\phi - B) + \left(F''(\bar{u}_\varepsilon) + \frac{P(\phi)}{\phi} \right) (\phi - B) \leq 0 \quad \text{in } \Omega.$$

By the maximum principle, for $f_{\varepsilon,1}(0) + r_0 < y < 0$, we have

$$\begin{aligned} |\phi(x, y)| &\leq C_1 e^{2\sqrt{2} f_{\varepsilon,1}(0)} \cosh(\iota_0 y) \\ &\leq C_1 e^{(2\sqrt{2} - \iota_0) f_{\varepsilon,1}(0)} e^{-\iota_0 |f_{\varepsilon,1}(0) - y|}. \end{aligned}$$

Therefore the lemma is true for $f_{\varepsilon,1}(0) + r_0 < y < 0$. For $f_{\varepsilon,1}(0) < y < f_{\varepsilon,1}(0) + r_0$, the lemma obviously holds since $\|\phi\|_{\mathscr{C}^0(\mathbb{R}^2)} \leq Ce^{-2\sqrt{2}|f_{\varepsilon,1}(0)|}$. □

Now let us go back to the Toda system (4-3) and recall that by $q_{\varepsilon,1}(x) < 0 < q_{\varepsilon,2}(x)$ we have denoted the solution of this system whose slope at ∞ is ε (this means the tangent of the angle between the asymptotic line of $y = q_{\varepsilon,2}(x)$ in the first quadrant and the x axis). We note that the curve $\widetilde{\mathcal{N}}_{\varepsilon,1} := \{y = q_{\varepsilon,1}(x)\}$ is contained in the lower half-plane.

In the rest of the paper we will also use α, β to denote general positive constants, which may change from step to step, but are always independent of ε.

Our aim is to show that the curves $\mathcal{N}_{\varepsilon,1}$ and $\widetilde{\mathcal{N}}_{\varepsilon,1}$ are close to each other. First of all, we prove the following.

Lemma 5.9. *There exists $\alpha_1 > 0$ such that $|f_{\varepsilon,1}(0) - q_{\varepsilon,1}(0)| \leq C\varepsilon^{\alpha_1}$.*

Proof. The idea of the proof is to relate the asymptotic behavior of u along vertical straight lines, as $\varepsilon \to 0$, using the Hamiltonian identity,

$$\int_{\mathbb{R}} \left\{ \tfrac{1}{2} u_y^2(0, y) - \tfrac{1}{2} u_x^2(0, y) + F(u(0, y)) \right\} dy = \int_{\mathbb{R}} \left\{ \tfrac{1}{2} u_y^2(x, y) - \tfrac{1}{2} u_x^2(x, y) + F(u(x, y)) \right\} dy \quad \text{for all } x, \tag{5-18}$$

and in particular take $x \to \infty$ on the right side of (5-18). Indeed, using the asymptotic behavior of a four-end solution, it is not hard to show that

$$\lim_{x \to \infty} \int_{\mathbb{R}} \left\{ \tfrac{1}{2} u_y^2(x, y) - \tfrac{1}{2} u_x^2(x, y) + F(u(x, y)) \right\} dy = 2\boldsymbol{e}_F \cos\theta(u),$$

where $\boldsymbol{e}_F = \int_{\mathbb{R}} [\frac{1}{2}(H')^2 + F(H)]$. Since u is an even function of x, we also have $u_x(0, y) = 0$, and thus it follows from (5-18) that

$$\int_{\mathbb{R}} \left\{ \tfrac{1}{2} u_y^2(0, y) + F(u(0, y)) \right\} dy = 2\boldsymbol{e}_F \cos\theta(u).$$

We will now calculate the left side of the above identity using the estimate of the error ϕ.

Recall that the heteroclinic solution has the asymptotic behavior

$$H(s) = 1 - a_F e^{-\sqrt{2}s} + \mathcal{O}(e^{-2\sqrt{2}s}) \quad \text{as } s \to +\infty,$$

which can also be differentiated. Set $t = f_{\varepsilon,1}(0) + h_\varepsilon(0)$. Let η_1, η_2 be cut off functions appearing in the definition of the approximate solution (4-1). For the points on the y-axis we have $(x_1, y_1) = (0, y - f_{\varepsilon,1}(x))$, where (x_1, y_1) are their Fermi coordinates with respect to $\mathcal{N}_{\varepsilon,1}$. Then, abusing the notation slightly, we can write

$$u(0,y) = \underbrace{H(y-t) - H(y+t) - 1 + \phi(0,y)}_{u_0(y)} + \underbrace{(1-\eta_1(0,y))\left[\frac{H(y-t)}{|H(y-t)|} - H(y-t)\right]}_{\psi_1(y)} - \underbrace{(1-\eta_2(0,y))\left[\frac{H(y+t)}{|H(y+t)|} - H(y+t)\right]}_{\psi_2(y)}.$$

We observe that $\psi_1(y) = 0$ for $|y_1| < \mathrm{d}_\varepsilon(0) - 1$ and

$$|\psi_1(y)| + |\psi_1'(y)| \le C e^{-\sqrt{2}|y_1|} \quad \text{for } |y_1| \ge \mathrm{d}_\varepsilon(0) - 1.$$

Therefore

$$\int_{\mathbb{R}} [|\psi_1(y)| + |\psi_1'(y)|]\, dy \le C e^{-\sqrt{2}\mathrm{d}_\varepsilon(0)} \le \|f_{\varepsilon,1}''\|^2_{C^0(\mathbb{R})} \le C e^{-4\sqrt{2}|f_{\varepsilon,1}(0)|}.$$

Similarly,

$$\int_{\mathbb{R}} [|\psi_2(y)| + |\psi_2'(y)|]\, dy \le C e^{-4\sqrt{2}|f_{\varepsilon,1}(0)|}.$$

This implies

$$\int_{\mathbb{R}} \left[\tfrac{1}{2} u_y^2(0,y) + F(u(0,y))\right] dy = \int_{\mathbb{R}} \left[\tfrac{1}{2}(u_0'(y))^2 + F(u_0(y))\right] dy + \mathcal{O}(e^{-4\sqrt{2}|f_{\varepsilon,1}(0)|}).$$

Now we calculate

$$\begin{aligned}\int_{-\infty}^0 \left[\tfrac{1}{2}(u_0'(y))^2 + F(u_0(y))\right] dy = {} & \underbrace{\int_{-\infty}^0 \left[\tfrac{1}{2}(H'(y-t))^2 + F(H(y-t))\right] dy}_{I_1} \\ & + \underbrace{\int_{-\infty}^0 \left[H'(y-t)(\partial_y\phi - H'(y+t)) + F'(H(y-t))(\phi - H(y+t) - 1)\right] dy}_{I_2} \\ & + \underbrace{\frac{1}{2}\int_{-\infty}^0 \left[(\partial_y\phi - H'(y+t))^2 + F''(H(y-t))(\phi - H(y+t) - 1)^2\right] dy}_{I_3} \\ & + \mathcal{O}\Big(\int_{-\infty}^0 (\phi - H(y+t) - 1)^3\, dy\Big). \end{aligned} \tag{5-19}$$

The first term on the right side of (5-19) is equal to

$$\begin{aligned} I_1 &= \int_{-\infty}^{-t} \left[\tfrac{1}{2}(H'(y))^2 + F(H(y))\right] dy \\ &= \boldsymbol{e}_F - \int_{-t}^{+\infty} \left[\tfrac{1}{2}(H'(y))^2 + F(H(y))\right] dy \\ &= \boldsymbol{e}_F - \int_{-t}^{+\infty} 2a_F^2 e^{-2\sqrt{2}y}\, dy + \mathbb{O}(e^{-3\sqrt{2}|t|}) \\ &= \boldsymbol{e}_F - \frac{\sqrt{2}}{2} a_F^2 e^{-2\sqrt{2}|t|} + \mathbb{O}(e^{-3\sqrt{2}|t|}). \end{aligned}$$

Next we analyze the second term I_2. We observe that after an integration by parts,

$$I_2 = H'(-t)(\phi(0) - H(t) - 1) = -\sqrt{2} a_F^2 e^{-2\sqrt{2}|t|} + \mathbb{O}(e^{-3\sqrt{2}|t|}).$$

On the other hand, using Lemma 5.8, we can estimate

$$\begin{aligned} I_3 &= \frac{1}{2} \int_{-\infty}^{0} [(H'(y+t))^2 + F''(H(y-t))(H(y+t)-1)^2]\, dy + \mathbb{O}(e^{-(3\sqrt{2}-\iota_0)|t|}) \\ &= \frac{\sqrt{2}a_F^2}{4} e^{-2\sqrt{2}|t|} + \frac{a_F^2}{2} \int_{-\infty}^{0} [F''(H(y-t)) e^{-2\sqrt{2}|y+t|}]\, dy + \mathbb{O}(e^{-(3\sqrt{2}-\iota_0)|t|}). \end{aligned}$$

But we have

$$\begin{aligned} \int_{-\infty}^{0} [F''(H(y-t)) e^{-2\sqrt{2}|y+t|}]\, dy &= \int_{-\infty}^{0} 2e^{2\sqrt{2}(y+t)}\, dy + \int_{-\infty}^{0} \{[F''(H(y-t)) - F''(1)] e^{-2\sqrt{2}|y+t|}\}\, dy \\ &= \frac{\sqrt{2}}{2} e^{-2\sqrt{2}|t|} + \mathbb{O}\Big(\int_{-\infty}^{0} e^{-\sqrt{2}|y-t| - 2\sqrt{2}|y+t|}\, dy\Big) \\ &= \frac{\sqrt{2}}{2} e^{-2\sqrt{2}|t|} + \mathbb{O}(e^{-3\sqrt{2}|t|}). \end{aligned}$$

Hence

$$I_3 = \frac{\sqrt{2}a_F^2}{2} e^{-2\sqrt{2}|t|} + \mathbb{O}(e^{-(3\sqrt{2}-\iota_0)|t|}).$$

Consequently,

$$I_0 := \int_{\mathbb{R}} \left[\tfrac{1}{2}u_y^2(0,y) + F(u(0,y))\right] dy = 2\boldsymbol{e}_F - 2\sqrt{2} a_F^2 e^{-2\sqrt{2}|f_{\varepsilon,1}(0) + h_\varepsilon(0)|} + \mathbb{O}(e^{-3|f_{\varepsilon,1}(0)|}).$$

According to the Hamiltonian identity (5-18),

$$I_0 = 2\boldsymbol{e}_F \cos\theta(u).$$

Now, let u_ε with $\varepsilon = \tan\theta(u)$ be a solution constructed in [del Pino et al. 2010] whose nodal line in the lower half-plane is given by the curve $y = q_{\varepsilon,1}(x) + r_{\varepsilon,1}(\varepsilon x)$, where $q_{\varepsilon,1}$ is the solution of the Toda system whose asymptotic angle at ∞ is ε, and $r_{\varepsilon,1}(x)$ satisfies, as we stated in Theorem 2.2, with some $\alpha > 0$,

$$\|r_{\varepsilon,1}\|_{\mathscr{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D} \le C\varepsilon^\alpha.$$

We recall that since we are working in the class of even functions, $|r_{\varepsilon,1}(x)| \le C\varepsilon^{\alpha}$, which implies that $r_{\varepsilon,1}$ is a bounded, small function. Now, the Hamiltonian identity (5-18) can be used for u_ε as well, and, by a similar computation as for I_0, we get

$$2\boldsymbol{e}_F \cos\theta(u_\varepsilon) = 2\boldsymbol{e}_F - 2\sqrt{2}a_F^2 e^{-2\sqrt{2}|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|} + \mathbb{O}(e^{-3|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|}),$$

where $r_{\varepsilon,1}(0) = \mathbb{O}(\varepsilon^{\alpha})$. Therefore,

$$I_0 = 2\boldsymbol{e}_F - 2\sqrt{2}a_F^2 e^{-2\sqrt{2}|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|} + \mathbb{O}(e^{-3|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|}).$$

That is,

$$e^{-2\sqrt{2}|f_{\varepsilon,1}(0)+h_\varepsilon(0)|} + \mathbb{O}(e^{-3|f_{\varepsilon,1}(0)|}) = e^{-2\sqrt{2}|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|} + \mathbb{O}(e^{-3|q_{\varepsilon,1}(0)+r_{\varepsilon,1}(0)|}).$$

This yields

$$f_{\varepsilon,1}(0) + h_\varepsilon(0) + \mathbb{O}(e^{-(3-2\sqrt{2})|f_{\varepsilon,1}(0)+h_\varepsilon(0)|}) = q_{\varepsilon,1}(0) + \mathbb{O}(\varepsilon^{\alpha}).$$

Since $q_{\varepsilon,1}(0) - (\sqrt{2}/2)\ln\varepsilon = \mathbb{O}(1)$, we get

$$f_{\varepsilon,1}(0) + h_\varepsilon(0) = \frac{\sqrt{2}}{2}\ln\varepsilon + \mathbb{O}(1),$$

which leads to

$$f_{\varepsilon,1}(0) + h_\varepsilon(0) - q_{\varepsilon,1}(0) = \mathbb{O}(\varepsilon^{\alpha}),$$

as claimed. □

Now we are in a position to prove Proposition 4.2. As we will see, the proof of Proposition 4.1 is obtained as an intermediate step.

Proof of Propositions 4.1 and 4.2. Our first goal is to show the estimate (4-6), and this will be done in a few steps. For brevity let us denote $p_{\varepsilon,1} = f_{\varepsilon,1} + h_\varepsilon$ and $\chi_{\varepsilon,1} = p_{\varepsilon,1} - q_{\varepsilon,1}$.

Step 1. We want to show that, in the interval $I := [\ln\varepsilon/\varepsilon, -\ln\varepsilon/\varepsilon]$,

$$|\chi_{\varepsilon,1}(x)| \le C\varepsilon^{\alpha}, \quad |\chi'_{\varepsilon,1}(x)| \le C\varepsilon^{1+\alpha}, \quad \text{and } \|\chi''_{\varepsilon,1}\|_{C^{0,\mu}(I)} \le C\varepsilon^{2+\alpha}.$$

Claim 1. If $I_a := [-a, a] \subset I$ is an interval where

$$|p_{\varepsilon,1}(x)| < 2|\ln\varepsilon|, \quad |p'_{\varepsilon,1}(x)| < 2\varepsilon, \quad x \in I_a, \tag{5-20}$$

then $p_{\varepsilon,1}$ satisfies a perturbed Toda equation in I_a, that is,

$$p''_{\varepsilon,1}(x) = -c_* e^{2\sqrt{2}p_{\varepsilon,1}(x)} + \lambda_1(x), \quad x \in I_a, \tag{5-21}$$

where λ_1 is a function satisfying

$$\|\lambda_1\|_{C^{0,\mu}(I_a)} \le C\varepsilon^{2+\beta_1} \tag{5-22}$$

for some constant $\beta_1 > 0$.

To begin the proof of the claim, let us consider a point $\boldsymbol{x} = (x_1, y_1)$ in the Fermi coordinates of $\Gamma_{\varepsilon,1}$ with $|y_1| \le |f_{\varepsilon,1}(0)|$, and denote it's Fermi coordinates relative to $\Gamma_{\varepsilon,2}$ by (x_2, y_2). Then, using (5-20) and elementary geometry, one can show that if $|x_1| \le a$, we have

$$y_1 - y_2 = -2 f_{\varepsilon,1}(x_1)(1 + \mathcal{O}(\varepsilon^2)). \tag{5-23}$$

Using this and (5-7) and calculating $\int_{\mathbb{R}} \boldsymbol{x}^*_{\varepsilon,1}[E(\bar{u}_\varepsilon)\rho_{\varepsilon,1} H'_{\varepsilon,1}]\, dy_1$ as in Lemma 5.7, we get

$$(1 + \mathcal{O}_{\mathscr{C}^{0,\mu}}(\varepsilon^\alpha)) f''_{\varepsilon,1}(x) + (1 + \mathcal{O}_{\mathscr{C}^{0,\mu}}(\varepsilon^\alpha)) h''_\varepsilon(x) = -c_* e^{2\sqrt{2} p_{\varepsilon,1}(x)}(1 + \mathcal{O}_{\mathscr{C}^{0,\mu}}(\varepsilon^\alpha)) + \mathcal{O}_{\mathscr{C}^{0,\mu}}(\varepsilon^{2+\alpha}). \tag{5-24}$$

This relation gives the claim. (For details, we refer the reader to [del Pino et al. 2010], where similar calculations can be found.) We note here that the term $e^{2\sqrt{2} p_{\varepsilon,1}(x)}$ essentially comes from the integral

$$\int_{\mathbb{R}} \boldsymbol{x}^*_1[(F''(1) - F''(H_{\varepsilon,1}))(H_{\varepsilon,2} + 1)\rho_{\varepsilon,1} H'_{\varepsilon,1}]\, dy_1,$$

and to calculate this integral we have used (5-23).

Next we will use Claim 1 to show

$$|\chi_{\varepsilon,1}| \le C\varepsilon^\alpha \quad \text{in } I_a. \tag{5-25}$$

In fact, from (5-21) we deduce that in I_a, as long as $\chi_{\varepsilon,1}$ is small,

$$\chi''_{\varepsilon,1} = -2\sqrt{2} c_* e^{2\sqrt{2} q_{\varepsilon,1}} \chi + \underbrace{\mathcal{O}(\chi^2_{\varepsilon,1}) e^{-2\sqrt{2} q_{\varepsilon,1}} + \lambda_1(x)}_{\lambda_2(x)}. \tag{5-26}$$

Let ς_i, $i = 1, 2$, be two linearly independent solutions of the linearized Toda equation

$$\varsigma''_i(x) = -2\sqrt{2} c_* e^{2\sqrt{2} q_{\varepsilon,1}(x)} \varsigma_i(x).$$

We can assume that ς_1 is even, ς_2 is odd, $\varsigma_1(0) = 1$, $\varsigma'_2(0) = \varepsilon$, and $|\varsigma'_i| \le C\varepsilon$, $i = 1, 2$. Since $\chi_{\varepsilon,1}$ is an even function, the variation of parameters formula tells us

$$\chi_{\varepsilon,1}(x) = \frac{\varsigma_2(x)}{\varepsilon} \int_0^x \varsigma_1(s)\lambda_2(s)\, ds - \frac{\varsigma_1(x)}{\varepsilon} \int_0^x \varsigma_2(s)\lambda_2(s)\, ds + (p_{\varepsilon,1}(0) - q_{\varepsilon,1}(0))\varsigma_1(x),$$

and

$$\chi'_{\varepsilon,1}(x) = \frac{\varsigma'_2(x)}{\varepsilon} \int_0^x \varsigma_1(s)\lambda_2(s)\, ds - \frac{\varsigma'_1(x)}{\varepsilon} \int_0^x \varsigma_2(s)\lambda_2(s)\, ds + (p_{\varepsilon,1}(0) - q_{\varepsilon,1}(0))\varsigma'_1(x).$$

Let β_2 be a fixed constant satisfying $0 < \beta_2 < \min(\beta_1, \alpha_1)$, where α_1 is the constant appearing in the assertion of Lemma 5.9. If $I_{a_1} := [-a_1, a_1] \subset I_a$ is an interval where $|\chi_{\varepsilon,1}| \le \varepsilon^{\beta_2}$, then, by (5-26),

$$\|\lambda_2\|_{\mathscr{C}^0(I_{a_1})} \le C\varepsilon^{2+\beta_1} + C\varepsilon^{2+2\beta_2}.$$

Recall that $|p_{\varepsilon,1}(0) - q_{\varepsilon,1}(0)| \le C\varepsilon^{\alpha_1}$. Therefore

$$\|\chi_{\varepsilon,1}\|_{\mathscr{C}^0(I_a)} \le C\varepsilon(\varepsilon^{\beta_1} + \varepsilon^{2\beta_2})\left(|\varsigma_2(x)| \int_0^x |\varsigma_1(s)|\, ds + |\varsigma_1(x)| \int_0^x |\varsigma_2(s)|\, ds\right) + C\varepsilon^{\alpha_1}|\varsigma_1(x)|.$$

Since $|\varsigma_1(s)| \le C\varepsilon|s|$ and $|\varsigma_2(s)| \le C$, we find that, for $x \in [\ln\varepsilon/\varepsilon, -\ln\varepsilon/\varepsilon]$,

$$|\varsigma_2(x)| \int_0^x |\varsigma_1(s)|\,ds + |\varsigma_1(x)| \int_0^x |\varsigma_2(s)|\,ds \le C|\ln\varepsilon|^2/\varepsilon.$$

Therefore, in I_{a_1}, if ε is small enough,

$$\|\chi_{\varepsilon,1}\|_{\mathscr{C}^0(I_{a_1})} \le C(\varepsilon^{\beta_1} + \varepsilon^{2\beta_2})|\ln\varepsilon|^2 + C\varepsilon^{\alpha_1}|\ln\varepsilon| \le \frac{\varepsilon^{\beta_2}}{2}.$$

From this we deduce $\|\chi_{\varepsilon,1}\|_{\mathscr{C}^0(I_a)} \le \varepsilon^{\beta_2}$, which proves (5-25).

Since $|\varsigma_i'(x)| \le C\varepsilon$, it then follows that, for $x \in I_a$,

$$\begin{aligned}|\chi_{\varepsilon,1}'(x)| &= C\varepsilon(\varepsilon^{\beta_1} + \varepsilon^{2\beta_2})\left(|\varsigma_2'(x)| \int_0^x |\varsigma_1(s)|\,ds + |\varsigma_1'(x)| \int_0^x |\varsigma_2(s)|\,ds\right) + C\varepsilon^{\alpha_1}|\varsigma_1'(x)| \\ &\le C\varepsilon(\varepsilon^{\beta_1} + \varepsilon^{2\beta_2})|\ln\varepsilon|^2 + C\varepsilon^{1+\alpha_1} \le C\varepsilon^{1+\beta_2}.\end{aligned} \tag{5-27}$$

Now recall that in I, $|q_{\varepsilon,1}(x)| < \frac{9}{5}|\ln\varepsilon|$ and $|q_{\varepsilon,1}'(x)| < \frac{3}{2}\varepsilon$. It then follows from Claim 1, (5-25), and (5-27) that, for ε small enough, the interval I satisfies the assumption of Claim 1. Therefore

$$|\chi_{\varepsilon,1}(x)| \le C\varepsilon^{\alpha} \quad \text{and} \quad |\chi_{\varepsilon,1}'(x)| \le C\varepsilon^{1+\alpha} \quad \text{for } x \in I.$$

Moreover, using (5-26), we get $\|\chi_{\varepsilon,1}''\|_{\mathscr{C}^{0,\mu}(I)} \le C\varepsilon^{2+\alpha}$.

Step 2. Next we will prove that $\|\chi_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R})} \to 0$ as $\varepsilon \to 0$. By Step 1, it suffices to show that

$$\|\chi_{\varepsilon,1}\|_{\mathscr{C}^0(\mathbb{R}\setminus I)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let the asymptotic line of u in the fourth quadrant be $y = -\varepsilon x - \mathscr{A}_\varepsilon$. Define

$$a_\varepsilon := \inf\{t \ge |\ln\varepsilon|/\varepsilon : |f_{\varepsilon,1}(x) + (\varepsilon x + \mathscr{A}_\varepsilon)| \le 1 \text{ for } x \in [t, +\infty)\}.$$

We wish to show that in fact $a_\varepsilon = |\ln\varepsilon|/\varepsilon$. For this purpose, we consider the domain

$$\Omega_L := \left\{(x, y) : y < 0,\ x > a_\varepsilon,\ y > \frac{x}{\varepsilon} - L\right\}.$$

Here $L > \varepsilon a_\varepsilon$ is large and indeed we will finally let it go to $+\infty$. We use the balancing formula in this domain and with the vector field $X := (f_{\varepsilon,1}(a_\varepsilon) - y, x - a_\varepsilon)$. This formula tells us that

$$\int_{\partial\Omega_L} \left\{\left(\tfrac{1}{2}|\nabla u|^2 + F(u)\right)X - X(u)\nabla u\right\} \cdot \nu\,dS = 0.$$

Let us estimate the relevant boundary integrals. First,

$$\int_{\partial\Omega_L \cap \{y=0\}} \left\{\left(\tfrac{1}{2}|\nabla u|^2 + F(u)\right)X - X(u)\nabla u\right\} \cdot \nu\,dS = \int_{a_\varepsilon}^{\varepsilon L} \left(\tfrac{1}{2}u_x^2 + F(u)\right)(x - a_\varepsilon)\,dx$$

whose limit as $L \to \infty$ is

$$\int_{a_\varepsilon}^{\infty} \left(\tfrac{1}{2}u_x^2 + F(u)\right)(x - a_\varepsilon).$$

To estimate this integral, let us recall that, by symmetry and (2-7), we have, for $\boldsymbol{x} = (x, y)$, $y \le 0$, with some $\kappa > 0$,

$$|(u(\boldsymbol{x}))^2 - 1| + |\nabla u(\boldsymbol{x})| \le C e^{-\kappa \operatorname{dist}(\Gamma_{\varepsilon,1}, \boldsymbol{x})}.$$

Now, using this and the fact that

$$|\varepsilon a_\varepsilon + \mathscr{A}_\varepsilon| \ge |f_{\varepsilon,1}(a_\varepsilon)| - 1 \ge \left(1 + \frac{\sqrt{2}}{2}\right) |\ln \varepsilon| - C,$$

after some calculation, we deduce that, as $\varepsilon \to 0$,

$$\int_{\partial\Omega_L \cap \{y=0\}} \left\{ \left(\tfrac{1}{2}|\nabla u|^2 + F(u)\right) X - X(u) \nabla u \right\} \cdot \nu \, dS \to 0.$$

On the other hand, using the asymptotic behavior of u in the lower half plane, we get

$$u = \bar{H} + o(1) e^{-\kappa \operatorname{dist}(\Gamma_{\varepsilon,1}, \boldsymbol{x})}, \quad (\boldsymbol{x}^*_{\varepsilon,1} \bar{H})(x_1, y_1) = H(y_1),$$

where (x_1, y_1) are the Fermi coordinates of the point $\boldsymbol{x}$. Since on the line $\{x = a_\varepsilon\}$ we have $X = (f_{\varepsilon,1}(a_\varepsilon) - y, 0)$, we get

$$\int_{\partial\Omega_L \cap \{x=a_\varepsilon\}} \left\{ \left(\tfrac{1}{2}|\nabla u|^2 + F(u)\right) X - X(u) \nabla u \right\} \cdot \nu \, dS = o(1).$$

Finally, we compute:

$$\left| \int_{\partial\Omega_L \cap \{y = x/\varepsilon - L\}} \left\{ \left(\tfrac{1}{2}|\nabla u|^2 + F(u)\right) X - X(u) \nabla u \right\} \cdot \nu \, dS \right| = \frac{|f_{\varepsilon,1}(a_\varepsilon) + \varepsilon a_\varepsilon + \mathscr{A}_\varepsilon|}{\sqrt{1 + \varepsilon^2}} + o(1).$$

Collecting all these estimates, we conclude

$$|f_{\varepsilon,1}(a_\varepsilon) + \varepsilon a_\varepsilon + \mathscr{A}_\varepsilon| = o(1).$$

Appealing to the definition of a_ε, this implies that $a_\varepsilon = |\ln \varepsilon|/\varepsilon$, and consequently,

$$|f_{\varepsilon,1}(x) + \varepsilon x + \mathscr{A}_\varepsilon| = o(1) \quad \text{for } x \in [|\ln \varepsilon|/\varepsilon, +\infty).$$

This implies that outside this interval, $\mathscr{N}_{\varepsilon,1}$ is close to a straight line, which combined with the estimates (4-6) yields the desired result. Indeed, now we have

$$\begin{aligned} q_{\varepsilon,1}(a_\varepsilon) &= f_{\varepsilon,1}(a_\varepsilon) + o(1) \\ &= -\varepsilon a_\varepsilon - \mathscr{A}_\varepsilon + o(1). \end{aligned}$$

On the other hand, since $q_{\varepsilon,1}$ is the solution of the Toda equation, we have

$$q_{\varepsilon,1}(x) = -\varepsilon x - \widetilde{\mathscr{A}}_\varepsilon + o(1) \quad \text{for } x \ge a_\varepsilon.$$

It follows that $\mathscr{A}_\varepsilon = \widetilde{\mathscr{A}}_\varepsilon + o(1)$. This ends the proof of Step 2.

Step 3. At this point we can use what we have just proven in Step 2 to get

$$f_{\varepsilon,1}(x) = \frac{\sqrt{2}}{2}\ln\varepsilon - \varepsilon|x| + \mathcal{O}(1), \quad |x| \gg 1.$$

As a consequence,

$$\|\exp(-\sqrt{2}\mathcal{D})\|_{\mathcal{C}^0_{\varepsilon\tau}(\mathbb{R})} \le C\varepsilon^2, \tag{5-28}$$

which, together with Lemma 5.7, yields

$$\|\phi\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R}^2)} + \|f''_{\varepsilon,1}\|_{\mathcal{C}^{0,\mu}_{\varepsilon\tau}(\mathbb{R})} + \|h_\varepsilon\|_{\mathcal{C}^{2,\mu}_{\varepsilon\tau}(\mathbb{R})} \le C\varepsilon^2. \tag{5-29}$$

Then, by a similar calculation to that of (5-20), we find that, in the half line $\mathbb{R}\backslash I = (|\ln\varepsilon|/\varepsilon, +\infty)$, the function $p_{\varepsilon,1}$ satisfies

$$\|p''_{\varepsilon,1}\|_{\mathcal{C}^{0,\mu}_{\varepsilon\hat{\tau}}(\mathbb{R}\backslash I)} = \mathcal{O}(\varepsilon^{2+\alpha}) \tag{5-30}$$

for some $\hat{\tau} > 0$ independent of ε. This implies that, in $\mathbb{R}\backslash I$,

$$\begin{aligned} |p'_{\varepsilon,1}(x) + \varepsilon| &\le C\int_x^{+\infty} \varepsilon^{2+\alpha} e^{-\varepsilon\hat{\tau}|s|}\, ds \\ &= C\varepsilon^{1+\alpha} e^{-\varepsilon\hat{\tau}|x|}. \end{aligned}$$

Therefore,

$$|p_{\varepsilon,1}(x) + \varepsilon x + \mathcal{A}_\varepsilon| \le C\varepsilon^\alpha e^{-\varepsilon\hat{\tau}|x|}, \quad x \in \mathbb{R}\setminus I. \tag{5-31}$$

On the other hand, by Step 1 and the fact that

$$|q_{\varepsilon,1}(x) + \varepsilon x + \widetilde{\mathcal{A}}_\varepsilon| \le C\varepsilon^\alpha e^{-\varepsilon\beta|x|}, \quad x \in \mathbb{R}\setminus I, \tag{5-32}$$

we get

$$|p_{\varepsilon,1}(|\ln\varepsilon|/\varepsilon) + |\ln\varepsilon| + \widetilde{\mathcal{A}}_\varepsilon| \le C\varepsilon^\alpha.$$

This together with (5-31) then yields $|\mathcal{A}_\varepsilon - \widetilde{\mathcal{A}}_\varepsilon| < C\varepsilon^\alpha$. Now, letting $j_\varepsilon = \mathcal{A}_\varepsilon - \widetilde{\mathcal{A}}_\varepsilon$, taking into account Step 1, (5-31), (5-32), and reducing $\hat{\tau}$ if necessary, the assertion of Proposition 4.2 follows. The conclusion of the proof of Proposition 4.1 is contained in (5-29). □

6. Uniqueness of solutions with almost parallel nodal lines

Parametrization of the family of solutions of (1-1) by the trajectories of the Toda system. Let us consider the curve $\widetilde{\mathcal{N}}_{\varepsilon,i}$ which is the graph of the function $y = q_{\varepsilon,i}(x)$. When $i = 1$, it is contained in the lower half-plane, and when $i = 2$, it is contained in the upper half-plane. We have $q_{\varepsilon,1}(x) = -q_{\varepsilon,2}(x)$. With these curves we will associate the Fermi coordinates $(\tilde{x}_i, \tilde{y}_i)$:

$$\boldsymbol{x} = (\tilde{x}_i, q_{\varepsilon,i}(\tilde{x}_i)) + \tilde{y}_i\tilde{n}_{\varepsilon,i}(\tilde{x}_i), \quad \tilde{n}_{\varepsilon,i}(x) = \frac{(-q'_{\varepsilon,i}(x), 1)}{\sqrt{1+q'_{\varepsilon,i}(x)^2}}, \quad i = 1, 2.$$

The change of variables $(\tilde{x}_i, \tilde{y}_i) \mapsto \boldsymbol{x} = (x, y)$ is a diffeomorphism in a neighborhood $\widetilde{\mathbb{O}}_i$ of $\widetilde{\mathcal{N}}_{\varepsilon,i}$. We denote this diffeomorphism by $\tilde{\boldsymbol{x}}_{\varepsilon,i}$ so that

$$\tilde{\boldsymbol{x}}_{\varepsilon,i}(\tilde{x}_i, \tilde{y}_i) = \boldsymbol{x} \in \widetilde{\mathbb{O}}_i.$$

For any function $w\colon \widetilde{\mathbb{O}}_i \to \mathbb{R}$ by $\tilde{\boldsymbol{x}}^*_{\varepsilon,i} w$ we denote its pullback by $\tilde{\boldsymbol{x}}_{\varepsilon,i}$:

$$(\tilde{\boldsymbol{x}}^*_{\varepsilon,i} w)(\tilde{x}_i, \tilde{y}_i) = (w \circ \tilde{\boldsymbol{x}}_{\varepsilon,i})(\tilde{x}_i, \tilde{y}_i).$$

Using basic properties (linear growth, scaling) of the trajectories of the solutions of the Toda system, one can check [del Pino et al. 2010] that there exists a constant C_1 such that we can choose $\widetilde{\mathbb{O}}_i$, i=1,2, to be the set

$$\{(x, y) \in \mathbb{R}^2 : |y| \le C_1 \varepsilon^{-1} \sqrt{1 + x^2}\}.$$

With these preparations, we would like to write locally any solution u, with $\tan\theta(u) = \varepsilon$ small, in the Fermi coordinates with respect to $\widetilde{\mathcal{N}}_{\varepsilon,i}$. To this end, we will construct a suitable approximation of u in $\widetilde{\mathbb{O}}_i$ based on the fact that the true solution is locally close to the heteroclinic one. By symmetry we may focus on the case $i = 1$, namely, consider the lower half plane. The nodal line $\mathcal{N}_{\varepsilon,1}$ of u in the lower half plane is the graph of $y = f_{\varepsilon,1}(x)$. Recall that $q_{\varepsilon,1}(x)$ is the solution of the Toda equation such that the assertions of Proposition 4.2 are satisfied. We let $\tilde{\eta}$ to be a smooth cut off function equal to 1 in $\widetilde{\mathbb{O}}_1 \cap \{\operatorname{dist}(\boldsymbol{x}, \partial\widetilde{\mathbb{O}}_1) > 1\}$ and equal to 0 in $\mathbb{R}^2 \setminus \widetilde{\mathbb{O}}_1$. A reasonable ansatz for an approximate solution is built defining the function $\widetilde{H}_{\varepsilon,1}$ by

$$\tilde{\boldsymbol{x}}^*_{\varepsilon,1} \widetilde{H}_{\varepsilon,1}(\tilde{x}_1, \tilde{y}_1) := \tilde{\boldsymbol{x}}^*_{\varepsilon,1} \tilde{\eta}(\tilde{x}_1, \tilde{y}_1) H(\tilde{y}_1 - \tilde{g}_\varepsilon(\tilde{x}_1)) + (1 - \tilde{\boldsymbol{x}}^*_{\varepsilon,1} \tilde{\eta}(\tilde{x}_1, \tilde{y}_1)) \frac{H(\tilde{y}_1 - \tilde{g}_\varepsilon(\tilde{x}_1))}{|H(\tilde{y}_1 - \tilde{g}_\varepsilon(\tilde{x}_1))|},$$

which is extended to the whole $\mathbb{R}^2$ by ± 1, setting $\widetilde{H}_{\varepsilon,2}(x, y) = -\widetilde{H}_{\varepsilon,1}(x, -y)$, and finally defining

$$\tilde{u}_\varepsilon := \widetilde{H}_{\varepsilon,1} - \widetilde{H}_{\varepsilon,2} - 1. \tag{6-1}$$

Note that the function $\tilde{g}_\varepsilon$ has not been specified so far. It turns out that, in order to have a good approximation of u by $\tilde{u}_\varepsilon$, we should impose the orthogonality condition

$$\int_{\mathbb{R}} \tilde{\boldsymbol{x}}^*_{\varepsilon,i} [(u - \tilde{u}_\varepsilon) \tilde{\rho}_{\varepsilon,i} \widetilde{H}'_{\varepsilon,i}](\tilde{x}_i, \tilde{y}_i)\, d\tilde{y}_i = 0 \quad \text{for all } \tilde{x}_i,\ i = 1, 2, \tag{6-2}$$

where smooth cutoff functions $\tilde{\rho}_{\varepsilon,i}$ are defined through

$$(\tilde{\boldsymbol{x}}^*_{\varepsilon,i} \tilde{\rho}_{\varepsilon,i})(\tilde{x}_i, \tilde{y}_i) = \tilde{\rho}(\tilde{y}_i - (-1)^{i+1} \tilde{g}_\varepsilon(\tilde{x}_i)),$$

and $\tilde{\rho}$ is an even cutoff function equal to 1 in the interval $(\sqrt{2}\ln\varepsilon/8, -\sqrt{2}\ln\varepsilon/8)$ and equal to 0 outside $(\sqrt{2}\ln\varepsilon/4, -\sqrt{2}\ln\varepsilon/4)$, while $\widetilde{H}'_{\varepsilon,i}$ is defined by

$$\tilde{\boldsymbol{x}}^*_{\varepsilon,i} \widetilde{H}'_{\varepsilon,i}(\tilde{x}_i, \tilde{y}_i) = H'(\tilde{y}_i - (-1)^{i+1} \tilde{g}_\varepsilon(\tilde{x}_i)).$$

To show the existence of the function $\tilde{g}_\varepsilon$, one can use an argument similar to the one in Lemma 5.1. However, since the graph of the function $y = q_{\varepsilon,i}(x)$ does not converge to the nodal set of the solution at infinity, the function $\tilde{g}_\varepsilon$ does not decay exponentially. To determine the behavior of the function $\tilde{g}_\varepsilon$ more precisely, we need the following.

Lemma 6.1. *There exist constants $\tilde{\tau} > 0$ and v_ε such that $|v_\varepsilon| \le C\varepsilon^\alpha$, and the function $\tilde{h}_\varepsilon(x) := \tilde{g}_\varepsilon(x) + v_\varepsilon$ satisfies*

$$\begin{aligned} \|\tilde{h}_\varepsilon\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} &\le C\varepsilon^\alpha, \\ \|\tilde{h}'_\varepsilon\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} &\le C\varepsilon^{1+\alpha}, \\ \|\tilde{h}''_\varepsilon\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})} &\le C\varepsilon^{2+\alpha}. \end{aligned} \tag{6-3}$$

Proof. The function $\tilde{g}_\varepsilon$ is determined by

$$\int_{\mathbb{R}} \tilde{\boldsymbol{x}}^*_{\varepsilon,1}[(u-\tilde{u}_\varepsilon)\tilde{\rho}_{\varepsilon,1}\widetilde{H}'_{\varepsilon,1}]\, d\tilde{y}_1 = 0.$$

Changing variables, this relation can also be written as

$$\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(u-\tilde{u}_\varepsilon)(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 = 0. \tag{6-4}$$

For this integral, it suffices to consider the points in the support of $\tilde{\rho}_{\varepsilon,1}$.

Recall that, by the definition of $\tilde{u}_\varepsilon$,

$$\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\tilde{u}_\varepsilon(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1)) = H(\tilde{y}_1) - \tilde{\boldsymbol{x}}^*_{\varepsilon,1}(\widetilde{H}_{\varepsilon,2}+1)(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1)).$$

It is not difficult to see that

$$\left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(\widetilde{H}_{\varepsilon,2}+1)(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^2$$

for some $\tilde{\tau} > 0$. This combined with (6-4) leads to

$$\left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}u(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^2. \tag{6-5}$$

On the other hand, $u = \bar{u}_\varepsilon + \phi$ with $\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R}^2)} \le C\varepsilon^2$. Hence, reducing $\tilde{\tau}$ if necessary, we get

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\bar{u}_\varepsilon(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \\ &\qquad \le \left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\phi(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tau_0}(\mathbb{R})} + C\varepsilon^2 \le C\varepsilon^2. \end{aligned} \tag{6-6}$$

Now, in the support of $\tilde{\rho}_{\varepsilon,1}$, $\bar{u}_\varepsilon = H(y_1 - h_\varepsilon(x_1)) - H(y_2 + h_\varepsilon(x_2)) - 1$. Denoting the function $(x, y) = \boldsymbol{x}_{\varepsilon,1}(x_1, y_1) \mapsto H(y_1 - h_\varepsilon(x_1))$ by $\mathfrak{R}$, it follows from (6-6) that (reducing $\tilde{\tau}$ if necessary)

$$\left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\mathfrak{R}(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^2. \tag{6-7}$$

To proceed, let us investigate the relation between the Fermi coordinates (x_1, y_1) and $(\tilde{x}_1, \tilde{y}_1)$. Using $|f'_{\varepsilon,1}| \le C\varepsilon$, $|f_{\varepsilon,1} - q_{\varepsilon,1}| \le C\varepsilon^\alpha$, $|y_1| \le C|\ln \varepsilon|$, and elementary geometry, one can verify that

$$|\tilde{x}_1 - x_1| \le C||y_1| + C\varepsilon^\alpha|\varepsilon \le C\varepsilon^\alpha. \tag{6-8}$$

Additionally, recall that by Proposition 4.2, $\| f_{\varepsilon,1} - q_{\varepsilon,1} + j_\varepsilon \|_{C^0_{\varepsilon\hat{\tau}}(\mathbb{R})} \le C\varepsilon^\alpha$. Using (6-8) , one can show

$$y_1 = \tilde{y}_1 + \big(\sqrt{1 + (q'_{\varepsilon,1}(\tilde{x}_1))^2}\,\big)^{-1} j_\varepsilon + \mathbb{O}(\varepsilon^\alpha e^{-\varepsilon\beta|\tilde{x}_1|}). \tag{6-9}$$

Inserting this into (6-7), we find (again reducing $\tilde{\tau}$ if necessary)

$$\left\| \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1) H'(\tilde{y}_1) H(\tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1) + \big(\sqrt{1 + (q'_{\varepsilon,1}(\tilde{x}_1))^2}\,\big)^{-1} j_\varepsilon)\, d\tilde{y}_1 \right\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^\alpha. \tag{6-10}$$

As a consequence,

$$\|\tilde{g}_\varepsilon + \big(\sqrt{1 + (q'_{\varepsilon,1})^2}\,\big)^{-1} j_\varepsilon \|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^\alpha, \tag{6-11}$$

which together with the behavior of $q'_{\varepsilon,1}$ implies that

$$\|\tilde{h}_\varepsilon\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^\alpha, \quad |v_\varepsilon| \le C\varepsilon^\alpha, \tag{6-12}$$

where $v_\varepsilon := j_\varepsilon / \sqrt{1+\varepsilon^2}$ and

$$\tilde{h}_\varepsilon(x) := \tilde{g}_\varepsilon(x) + v_\varepsilon. \tag{6-13}$$

Next we need to estimate the weighted norm of the first derivative of $\tilde{h}_\varepsilon$.

Let us denote the diffeomorphism $\boldsymbol{x}^{-1}_{\varepsilon,1} \circ \tilde{\boldsymbol{x}}_{\varepsilon,1}$ by $\Phi_{\varepsilon,1}$ and denote $\boldsymbol{x}^{-1}_{\varepsilon,2} \circ \tilde{\boldsymbol{x}}_{\varepsilon,1}$ by $\Phi_{\varepsilon,2}$. Then, using (6-8), (6-9), and formulas (3-10), after direct calculations, we find that

$$|D\Phi_{\varepsilon,1} - \mathrm{Id}_{2\times 2}| = \mathbb{O}(\varepsilon^{1+\alpha} e^{-\varepsilon\beta|\tilde{x}_1|}), \quad |D^2\Phi_{\varepsilon,1}| = \mathbb{O}(\varepsilon^{2+\alpha} e^{-\varepsilon\beta|\tilde{x}_1|}), \tag{6-14}$$

$$|D\Phi_{\varepsilon,2} - \mathrm{Id}_{2\times 2}| = \mathbb{O}(\varepsilon e^{-\varepsilon\beta|\tilde{x}_2|}), \qquad |D^2\Phi_{\varepsilon,2}| = \mathbb{O}(\varepsilon^2 e^{-\varepsilon\beta|\tilde{x}_2|}). \tag{6-15}$$

We now differentiate (6-4) with respect to $\tilde{x}_1$. Set

$$\mathfrak{R}_1 := \partial_{\tilde{x}_1} \tilde{\boldsymbol{x}}^*_{\varepsilon,1} u(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1)),$$
$$\mathfrak{R}_2 := \partial_{\tilde{y}_1} \tilde{\boldsymbol{x}}^*_{\varepsilon,1} u(\tilde{x}_1, \tilde{y}_1 + \tilde{g}_\varepsilon(\tilde{x}_1)).$$

By estimate (6-15), one has

$$\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1) H'(\tilde{y}_1) \{\mathfrak{R}_1 + \mathfrak{R}_2 \tilde{g}'_\varepsilon(\tilde{x}_1)\}\, d\tilde{y}_1 = O(\varepsilon^2 e^{-\varepsilon\beta|\tilde{x}_1|}).$$

Therefore, using (6-13),

$$\tilde{h}'_\varepsilon(\tilde{x}_1) = -\frac{\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1) H'(\tilde{y}_1) \mathfrak{R}_1\, d\tilde{y}_1}{\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1) H'(\tilde{y}_1) \mathfrak{R}_2\, d\tilde{y}_1} + \mathbb{O}(\varepsilon^2 e^{-\varepsilon\beta|\tilde{x}_1|}). \tag{6-16}$$

Keep in mind that

$$\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\mathfrak{R}_1\, d\tilde{y}_1 = \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\partial_{\tilde{x}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\bar{u}_\varepsilon(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1 + \int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\partial_{\tilde{x}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}\phi(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1))\, d\tilde{y}_1. \tag{6-17}$$

Equations (6-14), (6-15) and (6-17), together with $\|\phi\|_{\mathscr{C}^2_{\varepsilon\tau_0}(\mathbb{R}^2)} \le C\varepsilon^2$, yield

$$\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\mathfrak{R}_1\, d\tilde{y}_1 = \mathbb{O}(\varepsilon^{1+\alpha}e^{-\varepsilon\beta|\tilde{x}_1|}).$$

It then follows from (6-16) that (reducing $\tilde{\tau}$ if necessary)

$$\|\tilde{h}'_\varepsilon\|_{\mathscr{C}^0_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{1+\alpha}.$$

It remains to estimate $\tilde{h}''_\varepsilon$. Setting

$$\mathfrak{R}_3 = \partial^2_{\tilde{x}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(u-\tilde{u}_\varepsilon)(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1)), \qquad \mathfrak{R}_5 = \partial^2_{\tilde{y}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(u-\tilde{u}_\varepsilon)(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1)),$$
$$\mathfrak{R}_4 = \partial^2_{\tilde{x}_1\tilde{y}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(u-\tilde{u}_\varepsilon)(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1)), \qquad \mathfrak{R}_6 = \partial_{\tilde{y}_1}\tilde{\boldsymbol{x}}^*_{\varepsilon,1}(u-\tilde{u}_\varepsilon)(\tilde{x}_1,\tilde{y}_1+\tilde{g}_\varepsilon(\tilde{x}_1)),$$

from (6-4), one gets

$$\tilde{h}''_\varepsilon(\tilde{x}_1) = -\frac{\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)[\mathfrak{R}_3 + 2\mathfrak{R}_4\tilde{g}'_\varepsilon(\tilde{x}_1) + \mathfrak{R}_5(\tilde{g}'_\varepsilon(\tilde{x}_1))^2]\, d\tilde{y}_1}{\int_{\mathbb{R}} \tilde{\rho}(\tilde{y}_1)H'(\tilde{y}_1)\mathfrak{R}_6\, d\tilde{y}_1}. \tag{6-18}$$

Recall that

$$\|\phi\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R})} + \|h_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tau_0}(\mathbb{R})} \le C\varepsilon^2.$$

A refined argument which involves closer analysis of the main order of ϕ shows that in reality $\partial^2_x\phi$ and h''_ε have better estimates:

$$\|\partial^2_x\phi\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R})} + \|h''_\varepsilon\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tau_0}(\mathbb{R})} \le C\varepsilon^{2+\alpha}.$$

This estimate follows by observing first that the orthogonality relation for ϕ can be differentiated in x twice. Then we note that, furthermore, differentiating the equation satisfied by ϕ twice, we gain powers of ε in the main order term, namely, the right side will be of order at least $\mathbb{O}(\varepsilon^{2+\alpha})$. Then $\partial^2_x\phi$ and h''_ε can be estimated using the same orthogonal decomposition as in Section 5. Combining this with (6-14), (6-15), and (6-18), after some calculations, we get, reducing $\tilde{\tau}$ if necessary,

$$\|\tilde{h}''_\varepsilon\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{2+\alpha}. \qquad \square$$

Given a solution u of (1-1) such that $\tan\theta(u) = \varepsilon$, we can define an approximate solution $\tilde{u}_\varepsilon$ by (6-1) using the solution of the Toda system with the asymptotic slope ε. Then we can write

$$u = \tilde{u}_\varepsilon + \tilde{\phi}.$$

By the definition of $\tilde{g}_\varepsilon$, we know that $\tilde{\phi} = u - \tilde{u}_\varepsilon$ satisfies the orthogonality condition (6-2). This allows us to control the size of $\tilde{\phi}$ in the weighted norm in terms of the error of the approximation

$$E(\tilde{u}_\varepsilon) = \Delta\tilde{u}_\varepsilon - F'(\tilde{u}_\varepsilon),$$

following essentially the same approach as in Section 5, and, in particular, relying on a version of Proposition 5.5. In fact, one can prove that

$$\|\tilde{\phi}\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)} \le C\varepsilon^2. \tag{6-19}$$

Conclusion of the proof: the Lipschitz property of solutions. Based on the results of the previous section, we know that any solution with a small angle can be written in the following way:

$$u(\,\cdot\,;\tilde{g}_\varepsilon,\tilde{\phi}) = \tilde{u}_\varepsilon(\,\cdot\,;\tilde{g}_\varepsilon) + \tilde{\phi},$$

where $\tilde{u}_\varepsilon$ is the approximate solution defined in (6-1). Here and below we will indicate the dependence of this solution on the modulation function $\tilde{g}_\varepsilon$ as well as on $\tilde{\phi}$. Now let us consider two solutions $u^{(j)}$, $j = 1, 2$, with the same asymptotic angle $\theta(u^{(j)}) = \arctan\varepsilon$. Since the asymptotic angle is the same for both solutions, there is just one solution of the Toda system represented by the functions $q_{\varepsilon,1} = -q_{\varepsilon,2}$. On the other hand, it may happen that $\tilde{g}^{(1)}_\varepsilon \ne \tilde{g}^{(2)}_\varepsilon$ and $\tilde{\phi}^{(1)} \ne \tilde{\phi}^{(2)}$. In the notation of [del Pino et al. 2010], we have that $\tilde{g}^{(j)}_\varepsilon \in \mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}) \oplus D$ (see also the summary on pages 1684–1685). In the previous section we have shown that $\|\tilde{g}^{(j)}_\varepsilon\|_{\mathscr{C}^{0}_{\varepsilon\tilde{\tau}}(\mathbb{R})\oplus D} \le C\varepsilon^\alpha$, with corresponding estimates for the higher-order derivatives. In addition, for the functions $\tilde{\phi}^{(j)}$, we have (6-19). Without loss of generality, we can assume that $\tilde{\tau}$ is small but independent of ε.

To prove the uniqueness of solutions with small angles, it is enough to prove "local uniqueness" in the following sense. Given two four-end solutions associated to the same solution of the Toda system, we have $\tilde{\phi}^{(1)} = \tilde{\phi}^{(2)}$ and $\tilde{g}^{(1)}_\varepsilon = \tilde{g}^{(2)}_\varepsilon$. Our strategy to prove this fact follows in some sense the strategy used to prove the existence of solutions with small angles employed in [del Pino et al. 2010]. To explain this, let us introduce the scaled functions $\hat{g}^{(j)}_\varepsilon(x) := \tilde{g}^{(j)}_\varepsilon(x/\varepsilon)$, $j = 1, 2$. We show the Lipschitz property of the map $\hat{g}_\varepsilon \mapsto E(\tilde{u}_\varepsilon(\,\cdot\,;\tilde{g}_\varepsilon))$, and then we use the linearized equation to show that $\tilde{\phi}^{(1)} - \tilde{\phi}^{(2)}$ can be controlled by a small constant times $\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon$. As a final step we show that the function $\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon$ satisfies the linearized Toda system with the right side again controlled by a small constant times $\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon$. This leads us to conclude that $\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon = 0$, and as a result we infer the uniqueness.

Now we will present some details of the argument outlined above. Many of the calculations are quite similar to the ones in [del Pino et al. 2010].

Lemma 6.2. *The following estimates hold*:

$$\|E(\tilde{u}^{(1)}(\,\cdot\,;\tilde{g}^{(1)}_\varepsilon)) - E(\tilde{u}^{(2)}(\,\cdot\,;\tilde{g}^{(2)}_\varepsilon))\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)} \le C\varepsilon^2\|\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D}, \tag{6-20}$$

$$\|\tilde{\phi}^{(1)} - \tilde{\phi}^{(2)}\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)} \le C\varepsilon^2\|\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D}. \tag{6-21}$$

Remark 6.3. Essentially, up to some minor difference, this Lipschitz property has already been proven in [del Pino et al. 2010]. Here we give a sketch of the proof for completeness.

Proof. To begin with, let us mention that, for a function $g : \mathbb{R} \to \mathbb{R}$, we have the obvious estimates:

$$\|g(\varepsilon\cdot)\|_{C^{l,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\|g(\cdot)\|_{C^{l,\mu}_{\tilde{\tau}}(\mathbb{R})},$$
$$\|g(\cdot)\|_{C^{l,\mu}_{\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{-l-\mu}\|g(\varepsilon\cdot)\|_{C^{l,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})}.$$

To prove (6-20) we use essentially the formula (5-7) for the error, replacing $\bar{u}_\varepsilon$ by $\tilde{u}_\varepsilon^{(j)}$, $j=1,2$, and then take the difference of the resulting terms $E(\tilde{u}^{(j)}(\,\cdot\,;\tilde{g}_\varepsilon^{(j)}))$.

To show (6-21), we should consider the equation satisfied by the difference $\tilde{\psi}=\tilde{\phi}^{(1)}-\tilde{\phi}^{(2)}$ and use Proposition 5.5. The slight technical problem is that $\tilde{\psi}$ does not satisfy the orthogonality condition as in (6-2). To overcome this, we further define a function $\tilde{\psi}^\perp$ by

$$\tilde{\psi}^\perp := \tilde{\psi} - \sum_{i=1,2} \tilde{\psi}_i^\parallel,$$

where $\tilde{\psi}_i^\parallel:\mathbb{R}^2\to\mathbb{R}$ is equal to 0 outside $\widetilde{\mathcal{O}}_i$ and

$$\tilde{\boldsymbol{x}}_{\varepsilon,i}^*\tilde{\psi}_i^\parallel(\tilde{x}_i,\tilde{y}_i) := \tilde{c}_\varepsilon \tilde{\boldsymbol{x}}_{\varepsilon,i}^*(\tilde{\rho}_{\varepsilon,i}^{(1)}\widetilde{H}_{\varepsilon,i}^{(1)\prime})\int_{\mathbb{R}} \tilde{\boldsymbol{x}}_{\varepsilon,i}^*[\tilde{\psi}\tilde{\rho}_{\varepsilon,i}^{(1)}\widetilde{H}_{\varepsilon,i}^{(1)\prime}]\,d\tilde{y}_i \quad \text{in } \widetilde{\mathcal{O}}_i,$$

where $\tilde{c}_\varepsilon=\{\int_{\mathbb{R}}[\tilde{\rho}(y)H'(y)]^2\,dy\}^{-1}$.

Using the fact that $\|\tilde{\phi}^{(2)}\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)}\le C\varepsilon^2$ and

$$\int_{\mathbb{R}} \tilde{\boldsymbol{x}}_{\varepsilon,i}^*[\tilde{\phi}^{(2)}\tilde{\rho}_{\varepsilon,i}^{(2)}\widetilde{H}_{\varepsilon,i}^{(2)\prime}]\,d\tilde{y}_i=0, \quad i=1,2,$$

it is not hard to show that

$$\|\tilde{\psi}_i^\parallel\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)}\le C\varepsilon^2\|\hat{g}_\varepsilon^{(1)}-\hat{g}_\varepsilon^{(2)}\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D}.$$

Hence

$$\|\tilde{\psi}^\perp\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)}\ge \|\tilde{\psi}\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)} - C\varepsilon^2\|\hat{g}_\varepsilon^{(1)}-\hat{g}_\varepsilon^{(2)}\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D}. \tag{6-22}$$

On the other hand, setting

$$L^{(i)}=-\Delta+F''(\tilde{u}_\varepsilon^{(i)}),\quad P^{(i)}(\tilde{\phi}^{(i)})=F'(\tilde{u}_\varepsilon^{(i)}+\tilde{\phi}^{(i)})-F'(\tilde{u}_\varepsilon^{(i)})-F''(\tilde{u}_\varepsilon^{(i)})\tilde{\phi}^{(i)},\quad i=1,2,$$

we get

$$L^{(1)}\tilde{\psi}^\perp = \underbrace{E(\tilde{u}_\varepsilon^{(1)})-E(\tilde{u}_\varepsilon^{(2)})-P^{(1)}(\tilde{\phi}^{(1)})+P^{(2)}(\tilde{\phi}^{(2)})-(L^{(1)}-L^{(2)})\tilde{\phi}^{(2)}-L^{(1)}(\tilde{\psi}_1^\parallel+\tilde{\psi}_2^\parallel)}_{\tilde{f}}. \tag{6-23}$$

Applying Lemma 6.2, one can see that

$$\|\tilde{f}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)}\le o(1)\|\tilde{\psi}\|_{\mathscr{C}^{2,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R}^2)}+C\varepsilon^2\|\hat{g}_\varepsilon^{(1)}-\hat{g}_\varepsilon^{(2)}\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D}.$$

From this and (6-22), the required estimate follows. □

As we have already seen, the Toda system appears in the projected equation. It turns out that we also need to analyze the linearized Toda system. Recall that we are always working in the space of even functions. Suppose q is an even solution of the Toda system

$$q''(t)=-c_*e^{2\sqrt{2}q(t)},$$

and the linearized operator is

$$\mathcal{P}:\varphi\to\varphi''+2\sqrt{2}c_*e^{2\sqrt{2}q}\varphi.$$

We want to know the mapping property of this operator. Let $\mathscr{C}^{l,\mu}_{\tilde{\tau}}(\mathbb{R})_e$ be the space of even functions in $\mathscr{C}^{l,\mu}_{\tilde{\tau}}(\mathbb{R})$, and let D_0 be the one-dimensional deficiency space spanned by the constant function.

Lemma 6.4. *For small $\tilde{\tau} > 0$, the map $\mathscr{P} : C^{2,\mu}_{\tilde{\tau}}(\mathbb{R})_e \oplus D_0 \to C^{0,\mu}_{\tilde{\tau}}(\mathbb{R})_e$ is an isomorphism and therefore has a bounded inverse.*

This result has already been proven in [del Pino et al. 2010] and we omit the proof. With all these properties understood, we are ready to prove the uniqueness of solutions with given small angles.

Proof of Theorem 1.2. Let us consider the quantity (cf. the proof of Lemma 5.7)

$$\mathscr{T} = \int_{\mathbb{R}} \tilde{\boldsymbol{x}}^*_{\varepsilon,1}[E(\tilde{u}^{(1)}_\varepsilon)\tilde{\rho}^{(1)}_{\varepsilon,1}\tilde{H}^{(1)\prime}_{\varepsilon,1}]\,d\tilde{y}_1 - \int_{\mathbb{R}} \tilde{\boldsymbol{x}}^*_{\varepsilon,1}[E(\tilde{u}^{(2)}_\varepsilon)\tilde{\rho}^{(2)}_{\varepsilon,1}\tilde{H}^{(2)\prime}_{\varepsilon,1}]\,d\tilde{y}_1.$$

Recall that

$$E(\tilde{u}^{(i)}_\varepsilon) = -\Delta\tilde{\phi}^{(i)} + F''(\tilde{u}^{(i)}_\varepsilon)\tilde{\phi}^{(i)} + P^{(i)}(\tilde{\phi}^{(i)}).$$

Inserting this into the expression of $\mathscr{T}$, calculating as in Step 1 in the proof of Lemma 5.7, using the estimates in Lemma 6.2, we get

$$\|\mathscr{T}\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{2+\alpha}\|\hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D_0}. \tag{6-24}$$

For brevity set

$$\tilde{\boldsymbol{g}}_\varepsilon := \tilde{g}^{(1)}_\varepsilon - \tilde{g}^{(2)}_\varepsilon \quad \text{and} \quad \hat{\boldsymbol{g}}_\varepsilon := \hat{g}^{(1)}_\varepsilon - \hat{g}^{(2)}_\varepsilon.$$

Now we calculate $\mathscr{T}$ using the explicit expressions for $\tilde{u}^{(i)}_\varepsilon$ in a manner similar to Step 2 of Lemma 5.7, and, as a result, we get a formula similar to (5-24), which reads

$$\mathscr{T} = (1+\mathbb{O}_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})}(\varepsilon^\alpha))\tilde{\boldsymbol{g}}''_\varepsilon + 2\sqrt{2}c_*(1+\mathbb{O}_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})}(\varepsilon^\alpha))e^{2\sqrt{2}q_{\varepsilon,1}}\tilde{\boldsymbol{g}}_\varepsilon + \mathbb{O}_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})}(\varepsilon^{1+\alpha})\tilde{\boldsymbol{g}}'_\varepsilon + \mathbb{O}_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})}(\varepsilon^{2+\alpha})\tilde{\boldsymbol{g}}_\varepsilon.$$

Thus, calculating $\mathscr{T}$ in two ways, we get at the end that

$$\tilde{\boldsymbol{g}}''_\varepsilon + 2\sqrt{2}c_* e^{2\sqrt{2}q_{\varepsilon,1}}\tilde{\boldsymbol{g}}_\varepsilon = \mathbb{G}_\varepsilon, \tag{6-25}$$

where the term $\mathbb{G}_\varepsilon$ on the right satisfies

$$\|\mathbb{G}_\varepsilon\|_{\mathscr{C}^{0,\mu}_{\varepsilon\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{2+\alpha}\|\hat{\boldsymbol{g}}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D_0}. \tag{6-26}$$

(6-25) could be written as

$$\hat{\boldsymbol{g}}''_\varepsilon + 2\sqrt{2}c_* e^{2\sqrt{2}q_1}\hat{\boldsymbol{g}}_\varepsilon = \varepsilon^{-2}\mathbb{G}_\varepsilon(\varepsilon^{-1}\cdot),$$

where $q = (q_1, q_2)$ is the even solution of the Toda system whose asymptotic lines have slopes ∓ 1 (cf. the function U_0 in (4-5)). Now we adapt Lemma 6.4 to the present context and use (6-26) to get

$$\|\hat{\boldsymbol{g}}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D_0} \le C\varepsilon^{-2}\|\mathbb{G}_\varepsilon(\varepsilon^{-1}\cdot)\|_{\mathscr{C}^{0,\mu}_{\tilde{\tau}}(\mathbb{R})} \le C\varepsilon^{\alpha-\mu}\|\hat{\boldsymbol{g}}_\varepsilon\|_{\mathscr{C}^{2,\mu}_{\tilde{\tau}}(\mathbb{R})\oplus D_0}, \tag{6-27}$$

from which it follows that $\hat{\boldsymbol{g}}_\varepsilon = 0$, provided that we choose $\mu < \alpha$ and ε is taken small. This in turn implies $\tilde{g}^{(1)}_\varepsilon = \tilde{g}^{(2)}_\varepsilon$ and $\tilde{\phi}^{(1)} = \tilde{\phi}^{(2)}$, hence we get uniqueness. This ends the proof of Theorem 1.2. □

References

[Ambrosio and Cabré 2000] L. Ambrosio and X. Cabré, "Entire solutions of semilinear elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi", *J. Amer. Math. Soc.* **13**:4 (2000), 725–739. MR 2001g:35064 Zbl 0968.35041

[Berestycki et al. 1997] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, "Monotonicity for elliptic equations in unbounded Lipschitz domains", *Comm. Pure Appl. Math.* **50**:11 (1997), 1089–1111. MR 98k:35064 Zbl 0906.35035

[Dancer 2005] E. N. Dancer, "Stable and finite Morse index solutions on $\mathbf{R}^n$ or on bounded domains with small diffusion", *Trans. Amer. Math. Soc.* **357**:3 (2005), 1225–1243. MR 2006e:35093 Zbl 1145.35369

[Dang et al. 1992] H. Dang, P. C. Fife, and L. A. Peletier, "Saddle solutions of the bistable diffusion equation", *Z. Angew. Math. Phys.* **43**:6 (1992), 984–998. MR 94b:35041 Zbl 0764.35048

[Ghoussoub and Gui 1998] N. Ghoussoub and C. Gui, "On a conjecture of De Giorgi and some related problems", *Math. Ann.* **311**:3 (1998), 481–491. MR 99j:35049 Zbl 0918.35046

[Gui 2008] C. Gui, "Hamiltonian identities for elliptic partial differential equations", *J. Funct. Anal.* **254**:4 (2008), 904–933. MR 2009b:35112 Zbl 1148.35023

[Gui 2012] C. Gui, "Symmetry of some entire solutions to the Allen–Cahn equation in two dimensions", *J. Differential Equations* **252**:11 (2012), 5853–5874. MR 2911416 Zbl 1250.35078

[Karcher 1988] H. Karcher, "Embedded minimal surfaces derived from Scherk's examples", *Manuscripta Math.* **62**:1 (1988), 83–114. MR 89i:53009 Zbl 0658.53006

[Kowalczyk et al. 2012] M. Kowalczyk, Y. Liu, and F. Pacard, "The space of 4–ended solutions to the Allen–Cahn equation in the plane", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29**:5 (2012), 761–781. MR 2971030 Zbl 1254.35219

[Meeks and Wolf 2007] W. H. Meeks, III and M. Wolf, "Minimal surfaces with the area growth of two planes: the case of infinite symmetry", *J. Amer. Math. Soc.* **20**:2 (2007), 441–465. MR 2007m:53008 Zbl 1115.53008

[Pacard and Wei 2013] F. Pacard and J. Wei, "Stable solutions of the Allen–Cahn equation in dimension 8 and minimal cones", *J. Funct. Anal.* **264**:5 (2013), 1131–1167. MR 3010017 Zbl 06143073

[Pérez and Traizet 2007] J. Pérez and M. Traizet, "The classification of singly periodic minimal surfaces with genus zero and Scherk-type ends", *Trans. Amer. Math. Soc.* **359**:3 (2007), 965–990. MR 2007m:53010 Zbl 1110.53008

[del Pino et al. 2010] M. del Pino, M. Kowalczyk, F. Pacard, and J. Wei, "Multiple-end solutions to the Allen–Cahn equation in $\mathbb{R}^2$", *J. Funct. Anal.* **258**:2 (2010), 458–503. MR 2010i:35133 Zbl 1203.35108

[del Pino et al. 2011] M. del Pino, M. Kowalczyk, and J. Wei, "On De Giorgi's conjecture in dimension $N \geq 9$", *Ann. of Math.* (2) **174**:3 (2011), 1485–1569. MR 2012i:35133 Zbl 1238.35019

[del Pino et al. 2013] M. del Pino, M. Kowalczyk, and F. Pacard, "Moduli space theory for the Allen–Cahn equation in the plane", *Trans. Amer. Math. Soc.* **365**:2 (2013), 721–766. MR 2995371 Zbl 06154645

[Savin 2009] O. Savin, "Regularity of flat level sets in phase transitions", *Ann. of Math.* (2) **169**:1 (2009), 41–78. MR 2009m:58025 Zbl 1180.35499

[Schatzman 1995] M. Schatzman, "On the stability of the saddle solution of Allen–Cahn's equation", *Proc. Roy. Soc. Edinburgh Sect. A* **125**:6 (1995), 1241–1275. MR 96j:35085

Received 4 Jul 2012. Accepted 7 Jan 2013.

MICHAŁ KOWALCZYK: kowalczy@dim.uchile.cl
Departamento de Ingeníeria Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170 Correo 3, 00001 Santiago, Chile

YONG LIU: liuyong@ncepu.edu.cn
Departamento de Ingeníeria Matemática and Centro de Modelamiento Matemático, Universidad de Chile, Casilla 170 Correo 3, 00001 Santiago, Chile
Current address: Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, China

FRANK PACARD: frank.pacard@math.polytechnique.fr
Centre de Mathématiques Laurent Schwartz, UMR-CNRS 7640, École Polytechnique, 91128 Palaiseau, France

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2013 is US $160/year for the electronic version, and $310/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 6 No. 7 2013