ANALYSIS & PDE

Volume 12 No. 2 2019

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ANALYSIS AND PDE
Vol. 12, No. 2, 2019
dx.doi.org/10.2140/apde.2019.12.281

THE MUSKAT PROBLEM IN TWO DIMENSIONS: EQUIVALENCE OF FORMULATIONS, WELL-POSEDNESS, AND REGULARITY RESULTS

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We consider the Muskat problem describing the motion of two unbounded immiscible fluid layers with equal viscosities in vertical or horizontal two-dimensional geometries. We first prove that the mathematical model can be formulated as an evolution problem for the sharp interface separating the two fluids, which turns out to be, in a suitable functional-analytic setting, quasilinear and of parabolic type. Based upon these properties, we then establish the local well-posedness of the problem for arbitrary large initial data and show that the solutions become instantly real-analytic in time and space. Our method allows us to choose the initial data in the class H^s, $s \in \left(\frac{3}{2}, 2\right)$, when neglecting surface tension, respectively in H^s, $s \in (2, 3)$, when surface-tension effects are included. Besides, we provide new criteria for the global existence of solutions.

1. Introduction and the main results

The Muskat problem [1934] is a classical model describing the motion of two immiscible fluids in a porous medium or a Hele-Shaw cell. We consider here the particular case when the fluids have equal viscosities and we assume that the flows are two-dimensional. Furthermore, we consider an unbounded geometry corresponding to fluid layers that occupy the entire space, the fluid motion being localized and the fluid system close to the rest state far away from the origin. We further assume that the fluids are separated by a sharp interface which flattens out at infinity, evolves in time, and is unknown. We consider two different scenarios for this unconfined Muskat problem:

MSC2010: 35R37, 35K59, 35K93, 35Q35, 42B20.
Keywords: Muskat problem, surface tension, singular integral.

(a) In the absence of surface-tension effects at the free boundary, the Hele-Shaw cell is vertical and the fluid located below is more dense.

(b) In the presence of surface-tension effects, the Hele-Shaw cell is either vertical or horizontal and we make no restrictions on the densities of the fluids.

One big advantage of considering this setting is that the equations of motion can be very elegantly formulated as a single evolution equation for the interface between the fluids. Indeed, parametrizing this interface as the graph $[y=f(t,x)]$, the Muskat problem is equivalent in this setting to an evolution problem for the unknown function f, see Section 2, and it can be written as

$$\begin{cases} \partial_t f(t,x) = \dfrac{\sigma k}{2\pi\mu} f'(t,x)\,\mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{f(t,x)-f(t,x-y)}{y^2+(f(t,x)-f(t,x-y))^2}(\kappa(f))'(t,x-y)\,dy \\ \qquad\qquad + \dfrac{\sigma k}{2\pi\mu}\,\mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{y}{y^2+(f(t,x)-f(t,x-y))^2}(\kappa(f))'(t,x-y)\,dy \\ \qquad\qquad + \dfrac{\Delta_\rho k}{2\pi\mu}\,\mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{y(f'(t,x)-f'(t,x-y))}{y^2+(f(t,x)-f(t,x-y))^2}\,dy \qquad \text{for } t>0,\quad x\in\mathbb{R}, \\ f(0,\cdot)=f_0. \end{cases} \tag{1-1}$$

For brevity we write f' for the spatial derivative $\partial_x f$. We let k denote the permeability of the homogeneous porous medium, μ is the viscosity coefficient of the fluids, σ is the surface-tension coefficient at the free boundary, and

$$\Delta_\rho := g(\rho_- - \rho_+),$$

where g is the Earth's gravity and $\rho_\pm$ is the density of the fluid which occupies the domain $\Omega_\pm(t)$ defined by

$$\Omega_-(t) := [y<f(t,x)] \quad \text{and} \quad \Omega_+(t) := [y>f(t,x)].$$

Furthermore, $\kappa(f(t))$ is the curvature of the graph $[y=f(t,x)]$ and PV denotes the principal value which, depending on the regularity of the functions under the integral, is taken at zero and/or at infinity. Our analysis covers the following scenarios

$$\text{(a) } \sigma=0,\ \Delta_\rho>0 \quad \text{and} \quad \text{(b) } \sigma>0,\ \Delta_\rho\in\mathbb{R},$$

meaning that (a) corresponds to the stable case when the denser fluid is located below.

Due to its physical relevance [Bear 1972], the Muskat problem has been widely studied in the last decades in several geometries and physical settings and with various methods. When neglecting surface-tension effects the well-posedness of the Muskat problem is in strong relationship with the Rayleigh–Taylor condition, being implied by the latter. The Rayleigh–Taylor condition, which appears first in [Saffman and Taylor 1958], is a sign restriction on the jump of the pressure gradient in normal direction at the free boundary. For fluids with equal viscosities moving in a vertical geometry, it reduces to the simple relation

$$\Delta_\rho > 0;$$

see, e.g., [Córdoba and Gancedo 2010; Escher et al. 2018] and also (2-1a)–(2-1b). The first local existence result was established in [Yi 1996] by using Newton's iteration method; the analysis in [Ambrose 2004;

Berselli et al. 2014; Cheng et al. 2016; Gómez-Serrano and Granero-Belinchón 2014; Constantin et al. 2017; Córdoba et al. 2011; 2013; 2014; Córdoba and Gancedo 2007; 2010] is based on energy estimates and the energy method; the authors of [Siegel et al. 2004] use methods from complex analysis and a version of the Cauchy–Kowalewski theorem; a fixed-point argument is employed in [Bazaliy and Vasylyeva 2014] for nonregular initial data, and the approach in [Escher and Matioc 2011; Escher et al. 2012; 2018] relies on the formulation of the problem as a nonlinear and nonlocal parabolic equation together with an abstract well-posedness result from [Da Prato and Grisvard 1979] based on continuous maximal regularity. Other papers study the qualitative aspects of solutions to the Muskat problem for fluids with equal viscosities, such as global existence of strong and weak solutions [Constantin et al. 2013; 2017; Granero-Belinchón 2014], existence of initial data for which solutions turn over [Castro et al. 2011; 2012; 2013], and the absence of squirt or splash singularities [Córdoba and Gancedo 2010; Gancedo and Strain 2014].

Compared to the zero-surface-tension case, the Muskat problem with surface tension is less well-studied. When allowing for surface tension, the Rayleigh–Taylor condition is no longer needed and the problem is well-posed for general initial data. While some of the references require quite high regularity from the initial data, see [Ambrose 2014; Friedman and Tao 2003; Hong et al. 1997; Tofts 2017], optimal results are established in bounded or periodic geometries under the observation that the Muskat problem with surface tension can be formulated as a quasilinear parabolic evolution problem; see [Escher et al. 2018; Prüss and Simonett 2016a].

The stability properties of equilibria which are, depending on the physical scenario, horizontal lines [Cheng et al. 2016; Ehrnström et al. 2013; Escher et al. 2012; Escher and Matioc 2011], finger-shaped [Ehrnström et al. 2013; Escher et al. 2012; Escher and Matioc 2011], circular [Friedman and Tao 2003], or a union of disjoint circles/spheres [Prüss and Simonett 2016b] have been also addressed in the references just mentioned.

In this paper we first rigorously prove in Section 2 that the Muskat problem in the classical formulation (2-1) and the system (1-1) are equivalent for a certain class of solutions. Thereafter, the analysis of (1-1) starts from the obvious observation that the right-hand side of the first equation of (1-1) is linear with respect to the highest-order spatial derivative of f; that is, this particular Muskat problem has a quasilinear structure (also when neglecting surface tension). This property is not obvious in the particular geometry considered in [Escher et al. 2018; Yi 1996] (when $\sigma = 0$). In a suitable functional-analytic setting we then prove that (1-1) is additionally parabolic for general initial data. The parabolic character was established previously for bounded geometries [Escher et al. 2012; 2018; Escher and Matioc 2011; Prüss and Simonett 2016a; 2016b] (in the absence of surface-tension effects only when the Rayleigh–Taylor condition holds), but for (1-1) only for small initial data; see [Constantin et al. 2013; Córdoba and Gancedo 2007]. These two aspects, that is, the quasilinearity and the parabolicity, enable us to use abstract results for quasilinear parabolic problems due to H. Amann [1993, Section 12] to prove, by similar strategies, the well-posedness of the Muskat problem with and without surface tension.

It is worth emphasizing that for this particular Muskat problem the local well-posedness is established, in the zero-surface-tension case, only for initial data that are twice-weakly differentiable and which belong to $W^2_p(\mathbb{R})$ for some $p \in (1, \infty]$; see [Constantin et al. 2017]. Our first main result, i.e., Theorem 1.1, extends the local well-posedness to general initial data in $H^s(\mathbb{R})$ with $s \in \left(\frac{3}{2}, 2\right)$. For the unconfined Muskat problem with surface tension, the well-posedness is considered only in [Ambrose 2014; Tofts 2017] and in both papers the authors require that $f_0 \in H^s(\mathbb{X})$, with $\mathbb{X} \in \{\mathbb{R}, \mathbb{T}\}$ and $s \geq 6$. In our well-posedness result, i.e., Theorem 1.2, the curvature of the initial data may be even unbounded as we allow for general initial data $f_0 \in H^s(\mathbb{R})$ with $s \in (2, 3)$. Additionally, we also obtain new criteria for the existence of global solutions to the Muskat problem with and without surface tension and, as a consequence of the parabolic character of the equations, we show that the fluid interfaces become instantly real-analytic.

Our strategy is the following: we formulate, in a suitable functional-analytic setting, (1-1) as a quasilinear evolution problem of the form[1]

$$\dot{f} = \Phi_\sigma(f)[f], \quad t > 0, \quad f(0) = f_0,$$

and then we study the properties of the operator Φ_σ. We differentiate between the case $\sigma = 0$, studied in Sections 3–5, when we simply write $\Phi_\sigma =: \Phi$, and the case $\sigma > 0$, as in the first case $\Phi(f)$ is a nonlocal operator of order 1 and in the second case $\Phi_\sigma(f)$ has order 3 (for f appropriately chosen). At the core of our estimates lies the following deep result from harmonic analysis: given a Lipschitz function $a : \mathbb{R} \to \mathbb{R}$, the singular integral operator

$$h \mapsto \left[x \mapsto \mathrm{PV}\int_{\mathbb{R}} \frac{h(x-y)}{y} \exp\left(i\frac{a(x)-a(x-y)}{y}\right) dy\right] \tag{1-2}$$

belongs to $\mathcal{L}(L_2(\mathbb{R}))$ and its norm is bounded by $C(1 + \|a'\|_\infty)$, see [Murai 1986], with C denoting a universal constant independent of a. Relying on (1-2), we study the mapping properties of Φ_σ and show, for suitable f, that $\Phi_\sigma(f)$ is the generator of a strongly continuous and real-analytic semigroup. The main results of this paper, that is, Theorems 1.1–1.3, are then obtained by employing abstract results presented in [Amann 1993, Section 12], and which we briefly recall at the end of this section. The line of approach is close to the one we followed in [Escher et al. 2018]; however, the functional-analytic setting and the methods used to establish the needed estimates are substantially different. We expect that our method extends to the general case when $\mu_- \neq \mu_+$ and we believe to obtain, for periodic flows, a similar stability behavior of the — flat and finger shaped — equilibria, as in [Escher and Matioc 2011].

Our first main result is the following well-posedness theorem for the Muskat problem without surface-tension effects.

Theorem 1.1 (well-posedness: no surface tension). *Let $\sigma = 0$ and $\Delta_\rho > 0$. The problem (1-1) possesses for each $f_0 \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, a unique maximal classical solution*

$$f := f(\cdot\,; f_0) \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^2(\mathbb{R})) \cap C^1((0, T_+(f_0)), H^1(\mathbb{R})),$$

with $T_+(f_0) \in (0, \infty]$, and $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $H^s(\mathbb{R})$. Additionally, if

$$\sup_{[0,T_+(f_0))\cap[0,T]} \|f(t)\|_{H^s} < \infty \quad \textit{for all } T > 0,$$

then $T_+(f_0) = \infty$.

[1] We write $\dot{f}$ to denote the derivative df/dt.

The quasilinear character of the problem is enhanced by the presence of surface tension. For this reason we may consider, when $\sigma > 0$, initial data with unbounded curvature. We show in Theorem 1.3, however, that the curvature becomes instantly real-analytic and bounded.

Theorem 1.2 (well-posedness: with surface tension). *Let $\sigma > 0$ and $\Delta_\rho \in \mathbb{R}$. The problem (1-1) possesses for each $f_0 \in H^s(\mathbb{R})$, $s \in (2,3)$, a unique maximal classical solution*

$$f := f(\cdot\,; f_0) \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^3(\mathbb{R})) \cap C^1((0, T_+(f_0)), L_2(\mathbb{R})),$$

with $T_+(f_0) \in (0, \infty]$, and $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $H^s(\mathbb{R})$. Additionally, if

$$\sup_{[0,T_+(f_0))\cap[0,T]} \|f(t)\|_{H^s} < \infty \quad \text{for all } T > 0,$$

then $T_+(f_0) = \infty$.

These results reflect the fact that the Muskat problem without surface tension is a first-order evolution problem, while the Muskat problem with surface tension is of third order. The solutions obtained in Theorems 1.1 and 1.2 become instantly real-analytic.

Theorem 1.3. *Let $s \in \left(\frac{3}{2}, 2\right)$ if $\sigma = 0$ and $\Delta_\rho > 0$, and let $s \in (2, 3)$ if $\sigma > 0$. Given $f_0 \in H^s(\mathbb{R})$, let $f = f(\cdot\,; f_0)$ denote the unique maximal solution to (1-1) found in Theorems 1.1 and 1.2, respectively. Then*

$$[(t, x) \mapsto f(t, x)] : (0, T_+(f_0)) \times \mathbb{R} \to \mathbb{R}$$

is a real-analytic function. In particular, $f(t, \cdot)$ is real-analytic for each $t \in (0, T_+(f_0))$. Moreover, given $k \in \mathbb{N}$, it holds that

$$f \in C^\omega((0, T_+(f_0)), H^k(\mathbb{R})),$$

where C^ω denotes real-analyticity.

As a direct consequence of Theorems 1.1 and 1.3 and of [Constantin et al. 2013, Theorem 3.1], see also [Constantin et al. 2016, Remark 6.2], we obtain a global existence result for solutions to the Muskat problem without surface tension that correspond to initial data of medium size in $H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$. In the following $\mathcal{F}$ denotes the Fourier transform.

Corollary 1.4. *There exists a constant $c_0 \geq \frac{1}{5}$ such that for all $f_0 \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, with*

$$|||f_0||| := \int_{\mathbb{R}} |\xi|\,|\mathcal{F} f_0(\xi)|\, d\xi < c_0,$$

the solution found in Theorem 1.1 exists globally.

Proof. The claim follows from the inequality

$$|||f||| = \int_{\mathbb{R}} |\xi|\,|\mathcal{F} f(\xi)|\, d\xi \leq \|f\|_{H^s} \int_{\mathbb{R}} \frac{1}{(1+|\xi|^2)^{s-1}}\, d\xi \leq C\|f\|_{H^s}$$

for $s \in \left(\frac{3}{2}, 2\right)$ and $f \in H^s(\mathbb{R})$. □

An abstract setting for quasilinear parabolic evolution equations. In Theorem 1.5 we collect abstract results from [Amann 1993, Section 12] for a general class of abstract quasilinear parabolic evolution equations, which we use in an essential way in our analysis.

Given Banach spaces $\mathbb{E}_0, \mathbb{E}_1$ with dense embedding $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$, we define $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ as the subset of $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$ consisting of negative generators of strongly continuous analytic semigroups. More precisely, $\mathbb{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ if $-\mathbb{A}$, considered as an unbounded operator in $\mathbb{E}_0$ with domain $\mathbb{E}_1$, generates a strongly continuous and analytic semigroup in $\mathcal{L}(\mathbb{E}_0)$.

Theorem 1.5. *Let $\mathbb{E}_0, \mathbb{E}_1$ be Banach spaces with dense embedding $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$ and let $\mathbb{E}_\theta := [\mathbb{E}_0, \mathbb{E}_1]_\theta$ for $0 < \theta < 1$ be endowed with the $\|\cdot\|_\theta$-norm. Let further $0 < \beta < \alpha < 1$ and assume that*

$$-\Phi \in C^{1-}(\mathcal{O}_\beta, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)), \tag{1-3}$$

where $\mathcal{O}_\beta$ denotes an open subset of $\mathbb{E}_\beta$ and C^{1-} stands for local Lipschitz continuity. The following assertions hold for the quasilinear evolution problem

$$\dot{f} = \Phi(f)[f], \quad t > 0, \quad f(0) = f_0. \tag{QP}$$

Existence: *given $f_0 \in \mathcal{O}_\alpha := \mathcal{O}_\beta \cap \mathbb{E}_\alpha$, the problem (QP) possesses a maximal solution*

$$f := f(\cdot\,; f_0) \in C([0, T_+(f_0)), \mathcal{O}_\alpha) \cap C((0, T_+(f_0)), \mathbb{E}_1) \cap C^1((0, T_+(f_0)), \mathbb{E}_0) \cap C^{\alpha-\beta}([0, T], \mathbb{E}_\beta)$$

for all $T \in (0, T_+(f_0))$, with $T_+(f_0) \in (0, \infty]$.

Uniqueness: *if $\widetilde{T} \in (0, \infty]$, $\eta \in (0, \alpha - \beta]$, and $\tilde{f} \in C((0, \widetilde{T}), \mathbb{E}_1) \cap C^1((0, \widetilde{T}), \mathbb{E}_0)$ satisfies*

$$\tilde{f} \in C^\eta([0, T], \mathbb{E}_\beta) \quad \text{for all } T \in (0, \widetilde{T})$$

and solves (QP), then $\widetilde{T} \le T_+(f_0)$ and $\tilde{f} = f$ on $[0, \widetilde{T})$.

Criterion for global existence: *if $f : [0, T] \cap [0, T_+(f_0)) \to \mathcal{O}_\alpha$ is uniformly continuous for all $T > 0$, then*

$$T_+(f_0) = \infty \quad \text{or} \quad T_+(f_0) < \infty \text{ and } \operatorname{dist}(f(t), \partial\mathcal{O}_\alpha) \to 0 \text{ for } t \to T_+(f_0).$$

Continuous dependence of initial data: *the mapping $[(t, f_0) \mapsto f(t; f_0)]$ defines a semiflow on $\mathcal{O}_\alpha$ and, if $\Phi \in C^\omega(\mathcal{O}_\beta, \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0))$, then*

$$[(t, f_0) \mapsto f(t; f_0)] : \{(t, f_0) : f_0 \in \mathcal{O}_\alpha, t \in (0, T_+(f_0))\} \to \mathbb{E}_\alpha$$

is a real-analytic map too.

As usual, $[\cdot\,,\cdot]_\theta$ denotes the complex interpolation functor. We choose for our particular problem $\mathbb{E}_i \in \{H^s(\mathbb{R}) : 0 \le s \le 3\}$, $i = 1, 2$, and in this context we rely on the well-known interpolation property

$$[H^{s_0}(\mathbb{R}), H^{s_1}(\mathbb{R})]_\theta = H^{(1-\theta)s_0 + \theta s_1}(\mathbb{R}), \quad \theta \in (0, 1), \ -\infty < s_0 \le s_1 < \infty; \tag{1-4}$$

see, e.g., [Triebel 1978, Remark 2, Section 2.4.2].

The proof of Theorem 1.5 uses to a large extent the linear theory developed in [Amann 1995, Chapter II]. The main ideas of the proof of Theorem 1.5 can be found in [Amann 1986; 1988]. The uniqueness claim in Theorem 1.5 is slightly stronger compared to the result in [Amann 1993, Section 12] and it turns out to be quite useful when establishing the uniqueness in Theorems 1.1–1.2. For this reason we present in Appendix B the proof of Theorem 1.5.

In order to use Theorem 1.5 in the study of the Muskat problem (1-1), we have to write this evolution problem in the form (QP) and to establish then the property (1-3). With respect to this goal, we use the estimate provided in (1-2) and many techniques of nonlinear analysis.

2. The equations of motion and the equivalent formulation

We present the equations governing the dynamic of the fluids system and we prove, for a certain class of solutions, that the latter are equivalent to the system (1-1). The Muskat problem was originally proposed as a model for the encroachment of water into an oil sand, and therefore it is natural to assume that both fluids are incompressible, of Newtonian type, and immiscible. Since for flows in porous media the conservation of momentum equation can be replaced by Darcy's law, see, e.g., [Bear 1972], the equations governing the dynamic of the fluids are

$$\begin{cases} \operatorname{div} v_\pm(t) = 0 & \text{in } \Omega_\pm(t), \\ v_\pm(t) = -(k/\mu)(\nabla p_\pm(t) + (0, \rho_\pm g)) & \text{in } \Omega_\pm(t) \end{cases} \tag{2-1a}$$

for $t > 0$, where, using the subscript $\pm$ for the fluid located at $\Omega_\pm(t)$, we denote by $v_\pm(t) := (v^1_\pm(t), v^2_\pm(t))$ the velocity vector and $p_\pm(t)$ the pressure of the fluid $\pm$. These equations are supplemented by the natural boundary conditions at the free surface

$$\begin{cases} p_+(t) - p_-(t) = \sigma\kappa(f(t)) & \text{on } [y=f(t,x)], \\ \langle v_+(t) \mid \nu(t)\rangle = \langle v_-(t) \mid \nu(t)\rangle & \text{on } [y=f(t,x)], \end{cases} \tag{2-1b}$$

where $\nu(t)$ is the unit normal at $[y=f(t,x)]$ pointing into $\Omega_+(t)$ and $\langle\cdot\mid\cdot\rangle$ is the inner product on $\mathbb{R}^2$. Furthermore, the far-field boundary conditions

$$\begin{cases} f(t,x) \to 0 & \text{for } |x| \to \infty, \\ v_\pm(t,x,y) \to 0 & \text{for } |(x,y)| \to \infty \end{cases} \tag{2-1c}$$

state that the fluid motion is localized, the fluids being close to the rest state far away from the origin. The motion of the interface $[y=f(t,x)]$ is coupled to that of the fluids through the kinematic boundary condition

$$\partial_t f(t) = \langle v_\pm(t) \mid (-f'(t), 1)\rangle \quad \text{on } [y=f(t,x)]. \tag{2-1d}$$

Finally, the interface at time $t = 0$ is assumed to be known,

$$f(0) = f_0. \tag{2-1e}$$

The equations (2-1) are known as the Muskat problem and they determine completely the dynamic of the system. We now show that the Muskat problem (2-1) is equivalent to the system (1-1) presented in the Introduction. The proof uses classical results on Cauchy-type integrals defined on regular curves; see, e.g., [Lu 1993]. More precisely, we establish the following equivalence result.

Proposition 2.1 (equivalence of the two formulations). *Let $\sigma \geq 0$ and $T \in (0, \infty]$. The following are equivalent*:

(i) *The Muskat problem (2-1) for*[2]

$$f \in C^1((0,T), L_2(\mathbb{R})) \cap C([0,T), L_2(\mathbb{R})), \qquad f(t) \in H^5(\mathbb{R}) \quad \text{for all } t \in (0,T),$$
$$v_\pm(t) \in C(\overline{\Omega_\pm(t)}) \cap C^1(\Omega_\pm(t)), \quad p_\pm(t) \in C^1(\overline{\Omega_\pm(t)}) \cap C^2(\Omega_\pm(t)) \quad \text{for all } t \in (0,T).$$

(ii) *The evolution problem (1-1) for*

$$f \in C^1((0,T), L_2(\mathbb{R})) \cap C([0,T), L_2(\mathbb{R})), \qquad f(t) \in H^5(\mathbb{R}) \quad \text{for all } t \in (0,T).$$

Proof. We first establish the implication (i) $\Rightarrow$ (ii). Assuming that we are given a solution to (2-1) as in (i), we have to show that the first equation of (1-1) holds for each $t \in (0,T)$. Therefore, we fix $t \in (0,T)$ and we do not write in the arguments that follow the dependence of the physical variables of time t explicitly. In the following, $\mathbf{1}_E$ is the characteristic function of the set E. Introducing the global velocity field $v := (v^1, v^2) := v_- \mathbf{1}_{[y \leq f(x)]} + v_+ \mathbf{1}_{[y > f(x)]}$, Stokes' theorem together with (2-1a) and (2-1b) yields that the vorticity, which for two-dimensional flows corresponds to the scalar function $\omega := \partial_x v^2 - \partial_y v^1$, is supported on the free boundary, that is,

$$\langle \omega, \varphi \rangle = \int_{\mathbb{R}} \bar{\omega}(x) \varphi(x, f(x))\, dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^2),$$

where

$$\bar{\omega} := \frac{k}{\mu} [\sigma \kappa(f) - \Delta_\rho f]'.$$

We next prove that the velocity is defined by the Biot–Savart law, that is, $v = \tilde{v}$ in $\mathbb{R}^2 \setminus [y = f(x)]$, where

$$\tilde{v}(x,y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(-(y - f(s)), x - s)}{(x-s)^2 + (y - f(s))^2} \bar{\omega}(s)\, ds \quad \text{in } \mathbb{R}^2 \setminus [y = f(x)]. \tag{2-2}$$

To this end we compute the limits $\tilde{v}_-(x, f(x))$ and $\tilde{v}_+(x, f(x))$ of $\tilde{v}$ at $(x, f(x))$ when we approach this point from below the interface $[y = f(x)]$ or from above, respectively. Using the well-known Plemelj formula, see, e.g., [Lu 1993], due to the fact that $f \in H^4(\mathbb{R})$ and after changing variables, we find the expressions

$$\tilde{v}_\pm(x, f(x)) = \frac{1}{2\pi} \mathrm{PV}\!\int_{\mathbb{R}} \frac{(-(f(x) - f(x-s)), s)}{s^2 + (f(x) - f(x-s))^2} \bar{\omega}(x-s)\, ds \mp \frac{1}{2} \frac{(1, f'(x))\bar{\omega}(x)}{1 + f'^2(x)}, \quad x \in \mathbb{R}, \tag{2-3}$$

where the principal value needs to be taken only at 0. In view of Lemma A.2 and of $f \in H^5(\mathbb{R})$, the restrictions $\tilde{v}_\pm$ of $\tilde{v}$ to $\Omega_\pm$ satisfy $\tilde{v}_\pm \in C(\overline{\Omega}_\pm) \cap C^1(\Omega_\pm)$ and moreover $\tilde{v}_\pm$ vanish at infinity. Next, we define the pressures $\tilde{p}_\pm \in C^1(\overline{\Omega}_\pm) \cap C^2(\Omega_\pm)$ by the formula

$$\tilde{p}_\pm(x,y) := c_\pm - \frac{\mu}{k} \int_0^x \tilde{v}_\pm^1(s, \pm d)\, ds - \frac{\mu}{k} \int_{\pm d}^y \tilde{v}_\pm^2(x, s)\, ds - \rho_\pm g y, \quad (x,y) \in \overline{\Omega}_\pm, \tag{2-4}$$

[2]The regularity $f(t) \in H^5(\mathbb{R})$, $t \in (0,T)$, is not optimal; that is, the two formulations are still equivalent if $f(t) \in H^r(\mathbb{R})$, $t \in (0,T)$, for $r < 5$ suitably chosen. In fact, if $\sigma = 0$, we may take $r = 3$. However, as stated in Theorem 1.3, $f(t) \in H^\infty(\mathbb{R})$ for all $t \in (0,T)$, and there is no reason for seeking the optimal range for r.

where d is a positive constant satisfying $d > \|f\|_\infty$ and $c_\pm \in \mathbb{R}$. For a proper choice of the constants $c_\pm$, it is not difficult to see that the pair $(\tilde{p}_\pm, \tilde{v}_\pm)$ satisfies (2-1a)–(2-1c). Let $V_\pm := v_\pm - \tilde{v}_\pm$, $V := (V^1, V^2) := V_- \mathbf{1}_{[y \le f(x)]} + V_+ \mathbf{1}_{[y > f(x)]} \in C(\mathbb{R}^2)$, and

$$\psi_\pm(x, y) := \int_{f(x)}^{y} V_\pm^1(x, s)\, ds - \int_0^x \langle V_\pm(s, f(s)) | (-f'(s), 1) \rangle\, ds \quad \text{for } (x, y) \in \bar{\Omega}_\pm,$$

be the stream function associated to $V_\pm$. Recalling (2-1a)–(2-1c), we deduce that the function $\psi := \psi_- \mathbf{1}_{[y \le f]} + \psi_+ \mathbf{1}_{[y > f]}$ satisfies $\Delta \psi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence, ψ is the real part of a holomorphic function $u : \mathbb{C} \to \mathbb{C}$. Since u' is also holomorphic and $u' = \partial_x \psi - i \partial_y \psi = -(V^2, V^1)$ is bounded and vanishes for $|(x, y)| \to \infty$ it follows that $u' = 0$; hence $V = 0$. This proves that $v_\pm = \tilde{v}_\pm$.

We now infer from (2-1d) and (2-3) that the dynamic of the free boundary separating the fluids is described by the evolution equation

$$\begin{aligned}\partial_t f(t, x) = &\frac{k}{2\pi\mu} f'(t, x)\, \mathrm{PV}\int_{\mathbb{R}} \frac{f(t, x) - f(t, x - s)}{s^2 + (f(t, x) - f(t, x - s))^2} [\sigma\kappa(f) - \Delta_\rho f]'(t, x - s)\, ds \\ &+ \frac{k}{2\pi\mu}\, \mathrm{PV}\int_{\mathbb{R}} \frac{s}{s^2 + (f(t, x) - f(t, x - s))^2} [\sigma\kappa(f) - \Delta_\rho f]'(t, x - s)\, ds\end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}$. This equation can be further simplified by using the formula

$$\int_{\delta<|x|<1/\delta} \frac{\partial}{\partial s}\big(\ln(s^2 + (f(x) - f(x - s))^2)\big)\, ds = \ln \frac{1+\delta^2(f(x)-f(x-1/\delta))^2}{1+\delta^2(f(x)-f(x+1/\delta))^2} \frac{1+(f(x)-f(x+\delta))^2/\delta^2}{1+(f(x)-f(x-\delta))^2/\delta^2}$$

for $\delta \in (0, 1)$ and $x \in \mathbb{R}$. Letting $\delta \to 0$, we get

$$\begin{aligned}0 &= \tfrac{1}{2}\, \mathrm{PV}\int_{\mathbb{R}} \frac{\partial}{\partial s}\big(\ln(s^2 + (f(x) - f(x - s))^2)\big)\, ds \\ &= \mathrm{PV}\int_{\mathbb{R}} \frac{s}{s^2 + (f(t, x) - f(t, x - s))^2}\, ds + \mathrm{PV}\int_{\mathbb{R}} \frac{(f(t, x) - f(t, x - s) f'(t, x - s)}{s^2 + (f(t, x) - f(t, x - s))^2}\, ds,\end{aligned}$$

and now the principal value needs to be taken in the first integral at zero and at infinity. Using this identity, we have shown that the mapping $[t \to f(t)]$ satisfies the evolution problem (1-1).

The implication (ii) $\Rightarrow$ (i) is now obvious. □

3. The Muskat problem without surface tension: mapping properties

In Sections 3 and 4 we consider the stable case (a) mentioned on page 282. In this regime, after rescaling time, we may rewrite (1-1) in the abstract form

$$\dot{f} = \Phi(f)[f], \quad t > 0, \quad f(0) = f_0, \tag{3-1}$$

where $\Phi(f)$ is the linear operator formally defined by

$$\Phi(f)[h](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{y(h'(x) - h'(x - y))}{y^2 + (f(x) - f(x - y))^2}\, dy. \tag{3-2}$$

We show in the next two sections that the mapping Φ satisfies all the assumptions of Theorem 1.5 if we make the following choices: $\mathbb{E}_0 := H^1(\mathbb{R})$, $\mathbb{E}_1 := H^2(\mathbb{R})$, $\mathbb{E}_\alpha = H^s(\mathbb{R})$ with $s \in \left(\frac{3}{2}, 2\right)$, and $\mathcal{O}_\beta := H^{\bar{s}}(\mathbb{R})$ with $\bar{s} \in \left(\frac{3}{2}, s\right)$. The first goal is to prove that

$$\Phi \in C^{1-}(H^s(\mathbb{R}), \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))) \tag{3-3}$$

for each $s \in \left(\frac{3}{2}, 2\right)$. Because the property (3-3) holds for all $s \in \left(\frac{3}{2}, 2\right)$, the parameter $\bar{s}$ will appear only in the proof of Theorem 1.1, which we present at the end of Section 4.

For the sake of brevity we set

$$\delta_{[x,y]} f := f(x) - f(x-y) \quad \text{for } x, y \in \mathbb{R},$$

and therewith

$$\Phi(f)[h](x) = \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}h'/y}{1 + (\delta_{[x,y]}f/y)^2}\, dy.$$

Boundedness of some multilinear singular integral operators. We first consider some multilinear operators which are related to Φ.[3] The estimates in Lemmas 3.1 and 3.4 enable us in particular to establish the regularity property (3-3). Lemma 3.1 is reconsidered later on, see Lemma 5.3, in a particular context when showing that Φ is in fact real-analytic.

Lemma 3.1. *Given $n, m \in \mathbb{N}$, $r \in \left(\frac{3}{2}, 2\right)$, $a_1, \dots, a_{n+1}, b_1, \dots, b_m \in H^r(\mathbb{R})$, and a function $c \in L_2(\mathbb{R})$ we define*

$$A_{m,n}(a_1, \dots, a_{n+1})[b_1, \dots, b_m, c](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{\prod_{i=1}^{m}(\delta_{[x,y]}b_i/y)}{\prod_{i=1}^{n+1}[1 + (\delta_{[x,y]}a_i/y)^2]} \frac{\delta_{[x,y]}c}{y}\, dy.$$

Then:

(i) *There exists a constant C, depending only on r, n, m, and $\max_{i=1,\dots,n+1} \|a_i\|_{H^r}$, such that*

$$\|A_{m,n}(a_1, \dots, a_{n+1})[b_1, \dots, b_m, c]\|_2 \le C\|c\|_2 \prod_{i=1}^{m} \|b_i\|_{H^r} \tag{3-4}$$

for all $b_1, \dots, b_m \in H^r(\mathbb{R})$ and $c \in L_2(\mathbb{R})$.

(ii) $A_{m,n} \in C^{1-}\big((H^r(\mathbb{R}))^{n+1}, \mathcal{L}_{m+1}((H^r(\mathbb{R}))^m \times L_2(\mathbb{R}), L_2(\mathbb{R}))\big)$.

Remark 3.2. We note that

$$\Phi(f)[h] = A_{0,0}(f)[h'] \tag{3-5}$$

for all $f \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, and $h \in H^2(\mathbb{R})$, and

$$A_{0,0}(0)[c](x) = \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}c}{y}\, dy = -\,\mathrm{PV}\int_{\mathbb{R}} \frac{c(x-y)}{y}\, dy = -\pi Hc(x),$$

where H denotes the Hilbert transform [Stein 1993].

[3] As usual, the empty product is set to be equal to 1.

Remark 3.3. In the proof of Lemma 3.1 we split the operator $A_{m,n} := A_{m,n}(a_1, \dots, a_{n+1})$ into two operators

$$A_{m,n} = A^1_{m,n} - A^2_{m,n}.$$

If we keep $b_1, \dots, b_m$ fixed, then $A^1_{m,n}$ is a multiplication-type operator

$$A^1_{m,n}[b_1, \dots, b_m, c](x) := c(x)\,\mathrm{PV}\int_{\mathbb{R}} \frac{1}{y} \frac{\prod_{i=1}^m (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^{n+1}[1 + (\delta_{[x,y]} a_i/y)^2]}\,dy,$$

while $A^2_{m,n}$ is the singular integral operator

$$A^2_{m,n}[b_1, \dots, b_m, c](x) := \mathrm{PV}\int_{\mathbb{R}} K(x, y)c(x - y)\,dy,$$

with the kernel K defined by

$$K(x, y) := \frac{1}{y} \frac{\prod_{i=1}^m (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^{n+1}[1 + (\delta_{[x,y]} a_i/y)^2]} \quad \text{for } x \in \mathbb{R}, \quad y \neq 0.$$

Our proof shows that both operators $A^i_{m,n}$, $1 \le i \le 2$, satisfy (3-4). While the boundedness of $A^1_{m,n}$ follows by direct computation, the boundedness of $A^2_{m,n}$ follows from the estimate on the norm of operator defined in (1-2) and an argument due to Calderón as it appears in the proof of [Meyer and Coifman 1997, Theorem 9.7.11]. In fact, the arguments in the proof of Lemma 3.1 show that given Lipschitz functions $a_1, \dots, a_{n+m} : \mathbb{R} \to \mathbb{R}$, the singular integral operator

$$B_{n,m}(a_1, \dots, a_{n+m})[h](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{h(x-y)}{y} \frac{\prod_{i=1}^n (\delta_{[x,y]} a_i/y)}{\prod_{i=n+1}^{n+m}[1 + (\delta_{[x,y]} a_i/y)^2]}\,dy$$

belongs to $\mathcal{L}(L_2(\mathbb{R}))$ and $\|B_{n,m}(a_1, \dots, a_{n+m})\|_{\mathcal{L}(L_2(\mathbb{R}))} \le C \prod_{i=1}^n \|a_i'\|_\infty$, where C is a constant depending only on n, m and $\max_{i=n+1,\dots,n+m} \|a_i'\|_\infty$.

It is worth pointing out that $B_{0,0} = B_{0,1}(0) = \pi H$.

Proof of Lemma 3.1. The multilinear operator $A^1_{m,n}$ is bounded provided that the mapping

$$\left[x \mapsto \mathrm{PV}\int_{\mathbb{R}} \frac{1}{y} \frac{\prod_{i=1}^m (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^{n+1}[1 + (\delta_{[x,y]} a_i/y)^2]}\,dy\right]$$

belongs to $L_\infty(\mathbb{R})$. To establish this boundedness property we note that

$$\begin{aligned}\int_{\delta<|y|<1/\delta} \frac{1}{y} \frac{\prod_{i=1}^m (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^{n+1}[1+(\delta_{[x,y]} a_i/y)^2]}\,dy &= \int_\delta^{1/\delta} \frac{1}{y} \frac{\prod_{i=1}^m (\delta_{[x,y]} b_i/y)}{\prod_{i=1}^{n+1}[1+(\delta_{[x,y]} a_i/y)^2]} - \frac{1}{y} \frac{\prod_{i=1}^m (-\delta_{[x,-y]} b_i/y)}{\prod_{i=1}^{n+1}[1+(\delta_{[x,-y]} a_i/y)^2]}\,dy \\ &=: \int_\delta^{1/\delta} I(x, y)\,dy\end{aligned}$$

for $\delta\in(0,1)$ and $x\in\mathbb{R}$, where

$$I(x,y):=\frac{1}{y}\frac{1}{\prod_{i=1}^{n+1}[1+(\delta_{[x,y]}a_i/y)^2]}\biggl(\prod_{i=1}^{m}(\delta_{[x,y]}b_i/y)-\prod_{i=1}^{m}(-\delta_{[x,-y]}b_i/y)\biggr)$$
$$+\frac{1}{y}\biggl(\prod_{i=1}^{m}(-\delta_{[x,-y]}b_i/y)\biggr)\frac{\prod_{i=1}^{n+1}[1+(\delta_{[x,-y]}a_i/y)^2]-\prod_{i=1}^{n+1}[1+(\delta_{[x,y]}a_i/y)^2]}{\prod_{i=1}^{n+1}[1+(\delta_{[x,y]}a_i/y)^2][1+(\delta_{[x,-y]}a_i/y)^2]}.$$

We further have

$$\frac{1}{y}\biggl(\prod_{i=1}^{m}(\delta_{[x,y]}b_i/y)-\prod_{i=1}^{m}(-\delta_{[x,-y]}b_i/y)\biggr)$$
$$=\frac{1}{y}\prod_{i=1}^{m}\frac{b_i(x)-b_i(x-y)}{y}-\frac{1}{y}\prod_{i=1}^{m}\frac{b_i(x+y)-b_i(x)}{y}$$
$$=-\sum_{i=1}^{m}\frac{b_i(x+y)-2b_i(x)+b_i(x-y)}{y^2}\biggl[\prod_{j=1}^{i-1}(\delta_{[x,y]}b_j/y)\biggr]\biggl[\prod_{j=i+1}^{m}(-\delta_{[x,-y]}b_j/y)\biggr],$$

and similarly

$$\frac{1}{y}\biggl(\prod_{i=1}^{n+1}[1+(\delta_{[x,-y]}a_i/y)^2]-\prod_{i=1}^{n+1}[1+(\delta_{[x,y]}a_i/y)^2]\biggr)$$
$$=\sum_{i=1}^{n+1}\biggl[\prod_{j=1}^{i-1}[1+(\delta_{[x,-y]}a_j/y)^2]\biggr]\biggl[\prod_{j=i+1}^{n+1}[1+(\delta_{[x,y]}a_j/y)^2]\biggr]$$
$$\times\frac{a_i(x+y)-a_i(x-y)}{y}\,\frac{a_i(x+y)-2a_i(x)+a_i(x-y)}{y^2}.$$

Let us now observe that

$$|I(x,y)|\le\frac{2^{m+1}[1+4(n+1)\max_{i=1,\dots,n+1}\|a_i\|_\infty^2]}{y^2}\prod_{i=1}^{m}\|b_i\|_\infty\quad\text{for } x\in\mathbb{R},\quad y\ge 1.\tag{3-6}$$

Furthermore, since $r-\frac{1}{2}\in(1,2)$, we find, by taking advantage of $H^r(\mathbb{R})\hookrightarrow \mathrm{BC}^{r-1/2}(\mathbb{R})$, that

$$\frac{|f(x+y)-2f(x)+f(x-y)|}{y^{r-1/2}}\le 4[f']_{r-3/2}\le C\|f\|_{H^r}\quad\text{for all } f\in H^r(\mathbb{R}),\quad x\in\mathbb{R},\quad y>0;\tag{3-7}$$

see [Lunardi 1995, Relation (0.2.2)]. Here $[\,\cdot\,]_{r-3/2}$ denotes the usual Hölder seminorm. Using (3-7), it follows that

$$|I(x,y)|\le Cy^{r-5/2}\biggl[\sum_{i=1}^{m}\biggl(\|b_i\|_{H^r}\prod_{j=1,j\ne i}^{m}\|b_j'\|_\infty\biggr)+\biggl(\prod_{i=1}^{m}\|b_i'\|_\infty\biggr)\sum_{i=1}^{n+1}\|a_i'\|_\infty\|a_i\|_{H^r}\biggr]\tag{3-8}$$

for $x \in \mathbb{R}$, $y \in (0, 1)$. Combining (3-6) and (3-8) yields

$$\sup_{x\in\mathbb{R}} \int_0^\infty |I(x, y)|\,dy \le C \prod_{i=1}^{m} \|b_i\|_{H^r},$$

where C depends only on r, n, m, and $\max_{i=1,\dots,n+1} \|a_i\|_{H^r}$. The latter estimate shows that (3-4) is satisfied when $A_{m,n}$ is replaced by $A^1_{m,n}$.

To deal with $A^2_{m,n}$, we define the functions $F : \mathbb{R}^{n+m+1} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}^{n+m+1}$ by

$$F(u_1, \dots, u_{n+1}, v_1, \dots, v_m) = \frac{\prod_{i=1}^{m} v_i}{\prod_{i=1}^{n+1}(1 + u_i^2)} \quad \text{and} \quad A := (a_1, \dots, a_{n+1}, b_1, \dots, b_m),$$

where $b_i \in H^r(\mathbb{R})$ satisfy $\|b_i'\|_\infty \le 1$, $1 \le i \le m$. The function F is smooth and A is Lipschitz continuous with a Lipschitz constant $L := \sqrt{m + (n+1)\max_{i=1,n+1} \|a_i'\|^2_\infty} \ge \|A'\|_\infty$. We further observe that

$$K(x, y) = \frac{1}{y} F\left(\frac{\delta_{[x,y]}A}{y}\right),$$

with $|\delta_{[x,y]}A/y| \le L$. Let $\widetilde{F}$ be a smooth function on $\mathbb{R}^{n+m+1}$ which is $4L$-periodic in each variable and which matches F on $[-L, L]^{n+m+1}$. Expanding $\widetilde{F}$ by its Fourier series

$$\widetilde{F} = \sum_{p\in\mathbb{Z}^{n+m+1}} \alpha_p e^{i(\pi/2L)\langle p|\cdot\rangle},$$

the associated sequence $(\alpha_p)_p$ is rapidly decreasing. Furthermore, we can write the kernel K as

$$K(x, y) = \sum_{p\in\mathbb{Z}^{n+m+1}} \alpha_p K_p(x, y), \qquad x \in \mathbb{R}, \quad y \ne 0,$$

with

$$K_p(x, y) := \frac{1}{y} \exp\left(i \frac{\pi}{2L} \frac{\delta_{[x,y]}\langle p \mid A\rangle}{y}\right), \qquad x \in \mathbb{R}, \quad y \ne 0, \quad p \in \mathbb{Z}^{n+m+1}.$$

The kernels K_p, $p \in \mathbb{Z}^{n+m+1}$, define operators in $\mathcal{L}(L_2(\mathbb{R}))$ of the type (1-2) and with norms bounded by

$$C\left(1 + \frac{\pi}{2L}|p|\|A'\|_\infty\right) \le C(1 + |p|), \qquad p \in \mathbb{Z}^{n+m+1},$$

with a universal constant C independent of p. Hence, the associated series is absolutely convergent in $\mathcal{L}(L_2(\mathbb{R}))$, meaning that the operator $A^2_{m,n}(a_1, \dots, a_{n+1})[b_1, \dots, b_m, \cdot\,]$ belongs to $\mathcal{L}(L_2(\mathbb{R}))$ and

$$\|A^2_{m,n}(a_1, \dots, a_{n+1})[b_1, \dots, b_m, c]\|_2 \le C\big(n, m, \max_{i=1,\dots,n+1} \|a_i'\|_\infty\big)\|c\|_2$$

for all $c \in L_2(\mathbb{R})$ and for all $b_i \in H^r(\mathbb{R})$ that satisfy $\|b_i'\|_\infty \le 1$. The desired estimate (3-4) follows now by using the linearity of $A^2_{m,n}$ in each argument. The claim (i) is now obvious.

Concerning (ii), we note that

$$A_{m,n}(\tilde a_1,\dots,\tilde a_{n+1})[b_1,\dots,b_m,c]-A_{m,n}(a_1,\dots,a_{n+1})[b_1,\dots,b_m,c]$$
$$=\sum_{i=1}^{n+1}A_{m+2,n+1}(\tilde a_1,\dots,\tilde a_i,a_i,\dots,a_{n+1})[a_i+\tilde a_i,a_i-\tilde a_i,b_1,\dots,b_m,c],$$

and the desired assertion follows now from (i). □

We consider once more the operators $A_{m,n}$ defined in Lemma 3.5 in the case when $m\geq 1$, but defined on a different Hilbert-space product where a weaker regularity of b_m is balanced by a higher regularity of the variable c. The estimates in Lemma 3.4 are slightly related to the ones announced in [Calderon et al. 1978, Theorem 4] and, except for that reference, we did not find similar results.

Lemma 3.4. *Let $n\in\mathbb{N}$, $1\leq m\in\mathbb{N}$, $r\in\big(\frac{3}{2},2\big)$, $\tau\in\big(\frac{5}{2}-r,1\big)$, and $a_1,\dots,a_{n+1}\in H^r(\mathbb{R})$ be given. Then:*

(i) *There exists a constant C, depending only on r and τ, such that*

$$\|A_{m,n}(a_1,\dots,a_{n+1})[b_1,\dots,b_m,c]\|_2\leq C\|c\|_{H^\tau}\|b_m\|_{H^{r-1}}\prod_{i=1}^{m-1}\|b_i'\|_\infty$$

for all $b_1,\dots,b_m\in H^r(\mathbb{R})$ and all $c\in H^1(\mathbb{R})$. In particular, $A_{m,n}(a_1,\dots,a_{n+1})$ extends to a bounded operator

$$A_{m,n}(a_1,\dots,a_{n+1})\in\mathcal{L}_{m+1}((H^r(\mathbb{R}))^{m-1}\times H^{r-1}(\mathbb{R})\times H^\tau(\mathbb{R}),L_2(\mathbb{R})).$$

(ii) $A_{m,n}\in C^{1-}\big((H^r(\mathbb{R}))^{n+1},\mathcal{L}_{m+1}((H^r(\mathbb{R}))^{m-1}\times H^{r-1}(\mathbb{R})\times H^\tau(\mathbb{R}),L_2(\mathbb{R}))\big)$.

Proof. The claim (ii) is again a direct consequence of (i), so that we are left to prove the first claim. To this end we write

$$A_{m,n}(a_1,\dots,a_{n+1})[b_1,\dots,b_m,c](x)=\int_{\mathbb{R}}K(x,y)\,dy,$$

where

$$K(x,y):=\frac{\prod_{i=1}^{m-1}(\delta_{[x,y]}b_i/y)}{\prod_{i=1}^{n+1}[1+(\delta_{[x,y]}a_i/y)^2]}\,\frac{\delta_{[x,y]}b_m}{y}\,\frac{\delta_{[x,y]}c}{y}\quad\text{for } x\in\mathbb{R},\ \ y\neq 0.$$

Using Minkowski's integral inequality, we compute

$$\bigg(\int_{\mathbb{R}}\bigg|\int_{\mathbb{R}}K(x,y)\,dy\bigg|^2dx\bigg)^{1/2}\leq\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}|K(x,y)|^2\,dx\bigg)^{1/2}dy,$$

and exploiting the fact $H^{r-1}(\mathbb{R})\hookrightarrow \mathrm{BC}^{r-3/2}(\mathbb{R})$, we get

$$\int_{\mathbb{R}}|K(x,y)|^2\,dx\leq\frac{C}{y^{7-2r}}\|b_m\|_{H^{r-1}}^2\bigg(\prod_{i=1}^{m-1}\|b_i'\|_\infty^2\bigg)\int_{\mathbb{R}}|c-\tau_yc|^2\,dx$$
$$=\frac{C}{y^{7-2r}}\|b_m\|_{H^{r-1}}^2\bigg(\prod_{i=1}^{m-1}\|b_i'\|_\infty^2\bigg)\int_{\mathbb{R}}|\mathcal{F}c(\xi)|^2|e^{iy\xi}-1|^2\,d\xi.$$

Since

$$|e^{iy\xi}-1|^2 \le C[(1+|\xi|^2)^\tau y^{2\tau}\mathbf{1}_{(-1,1)}(y)+\mathbf{1}_{[|y|\ge 1]}(y)], \qquad y,\xi\in\mathbb{R},$$

it follows that

$$\int_{\mathbb{R}} |K(x,y)|^2\,dx \le C\|c\|_{H^\tau}^2\|b_m\|_{H^{r-1}}^2\Bigl(\prod_{i=1}^{m-1}\|b_i'\|_\infty^2\Bigr)\Bigl[y^{2(r+\tau)-7}\mathbf{1}_{(-1,1)}(y)+\frac{1}{y^{7-2r}}\mathbf{1}_{[|y|\ge 1]}(y)\Bigr],$$

and we conclude that

$$\int_{\mathbb{R}}\Bigl(\int_{\mathbb{R}}|K(x,y)|^2\,dx\Bigr)^{1/2}dy \le C\|c\|_{H^\tau}\|b_m\|_{H^{r-1}}\prod_{i=1}^{m-1}\|b_i'\|_\infty.$$

The claim (i) follows at once. □

Mapping properties. We now use Lemmas 3.1 and 3.4 to prove that the mapping Φ defined by (3-2) is well-defined and locally Lipschitz continuous as an operator from $H^s(\mathbb{R})$ into the Banach space $\mathcal{L}(H^2(\mathbb{R}),H^1(\mathbb{R}))$ for each $s\in\bigl(\frac{3}{2},2\bigr)$.

Lemma 3.5. *Given $s\in\bigl(\frac{3}{2},2\bigr)$, it holds that*

$$\Phi\in C^{1-}(H^s(\mathbb{R}),\mathcal{L}(H^2(\mathbb{R}),H^1(\mathbb{R}))).$$

Proof. We first prove that $\Phi(f)\in\mathcal{L}(H^2(\mathbb{R}),H^1(\mathbb{R}))$ for each $f\in H^s(\mathbb{R})$. Remark 3.2 and Lemma 3.1 (with $r=s$) yield that $\Phi(f)\in\mathcal{L}(H^2(\mathbb{R}),L_2(\mathbb{R}))$. In order to establish that $\Phi(f)[h]\in H^1(\mathbb{R})$, we let $\{\tau_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ denote the C_0-group of right translations on $L_2(\mathbb{R})$, that is, $\tau_\varepsilon f(x):=f(x-\varepsilon)$ for $f\in L_2(\mathbb{R})$ and $x,\varepsilon\in\mathbb{R}$. Given $\varepsilon\in(0,1)$, it holds that

$$\begin{aligned}\frac{\tau_\varepsilon(\Phi(f)[h])-\Phi(f)[h]}{\varepsilon}&=\frac{\tau_\varepsilon(A_{0,0}[f][h'])-A_{0,0}(f)[h']}{\varepsilon}=\frac{A_{0,0}(\tau_\varepsilon f)[\tau_\varepsilon h']-A_{0,0}(f)[h']}{\varepsilon}\\&=A_{0,0}(\tau_\varepsilon f)\Bigl[\frac{\tau_\varepsilon h'-h'}{\varepsilon}\Bigr]-A_{2,1}(\tau_\varepsilon f,f)\Bigl[\tau_\varepsilon f+f,\frac{\tau_\varepsilon f-f}{\varepsilon},h'\Bigr]\end{aligned}$$

and the convergences

$$\tau_\varepsilon f\xrightarrow[\varepsilon\to 0]{} f \quad\text{in } H^s(\mathbb{R}),\qquad \frac{\tau_\varepsilon f-f}{\varepsilon}\xrightarrow[\varepsilon\to 0]{}-f'\quad\text{in } H^{s-1}(\mathbb{R}),\qquad \frac{\tau_\varepsilon h-h}{\varepsilon}\xrightarrow[\varepsilon\to 0]{}-h'\quad\text{in } H^1(\mathbb{R}),$$

together with Lemma 3.1 (with $r=s$) and Lemma 3.4 $\bigl($with $r=s$, $\tau\in\bigl(\frac{5}{2}-s,1\bigr)\bigr)$ imply that $\Phi(f)[h]\in H^1(\mathbb{R})$ and

$$(\Phi(f)[h])'=A_{0,0}(f)[h'']-2A_{2,1}(f,f)[f,f',h']. \tag{3-9}$$

This proves that $\Phi(f)\in\mathcal{L}(H^2(\mathbb{R}),H^1(\mathbb{R}))$. Finally, the local Lipschitz continuity of Φ follows from the local Lipschitz continuity properties established in Lemmas 3.1 and 3.4. □

4. The Muskat problem without surface tension: the generator property

We now fix $f \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$. The goal of this section is to prove that $\Phi(f)$, regarded as an unbounded operator in $H^1(\mathbb{R})$ with definition domain $H^2(\mathbb{R})$, is the generator of a strongly continuous and analytic semigroup in $\mathcal{L}(H^1(\mathbb{R}))$, that is,

$$-\Phi(f) \in \mathcal{H}(H^2(\mathbb{R}), H^1(\mathbb{R})).$$

In order to establish this property we first approximate locally the operator $\Phi(f)$, in a sense to be made precise in Theorem 4.2, by Fourier multipliers and carry then the desired generator property, which we establish for the Fourier multipliers, back to the original operator, see Theorem 4.4. A similar approach was followed in [Escher 1994; Escher et al. 2018; Escher and Simonett 1995; 1997] in the context of spaces of continuous functions. The situation here is different as we consider Sobolev spaces on the line. The method though can be adapted to this setting after exploiting the structure of the operator $\Phi(f)$, especially the fact that the functions f and f' both vanish at infinity. As a result of this decay property we can use localization families with a finite number of elements, and this fact enables us to introduce for each localization family an equivalent norm on the Sobolev spaces $H^k(\mathbb{R})$, $k \in \mathbb{N}$, which is suitable for the further analysis, see Lemma 4.1. We start by choosing for each $\varepsilon \in (0, 1)$, a finite ε-localization family, that is, a family

$$\{\pi_j^\varepsilon : -N+1 \le j \le N\} \subset C^\infty(\mathbb{R}, [0, 1]),$$

with $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large, such that

- $\operatorname{supp} \pi_j^\varepsilon$ is an interval of length less or equal to ε for all $|j| \le N-1$; (4-1)
- $\operatorname{supp} \pi_N^\varepsilon \subset (-\infty, -x_N] \cup [x_N, \infty)$ and $x_N \ge \varepsilon^{-1}$; (4-2)
- $\operatorname{supp} \pi_j^\varepsilon \cap \operatorname{supp} \pi_l^\varepsilon = \emptyset$ if $[|j-l| \ge 2, \max\{|j|, |l|\} \le N-1]$ or $[|l| \le N-2, j = N]$; (4-3)
- $\sum_{j=-N+1}^{N} (\pi_j^\varepsilon)^2 = 1$; (4-4)
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \le C\varepsilon^{-k}$ for all $k \in \mathbb{N}$, $-N+1 \le j \le N$. (4-5)

Such ε-localization families can be easily constructed. Additionally, we choose for each $\varepsilon \in (0, 1)$ a second family

$$\{\chi_j^\varepsilon : -N+1 \le j \le N\} \subset C^\infty(\mathbb{R}, [0, 1])$$

with the properties

- $\chi_j^\varepsilon = 1$ on $\operatorname{supp} \pi_j^\varepsilon$; (4-6)
- $\operatorname{supp} \chi_j^\varepsilon$ is an interval of length less or equal to 3ε for $|j| \le N-1$; (4-7)
- $\operatorname{supp} \chi_N^\varepsilon \subset [|x| \ge x_N - \varepsilon]$. (4-8)

Each ε-localization family $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ defines a norm on $H^k(\mathbb{R})$, $k \in \mathbb{N}$, which is equivalent to the standard H^k-norm.

Lemma 4.1. *Given $\varepsilon \in (0, 1)$, let $\{\pi_j^\varepsilon : -N+1 \le j \le N\} \subset C^\infty(\mathbb{R}, [0, 1])$ be a family with the properties (4-1)–(4-5). Then, for each $k \in \mathbb{N}$, the mapping*

$$\left[h \mapsto \sum_{j=-N+1}^{N} \|\pi_j^\varepsilon h\|_{H^k}\right] : H^k(\mathbb{R}) \to [0, \infty)$$

defines a norm on $H^k(\mathbb{R})$ which is equivalent to the standard H^k-norm.

Proof. The proof is a simple exercise. □

We now consider the mapping

$$[\tau \mapsto \Phi(\tau f)] : [0, 1] \to \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R})).$$

As a consequence of Lemma 3.5, this mapping continuously transforms the operator $\Phi(f)$, for which we want to establish the generator property, into the operator $\Phi(0) = -\pi(-\partial_x^2)^{1/2}$. Indeed, since the Hilbert transform is a Fourier multiplier with symbol $[\xi \mapsto -i\,\mathrm{sign}(\xi)]$, we obtain together with Remark 3.2 that

$$\mathcal{F}(\Phi(0)[h])(\xi) = -\pi\mathcal{F}(Hh')(\xi) = i\pi\,\mathrm{sign}(\xi)\mathcal{F}(h')(\xi) = -\pi|\xi|(\mathcal{F}h)(\xi) = -\pi\mathcal{F}((-\partial_x^2)^{1/2}h)(\xi)$$

for $\xi \in \mathbb{R}$. The parameter τ will allow us to use a continuity argument when showing that the resolvent set of $\Phi(f)$ contains a positive real number; see the proof Theorem 4.4.

Our next goal is to prove that the operator $\Phi(\tau f)$ can be locally approximated for each $\tau \in [0, 1]$ by Fourier multipliers, as stated below. The estimate (4-9) with $j = N$ uses to a large extent the fact that f and f' vanish at infinity.

Theorem 4.2. *Let $f \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, and $\mu > 0$ be given.*

Then, there exist $\varepsilon \in (0, 1)$, a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ satisfying (4-1)–(4-5), a constant $K = K(\varepsilon)$, and for each $j \in \{-N+1, \dots, N\}$ and $\tau \in [0, 1]$ there exist operators

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$$

such that

$$\|\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_{H^1} \le \mu\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^{(11-2s)/4}} \tag{4-9}$$

for all $j \in \{-N+1, \dots, N\}$, $\tau \in [0, 1]$, and $h \in H^2(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := \left[\mathrm{PV}\int_{\mathbb{R}} \frac{1}{y}\frac{1}{1+\tau^2(\delta_{[x_j^\varepsilon, y]}f/y)^2}\,dy\right]\partial_x - \frac{\pi}{1+(\tau f'(x_j^\varepsilon))^2}(-\partial_x^2)^{1/2}, \quad |j| \le N-1, \tag{4-10}$$

where x_j^ε is a point belonging to $\operatorname{supp}\pi_j^\varepsilon$, and

$$\mathbb{A}_{N,\tau} := -\pi(-\partial_x^2)^{1/2}. \tag{4-11}$$

Proof. Let $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ be an ε-localization family satisfying the properties (4-1)–(4-5) and $\{\chi_j^\varepsilon : -N+1 \le j \le N\}$ be an associated family satisfying (4-6)–(4-8), with $\varepsilon \in (0, 1)$ which will be fixed below. We first infer from Lemma A.1 that for each $\tau \in [0, 1]$ the function

$$a_\tau(x) := \mathrm{PV}\int_{\mathbb{R}} \frac{1}{y}\frac{1}{1+\tau^2(\delta_{[x,y]}f/y)^2}\,dy, \qquad x \in \mathbb{R},$$

belongs to $\mathrm{BC}^\alpha(\mathbb{R}) \cap C_0(\mathbb{R})$, with $\alpha := \frac{1}{2}s - \frac{3}{4}$. We now write

$$\mathbb{A}_{j,\tau} := \mathbb{A}^1_{j,\tau} - \mathbb{A}^2_{j,\tau},$$

where

$$\mathbb{A}^1_{j,\tau} := \alpha_{j,\tau}\partial_x, \quad \mathbb{A}^2_{j,\tau} := \beta_{j,\tau}(-\partial_x^2)^{1/2},$$

and

$$\alpha_{j,\tau} := \begin{cases} a_\tau(x_j^\varepsilon), & |j| \le N-1, \\ 0, & j = N, \end{cases} \qquad \beta_{j,\tau} := \begin{cases} \dfrac{\pi}{1+(\tau f'(x_j^\varepsilon))^2}, & |j| \le N-1, \\ \pi, & j = N. \end{cases} \tag{4-12}$$

Let now $h \in H^2(\mathbb{R})$ be arbitrary. In the following we shall denote by C constants which are independent of ε (and, of course, of $h \in H^2(\mathbb{R})$, $\tau \in [0, 1]$, and $j \in \{-N+1, \dots, N\}$), while the constants that we denote by K may depend only upon ε.

Step 1: We first infer from Lemma 3.5 that

$$\begin{aligned} \|\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_{H^1} &\le \|\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2 + \|(\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h])'\|_2 \\ &\le (1+\|(\pi_j^\varepsilon)'\|_\infty)\|A_{0,0}(\tau f)[h']\|_2 + \|\mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2 \\ &\quad + 2\|A_{2,1}(\tau f, \tau f)[f, f', h']\|_2 + \|\pi_j^\varepsilon A_{0,0}(\tau f)[h''] - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon h)']\|_2. \end{aligned}$$

Using Lemma 3.4 $\big($with $r = s$ and $\tau = \frac{7}{4} - \frac{1}{2}s\big)$ and Lemma 3.1 (with $r = s$), it follows that

$$\|\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_{H^1} \le K\|h\|_{H^{(11-2s)/4}} + \|\pi_j^\varepsilon A_{0,0}(\tau f)[h''] - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon h)']\|_2. \tag{4-13}$$

We are left to estimate the L_2-norm of the highest-order term $\pi_j^\varepsilon A_{0,0}(\tau f)[h''] - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon h)']$, and for this we need several steps.

Step 2: With the notation introduced in Remark 3.3 we have

$$A_{0,0}(\tau f)[h''] = a_\tau h'' - B_{0,1}(\tau f)[h''],$$

and therewith

$$\begin{aligned} &\|\pi_j^\varepsilon A_{0,0}(\tau f)[h''] - \mathbb{A}_{j,\tau}[(\pi_j^\varepsilon h)']\|_2 \\ &\qquad \le \|\pi_j^\varepsilon a_\tau h'' - \mathbb{A}^1_{j,\tau}[(\pi_j^\varepsilon h)']\|_2 + \|\pi_j^\varepsilon B_{0,1}(\tau f)[h''] - \mathbb{A}^2_{j,\tau}[(\pi_j^\varepsilon h)']\|_2. \end{aligned} \tag{4-14}$$

By virtue of Lemma A.1, in particular of the estimate (A-1), and of $\chi_j^\varepsilon = 1$ on $\operatorname{supp}\pi_j^\varepsilon$, we get for $|j| \le N-1$

$$\begin{aligned} \|\pi_j^\varepsilon a_\tau h'' - \mathbb{A}^1_{j,\tau}[(\pi_j^\varepsilon h)']\|_2 &= \|a_\tau \pi_j^\varepsilon h'' - a_\tau(x_j^\varepsilon)(\pi_j^\varepsilon h)''\|_2 \\ &\le \|(a_\tau - a_\tau(x_j^\varepsilon))(\pi_j^\varepsilon h)''\|_2 + K\|h\|_{H^1} \\ &= \|(a_\tau - a_\tau(x_j^\varepsilon))\chi_j^\varepsilon(\pi_j^\varepsilon h)''\|_2 + K\|h\|_{H^1} \\ &= \|(a_\tau - a_\tau(x_j^\varepsilon))\chi_j^\varepsilon\|_\infty \|(\pi_j^\varepsilon h)''\|_2 + K\|h\|_{H^1} \\ &\le \tfrac{1}{2}\mu\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^1}, \end{aligned} \tag{4-15}$$

provided that ε is sufficiently small. We have used here (and also later on without explicit mentioning) the fact that $|\operatorname{supp}\chi_j^\varepsilon|\le 3\varepsilon$. Since $\mathbb{A}^1_{N,\tau}=0$, we obtain from (A-2) for ε sufficiently small that

$$\begin{aligned}\|\pi_N^\varepsilon a_\tau h''-\mathbb{A}^1_{N,\tau}[(\pi_N^\varepsilon h)'\|_2=\|\pi_N^\varepsilon a_\tau h''\|_2&\le\|a_\tau\chi_N^\varepsilon\|_\infty\|(\pi_N^\varepsilon h)''\|_2+K\|h\|_{H^1}\\&\le\tfrac12\mu\|\pi_N^\varepsilon h\|_{H^2}+K\|h\|_{H^1}.\end{aligned}\tag{4-16}$$

Step 3: We are left with the term $\|\pi_j^\varepsilon B_{0,1}(\tau f)[h'']-\mathbb{A}^2_{j,\tau}[(\pi_j^\varepsilon h)']\|_2$, and we consider first the case $|j|\le N-1$ (see Step 4 for $j=N$). Observing that $\pi(-\partial_x^2)^{1/2}=B_{0,1}(0)\circ\partial_x$, it follows that

$$\pi_j^\varepsilon B_{0,1}(\tau f)[h'']-\mathbb{A}^2_{j,\tau}[(\pi_j^\varepsilon h)']=T_1[h]-T_2[h],$$

where

$$\begin{aligned}T_1[h]&:=\pi_j^\varepsilon B_{0,1}(\tau f)[h'']-\frac{1}{1+(\tau f'(x_j^\varepsilon))^2}B_{0,1}(0)[\pi_j^\varepsilon h''],\\T_2[h]&:=\frac{1}{1+(\tau f'(x_j^\varepsilon))^2}B_{0,1}(0)[(\pi_j^\varepsilon)''h+2(\pi_j^\varepsilon)'h'].\end{aligned}$$

Since by Remark 3.3

$$\|T_2[h]\|_2\le K\|h\|_{H^1},\tag{4-17}$$

we are left to estimate $T_1[h]$, which is further decomposed as

$$T_1[h]=T_{11}[h]-T_{12}[h],$$

with

$$\begin{aligned}T_{11}[h](x)&:=\mathrm{PV}\int_{\mathbb{R}}\Big[\frac{1}{1+\tau^2(\delta_{[x,y]}f/y)^2}-\frac{1}{1+(\tau f'(x_j^\varepsilon))^2}\Big]\frac{(\chi_j^\varepsilon\pi_j^\varepsilon h'')(x-y)}{y}\,dy,\\T_{12}[h](x)&:=\mathrm{PV}\int_{\mathbb{R}}\frac{\delta_{[x,y]}\pi_j^\varepsilon/y}{1+\tau^2(\delta_{[x,y]}f/y)^2}h''(x-y)\,dy.\end{aligned}$$

Integrating by parts, we obtain the relation

$$\begin{aligned}T_{12}[h]=B_{0,1}(\tau f)[(\pi_j^\varepsilon)'h']-B_{1,1}(\pi_j^\varepsilon,\tau f)[h']-2\tau^2B_{2,2}(\pi_j^\varepsilon,f,\tau f,\tau f)[f'h']&\\+2\tau^2B_{3,2}(\pi_j^\varepsilon,f,f,\tau f,\tau f)[h'],&\end{aligned}$$

and Remark 3.3 leads us to

$$\|T_{12}[h]\|_2\le K\|h\|_{H^1}.\tag{4-18}$$

In order to deal with the term $T_{11}[h]$ we let $F_j\in C(\mathbb{R})$ denote the Lipschitz function that satisfies

$$F_j=f\quad\text{on }\operatorname{supp}\chi_j^\varepsilon,\qquad F_j'=f'(x_j^\varepsilon)\quad\text{on }\mathbb{R}\setminus\operatorname{supp}\chi_j^\varepsilon,\tag{4-19}$$

and we observe that

$$\begin{aligned}T_{11}[h](x)&:=\tau^2\,\mathrm{PV}\int_{\mathbb{R}}\frac{[\delta_{[x,y]}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}}-f)/y][\delta_{[x,y]}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}}+f)/y]}{[1+\tau^2(\delta_{[x,y]}f/y)^2][1+(\tau f'(x_j^\varepsilon))^2]}\frac{(\chi_j^\varepsilon\pi_j^\varepsilon h'')(x-y)}{y}\,dy\\&=\frac{\tau^2}{1+(\tau f'(x_j^\varepsilon))^2}(T_{111}[h]-T_{112}[h])(x),\end{aligned}$$

where

$$T_{111}[h] := \chi_j^\varepsilon B_{2,1}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} - f,\ f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} + f,\ \tau f)[\pi_j^\varepsilon h''],$$

$$T_{112}[h](x) := \mathrm{PV}\int_{\mathbb{R}} \frac{[\delta_{[x,y]}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} - f)/y][\delta_{[x,y]}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} + f)/y](\delta_{[x,y]}\chi_j^\varepsilon/y)}{1+\tau^2(\delta_{[x,y]}f/y)^2}(\pi_j^\varepsilon h'')(x-y)\,dy.$$

Integrating by parts as in the case of $T_{12}[h]$, it follows from Remark 3.3 that

$$\|T_{112}[h]\|_2 \le K\|h\|_{H^1}. \tag{4-20}$$

On the other hand, (4-19), Remark 3.3 and the Hölder continuity of f' yield

$$\begin{aligned}
\|T_{111}[h]\|_2 &= \|\chi_j^\varepsilon B_{2,1}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} - f,\ f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} + f,\ \tau f)[\pi_j^\varepsilon h'']\|_2 \\
&= \|\chi_j^\varepsilon B_{2,1}(f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} - F_j,\ f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}} + F_j,\ \tau f)[\pi_j^\varepsilon h'']\|_2 \\
&\le C\|f'(x_j^\varepsilon) - F_j'\|_\infty \|\pi_j^\varepsilon h''\|_2 \\
&= C\|f'(x_j^\varepsilon) - f'\|_{L_\infty(\operatorname{supp}\chi_j^\varepsilon)} \|\pi_j^\varepsilon h''\|_2 \\
&\le \tfrac{1}{2}\mu\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^1}.
\end{aligned} \tag{4-21}$$

The desired estimate (4-9) follows for $|j| \le N-1$ from (4-13)–(4-15) and (4-17), (4-18), (4-20), and (4-21).

Step 4: We are left with the term $\|\pi_N^\varepsilon B_{0,1}(\tau f)[h''] - \mathbb{A}_{N,\tau}^2[(\pi_N^\varepsilon h)']\|_2$, which we decompose as

$$\begin{aligned}
(\pi_N^\varepsilon B_{0,1}(\tau f)[h''] &- \mathbb{A}_{N,\tau}^2[(\pi_N^\varepsilon h)'])(x) \\
&= \pi_N^\varepsilon(x)\,\mathrm{PV}\int_{\mathbb{R}} \frac{h''(x-y)}{y}\frac{1}{1+\tau^2(\delta_{[x,y]}f/y)^2}\,dy - \mathrm{PV}\int_{\mathbb{R}} \frac{(\pi_N^\varepsilon h)''(x-y)}{y}\,dy \\
&=: T_1[h](x) + T_2[h](x) - T_3[h](x),
\end{aligned}$$

where

$$\begin{aligned}
T_1[h] &:= -\tau^2 B_{2,1}(f, f, \tau f)[\pi_N^\varepsilon h''], \\
T_2[h](x) &:= \mathrm{PV}\int_{\mathbb{R}} h''(x-y)\frac{\delta_{[x,y]}\pi_N^\varepsilon}{y}\frac{1}{1+\tau^2(\delta_{[x,y]}f/y)^2}\,dy, \\
T_3[h] &:= B_{0,1}(0)[(\pi_N^\varepsilon)''h + 2(\pi_N^\varepsilon)'h'].
\end{aligned}$$

For the difference $T_2[h] - T_3[h]$ we find, as in the previous step (see (4-17) and (4-18)), that

$$\|T_2[h] - T_3[h]\|_2 \le K\|h\|_{H^1}. \tag{4-22}$$

When dealing with $T_1[h]$, we introduce the function $F_N \in W_\infty^1(\mathbb{R})$ by the formula

$$F_N(x) := \begin{cases} f(x), & |x| \ge x_N - \varepsilon, \\ \dfrac{x + x_N - \varepsilon}{2(x_N-\varepsilon)} f(x_N - \varepsilon) + \dfrac{x_N - \varepsilon - x}{2(x_N-\varepsilon)} f(-x_N+\varepsilon), & |x| \le x_N - \varepsilon. \end{cases}$$

The relation (4-2) implies $\|F_N\|_\infty + \|F_N'\|_\infty \to 0$ for $\varepsilon \to 0$. Moreover, it holds that

$$T_1[h](x) = -\tau^2\,\mathrm{PV}\int_{\mathbb{R}} \frac{(\chi_N^\varepsilon \pi_N^\varepsilon h'')(x-y)}{y}\frac{(\delta_{[x,y]}f/y)^2}{1+\tau^2(\delta_{[x,y]}f/y)^2} =: T_{11}[h](x) - T_{12}[h](x),$$

where

$$T_{11}[h](x) := \tau^2\,\mathrm{PV}\int_{\mathbb{R}} (\pi_N^\varepsilon h'')(x-y)\frac{(\delta_{[x,y]}f/y)^2(\delta_{[x,y]}\chi_N^\varepsilon/y)}{1+\tau^2(\delta_{[x,y]}f/y)^2}\,dy,$$
$$T_{12}[h] := \tau^2\chi_N^\varepsilon B_{2,1}(f,f,\tau f)[\pi_N^\varepsilon h''].$$

Recalling that $\operatorname{supp}\pi_N^\varepsilon \subset \operatorname{supp}\chi_N^\varepsilon \subset [|x| \ge x_N - \varepsilon]$ and that $f = F_N$ on $\operatorname{supp}\chi_N^\varepsilon$, it follows by Remark 3.3 that

$$\begin{aligned}\|T_{12}[h]\|_2 &= \|\tau^2\chi_N^\varepsilon B_{2,1}(F_N,F_N,\tau f)[\pi_N^\varepsilon h'']\|_2 \le \|B_{2,1}(F_N,F_N,\tau f)[\pi_N^\varepsilon h'']\|_2 \le C\|F_N'\|_\infty^2\|\pi_N^\varepsilon h''\|_2\\ &\le \tfrac{1}{2}\mu\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^1}\end{aligned} \tag{4-23}$$

for small ε. As $T_{11}[h]$ can be estimated in the same manner as the term $T_{112}[h]$ in the previous step, we obtain together with (4-22) and (4-23) that

$$\|\pi_N^\varepsilon B_{0,1}(\tau f)[h''] - \mathbb{A}_{N,\tau}^2[(\pi_N^\varepsilon h)']\|_2 \le \tfrac{1}{2}\mu\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^1} \tag{4-24}$$

if ε is sufficiently small. The claim (4-9) follows for $j = N$ from (4-13)–(4-14), (4-16), and (4-24). □

The operators $\mathbb{A}_{\tau,j}$ found in Theorem 4.2 are generators of strongly continuous analytic semigroups in $\mathcal{L}(H^1(\mathbb{R}))$ and they satisfy resolvent estimates which are uniform with respect to $x_j^\varepsilon \in \mathbb{R}$ and $\tau \in [0,1]$; see Proposition 4.3 below. To be more precise, in Proposition 4.3 and in the proof of Theorem 4.4, the Sobolev spaces $H^k(\mathbb{R})$, $k \in \{1,2\}$, consist of complex-valued functions and $\mathbb{A}_{j,\tau}$ are the natural extensions (complexifications) of the operators introduced in Theorem 4.2.

Proposition 4.3. *Let $f \in H^s(\mathbb{R})$, $s \in \left(\tfrac{3}{2}, 2\right)$, be fixed. Given $x_0 \in \mathbb{R}$ and $\tau \in [0,1]$, let*

$$\mathbb{A}_{x_0,\tau} := \alpha_\tau\partial_x - \beta_\tau(-\partial_x^2)^{1/2},$$

where

$$\alpha_\tau \in \{0, a_\tau(x_0)\} \quad \text{and} \quad \beta_\tau \in \left\{\pi, \frac{\pi}{1+(\tau f'(x_0))^2}\right\},$$

with a_τ denoting the function defined in Lemma A.1. Then, there exists a constant $\kappa_0 \ge 1$ such that

$$\lambda - \mathbb{A}_{x_0,\tau} \in \mathrm{Isom}(H^2(\mathbb{R}), H^1(\mathbb{R})), \tag{4-25}$$
$$\kappa_0\|(\lambda - \mathbb{A}_{x_0,\tau})[h]\|_{H^1} \ge |\lambda|\cdot\|h\|_{H^1} + \|h\|_{H^2} \tag{4-26}$$

for all $x_0 \in \mathbb{R}$, $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^2(\mathbb{R})$.

Proof. The constants α_τ, β_τ defined above satisfy, in view of (A-3),

$$|\alpha_\tau| \le 4\left(\|f\|_\infty^2 + \frac{2\|f'\|_\infty[f']_{s-3/2}}{s-\frac{3}{2}}\right) \quad \text{and} \quad \beta_\tau \in \left[\frac{\pi}{1+\max|f'|^2}, \pi\right]. \tag{4-27}$$

Furthermore, the operator $\mathbb{A}_{x_0,\tau}$ is a Fourier multiplier with symbol

$$m_\tau(\xi) := -\beta_\tau|\xi| + i\alpha_\tau\xi, \qquad \xi \in \mathbb{R}.$$

Given $\operatorname{Re}\lambda \geq 1$, it is easy to see that the operator $R(\lambda, \mathbb{A}_{x_0,\tau})$ defined by

$$\mathcal{F}(R(\lambda, \mathbb{A}_{x_0,\tau})[h]) = \frac{1}{\lambda - m_\tau}\mathcal{F}h, \qquad h \in H^1(\mathbb{R}),$$

belongs to $\mathcal{L}(H^1(\mathbb{R}), H^2(\mathbb{R}))$ and that it is the inverse of $\lambda - \mathbb{A}_{x_0,\tau}$. Moreover, for each $\operatorname{Re}\lambda \geq 1$ and $h \in H^2(\mathbb{R})$, we have

$$\begin{aligned}\|(\lambda - \mathbb{A}_{x_0,\tau})[h]\|_{H^1}^2 &= \int_{\mathbb{R}} (1+|\xi|^2)|\mathcal{F}((\lambda - \mathbb{A}_{x_0,\tau})[h])|^2(\xi)\,d\xi = \int_{\mathbb{R}} (1+|\xi|^2)|\lambda - m_\tau(\xi)|^2|\mathcal{F}h|^2(\xi)\,d\xi \\ &\geq \min\{1, \beta_\tau^2\}\int_{\mathbb{R}} (1+|\xi|^2)^2|\mathcal{F}h|^2(\xi)\,d\xi = \min\{1, \beta_\tau^2\}\|h\|_{H^2}^2. \end{aligned} \tag{4-28}$$

Appealing to the inequality

$$\begin{aligned}\frac{|\lambda|^2}{|\lambda - m_\tau(\xi)|^2} &= \frac{(\operatorname{Re}\lambda)^2}{(\operatorname{Re}\lambda + \beta_\tau|\xi|)^2 + (\operatorname{Im}\lambda - \alpha_\tau\xi)^2} + \frac{(\operatorname{Im}\lambda)^2}{(\operatorname{Re}\lambda + \beta_\tau|\xi|)^2 + (\operatorname{Im}\lambda - \alpha_\tau\xi)^2} \\ &\leq 1 + \frac{2(\operatorname{Im}\lambda - \alpha_\tau\xi)^2 + 2\alpha_\tau^2\xi^2}{(\operatorname{Re}\lambda + \beta_\tau|\xi|)^2 + (\operatorname{Im}\lambda - \alpha_\tau\xi)^2} \leq 1 + 2\left[1 + \left(\frac{\alpha_\tau}{\beta_\tau}\right)^2\right] \leq 3\left[1 + \left(\frac{\alpha_\tau}{\beta_\tau}\right)^2\right]\end{aligned}$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$, the estimate (4-26) follows from the relations (4-27) and (4-28). ☐

We now establish the desired generation result.

Theorem 4.4. *Let $f \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, be given. Then*

$$-\Phi(f) \in \mathcal{H}(H^2(\mathbb{R}), H^1(\mathbb{R})). \tag{4-29}$$

Proof. Let $\kappa_0 \geq 1$ be the constant determined in Proposition 4.3. Setting $\mu := \frac{1}{2}\kappa_0$, we deduce from Theorem 4.2 that there exists a constant $\varepsilon \in (0, 1)$, an ε-localization family $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$ that satisfies (4-1)–(4-5), a constant $K = K(\varepsilon)$, and for each $-N+1 \leq j \leq N$ and $\tau \in [0, 1]$ operators $\mathbb{A}_{j,\tau} \in \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))$ such that

$$\|\pi_j^\varepsilon \Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_{H^1} \leq \frac{1}{2\kappa_0}\|\pi_j^\varepsilon h\|_{H^2} + K\|h\|_{H^{(11-2s)/4}} \tag{4-30}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0, 1]$, and $h \in H^2(\mathbb{R})$. In view of Proposition 4.3, it holds that

$$\kappa_0\|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon h]\|_{H^1} \geq |\lambda| \cdot \|\pi_j^\varepsilon h\|_{H^1} + \|\pi_j^\varepsilon h\|_{H^2} \tag{4-31}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0, 1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$, and $h \in H^2(\mathbb{R})$. The relations (4-30)–(4-31) lead us to

$$\begin{aligned}\kappa_0\|\pi_j^\varepsilon(\lambda - \Phi(\tau f))[h]\|_{H^1} &\geq \kappa_0\|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon h]\|_{H^1} - \kappa_0\|\pi_j^\varepsilon\Phi(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_{H^1} \\ &\geq |\lambda| \cdot \|\pi_j^\varepsilon h\|_{H^1} + \tfrac{1}{2}\|\pi_j^\varepsilon h\|_{H^2} - \kappa_0 K\|h\|_{H^{(11-2s)/4}}\end{aligned}$$

for all $-N+1 \le j \le N$, $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^2(\mathbb{R})$. Summing up over $j \in \{-N+1, \dots, N\}$, we infer from Lemma 4.1 that there exists a constant $C \ge 1$ with the property that

$$C\|h\|_{H^{(11-2s)/4}} + C\|(\lambda - \Phi(\tau f))[h]\|_{H^1} \ge |\lambda| \cdot \|h\|_{H^1} + \|h\|_{H^2}$$

for all $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^2(\mathbb{R})$. Using (1-4) together with Young's inequality, we may find constants $\kappa \ge 1$ and $\omega > 0$ such that

$$\kappa\|(\lambda - \Phi(\tau f))[h]\|_{H^1} \ge |\lambda| \cdot \|h\|_{H^1} + \|h\|_{H^2} \tag{4-32}$$

for all $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge \omega$, and $h \in H^2(\mathbb{R})$. Furthermore, combining the property

$$(\omega - \Phi(\tau f))|_{\tau=0} = \omega - \Phi(0) = \omega + \pi(-\partial_x^2)^{1/2} \in \operatorname{Isom}(H^2(\mathbb{R}), H^1(\mathbb{R}))$$

with (4-32), the method of continuity, see, e.g., [Gilbarg and Trudinger 1998, Theorem 5.2], yields that

$$\omega - \Phi(f) \in \operatorname{Isom}(H^2(\mathbb{R}), H^1(\mathbb{R})). \tag{4-33}$$

The relations (4-32) (with $\tau = 1$), (4-33), and [Lunardi 1995, Corollary 2.1.3] lead us to the desired claim (4-29). $\square$

We are now in a position to prove the well-posedness result Theorem 1.1.

Proof of Theorem 1.1. Let $s \in \left(\frac{3}{2}, 2\right)$ and $\bar{s} \in \left(\frac{3}{2}, s\right)$ be given. Combining Lemma 3.5 and Theorem 4.4 yields

$$-\Phi \in C^{1-}(H^{\bar{s}}(\mathbb{R}), \mathcal{H}(H^2(\mathbb{R}), H^1(\mathbb{R}))).$$

Setting $\alpha := s-1$ and $\beta := \bar{s}-1$, we have $0 < \beta < \alpha < 1$ and (1-4) yields

$$H^{\bar{s}}(\mathbb{R}) = [H^1(\mathbb{R}), H^2(\mathbb{R})]_\beta \quad \text{and} \quad H^s(\mathbb{R}) = [H^1(\mathbb{R}), H^2(\mathbb{R})]_\alpha.$$

It follows now from Theorem 1.5 that (1-1), or equivalently (3-1), possesses a maximally defined solution

$$f := f(\cdot\,; f_0) \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^2(\mathbb{R})) \cap C^1((0, T_+(f_0)), H^1(\mathbb{R}))$$

with

$$f \in C^{s-\bar{s}}([0,T], H^{\bar{s}}(\mathbb{R})) \quad \text{for all } T < T_+(f_0).$$

Concerning uniqueness, we now show that any classical solution

$$\tilde{f} \in C([0,\widetilde{T}), H^s(\mathbb{R})) \cap C((0,\widetilde{T}), H^2(\mathbb{R})) \cap C^1((0,\widetilde{T})), H^1(\mathbb{R})), \quad \widetilde{T} \in (0,\infty],$$

satisfies

$$\tilde{f} \in C^\eta([0,T], H^{\bar{s}}(\mathbb{R})) \quad \text{for all } T \in (0,\widetilde{T}), \tag{4-34}$$

where $\eta := (s-\bar{s})/s \in (0, s-\bar{s})$. This proves then the uniqueness claim of Theorem 1.1. We pick thus $T \in (0,\widetilde{T})$ arbitrarily. Then it follows directly from Lemma 3.1(i) that

$$\sup_{(0,T]} \|\partial_t \tilde{f}\|_2 \le C;$$

hence $\tilde{f} \in \mathrm{BC}^1((0,T], L_2(\mathbb{R}))$. Since $\tilde{f} \in C([0,T], H^s(\mathbb{R}))$, we conclude form (1-4), the previous bound, and the mean value theorem, that

$$\|\tilde{f}(t)-\tilde{f}(s)\|_{H^{\bar{s}}} \le \|\tilde{f}(t)-\tilde{f}(s)\|_2^{1-\bar{s}/s}\|\tilde{f}(t)-\tilde{f}(s)\|_{H^s}^{\bar{s}/s} \le C|t-s|^{\eta}, \quad t,s\in[0,T],$$

which proves (4-34).

Assume now that $T_+(f_0)<\infty$ and

$$\sup_{[0,T_+(f_0))} \|f(t)\|_{H^s} < \infty.$$

Arguing as above, we find that

$$\|f(t)-f(s)\|_{H^{(s+\bar{s})/2}} \le C|t-s|^{(s-\bar{s})/2s}, \quad t,s\in[0,T_+(f_0)).$$

The criterion for global existence in Theorem 1.5 applied for $\alpha := (s+\bar{s}-2)/2$ and $\beta := \bar{s}-1$ implies that the solution can be continued on an interval $[0,\tau)$ with $\tau > T_+(f_0)$. Moreover, it holds that

$$f \in C^{(s-\bar{s})/2}([0,T], H^{\bar{s}}(\mathbb{R})) \quad \text{for all } T\in(0,\tau).$$

The uniqueness claim in Theorem 1.5 leads us to a contradiction. Hence our assumption was false and $T_+(f_0)=\infty$. □

5. Instantaneous real-analyticity

We now improve the regularity of the solutions found in Theorems 1.1 and 1.2. To this end we first show that the mapping Φ defined by (3-2) is actually real-analytic; see Proposition 5.1. As $[f \mapsto \Phi(f)]$ is not a Nemytskij-type operator, we cannot use classical results for such operators, as presented, e.g., in [Runst and Sickel 1996]. Instead, we directly estimate the rest of the associated Taylor series. We conclude the section with the proof of Theorem 1.3, which is obtained, via Proposition 5.1, from the real-analyticity property of the semiflow as stated in Theorem 1.5, applied in the context of a nonlinear evolution problem related to (1-1).

Proposition 5.1. *Given $s\in\left(\frac{3}{2},2\right)$, it holds that*

$$\Phi \in C^{\omega}(H^s(\mathbb{R}), \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))). \tag{5-1}$$

Proof. Let $\phi:\mathbb{R}\to\mathbb{R}$ be the map defined by $\phi(x) := (1+x^2)^{-1}$, $x\in\mathbb{R}$. Then, given $f_0\in H^s(\mathbb{R})$, it holds that

$$\Phi(f_0)[h](x) = \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}h'}{y}\phi\left(\frac{\delta_{[x,y]}f_0}{y}\right)dy, \quad h\in H^2(\mathbb{R}).$$

Given $n\in\mathbb{N}$, we let

$$\begin{aligned}\partial^n\Phi(f_0)[f_1,\dots,f_n][h](x) &:= \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}h'}{y}\left(\prod_{i=1}^n \frac{\delta_{[x,y]}f_i}{y}\right)\phi^{(n)}\left(\frac{\delta_{[x,y]}f_0}{y}\right)dy \\ &= \sum_{k=0,\, n+k\in2\mathbb{N}}^{n} a_k^n A_{n+k,n}(f_0,\dots,f_0)[\underbrace{f_0,\dots,f_0}_{k \text{ times}}, f_1,\dots,f_n,h'](x)\end{aligned}$$

for $f_i \in H^s(\mathbb{R})$, $1 \le i \le n$, $h \in H^2(\mathbb{R})$, and $x \in \mathbb{R}$, where $a_k^n, n \in \mathbb{N}, 0 \le k \le n$, are defined in Lemma 5.2. Arguing as in the proof of Lemma 3.5, it follows from Lemmas 3.1 and 3.4 that $\partial^n \Phi(f_0) \in \mathcal{L}^n_{\mathrm{sym}}(H^s(\mathbb{R}), \mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R})))$; that is, $\partial^n \Phi(f_0)$ is a bounded n-linear and symmetric operator.

Moreover, given $f_0, f \in H^s(\mathbb{R})$, $n \in \mathbb{N}^*$, and $h \in H^2(\mathbb{R})$, Fubini's theorem combined with Lebesgue's dominated convergence theorem and the continuity of the mapping

$$\left[\tau \mapsto \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[\,\cdot\,,y]}h'}{y}\left(\frac{\delta_{[\,\cdot\,,y]}f}{y}\right)^{n+1} \phi^{(n+1)}\left(\frac{\delta_{[\,\cdot\,,y]}(f_0+\tau f)}{y}\right)dy\right] : [0,1] \to H^1(\mathbb{R}),$$

yield that

$$\begin{aligned}
\Phi(f_0+f)[h](x) - \sum_{k=0}^{n} \frac{\partial^k \Phi(f_0)[f]^k[h](x)}{k!} \\
&= \mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}h'}{y}\left(\frac{\delta_{[x,y]}f}{y}\right)^{n+1} \int_0^1 \frac{(1-\tau)^n}{n!}\phi^{(n+1)}\left(\frac{\delta_{[x,y]}(f_0+\tau f)}{y}\right)d\tau\, dy \\
&= \int_0^1 \frac{(1-\tau)^n}{n!}\,\mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}h'}{y}\left(\frac{\delta_{[x,y]}f}{y}\right)^{n+1} \phi^{(n+1)}\left(\frac{\delta_{[x,y]}(f_0+\tau f)}{y}\right)dy\, d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\Phi(f_0+f)[h] - \sum_{k=0}^{n} \frac{\partial^k \Phi(f_0)[f]^k[h]}{k!}\right\|_{H^1} \\
&\le \frac{1}{n!}\max_{\tau\in[0,1]}\left\|\mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[\,\cdot\,,y]}h'}{y}\left(\frac{\delta_{[\,\cdot\,,y]}f}{y}\right)^{n+1} \phi^{(n+1)}(\delta_{[\,\cdot\,,y]}f_\tau/y)\,dy\right\|_{H^1},
\end{aligned} \tag{5-2}$$

where $f_\tau := f_0 + \tau f$, $0 \le \tau \le 1$. In order to estimate the right-hand side of (5-2) we note that

$$\begin{aligned}
&\left\|\mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[\,\cdot\,,y]}h'}{y}\left(\frac{\delta_{[\,\cdot\,,y]}f}{y}\right)^{n+1} \phi^{(n+1)}(\delta_{[\,\cdot\,,y]}f_\tau/y)\,dy\right\|_{H^1} \\
&\quad\le \sum_{k=0,\,n+k+1\in 2\mathbb{N}}^{n+1} |a_k^{n+1}|\,\big\|A_{n+k+1,n+1}(f_\tau,\dots,f_\tau)[\underbrace{f_\tau,\dots,f_\tau}_{k \text{ times}},\underbrace{f,\dots,f}_{n+1 \text{ times}},h']\big\|_{H^1} \\
&\quad\le \sum_{k=0,\,n+k+1\in 2\mathbb{N}}^{n+1} |a_k^{n+1}|\,\big\|A_{n+k+1,n+1}(f_\tau,\dots,f_\tau)[\underbrace{f_\tau,\dots,f_\tau}_{k \text{ times}},\underbrace{f,\dots,f}_{n+1 \text{ times}},h']\big\|_2 \\
&\qquad+ \sum_{k=0,\,n+k+1\in 2\mathbb{N}}^{n+1} |a_k^{n+1}|\,\big\|A_{n+k+1,n+1}(f_\tau,\dots,f_\tau)[\underbrace{f_\tau,\dots,f_\tau}_{k \text{ times}},\underbrace{f,\dots,f}_{n+1 \text{ times}},h'']\big\|_2 \\
&\qquad+k \sum_{k=0,\,n+k+1\in 2\mathbb{N}}^{n+1} |a_k^{n+1}|\,\big\|A_{n+k+1,n+1}(f_\tau,\dots,f_\tau)[\underbrace{f_\tau,\dots,f_\tau}_{k-1 \text{ times}},\underbrace{f,\dots,f}_{n+1 \text{ times}},f_\tau',h']\big\|_2 \\
&\qquad-2(n+2) \sum_{k=0,\,n+k+1\in 2\mathbb{N}}^{n+1} |a_k^{n+1}|\,\big\|A_{n+k+3,n+2}(f_\tau,\dots,f_\tau)[\underbrace{f_\tau,\dots,f_\tau}_{k+1 \text{ times}},\underbrace{f,\dots,f}_{n+1 \text{ times}},f_\tau',h']\big\|_2.
\end{aligned} \tag{5-3}$$

Combining the results of Lemmas 5.2–5.4, we conclude that there exists an integer $p > 0$ and a positive constant C (depending only on $\|f_0\|_{H^s}$) such that for all $f \in H^s(\mathbb{R})$ with $\|f\|_{H^s} \le 1$ and all $n \ge 3$ we have

$$\left\| \Phi(f_0 + f) - \sum_{k=0}^{n} \frac{\partial^k \Phi(f_0)[f]^k}{k!} \right\|_{\mathcal{L}(H^2(\mathbb{R}), H^1(\mathbb{R}))} \le C^{n+1} n^p \|f\|_{H^s}^{n+1}.$$

The claim follows. □

The following technical results are used in the proof of Proposition 5.1.

Lemma 5.2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) := (1 + x^2)^{-1}$, $x \in \mathbb{R}$. Given $n \in \mathbb{N}$, it holds that*

$$\phi^{(n)}(x) = \frac{1}{(1 + x^2)^{n+1}} \sum_{k=0}^{n} a_k^n x^k,$$

where the coefficients $a_k^n \in \mathbb{R}$ satisfy $|a_k^n| \le 4^n (n + 2)!$ for all $0 \le k \le n$. Moreover, $a_k^n = 0$ if $n + k \notin 2\mathbb{N}$.

Proof. The claim for $n \in \{0, 1, 2, 3\}$ is obvious. Assume that the claim holds for some integer $n \ge 3$. Since

$$(1 + x^2)^{n+2} \phi^{(n+1)}(x) = (1 + x^2) \sum_{k=1}^{n} k a_k^n x^{k-1} - 2(n + 1)x \sum_{k=0}^{n} a_k^n x^k,$$

the coefficient a_k^{n+1}, $0 \le k \le n + 1$, of x^k satisfies

$$\begin{aligned}
|a_{n+1}^{n+1}| &\le n|a_n^n| + 2(n + 1)|a_n^n| \le 4(n + 1)|a_n^n| \le 4^{n+1}(n + 3)!\,,\\
|a_n^{n+1}| &\le (n - 1)|a_{n-1}^n| + 2(n + 1)|a_{n-1}^n| = 0,
\end{aligned}$$

and for $n - 1 \ge k \ge 2$ we have

$$|a_k^{n+1}| \le (k + 1)|a_{k+1}^n| + (k - 1)|a_{k-1}^n| + 2(n + 1)|a_{k-1}^n| \le 4^{n+1}(n + 3)!\,,$$

while

$$\begin{aligned}
|a_1^{n+1}| &\le 2|a_2^n| + 2(n + 1)|a_0^n| \le 4^{n+1}(n + 3)!\,,\\
|a_0^{n+1}| &\le |a_1^n| \le 4^{n+1}(n + 3)!\,.
\end{aligned}$$

The conclusion is now obvious. □

In the next lemma we estimate the first two terms that appear on the right-hand side of (5-3).

Lemma 5.3. *Let $n, k \in \mathbb{N}$ satisfy $n \ge 3$ and $0 \le k \le n + 1$, and let $s \in \left(\frac{3}{2}, 2\right)$. Given $f, f_\tau \in H^s(\mathbb{R})$, it holds that*

$$\big\| A_{n+k+1,n+1}(f_\tau, \ldots, f_\tau)[\underbrace{f_\tau, \ldots, f_\tau}_{k \text{ times}}, \underbrace{f, \ldots, f}_{n+1 \text{ times}}, \cdot\,] \big\|_{\mathcal{L}(L_2(\mathbb{R}))} \le C^n n^4 \max\{1, \|f_\tau\|_{H^s}^4\} \|f\|_{H^s}^{n+1}, \tag{5-4}$$

with a constant $C \ge 1$ independent of n, k, f, and f_τ.

Proof. Much as in the proof of Lemma 3.1 we write

$$A_{n+k+1,n+1}(f_\tau, \ldots, f_\tau)[f_\tau, \ldots, f_\tau, f, \ldots, f, \cdot\,] = M - S,$$

where M is the multiplication operator

$$M[h](x) := h(x)\,\mathrm{PV}\int_{\mathbb{R}} \frac{1}{y}\Big(\frac{\delta_{[x,y]}f}{y}\Big)^{n+1} \frac{(\delta_{[x,y]}f_\tau/y)^k}{[1+(\delta_{[x,y]}f_\tau/y)^2]^{n+2}}\,dy$$

and S is the singular integral operator

$$S[h](x) := \mathrm{PV}\int_{\mathbb{R}} \Big(\frac{\delta_{[x,y]}f}{y}\Big)^{n+1} \frac{(\delta_{[x,y]}f_\tau/y)^k}{[1+(\delta_{[x,y]}f_\tau/y)^2]^{n+2}}\,\frac{h(x-y)}{y}\,dy$$

for $h \in L_2(\mathbb{R})$. Arguing as in proof of Lemma 3.1, it follows that

$$\|M\|_{\mathcal{L}(L_2(\mathbb{R}))} \le nC^n \max\{1, \|f_\tau\|_{H^s}\}\|f\|_{H^s}^{n+1} \tag{5-5}$$

with a constant $C \ge 1$ independent of n, k, f, and f_τ.

In order to deal with the operator S we consider the functions $F : \mathbb{R}^2 \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}^2$ defined by

$$F(x_1, x_2) := \frac{x_1^{n+1}x_2^k}{(1+x_2^2)^{n+2}}, \qquad A := (A_1, A_2) := (f, f_\tau).$$

The function F is smooth, A is Lipschitz continuous, and we set

$$a_j := \|A_j'\|_\infty, \qquad 1 \le j \le 2.$$

Since S is the singular integral operator with kernel

$$K(x, y) := \frac{1}{y}F\Big(\frac{\delta_{[x,y]}A}{y}\Big), \qquad x \in \mathbb{R}, \ \ y \ne 0,$$

and $|\delta_{[x,y]}A_j/y| \le a_j$ for $1 \le j \le 2$, it is natural to introduce a smooth periodic function $\widetilde{F}$ on $\mathbb{R}^2$, which is $4a_j$-periodic in the variable x_j, $1 \le j \le 2$, and which matches F on $\prod_{j=1}^2[-a_j, a_j]$. More precisely, we choose $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ with $\varphi = 1$ on $[|x| \le 1]$ and $\varphi = 0$ on $[|x| \ge 2]$ and we define $\widetilde{F}$ to be the periodic extension of

$$\Big[(x_1, x_2) \mapsto F(x_1, x_2)\prod_{j=1}^{2}\varphi\Big(\frac{x_j}{a_j}\Big)\Big] : Q \to \mathbb{R},$$

where $Q := \prod_{j=1}^2[-2a_j, 2a_j]$. We now expand $\widetilde{F}$ by its Fourier series

$$\widetilde{F}(x_1, x_2) = \sum_{p\in\mathbb{Z}^2} \alpha_p \exp\Big(i\sum_{j=1}^{2}\frac{p_jx_j}{T_j}\Big),$$

where

$$T_j := \frac{2a_j}{\pi}, \qquad \alpha_p := \frac{1}{4^2a_1a_2}\int_Q \widetilde{F}(x_1, x_2)\exp\Big(-i\sum_{j=1}^{2}\frac{p_jx_j}{T_j}\Big)\,d(x_1, x_2), \qquad p \in \mathbb{Z}^2,$$

and observe that

$$K(x, y) = \frac{1}{y}\widetilde{F}\Big(\frac{\delta_{[x,y]}A}{y}\Big) = \sum_{p\in\mathbb{Z}^2}\alpha_pK_p(x, y), \qquad x \in \mathbb{R}, \ \ y \ne 0,$$

with

$$K_p(x,y) := \frac{1}{y}\exp\Big(i\,\frac{\delta_{[x,y]}\big(\sum_{j=1}^{2}(p_j/T_j)A_j\big)}{y}\Big), \quad x\in\mathbb{R},\ \ y\neq 0,\ \ p\in\mathbb{Z}^2.$$

The kernels K_p define operators in $\mathcal{L}(L_2(\mathbb{R}))$ of type (1-2) and the norms of these operators can be estimated from above by

$$C\Big(1+\Big\|\sum_{j=1}^{2}\frac{p_j}{T_j}A_j'\Big\|_\infty\Big)\le C(1+|p|), \quad p\in\mathbb{Z}^2.$$

Since $\sum_{p\in\mathbb{Z}^2}(1+|p|^3)^{-1}<\infty$ we get

$$\|S\|_{\mathcal{L}(L_2(\mathbb{R}))}\le C\sum_{p\in\mathbb{Z}^2}|\alpha_p|(1+|p|)\le C\sup_{p\in\mathbb{Z}^2}[(1+|p|^4)|\alpha_p|].$$

We estimate next the quantity $\sup_{p\in\mathbb{Z}^2}(1+|p|^4)|\alpha_p|$. To this end we write

$$\alpha_p=\frac{1}{4^2}\prod_{j=1}^{2}\frac{I_j}{a_j},$$

where

$$I_1:=\int_{-2a_1}^{2a_1}x_1^{n+1}\varphi\Big(\frac{x_1}{a_1}\Big)e^{-ip_1x_1/T_1}\,dx_1, \quad I_2:=\int_{-2a_2}^{2a_2}\frac{x_2^k}{(1+x_2^2)^{n+2}}\varphi\Big(\frac{x_2}{a_2}\Big)e^{-ip_2x_2/T_2}\,dx_2.$$

Since $\varphi=0$ in $[|x|\ge 2]$ and $n\ge 3$, integration by parts leads us, in the case when $p_1\neq 0$, to

$$|I_1|\le\Big(\frac{T_1}{|p_1|}\Big)^4\int_{-2a_1}^{2a_1}\Big|\Big(x_1^{n+1}\varphi\Big(\frac{x_1}{a_1}\Big)\Big)^{(4)}\Big|\,dx_1\le C\,\frac{2^n n^4 a_1^{n+2}}{p_1^4}, \tag{5-6}$$

and similarly, since $x_2\le 1+x_2^2$, we find for $p_2\neq 0$ that

$$|I_2|\le C\,\frac{n^4\max\{a_2,a_2^5\}}{p_2^4}. \tag{5-7}$$

The estimates

$$|I_1|\le C2^n a_1^{n+2}, \quad |I_2|\le Ca_2, \tag{5-8}$$

are valid for all $p\in\mathbb{Z}^2$. Combining (5-6)–(5-8), we arrive at

$$\sup_{p\in\mathbb{Z}^2}(1+|p|^4)|\alpha_p|\le C2^n n^4\max\{1,a_2^4\}a_1^{n+1},$$

which leads us to

$$\|S\|_{\mathcal{L}(L_2(\mathbb{R}))}\le C2^n n^4\max\{1,\|f_\tau'\|_\infty^4\}\|f'\|_\infty^{n+1}\le n^4C^n\max\{1,\|f_\tau\|_{H^s}^4\}\|f\|_{H^s}^{n+1}.$$

This inequality together with (5-5) proves the desired claim. □

In the next lemma we estimate the last two terms on the right-hand side of (5-3) in the proof of Proposition 5.1.

Lemma 5.4. *Let $n, k \in \mathbb{N}$ satisfy $n \geq 1$ and $0 \leq k \leq n+1$. Let further $l \in \{0, 1\}$ and $s \in \left(\frac{3}{2}, 2\right)$. Given $f, f_\tau \in H^s(\mathbb{R})$, it holds that*

$$\left\| A_{n+k+1+2l,n+1+l}(f_\tau, \dots, f_\tau)[\underbrace{f_\tau, \dots, f_\tau}_{k-1+2l \text{ times}}, \underbrace{f, \dots, f}_{n+1 \text{ times}}, f_\tau', \cdot\,] \right\|_{\mathcal{L}(H^1(\mathbb{R}), L_2(\mathbb{R}))} \leq C^n \|f_\tau\|_{H^s} \|f\|_{H^s}^{n+1}, \tag{5-9}$$

with a constant $C \geq 1$ independent of n, k, f, and f_τ.

Proof. The proof is similar to that of Lemma 3.4. □

We are now in a position to prove Theorem 1.3 when $\sigma = 0$, where we use a parameter trick which appears, in other forms, also in [Angenent 1990; Escher and Simonett 1996; Prüss et al. 2015]. We present a new idea which uses only the abstract result Theorem 1.5 in the context of an evolution problem related to (1-1), and not explicitly the maximal regularity property as in the above-mentioned papers. The proof when $\sigma > 0$ is almost identical and is also discussed below, but it relies on some properties established in Section 6.

Proof of Theorem 1.3. Assume first that $\sigma = 0$. We then pick $f_0 \in H^s(\mathbb{R})$, $s \in \left(\frac{3}{2}, 2\right)$, and we let $f = f(\cdot\,; f_0) : [0, T_+(f_0)) \to H^s(\mathbb{R})$ denote the unique maximal solution to (1-1), whose existence is guaranteed by Theorem 1.1. We further choose $\lambda_1, \lambda_2 \in (0, \infty)$ and we define

$$f_{\lambda_1,\lambda_2}(t, x) := f(\lambda_1 t, x + \lambda_2 t), \quad x \in \mathbb{R}, \ \ 0 \leq t < T_+ := T_+(f_0)/\lambda_1.$$

Classical arguments show that

$$f_{\lambda_1,\lambda_2} \in C([0, T_+), H^s(\mathbb{R})) \cap C((0, T_+), H^2(\mathbb{R})) \cap C^1((0, T_+), H^1(\mathbb{R})).$$

We next introduce the function $u := (u_1, u_2, u_3) : [0, T_+) \to \mathbb{R}^2 \times H^s(\mathbb{R})$, where

$$(u_1, u_2)(t) := (\lambda_1, \lambda_2), \quad u_3(t) := f_{\lambda_1,\lambda_2}(t), \quad 0 \leq t < T_+,$$

and we note that u solves the quasilinear evolution problem

$$\dot{u} = \Psi(u)[u], \quad t > 0, \quad u(0) = (\lambda_1, \lambda_2, f_0), \tag{5-10}$$

with $\Psi : (0, \infty)^2 \times H^s(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^2 \times H^2(\mathbb{R}), \mathbb{R}^2 \times H^1(\mathbb{R}))$ denoting the operator defined by

$$\Psi((v_1, v_2, v_3))[(u_1, u_2, u_3)] := (0, 0, v_1 \Phi(v_3)[u_3] + v_2 \partial_x u_3). \tag{5-11}$$

Proposition 5.1 immediately yields

$$\Psi \in C^\omega\big((0, \infty)^2 \times H^s(\mathbb{R}), \mathcal{L}(\mathbb{R}^2 \times H^2(\mathbb{R}), \mathbb{R}^2 \times H^1(\mathbb{R}))\big) \quad \text{for all } s \in \left(\tfrac{3}{2}, 2\right).$$

Given $v := (v_1, v_2, v_3) \in (0, \infty)^2 \times H^s(\mathbb{R})$, the operator $\Psi(v)$ can be represented as a matrix

$$\Psi(v) = \begin{pmatrix} 0 & 0 \\ 0 & v_1 \Phi(v_3) + v_2 \partial_x \end{pmatrix} : \mathbb{R}^2 \times H^2(\mathbb{R}) \to \mathbb{R}^2 \times H^1(\mathbb{R}),$$

and we infer from [Amann 1995, Corollary I.1.6.3] that $-\Psi(v)\in\mathcal{H}(\mathbb{R}^2\times H^2(\mathbb{R}),\mathbb{R}^2\times H^1(\mathbb{R}))$ if and only if

$$-(v_1\Phi(v_3)+v_2\partial_x)\in\mathcal{H}(H^2(\mathbb{R}),H^1(\mathbb{R})). \tag{5-12}$$

We note that $v_2\partial_x$ is a first-order Fourier multiplier and its symbol is purely imaginary. Therefore, obvious modifications of the arguments presented in the proofs of Theorem 4.2 and Proposition 4.3 enable us to conclude that the property (5-12) is satisfied for each $(v_1,v_2,v_3)\in(0,\infty)^2\times H^s(\mathbb{R})$ and $s\in\left(\frac{3}{2},2\right)$. Setting $\mathbb{F}_0:=\mathbb{R}^2\times H^1(\mathbb{R})$ and $\mathbb{F}_1:=\mathbb{R}^2\times H^2(\mathbb{R})$, it holds that

$$[\mathbb{F}_0,\mathbb{F}_1]_\theta=\mathbb{R}^2\times H^{1+\theta}(\mathbb{R}),\qquad \theta\in(0,1),$$

and we may now apply Theorem 1.5 in the context of the quasilinear parabolic problem (5-10) to conclude (much as in the proof of Theorem 1.1), for each $u_0=(\lambda_1,\lambda_2,f_0)\in(0,\infty)^2\times H^s(\mathbb{R})$, $s\in\left(\frac{3}{2},2\right)$, the existence of a unique maximal solution

$$u:=u(\cdot\,;u_0)\in C([0,T_+(u_0)),(0,\infty)^2\times H^s(\mathbb{R}))\cap C((0,T_+(u_0)),\mathbb{F}_1)\cap C^1((0,T_+(u_0)),\mathbb{F}_0).$$

Additionally, the set

$$\Omega:=\{(\lambda_1,\lambda_2,f_0,t):t\in(0,T_+((\lambda_1,\lambda_2,f_0)))\}$$

is open in $(0,\infty)^2\times H^s(\mathbb{R})\times(0,\infty)$ and

$$[(\lambda_1,\lambda_2,f_0,t)\mapsto u(t;(\lambda_1,\lambda_2,f_0))]:\Omega\to\mathbb{R}^2\times H^s(\mathbb{R})$$

is a real-analytic map.

So, if we fix $f_0\in H^s(\mathbb{R})$, then we may conclude that

$$\frac{T_+(f_0)}{\lambda_1}=T_+((\lambda_1,\lambda_2,f_0))\quad\text{for all }(\lambda_1,\lambda_2)\in(0,\infty)^2.$$

As we want to prove that $f=f(\cdot\,;f_0)$ is real-analytic in $(0,T_+(f_0))\times\mathbb{R}$, it suffices to establish the real-analyticity property in a small ball around (t_0,x_0) for each $x_0\in\mathbb{R}$ and $t_0\in(0,T_+(f_0))$. Let thus $(t_0,x_0)\in(0,T_+(f_0))\times\mathbb{R}$ be arbitrary. For $(\lambda_1,\lambda_2)\in\mathbb{B}((1,1),\varepsilon)\subset(0,\infty)^2$, with ε chosen suitably small, we have that

$$t_0<T_+((\lambda_1,\lambda_2,f_0))\quad\text{for all }(\lambda_1,\lambda_2)\in\mathbb{B}((1,1),\varepsilon),$$

and therewith

$$\mathbb{B}((1,1),\varepsilon)\times\{f_0\}\times\{t_0\}\subset\Omega.$$

Moreover, since $u_3(\cdot\,;u_0)=f_{\lambda_1,\lambda_2}$, the restriction

$$[(\lambda_1,\lambda_2)\mapsto f_{\lambda_1,\lambda_2}(t_0)]:\mathbb{B}((1,1),\varepsilon)\to H^s(\mathbb{R}) \tag{5-13}$$

is a real-analytic map. Since $[h\mapsto h(x_0-t_0)]:H^s(\mathbb{R})\to\mathbb{R}$ is a linear operator, the composition

$$[(\lambda_1,\lambda_2)\mapsto f(\lambda_1t_0,x_0-t_0+\lambda_2t_0)]:\mathbb{B}((1,1),\varepsilon)\to\mathbb{R} \tag{5-14}$$

is real-analytic too. Furthermore, for $\delta > 0$ small, the mapping $\varphi : \mathbb{B}((t_0, x_0), \delta) \to \mathbb{B}((1,1), \varepsilon)$ with

$$\varphi(t,x) := \left(\frac{t}{t_0}, \frac{x - x_0 + t_0}{t_0}\right) \tag{5-15}$$

is well-defined and real-analytic, and therefore the composition of the functions defined by (5-14) and (5-15), that is, the mapping

$$[(t,x) \mapsto f(t,x)] : \mathbb{B}((t_0, x_0), \delta) \to \mathbb{R},$$

is also real-analytic. This proves the first claim.

Finally, the property $f \in C^\omega((0, T_+(f_0)), H^k(\mathbb{R}))$ for arbitrary $k \in \mathbb{N}$, is an immediate consequence of (5-13).

The arguments presented above carry over to the case when $\sigma > 0$ (with the obvious modifications). If $\sigma > 0$, the operator $v_2 \partial_x$ appearing in (5-12) can be regarded as being a lower-order perturbation and therefore the generator property of $\Psi(v)$ follows in this case directly from the corresponding property of the original operator; see Theorem 6.3. □

6. The Muskat problem with surface tension and gravity effects

We now consider surface-tension forces acting at the interface between the fluids; that is, we take $\sigma > 0$. The motion of the fluids may also be influenced by gravity, but we make no restrictions on Δ_ρ, which is now an arbitrary real number. If we model flows in a vertical Hele-Shaw cell, this means in particular that the lower fluid may be less dense than the fluid above. Since Δ_ρ can be zero, (1-1) is also a model for fluid motions in a horizontal Hele-Shaw cell as for these flows the effects due to gravity are usually neglected, that is, $g = 0$. Again, we rescale the time appropriately and rewrite (1-1) as the system

$$\begin{cases} \partial_t f(t,x) = f'(t,x)\,\mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{f(t,x) - f(t,x-y)}{y^2 + (f(t,x) - f(t,x-y))^2} (\kappa(f))'(t, x-y)\,dy \\ \qquad\qquad + \mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{y}{y^2 + (f(t,x) - f(t,x-y))^2} (\kappa(f))'(t, x-y)\,dy \\ \qquad\qquad + \Theta\,\mathrm{PV}\displaystyle\int_{\mathbb{R}} \frac{y(f'(t,x) - f'(t,x-y))}{y^2 + (f(t,x) - f(t,x-y))^2}\,dy \qquad \text{for } t > 0,\ x \in \mathbb{R}, \\ f(0,\cdot) = f_0, \end{cases} \tag{6-1}$$

with

$$\Theta := \frac{\Delta\rho}{\sigma} \in \mathbb{R}.$$

Since

$$(\kappa(f))' = \frac{f'''}{(1 + f'^2)^{3/2}} - 3\frac{f' f''^2}{(1 + f'^2)^{5/2}},$$

we observe that the first equation of (6-1) is again quasilinear, but now this property is due to the fact that $(\kappa(f))'$ is an affine function in the variable f'''.

To be more precise we set

$$\begin{aligned}\Phi_\sigma(f)[h](x) := f'(x)\,\mathrm{PV}\int_{\mathbb{R}} \frac{\delta_{[x,y]}f}{y^2+(\delta_{[x,y]}f)^2}\Big(\frac{h'''}{(1+f'^2)^{3/2}}-3\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big)(x-y)\,dy\\ +\mathrm{PV}\int_{\mathbb{R}} \frac{y}{y^2+(\delta_{[x,y]}f)^2}\Big(\frac{h'''}{(1+f'^2)^{3/2}}-3\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big)(x-y)\,dy\\ +\Theta\,\mathrm{PV}\int_{\mathbb{R}} \frac{y(\delta_{[x,y]}h')}{y^2+(\delta_{[x,y]}f)^2}\,dy,\end{aligned}\tag{6-2}$$

and we recast the problem (6-1) in the compact form

$$\dot f = \Phi_\sigma(f)[f], \qquad t>0, \qquad f(0)=f_0. \tag{6-3}$$

We emphasize that there are also other ways to write (6-1) as a quasilinear problem. For example the terms containing only $f^{(l)}$, $0\le l\le 2$, can be viewed as a nonlinear function $[f\mapsto F(f)]$ which would appear as an additive term to the right-hand side of (6-3), with $\Phi_\sigma(f)[h]$ modified accordingly. However, the formulation (6-2)–(6-3) appears to us as being optimal as it allows us to consider the largest set of initial data among all formulations. To be more precise, the operator introduced by (6-2) satisfies, with the notation in Lemma 3.1 and Remark 3.3, the relation

$$\begin{aligned}&\Phi_\sigma(f)[h]\\ &= f'B_{1,1}(f,f)\Big[\frac{h'''}{(1+f'^2)^{3/2}}-3\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big]+B_{0,1}(f)\Big[\frac{h'''}{(1+f'^2)^{3/2}}-3\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big]+\Theta A_{0,0}(f)[h'],\end{aligned}$$

and we now claim, based on the results in Section 5, that

$$\Phi_\sigma\in C^\omega(H^2(\mathbb{R}),\mathcal{L}(H^3(\mathbb{R}),L_2(\mathbb{R}))). \tag{6-4}$$

Indeed, arguing as in Section 5, it follows that

$$\begin{aligned}\Big[f\mapsto\Big[h\mapsto \mathrm{PV}\int_{\mathbb{R}}\frac{\delta_{[\cdot,y]}f}{y^2+(\delta_{[\cdot,y]}f)^2}h(\cdot-y)\,dy\Big]\Big]&\in C^\omega(H^2(\mathbb{R}),\mathcal{L}(L_2(\mathbb{R}))),\\ \Big[f\mapsto\Big[h\mapsto \mathrm{PV}\int_{\mathbb{R}}\frac{y}{y^2+(\delta_{[\cdot,y]}f)^2}h(\cdot-y)\,dy\Big]\Big]&\in C^\omega(H^2(\mathbb{R}),\mathcal{L}(L_2(\mathbb{R}))),\\ \Big[f\mapsto\Big[h\mapsto \mathrm{PV}\int_{\mathbb{R}}\frac{y(\delta_{[\cdot,y]}h')}{y^2+(\delta_{[\cdot,y]}f)^2}\,dy\Big]\Big]&\in C^\omega(H^2(\mathbb{R}),\mathcal{L}(H^3(\mathbb{R}),L_2(\mathbb{R}))).\end{aligned}\tag{6-5}$$

Moreover, classical arguments, see, e.g., [Runst and Sickel 1996, Theorem 5.5.3/4], yield that

$$\Big[f\mapsto\Big[h\mapsto\frac{h'''}{(1+f'^2)^{3/2}}-3\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big]\Big]\in C^\omega(H^2(\mathbb{R}),\mathcal{L}(H^3(\mathbb{R}),L_2(\mathbb{R}))). \tag{6-6}$$

The relations (6-5)–(6-6) immediately imply (6-4).

In the following, we prove that $\Phi_\sigma(f)$ is, for each $f\in H^2(\mathbb{R})$, the generator of a strongly continuous and analytic semigroup in $\mathcal{L}(L_2(\mathbb{R}))$, that is,

$$-\Phi_\sigma(f)\in\mathcal{H}(H^3(\mathbb{R}),L_2(\mathbb{R})).$$

To this end we write

$$\Phi_\sigma = \Phi_{\sigma,1} + \Phi_{\sigma,2}, \tag{6-7}$$

where

$$\Phi_{\sigma,1}(f)[h] := f' B_{1,1}(f, f)\Big[\frac{h'''}{(1+f'^2)^{3/2}}\Big] + B_{0,1}(f)\Big[\frac{h'''}{(1+f'^2)^{3/2}}\Big], \tag{6-8}$$

$$\Phi_{\sigma,2}(f)[h] := -3f' B_{1,1}(f, f)\Big[\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big] - 3B_{0,1}(f)\Big[\frac{f'f''h''}{(1+f'^2)^{5/2}}\Big] + \Theta A_{0,0}(f)[h'] \tag{6-9}$$

for $f \in H^2(\mathbb{R})$, $h \in H^3(\mathbb{R})$. Since $\Phi_{\sigma,2}(f) \in \mathcal{L}(H^{8/3}(\mathbb{R}), L_2(\mathbb{R}))$ and $[L_2(\mathbb{R}), H^3(\mathbb{R})]_{8/9} = H^{8/3}(\mathbb{R})$, we can view $\Phi_{\sigma,2}(f)$ as being a lower-order perturbation, see [Lunardi 1995, Proposition 2.4.1], and we only need to establish the generator property for the leading-order term $\Phi_{\sigma,1}(f)$. Much as in Section 4, we consider a continuous mapping

$$[\tau \mapsto \Phi_{\sigma,1}(\tau f)] : [0, 1] \to \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R})),$$

which transforms the operator $\Phi_{\sigma,1}(f)$ into the operator

$$\Phi_{\sigma,1}(0) = B_{0,1}(0) \circ \partial_x^3 = -\pi(\partial_x^4)^{3/4},$$

where $(\partial_x^4)^{3/4}$ is the Fourier multiplier with symbol $[\xi \mapsto |\xi|^3]$. We now establish the following result.

Theorem 6.1. *Let $f \in H^2(\mathbb{R})$ and $\mu > 0$ be given.*

Then, there exist $\varepsilon \in (0, 1)$, a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ satisfying (4-1)–(4-5), a constant $K = K(\varepsilon)$, and for each $j \in \{-N+1, \dots, N\}$ and $\tau \in [0, 1]$ there exist operators

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))$$

such that

$$\|\pi_j^\varepsilon \Phi_{\sigma,1}(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2 \le \mu \|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^2} \tag{6-10}$$

for all $j \in \{-N+1, \dots, N\}$, $\tau \in [0, 1]$, and $h \in H^3(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := -\frac{\pi}{(1+\tau^2 f'^2(x_j^\varepsilon))^{3/2}}(\partial_x^4)^{3/4}, \quad |j| \le N-1, \tag{6-11}$$

where x_j^ε is a point belonging to $\operatorname{supp}\pi_j^\varepsilon$, *and*

$$\mathbb{A}_{N,\tau} := -\pi(\partial_x^4)^{3/4}. \tag{6-12}$$

Proof. Let $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ be an ε-localization family satisfying the properties (4-1)–(4-5) and $\{\chi_j^\varepsilon : -N+1 \le j \le N\}$ be an associated family satisfying (4-6)–(4-8), with $\varepsilon \in (0, 1)$ which will be fixed below.

To deal with both terms of $\Phi_{\sigma,1}(\tau f)$, see (6-8), at once, we consider the operator

$$K_a(\tau f)[h] := f'_{a,\tau} B_{1,1}(f_{a,\tau}, \tau f)\Big[\frac{h'''}{(1+\tau^2 f'^2)^{3/2}}\Big],$$

where, for $a \in \{0, 1\}$, we set

$$f_{a,\tau} := (1-a)\tau f + a\,\mathrm{id}_{\mathbb{R}}.$$

For $a = 0$ we recover the first term in the definition of $\Phi_{\sigma,1}(\tau f)[h]$, while for $a = 1$ the expression matches the second one.

In the following, $h \in H^3(\mathbb{R})$ is arbitrary. Again, constants which are independent of ε (and, of course, of $h \in H^3(\mathbb{R})$, $\tau \in [0, 1]$, $a \in \{0, 1\}$, and $j \in \{-N+1, \dots, N\}$) are denoted by C, while the constants that we denote by K may depend only upon ε. We further let

$$\mathbb{A}^a_{j,\tau} := -\pi \frac{f'^2_{a,\tau}(x^\varepsilon_j)}{(1+\tau^2 f'^2(x^\varepsilon_j))^{5/2}} (\partial_x^4)^{3/4} \quad \text{for } |j| \le N-1,$$

and

$$\mathbb{A}^a_{N,\tau} := -\pi a^2 (\partial_x^4)^{3/4}.$$

We analyze the cases $j = N$ and $|j| \le N-1$ separately.

The case $|j| \le N-1$. For $|j| \le N-1$ we write

$$\pi^\varepsilon_j K_a(\tau f)[h] - \mathbb{A}^a_{j,\tau}[\pi^\varepsilon_j h] := T_1[h] + T_2[h] + T_3[h], \tag{6-13}$$

where

$$\begin{aligned}
T_1[h] &:= \pi^\varepsilon_j K_a(\tau f)[h] - f'_{a,\tau}(x^\varepsilon_j) B_{1,1}(f_{a,\tau}, \tau f)\Big[\frac{\pi^\varepsilon_j h'''}{(1+\tau^2 f'^2)^{3/2}}\Big],\\
T_2[h] &:= f'_{a,\tau}(x^\varepsilon_j) B_{1,1}(f_{a,\tau}, \tau f)\Big[\frac{\pi^\varepsilon_j h'''}{(1+\tau^2 f'^2)^{3/2}}\Big] - \frac{f'_{a,\tau}(x^\varepsilon_j)}{(1+\tau^2 f'^2(x^\varepsilon_j))^{3/2}} B_{1,1}(f_{a,\tau}, \tau f)[\pi^\varepsilon_j h'''],\\
T_3[h] &:= \frac{f'_{a,\tau}(x^\varepsilon_j)}{(1+\tau^2 f'^2(x^\varepsilon_j))^{3/2}} B_{1,1}(f_{a,\tau}, \tau f)[\pi^\varepsilon_j h'''] - \mathbb{A}^a_{j,\tau}[\pi^\varepsilon_j h].
\end{aligned}$$

We consider first the term $T_1[h]$. The identity $\chi^\varepsilon_j \pi^\varepsilon_j = 1$ on $\operatorname{supp} \pi^\varepsilon_j$ and integration by parts lead us to the relation

$$\begin{aligned}
T_1[h] = {}& \chi^\varepsilon_j (f'_{a,\tau} - f'_{a,\tau}(x^\varepsilon_j)) B_{1,1}(f_{a,\tau}, \tau f)\Big[\frac{\pi^\varepsilon_j h'''}{(1+\tau^2 f'^2)^{3/2}}\Big]\\
&+ (1-\chi^\varepsilon_j)(f'_{a,\tau}(x^\varepsilon_j) - f'_{a,\tau}) B_{1,1}(f_{a,\tau}, \tau f)\Big[\frac{\pi^{\varepsilon\prime}_j h''}{(1+\tau^2 f'^2)^{3/2}} - 3\tau^2 \frac{\pi^\varepsilon_j f' f'' h''}{(1+\tau^2 f'^2)^{5/2}}\Big]\\
&+ (f'_{a,\tau}(x^\varepsilon_j) - f'_{a,\tau}) \Big\{ B_{1,1}(\chi^\varepsilon_j, \tau f)\Big[\frac{\pi^\varepsilon_j f'_{a,\tau} h''}{(1+\tau^2 f'^2)^{3/2}}\Big] - 2 B_{2,1}(f_{a,\tau}, \chi^\varepsilon_j, \tau f)\Big[\frac{\pi^\varepsilon_j h''}{(1+\tau^2 f'^2)^{3/2}}\Big]\Big\}\\
&- 2\tau^2 (f'_{a,\tau}(x^\varepsilon_j) - f'_{a,\tau}) B_{3,2}(f_{a,\tau}, \chi^\varepsilon_j, f, \tau f, \tau f)\Big[\frac{\pi^\varepsilon_j f' h''}{(1+\tau^2 f'^2)^{3/2}}\Big]\\
&+ 2\tau^2 (f'_{a,\tau}(x^\varepsilon_j) - f'_{a,\tau}) B_{4,2}(f_{a,\tau}, \chi^\varepsilon_j, f, f, \tau f, \tau f)\Big[\frac{\pi^\varepsilon_j h''}{(1+\tau^2 f'^2)^{3/2}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+3\tau^2 f'_{a,\tau}\bigg\{\pi_j^\varepsilon B_{1,1}(f_{a,\tau},\tau f)\bigg[\frac{f'f''h''}{(1+\tau^2 f'^2)^{5/2}}\bigg]-B_{1,1}(f_{a,\tau},\tau f)\bigg[\frac{\pi_j^\varepsilon f'f''h''}{(1+\tau^2 f'^2)^{5/2}}\bigg]\bigg\}\\
&+f'_{a,\tau}\bigg\{B_{1,1}(\pi_j^\varepsilon,\tau f)\bigg[\frac{f'_{a,\tau}h''}{(1+\tau^2 f'^2)^{3/2}}\bigg]+B_{1,1}(f_{a,\tau},\tau f)\bigg[\frac{\pi_j^{\varepsilon\prime} h''}{(1+\tau^2 f'^2)^{3/2}}\bigg]\bigg\}\\
&-2f'_{a,\tau}B_{2,1}(\pi_j^\varepsilon,f_{a,\tau},\tau f)\bigg[\frac{h''}{(1+\tau^2 f'^2)^{3/2}}\bigg]-2\tau^2 f'_{a,\tau}B_{3,2}(\pi_j^\varepsilon,f_{a,\tau},f,\tau f,\tau f)\bigg[\frac{f'h''}{(1+\tau^2 f'^2)^{3/2}}\bigg]\\
&+2\tau^2 f'_{a,\tau}B_{4,2}(\pi_j^\varepsilon,f_{a,\tau},f,f,\tau f,\tau f)\bigg[\frac{h''}{(1+\tau^2 f'^2)^{3/2}}\bigg].
\end{aligned}
$$

Using Remark 3.3, the interpolation property (1-4), Young's inequality, and the Hölder continuity of $f'_{a,\tau}$, it follows that

$$
\begin{aligned}
\|T_1[h]\|_2 &\le C[\|\chi_j^\varepsilon(f'_{a,\tau}-f'_{a,\tau}(x_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon h'''\|_2+\|\pi_j^\varepsilon h''\|_\infty]+K\|h\|_{H^2}\\
&\le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3}+K\|h\|_{H^2}
\end{aligned}
\tag{6-14}
$$

provided that ε is sufficiently small.

Furthermore, we have

$$
T_2[h]=\frac{\tau^2 f'_{a,\tau}(x_j^\varepsilon)}{(1+\tau^2 f'^2(x_j^\varepsilon))^{3/2}}B_{1,1}(f_{a,\tau},\tau f)[Q(f'(x_j^\varepsilon)-f')\pi_j^\varepsilon h'''],
$$

where

$$
Q:=\frac{(f'(x_j^\varepsilon)+f')[(1+\tau^2 f'^2)^2+(1+\tau^2 f'^2)(1+\tau^2 f'^2(x_j^\varepsilon))+(1+\tau^2 f'^2(x_j^\varepsilon))^2]}{(1+\tau^2 f'^2)^{3/2}[(1+\tau^2 f'^2)^{3/2}+(1+\tau^2 f'^2(x_j^\varepsilon))^{3/2}]},
$$

and therewith

$$
\|T_2[h]\|_2\le C\|\chi_j^\varepsilon(f'_{a,\tau}-f'_{a,\tau}(x_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon h'''\|_2\le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3}+K\|h\|_{H^2}
\tag{6-15}
$$

if ε is sufficiently small.

Finally, arguing as in Step 3 of the proof of Theorem 4.4, we deduce that for ε sufficiently small we have

$$
\|T_3[h]\|_2\le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3}+K\|h\|_{H^2}.
\tag{6-16}
$$

Gathering (6-13)–(6-16), we have established the desired estimate (6-10) for $|j|\le N-1$.

The case $j=N$. For $j=N$ we write

$$
\pi_N^\varepsilon K_a(\tau f)[h]-\mathbb{A}^a_{N,\tau}[\pi_N^\varepsilon h]=:S_1[h]+S_2[h]+S_3[h]+S_4[h],
\tag{6-17}
$$

where

$$
\begin{aligned}
S_1[h]&:=\pi_N^\varepsilon K_a(\tau f)[h]-aB_{1,1}(f_{a,\tau},\tau f)\bigg[\frac{\pi_N^\varepsilon h'''}{(1+\tau^2 f'^2)^{3/2}}\bigg],\\
S_2[h]&:=aB_{1,1}(f_{a,\tau},\tau f)\bigg[\frac{\pi_N^\varepsilon h'''}{(1+\tau^2 f'^2)^{3/2}}\bigg]-aB_{1,1}(f_{a,\tau},\tau f)[\pi_N^\varepsilon h'''],
\end{aligned}
$$

$$S_3[h] := aB_{1,1}(f_{a,\tau},\tau f)[\pi_N^\varepsilon h'''] - a^2B_{0,1}(0)[\pi_N^\varepsilon h'''],$$
$$S_4[h] := a^2B_{0,1}(0)[\pi_N^\varepsilon h'''] - \mathbb{A}_{N,\tau}^a[\pi_N^\varepsilon h].$$

Much as for $T_1[h]$, we derive the identity

$$\begin{aligned}
S_1[h] = {} & \tau(1-a)\chi_N^\varepsilon f' B_{1,1}(f_{a,\tau},\tau f)\Big[\frac{\pi_N^\varepsilon h'''}{(1+\tau^2 f'^2)^{3/2}}\Big] \\
& -\tau(1-a)f'(1-\chi_N^\varepsilon)B_{1,1}(f_{a,\tau},\tau f)\Big[\frac{\pi_N^\varepsilon{}' h''}{(1+\tau^2 f'^2)^{3/2}} - 3\tau^2\frac{\pi_N^\varepsilon f' f'' h''}{(1+\tau^2 f'^2)^{5/2}}\Big] \\
& -\tau(1-a)f'\Big\{B_{1,1}(\chi_N^\varepsilon,\tau f)\Big[\frac{\pi_N^\varepsilon f'_{a,\tau}h''}{(1+\tau^2 f'^2)^{3/2}}\Big] - 2B_{2,1}(f_{a,\tau},\chi_N^\varepsilon,\tau f)\Big[\frac{\pi_N^\varepsilon h''}{(1+\tau^2 f'^2)^{3/2}}\Big]\Big\} \\
& +2\tau^3(1-a)f' B_{3,2}(f_{a,\tau},\chi_N^\varepsilon, f,\tau f,\tau f)\Big[\frac{\pi_N^\varepsilon f' h''}{(1+\tau^2 f'^2)^{3/2}}\Big] \\
& -2\tau^3(1-a)f' B_{4,2}(f_{a,\tau},\chi_N^\varepsilon, f, f,\tau f,\tau f)\Big[\frac{\pi_N^\varepsilon h''}{(1+\tau^2 f'^2)^{3/2}}\Big] \\
& +3\tau^2 f'_{a,\tau}\Big\{\pi_N^\varepsilon B_{1,1}(f_{a,\tau},\tau f)\Big[\frac{f' f'' h''}{(1+\tau^2 f'^2)^{5/2}}\Big] - B_{1,1}(f_{a,\tau},\tau f)\Big[\frac{\pi_N^\varepsilon f' f'' h''}{(1+\tau^2 f'^2)^{5/2}}\Big]\Big\} \\
& + f'_{a,\tau}\Big\{B_{1,1}(\pi_N^\varepsilon,\tau f)\Big[\frac{f'_{a,\tau}h''}{(1+\tau^2 f'^2)^{3/2}}\Big] + B_{1,1}(f_{a,\tau},\tau f)\Big[\frac{\pi_N^\varepsilon{}' h''}{(1+\tau^2 f'^2)^{3/2}}\Big]\Big\} \\
& -2f'_{a,\tau}B_{2,1}(\pi_N^\varepsilon, f_{a,\tau},\tau f)\Big[\frac{h''}{(1+\tau^2 f'^2)^{3/2}}\Big] - 2\tau^2 f'_{a,\tau}B_{3,2}(\pi_N^\varepsilon, f_{a,\tau}, f,\tau f,\tau f)\Big[\frac{f' h''}{(1+\tau^2 f'^2)^{3/2}}\Big] \\
& +2\tau^2 f'_{a,\tau}B_{4,2}(\pi_N^\varepsilon, f_{a,\tau}, f, f,\tau f,\tau f)\Big[\frac{h''}{(1+\tau^2 f'^2)^{3/2}}\Big].
\end{aligned}$$

Recalling that f' vanishes at infinity, we obtain by virtue of Remark 3.3, the interpolation property (1-4), and Young's inequality that

$$\|S_1[h]\|_2 \le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^2} \tag{6-18}$$

provided that ε is sufficiently small. Furthermore, Remark 3.3 implies that for ε sufficiently small

$$\begin{aligned}
\|S_2[h]\|_2 &= a\Big\|B_{11}(f_{a,\tau},\tau f)\Big[\frac{\pi_N^\varepsilon h'''}{(1+\tau^2 f'^2)^{3/2}}[1-(1+\tau^2 f'^2)^{3/2}]\Big]\Big\|_2 \\
&\le C\|\pi_N^\varepsilon h'''\|_2\|\chi_N^\varepsilon[1-(1+\tau^2 f'^2)^{3/2}]\|_\infty \le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^2}.
\end{aligned} \tag{6-19}$$

Since $a(1-a)=0$, we compute that

$$S_3[h] = -a^2B_{2,1}(f, f,\tau f)[\pi_\varepsilon^N h'''],$$

and the arguments presented in Step 4 of the proof of Theorem 4.2 yield

$$\|S_3[h]\|_2 \le \tfrac{1}{3}\mu\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^2} \tag{6-20}$$

for ε sufficiently small. Finally,

$$\|S_4[h]\|_2 = a^2\|B_{0,1}(0)[3(\pi_N^\varepsilon)'h'' + 3(\pi_N^\varepsilon)''h' + (\pi_N^\varepsilon)'''h]\|_2 \le K\|h\|_{H^2}, \tag{6-21}$$

and combining (6-17)–(6-21) we obtain the estimate (6-10) for $j = N$. $\square$

The Fourier multipliers defined by (6-11)–(6-12) are generators of strongly continuous analytic semigroups in $\mathcal{L}(L_2(\mathbb{R}))$ and they satisfy resolvent estimates which are uniform with respect to $x_j^\varepsilon \in \mathbb{R}$ and $\tau \in [0, 1]$. More precisely, we have the following result.

Proposition 6.2. *Let $f \in H^2(\mathbb{R})$ be fixed. Given $x_0 \in \mathbb{R}$ and $\tau \in [0, 1]$, let*

$$\mathbb{A}_{x_0,\tau} := -\frac{\pi}{(1+\tau^2 f'^2(x_0))^{3/2}}(\partial_x^4)^{3/4}.$$

Then, there exists a constant $\kappa_0 \ge 1$ such that

$$\lambda - \mathbb{A}_{x_0,\tau} \in \mathrm{Isom}(H^3(\mathbb{R}), L_2(\mathbb{R})), \tag{6-22}$$

$$\kappa_0\|(\lambda - \mathbb{A}_{x_0,\tau})[h]\|_2 \ge |\lambda| \cdot \|h\|_2 + \|h\|_{H^3} \tag{6-23}$$

for all $x_0 \in \mathbb{R}$, $\tau \in [0, 1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^3(\mathbb{R})$.

Proof. The proof is similar to that of Proposition 4.3 and therefore we omit it. $\square$

We now conclude with the following general result.

Theorem 6.3. *Let $f \in H^2(\mathbb{R})$ be given. Then*

$$-\Phi_\sigma(f) \in \mathcal{H}(H^3(\mathbb{R}), L_2(\mathbb{R})).$$

Proof. As mentioned in the discussion preceding Theorem 6.1, we only need to prove the claim for the leading-order term $\Phi_{\sigma,1}(f)$. Let $\kappa_0 \ge 1$ be the constant determined in Proposition 6.2 and let $\mu := \frac{1}{2}\kappa_0$. By virtue of Theorem 6.1 there exist constants $\varepsilon \in (0, 1)$ and $K = K(\varepsilon) > 0$, an ε-localization family $\{\pi_j^\varepsilon : -N+1 \le j \le N\}$ that satisfies (4-1)–(4-5), and for each $-N+1 \le j \le N$ and $\tau \in [0, 1]$ operators $\mathbb{A}_{j,\tau} \in \mathcal{L}(H^3(\mathbb{R}), L_2(\mathbb{R}))$ such that

$$\|\pi_j^\varepsilon \Phi_{\sigma,1}(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2 \le \frac{1}{2\kappa_0}\|\pi_j^\varepsilon h\|_{H^3} + K\|h\|_{H^2} \tag{6-24}$$

for all $-N+1 \le j \le N$, $\tau \in [0, 1]$, and $h \in H^3(\mathbb{R})$. Furthermore, Proposition 6.2 implies

$$\kappa_0\|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon h]\|_2 \ge |\lambda| \cdot \|\pi_j^\varepsilon h\|_2 + \|\pi_j^\varepsilon h\|_{H^3} \tag{6-25}$$

for all $-N+1 \le j \le N$, $\tau \in [0, 1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^3(\mathbb{R})$. Combining (6-24)–(6-25), we find

$$\begin{aligned}\kappa_0\|\pi_j^\varepsilon(\lambda - \Phi_{\sigma,1}(\tau f))[h]\|_2 &\ge \kappa_0\|(\lambda - \mathbb{A}_{j,\tau})[\pi_j^\varepsilon h]\|_2 - \kappa_0\|\pi_j^\varepsilon \Phi_{\sigma,1}(\tau f)[h] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon h]\|_2\\ &\ge |\lambda| \cdot \|\pi_j^\varepsilon h\|_2 + \tfrac{1}{2}\|\pi_j^\varepsilon h\|_{H^3} - \kappa_0 K\|h\|_{H^2}\end{aligned}$$

for all $-N+1 \le j \le N$, $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^3(\mathbb{R})$. Summing up over $j \in \{-N+1, \dots, N\}$, we infer from Lemma 4.1 that there exists a constant $C \ge 1$ with the property that

$$C\|h\|_{H^2} + C\|(\lambda - \Phi(\tau f))[h]\|_2 \ge |\lambda| \cdot \|h\|_2 + \|h\|_{H^3}$$

for all $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$, and $h \in H^3(\mathbb{R})$. Using (1-4) and Young's inequality, it follows that there exist constants $\kappa \ge 1$ and $\omega > 0$ with the property that

$$\kappa\|(\lambda - \Phi_{\sigma,1}(\tau f))[h]\|_2 \ge |\lambda| \cdot \|h\|_2 + \|h\|_{H^3} \tag{6-26}$$

for all $\tau \in [0,1]$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge \omega$, $h \in H^3(\mathbb{R})$. Since $(\omega - \Phi_{\sigma,1}(\tau f))|_{\tau=0} \in \operatorname{Isom}(H^3(\mathbb{R}), L_2(\mathbb{R}))$, the method of continuity together with (6-26) yields that

$$\omega - \Phi_{\sigma,1}(f) \in \operatorname{Isom}(H^3(\mathbb{R}), L_2(\mathbb{R})). \tag{6-27}$$

The claim follows from (6-26) (with $\tau = 1$), (6-27), and [Lunardi 1995, Proposition 2.4.1 and Corollary 2.1.3]. □

We are now in a position to prove the well-posedness for the Muskat problem with surface tension.

Proof of Theorem 1.2. Let $s \in (2,3)$, $\bar{s} = 2$, $1 > \alpha := \frac{1}{3}s > \beta := \frac{2}{3} > 0$. Combining (6-4) and Theorem 6.3, it follows that

$$-\Phi_\sigma \in C^\omega(H^2(\mathbb{R}), \mathcal{H}(H^3(\mathbb{R}), L_2(\mathbb{R}))).$$

Since

$$H^2(\mathbb{R}) = [L_2(\mathbb{R}), H^3(\mathbb{R})]_\beta \quad \text{and} \quad H^s(\mathbb{R}) = [L_2(\mathbb{R}), H^3(\mathbb{R})]_\alpha,$$

we now infer from Theorem 1.5 that (1-1), or equivalently (6-3), possesses a maximally defined solution

$$f := f(\cdot\,; f_0) \in C([0, T_+(f_0)), H^s(\mathbb{R})) \cap C((0, T_+(f_0)), H^3(\mathbb{R})) \cap C^1((0, T_+(f_0)), L_2(\mathbb{R}))$$

with

$$f \in C^{(s-2)/3}([0,T], H^2(\mathbb{R})) \quad \text{for all } T < T_+(f_0).$$

Concerning the uniqueness of solutions, we next show that any classical solution

$$\tilde{f} \in C([0,\widetilde{T}), H^s(\mathbb{R})) \cap C((0,\widetilde{T}), H^3(\mathbb{R})) \cap C^1((0,\widetilde{T}), L_2(\mathbb{R})), \qquad \widetilde{T} \in (0,\infty],$$

to (6-3) satisfies

$$\tilde{f} \in C^\eta([0,T], H^2(\mathbb{R})) \quad \text{for all } T \in (0,\widetilde{T}), \tag{6-28}$$

where $\eta := (s-2)/(s+1)$. To this end, we recall that

$$\Phi_\sigma(f)[f] = f' B_{1,1}(f,f)[(\kappa(f))'] + B_{0,1}(f)[(\kappa(f))'] + \Theta A_{0,0}(f)[f'] \quad \text{for } f \in H^3(\mathbb{R}). \tag{6-29}$$

Let $T \in (0,\widetilde{T})$ be fixed. Lemma 3.1(i) implies that

$$\sup_{[0,T]} \|A_{0,0}(\tilde{f})[\tilde{f}']\|_2 \le C. \tag{6-30}$$

We now consider the highest-order terms in (6-29). Arguing as in Lemma 3.5, it follows from Remark 3.3 that $B_{0,1}(f)[\kappa(f)],\ B_{1,1}(f,f)[\kappa(f)] \in H^1(\mathbb{R})$ for all $f \in H^3(\mathbb{R})$, with

$$(B_{0,1}(f)[\kappa(f)])' = B_{0,1}(f)[(\kappa(f))'] - 2B_{2,2}(f', f, f, f)[\kappa(f)],$$
$$(B_{1,1}(f,f)[\kappa(f)])' = B_{1,1}(f,f)[(\kappa(f))'] + B_{1,1}(f', f)[\kappa(f)] - 2B_{3,2}(f', f, f, f, f)[\kappa(f)].$$

Furthermore, given $t \in (0, T]$ and $\varphi \in H^1(\mathbb{R})$, integration by parts together with $\tilde{f} \in C([0,T], H^s(\mathbb{R}))$ leads us to

$$\begin{aligned}&\left|\int_{\mathbb{R}} \tilde{f}'(t)(B_{1,1}(\tilde{f}(t), \tilde{f}(t))[\kappa(\tilde{f}(t))])'\varphi\,dx\right|\\ &\quad= \left|\int_{\mathbb{R}} \tilde{f}''(t)B_{1,1}(\tilde{f}(t), \tilde{f}(t))[\kappa(\tilde{f}(t))]\varphi\,dx\right| + \left|\int_{\mathbb{R}} \tilde{f}'(t)B_{1,1}(\tilde{f}(t), \tilde{f}(t))[\kappa(\tilde{f}(t))]\varphi'\,dx\right| \le C\|\varphi\|_{H^1},\end{aligned}$$

so that

$$\sup_{(0,T]} \|\tilde{f}'(B_{1,1}(\tilde{f}, \tilde{f})[\kappa(\tilde{f})])'\|_{H^{-1}} \le C, \tag{6-31}$$

and similarly

$$\sup_{(0,T]} \|(B_{0,1}(\tilde{f})[\kappa(\tilde{f})])'\|_{H^{-1}} \le C. \tag{6-32}$$

We now estimate the term $f'B_{1,1}(f', f)[\kappa(f)]$ with $f \in H^3(\mathbb{R})$ in the $H^{-1}(\mathbb{R})$-norm. To this end, we rely on the formula

$$B_{1,1}(f', f)[\kappa(f)] = T_1(f) - T_2(f) - T_3(f),$$

where

$$T_1(f)(x) := \int_0^\infty \frac{\kappa(f)(x-y) - \kappa(f)(x+y)}{y}\,\frac{f'(x) - f'(x-y)}{y}\,\frac{1}{1+(\delta_{[x,y]}f/y)^2}\,dy,$$

$$T_2(f)(x) := \int_0^\infty \frac{\kappa(f)(x+y)}{y}\,\frac{f'(x+y) - 2f'(x) + f'(x-y)}{y}\,\frac{1}{1+(\delta_{[x,y]}f/y)^2}\,dy,$$

$$\begin{aligned}T_3(f)(x) := \int_0^\infty &\frac{\kappa(f)(x+y)}{y}\,\frac{f'(x) - f'(x+y)}{y}\,\frac{1}{[1+(\delta_{[x,y]}f/y)^2][1+(\delta_{[x,-y]}f/y)^2]}\\ &\times \frac{f(x+y) - f(x-y)}{y}\,\frac{f(x+y) - 2f(x) + f(x-y)}{y}\,dy.\end{aligned}$$

We estimate the terms $\tilde{f}'T_i(\tilde{f})$, $1 \le i \le 3$, separately. Given $t \in (0,T]$ and $\varphi \in H^1(\mathbb{R})$, we compute

$$\begin{aligned}&\left|\int_{\mathbb{R}} \tilde{f}'(t)T_1(\tilde{f}(t))\varphi\,dx\right|\\ &\le C\|\varphi\|_\infty \int_0^\infty\!\!\int_{\mathbb{R}} \frac{|\kappa(\tilde{f}(t))(x-y) - \kappa(\tilde{f}(t))(x+y)|}{y}\,\frac{|\tilde{f}'(t,x) - \tilde{f}'(t,x-y)|}{y}\,dx\,dy\\ &\le C\|\varphi\|_\infty \int_0^\infty \frac{1}{y^2}\left(\int_{\mathbb{R}} |\kappa(\tilde{f}(t))(x-y) - \kappa(\tilde{f}(t))(x+y)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}} |\tilde{f}'(t,x) - \tilde{f}'(t,x-y)|^2\,dx\right)^{1/2} dy\\ &= C\|\varphi\|_\infty \int_0^\infty \frac{1}{y^2}\left(\int_{\mathbb{R}} |\mathcal{F}(\kappa(\tilde{f}(t)))|^2(\xi)|e^{i2\xi y} - 1|^2\,d\xi\right)^{1/2}\left(\int_{\mathbb{R}} |\mathcal{F}(\tilde{f}'(t))|^2(\xi)|e^{iy\xi} - 1|^2\,d\xi\right)^{1/2} dy,\end{aligned}$$

and since

$$\begin{aligned}|e^{iy\xi}-1|^2&\le C(1+|\xi|^2)[y^2\mathbf{1}_{(0,1)}(y)+\mathbf{1}_{[y\ge 1]}(y)],\\ |e^{i2y\xi}-1|^2&\le C(1+|\xi|^2)^{s-2}[y^{2(s-2)}\mathbf{1}_{(0,1)}(y)+\mathbf{1}_{[y\ge 1]}(y)],\end{aligned}\qquad y>0,\ \ \xi\in\mathbb{R},$$

it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}}\tilde f'(t)T_1(\tilde f(t))\varphi\,dx\right|&\le C\|\varphi\|_\infty\|\kappa(\tilde f(t))|_{H^{s-2}}\|\tilde f(t)\|_{H^1}\int_0^\infty y^{s-3}\mathbf{1}_{(0,1)}(y)+y^{-2}\mathbf{1}_{[y\ge 1]}(y)\,dy\\ &\le C\|\varphi\|_{H^1}.\end{aligned}\tag{6-33}$$

To bound the curvature term in the $H^{s-2}(\mathbb{R})$-norm we have use the inequality

$$\|\kappa(f)\|_{H^{s-2}}\le C\|(1+f'^2)^{-3/2}\|_{BC^{s-3/2}}\|f\|_{H^s}\quad\text{for all } f\in H^s(\mathbb{R}).$$

Similarly we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\tilde f'(t)T_2(\tilde f(t))\varphi\,dx\right|&\le C\|\varphi\|_\infty\int_0^\infty\frac{1}{y^2}\left(\int_{\mathbb{R}}|\kappa(\tilde f(t))(x+y)|^2\,dx\right)^{1/2}\\ &\qquad\times\left(\int_{\mathbb{R}}|\tilde f'(t,x+y)-2\tilde f'(t,x)+\tilde f'(t,x-y)|^2\,dx\right)^{1/2}dy\\ &\le C\|\varphi\|_\infty\int_0^\infty\frac{1}{y^2}\left(\int_{\mathbb{R}}|\mathcal{F}(\tilde f'(t))|^2(\xi)|e^{iy\xi}-2+e^{-iy\xi}|^2\,d\xi\right)^{1/2}dy\\ &\le C\|\varphi\|_\infty\int_0^\infty y^{s-3}\mathbf{1}_{(0,1)}(y)+y^{-2}\mathbf{1}_{[y\ge 1]}(y)\,dy\\ &\le C\|\varphi\|_{H^1}\end{aligned}\tag{6-34}$$

by virtue of

$$|e^{iy\xi}-2+e^{-iy\xi}|^2\le C(1+|\xi|^2)^{s-1}[y^{2(s-1)}\mathbf{1}_{(0,1)}(y)+\mathbf{1}_{[y\ge 1]}(y)],\qquad y>0,\ \ \xi\in\mathbb{R}.$$

Lastly, since $H^{s-1}(\mathbb{R})\hookrightarrow \mathrm{BC}^{s-3/2}(\mathbb{R})$ for $s\neq\frac52$ $\big($the estimate (6-35) holds though also for $s=\frac52\big)$ and $H^s(\mathbb{R})\hookrightarrow \mathrm{BC}^1(\mathbb{R})$, the inequality

$$|e^{iy\xi}-2+e^{-iy\xi}|^2\le C(1+|\xi|^2)^2[y^4\mathbf{1}_{(0,1)}(y)+\mathbf{1}_{[y\ge 1]}(y)],\qquad y>0,\ \ \xi\in\mathbb{R},$$

leads us to

$$\begin{aligned}\left|\int_{\mathbb{R}}\tilde f'(t)T_3(\tilde f(t))\varphi\,dx\right|&\le C\|\varphi\|_\infty\int_0^\infty\frac{y^{\min\{1,s-3/2\}}}{y^3}\left(\int_{\mathbb{R}}|\mathcal{F}(\tilde f(t))|^2(\xi)|e^{iy\xi}-2+e^{-iy\xi}|^2\,d\xi\right)^{1/2}dy\\ &\le C\|\varphi\|_\infty\int_0^\infty y^{\min\{0,s-5/2\}}\mathbf{1}_{(0,1)}(y)+y^{-2}\mathbf{1}_{[y\ge 1]}(y)\,dy\\ &\le C\|\varphi\|_{H^1}.\end{aligned}\tag{6-35}$$

Gathering (6-33)–(6-35), we conclude that

$$\sup_{(0,T]}\|\tilde f'B_{1,1}(\tilde f',\tilde f)[\kappa(\tilde f)]\|_{H^{-1}}\le C,\tag{6-36}$$

and similarly we obtain

$$\sup_{(0,T]}\big[\|\tilde f' B_{3,2}(\tilde f', \tilde f, \tilde f, \tilde f, \tilde f)[\kappa(\tilde f)]\|_{H^{-1}} + \|B_{2,2}(\tilde f', \tilde f, \tilde f, \tilde f)[\kappa(\tilde f)]\|_{H^{-1}}\big] \le C. \tag{6-37}$$

Combining (6-30)–(6-32), (6-36), and (6-37), it follows that $\tilde f \in \mathrm{BC}^1((0,T], H^{-1}(\mathbb{R}))$. Recalling that $\eta = (s-2)/(s+1)$, (1-4) together with the mean value theorem yields

$$\|\tilde f(t) - \tilde f(s)\|_{H^2} \le \|\tilde f(t) - \tilde f(s)\|_{H^{-1}}^{\eta}\|\tilde f(t) - \tilde f(s)\|_{H^s}^{1-\eta} \le C|t-s|^{\eta}, \quad t, s \in [0,T],$$

which proves (6-28) and the uniqueness claim in Theorem 1.2.

Finally, let us assume that $T_+(f_0) < \infty$ and that

$$\sup_{[0,T_+(f_0))}\|f(t)\|_{H^s} < \infty.$$

Arguing as above, we find that

$$\|f(t) - f(s)\|_{H^{(s+2)/2}} \le C|t-s|^{(s-2)/(2s+2)}, \quad t, s \in [0, T_+(f_0)).$$

The criterion for global existence in Theorem 1.5 applied for $\alpha := \frac16(s+2)$ and $\beta := \frac23$ implies that the solution can be continued on an interval $[0,\tau)$ with $\tau > T_+(f_0)$ and that

$$f \in C^{(s-2)/6}([0,T], H^2(\mathbb{R})) \quad \text{for all } T \in (0,\tau).$$

The uniqueness claim in Theorem 1.5 leads us to a contradiction. Hence our assumption was false and $T_+(f_0) = \infty$. □

Appendix A: Some technical results

The following lemma is used in the proof of Theorem 4.2.

Lemma A.1. *Given $f \in H^s(\mathbb{R})$, $s \in \left(\frac32, 2\right)$, and $\tau \in [0,1]$, let $a_\tau : \mathbb{R} \to \mathbb{R}$ be defined by*

$$a_\tau(x) := \mathrm{PV}\int_{\mathbb{R}} \frac{y}{y^2 + \tau^2(\delta_{[x,y]}f)^2}\,dy, \quad x \in \mathbb{R}.$$

Let further $\alpha := \frac12 s - \frac34 \in (0,1)$. Then, $a_\tau \in \mathrm{BC}^\alpha(\mathbb{R}) \cap C_0(\mathbb{R})$,

$$\sup_{\tau\in[0,1]}\|a_\tau\|_{\mathrm{BC}^\alpha} < \infty, \tag{A-1}$$

and, given $\varepsilon_0 > 0$, there exists $\eta > 0$ such that

$$\sup_{\tau\in[0,1]}\sup_{|x|\ge\eta}|a_\tau(x)| \le \varepsilon_0. \tag{A-2}$$

Proof. It holds that

$$a_\tau(x) = \tau^2\lim_{\delta\to0}\int_\delta^{1/\delta}\frac{f(x+y)-2f(x)+f(x-y)}{y^2}\,\frac{f(x+y)-f(x-y)}{y}\,\frac{y^4}{\Pi(x,y)}\,dy, \quad x \in \mathbb{R},$$

with

$$\Pi(x,y) := [y^2+\tau^2(\delta_{[x,-y]}f)^2][y^2+\tau^2(\delta_{[x,y]}f)^2].$$

Letting

$$I(x,y) := \tau^2 \frac{f(x+y)-2f(x)+f(x-y)}{y^2}\,\frac{f(x+y)-f(x-y)}{y}\,\frac{y^4}{\Pi(x,y)}, \qquad (x,y)\in\mathbb{R}\times(0,\infty),$$

it follows that

$$|I(x,y)| \le 8\Big(\|f\|_\infty^2\frac{1}{y^3}\mathbf{1}_{[1,\infty)}(y)+\|f'\|_\infty[f']_{s-3/2}\frac{1}{y^{5/2-s}}\mathbf{1}_{(0,1)}(y)\Big), \qquad (x,y)\in\mathbb{R}\times(0,\infty). \tag{A-3}$$

The latter estimate was obtained by using the fact that $f\in \mathrm{BC}^{s-1/2}(\mathbb{R})$, $s-\frac{1}{2}\in(1,2)$, together with (3-7). Hence,

$$a_\tau(x) = \int_0^\infty I(x,y)\,dy, \qquad x\in\mathbb{R},$$

and $\sup_{\tau\in[0,1]}\|a_\tau\|_\infty<\infty$. To estimate the Hölder seminorm of a_τ, we compute for $x,x'\in\mathbb{R}$ that

$$|a_\tau(x)-a_\tau(x')| \le \int_0^\infty |I(x,y)-I(x',y)|\,dy \le T_1+T_2+T_3, \tag{A-4}$$

where

$$T_1 := \int_0^\infty \frac{|f(x+y)-2f(x)+f(x-y)|}{y^2}\,\frac{|[f(x+y)-f(x-y)]-[f(x'+y)-f(x'-y)]|}{y}\,\frac{y^4}{\Pi(x,y)}\,dy,$$

$$T_2 := \int_0^\infty \frac{|[f(x+y)-2f(x)+f(x-y)]-[f(x'+y)-2f(x')+f(x'-y)]|}{y^2} \times \frac{|f(x'+y)-f(x'-y)|}{y}\,\frac{y^4}{\Pi(x,y)}\,dy,$$

$$T_3 := \int_0^\infty \frac{|f(x'+y)-2f(x')+f(x'-y)|}{y^2}\,\frac{|f(x'+y)-f(x'-y)|}{y}\,\frac{|\Pi(x,y)-\Pi(x',y)|}{y^4}\,dy.$$

Using the mean value theorem, we have

$$\begin{aligned}\frac{|[f(x+y)-f(x-y)]-[f(x'+y)-f(x'-y)]|}{y} &\le 2\int_0^1 |f'(x+(2\tau-1)y)-f'(x'+(2\tau-1)y)|\,d\tau\\ &\le 2[f']_{s-3/2}|x-x'|^{s-3/2}, \qquad y>0,\end{aligned}$$

and, much as above, we find that

$$|T_1| \le C\|f\|_{H^s}^2|x-x'|^{2\alpha}. \tag{A-5}$$

To deal with the second term we appeal to the formula

$$\begin{aligned}f(x+y)-2f(x)+f(x-y) = y[f'(x+y)-f'(x-y)] + y\int_0^1 f'(x+\tau y)-f'(x+y)\,d\tau\\ - y\int_0^1 f'(x-\tau y)-f'(x-y)\,d\tau \quad \text{for } x,y\in\mathbb{R},\end{aligned}$$

and we get

$$\frac{|[f(x+y)-2f(x)+f(x-y)]-[f(x'+y)-2f(x')+f(x'-y)]|}{y^2} \le T_{2a}+T_{2b}+T_{2c},$$

where

$$\begin{aligned} T_{2a} &:= \frac{|[f'(x+y)-f'(x-y)]-[f'(x'+y)-f'(x'-y)]|}{y} \\ &\le 2[f']_{2\alpha}\Big(\frac{1}{y}\mathbf{1}_{[1,\infty)}(y)|x-x'|^{2\alpha}+2\frac{1}{y^{1-\alpha}}\mathbf{1}_{(0,1)}(y)|x-x'|^{\alpha}\Big), \\ T_{2b} &:= \frac{1}{y}\int_0^1 \big|[f'(x+\tau y)-f'(x+y)]-[f'(x'+\tau y)-f'(x'+y)]\big|\,d\tau \\ &\le 2[f']_{2\alpha}\Big(\frac{1}{y}\mathbf{1}_{[1,\infty)}(y)|x-x'|^{2\alpha}+\frac{1}{y^{1-\alpha}}\mathbf{1}_{(0,1)}(y)|x-x'|^{\alpha}\Big), \\ T_{2c} &:= \frac{1}{y}\int_0^1 \big|[f'(x-\tau y)-f'(x-y)]-[f'(x'-\tau y)-f'(x'-y)]\big|\,d\tau \\ &\le 2[f']_{2\alpha}\Big(\frac{1}{y}\mathbf{1}_{[1,\infty)}(y)|x-x'|^{2\alpha}+\frac{1}{y^{1-\alpha}}\mathbf{1}_{(0,1)}(y)|x-x'|^{\alpha}\Big), \end{aligned}$$

and therewith

$$|T_2| \le C\|f\|_{H^s}^2(|x-x'|^{\alpha}+|x-x'|^{2\alpha}). \tag{A-6}$$

Finally, since

$$\frac{|\Pi(x,y)-\Pi(x',y)|}{y^4} \le 4\|f'\|_\infty(1+\|f'\|_\infty^2)[f']_{2\alpha}|x-x'|^{2\alpha},$$

we infer from (A-3) that

$$|T_3| \le C\|f\|_{H^s}^4(1+\|f\|_{H^s}^2)(|x-x'|^{\alpha}+|x-x'|^{2\alpha}). \tag{A-7}$$

The relation (A-1) is a simple consequence of (A-4)–(A-7) and of $\sup_{\tau\in[0,1]}\|a_\tau\|_\infty<\infty$.

To prove that a_τ vanishes at infinity, let $\varepsilon_0>0$ be arbitrary. We write

$$a_\tau(x)=\int_0^M I(x,y)\,dy+\int_M^\infty I(x,y)\,dy, \qquad x\in\mathbb{R},$$

for some $M>1$ with

$$\frac{4\|f\|_\infty^2}{M^2}\le\frac{\varepsilon_0}{2}.$$

Recalling (A-3), it follows that for all $x\in\mathbb{R}$ we have

$$\int_M^\infty |I(x,y)|\,dy \le 8\|f\|_\infty^2\int_M^\infty \frac{1}{y^3}\,dy=\frac{4\|f\|_\infty^2}{M^2}\le\frac{\varepsilon_0}{2}.$$

Let $\beta\in(0,1)$ be chosen such that $\frac{3}{2}+\beta<s$. Since $f\in C_0(\mathbb{R})$, there exists $\eta>M$ with

$$|f(y)| \le \left[\frac{\big(s-\frac{3}{2}-\beta\big)\varepsilon_0 M^{3/2+\beta-s}}{32([f']_{s-3/2}\|f'\|_\infty^{1-\beta}+1)}\right]^{1/\beta} \qquad \text{for all } |y|\ge\eta-M.$$

Using this inequality, it follows that for all $|x| \geq \eta$ we have

$$|a_\tau(x)| \leq \int_0^M |I(x,y)|\,dy + \tfrac{1}{2}\varepsilon_0 \leq \tfrac{1}{2}\varepsilon_0 + 8[f']_{s-3/2}\|f'\|_\infty^{1-\beta} \int_0^M \frac{|f(x+y)-f(x-y)|^\beta}{y^{5/2+\beta-s}}\,dy \leq \varepsilon_0;$$

hence $a_\tau \in C_0(\mathbb{R})$ and (A-2) holds true. □

The next result is used in Proposition 2.1.

Lemma A.2. *Given $f \in H^5(\mathbb{R})$ and $\bar{\omega} \in H^2(\mathbb{R})$, set*

$$\tilde{v}(x,y) := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(-(y-f(s)),\, x-s)}{(x-s)^2+(y-f(s))^2}\,\bar{\omega}(s)\,ds \quad \text{in } \mathbb{R}^2 \setminus [y=f(x)]. \tag{A-8}$$

Let further $\Omega_- := [y<f(x)]$, $\Omega_+ := [y>f(x)]$, and $\tilde{v}_\pm := \tilde{v}|_{\Omega_\pm}$. Then, $\tilde{v}_\pm \in C(\overline{\Omega}_\pm) \cap C^1(\Omega_\pm)$ and

$$\tilde{v}_\pm(x,y) \to 0 \quad \text{for} \quad |(x,y)| \to \infty. \tag{A-9}$$

Proof. It is easy to see that $\tilde{v}_\pm \in C^1(\Omega_\pm)$. Plemelj's formula further shows that $\tilde{v}_\pm \in C(\overline{\Omega}_\pm)$ and

$$\tilde{v}_\pm(x,f(x)) = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}} \frac{(-(f(x)-f(x-s)),\, s)}{s^2+(f(x)-f(x-s))^2}\,\bar{\omega}(x-s)\,ds \mp \frac{1}{2}\frac{(1,f'(x))\bar{\omega}(x)}{1+f'^2(x)}, \qquad x \in \mathbb{R},$$

or equivalently, with the notation in Remark 3.3,

$$\tilde{v}_\pm|_{[y=f(x)]} = \pm\frac{1}{2}\frac{(1,f')\bar{\omega}}{1+f'^2} - \frac{1}{2\pi}B_{1,1}(f,f)[\bar{\omega}] + \frac{i}{2\pi}B_{0,1}(f)[\bar{\omega}].$$

Recalling that $f \in H^5(\mathbb{R})$ and $\bar{\omega} \in H^2(\mathbb{R})$, the arguments in the proof of Lemma 3.5 show that $B_{1,1}(f,f)[\bar{\omega}]$ and $B_{0,1}(f)[\bar{\omega}]$ belong to $H^1(\mathbb{R})$; thus

$$\tilde{v}_\pm(x,f(x)) \to 0 \quad \text{for } |x| \to \infty. \tag{A-10}$$

Furthermore, since f and $\bar{\omega}$ vanish at infinity, we find, much as in the proof of (A-2), that

$$\sup_{[y\geq n]} |\tilde{v}_+| + \sup_{[y\leq -n]} |\tilde{v}_-| \to 0 \quad \text{for } n \to \infty \tag{A-11}$$

and that, for arbitrary $0<a<b$,

$$\sup_{[a\leq y\leq b]\cap[|x|\geq n]} |\tilde{v}_+| + \sup_{[-b\leq y\leq -a]\cap[|x|\geq n]} |\tilde{v}_-| \to 0 \quad \text{for } n \to \infty. \tag{A-12}$$

Finally, arguing along the lines of the proof of Privalov's theorem, see [Lu 1993, Theorem 3.1.1] (the lengthy details, which for $\bar{\omega} \in W^1_\infty(\mathbb{R})$ are simpler than in that book, are left to the interested reader), it follows that there exists a constant C, which depends only on f and $\bar{\omega}$, such that for each $z=(x,y) \in \mathbb{R}^2 \setminus [y=f(x)]$ with $y \in [-\|f\|_\infty - 1, \|f\|_\infty + 1]$ the following inequalities hold:

$$\begin{aligned} |\tilde{v}_+(z) - \tilde{v}_+(x,f(x))| &\leq C|y-f(x)|^{1/2} \quad \text{if } y > f(x),\\ |\tilde{v}_-(z) - \tilde{v}_-(x,f(x))| &\leq C|y-f(x)|^{1/2} \quad \text{if } y < f(x). \end{aligned} \tag{A-13}$$

The relation (A-9) is an obvious consequence of (A-10)–(A-13). □

Appendix B: The proof of Theorem 1.5

This section is dedicated to the proof of Theorem 1.5. In the following $\mathbb{E}_0$ and $\mathbb{E}_1$ denote complex Banach spaces[4] and we assume that the embedding $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$ is dense. In view of [Amann 1995, Theorem I.1.2.2], we may represent the set $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ of negative analytic generators as

$$\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) = \bigcup_{\substack{\kappa \geq 1 \\ \omega > 0}} \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0, \kappa, \omega),$$

where, given $\kappa \geq 1$ and $\omega > 0$, the class $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0, \kappa, \omega)$ consists of the operators $\mathbb{A} \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$ having the properties

- $\omega + \mathbb{A} \in \mathrm{Isom}(\mathbb{E}_1, \mathbb{E}_0)$, and
- $\kappa^{-1} \leq \dfrac{\|(\lambda + \mathbb{A})x\|_0}{|\lambda| \cdot \|x\|_0 + \|x\|_1} \leq \kappa$ for all $0 \neq x \in \mathbb{E}_1$ and all $\mathrm{Re}\,\lambda \geq \omega$.

Given $\mathbb{A} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0, \kappa, \omega)$ and $r \in (0, \kappa^{-1})$, it follows from [Amann 1995, Theorem I.1.3.1(i)] that

$$\mathbb{A} + B \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0, \kappa/(1 - \kappa r), \omega) \quad \text{for all } \|B\|_{\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)} \leq r. \tag{B-1}$$

This property shows in particular that $\mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ is an open subset of $\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$.

The proof of Theorem 1.5 uses to a large extent the powerful theory of parabolic evolution operators developed in [Amann 1995]. The following result is a direct consequence of Theorem II.5.1.1, Lemma II.5.1.3 and Lemma II.5.1.4 in that paper.

Proposition B.1. *Let $T > 0$, $\rho \in (0, 1)$, $L \geq 0$, $\kappa \geq 1$, and $\omega > 0$ be given constants. Moreover, let $\mathcal{A} \subset C^{\rho}([0, T], \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0))$ be a family satisfying*

- $[\mathbb{A}]_{\rho,[0,T]} := \sup\limits_{t \neq s \in [0,T]} \dfrac{\|\mathbb{A}(t) - \mathbb{A}(s)\|}{|t - s|^{\rho}} \leq L$ *for all $\mathbb{A} \in \mathcal{A}$, and*
- $\mathbb{A}(t) \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0, \kappa, \omega)$ *for all $\mathbb{A} \in \mathcal{A}$ and $t \in [0, T]$.*

Then, given $\mathbb{A} \in \mathcal{A}$, there exists a unique parabolic evolution operator[5] *$U_{\mathbb{A}}$ for $\mathbb{A}$ possessing $\mathbb{E}_1$ as a regularity subspace. Moreover, the following hold:*

(i) *There exists a constant $C > 0$ such that*

$$\|U_{\mathbb{A}}(t, s)\|_{\mathcal{L}(\mathbb{E}_j)} + (t - s)\|U_{\mathbb{A}}(t, s)\|_{\mathcal{L}(\mathbb{E}_0, \mathbb{E}_1)} \leq C \tag{B-2}$$

for all $(t, s) \in \Delta_T^* := \{(t, s) \in [0, T]^2 : 0 \leq s < t \leq T\}$, $j \in \{0, 1\}$, and all $\mathbb{A} \in \mathcal{A}$.

(ii) *Let $\Delta_T := \{(t, s) \in [0, T]^2 : 0 \leq s \leq t \leq T\}$ and $0 \leq \beta \leq \alpha \leq 1$. Then, given $x \in \mathbb{E}_\alpha$, it holds that $U_{\mathbb{A}}(\cdot\,, \cdot)x \in C(\Delta_T, \mathbb{E}_\alpha)$. Moreover, $U_{\mathbb{A}} \in C(\Delta_T^*, \mathcal{L}(\mathbb{E}_\beta, \mathbb{E}_\alpha))$, and there exists a constant $C > 0$ such that*

$$(t - s)^{\alpha - \beta}\|U_{\mathbb{A}}(t, s)\|_{\mathcal{L}(\mathbb{E}_\beta, \mathbb{E}_\alpha)} \leq C \tag{B-3}$$

for all $(t, s) \in \Delta_T^*$ and all $\mathbb{A} \in \mathcal{A}$.

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[4]The proof of Theorem 1.5 in the context of real Banach spaces is identical.

[5]In the sense of [Amann 1995, Section II].

(iii) *Given* $0 \le \beta < 1$ *and* $0 < \alpha \le 1$, *there exists a constant* $C > 0$ *such that*

$$(t-s)^{\beta-\alpha}\|(U_{\mathbb{A}} - U_{\mathbb{B}})(t,s)\|_{\mathcal{L}(\mathbb{E}_\alpha,\mathbb{E}_\beta)} \le C \max_{\tau\in[s,t]} \|\mathbb{A}(\tau) - \mathbb{B}(\tau)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)} \tag{B-4}$$

for all $(t,s) \in \Delta_T^*$ *and all* $\mathbb{A},\ \mathbb{B} \in \mathcal{A}$.

Let now $\mathcal{A}$ be a family as in Proposition B.1. Given $\mathbb{A} \in \mathcal{A}$ and $x \in \mathbb{E}_0$, we consider the linear problem

$$\dot{u} + \mathbb{A}(t)u = 0, \qquad t \in (0,T], \qquad u(0) = x. \tag{B-5}$$

Using the fundamental properties of the parabolic evolution operator $U_{\mathbb{A}}$ associated to $\mathbb{A}$, it follows from [Amann 1995, Remark II.2.1.2] that (B-5) has a unique classical solution $u := u(\,\cdot\,; x, \mathbb{A})$, that is,

$$u := u(\,\cdot\,; x, \mathbb{A}) \in C^1((0,T], \mathbb{E}_0) \cap C((0,T], \mathbb{E}_1) \cap C([0,T], \mathbb{E}_0)$$

and u satisfies the equation of (B-5) pointwise. This solution is given by the expression

$$u(t) = U_{\mathbb{A}}(t,0)x, \qquad t \in [0,T].$$

If $x \in \mathbb{E}_\alpha$ for some $\alpha \in (0,1)$, we may use the relations (B-2)–(B-4) to derive additional regularity properties for the solution, as stated below.

Proposition B.2. *Let* $\mathcal{A}$ *be a family as in Proposition B.1. The following hold true*:

(i) *Let* $0 \le \beta \le \alpha < 1$ *and* $x \in \mathbb{E}_\alpha$. *Then* $u \in C^{\alpha-\beta}([0,T], \mathbb{E}_\beta)$ *and there exists* $C > 0$ *such that*

$$\|u(t) - u(s)\|_\beta \le C(t-s)^{\alpha-\beta}\|x\|_\alpha \tag{B-6}$$

for all $(t,s) \in \Delta_T,\ x \in \mathbb{E}_\alpha$, *and* $\mathbb{A} \in \mathcal{A}$.

(ii) *Let* $0 \le \beta < \alpha \le 1$. *Then, there exists* $C > 0$ *such that*

$$\|u(t; x, \mathbb{A}) - u(t; x, \mathbb{B})\|_\beta \le Ct^{\alpha-\beta} \max_{\tau\in[0,t]} \|\mathbb{A}(\tau) - \mathbb{B}(\tau)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)}\|x\|_\alpha \tag{B-7}$$

for all $t \in [0,T],\ x \in \mathbb{E}_\alpha$, *and* $\mathbb{A},\ \mathbb{B} \in \mathcal{A}$.

Proof. The claim (i) follows from [Amann 1995, Theorem II.5.3.1], while (ii) is a consequence of Theorem II.5.2.1 of the same book. □

By means of a contraction argument we now obtain as a preliminary result the following (uniform) local existence theorem, which stays at the basis of Theorem 1.5.

Proposition B.3. *Let the assumptions of Theorem 1.5 be satisfied and let* $\bar{f} \in \mathcal{O}_\alpha := \mathcal{O}_\beta \cap \mathbb{E}_\alpha$. *Then, there exist constants* $\delta = \delta(\bar{f}) > 0$ *and* $r = r(\bar{f}) > 0$ *with the property that for all* $f_0 \in \mathcal{O}_\alpha$ *with* $\|f_0 - \bar{f}\|_\alpha \le r$ *the problem*

$$\dot{f} = \Phi(f)[f], \qquad t > 0, \qquad f(0) = f_0, \tag{QP}$$

possesses a classical solution

$$f \in C([0,\delta], \mathcal{O}_\alpha) \cap C((0,\delta], \mathbb{E}_1) \cap C^1((0,\delta], \mathbb{E}_0) \cap C^{\alpha-\beta}([0,\delta], \mathbb{E}_\beta).$$

Moreover, if h is a further solution to (QP) with

$$h \in C((0,\delta],\mathbb{E}_1)\cap C^1((0,\delta],\mathbb{E}_0)\cap C^\eta([0,\delta],\mathcal{O}_\beta) \quad \textit{for some } \eta\in(0,\alpha-\beta],$$

then $f\equiv h$.

Proof. Existence: We first note that $\mathcal{O}_\alpha$ is an open subset of $\mathbb{E}_\alpha$; see, e.g., [Amann 1995, Section I.2.11]. Since by assumption $-\Phi\in C^{1-}(\mathcal{O}_\beta,\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0))$, it follows from (B-1) there exist constant $R>0,\ L>0$, $\kappa\geq 1$, and $\omega>0$ such that

$$\|\Phi(f)-\Phi(g)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)}\leq L\|f-g\|_\beta \quad \text{for all } f,\ g\in\overline{\mathbb{B}}_{\mathbb{E}_\beta}(\bar f,R)\subset\mathcal{O}_\beta, \tag{B-8}$$

$$-\Phi(f)\in\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0,\kappa,\omega) \quad \text{for all } f\in\overline{\mathbb{B}}_{\mathbb{E}_\beta}(\bar f,R). \tag{B-9}$$

Let $\rho\in(0,\alpha-\beta)$ be fixed. If $r>0$ is sufficiently small, it holds that

$$\overline{\mathbb{B}}_{\mathbb{E}_\alpha}(\bar f,r)\subset\overline{\mathbb{B}}_{\mathbb{E}_\beta}(\bar f,R)\cap\mathcal{O}_\alpha. \tag{B-10}$$

Given $\delta>0,\ r>0$ such that (B-10) holds (r and δ will be fixed later on) and $f_0\in\overline{\mathbb{B}}_{\mathbb{E}_\alpha}(\bar f,r)$, we define the set

$$\mathbb{M}:=\bigl\{f\in C([0,\delta],\overline{\mathbb{B}}_{\mathbb{E}_\beta}(\bar f,R)): f(0)=f_0 \text{ and } \|f(t)-f(s)\|_\beta\leq|t-s|^\rho \text{ for all } t,s\in[0,\delta]\bigr\}.$$

Since $\mathbb{M}$ is a closed subset of $C([0,\delta],\mathbb{E}_\beta)$, it is also a (nonempty) complete metric space. Given $f\in\mathbb{M}$, we define

$$\mathbb{A}_f(t):=-\Phi(f(t)),\qquad t\in[0,\delta].$$

As a direct consequence of (B-8) and of the definition of $\mathbb{M}$, it follows that

$$\|\mathbb{A}_f(t)-\mathbb{A}_f(s)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)}\leq L\|f(t)-f(s)\|_\beta\leq L|t-s|^\rho,\qquad t,s\in[0,\delta],$$

and (B-9) yields that $\mathbb{A}_f(t)\in\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0,\kappa,\omega)$ for all $f\in\mathbb{M}$ and all $t\in[0,\delta]$. Proposition B.1 ensures the existence of a parabolic evolution operator $U_{\mathbb{A}_f}$ for $\mathbb{A}_f$. Given $f\in\mathbb{M}$, it is natural to consider the linear evolution problem

$$\dot g+\mathbb{A}_f(t)g=0,\qquad t\in(0,\delta],\qquad g(0)=f_0, \tag{B-11}$$

which has, in view of Proposition B.2, a unique classical solution

$$g:=\Gamma(f):=U_{\mathbb{A}_f}(\,\cdot\,,0)f_0\in C^{\alpha-\beta}([0,\delta],\mathbb{E}_\beta)\cap C([0,\delta],\mathbb{E}_\alpha).$$

The existence part of Proposition B.1 reduces to proving that $\Gamma:\mathbb{M}\to\mathbb{M}$ is a strict contraction for suitable r and δ. Clearly $\Gamma(f)(0)=f_0$. Moreover, (B-6) yields

$$\|\Gamma(f)(t)-\Gamma(f)(s)\|_\beta\leq C|t-s|^{\alpha-\beta}\|f_0\|_\alpha\leq C\delta^{\alpha-\beta-\rho}(\|\bar f\|_\alpha+r)|t-s|^\rho\leq|t-s|^\rho \quad \text{for all } t,s\in[0,\delta],$$

provided that

$$C\delta^{\alpha-\beta-\rho}(\|\bar f\|_\alpha+r)\leq 1. \tag{B-12}$$

The latter estimate (with $s=0$) yields

$$\|\Gamma(f)(t)-\bar{f}\|_\beta \le \|\Gamma(f)(t)-\Gamma(f)(0)\|_\beta + \|f_0-\bar{f}\|_\beta \le \delta^\rho + r\|i_{\mathbb{E}_\alpha\hookrightarrow\mathbb{E}_\beta}\|_{\mathcal{L}(\mathbb{E}_\alpha,\mathbb{E}_\beta)} \le R \quad \text{for all } t\in[0,\delta],$$

if we additionally require that

$$\delta^\rho + r\|i_{\mathbb{E}_\alpha\hookrightarrow\mathbb{E}_\beta}\|_{\mathcal{L}(\mathbb{E}_\alpha,\mathbb{E}_\beta)} \le R. \tag{B-13}$$

We now assume that r and δ are chosen such that (B-12)–(B-13) hold true. It then follows that $\Gamma:\mathbb{M}\to\mathbb{M}$ is a well-defined map. Furthermore, given $f,\ h\in\mathbb{M}$, the estimate (B-7) together with (B-8) yields

$$\begin{aligned}\|\Gamma(f)(t)-\Gamma(h)(t)\|_\beta &= \|U_{\mathbb{A}_f}(t,0)f_0-U_{\mathbb{A}_h}(t,0)f_0\|_\beta\\ &\le Ct^{\alpha-\beta}\max_{\tau\in[0,t]}\|\mathbb{A}_f(\tau)-\mathbb{A}_h(\tau)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)}\|f_0\|_\alpha\\ &\le C\delta^{\alpha-\beta}L(\|\bar{f}\|_\alpha+r)\max_{t\in[0,\delta]}\|f(t)-h(t)\|_\beta\\ &\le \tfrac{1}{2}\max_{t\in[0,\delta]}\|f(t)-h(t)\|_\beta \quad \text{for all } t\in[0,\delta],\end{aligned}$$

provided that

$$C\delta^{\alpha-\beta}L(\|\bar{f}\|_\alpha+r)\le \tfrac{1}{2}. \tag{B-14}$$

Hence, if r and δ are chosen such that also (B-14) is satisfied, then Γ is a strict contraction and Banach's fixed-point theorem ensures that Γ has a fixed point. This proves the existence part.

Uniqueness: Let f be a solution to (QP) as found above and let $h\not\equiv f$ be a further classical solution such that $h\in C^\eta([0,\delta],\mathbb{E}_\beta)$ for some $\eta\in(0,\alpha-\beta]$. The real number

$$t_0 := \max\{t\in[0,\delta] : f|_{[0,t]}=h|_{[0,t]}\}$$

satisfies $0\le t_0<\delta$ and $f=h$ on $[0,t_0]$. Since $f(t_0)\in\mathcal{O}_\alpha$, there exist $R>0$, $L>0$, $\kappa\ge 1$, and $\omega>0$ such that

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{\mathcal{L}(\mathbb{E}_1,\mathbb{E}_0)} \le L\|u-v\|_\beta &\quad \text{for all } u,\ v\in\overline{\mathbb{B}}_{\mathbb{E}_\beta}(f(t_0),R)\subset\mathcal{O}_\beta,\\ -\Phi(u)\in\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0,\kappa,\omega) &\quad \text{for all } u\in\overline{\mathbb{B}}_{\mathbb{E}_\beta}(f(t_0),R).\end{aligned}$$

Given $\delta_0\in(t_0,\delta]$, the set

$$\mathbb{M}_0 := \left\{h\in C([0,\delta_0-t_0],\overline{\mathbb{B}}_{\mathbb{E}_\beta}(f(t_0),R)) : h(0)=f(t_0),\ \frac{\|h(t)-h(s)\|_\beta}{|t-s|^{\eta/2}}\le 1 \text{ for all } t\ne s\in[0,\delta_0-t_0]\right\}$$

is a (nonempty) complete metric space. Letting $\mathbb{A}_h(t):=-\Phi(h(t))$ for $h\in\mathbb{M}_0$ and $t\in[0,\delta_0-t_0]$, we may argue as in the existence part of this proof to conclude that the linear problem

$$\dot{u}+\mathbb{A}_h(t)u=0,\quad t\in(0,\delta_0-t_0],\quad h(0)=f(t_0)$$

has a unique classical solution $\Gamma_0(h)\in C^{\alpha-\beta}([0,\delta_0-t_0],\mathbb{E}_\beta)\cap C([0,\delta_0-t_0],\mathbb{E}_\alpha)$. Furthermore, $\Gamma_0:\mathbb{M}_0\to\mathbb{M}_0$ is a $\frac{1}{2}$-contraction provided that δ_0 is sufficiently close to t_0; hence Γ_0 has a unique fixed point. But, if δ_0-t_0 is sufficiently small, then it can be easily seen that $f(\cdot+t_0)|_{[0,\delta_0-t_0]}$ and $h(\cdot+t_0)|_{[0,\delta_0-t_0]}$ both belong to $\mathbb{M}_0$ and these functions are therefore fixed points of Γ_0. This implies $f=h$ on $[0,\delta_0]$ for some $\delta_0>t_0$, in contradiction with the definition of t_0. This proves the uniqueness claim. ☐

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Let $f_0 \in \mathcal{O}_\alpha$ be given. According to Proposition B.3 (with $\bar{f} := f_0$), there exists $\delta > 0$ and a classical solution

$$f \in C([0,\delta], \mathcal{O}_\alpha) \cap C((0,\delta], \mathbb{E}_1) \cap C^1((0,\delta], \mathbb{E}_0) \cap C^{\alpha-\beta}([0,\delta], \mathbb{E}_\beta)$$

to (QP). This solution can be continued as follows. Applying Proposition B.3 (with $\bar{f} := f(\delta)$), we find $r > 0$ and $\delta_1 > 0$ such that

$$\dot{h} = \Phi(h)[h], \quad t \in (0, \delta_1], \quad h(0) = f_1 \tag{B-15}$$

has a classical solution $h \in C([0,\delta_1], \mathcal{O}_\alpha) \cap C((0,\delta_1], \mathbb{E}_1) \cap C^1((0,\delta_1], \mathbb{E}_0) \cap C^{\alpha-\beta}([0,\delta_1], \mathbb{E}_\beta)$ for each $f_1 \in \mathcal{O}_\alpha$ with $\|f_1 - f(t_0)\|_\alpha \le r$. Let $t_0 \in (0,\delta)$ be such that

$$t_0 + \delta_1 > \delta \quad \text{and} \quad \|f(t_0) - \bar{f}\|_\alpha \le r.$$

Hence it is possible to choose $f_1 := f(t_0)$ as an initial value in (B-15). Since $f(\cdot + t_0) : [0, \delta - t_0] \to \mathcal{O}_\alpha$ and $h : [0, \delta - t_0] \to \mathcal{O}_\alpha$ are both classical solutions to

$$\dot{h} = \Phi(h)[h], \quad t \in (0, \delta - t_0], \quad h(0) = f(t_0),$$

by Proposition B.3 they must coincide. Consequently, the function $F : [0, t_0 + \delta_1] \to \mathcal{O}_\alpha$ defined by

$$F(t) := \begin{cases} f(t), & t \in [0,\delta], \\ h(t - t_0), & t \in [\delta, t_0 + \delta_1] \end{cases}$$

is a classical solution to (QP) which extends f. The maximal solution $f = f(\cdot\,; f_0) : I(f_0) \to \mathcal{O}_\alpha$ in Theorem 1.5 is defined by setting

$$I(f_0) := \bigcup \big\{[0,\delta] : \text{(QP) has a classical solution } f_\delta \text{ on } [0,\delta] \text{ with } f_\delta \in C^{\alpha-\beta}([0,\delta], \mathbb{E}_\beta)\big\},$$
$$f(t) := f_\delta(t) \quad \text{for } t \in [0,\delta].$$

The construction above shows that f is well-defined and that $I(f_0) = [0, T_+(f_0))$ with $T_+(f_0) \le \infty$. This proves the existence claim in Theorem 1.5. The uniqueness assertion is an immediate consequence of Proposition B.3.

We now prove the criterion for global existence. Hence, let us assume that the unique classical maximal solution $f = f(\cdot\,; f_0) : [0, T_+(f_0)) \to \mathcal{O}_\alpha$ to (QP) is uniformly continuous when restricted to each interval $[0,T] \cap [0, T_+(f_0))$, with $T > 0$ arbitrary. We further assume that $\tau := T_+(f_0) < \infty$; otherwise we are done. Then, since f is uniformly continuous on $[0,\tau)$, it is straightforward to see that the limit

$$f(\tau) := \lim_{t \nearrow \tau} f(t)$$

exists in $\overline{\mathcal{O}}_\alpha$. If $\operatorname{dist}(f(t), \partial\mathcal{O}_\alpha) \not\to 0$ for $t \to \tau$, it must hold that $f(\tau) \in \mathcal{O}_\alpha$. Proceeding as above, we may extend in view of Proposition B.3 this maximal solution to an interval $[0, \tau + \delta_1)$ for some $\delta_1 > 0$, which is a contradiction and we are done.

Finally, the semiflow property of the solution map $[(t, f_0) \mapsto f(t; f_0)]$ stated at the end of Theorem 1.5 is proven in detail in [Amann 1988, Theorem 8.1]. Furthermore, if Φ is additionally smooth, then proceeding as in Theorem 11.3 of the same paper one may show that the semiflow map is also smooth. For real-analytic Φ, the real-analyticity of $[(t, f_0) \mapsto f(t; f_0)]$ follows by estimating the Fréchet derivatives of the flow map, which is a rather tedious and lengthy procedure which we refrain from presenting here. □

Acknowledgement

The author would like to thank Christoph Thiele for the discussion on an issue related to the analysis in Section 3.

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Received 18 Oct 2016. Revised 17 Jan 2018. Accepted 7 May 2018.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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Analysis & PDE

Volume 12 No. 2 2019