ANALYSIS & PDE

Volume 12 No. 3 2019

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ANALYSIS AND PDE
Vol. 12, No. 3, 2019
dx.doi.org/10.2140/apde.2019.12.789

GENERALIZED CRYSTALLINE EVOLUTIONS AS LIMITS OF FLOWS WITH SMOOTH ANISOTROPIES

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We prove existence and uniqueness of weak solutions to anisotropic and crystalline mean curvature flows, obtained as a limit of the viscosity solutions to flows with smooth anisotropies.

1. Introduction

In this note we deal with anisotropic, and possibly crystalline, mean curvature flows, that is, flows of sets $t \mapsto E(t)$ governed by the law

$$V(x,t) = -\psi(\nu^{E(t)}(x))(\kappa_\phi^{E(t)}(x) + g(x,t)), \tag{1-1}$$

where

- $V(x,t)$ stands for the outer normal velocity of the boundary $\partial E(t)$ at x,
- ϕ is a given norm on $\mathbb{R}^N$ representing the *surface tension*,
- $\kappa_\phi^{E(t)}$ is the *anisotropic mean curvature* of $\partial E(t)$ associated with the anisotropy ϕ,
- ψ is a norm evaluated at the outer unit normal $\nu^{E(t)}$ to $\partial E(t)$, and g is a forcing term.

The factor ψ plays the role of a *mobility*.

We refer to [Chambolle et al. 2017a] for the motivations to study this flow, which originate in problems from phase transitions and materials science; see for instance [Taylor 1978; Gurtin 1993]. Its mathematical well-posedness is established in the smooth setting, that is, when ϕ, ψ, g and the initial set are sufficiently smooth and ϕ satisfies suitable ellipticity conditions. However, it is also well known that in dimensions $N \geq 3$ singularities may form in finite time even in the smooth case and for regular initial sets. When this occurs, the strong formulation of (1-1) ceases to be meaningful and thus needs to be replaced by weaker notions of global-in-time solution.

Among the different weak approaches that have been proposed in the literature for the classical mean curvature flow (and for several other "regular" flows) here we recall the so-called *level-set formulation* [Osher and Sethian 1988; Evans and Spruck 1991; 1992a; Chen et al. 1991; Giga 2006] and the *flat flow formulation*, proposed by Almgren, Taylor and Wang [Almgren et al. 1993] and based on the *minimizing movements* variational scheme (referred to as the ATW scheme).

MSC2010: 53C44, 49M25, 35D40.
Keywords: geometric evolution equations, crystalline mean curvature flow, level-set formulation, nonlocal curvature flows, nonlocal geometric flows, minimizing movements, viscosity solutions.

However, when the anisotropy ϕ in (1-1) is nondifferentiable or crystalline, the lack of smoothness of the involved differential operators makes it much harder to pursue the aforementioned approaches. In fact, in the crystalline case the problem of finding a suitable weak formulation of (1-1) in dimension $N \geq 3$ leading to a unique global-in-time solution for general initial sets has remained open until the very recent works [Chambolle et al. 2017a; 2017b; Giga and Požár 2016; 2018].

We refer also to [Giga et al. 1998; Caselles and Chambolle 2006; Bellettini et al. 2006] for previous results holding for special classes of initial data, and to [Giga et al. 2014] for a well-posedness result dealing with a very specific anisotropy. The two-dimensional case is somewhat easier and has been essentially settled in [Giga and Giga 2001] (when g is constant) by developing a crystalline version of the viscosity approach for the level-set equation; see also [Taylor 1978; Almgren and Taylor 1995; Angenent and Gurtin 1989; Giga and Giga 1998; Giga and Gurtin 1996] for relevant former work. We also mention the recent papers [Chambolle and Novaga 2015; Mercier et al. 2016], where short time existence and uniqueness of strong solutions for initial "regular" sets (in a suitable sense) is shown.

Let us now briefly describe the most recent progress on the problem. In [Chambolle et al. 2017b], the first global-in-time existence and uniqueness result for the level-set flow associated to (1-1), valid in all dimensions, for arbitrary (possibly unbounded) initial sets, and for general (including crystalline) anisotropies ϕ was established, but under the particular choice $\psi = \phi$ (and $g = 0$). The main contribution of that work is the observation that the variant of the ATW scheme proposed in [Chambolle 2004; Caselles and Chambolle 2006] converges to solutions that satisfy a new stronger *distributional* formulation of the problem in terms of distance functions. Such a formulation is only reminiscent of, but not quite the same as, the distance formulation studied in [Soner 1993], see also [Barles et al. 1993; Ambrosio and Soner 1996; Caselles and Chambolle 2006; Ambrosio 2000], and because of its distributional character it enables the use of parabolic PDE's arguments in order to establish a comparison result yielding uniqueness.

In [Chambolle et al. 2017a], we first observe that the methods of [Chambolle et al. 2017b] can be pushed to treat bounded spatially Lipschitz continuous forcing terms g and more general mobilities ψ, which are "regular" with respect to the anisotropy ϕ. More precisely, a norm ψ is said to be ϕ-regular if the associated ψ-Wulff shape W^ψ satisfies a uniform inner ϕ-Wulff shape condition at all points of its boundary. Such a condition implies that the ϕ-curvature κ_ϕ of ∂W^ψ is bounded above and it enables us to show that a distributional formulation in the spirit of [Chambolle et al. 2017b] still holds true. Next, owing to the simple observation that the ϕ-regular mobilities are dense, we succeed in extending the notion of solution to general mobilities by an approximation procedure. More precisely, by establishing delicate stability estimates on the ATW scheme, we show that if ψ is any norm and $\psi_n \to \psi$, with ψ_n a ϕ-regular mobility for every n, then the corresponding distributional level-set solutions u^{ψ_n}, with the given initial datum u^0, admit a unique limit u^ψ (independent of the choice of the approximating ψ_n), which we may therefore regard as the unique solution to the level-set flow with mobility ψ and initial datum u^0. As a byproduct of this analysis, we also settle the problem of the uniqueness (up to fattening) of flat flows for general mobilities. Once again, our results hold in all dimensions, for arbitrary (possibly unbounded) initial sets and general, possibly crystalline, anisotropies ϕ.

By completely different methods, in [Giga and Požár 2016], and more recently in [Giga and Požár 2018], the authors succeed in extending the viscosity approach of [Giga and Giga 2001] to the case $N = 3$ and to the general case $N \geq 3$, respectively. In fact, as in [Giga and Giga 2001], they are able to deal with very general equations of the form

$$V = f(\nu^E, -\kappa_\phi^E),$$

with f continuous and nondecreasing with respect to the second variable, but without spatial dependence, establishing existence and uniqueness for the corresponding level-set formulation. Important achievements in their work are the definition of a crystalline curvature for sets with appropriate regularity, a comparison result for such a curvature, and an approximation result showing that any compact set is arbitrarily close to sets with well-defined curvature. They can also deduce stability results, [Giga and Požár 2016, Theorem 8.9; 2018, Theorem 1.5], which ensure in particular that the viscosity solution of the nonsmooth problem can be built as the limit of a sequence of classical viscosity solutions of the problem with smooth regularized anisotropies. As our approach in the current paper is based on the same idea, a by-product is that when both are defined, their evolutions and ours coincide (Remarks 3.8 and 3.9). On the other hand, their method currently works only for purely crystalline anisotropies ϕ, bounded initial sets, and constant forcing terms.

As said, we propose here an approach different from our previous work [Chambolle et al. 2017a]: Following [Giga and Požár 2016; 2018], we derive existence, uniqueness and some properties of anisotropic and crystalline flows directly from the corresponding properties of smooth (i.e., with smooth anisotropies) flows, appropriately defined as viscosity solutions of a geometric PDE. This leads to a more direct and easier proof of the well-posedness of (1-1) for general mobilities and anisotropies, relying on purely viscosity methods. On the other hand, our new estimates are too weak to provide information about the uniqueness of flat flows, shown in [Chambolle et al. 2017a].

Let us describe the new approach in more detail. The starting point is the observation that when the anisotropy is smooth, the distributional formulation of [Chambolle et al. 2017a; 2017b] is equivalent to the classical viscosity formulation; see Section 2B. Next, in Section 2C we show that if $\phi_n \to \phi$, with ϕ_n smooth, and if $\psi_n \to \psi$, with ψ_n ϕ_n-regular "uniformly" with respect to n (see the statement of Theorem 2.8 below for the precise meaning), then the corresponding viscosity (and thus distributional) level-set solutions u_n converge locally uniformly to the unique distributional level-set flow with anisotropy ϕ and (ϕ-regular) mobility ψ. This leads to a new proof of the existence of distributional level-set solutions for ϕ-regular mobilities, without using the ATW scheme as in [Chambolle et al. 2017a].

In Sections 3A and 3B we establish the crucial stability estimates of the flow with respect to changing ϕ-regular mobilities. This is achieved once again by exploiting the viscosity formulation in order to prove first the estimates in the case of smooth anisotropies and to conclude by approximation.

Finally, in Section 3C we prove the main existence and uniqueness result for the level-set formulation of (1-1), in the case of general anisotropies and mobilities. In this last step we proceed essentially as in [Chambolle et al. 2017a]: we approximate any mobility ψ by a sequence ϕ-regular mobilities ψ_n and show, by means of the stability estimates of the previous sections, that the corresponding solutions admit a unique limit.

2. Distributional mean curvature flows

Given a norm η on $\mathbb{R}^N$ (a convex, even, one-homogeneous real-valued function with $\eta(\nu) > 0$ if $\nu \neq 0$), we define a polar norm η° by $\eta^\circ(\xi) := \sup_{\eta(\nu)\le 1} \nu \cdot \xi$ and an associated anisotropic perimeter P_η as

$$P_\eta(E) := \sup\left\{ \int_E \operatorname{div} \zeta \, dx : \zeta \in C^1_c(\mathbb{R}^N; \mathbb{R}^N),\ \eta^\circ(\zeta) \le 1 \right\}.$$

As is well known, $(\eta^\circ)^\circ = \eta$ so that when the set E is smooth enough one has

$$P_\eta(E) = \int_{\partial E} \eta(\nu^E) \, d\mathcal{H}^{N-1},$$

which is the perimeter of E weighted by the surface tension $\eta(\nu)$.

We will make repeated use of the identities

$$\begin{aligned} \partial\eta(\nu) &= \{\xi : \eta^\circ(\xi) \le 1 \text{ and } \xi \cdot \nu \ge \eta(\nu)\} \\ &= \{\xi : \eta^\circ(\xi) = 1 \text{ and } \xi \cdot \nu = \eta(\nu)\} \end{aligned} \tag{2-1}$$

(and the symmetric statement for η°) for $\nu \neq 0$, where $\partial\eta(\nu)$ denotes the subdifferential of η at ν. Moreover, $\partial\eta(0) = \{\xi : \eta^\circ(\xi) \le 1\}$, while $\partial\eta^\circ(0) = \{\xi : \eta(\xi) \le 1\}$. For $R > 0$ we define

$$W^\eta(x, R) := \{y : \eta^\circ(y - x) \le R\}.$$

Such a set is called the *Wulff shape* (of radius R and center x) associated with the norm η and represents the unique (up to translations) solution of the anisotropic isoperimetric problem

$$\min\{P_\eta(E) : |E| = |W^\eta(0, R)|\};$$

see for instance [Fonseca and Müller 1991].

We denote by $\operatorname{dist}^\eta(\,\cdot\,, E)$ the distance from E induced by the norm η; that is, for any $x \in \mathbb{R}^N$,

$$\operatorname{dist}^\eta(x, E) := \inf_{y \in E} \eta(x - y) \tag{2-2}$$

if $E \neq \varnothing$ and $\operatorname{dist}^\eta(x, \varnothing) := +\infty$. Moreover, we denote by d^η_E the signed distance from E induced by η, i.e.,

$$d^\eta_E(x) := \operatorname{dist}^\eta(x, E) - \operatorname{dist}^\eta(x, E^c).$$

so that $\operatorname{dist}^\eta(x, E) = d^\eta_E(x)^+$ and $\operatorname{dist}^\eta(x, E^c) = d^\eta_E(x)^-$, where we adopt the standard notation $t^+ := t \vee 0$ and $t^- := (-t)^+$. Note that by (2-1) we have $\eta(\nabla d^{\eta^\circ}_E) = \eta^\circ(\nabla d^\eta_E) = 1$ a.e. in $\mathbb{R}^N \setminus \partial E$.

Finally we recall that a sequence of closed sets E_n in $\mathbb{R}^m$ converges to a closed set E in the *Kuratowski sense* if the following conditions are satisfied:

(i) if $x_n \in E_n$, any limit point of $\{x_n\}$ belongs to E,

(ii) any $x \in E$ is the limit of a sequence $\{x_n\}$, with $x_n \in E_n$,

and we write

$$E_n \xrightarrow{\mathcal{K}} E.$$

Since $E_n \xrightarrow{\mathcal{K}} E$ if and only if (for any norm η) $\mathrm{dist}^\eta(\,\cdot\,, E_n) \to \mathrm{dist}^\eta(\,\cdot\,, E)$ locally uniformly in $\mathbb{R}^m$, by the Ascoli–Arzelà theorem any sequence of closed sets admits a converging subsequence in the Kuratowski sense.

2A. *The weak formulation of the crystalline flow.* In this section we recall the weak formulation of the crystalline mean curvature flow introduced in [Chambolle et al. 2017a; 2017b].

In what follows, we will consider forcing terms $g : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ satisfying the following two hypotheses:

(H1) $g \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.

(H2) There exists $L > 0$ such that $g(\,\cdot\,, t)$ is L-Lipschitz continuous with respect to the metric ψ° for a.e. $t > 0$. Here ψ is the norm representing the mobility in (1-1).

Remark 2.1. Assumption (H1) can be in fact weakened and replaced by

(H1′) for every $T > 0$, we have $g \in L^\infty(\mathbb{R}^N \times (0, T))$.

Indeed under the weaker assumption (H1′), all the arguments and the estimates presented throughout the paper continue to work in any time interval $(0, T)$, with some of the constants involved possibly depending on T. In the same way, if one restricts our study to the evolution of sets with compact boundary, then one could assume that g is only locally bounded in space. We assume (H1) instead of (H1′) only to simplify the presentation.

Let ϕ, ψ be two (possibly crystalline) norms representing the anisotropy and the mobility in (1-1), respectively. We recall the following distributional formulation of (1-1).

Definition 2.2 [Chambolle et al. 2017a]. Let $E^0 \subset \mathbb{R}^N$ be a closed set. Let E be a closed set in $\mathbb{R}^N \times [0, +\infty)$ and for each $t \geq 0$ define $E(t) := \{x \in \mathbb{R}^N : (x, t) \in E\}$. We say that E is a *superflow* of (1-1) with initial datum E^0 if:

(a) (initial condition) $E(0) \subseteq E^0$.

(b) (left continuity) $E(s) \xrightarrow{\mathcal{K}} E(t)$ as $s \nearrow t$ for all $t > 0$.

(c) (extinction time) If $E(t) = \varnothing$ for $t \geq 0$, then $E(s) = \varnothing$ for all $s > t$.

(d) (differential inequality) Set $T^* := \inf\{t > 0 : E(s) = \varnothing \text{ for } s \geq t\}$, and

$$d(x, t) := \mathrm{dist}^{\psi^\circ}(x, E(t)) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T^*) \setminus E.$$

Then there exists $M > 0$ such that the inequality

$$\partial_t d \geq \operatorname{div} z + g - Md \tag{2-3}$$

holds in the distributional sense in $\mathbb{R}^N \times (0, T^*) \setminus E$ for a suitable $z \in L^\infty(\mathbb{R}^N \times (0, T^*))$ such that $z \in \partial\phi(\nabla d)$ a.e., $\operatorname{div} z$ is a Radon measure in $\mathbb{R}^N \times (0, T^*) \setminus E$, and

$$(\operatorname{div} z)^+ \in L^\infty(\{(x, t) \in \mathbb{R}^N \times (0, T^*) : d(x, t) \geq \delta\}) \quad \text{for every } \delta \in (0, 1).$$

We say that A, an open set in $\mathbb{R}^N \times [0, +\infty)$, is a subflow of (1-1) with initial datum E^0 if A^c is a superflow of (1-1) with g replaced by $-g$ and with initial datum $(\mathring{E}^0)^c$.

Finally, we say that E, a closed set in $\mathbb{R}^N \times [0, +\infty)$, is a solution of (1-1) with initial datum E^0 if it is a superflow and if $\mathring{E}$ is a subflow, both with initial datum E^0.

It is shown in [Chambolle et al. 2017a] (see also [Chambolle et al. 2017b] for a simpler equation), using quite standard parabolic comparison arguments, that such evolutions satisfy a comparison principle:

Theorem 2.3 [Chambolle et al. 2017a, Theorem 2.7]. *Let E be a superflow with initial datum E^0 and F be a subflow with initial datum F^0 in the sense of Definition 2.2. Assume that* $\operatorname{dist}^{\psi^\circ}(E^0, \mathbb{R}^N \setminus F^0) =: \Delta > 0$. *Then,*

$$\operatorname{dist}^{\psi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta e^{-Mt} \quad \text{for all } t \geq 0,$$

where $M > 0$ is as in (2-3) for both E and F.

We now recall the corresponding notion of sub- and supersolution to the level-set flow associated with (1-1); see again [Chambolle et al. 2017a].

Definition 2.4 (level-set subsolutions and supersolutions). Let u^0 be a uniformly continuous function on $\mathbb{R}^N$. We will say that a lower semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *level-set supersolution* corresponding to (1-1), with initial datum u^0, if $u(\,\cdot\,, 0) \geq u^0$ and if for a.e. $\lambda \in \mathbb{R}$, the closed sublevel set $\{(x, t) : u(x, t) \leq \lambda\}$ is a superflow of (1-1) in the sense of Definition 2.2, with initial datum $\{u_0 \leq \lambda\}$.

We will say that an upper-semicontinuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *level-set subsolution* corresponding to (1-1), with initial datum u^0, if $-u$ is a superlevel-set flow in the previous sense, with initial datum $-u_0$ and with g replaced by $-g$.

Finally, we will say that a continuous function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *solution* to the level-set flow corresponding to (1-1) if it is both a level-set subsolution and supersolution.

As shown in [Chambolle et al. 2017a], Theorem 2.3 easily yields that almost all closed sublevels of a solution of the level-set flows are solutions of (1-1) in the sense of Definition 2.2. Moreover, the following comparison principle between level-set subsolutions and supersolutions holds true.

Theorem 2.5 [Chambolle et al. 2017a, Theorem 2.8]. *Let u^0, v^0 be uniformly continuous functions on $\mathbb{R}^N$ and let u, v be respectively a level-set subsolution with initial datum u^0 and a level-set supersolution with initial datum v^0, in the sense of Definition 2.4. If $u^0 \leq v^0$, then $u \leq v$.*

For smooth anisotropies, solutions to the level-set flow and (minus the characteristic function of) solutions of the geometric flow in the sense of Definition 2.2 are in fact viscosity solutions of the (degenerate) parabolic equation (2-4) below. This classical fact will be shown and exploited to some extent to nonsmooth anisotropies in the next two sections.

2B. *Viscosity solutions.* We show here that in the smooth cases, the notion of solution in Definition 2.2 coincides with the definition of standard viscosity solutions for geometric motions, as for instance in [Barles and Souganidis 1998]. This property will be helpful to establish estimates using standard approaches for viscosity solutions.

Lemma 2.6. *Assume that ϕ, ψ, $\psi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, and that g is continuous. Let E be a superflow in the sense of Definition 2.2. Then, $-\chi_E$ is a viscosity supersolution of*

$$u_t = \psi(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g) \tag{2-4}$$

in $\mathbb{R}^N \times (0, T^*]$, where T^* is the possible extinction time of E.

Conversely, a viscosity supersolution $-\chi_E$ of (2-4) defines a superflow in the sense of Definition 2.2.

Proof. A similar statement (in a simpler context) is proved in [Chambolle et al. 2017b, Appendix], while it is proved in [Chambolle et al. 2017a] that a superflow defines a viscosity supersolution. We therefore here focus on the converse: Given an evolving set $E(t)$ such that $-\chi_E$ is a viscosity supersolution of (2-4), we show that $E(t)$ is a superflow in the sense of Definition 2.2, with the constant M in (2-3) equal to the Lipschitz constant L of $g(\cdot\,, t)$ appearing in the assumption (H2).

Step 1: left continuity and extinction time. Let $T^* \in [0, +\infty]$ be the (first) extinction time of E, and assume without loss of generality $T^* > 0$. Let $d(x, t) := \operatorname{dist}^{\psi^\circ}(x, E(t))$. We fix $\delta > 0$ and we set $A = (\mathbb{R}^N \times [0, T^*)) \setminus E$ and $A^\delta = A \cap \{d > \delta\}$. Let (x, t) with $d(x, t) = R > \delta > 0$. Then $W^\psi(x, R - \varepsilon) \cap E(t) = \varnothing$ for any $\varepsilon > 0$ (small). There exists a constant C (depending on ϕ, ψ) such that, letting

$$W(s) = \mathbb{R}^N \setminus W^\psi\Big(x, R - \varepsilon - \Big(\frac{C}{R} + \|g\|_\infty\Big)s\Big),$$

$-\chi_{W(s)}$ is a viscosity subsolution of (2-6) for $s \le R^2/(2(C + R\|g\|_\infty))$ and $\varepsilon \le R/4$. By standard comparison results [Barles et al. 1993], it follows that $E(t + s) \subset W(s)$ for such times s, so that $d(x, t + s) \ge R - \varepsilon - (C/R + \|g\|_\infty)s$. Hence, letting $\varepsilon \to 0$, we find that

$$d(x, t + s) \ge d(x, t) - \Big(\frac{C}{\delta} + \|g\|_\infty\Big)s \quad \text{if } (x, t) \in A^\delta. \tag{2-5}$$

In particular, it follows that $\partial_t d$ is bounded from below in such sets and hence is a measure. By (2-5) and the fact that E is closed we deduce that the left continuity (b) of Definition 2.2 holds for $E(t)$. Moreover, the same argument shows that if $t > T^*$ then $d(x, t) = +\infty$, showing also point (c).

Step 2: the distance function is a viscosity supersolution. We now show that the function $d(x, t)$ is a viscosity supersolution of

$$u_t = \psi(\nabla u)(D^2\phi(\nabla u) : D^2 u + g - Lu). \tag{2-6}$$

In fact, this is essentially classical [Soner 1993]; however the proof in this reference needs to be adapted to deal with the forcing term. An elementary proof is as follows: Let η be a smooth test function and assume $(\bar{x}, \bar{t})$ is a contact point, where $\eta(\bar{x}, \bar{t}) = d(\bar{x}, \bar{t})$ and $\eta \le d$. If the common value of η, d at $(\bar{x}, \bar{t})$ is zero then it is also a contact point of $1 - \chi_E$ and η, so that

$$\partial_t \eta(\bar{x}, \bar{t}) \ge \psi(\nabla\eta(\bar{x}, \bar{t}))\big(D^2\phi(\nabla\eta(\bar{x}, \bar{t})) : D^2\eta(\bar{x}, \bar{t}) + g(\bar{x}, \bar{t}) - L\eta(\bar{x}, \bar{t})\big) \tag{2-7}$$

obviously holds, by definition (recalling (2-4) and that $\eta(\bar{x}, \bar{t}) = 0$). Hence we consider the case where $R = d(\bar{x}, \bar{t}) > 0$. Let $\bar{y} \in \partial E(\bar{t})$ such that $R = \psi^\circ(\bar{x} - \bar{y})$. We let

$$\eta'(y, t) := \eta(y + \bar{x} - \bar{y}, t) - R \le d(y + \bar{x} - \bar{y}, t) - R \le d(y, t)$$

since d is 1-Lipschitz in the ψ° norm. In particular, in a neighborhood of $(\bar{y}, \bar{t})$ we have $\eta'(y,t) \le 1 - \chi_{E(t)}(y)$. On the other hand, $\eta'(\bar{y}, \bar{t}) = 0 = d(\bar{y}, \bar{t}) = 1 - \chi_{E(\bar{t})}(\bar{y})$. Hence, by (2-4)

$$\partial_t \eta(\bar{x}, \bar{t}) = \partial_t \eta'(\bar{y}, \bar{t}) \ge \psi(\nabla \eta'(\bar{y}, \bar{t}))\big(D^2\phi(\nabla \eta'(\bar{y}, \bar{t})) : D^2\eta'(\bar{y}, \bar{t}) + g(\bar{y}, \bar{t})\big).$$

Since $g(\bar{y}, \bar{t}) \ge g(\bar{x}, \bar{t}) - L\eta(\bar{x}, \bar{t})$, (2-7) follows.

Step 3: differential inequality. A classical remark is that d^2, as an infimum of the uniformly semiconcave functions $\psi^\circ(\cdot - y)^2$, $y \in E(t)$, is semiconcave; hence in A^δ one has $D^2 d \le C/\delta I$ in the sense of measures for some constant C depending only on ψ°. In particular, $\operatorname{div} \nabla\phi(\nabla d) = D^2\phi(\nabla d) : D^2 d \le C/\delta$ in A^δ in the sense of measures.

We proceed as in [Chambolle et al. 2017b]: For $n \ge 1$, let $d_n(x,t) := \min_s (d(x, t-s) + ns^2)$, which is semiconcave and converges to d as $n \to \infty$. Moreover, one can easily check that $d_n(\cdot, t) \to d(\cdot, t)$ locally uniformly if t is a continuity point of d. Let $B \subset A^\delta$ be an open ball (where in particular d is bounded from above and it is bounded from below by δ) and observe that d_n is still a supersolution of (2-6), provided $g(x,t)$ is replaced with $g(x,t) - \omega_n$ for some $\omega_n \to 0$ as $n \to +\infty$. Since d_n, which is semiconcave, has a second-order jet a.e. in B, (2-6) holds for d_n a.e. in B. Reasoning as in [Chambolle et al. 2017b, Appendix], we deduce that

$$\partial_t d_n \ge \psi(\nabla d_n)(\operatorname{div} z_n + g - \omega_n - L d_n) \tag{2-8}$$

in the distributional sense (or as measures) in B, where $z_n := \nabla\phi(\nabla d_n)$. It remains to send $n \to \infty$: Clearly, $\partial_t d_n \to \partial_t d$ in the distributional sense. Consider (x,t) a point where $\nabla d(x,t)$ and $\nabla d_n(x,t)$ exist for all n. First, if $d(x, t-s) + ns^2$ attains the minimum at s_n, one has for any $p \in \partial^+ d(x, t - s_n)$ (the spatial supergradient of the semiconcave function $d(\cdot, t - s_n)$) that

$$\begin{aligned} d_n(x+h, t) &\le d(x+h, t-s_n) + ns_n^2 \\ &\le d(x, t-s_n) + p \cdot h + \frac{C}{\delta}|h|^2 + ns_n^2 = d_n(x,t) + p\cdot h + \frac{C}{\delta}|h|^2, \end{aligned}$$

showing that $p \in \partial^+ d_n(x,t) = \{\nabla d_n(x,t)\}$. We deduce that $d(\cdot, t-s_n)$ is differentiable at x, with gradient $\nabla d_n(x,t)$, and in particular that $\psi(\nabla d_n(x,t)) = 1$.

Assume now that in addition d is continuous at t. Then $d_n(\cdot, t) \to d(\cdot, t)$ uniformly in $B \cap (\mathbb{R}^N \times \{t\})$, and using the (uniform) semiconcavity of these functions, one also deduces that $\nabla d_n(x,t) \to \nabla d(x,t)$ a.e.; hence, $z_n(x,t) = \nabla\phi(\nabla d_n(x,t))$ converges to $z(x,t) = \nabla\phi(\nabla d(x,t))$ a.e. Hence we may send n to ∞ in (2-8) to find that

$$\partial_t d \ge \operatorname{div} z + g - Ld$$

in the distributional sense in B, with $z = \nabla\phi(\nabla d)$ a.e.

This shows the lemma. □

2C. ***The level-set formulation.*** Let u^0 be a bounded, uniformly continuous function on $\mathbb{R}^N$. Then, it is well known [Chen et al. 1991] that if $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$ and ψ, g are continuous, then there exists a unique viscosity solution u of (2-4) with initial datum u^0. Moreover, for all $\lambda \in \mathbb{R}$, we know $-\chi_{\{u<\lambda\}}$

is a viscosity supersolution and $-\chi_{\{u\leq\lambda\}}$ is a viscosity subsolution of the same equation. If in addition $\psi, \psi^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$, it follows from Lemma 2.6 that $E_\lambda(t) := \{u(\cdot, t) \leq \lambda\}$ is a superflow in the sense of Definition 2.2, while $A_\lambda(t) := \{u(\cdot, t) < \lambda\}$ is a subflow.[1]

In what follows we will say that a given norm η is *smooth and elliptic* if both η and η° belong to $C^2(\mathbb{R}^N \setminus \{0\})$.

We now consider sequences ϕ_n, ψ_n of smooth and elliptic anisotropies/mobilities converging to ϕ, ψ. We also consider $g_n(x,t)$ a smooth forcing term, which converges to $g(x,t)$ weakly-$*$ in $L^\infty(\mathbb{R}^N \times [0,+\infty))$. We assume also that g_n is uniformly spatially Lipschitz continuous and we denote by L, M the (uniform) Lipschitz constants of g_n with respect to ψ_n° and ϕ_n°, respectively. Given u_n, the corresponding unique viscosity solution of (2-4) (with ψ_n, ϕ_n, g_n instead of ψ, ϕ, g) with initial datum u^0, we want to study the possible limits of u_n. If the limiting anisotropies and forcing term are still smooth enough, it is well known that the limiting u is the unique viscosity solution of the corresponding limit problem. If not, we will show that the limit is still unique. We recall, see [Chambolle et al. 2017a], the following:

Definition 2.7. We will say that a norm ψ is ϕ-regular if the associated Wulff shape $W^\psi(0,1)$ satisfies a uniform interior ϕ-Wulff shape condition, that is, if there exists $\varepsilon_0 > 0$ with the following property: for every $x \in \partial W^\psi(0,1)$ there exists $y \in W^\psi(0,1)$ such that $W^\phi(y,\varepsilon_0) \subseteq W^\psi(0,1)$ and $x \in \partial W^\phi(y,\varepsilon_0)$.

Notice that it is equivalent to saying that $W^\psi(0,1)$ is the sum of a convex set and $W^\phi(0,\varepsilon_0)$, or equivalently that $\psi(\nu) = \psi_0(\nu) + \varepsilon_0\phi(\nu)$ for some convex function ψ_0.

We now show the following result.

Theorem 2.8. *Let $(\psi_n)_n$, $(\phi_n)_n$ and $(g_n)_n$ be as above, and, in addition, assume that the mobilities $(\psi_n)_n$ are uniformly ϕ_n-regular, meaning that $\varepsilon_0 > 0$ in Definition 2.7 does not depend on n. Let u_n be the level-set solutions to (1-1) in the sense of Definition 2.4, with initial datum u^0, anisotropy $(\psi_n)_n$, mobility $(\phi_n)_n$ and forcing term $(g_n)_n$. Then, the u_n converge locally uniformly to the unique level-set solution u to (1-1) in the sense of Definition 2.4, with initial datum u^0, anisotropy ψ, mobility ϕ and forcing term g.*

Proof. A first observation is that the functions u_n remain uniformly continuous in space and time on $\mathbb{R}^N \times [0,T]$ for all $T > 0$, with a modulus depending only on the modulus of continuity ω of u^0 and the Lipschitz constant M. Indeed, by Proposition 3.4 below it follows that for any $\lambda < \lambda'$

$$\operatorname{dist}^{\phi_n^\circ}\big(\{u_n(\cdot,t) \leq \lambda\}, \{u_n(\cdot,t) \geq \lambda'\}\big) \geq \Delta e^{-\beta M t},$$

where $\Delta := \omega^{-1}(\lambda' - \lambda) \geq \operatorname{dist}^{\phi^\circ}(\{u^0 \leq \lambda\}, \{u^0 \geq \lambda'\}) > 0$, and $\beta > 0$ depends (for large n) only on ϕ and ψ; see (3-16). Therefore, $u_n(\cdot,t)$ is uniformly continuous with modulus of continuity with respect to the norm ϕ_n° given by $\omega(e^{\beta M t}\,\cdot)$. As for the equicontinuity in time, we set $\omega_T(s) := \omega(e^{\beta M T}s)$ and we start by observing that for any $x \in \mathbb{R}^N$, $\varepsilon > 0$, $t \in (0,T]$, and $n \in N$ we have

$$W^{\phi_n}(x, \omega_T^{-1}(\varepsilon)) \subseteq \{y : u_n(y,t) > u_n(x,t) - \varepsilon\}.$$

[1] In the case of "fattening", also $\overline{\{u < \lambda\}}$ is a superflow, and the interior of $\{u \leq \lambda\}$ a subflow.

Therefore, by standard comparison results we have $u_n(x,t') > u_n(x,t) - \varepsilon$ provided that $0 < t' - t < \tau$, where τ is the extinction time for $W^{\phi_n}(x, \omega_T^{-1}(\varepsilon))$ under the evolution (1-1). Analogously, one shows that $u_n(x,t') < u_n(x,t) + \varepsilon$ if $0 < t' - t < \tau$. Since τ is bounded away from zero by a quantity independent of n (depending only on ε, $\sup_n \|g_n\|_\infty$ and, for n large, on ϕ and ψ); see for instance [Chambolle et al. 2017a, Remark 4.6]. This establishes the equicontinuity in time.

Hence, up to a subsequence (not relabeled), we may assume that u_n converges locally uniformly to some u. In view of Theorem 2.5, it is enough to show that u is a solution in the sense of Definition 2.4, that is, that for a.e. $\lambda \in \mathbb{R}$ the set $E_\lambda := \{u \le \lambda\}$ is a superflow in the sense of Definition 2.2 and $A_\lambda := \{u < \lambda\}$ a subflow.

We prove the assertion for E_λ. We first notice that since $u_n \to u$ locally uniformly, the Kuratowski limit superior of the sets $E_n := \{u_n \le \lambda\}$ as $n \to \infty$ is contained in E_λ.

By Lemma 2.6, the sets E_n are superflows in the sense of Definition 2.2. We consider $d_n(x,t) := \operatorname{dist}^{\psi_n^\circ}(x, E_n(t))$ and $d(x,t) := \operatorname{dist}^{\psi^\circ}(x, E_\lambda(t))$, the corresponding distance functions, which are finite up to some time $T_n^*, T^* \in (0, +\infty]$ respectively, where T^* is defined according with Definition 2.2. Notice that T^* is increasing with respect to λ, and that if λ is a continuity point, then we have $T_n^* \to T^*$, as $n \to \infty$.

Recalling (2-5), one can deduce that for $s > 0$ small,

$$\frac{d_n(t+s) - d_n(t)}{s} \ge -\frac{C}{d_n(t)} - \|g\|_\infty,$$

where the constant C does not depend on n, as it is essentially the maximal speed, without forcing, of the Wulff shape $W^{\psi_n} := W^{\psi_n}(0,1)$, which is bounded by $(\max_\xi \psi_n) \times (\max_{\partial W^{\psi_n}} \kappa_{\phi_n})$. The curvature κ_{ϕ_n} of ∂W^{ψ_n} is in $[0, (N-1)/\varepsilon_0]$, thanks to the assumption that $\psi_n' := \psi_n - \varepsilon_0 \phi_n$ is convex, which yields that $W^{\psi_n} = W^{\psi_n'} + \varepsilon_0 W^{\phi_n}$. We deduce $\partial_t d_n \ge -C/d_n - \|g\|_\infty$, which yields that there is an increasing function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Theta(d_n(t+s)) \ge \Theta(d_n(t)) - \|g\|_\infty s \quad \text{for all } t, s > 0. \tag{2-9}$$

Actually Θ is explicitly given by

$$\Theta(d) = d - \frac{C}{\|g\|_\infty} \log\left(1 + \frac{\|g\|_\infty}{C} d\right)$$

for $d \ge 0$. Notice that, for small $d \ge 0$, we have $\Theta(d) \approx \|g\|_\infty d^2/(2C)$, while $\Theta(d) \approx d$ for large d.

It follows from (2-9) (see for instance details in the proof of [Chambolle et al. 2017b, Proposition 4.4], which is an adaptation of Helly's selection theorem) that one can find an at most countable set $\mathcal{N} \subset (0, T^*)$ such that for all $t \notin \mathcal{N}$, $d_n(\cdot, t) \to d(\cdot, t)$ locally uniformly. If $B \Subset (\mathbb{R}^N \times (0, T^*)) \setminus E_\lambda$, one has $B \cap E_n = \varnothing$ for n large enough and

$$\partial_t d_n \ge \operatorname{div} z_n + g_n - L d_n$$

in the distributional sense in B, thanks to (2-3) and Lemma 2.6. Here, $z_n = \nabla\phi_n(\nabla d_n)$. Notice that the z_n are (for n large) well-defined and bounded in $L^\infty(\mathbb{R}^N \times (0,T))$ for any $T < T^*$. In the limit, we find that (2-3) holds for d, with z the weak-$*$ (local in time) limit of $(z_n)_n$ (or rather, in fact, a subsequence). It remains to show that $z \in \partial\phi(\nabla d)$ a.e. in B. An important observation is that, using again the ϕ_n-regularity of ψ_n, one can show that $\operatorname{div} \nabla\phi_n(\nabla d_n) \le (N-1)/(\varepsilon_0 d_n)$; hence it is bounded in $\{d_n > \delta\}$. In particular, in the limit, $(\operatorname{div} z)^+ \chi_{\{d>\delta\}} \in L^\infty(\mathbb{R}^N \times (0, T^*))$.

To show $z \in \partial\phi(\nabla d)$ a.e. in B, we establish that $z \cdot \nabla d \ge \phi(\nabla d)$ a.e. in B. The proof here is as in [Chambolle et al. 2017b]. There exists δ such that for all n large enough, $d_n \ge \delta$ in B; hence $\operatorname{div} z_n \le (N-1)/(\varepsilon_0 \delta)$. Let $\eta \in C_c^\infty(B; \mathbb{R}_+)$; then

$$\int_B \phi(\nabla d)\eta \, dx \, dt \le \liminf_n \int_B \phi_n(\nabla d_n)\eta \, dx \, dt = \liminf_n \int_B (z_n \cdot \nabla d_n)\eta \, dx \, dt.$$

On the other hand,

$$\int_B (z_n \cdot \nabla d_n)\eta \, dx \, dt = \int_B (z_n \cdot \nabla d)\eta \, dx \, dt + \int_B (z_n \cdot \nabla(d_n - d))\eta \, dx \, dt, \tag{2-10}$$

and $\lim_n \int_B (z_n \cdot \nabla d)\eta \, dx \, dt = \int_B (z \cdot \nabla d)\eta \, dx \, dt$ since $z_n \overset{*}{\rightharpoonup} z$.

It remains to prove that the second addend in the right hand side of (2-10) tends to zero as $n \to +\infty$. Set

$$m_n(t) = \min_{x:(x,t)\in \bar{B}} (d_n(x,t) - d(x,t)), \qquad M_n(t) = \max_{x:(x,t)\in \bar{B}} (d_n(x,t) - d(x,t)).$$

Then $M_n(t) - m_n(t) \to 0$ for all $t \notin \mathcal{N}$. One has

$$\begin{aligned}\int_B (z_n \cdot \nabla(d_n - d))\eta \, dx \, dt &= \int_B (z_n \cdot \nabla(d_n - d - m_n(t)))\eta \, dx \, dt \\ &= -\int_B (d_n - d - m_n)\eta \operatorname{div} z_n \, dx \, dt - \int_B (d_n - d - m_n) z_n \cdot \nabla\eta \, dx \, dt.\end{aligned}$$

The last integral goes to zero as $n \to \infty$. Since $(d_n - d - m_n(t))\eta \ge 0$ we have

$$-\int_B (d_n - d - m_n)\eta \operatorname{div} z_n \, dx \, dt \ge -\frac{N-1}{\varepsilon_0 \delta} \int_B (d_n - d - m_n)\eta \, dx \, dt \xrightarrow{n\to\infty} 0.$$

Using instead $d_n - d - M_n$, we show the reverse inequality, and we deduce

$$\int_B \phi(\nabla d)\eta \, dx \, dt \le \int_B (z \cdot \nabla d)\eta \, dx \, dt,$$

which concludes the proof. □

3. Existence by approximation

3A. ***A useful estimate*: *comparison with different forcing terms.*** We prove in this section and the following a series of comparison results, which will then be combined together to deduce a global comparison result for flows with possibly different mobilities. In this section, we shall assume that the surface tensions ϕ, ψ are smooth and elliptic, so that we can work in the classical viscosity setting. In the limit, our main estimate will also hold for crystalline flows in the sense of Definition 2.2.

We start by recalling standard comparison results for flows with constant velocities; however, we pay special attention to the particular metrics in which our velocities are expressed. We first consider the equation

$$u_t = \psi(\nabla u) g(x,t). \tag{3-1}$$

The following is a slight variant of the well-known result [Barles 2013, Theorem 8.1]:

Lemma 3.1. *Consider $u^0 : \mathbb{R}^N \to \mathbb{R}$, bounded and Λ-Lipschitz continuous with respect to a norm η, smooth and elliptic, and let $\beta > 0$ be such that*

$$\psi \leq \beta \eta^\circ. \tag{3-2}$$

Assume g is bounded, continuous and M-Lipschitz in space in the norm η. Let $u(x,t)$ be a viscosity solution of (3-1) with initial datum u_0. Then for all $t \geq 0$, the function $u(\cdot,t)$ is $\Lambda e^{\beta M t}$-Lipschitz continuous in the norm η.

Proof. We start by observing that by classical results the solution u is uniformly continuous locally in time; see for instance [Giga et al. 1991]. The rest of the proof is an adaptation of the argument in [Barles 2013, proof of Theorem 8.1]. Let $\delta > 0$ be given, and let C be a smooth function such that

$$C' - \beta M C \geq \beta M \delta > 0, \tag{3-3}$$

with $C(0) = \Lambda$. Set

$$\sigma := \sup_{\substack{x,y \in \mathbb{R}^N \\ t \in [0,T]}} u(x,t) - u(y,t) - C(t)\eta(x-y).$$

We claim that $\sigma = 0$. Using this claim, we have

$$u(x,t) - u(y,t) \leq (\Lambda e^{\beta M t} + \delta(e^{\beta M t} - 1))\eta(x-y)$$

for all $x, y, t \leq T$, and sending $\delta \to 0$ we conclude the proof of the lemma.

We are left to prove the claim that $\sigma = 0$. Arguing by contradiction, assume that $\sigma > 0$. Consider a maximum point $(\bar{x}, \bar{y}, \bar{t}, \bar{s})$ in $\mathbb{R}^{2N} \times [0,T]^2$ for the function

$$\varphi(x,y,s,t) = u(x,t) - u(y,s) - C(t)\eta(x-y) - \frac{|t-s|^2}{2a} - b\frac{|x|^2 + |y|^2}{2},$$

where $a, b > 0$ are small parameters (notice that $\varphi(x,y,0,0) \leq 0$). For b small enough, $\varphi(\bar{x}, \bar{y}, \bar{t}, \bar{s}) \geq \sigma/2 > 0$, and then by standard arguments (using in particular that $|\bar{x}|, |\bar{y}| \leq c/\sqrt{b}$, and that for fixed b, both $\bar{t}$ and $\bar{s}$ converge, up to a subsequence, to the same positive value as $a \to 0$, see for instance [Barles 2013, Lemma 5.2]) we may assume $0 < \bar{t}, \bar{s} \leq T$, so that

$$C'(\bar{t})\eta(\bar{x}-\bar{y}) + \frac{\bar{t}-\bar{s}}{a} \leq \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) + b\bar{x}) g(\bar{x},\bar{t}),$$

$$\frac{\bar{t}-\bar{s}}{a} \geq \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) - b\bar{y}) g(\bar{y},\bar{s}).$$

Evaluating the difference and recalling (3-3) we obtain

$$\beta M(C(\bar{t}) + \delta)\eta(\bar{x}-\bar{y}) \leq \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) + b\bar{x}) g(\bar{x},\bar{t}) - \psi(C(\bar{t})\nabla\eta(\bar{x}-\bar{y}) - b\bar{y}) g(\bar{y},\bar{s}).$$

For fixed $b > 0$, we can then let $a \to 0$ and denote by $\tilde{t} \in (0, T]$ the common limit (along a subsequence) of $\bar{t}$ and $\bar{s}$ as $a \to 0$, and by $\tilde{x}$ and $\tilde{y}$ the limits (along a subsequence) of $\bar{x}$ and $\bar{y}$, respectively. Thus, using (3-2), we obtain

$$\begin{aligned} &M(C(\tilde{t})+\delta)\eta(\tilde{x}-\tilde{y}) \\ &\leq \frac{1}{\beta}\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})g(\tilde{x},\tilde{t})-\frac{1}{\beta}\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})g(\tilde{y},\tilde{t}) \\ &\leq \frac{1}{\beta}\big(\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})-\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})\big)g(\tilde{y},\tilde{t})+\eta^\circ(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+bx)M\eta(\tilde{x}-\tilde{y}). \end{aligned}$$

We deduce

$$C(\tilde{t})+\delta \leq \eta^\circ(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})+\frac{\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})+b\tilde{x})-\psi(C(\tilde{t})\nabla\eta(\tilde{x}-\tilde{y})-b\tilde{y})}{\beta M\eta(\tilde{x}-\tilde{y})}\|g\|_\infty,$$

and sending $b \to 0$ (and observing that $\eta(\tilde{x}-\tilde{y}) \not\to 0$ as $\sigma > 0$ and u is uniformly continuous), we find that if $\hat{t}$ is a limit point of $\tilde{t}$, then $C(\hat{t})+\delta \leq C(\hat{t})$, which gives a contradiction. Hence one must have $\sigma = 0$. □

In the next lemma we show that if $E^0 \subset F^0$ are initial sets and $-\chi_E$, $-\chi_F$ are viscosity solutions of (3-1), starting from $-\chi_{E^0}$ and $-\chi_{F^0}$, respectively, then $\operatorname{dist}^\eta(\partial E(t), \partial F(t)) \geq \operatorname{dist}^\eta(\partial E^0, \partial F^0)e^{-\beta M t}$. A splitting strategy will then extend this result to the solutions of (2-4).

Lemma 3.2. *Let η be a smooth and elliptic norm satisfying (3-2). Let g_1, g_2 be two admissible forcing terms satisfying assumptions (H1), (H2) of Section 2A, and both M-Lipschitz in the η norm. Assume*

$$g_2 - g_1 \leq c < +\infty \quad \text{in } \mathbb{R}^N \times [0, +\infty). \tag{3-4}$$

Let $E^0 \subset F^0$ be two closed sets with $\operatorname{dist}^\eta(E^0, \mathbb{R}^N \setminus F^0) := \Delta > 0$. Assume that $-\chi_{E(t)}$ is a viscosity supersolution of $u_t = \psi(\nabla u)g_1(x,t)$ starting from $-\chi_{E^0}$, and $-\chi_{F(t)}$ is a subsolution of $v_t = \psi(\nabla v)g_2(x,t)$ starting from $-\chi_{F^0}$. Then at any time $t \geq 0$,

$$\operatorname{dist}^\eta(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta e^{-\beta M t} - c\frac{1-e^{-\beta M t}}{M}. \tag{3-5}$$

Proof. With Lemma 3.1 at hand, this is a straightforward application of standard comparison principles. We consider first $u_0(x) := -\Delta \vee (2\Delta \wedge d_E^\eta(x))$ and $v_0(x) := -2\Delta \vee (\Delta \wedge d_F^\eta(x))$, so that $v_0 + \Delta \leq u_0$. These functions are both 1-Lipschitz in the norm η. We then consider the viscosity solutions u of $u_t = \psi(\nabla u)g_1(x,t)$ starting from u_0, and v of $v_t = \psi(\nabla v)g_2(x,t)$, starting from v_0. By standard comparison results, $E(t) \subseteq \{u(t) \leq 0\}$ and $F(t) \supseteq \{v(t) \leq 0\}$ for all $t \geq 0$.

Thanks to Lemma 3.1, $u(\cdot, t)$, $v(\cdot, t)$ are $e^{\beta M t}$-Lipschitz. Let now

$$w(\cdot, t) = v(\cdot, t) + \Delta - c\frac{e^{\beta M t}-1}{M}.$$

Then at $t = 0$, we have $w(\cdot, 0) = v_0 + \Delta \leq u_0$. We show that w is a subsolution of $u_t = \psi(\nabla u)g_1(x,t)$, so that $w \leq u$. Indeed, if φ is a smooth test function and $(\bar{x}, \bar{t})$ is a point of maximum of $w - \varphi$, then it is a point of maximum of $v-(\varphi-\Delta+c\beta(e^{\beta Mt}-1)/M)$ so that, using (3-4) and the fact that v is a subsolution, we get

$$\partial_t\varphi(\bar x,\bar t)+c\beta e^{\beta M\bar t}\le\psi(\nabla\varphi(\bar x,\bar t))g_2(\bar x,\bar t)\le\psi(\nabla\varphi(\bar x,\bar t))g_1(\bar x,\bar t)+c\psi(\nabla\varphi(\bar x,\bar t)).$$

Since $\bar x$ is a contact point of the smooth function $\varphi(\,\cdot\,,\bar t)$ and the $e^{\beta M\bar t}$-Lipschitz function $w(\,\cdot\,,\bar t)$ (in the η norm), we have $\eta^\circ(\nabla\varphi)\le e^{\beta M\bar t}$ at $(\bar x,\bar t)$. By (3-2), $c\psi(\nabla\varphi(\bar x,\bar t))\le c\beta e^{\beta M\bar t}$, whence

$$\partial_t\varphi\le\psi(\nabla\varphi)g_1$$

and this shows that w is a subsolution of this equation, and hence that $w\le u$. Therefore, for all x,t, $v(x,t)\le u(x,t)-\Delta+c(e^{\beta Mt}-1)/M$. Thus, for $t\ge 0$ and $x,\,y\in\mathbb{R}^N$, recalling that v is $e^{\beta M\bar t}$-Lipschitz,

$$v(y,t)\le u(x,t)-e^{\beta Mt}\Bigl(\Delta e^{-\beta Mt}-c\frac{1-e^{-\beta Mt}}{M}-\eta(x-y)\Bigr).$$

It follows that if $\operatorname{dist}^\eta(y,E(t))\le\Delta e^{-\beta Mt}-c(1-e^{-\beta Mt})/M$, then $v(y,t)\le 0$, and hence $y\in F(t)$, which shows the lemma. □

3B. *Comparison for different mobilities.* In this section we provide the crucial stability estimates with respect to varying mobilities, not necessarily smooth and elliptic.

3B1. *A comparison result with a constant forcing term.* In this subsection we shall assume that ϕ, ψ_1, ψ_2 are smooth and elliptic, and that

$$(1-\delta)\psi_2(\xi)\le\psi_1(\xi)\le(1+\delta)\psi_2(\xi)\quad\text{for all }\xi\in\mathbb{R}^N,\tag{3-6}$$

for some (small) $\delta>0$. We first show the following:

Lemma 3.3. *There exists a constant $c_0>0$ depending only on N such that the following holds: Let $\Delta>0$, and let E be a superflow for the equation $V=-\psi_1(\nu)\kappa_\phi$ and F be a subflow for the equation $V=-\psi_2(\nu)(\kappa_\phi-c_0\delta/\Delta)$, with* $\operatorname{dist}^{\phi^\circ}(E(0),\mathbb{R}^N\setminus F(0))=\Delta$. *Then for all t until extinction of E or F^c, we have* $\operatorname{dist}^{\phi^\circ}(E(t),\mathbb{R}^N\setminus F(t))\ge\Delta$.

Proof. We first assume that $\partial E(t)$, $\partial F(t)$ are bounded for all t.

We shall use the fact that $u(x,t)=-\chi_E(x,t)$ is a viscosity supersolution of

$$\partial_t u=\psi_1(\nabla u)\operatorname{div}\nabla\phi(\nabla u),\tag{3-7}$$

while $v(x,t)=-\chi_F(x,t)$ is a viscosity subsolution of (see Lemma 2.6)

$$\partial_t v=\psi_2(\nabla v)\Bigl(\operatorname{div}\nabla\phi(\nabla v)-c_0\frac{\delta}{\Delta}\Bigr).\tag{3-8}$$

A first remark is that since the equations are translationally invariant, we also have

$$u'(x,t)=\inf_{\phi^\circ(z)\le\Delta/4}u(x+z,t)$$

is a supersolution of (3-7), and similarly,

$$v'(x,t) = \sup_{\phi^\circ(z)\le \Delta/4} v(x+z,t)$$

is a subsolution of (3-8). Note that $u' = -\chi_{E'}$ and $v' = -\chi_{F'}$, with the tubes E', F' defined by

$$E'(t) = E(t) + W^\phi\big(0, \tfrac14\Delta\big),$$
$$\mathbb{R}^N \setminus F'(t) = (\mathbb{R}^N \setminus F(t)) + W^\phi\big(0, \tfrac14\Delta\big)$$

until their respective extinction times. We denote by t^* the minimum extinction time of these sets. In particular,

$$\operatorname{dist}^{\phi^\circ}(E'(0), \mathbb{R}^N \setminus F'(0)) = \tfrac12\Delta.$$

Using [Chambolle et al. 2017a, Lemma 2.6], there is a time t_0 such that for $t \le t_0$,

$$\operatorname{dist}^{\phi^\circ}(E'(t), \mathbb{R}^N \setminus F'(t)) \ge \tfrac14\Delta.$$

Let $\varepsilon > 0$, and consider a point $(\bar x, \bar t, \bar y, \bar s)$ (depending on ε) which attains

$$M_\varepsilon = \min_{\substack{x,\,y\in\mathbb{R}^N\\ 0\le s,\,t<t_0}} \frac{1}{\varepsilon}(1+u'(x,t)-v'(y,s)) + \frac{\phi^\circ(x-y)^2}{2} + \frac{(t-s)^2}{2\varepsilon} + \frac{\varepsilon}{t_0-t} + \frac{\varepsilon}{t_0-s}. \tag{3-9}$$

Observe that for every fixed $x \in E'(0)$, $y \notin F'(0)$ and $s = t = 0$, this quantity is less than

$$\frac{\phi^\circ(x-y)^2}{2} + 2\frac{\varepsilon}{t_0}$$

and in particular, $M_\varepsilon \le \Delta^2/8 + 2\varepsilon/t_0$. If ε is small enough, one must have $1 + u'(\bar x, \bar t) - v'(\bar y, \bar s) = 0$, that is, $\bar x \in E'(\bar t)$ and $\bar y \notin F'(\bar s)$; hence

$$\phi^\circ(\bar x - \bar y) = \operatorname{dist}^{\phi^\circ}(E'(\bar t), \mathbb{R}^N \setminus F'(\bar s)).$$

If both $\bar t, \bar s > 0$, then from [Crandall et al. 1992, Theorem 3.2] (with $\varepsilon = 1$ in their notation), there exist $(N+1)\times(N+1)$ symmetric matrices

$$\widetilde X = \begin{pmatrix} X & \zeta \\ \zeta^T & \zeta_0 \end{pmatrix}, \quad \widetilde Y = \begin{pmatrix} Y & \eta \\ \eta^T & \eta_0 \end{pmatrix} \tag{3-10}$$

such that

$$\begin{aligned} &\left(\frac{\bar s - \bar t}{\varepsilon} - \frac{\varepsilon}{(t_0-\bar t)^2}, \nabla\phi^\circ(\bar y - \bar x), \widetilde X\right) \in \overline{\mathcal{P}^{2,-}}\frac{u'}{\varepsilon}(\bar x, \bar t),\\ &\left(\frac{\bar s - \bar t}{\varepsilon} + \frac{\varepsilon}{(t_0-\bar s)^2}, \nabla\phi^\circ(\bar y - \bar x), \widetilde Y\right) \in \overline{\mathcal{P}^{2,+}}\frac{v'}{\varepsilon}(\bar y, \bar s), \end{aligned} \tag{3-11}$$

and such that

$$-(1+\|A\|)\mathrm{Id} \le \begin{pmatrix} -\widetilde X & 0 \\ 0 & \widetilde Y \end{pmatrix} \le A + A^2, \tag{3-12}$$

where in (3-11) we used the standard notation for the (closed) parabolic second-order sub/superjets, see [Crandall et al. 1992], and

$$A=\begin{pmatrix} D^2\phi^\circ(\bar x-\bar y) & 0 & -D^2\phi^\circ(\bar x-\bar y) & 0\\ 0 & 1/\varepsilon-2\varepsilon/(t_0-\bar t)^3 & 0 & -1/\varepsilon\\ -D^2\phi^\circ(\bar x-\bar y) & 0 & D^2\phi^\circ(\bar x-\bar y) & 0\\ 0 & -1/\varepsilon & 0 & 1/\varepsilon-2\varepsilon/(t_0-\bar s)^3\end{pmatrix}.$$

In particular, for all $\xi\in\mathbb{R}^N$, letting $\tilde\xi=(\xi,0,\xi,0)\in\mathbb{R}^{2N+2}$, from (3-12) and (3-10) we get

$$-\xi^T X\xi+\xi^T Y\xi\le\tilde\xi^T A\tilde\xi+\tilde\xi^T A^2\tilde\xi=0,$$

which gives the inequality

$$X\ge Y. \tag{3-13}$$

Recall that u'/ε is a supersolution and v'/ε is a subsolution. Thanks to (3-11), letting $p=\nabla\phi^\circ(\bar y-\bar x)$ and $a=(\bar s-\bar t)/\varepsilon$, one has

$$a-\frac{\varepsilon}{(t_0-\bar t)^2}\ge\psi_1(p)D^2\phi(p):X,$$
$$a+\frac{\varepsilon}{(t_0-\bar s)^2}\le\psi_2(p)\Big(D^2\phi(p):Y-c_0\frac{\delta}{\Delta}\Big),$$

yielding

$$0<\frac{\varepsilon}{(t_0-\bar t)^2}+\frac{\varepsilon}{(t_0-\bar s)^2}\le\psi_2(p)\Big(D^2\phi(p):Y-c_0\frac{\delta}{\Delta}\Big)-\psi_1(p)D^2\phi(p):X. \tag{3-14}$$

Now, we observe that as $E'(\bar t)=E(\bar t)+W^\phi(0,\Delta/4)$ and (necessarily) $\bar x\in\partial E'(\bar t)$, we find that (p,X) is also a subjet of $-\chi_{W^\phi(x',\Delta/4)}$ for some $x'\in E(\bar t)$ with $\phi^\circ(\bar x-x')=\Delta/4$. In particular, it follows that $D^2\phi(p):X\le 4(N-1)/\Delta$. In the same way, $D^2\phi(p):Y\ge -4(N-1)/\Delta$ and using (3-13), we obtain

$$-4\,\frac{N-1}{\Delta}\le D^2\phi(p):Y\le D^2\phi(p):X\le 4\,\frac{N-1}{\Delta}. \tag{3-15}$$

Thanks to (3-6) and (3-15),

$$\begin{aligned}-\psi_1(p)D^2\phi(p):X&\le-\psi_2(p)D^2\phi(p):X+\delta\psi_2(p)|D^2\phi(p):X|\\ &\le-\psi_2(p)D^2\phi(p):X+4(N-1)\frac{\delta}{\Delta}\psi_2(p),\end{aligned}$$

so that (3-14) and (3-13) yield

$$\begin{aligned}0&<\psi_2(p)\Big(D^2\phi(p):Y-c_0\frac{\delta}{\Delta}\Big)-\psi_1(p)D^2\phi(p):X\\ &=\psi_2(p)\Big(D^2\phi(p):(Y-X)-c_0\frac{\delta}{\Delta}\Big)+(\psi_1(p)-\psi_2(p))D^2\phi(p):X\\ &\le\psi_2(p)\Big(D^2\phi(p):(Y-X)-(c_0-4(N-1))\frac{\delta}{\Delta}\Big)\le 0\end{aligned}$$

as soon as $c_0\ge 4(N-1)$, yielding a contradiction.

We deduce that at least one of $\bar t$ or $\bar s$ is zero; without loss of generality let us assume $\bar s = 0$. For any $t < t_0$, thanks to (3-9) (choosing $s = t$), if ε is small enough one has

$$\tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(t), \mathbb{R}^N \setminus F'(t))^2 + 2\frac{\varepsilon}{t_0 - t} \geq \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(\bar t), \mathbb{R}^N \setminus F'(0))^2 + \frac{\bar t^2}{2\varepsilon} + \frac{\varepsilon}{t_0 - \bar t} + \frac{\varepsilon}{t_0},$$

from which we see, in particular, that $\bar t \to 0$ as $\varepsilon \to 0$. Hence, in the limit $\varepsilon \to 0$, using also that E is closed, see [Chambolle et al. 2017a, Remark 2.3] for more details, we deduce

$$\begin{aligned}\tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(t), \mathbb{R}^N \setminus F'(t))^2 &\geq \liminf_{\bar t \to 0} \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(\bar t), \mathbb{R}^N \setminus F'(0))^2 \\ &\geq \tfrac{1}{2}\operatorname{dist}^{\phi^\circ}(E'(0), \mathbb{R}^N \setminus F'(0))^2 = \tfrac{1}{8}\Delta^2,\end{aligned}$$

which shows the thesis of the lemma, until $t = t_0$ (thanks to the continuity property (b)). Starting again from t_0, we have proven the lemma for bounded sets (or sets with bounded boundary).

If $\partial E(0)$ or $\partial F(0)$ is unbounded, we proceed as follows: We first consider, for $\varepsilon > 0$, the sets

$$\begin{aligned} E_0^\varepsilon &:= E(0) + W^\phi(0, \varepsilon), \\ F_0^\varepsilon &:= \mathbb{R}^N \setminus \big((\mathbb{R}^N \setminus F(0)) + W^\phi(0, \varepsilon)\big), \end{aligned}$$

which satisfy $\operatorname{dist}^{\phi^\circ}(E_0^\varepsilon, \mathbb{R}^N \setminus F_0^\varepsilon) \geq \Delta - 2\varepsilon$.

Then, for $R > 0$, we consider the initial sets $E_0^{\varepsilon,R} = E_0^\varepsilon \cap B_R$ and $F_0^{\varepsilon,R} = F_0^\varepsilon \cap (B_R + W^\phi(0, \Delta))$. The result holds for the evolutions starting from these two sets, with the distance $\Delta - 2\varepsilon$. Hence in the limit $R \to \infty$, it must hold for the (viscosity) evolutions starting from E_0^ε and F_0^ε (which are unique for almost all ε).

By standard comparison results for discontinuous viscosity solutions [Barles 1994; Barles and Souganidis 1998; Barles et al. 1993], it then follows that the superflow E (which is also a viscosity superflow) is contained in the evolution starting from E_0^ε, while F contains the evolution starting from F_0^ε (the ε-regularization has been introduced to avoid issues due to the possible nonuniqueness of viscosity solutions).

We deduce that $\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta - 2\varepsilon$ for all t, until extinction. Since this is true for any $\varepsilon > 0$, the lemma is proven. □

3B2. *Comparison with a nonconstant forcing term.* In this section we prove the crucial stability estimate for motions corresponding to different but close mobilities. We start with the following:

Proposition 3.4. *Assume that ϕ, ψ_1, ψ_2 are smooth and elliptic, that ψ_1, ψ_2 satisfy (3-6), and that $\beta > 0$ is such that*

$$\psi_2(\xi) \leq \beta\phi(\xi) \quad \textit{for all } \xi \in \mathbb{R}^N. \tag{3-16}$$

Let $E_0 \subset F_0$ be a closed and an open set, respectively, such that $\operatorname{dist}^{\phi^\circ}(E_0, \mathbb{R}^N \setminus F_0) =: \Delta > 0$, *and let E, F be a closed and an open "tube" in $\mathbb{R}^N \times [0, \infty)$, respectively, with $E(0) = E_0$, $F(0) = F_0$, such that $-\chi_E$ is a supersolution of*

$$u_t = \psi_1(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g), \tag{3-17}$$

and $-\chi_F$ is a subsolution of

$$u_t = \psi_2(\nabla u)(\operatorname{div} \nabla\phi(\nabla u) + g). \tag{3-18}$$

Then,

$$\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \geq \Delta e^{-\beta M t} - \delta \frac{2c_0/\Delta + \|g\|_\infty}{M}(1 - e^{-\beta M t}) \tag{3-19}$$

as long as this quantity is larger than $\Delta/2$, where c_0 is as in Lemma 3.3 and M is the Lipschitz constant of g with respect to ϕ°.

Proof. In order to obtain the estimate, we combine the results of Lemmas 3.3 and 3.2 (with $\eta = \phi^\circ$), together with a splitting result which follows from [Barles and Souganidis 1991]; see Example 1 of that paper, as well as [Barles 2006].

As before, we may need to slightly perturb the initial sets, considering rather $E_0^s = E_0 + W^\phi(0, s)$ and $F_0^s = \mathbb{R}^N \setminus (\mathbb{R}^N \setminus F_0 + W^\phi(0, s))$ for a small s (which eventually will go to 0).

Given $s > 0$ small, we start with building, for $\varepsilon > 0$ given, the motions $u^\varepsilon(x, t)$, $v^\varepsilon(x, t)$ defined as follows: We let $u^\varepsilon(x, 0) = -\chi_{E_0^s}$ and define recursively u^ε for $j \geq 0$ as a viscosity solution of

$$u_t^\varepsilon = \begin{cases} 2\psi_1(\nabla u^\varepsilon) \operatorname{div} \nabla\phi(\nabla v^\varepsilon), & 2j\varepsilon < t \leq 2j\varepsilon + \varepsilon, \\ 2\psi_1(\nabla u^\varepsilon) \fint_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds, & 2j\varepsilon + \varepsilon < t \leq 2(j+1)\varepsilon. \end{cases}$$

(In the case of nonuniqueness, we select for instance the smallest (super)solution, corresponding to the largest set $E^\varepsilon(t) = \{u^\varepsilon = -1\}$.) Similarly, we let $v^\varepsilon(x, 0) = -\chi_{F_0^s}$ and let $v^\varepsilon(x, t)$ be the largest (sub)solution of

$$v_t^\varepsilon = \begin{cases} 2\psi_2(\nabla v^\varepsilon)\Big(\operatorname{div} \nabla\phi(\nabla v^\varepsilon) - 2c_0 \dfrac{\delta}{\Delta}\Big), & 2j\varepsilon < t \leq 2j\varepsilon + \varepsilon, \\ 2\psi_2(\nabla v^\varepsilon)\Big(\fint_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds + 2c_0 \dfrac{\delta}{\Delta}\Big), & 2j\varepsilon + \varepsilon < t \leq 2(j+1)\varepsilon, \end{cases}$$

where c_0 is as in Lemma 3.3. Thanks to [Barles and Souganidis 1991; Barles 2006], as $\varepsilon \to 0$ these functions converge to the viscosity solutions of (3-17) and (3-18), respectively, starting from $-\chi_{E_0^\varepsilon}$ and $-\chi_{F_0^\varepsilon}$, provided these solutions are uniquely defined, which is known to be true for almost all ε (in fact all but a countable set of values), in which case it is also known that they are (negative of) characteristic functions.

We now show that we can estimate the distance between the corresponding geometric evolutions, using Lemmas 3.3 and 3.2.

Let δ be as in (3-6). A first observation is that, for $j \geq 0$, if we consider on the interval $[2j\varepsilon+\varepsilon, 2(j+1)\varepsilon]$ the smallest solution $\tilde{u}^\varepsilon(x, t)$ of

$$\tilde{u}_t^\varepsilon = 2\psi_2(\nabla\tilde{u}^\varepsilon)\Big(\fint_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds - \delta\|g\|_\infty\Big), \quad \tilde{u}^\varepsilon(\,\cdot\,, 2j\varepsilon + \varepsilon) = u^\varepsilon(\,\cdot\,, 2j\varepsilon + \varepsilon),$$

then, since for any $p \in \mathbb{R}^N$,

$$\psi_1(p) \fint_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds \geq \psi_2(p) \fint_{2j\varepsilon}^{2(j+1)\varepsilon} g(x, s)\, ds - \delta\psi_2(p)\|g\|_\infty,$$

one has $\tilde{u}^\varepsilon(x,t) \le u^\varepsilon(x,t)$ for $2j\varepsilon+\varepsilon \le t \le 2(j+1)\varepsilon$, and thus $E^\varepsilon(t) \subseteq \{\tilde{u}^\varepsilon(\cdot,t) = -1\}$. Hence, Lemma 3.2 yields that for $2j\varepsilon+\varepsilon \le t \le 2(j+1)\varepsilon$,

$$\begin{aligned}\operatorname{dist}^{\phi^\circ}(E^\varepsilon(t), \mathbb{R}^N \setminus F^\varepsilon(t)) &\ge \operatorname{dist}^{\phi^\circ}\big(\{\tilde{u}^\varepsilon(\cdot,t) = -1\}, \mathbb{R}^N \setminus F^\varepsilon(t)\}\big) \\ &\ge \Big(\operatorname{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon+\varepsilon), \mathbb{R}^N \setminus F^\varepsilon(2j\varepsilon+\varepsilon)) - \frac{c}{M}\Big)e^{-2\beta M(t-2j\varepsilon-\varepsilon)} + \frac{c}{M}\end{aligned}$$

for $c = -\delta(2c_0/\Delta + \|g\|_\infty)$. Note that here we use the fact that the mobility $2\psi_2$ satisfies $2\psi_2 \le 2\beta\phi$; see (3-16).

On the other hand, Lemma 3.3 yields that for all $j \ge 0$ and $2j\varepsilon \le t \le 2j\varepsilon+\varepsilon$,

$$\operatorname{dist}^{\phi^\circ}(E^\varepsilon(t), \mathbb{R}^N \setminus F^\varepsilon(t)) \ge \operatorname{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N \setminus F^\varepsilon(2j\varepsilon))$$

as long as $\operatorname{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N \setminus F^\varepsilon(2j\varepsilon)) \ge \Delta/2$.

In particular, setting $d_j = \operatorname{dist}^{\phi^\circ}(E^\varepsilon(2j\varepsilon), \mathbb{R}^N \setminus F^\varepsilon(2j\varepsilon))$, one obtains by induction that

$$d_{j+1} \ge \Big(d_j - \frac{c}{M}\Big)e^{-2\beta M\varepsilon} + \frac{c}{M} \ge \Big(d_0 - \frac{c}{M}\Big)e^{-2\beta M(j+1)\varepsilon} + \frac{c}{M},$$

as long as $d_j \ge \Delta/2$. In the limit, we find that, letting $E^s(t) = \{u(\cdot,t) = -1\}$ and $F^s(t) = \{v(\cdot,t) = -1\}$ and recalling that $\operatorname{dist}^{\phi^\circ}(E^s_0, \mathbb{R}^N \setminus F^s_0) \ge \Delta - 2s$,

$$\operatorname{dist}^{\phi^\circ}(E^s(t), \mathbb{R}^N \setminus F^s(t)) \ge (\Delta - 2s)e^{-\beta Mt} - \delta\frac{2c_0/\Delta + \|g\|_\infty}{M}(1 - e^{-\beta Mt})$$

as long as this quantity is larger than $\Delta/2$.

By comparison, it is clear that $E \subset E^s$ and $F^s \subset F$; hence (letting eventually $s \to 0$), we deduce that (3-19) holds as long as the right-hand side is larger than $\Delta/2$. □

We are now ready to state and prove the main result of the section.

Theorem 3.5. *Let ψ_1, ψ_2 and ϕ satisfy (3-6) and (3-16). Assume also that ψ_1, ψ_2 are ϕ-regular in the sense of Definition 2.7. Let the forcing term $g(x,t)$ be continuous, bounded, and spatially M-Lipschitz continuous with respect to the distance ϕ°, and denote by E a superflow for $V = -\psi_1(\nu)(\kappa_\phi + g)$ and by F a subflow for $V = -\psi_2(\nu)(\kappa_\phi + g)$, both in the sense of Definition 2.2. Finally, assume that* $\operatorname{dist}^{\phi^\circ}(E(0), \mathbb{R}^N \setminus F(0)) \ge \Delta > 0$. *Then for all t,*

$$\operatorname{dist}^{\phi^\circ}(E(t), \mathbb{R}^N \setminus F(t)) \ge \Delta e^{-\beta Mt} - \delta\frac{2c_0/\Delta + \|g\|_\infty}{M}(1 - e^{-M\beta t}) \tag{3-20}$$

as long as this quantity is larger than $\Delta/2$.

Proof. Consider smooth, elliptic approximations of ψ_i $(i = 1, 2)$, ϕ, denoted by ψ_i^n, ϕ^n, such that (3-6)–(3-16) hold also for ψ_i^n, ϕ^n (with slightly larger constants δ and β that, with a small abuse of notation, will not be relabeled) and with $\psi_i^n - \varepsilon\phi^n$ convex $(i = 1, 2)$, that is, ψ_i^n are uniformly ϕ^n-regular (see the statement of Theorem 2.8).

Consider as before, for $s > 0$ small, the initial sets $E^s_0 := E_0 + W^{\phi^n}(0, s)$ and $F^s_0 := \mathbb{R}^N \setminus [(\mathbb{R}^N \setminus F_0) + W^{\phi^n}(0, s)]$. As in Theorem 2.8 we can build subflows A^s_n and superflows B^s_n for the evolution $V = -\psi_1^n(\nu)(\kappa_{\phi^n} + g)$, both starting from E^s_0, such that $A^s_n \subset B^s_n$, and a subflow A'^s_n and superflow B'^s_n for the evolution $V = -\psi_2^n(\nu)(\kappa_{\phi^n} + g)$, both starting from F_0^s, such that $A_n'^s \subset B_n'^s$. Thanks to Lemma 2.6, $-\chi_{B_n^s}$ is a viscosity supersolution and $-\chi_{A_n'^s}$ is a viscosity subsolution, so that we can apply Proposition 3.4 and estimate their $(\phi^n)^\circ$-distance according to (3-19).

Again thanks to Theorem 2.8, $\mathbb{R}^N \setminus A_n^s$ converges in the Kuratowski sense as $n \to \infty$ to the complement of a subflow, which contains E thanks to Theorem 2.3, and analogously $B_n'^s$ converges to a superflow contained in F. We deduce (3-20), letting $s \to 0$. □

3C. *Existence and uniqueness by approximation.* We recall that the existence theory for level-set flows (in the sense of Definition 2.4) that we have so far works only for ϕ-regular mobilities. The goal of this section is to extend the existence theory to general mobilities. To this aim, we consider the following notion of *solution via approximation*:

Definition 3.6 (level-set flows via approximation). Let ψ be a mobility, g an admissible forcing term and u^0 a uniformly continuous function on $\mathbb{R}^N$.

We will say that a continuous function $u^\psi : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ is a *solution via approximation* to the level-set flow corresponding to (1-1), with initial datum u^0, if $u^\psi(\cdot, 0) = u^0$ and if there exists a sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$ and, denoting by u^{ψ_n} the unique solution to (1-1) (in the sense of Definition 2.4) with mobility ψ_n and initial datum u^0, we have $u^{\psi_n} \to u^\psi$ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$.

The next theorem is the main result of this section: it shows that for any mobility ψ, a solution via approximation u^ψ in the sense of the previous definition always exists; such a solution is also unique in that it is independent of the choice of the approximating sequence of ϕ-regular mobilities $\{\psi_n\}$. In particular, in the case of a ϕ-regular mobility, the notion of solution via approximation is consistent with that of Definition 2.4.

Theorem 3.7. *Let ψ, g, and u^0 be as in Definition 3.6. Then, there exists a unique solution u^ψ in the sense of Definition 3.6 with initial datum u^0.*

Proof. We have to prove that for any sequence $\{\psi_n\}$ of ϕ-regular mobilities such that $\psi_n \to \psi$, the corresponding solutions u^{ψ_n} to (1-1) with initial datum u^0 converge to some function u locally uniformly in $\mathbb{R}^N \times [0, +\infty)$. We split the proof of the theorem into two steps.

Step 1. Let β be as in (3-16). Let $T_0 > 0$ be defined by $e^{-2\beta M T_0} = \frac{3}{4}$, where as usual M is the spatial Lipschitz constant of the forcing term g with respect to the distance induced by ϕ°. We claim that for every $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\|u^{\psi_n} - u^{\psi_m}\|_{L^\infty(\mathbb{R}^N \times [0,T_0])} \le \varepsilon \quad \text{for all } n, m \ge \bar{n}. \tag{3-21}$$

To this aim, we observe that since $\psi_n \to \psi$, for n large enough

$$\psi_n(\xi) \le 2\beta\phi(\xi) \quad \text{for all } \xi \in \mathbb{R}^N, \tag{3-22}$$

and there exists $\delta_j \to 0$ such that

$$(1 - \delta_j)\psi_n \le \psi_m \le (1 + \delta_j)\psi_n \quad \text{for all } m, n \ge j. \tag{3-23}$$

Set $E_\lambda^{\psi_n}(t) := \{u^{\psi_n}(\,\cdot\,, t) \le \lambda\}$, $F_\lambda^{\psi_n}(t) := \{u^{\psi_n}(\,\cdot\,, t) < \lambda\}$ and recall that $E_\lambda^{\psi_n}$ is a superflow, while $F_\lambda^{\psi_n}$ is a subflow in the sense of Definition 2.2.

Let ω be a modulus of continuity for u^0 with respect to ϕ° and recall that for any $\lambda \in \mathbb{R}$

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(0), \mathbb{R}^N \setminus F_{\lambda+\varepsilon}^{\psi_n}(0)) = \mathrm{dist}^{\phi^\circ}(\{u^0 \le \lambda\}, \{u^0 \ge \lambda+\varepsilon\}) \ge \omega^{-1}(\varepsilon).$$

By (3-22), (3-23) and Theorem 3.5, for all $n, m \ge j$ we have

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(t), \mathbb{R}^N \setminus F_{\lambda+\varepsilon}^{\psi_n}(t)) \ge \omega^{-1}(\varepsilon)e^{-2\beta M t} - \delta_j \frac{2c_0/\omega^{-1}(\varepsilon) + \|g\|_\infty}{M}(1 - e^{-2\beta M t}),$$

as long as the right-hand side is larger than $\omega^{-1}(\varepsilon)/2$, that is, for all $t \in [0, T_0]$, provided j is large enough. In particular, for n, m large enough $E_\lambda^{\psi_m}(t) \subset F_{\lambda+\varepsilon}^{\psi_n}(t)$ for all $t \in [0, T_0]$, which yields

$$u^{\psi_n}(\,\cdot\,, t) \le u^{\psi_m}(\,\cdot\,, t) + \varepsilon \quad \text{for all } t \in [0, T_0].$$

By switching the roles of n and m we deduce (3-21).

Step 2. First arguing as in the proof of Theorem 2.8 and using (3-22) we see that $\omega(e^{2\beta M t}\,\cdot\,)$ is a spatial modulus of continuity for $u^{\psi_n}(\,\cdot\,, t)$ for all n. Observe that from (3-21) it follows that for n, m large enough we have

$$E_\lambda^{\psi_m}(T_0) \subseteq E_{\lambda+\varepsilon}^{\psi_n}(T_0),$$

which in turn implies

$$\mathrm{dist}^{\phi^\circ}(E_\lambda^{\psi_m}(T_0), \mathbb{R}^N \setminus F_{\lambda+2\varepsilon}^{\psi_n}(T_0)) \ge \mathrm{dist}^{\phi^\circ}(E_{\lambda+\varepsilon}^{\psi_n}(T_0), \mathbb{R}^N \setminus F_{\lambda+2\varepsilon}^{\psi_n}(T_0)) \ge \omega^{-1}(e^{2\beta M T_0}\varepsilon).$$

We can now argue as in Step 1 to conclude that, for n, m large enough,

$$\|u^{\psi_n} - u^{\psi_m}\|_{L^\infty(\mathbb{R}^N \times [T_0, 2T_0])} \le 2\varepsilon.$$

Therefore, by an easy iteration argument we can show that, for every given $T > 0$, the sequence $\{u^{\psi_n}\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^N \times [0, T])$. □

We conclude by recalling the following remarks, referring to [Chambolle et al. 2017a] for the details.

Remark 3.8 (stability). As a byproduct of the previous theorem, and a standard diagonalization argument, we have the following stability property for solutions to (1-1): Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of mobilities and ϕ_n a sequence of anisotropies such that $\psi_n \to \psi$ and $\phi_n \to \phi$ as $n \to +\infty$. Then u^{ψ_n} converge to u^ψ locally uniformly in $\mathbb{R}^N \times [0, +\infty)$ as $h \to 0$ (where u^{ψ_n} is the solution to (1-1) with ψ replaced by ψ_n and ϕ replaced by ϕ_n).

Remark 3.9 (comparison with the Giga–Požár solution). When ϕ is purely crystalline and $g \equiv c$ for some $c \in \mathbb{R}$, the unique level-set solution in the sense of Definition 3.6 coincides with the viscosity solution constructed in [Giga and Požár 2016; 2018].

We also recall that when g is constant, (1-1) admits a phase-field approximation by means of the anisotropic Allen–Cahn equation; see [Chambolle et al. 2017a, Remark 6.2] for the details.

In the next theorem we recall the main properties of the level-set solutions introduced in Definition 3.6. In the statement of the theorem, we will say that a uniformly continuous initial function u^0 is *well-prepared* at $\lambda \in \mathbb{R}$ if the following two conditions hold:

(a) If $H \subset \mathbb{R}^N$ is a closed set such that $\operatorname{dist}(H, \{u_0 \geq \lambda\}) > 0$, then there exists $\lambda' < \lambda$ such that $H \subseteq \{u_0 < \lambda'\}$.

(b) If $A \subset \mathbb{R}^N$ is an open set such that $\operatorname{dist}(\{u_0 \leq \lambda\}, \mathbb{R}^N \setminus A) > 0$, then there exists $\lambda' > \lambda$ such that $\{u_0 \leq \lambda'\} \subset A$.

Here $\operatorname{dist}(\,\cdot\,,\,\cdot\,)$ denotes the distance function with respect to a given norm. Clearly, the properties stated in (a) and (b) above do not depend on the choice of such a norm.

Remark 3.10. Note that the above assumption of well-preparedness is automatically satisfied if the set $\{u_0 \leq \lambda\}$ is bounded.

Theorem 3.11 (properties of the level-set flow). *Let u^ψ be a solution in the sense of Definition 3.6, with initial datum u^0. The following properties hold true*:

(i) (*nonfattening of level sets*) *There exists a countable set $N \subset \mathbb{R}$ such that for all $t \in [0, +\infty)$, $\lambda \notin N$,*

$$\begin{aligned} \{(x,t) : u^\psi(x,t) < \lambda\} &= \operatorname{Int}(\{(x,t) : u^\psi(x,t) \leq \lambda\}), \\ \overline{\{(x,t) : u^\psi(x,t) < \lambda\}} &= \{(x,t) : u^\psi(x,t) \leq \lambda\}. \end{aligned} \tag{3-24}$$

(ii) (*distributional formulation when ψ is ϕ-regular*) *If ψ is ϕ-regular, then u^ψ coincides with the distributional solution in the sense of Definition 2.4.*

(iii) (*comparison*) *Assume that $u^0 \leq v^0$ and denote the corresponding level-set flows by u^ψ and v^ψ, respectively. Then $u^\psi \leq v^\psi$.*

(iv) (*geometricity*) *Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and uniformly continuous. Then u^ψ is a solution with initial datum u^0 if and only if $f \circ u^\psi$ is a solution with initial datum $f \circ u^0$.*

(v) (*independence of the initial level-set function*) *Assume that u^0 and v^0 are well-prepared at λ. If $\{u^0 < \lambda\} = \{v^0 < \lambda\}$, then $\{u^\psi(\,\cdot\,,t) < \lambda\} = \{v^\psi(\,\cdot\,,t) < \lambda\}$ for all $t > 0$. Analogously, if $\{u^0 \leq \lambda\} = \{v^0 \leq \lambda\}$, then $\{u^\psi(\,\cdot\,,t) \leq \lambda\} = \{v^\psi(\,\cdot\,,t) \leq \lambda\}$ for all $t > 0$.*

For the proof we refer to [Chambolle et al. 2017a, Theorem 5.9].

We conclude with a remark about conditions that prevent the occurrence of fattening.

Remark 3.12 (star-shaped sets, convex sets and graphs). It is well-known [Soner 1993, Section 9] that for the motion without forcing, strictly star-shaped sets do not develop fattening so that, in particular, their evolution is unique. The proof of this fact, given for instance in [Soner 1993] for the mean curvature flow, works also for solutions in the sense of Definition 2.2 when the mobility ψ is ϕ-regular, and in turn, by approximation, also for the *generalized motion* associated to level-set solutions in the sense of Definition 3.6, when ψ is general. Uniqueness also holds for motions with a time-dependent forcing $g(t)$ [Bellettini et al. 2009, Theorem 5] as long as the set remains strictly star-shaped. This remark obviously applies to initial convex sets, which, in addition, remain convex for all times, as was shown in [Bellettini et al. 2006; 2009; Caselles and Chambolle 2006] with a spatially constant forcing term.[2] The case of unbounded initial convex sets was not considered in these references but can be easily addressed by approximation (and uniqueness still holds with the same proof).

In the same way, if the initial set $E_0 = \{x_N \le v^0(x_1, \dots, x_{N-1})\}$ is the subgraph of a uniformly continuous function v^0, and the forcing term does not depend on x_N, then one can show that fattening does not develop and $E(t)$ is still the subgraph of a uniformly continuous function for all $t > 0$, as in the classical case [Ecker and Huisken 1989; Evans and Spruck 1992b]; see also [Giga and Giga 1998] for the two-dimensional crystalline case.

Acknowledgements

Chambolle was partially supported by the ANR projects ANR-12-BS01-0014-01 "GEOMETRYA" and ANR-12-BS01-0008-01 "HJnet" at the beginning of this project. Part of this work was done while he was then in residence at the Isaac Newton Institute for Mathematical Sciences, Cambridge, supported by EPSRC grant no. EP/K032208/1, and partially supported by a grant of the Simons Foundation. Novaga was partially supported by the GNAMPA and by the University of Pisa via grant PRA 2017 "Problemi di ottimizzazione e di evoluzione in ambito variazionale". Morini and Ponsiglione were partially supported by the GNAMPA 2016 grant "Variational methods for nonlocal geometric flows". The authors thank the referees for their careful reading of our manuscript and their numerous constructive remarks and observations.

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Received 10 Nov 2017. Revised 4 May 2018. Accepted 29 Jun 2018.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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Analysis & PDE

Volume 12 No. 3 2019
