ANALYSIS & PDE

Volume 11 No. 4 2018

IVICA NAKIĆ, MATTHIAS TÄUFER,
MARTIN TAUTENHAHN AND IVAN VESELIĆ

SCALE-FREE UNIQUE CONTINUATION PRINCIPLE FOR SPECTRAL PROJECTORS, EIGENVALUE-LIFTING AND WEGNER ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS

msp

ANALYSIS AND PDE
Vol. 11, No. 4, 2018
dx.doi.org/10.2140/apde.2018.11.1049

SCALE-FREE UNIQUE CONTINUATION PRINCIPLE FOR SPECTRAL PROJECTORS, EIGENVALUE-LIFTING AND WEGNER ESTIMATES FOR RANDOM SCHRÖDINGER OPERATORS

IVICA NAKIĆ, MATTHIAS TÄUFER, MARTIN TAUTENHAHN AND IVAN VESELIĆ

We prove a scale-free, quantitative unique continuation principle for functions in the range of the spectral projector $\chi_{(-\infty,E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$, with bounded potential. Previously, such estimates were known only for individual eigenfunctions and for spectral projectors $\chi_{(E-\gamma,E]}(H_L)$ with small γ. Such estimates are also called, depending on the context, uncertainty principles, observability estimates, or spectral inequalities. Our main application of such an estimate is to find lower bounds for the lifting of eigenvalues under semidefinite positive perturbations, which in turn can be applied to derive a Wegner estimate for random Schrödinger operators with nonlinear parameter-dependence. Another application is an estimate of the control cost for the heat equation in a multiscale domain in terms of geometric model parameters. Let us emphasize that previous uncertainty principles for individual eigenfunctions or spectral projectors onto small intervals were not sufficient to study such applications.

1. Introduction

We prove a *quantitative unique continuation* inequality, announced in [Nakić et al. 2015b], for functions in the range of the spectral projector $\chi_{(-\infty,E]}(H_L)$ of a Schrödinger operator H_L on a cube of side $L \in \mathbb{N}$. Depending on the area of mathematics and the context, estimates of this type have various names: quantitative unique continuation principles (UCP), uncertainty principles, spectral inequalities, observability or sampling estimates, or bounds on the vanishing order. For our applications it is crucial (i) to exhibit explicitly the dependence of the quantitative unique continuation inequality on the model parameters, and (ii) to allow energy intervals $(-\infty, E]$ of arbitrary length, that is, for arbitrary E. If the observability or sampling set respects in a certain way the underlying lattice structure, our estimate is independent of L; for this reason we call it *scale-free*. This property is crucial for applications where one studies spectral properties of the Schrödinger operator H_L in the thermodynamic limit $L \nearrow \infty$.

A key motivation to study scale-free quantitative unique continuation estimates comes from the theory of random Schrödinger operators, in particular with *nonlinear dependence* on the random variables. The class of operators considered here includes the random breather model as studied in [Combes et al. 1996; 2001; Täufer and Veselić 2015; 2016]. Models with nonlinear randomness constitute a step towards a better understanding of the universality of Anderson localization.

MSC2010: primary 35J10, 35P15, 35Q82, 35R60, 81Q10; secondary 81Q15.
Keywords: uncertainty relation, spectral inequality, Wegner estimate, control of heat equation, random Schroedinger operator.

We establish eigenvalue-lifting estimates, Wegner bounds, and the continuity of the integrated density of states. (We defer precise definitions to Section 2.) In fact, there are a number of previous papers which have derived a scale-free UCP and eigenvalue-lifting estimates under special assumptions.

Naturally, the first situation to be considered was the case where the Schrödinger operator is the pure Laplacian $H = -\Delta$, i.e., the background potential V vanishes identically. For instance, [Kirsch 1996] derives a UCP which is valid for energies in an interval at zero, i.e., the bottom of the spectrum, if one has a periodic arrangement of sampling sets. The proof uses detailed information about hitting probabilities of Brownian motion paths, and is related to Harnack inequalities. A very elementary approach to eigenvalue-lifting estimates is provided by the spatial averaging trick, used in [Bourgain and Kenig 2005; Germinet et al. 2007] in periodic situations, and extended to nonperiodic situations in [Germinet 2008]. It is applicable to energies near zero. A different approach for eigenvalue-lifting was derived in [Boutet de Monvel et al. 2006]. In [Boutet de Monvel et al. 2011] it was shown how one can conclude an uncertainty principle at low energies based on an eigenvalue-lifting estimate. Related results have been derived for energies near spectral edges in [Kirsch et al. 1998; Combes et al. 2001] using resolvent comparison. In one space dimension, eigenvalue-lifting results and Wegner estimates have been proven in [Veselić 1996; Kirsch and Veselić 2002]. There a periodic arrangement of the sampling set is assumed. The proof carries over to the case of nonperiodic arrangements verbatim, which has been spelled out and used in the context of quantum graphs in [Helm and Veselić 2007]. In the case that both the deterministic background potential and the sampling set are periodic, an uncertainty principle and a Wegner estimate, which are valid for arbitrary bounded energy regions, have been proven in [Combes et al. 2003; 2007]. These papers make use of Floquet theory; hence they are a priori restricted to periodic background potentials as well as periodic sampling sets. An alternative proof for the result in [Combes et al. 2007], with more explicit control of constants, has been worked out in [Germinet and Klein 2013]. The case where the background potential is periodic but the impurities need not be periodically arranged has been considered in [Boutet de Monvel et al. 2006; Germinet 2008] for low energies. Our main theorem *unifies and generalizes all the results* mentioned so far and makes the dependence on the model parameters quantitative. Indeed, our scale-free unique continuation principle answers positively a question asked in [Rojas-Molina and Veselić 2013]. A partial answer was given already in [Klein 2013]. While [Rojas-Molina and Veselić 2013] concerns the case of a single eigenfunctions, [Klein 2013] uses a very nice perturbation argument to treat linear combinations of eigenfunctions corresponding to eigenvalues which lie in an interval whose size is smaller than an explicitly determined number. For a broader discussion we refer to the summer school notes [Täufer et al. 2016].

A second application of our scale-free UCP is in the control theory of the heat equation. Here one asks whether one can drive a given initial state to a desired state with a control function living in a specified subset, and what the minimal L^2-norm of the control function (called control cost) is. Recently, the search for optimal placement of the control set and the dependence of the control cost on geometric features of this set has received much attention; see, e.g., [Privat et al. 2015b; 2015a]. Our scale-free UCP gives an explicit estimate of the control cost with respect to the model parameters in multiscale domains. While this is of interest in itself, *our main motivation* to include the application to control theory in our paper is to bring to attention the relation between methods and ideas from this field and the theory of random Schrödinger operators. This relation has not been explored before and it seems that it can be effectively used in other problems of random operators.

Other authors have applied our main result, as announced in [Nakić et al. 2015b], to prove decorrelation estimates for eigenvalues of random Schrödinger operators [Shirley 2015] and lower bounds on averaged spectral shift functions [Dietlein et al. 2017]. We will generalize the methods of the present paper to certain unbounded domains in $\mathbb{R}^d$ in a forthcoming paper, while two of us have extended the results to certain infinite-dimensional spectral subspaces in [Täufer and Tautenhahn 2017].

Our proof of the scale-free unique continuation estimate uses two Carleman and nested interpolation bounds to obtain propagation of smallness estimates, an idea used before, e.g., in [Lebeau and Robbiano 1995; Jerison and Lebeau 1999]. Roughly speaking, one of the Carleman estimates establishes propagation of smallness from a set of codimension one to a small ball, and the other one from a small ball to a larger ball. To obtain explicit estimates we need explicit weight functions. The first Carleman estimate includes a boundary term and uses a parabolic weight function as proposed in [Jerison and Lebeau 1999]. The second Carleman estimate is similar to the ones in [Escauriaza and Vessella 2003; Bourgain and Kenig 2005]. However, neither of the two is quite sufficient for our purposes, so we use a variant developed in [Nakić et al. 2015a]. A similar result was established recently in [Davey 2014]. Moreover, at first sight it seems that one can get our result simply by summing up doubling estimates (which are a standard consequence of Carleman estimates). However, the prefactor in the doubling estimate depends on the ambient space, in particular its diameter. In our case we consider a family of domains Λ_L, $L \in \mathbb{N}$, and the diameter grows unboundedly in L; hence the constant in the doubling estimate becomes worse and worse. Thus, to eliminate the L-dependence we have to use techniques developed in the context of random Schrödinger operators to accommodate for the multiscale structure of the underlying domain and sampling set.

In the next section we state our main results. Section 3 is devoted to the proof of the scale-free unique continuation principle, Section 4 to proofs concerning random Schrödinger operators, and Section 5 to the observability estimate of the control equation, while certain technical aspects are deferred to the Appendix.

2. Results

Scale-free unique continuation and eigenvalue lifting. Let $d \in \mathbb{N}$. For $L > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d \subset \mathbb{R}^d$ the cube with side length L, and by Δ_L the Laplace operator on $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions. Moreover, for a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ we denote by $V_L : \Lambda_L \to \mathbb{R}$ its restriction to Λ_L given by $V_L(x) = V(x)$ for $x \in \Lambda_L$, and by

$$H_L = -\Delta_L + V_L \quad \text{on } L^2(\Lambda_L)$$

the corresponding Schrödinger operator. Note that H_L has purely discrete spectrum. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B(x, r)$ the ball with center x and radius r with respect to Euclidean norm. If the ball is centered at zero we write $B(r) = B(0, r)$.

Definition 2.1. Let $G > 0$ and $\delta > 0$. We say that a sequence $z_j \in \mathbb{R}^d$, $j \in (G\mathbb{Z})^d$, is (G,δ)-*equidistributed*, if

$$\text{for all } j \in (G\mathbb{Z})^d \text{ we have } B(z_j,\delta) \subset \Lambda_G + j.$$

Corresponding to a (G,δ)-equidistributed sequence we define for $L \in G\mathbb{N}$ the set

$$W_\delta(L) = \bigcup_{j\in(G\mathbb{Z})^d} B(z_j,\delta) \cap \Lambda_L.$$

Theorem 2.2. *There is $N = N(d)$ such that for all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1,\delta)$-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$ we have*

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq C_{\text{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)}, \tag{1}$$

where

$$C_{\text{sfuc}} = C_{\text{sfuc}}(d,\delta,E,\|V\|_\infty) := \delta^{N(1+\|V\|_\infty^{2/3}+\sqrt{E})}.$$

The result can be formulated in terms of spectral projectors. This is the convenient form to use in the context of random Schrödinger operators.

Corollary 2.3. *Under the same assumptions as in the above theorem, we have in the sense of quadratic forms*

$$\chi_{(-\infty,E]}(H_L)\ \chi_{W_\delta(L)}\ \chi_{(-\infty,E]}(H_L) \geq \delta^{N(1+\|V\|_\infty^{2/3}+\sqrt{E})} \chi_{(-\infty,E]}(H_L). \tag{2}$$

Here $\chi_{W_\delta(L)}$ denotes the multiplication operator with a characteristic function, and $\chi_{(-\infty,E]}(H_L)$ denotes a spectral projector.

The crucial point here is that we allow energy intervals $(-\infty, E]$ of arbitrary length. It is not possible to achieve this result with the methods of [Rojas-Molina and Veselić 2013; Klein 2013]. For $t, L > 0$ and a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ we define the Schrödinger operator $H_{t,L} = -t\Delta_L + V_L$ on $L^2(\Lambda_L)$. By scaling we obtain the following corollary.

Corollary 2.4. *Let $N = N(d)$ be the constant from Theorem 2.2. Then, for all $G, t > 0$, all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in G\mathbb{N}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_{t,L}))$ we have*

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq C^{G,t}_{\text{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)},$$

where

$$C^{G,t}_{\text{sfuc}} = C^{G,t}_{\text{sfuc}}(d,\delta,E,\|V\|_\infty) := \left(\frac{\delta}{G}\right)^{N(1+G^{4/3}\|V\|_\infty^{2/3}/t^{2/3}+G\sqrt{E/t})}.$$

Note that the set $W_\delta(L)$ depends on G and the choice of the (G,δ)-equidistributed sequence. In particular, there is a constant $M = M(d,G,t) \geq 1$ such that

$$C^{G,t}_{\text{sfuc}} \geq \delta^{M(1+\|V\|_\infty^{2/3}+\sqrt{|E|})}. \tag{3}$$

We also emphasize that Theorem 2.2 and Corollary 2.4 also hold for $E < 0$, since

$$\operatorname{Ran}(\chi_{(-\infty,E]}(H)) \subset \operatorname{Ran}(\chi_{(-\infty,0]}(H))$$

for any self-adjoint operator H.

Remark 2.5 (previous results). If $L = G$ the result is closely related to doubling estimates and bounds on the vanishing order; see [Lebeau and Robbiano 1995; Kukavica 1998; Jerison and Lebeau 1999; Bakri 2013]. These results, however, do not study the dependence of the bound on geometric data, e.g., the diameter of the domain or manifold. In the context of random Schrödinger operators results like (1) have been proven before under additional assumptions and using other methods: for $V \equiv 0$ and energies close to the minimum of the spectrum in [Kirsch 1996; Bourgain and Kenig 2005], near spectral edges of periodic Schrödinger operators in [Kirsch et al. 1998], and for periodic geometries $W_\delta(L)$ and potentials in [Combes et al. 2003]. More recently and using similar methods to ours, bounds like (1) have been established for individual eigenfunctions in [Rojas-Molina and Veselić 2013]. This has then been extended in [Klein 2013] to linear combinations of eigenfunctions corresponding to eigenvalues which are close to each other. For more references and a broader discussion of the history see, e.g., [Rojas-Molina and Veselić 2013; Klein 2013; Täufer et al. 2016].

As an application to spectral theory we have the following corollary. A proof is given at the end of Section 3.

Corollary 2.6. *Let $E, \alpha, G > 0$, $\delta \in (0, G/2)$, $L \in G\mathbb{N}$ and $A, B : \Lambda_L \to \mathbb{R}$ be measurable, bounded potentials and assume that*

$$B \geq \alpha \chi_{W_\delta(L)}$$

for a (G, δ)-equidistributed sequence. Denote the eigenvalues of a self-adjoint operator H with discrete spectrum by $\lambda_i(H)$, enumerated increasingly and counting multiplicities. Then for all $i \in \mathbb{N}$ with $\lambda_i(-\Delta + A + B) \leq E$, we have

$$\lambda_i(-\Delta_L + A + B) \geq \lambda_i(-\Delta_L + A) + \alpha C^{G,1}_{\mathrm{sfuc}}(d, \delta, E, \|A + B\|_\infty).$$

Remark 2.7 (generalizations). In [Täufer and Tautenhahn 2017] it has been proven that Corollary 2.4 holds also if $\chi_{(-\infty,E]}(H_L)$ is replaced by $\exp(-tH_L)$ for sufficiently large $t > 0$. An adaptation of our methods allows us to treat Schrödinger operators H on the whole of $\mathbb{R}^d$ instead on cubes. This will be discussed in our forthcoming paper. An important consequence of this result is a lifting estimate for boundaries of the essential spectrum, quite analogous to Corollary 2.6. Finally, let us remark that an analog of Theorem 2.2 for the case $V \equiv 0$ where the equidistributed set needs only to be measurable (and not open) has been established in [Egidi and Veselić 2016] using different methods.

Application to random breather Schrödinger operators. An important application of our result is in the spectral theory of random Schrödinger operators. The above scale-free unique continuation estimate is the key for proving the Wegner estimate formulated below, which is a bound on the expected number of eigenvalues in a short energy interval of a finite box restriction of our random Hamiltonian. Together with a so-called initial scale estimate, Wegner estimates facilitate a proof of Anderson localization via multiscale analysis. For more background on multiscale analysis and localization and on Wegner estimates consult, e.g., the monographs [Stollmann 2001] and [Veselić 2008], respectively.

The main point is that the potentials we are dealing with here exhibit a *nonlinear dependence* on the random parameters ω_j. Due to this challenge, it is not clear how to apply previously established versions of (1), as discussed in Remark 2.5, to such models. We emphasize that our scale-free unique continuation principle and Wegner estimate are valid for all bounded energy intervals, not only near the bottom of the spectrum.

Let us introduce a simple, but paradigmatic example of the models we are considering. (The general case will be studied in the next paragraph.)

Let $\mathcal{D}$ be a countable set to be specified later. For $0 \le \omega_- < \omega_+ < 1$ we define the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with

$$\Omega = \bigtimes_{j\in\mathcal{D}} \mathbb{R}, \quad \mathcal{A} = \bigotimes_{j\in\mathcal{D}} \mathcal{B}(\mathbb{R}) \quad \text{and} \quad \mathbb{P} = \bigotimes_{j\in\mathcal{D}} \mu,$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra and μ is a probability measure with $\operatorname{supp}\mu \subset [\omega_-, \omega_+]$ and a bounded density ν_μ. Hence, the projections $\omega \mapsto \omega_k$ give rise to a sequence of independent and identically distributed random variables ω_j, $j \in \mathcal{D}$. We denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$. The standard random breather model is defined as

$$H_\omega = -\Delta + V_\omega(x) \quad \text{with } V_\omega(x) = \sum_{j\in\mathbb{Z}^d} \chi_{B_{\omega_j}}(x-j), \tag{4}$$

and the restriction of H_ω to the box Λ_L is denoted by $H_{\omega,L}$. Here obviously $\mathcal{D} = \mathbb{Z}^d$. Denote by $\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})$ the spectral projector of $H_{\omega,L}$. We formulate now a version of our general Theorem 2.10 applied to the standard random breather model.

Theorem 2.8 (Wegner estimate for the standard random breather model). *Assume that* $[\omega_-, \omega_+] \subset \left[0, \frac{1}{4}\right]$, *fix* $E_0 \in \mathbb{R}$, *and set* $\varepsilon_{\max} = \frac{1}{4} \cdot 8^{-N(2+|E_0+1|^{1/2})}$, *where* N *is the constant from Theorem 2.2. Then there is* $C = C(d, E_0) \in (0, \infty)$ *such that for all* $\varepsilon \in (0, \varepsilon_{\max}]$ *and* $E \ge 0$ *with* $[E-\varepsilon, E+\varepsilon] \subset (-\infty, E_0]$, *we have*

$$\mathbb{E}\bigl[\operatorname{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})]\bigr] \le C \|\nu\|_\infty \varepsilon^{[N(2+|E_0+1|^{1/2})]^{-1}} |\ln\varepsilon|^d L^d.$$

Theorem 2.8 implies local Hölder continuity of the integrated density of states (IDS) and is sufficient for the multiscale analysis proof of spectral localization; see the next paragraph.

Remark 2.9 (previous results on the random breather model). The paper [Combes et al. 1996] introduced random breather potentials, while a Wegner estimate was proven in [Combes et al. 2001], however, excluding any bounded and any continuous single site potential; see the Appendix. Lifshitz tails for random breather Schrödinger operators were proven in [Kirsch and Veselić 2010]. All of the papers mentioned so far approached the breather model using techniques which have been developed for the alloy-type model. Consequently, at some stage the nonlinear dependence on the random variables was linearized, giving rise to certain differentiability conditions. As a result, characteristic functions of cubes or balls, which would be the most basic example one can think of, were excluded as single-site potentials. Only [Veselić 2007] considers a simple nondifferentiable example, namely the standard random breather potential in one dimension, and proves a Lifshitz tail estimate. This will be extended to multidimensional models in the forthcoming paper [Schumacher and Veselić 2017].

More general nonlinear models and localization. We formulate now a Wegner estimate for a general class of models, which includes the standard random breather potential, considered in the last paragraph as a special case. We state also an initial scale estimate which implies localization.

Here, in the general setting, we assume that $\mathcal{D} \subset \mathbb{R}^d$ is a Delone set; i.e., there are $0 < G_1 < G_2$ such that for any $x \in \mathbb{R}^d$ we have $\sharp\{\mathcal{D} \cap (\Lambda_{G_1} + x)\} \leq 1$ and $\sharp\{\mathcal{D} \cap (\Lambda_{G_2} + x)\} \geq 1$. Here, $\sharp\{\cdot\}$ stands for the cardinality. In other words, Delone sets are relatively dense and uniformly discrete subsets of $\mathbb{R}^d$. For more background about Delone sets, see, for example, the contributions in [Kellendonk et al. 2015]. The reader unacquainted with the concept of a Delone set can always think of $\mathcal{D} = \mathbb{Z}^d$.

Furthermore, let $\{u_t : t \in [0, 1]\} \subset L_0^\infty(\mathbb{R}^d)$ be functions such that there are $G_u \in \mathbb{N}$, $u_{\max} \geq 0$, $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ with

$$\begin{aligned} \forall t \in [0,1], &\quad \operatorname{supp} u_t \subset \Lambda_{G_u}, \\ \forall t \in [0,1], &\quad \|u_t\|_\infty \leq u_{\max}, \\ \forall t \in [\omega_-, \omega_+],\ \delta \leq 1-\omega_+,\ \exists x_0 \in \Lambda_{G_u}, &\quad u_{t+\delta} - u_t \geq \alpha_1 \delta^{\alpha_2} \chi_{B(x_0, \beta_1 \delta^{\beta_2})}. \end{aligned} \tag{5}$$

We define the family of Schrödinger operators H_ω, $\omega \in \Omega$, on $L^2(\mathbb{R}^d)$ given by

$$H_\omega := -\Delta + V_\omega, \quad \text{where } V_\omega(x) = \sum_{j \in \mathcal{D}} u_{\omega_j}(x - j).$$

Note that for all $\omega \in [0, 1]^{\mathcal{D}}$ we have

$$\|V_\omega\|_\infty \leq K_u := u_{\max} \lceil G_u / G_1 \rceil^d;$$

see Lemma 4.1. Assumption (5) includes many prominent models of random Schrödinger operators — linear and nonlinear. We give some examples.

Standard random breather model: Let μ be the uniform distribution on $\left[0, \frac{1}{4}\right]$ and let $u_t(x) = \chi_{B(0,t)}$, $j \in \mathbb{Z}^d$. Then $V_\omega = \sum_{j \in \mathbb{Z}^d} \chi_{B(j, \omega_j)}$ is the characteristic function of a disjoint union of balls with random radii. This model was introduced in the previous subsection.

General random breather models: Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$ and define $u_t(x) := u(x/t)$ for $t > 0$ and $u_0 :\equiv 0$ and assume that the family $\{u_t : t \in [0, 1]\}$ satisfies (5). Natural examples are discussed in the Appendix. They include the characteristic function of bounded convex sets, the hat-potential $(1 - |x|)\chi_{\{|x|<1\}}$ or the bump function $\exp(1/(|x|^2 - 1))\chi_{\{|x|<1\}}$. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} u_{\omega_j}(x - j)$ is a sum of random dilations of a single-site potential u at each lattice site $j \in \mathbb{Z}^d$.

Alloy-type model: Let $0 \leq u \in L_0^\infty(\mathbb{R}^d)$, $u \geq \alpha > 0$, on some open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \omega_j u(x - j)$ is a sum of copies of u at all lattice sites $j \in \mathbb{Z}^d$, multiplied with ω_j.

Delone-alloy-type model: Let $\mathcal{D} \subset \mathbb{R}^d$ be a Delone set, $0 \leq u \in L_0^\infty(\mathbb{R}^d)$, $u \geq \alpha > 0$, on some nonempty open set and let $u_t(x) := tu(x)$. Then $V_\omega(x) = \sum_{j \in \mathcal{D}} \omega_j u(x - j)$ is a sum of copies of u at all Delone points $j \in \mathcal{D}$, multiplied with ω_j. See [Germinet et al. 2015] for background on such models.

For $L > 0$ we denote by $H_{\omega,L}$ the restriction of H_ω to $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Following the methods developed in [Hundertmark et al. 2006], we obtain a Wegner estimate under our general assumption (5).

Theorem 2.10 (Wegner estimate). *For all $E_0 \in \mathbb{R}$ there are constants $C, \kappa, \varepsilon_{\max} > 0$, depending only on d, E_0, K_u, G_u, G_2, α_1, α_2, β_1, β_2, ω_+ and $\|\nu_\mu\|_\infty$, such that for all $L \in (G_2 + G_u)\mathbb{N}$, all $E \in \mathbb{R}$ and $\varepsilon \le \varepsilon_{\max}$ with $[E-\varepsilon, E+\varepsilon] \subset (-\infty, E_0]$ we have*

$$\mathbb{E}\big[\mathrm{Tr}[\chi_{[E-\varepsilon,E+\varepsilon]}(H_{\omega,L})]\big] \le C\varepsilon^{1/\kappa}|\ln\varepsilon|^d L^d. \tag{6}$$

Theorem 2.11 (initial scale estimate). *Let κ be as in Theorem 2.10 for $E_0 = d\pi^2 + K_u$. Assume that there are $t_0, C > 0$ such that*

$$0 \in \operatorname{supp}\mu \qquad \textit{and} \qquad \textit{for all } t \in [0, t_0], \quad \mu([0,t]) \le Ct^{d\kappa}.$$

Then there is $L_0 = L_0(t_0, \delta_{\max}, \kappa, G_u, G_1) \ge 1$ such that for all $L \in (G_2 + G_u)\mathbb{N}$, $L \ge L_0$, we have

$$\mathbb{P}\left(\left\{\omega \in \Omega : \lambda_1(H_{\omega,L}) - \lambda_1(H_{0,L}) \ge \frac{1}{L^{3/2}}\right\}\right) \ge 1 - \frac{C}{L^{d/2}},$$

where $H_{0,L}$ is obtained from $H_{\omega,L}$ by setting ω_j to zero for all $j \in \mathcal{D}$.

Remark 2.12 (discussion on initial scale estimate). Theorem 2.11 may serve as an initial scale estimate for a proof of localization via multiscale analysis. More precisely, by using the Combes–Thomas estimate, an initial scale estimate in some neighborhood of $a := \inf\sigma(H_0)$ follows. Note that the exponents $\frac{3}{2}$ and $\frac{d}{2}$ in Theorem 2.11 can be modified to some extent by adapting the proof and the assumption on the measure μ. Localization in a neighborhood I_a of a follows via multiscale analysis, e.g., à la [Stollmann 2001]. The question of whether $\sigma(H_\omega) \cap I_a \ne \varnothing$ for almost all $\omega \in \Omega$ has to be settled. This is, however, satisfied for all examples mentioned above. In the special case of the standard random breather model one can get rid of the assumption on μ by proving and using the Lifshitz tail behavior of the integrated density of states; see [Veselić 2007] for the one-dimensional case, and the forthcoming paper of Schumacher and Veselić for the multidimensional one.

Application to control theory. We consider the controlled heat equation with heat generation term $(-V)$

$$\begin{cases} \partial_t u - \Delta u + Vu = f\chi_\omega, & u \in L^2([0,T]\times\Omega), \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Omega), \end{cases} \tag{7}$$

where ω is an open subset of the connected $\Omega \subset \mathbb{R}^d$, $T > 0$ and $V \in L^\infty(\Omega)$. In (7) u is the state and f is the control function which acts on the system through the control set ω.

Definition 2.13. For initial data $u_0 \in L^2(\Omega)$ and time $T > 0$, the set of reachable states $R(T, u_0)$ is

$$R(T,u_0) = \big\{u(T,\cdot) : \text{there exists } f \in L^2([0,T]\times\omega) \text{ such that } u \text{ is solution of (7) with RHS}\big\}.$$

The system (7) is called null controllable at time T if $0 \in R(T; u_0)$ for all $u_0 \in L^2(\Omega)$. The controllability cost $\mathcal{C}(T, u_0)$ at time T for the initial state u_0 is

$$\mathcal{C}(T,u_0) = \inf\big\{\|f\|_{L^2([0,T]\times\omega)} : u \text{ is solution of (7) and } u(T,\cdot) = 0\big\}.$$

Since the system is linear, null controllability implies that the range of the semigroup generated by the heat equation is reachable too. It is well known that null controllability holds for any time $T > 0$, connected Ω and any nonempty and open set $\omega \subset \Omega$ on which the control acts; see [Fursikov and Imanuvilov 1996].

It is also known, see for instance [Tucsnak and Weiss 2009, Theorem 11.2.1], that null controllability of the system (7) at time T is equivalent to final state observability on the set ω at time T of the system

$$\begin{cases} \partial_t u - \Delta u + Vu = 0, & u \in L^2([0,T] \times \Omega), \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Omega). \end{cases} \tag{8}$$

Definition 2.14. The system (8) is called final state observable on the set ω at time T if there exists $\kappa_T = \kappa_T(\Omega, \omega, V)$ such that for every initial state $u_0 \in L^2(\Omega)$ the solution $u \in L^2([0,T] \times \Omega)$ of (8) satisfies

$$\|u(T,\cdot)\|_\Omega^2 \le \kappa_T \|u\|_{L^2([0,T]\times\omega)}^2. \tag{9}$$

Moreover, the controllability cost $\mathcal{C}(T, u_0)$ of (7) coincides with the infimum over all observability costs $\sqrt{\kappa_T}$ in (9) times $\|u_0\|_\Omega$; see, for example, the proof of [Tucsnak and Weiss 2009, Theorem 11.2.1].

The problem of obtaining explicit bounds on $\mathcal{C}(T, u_0)$ received much consideration in the literature, see, for example, [Güichal 1985; Fernández-Cara and Zuazua 2000; Phung 2004; Tenenbaum and Tucsnak 2007; Miller 2006; 2004; 2010; Ervedoza and Zuazua 2011; Lissy 2012], especially the case of small time, i.e., when T goes to zero. The dependencies of the controllability cost on T and $\|V\|_\infty$ are today well understood; see, for example, [Zuazua 2007]. However, the dependence on the geometry of the control set is less clear: in the known estimates the geometry enters only in terms of the distance to the boundary or in terms of the geometrical optics condition. To find an optimal control set is a very difficult problem; see for instance the recent articles [Privat et al. 2015a; 2015b].

We are interested in the situation $\Omega = \Lambda_L \subset \mathbb{R}^d$ and $\omega = W_\delta(L)$ for a (G,δ)-equidistributed sequence with $L \in G\mathbb{N}$, $G > 0$ and $\delta < G/2$. In this specific setting we will give an estimate on the controllability cost. The novelty of our result is that the observability cost is independent of the scale L and the specific choice of the (G,δ)-equidistributed sequence. Moreover, the dependencies on $\|V\|_\infty$ and on the size of the control set via δ are known explicitly. As far as we are aware, this is the first time that such a scale-free estimate is obtained.

By the equivalence between null-controllability and final state observability, it is sufficient to construct an estimate of the form (9). In order to find such an estimate, we will combine Corollary 2.4 with results from [Miller 2010] to obtain the following theorem.

Theorem 2.15. *For every $G > 0$, $\delta \in (0, G/2)$ and $K_V \ge 0$ there is $T' = T'(G,\delta,K_V) > 0$ such that for all $T \in (0, T']$, all (G,δ)-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}^d$ with $\|V\|_\infty \le K_V$ and all $L \in G\mathbb{N}$, the system*

$$\begin{cases} \partial_t u - \Delta_L u + V_L u = 0, & u \in L^2([0,T] \times \Lambda_L), \\ u = 0 & \textit{on } (0,T) \times \partial\Lambda_L, \\ u(0,\cdot) = u_0, & u_0 \in L^2(\Lambda_L) \end{cases}$$

is final state observable on the set $W_\delta(L)$ with cost κ_T satisfying

$$\kappa_T \leq 4 a_0 b_0 \mathrm{e}^{2c_*/T},$$

where $a_0 = (\delta/G)^{-N(1+G^{4/3}\|V\|_\infty^{2/3})}$, $b_0 = \mathrm{e}^{2\|V\|_\infty}$, $c_* \leq \ln(G/\delta)^2 (NG + 4/\ln 2)^2$ *and* $N = N(d)$ *is the constant from Theorem 2.2.*

Remark 2.16. (i) The same result holds also in the case of the controlled heat equation with periodic or Neumann boundary conditions with obvious modifications.

(ii) Null controllability of the heat equation implies a stronger type of controllability, so-called approximate controllability. Following [Fernández-Cara and Zuazua 2000], one can find an estimate for the cost of approximate controllability from the proof of Theorem 2.15. We will not pursue it in this paper.

3. Proof of the scale-free unique continuation principle

Carleman inequalities. We denote by $\mathbb{R}^{d+1}_+ := \{x \in \mathbb{R}^{d+1} : x_{d+1} \geq 0\}$ the $(d+1)$-dimensional half-space and by $B_r^+ := \{x \in \mathbb{R}^{d+1}_+ : |x| < r\}$ the $(d+1)$-dimensional half-ball. For $x \in \mathbb{R}^{d+1}$ we denote by x' the projection on the first d coordinates; i.e., for $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we use the notation $x' = (x_1, \dots, x_d) \in \mathbb{R}^d$. By $|x|$ and $|x'|$ we denote the Euclidean norms and by Δ the Laplacian on $\mathbb{R}^{d+1}$. For functions $f \in C^\infty(\mathbb{R}^{d+1}_+)$ we use the notation $f_0 = f|_{x_{d+1}=0}$.

In the appendix of [Lebeau and Robbiano 1995], the authors state a Carleman estimate for complex-valued functions with support in B_r^+ by using a real-valued weight function $\psi \in C^\infty(\mathbb{R}^{d+1})$ satisfying the two conditions

$$\text{for all } x \in B_r^+ \text{ we have } (\partial_{d+1}\psi)(x) \neq 0, \tag{10}$$

and for all $\xi \in \mathbb{R}^{d+1}$ and $x \in B_r^+$ there holds

$$\left.\begin{aligned} 2\langle \xi, \nabla\psi\rangle = 0, \\ |\xi|^2 = |\nabla\psi|^2 \end{aligned}\right\} \quad \Longrightarrow \quad \sum_{j,k=1}^{d+1} (\partial_{jk}\psi)\big(\xi_j\xi_k + (\partial_j\psi)(\partial_k\psi)\big) > 0. \tag{11}$$

As proposed in [Jerison and Lebeau 1999] we choose $r < 2 - \sqrt{2}$ and the special weight function $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$,

$$\psi(x) = -x_{d+1} + \tfrac{1}{2}x_{d+1}^2 - \tfrac{1}{4}|x'|^2. \tag{12}$$

Note that $\psi(x) \leq 0$ for all $x \in B_2^+$. This function ψ indeed satisfies the assumptions (10) and (11). Condition (10) is trivial for $r < 1$. In order to show the implication (11) we show

$$|\xi|^2 = |\nabla\psi|^2 \quad \Longrightarrow \quad \sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\,\partial_k\psi) > 0. \tag{13}$$

We use the hypothesis of (13) and calculate

$$\sum_{j,k=1}^{d+1} \partial_{jk}\psi(\xi_j\xi_k + \partial_j\psi\,\partial_k\psi) = -\frac{1}{2}\sum_{i=1}^{d} \xi_i^2 + \xi_{d+1}^2 - \tfrac{1}{8}|x'|^2 + (x_{d+1}-1)^2 = \tfrac{3}{2}\xi_{d+1}^2 - \tfrac{1}{4}|x'|^2 + \tfrac{1}{2}(x_{d+1}-1)^2.$$

Since $|x'|^2 \leq r^2$ and $(x_{d+1}-1)^2 \geq (1-r)^2$, assumption (13) is satisfied if $r < 2-\sqrt{2}$. Now let

$$C^\infty_{c,0}(B_r^+) = \Big\{ g : \mathbb{R}^{d+1}_+ \to \mathbb{C} : g \equiv 0 \text{ on } \{x_{d+1}=0\},\ \exists \phi \in C^\infty(\mathbb{R}^{d+1}) \text{ with } \operatorname{supp}\phi \subset \{x \in \mathbb{R}^{d+1} : |x| < r\} \text{ and } g \equiv \phi \text{ on } \mathbb{R}^{d+1}_+ \Big\}.$$

Hence, as a corollary of Proposition 1 in the appendix of [Lebeau and Robbiano 1995] we have:

Proposition 3.1. *Let $\psi \in C^\infty(\mathbb{R}^{d+1};\mathbb{R})$ be as in (12) and $\rho \in (0, 2-\sqrt{2})$. Then there are constants $\beta_0, C_1 \geq 1$ such that for all $\beta \geq \beta_0$, and all $g \in C^\infty_{c,0}(B_\rho^+)$ we have*

$$\int_{\mathbb{R}^{d+1}} e^{2\beta\psi}(\beta|\nabla g|^2 + \beta^3|g|^2) \leq C_1\bigg(\int_{\mathbb{R}^{d+1}} e^{2\beta\psi}|\Delta g|^2 + \beta\int_{\mathbb{R}^d} e^{2\beta\psi_0}|(\partial_{d+1}g)_0|^2\bigg).$$

We will need another Carleman estimate with a weight function whose level sets can be explicitly controlled.

Proposition 3.2 [Nakić et al. 2015a]. *Let $\rho > 0$ and $w : \mathbb{R}^d \to \mathbb{R}$,*

$$w(x) = \frac{|x|}{\rho}\int_0^{|x|/\rho} \frac{1-e^{-t}}{t}\,dt.$$

In particular,

$$\text{for all } x \in B(\rho), \quad \frac{|x|}{\rho e} \leq w(x) \leq \frac{|x|}{\rho}.$$

Then there are constants $\alpha_0, C_2 \geq 1$ depending only on the dimension such that for all $\alpha \geq \alpha_0$, and all $u \in W^{2,2}(\mathbb{R}^d)$ with support in $B(\rho)\setminus\{0\}$ we have

$$\int_{\mathbb{R}^d}\big(\alpha\rho^2 w^{1-2\alpha}|\nabla u|^2 + \alpha^3 w^{-1-2\alpha}|u|^2\big)\,dx \leq C_2\rho^4\int_{\mathbb{R}^d} w^{2-2\alpha}|\Delta u|^2\,dx.$$

This variant of the Carleman estimate is essentially given in [Escauriaza and Vessella 2003], albeit that paper concerns parabolic operators. For elliptic operators, in [Bourgain and Kenig 2005] a weaker statement than Proposition 3.2, without the gradient term on the left-hand side, was spelled out and proven explicitly. A version of Proposition 3.2 for divergence-type elliptic operators is stated in [Kenig et al. 2011]. While this covers more general operators than we are interested in here, it lacks a quantitative statement about the admissible functions u. An explicit proof of Proposition 3.2, i.e., for the pure Laplacian, was first given in [Klein and Tsang 2016]. See also [Nakić et al. 2015a] for the case of divergence-type elliptic operators. The paper [Davey 2014] also contains a Carleman estimate which is less explicit than Proposition 3.2, but would still be sufficient for the purpose of the proof of Theorem 2.2.

Extension to larger boxes. For each measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ and each $L \in \mathbb{N}$ we denote the eigenvalues of the corresponding operator H_L by E_k, $k \in \mathbb{N}$, enumerated in increasing order and counting multiplicities, and fix a corresponding sequence ϕ_k, $k \in \mathbb{N}$, of normalized eigenfunctions. Note that we suppress the dependence of E_k and ϕ_k on V and L.

Given V and L we define an extension of the potential V_L and the eigenfunctions ϕ_k to the set Λ_{RL} for some $R \in \mathbb{N}_{\text{odd}} = \{1, 3, 5, \dots\}$ to be chosen later on. The extension will depend on the type of boundary conditions we are considering for the Laplace operator.

Extension for periodic boundary conditions: We extend the potential V_L as well as the function ϕ_k, defined on the box Λ_L, periodically to $\widetilde{V}, \tilde{\psi} : \mathbb{R}^d \to \mathbb{R}$ and then restrict them to Λ_{RL}. By the very definition of the operator domain of Δ_{Λ_L} with periodic boundary conditions the extension $\tilde{\psi}$ is locally in the Sobolev space $W^{2,2}(\mathbb{R}^d)$.

Extension for Dirichlet and Neumann boundary conditions: The potential V_L will be extended by symmetric reflections with respect to the hypersurfaces forming the boundaries of Λ_L. In the first step we extend $V_L : \Lambda_L \to \mathbb{R}$ to the set

$$\{x \in \Lambda_{3L} : x_i \in (-L/2, L/2),\ i \in \{2, \dots, d\}\}$$

by

$$V_L(x) = \begin{cases} V_L(x) & \text{if } x \in \Lambda_L, \\ 0 & \text{if } x_1 \in \{-L/2, L/2\}, \\ V_L(L - x_1, x_2, \dots, x_d) & \text{if } x_1 > L/2, \\ V_L(-L - x_1, x_2, \dots, x_d) & \text{if } x_1 < -L/2. \end{cases}$$

Now we iteratively extend V_L in the remaining $d - 1$ directions using the same procedure and obtain a function $V_L : \Lambda_{3L} \to \mathbb{R}$. Iterating this procedure we obtain a function $V_L : \Lambda_{RL} \to \mathbb{R}$. The extensions of the eigenfunctions will depend on the boundary conditions. In the case of Dirichlet boundary conditions, we extend an eigenfunction similarly to the potential by antisymmetric reflections, while in the case of Neumann boundary conditions, we extend by symmetric reflections.

The extensions of the functions and V_L and $\phi_k,\ k \in \mathbb{N}$, to the set Λ_{RL} will again be denoted by V_L and $\phi_k,\ k \in \mathbb{N}$. The reader should be reminded that (the extended) $V_L : \Lambda_{RL} \to \mathbb{R}$ does in general not coincide with $V_{RL} : \Lambda_{RL} \to \mathbb{R}$. Note that for all three boundary conditions, $V_L : \Lambda_{RL} \to \mathbb{R}$ takes values in $[-\|V\|_\infty, \|V\|_\infty]$, the extended ϕ_k are elements of $W^{2,2}(\Lambda_{RL})$ with corresponding boundary conditions and they satisfy $\Delta\phi_k = (V_L - E_k)\phi_k$ on Λ_{RL}. Furthermore, the orthogonality relations remain valid.

Ghost dimension. For a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R},\ L \in \mathbb{N},\ E \geq 0$ and $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$ we have

$$\phi = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq E}} \alpha_k \phi_k, \quad \text{with} \quad \alpha_k = \langle \phi_k, \phi \rangle.$$

Since the ϕ_k extend to Λ_{RL} as explained in the previous subsection, the function ϕ also extends to Λ_{RL}. We set $\omega_k := \sqrt{|E_k|}$ and define the function $F : \Lambda_{RL} \times \mathbb{R} \to \mathbb{C}$ by

$$F(x, x_{d+1}) = \sum_{\substack{k \in \mathbb{N} \\ E_k \leq E}} \alpha_k \phi_k(x)\, \mathrm{s}_k(x_{d+1}),$$

where $s_k : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{s}_k(t) = \begin{cases} \sinh(\omega_k t)/\omega_k, & E_k > 0, \\ t, & E_k = 0, \\ \sin(\omega_k t)/\omega_k, & E_k < 0. \end{cases}$$

Note that we suppress the dependence of ϕ and ϕ_k on V, L, E. Furthermore, the sums are finite since H_L is lower semibounded with purely discrete spectrum. The function F satisfies

$$\Delta F = \sum_{i=1}^{d+1} \partial_i^2 F = V_L F \quad \text{on } \Lambda_{RL} \times \mathbb{R}$$

and

$$\partial_{d+1} F(x, 0) = \sum_{\substack{k \in \mathbb{N} \\ E_k \le E}} \alpha_k \phi_k(x) \quad \text{for } x \in \Lambda_{RL}.$$

In particular, for all $x \in \Lambda_L$ we have $\partial_{d+1} F(x, 0) = \phi$. This way we recover the original function we are interested in.

Let us also fix the geometry. For $\delta \in \left(0, \frac{1}{2}\right)$ we choose

$$\begin{aligned} \psi_1 &= -\tfrac{1}{16}\delta^2, & \psi_2 &= -\tfrac{1}{8}\delta^2, & \psi_3 &= -\tfrac{1}{4}\delta^2, \\ r_1 &= \tfrac{1}{2} - \tfrac{1}{8}\sqrt{16-\delta^2}, & r_2 &= 1, & r_3 &= 6\mathrm{e}\sqrt{d}, \\ R_1 &= 1 - \tfrac{1}{4}\sqrt{16-\delta^2}, & R_2 &= 3\sqrt{d}, & R_3 &= 9\mathrm{e}\sqrt{d}, \end{aligned}$$

and define for $i \in \{1, 2, 3\}$ the sets

$$\begin{aligned} S_i &:= \left\{x \in \mathbb{R}^{d+1} : \psi(x) > \psi_i,\, x_{d+1} \in [0, 1]\right\} \subset \mathbb{R}^{d+1}_+, \\ V_i &:= B(R_i) \setminus \overline{B(r_i)} \subset \mathbb{R}^{d+1}. \end{aligned}$$

Let $R \in \mathbb{N}$ be the least power of 3 larger than $2R_3 + 2$. For $i \in \{1, 2, 3\}$ and $x \in \mathbb{R}^d$ we denote by $S_i(x) = S_i + (x, 0)$ and $V_i(x) = V_i + (x, 0)$ the translates of the sets $S_i \subset \mathbb{R}^{d+1}$ and $V_i \subset \mathbb{R}^{d+1}$. Moreover, for $L \in \mathbb{N}$ and a $(1, \delta)$-equidistributed sequence $z_j \in \mathbb{R}^d$, $j \in \mathbb{Z}^d$, we define $Q_L = \mathbb{Z}^d \cap \Lambda_L$, $U_i(L) = \bigcup_{j \in Q_L} S_i(z_j)$, $X_1 = \Lambda_L \times [-1, 1]$ and $\widetilde{X}_{R_3} = \Lambda_{L+2R_3} \times [-R_3, R_3]$. Note that $W_\delta(L)$ is a disjoint union. In the following lemma we collect some consequences of our geometric setting. We will first restrict our attention to the case $L \in \mathbb{N}_{\mathrm{odd}}$, and consider the case of even integers thereafter.

Lemma 3.3. (i) *For all $\delta \in \left(0, \frac{1}{2}\right)$ we have $S_1 \subset S_2 \subset S_3 \subset B_\delta^+ \subset \mathbb{R}^{d+1}_+$.*

(ii) *For all $L \in \mathbb{N}_{\mathrm{odd}}$ with $L \ge 5$, all $\delta \in \left(0, \frac{1}{2}\right)$ and all $(1, \delta)$-equidistributed sequences z_j we have $\bigcup_{j \in Q_L} V_2(z_j) \supset X_1$.*

(iii) *There is a constant K_d, depending only on d, such that for all $L \in \mathbb{N}_{\mathrm{odd}}$, all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1, \delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $E \ge 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty, E]}(H_L))$ we have*

$$\sum_{j \in Q_L} \|F\|^2_{H^1(V_3(z_j))} \le K_d \|F\|^2_{H^1\left(\bigcup_{j \in Q_L} V_3(z_j)\right)}.$$

(iv) *For all $L \in \mathbb{N}_{\mathrm{odd}}$, $\delta \in \left(0, \frac{1}{2}\right)$ and all $(1, \delta)$-equidistributed sequences z_j we have $\bigcup_{j \in Q_L} V_3(z_j) \subset X_{R_3}$.*

We note that part (ii) of Lemma 3.3 will be applied with L replaced by $5L$.

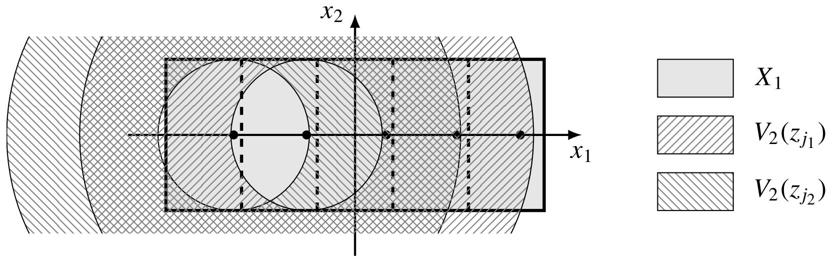

Figure 1. Illustration for (ii) in the case $d = 1$, $L = 5$ and for some configuration z_j, $j \in Q_L$. The set $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]$ is covered by $V_2(z_{j_1})$ and $V_2(z_{j_2})$.

Proof. Parts (i) and (iv) are obvious.

To show (ii), we first prove that $\left[-\frac{1}{2}, \frac{1}{2}\right]^d \times [-1, 1]$ can be covered by the sets $V_2(z_j)$. Let us take $j_1 = (-1, 0, \ldots, 0)$, $j_2 = (-2, 0, \ldots, 0)$, $j_1, j_2 \in Q_L$.

Then

$$\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^d \times [-1, 1] \subset V_2(z_{j_1}) \cup V_2(z_{j_2}); \tag{14}$$

see Figure 1. Indeed, let $x = (x_1, \ldots, x_{d+1})$ be an arbitrary point from $\left[-\frac{1}{2}, \frac{1}{2}\right]^d \times [-1, 1]$. Then (14) is not satisfied only if $|(z_{j_1}, 0) - x|^2 < 1$ and $|(z_{j_2}, 0) - x|^2 > R_2^2$. Since $z_{j_1} \in \left(-\frac{3}{2}+\delta, -\frac{1}{2}-\delta\right) \times \left(-\frac{1}{2}+\delta, \frac{1}{2}-\delta\right)^{d-1}$ and $z_{j_2} \in \left(-\frac{5}{2}+\delta, -\frac{3}{2}-\delta\right) \times \left(-\frac{1}{2}+\delta, \frac{1}{2}-\delta\right)^{d-1}$, it follows that

$$\left(-\tfrac{1}{2}-\delta-x_1\right)^2 + x_{d+1}^2 < 1 \quad \text{and} \quad \left(-\tfrac{5}{2}+\delta-x_1\right)^2 + (d-1)(1-\delta)^2 + x_{d+1}^2 > 9d.$$

Plugging the first relation into the second, we obtain

$$9d < (d-1)(1-\delta)^2 + 2(1-\delta)(3+2x_1) + 1 \le (d-1)(1-\delta)^2 + 8(1-\delta) + 1.$$

But this relation is satisfied only for $d < 1$. Since $L \ge 5$ the same argument applies to cover every elementary cell $\left(\left[-\frac{1}{2}, \frac{1}{2}\right] + i\right) \times [-1, 1]$, $i \in Q_L$, by two neighboring sets $V_2(z_j)$.

Now we turn to the proof of (iii). Since $R \ge 2R_3 + 2$, the function F is defined on $V_3(z_j)$ for all $j \in Q_L$. For all $x \in \bigcup_{j \in Q_L} V_3(z_j)$, the number of indices $j \in Q_L$ such that $V_3(z_j) \ni x$ is bounded from above by $(2R_3 + 2)^d$. Hence,

$$\text{for all } x \in \widetilde{X}_{R_3}, \qquad \sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \le (2R_3 + 2)^d \chi_{\bigcup_{j \in Q_L} V_3(z_j)}(x),$$

and thus

$$\begin{aligned} \sum_{j \in Q_L} \|F\|^2_{H^1(V_3(z_j))} &= \int_{\widetilde{X}_{R_3}} \left(\sum_{j \in Q_L} \chi_{V_3(z_j)}(x) \right) (|F(x)|^2 + |\nabla F(x)|^2) \, \mathrm{d}x \\ &\le (2R_3 + 2)^d \|F\|^2_{H^1(\bigcup_{j \in Q_L} V_3(z_j))}. \end{aligned}$$

Hence we can take $K_d = (2R_3 + 2)^d$. □

Interpolation inequalities.

Proposition 3.4. *For all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1,\delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \geq 0$ and all $\phi \in \operatorname{Ran}(\chi_{(-\infty,E]}(H_L))$:*

(a) *There is $\beta_1 = \beta_1(d, \|V\|_\infty) \geq 1$ such that for all $\beta \geq \beta_1$ we have*

$$\|F\|^2_{H^1(U_1(L))} \leq \widetilde{D}_1(\beta)\|F\|^2_{H^1(U_3(L))} + \widehat{D}_1(\beta)\|(\partial_{d+1}F)_0\|^2_{L^2(W_\delta(L))},$$

where β_1 is given in (16), and $\widetilde{D}_1(\beta)$ and $\widehat{D}_1(\beta)$ are given in (17).

(b) *We have*

$$\|F\|_{H^1(U_1(L))} \leq D_1\|(\partial_{d+1}F)_0\|^{1/2}_{L^2(W_\delta(L))}\|F\|^{1/2}_{H^1(U_3(L))},$$

where D_1 is given in (21).

Proof. First we recall that $\Delta F = V_L F$, $\partial_{d+1}F(x',0) = \phi(x')$ and $B_\delta^+ \supset S_3$. Now we choose a cutoff function $\chi \in C^\infty(\mathbb{R}^{d+1}; [0,1])$ with $\operatorname{supp}\chi \subset \bar{S}_3$, $\chi(x) = 1$ if $x \in S_2$ and

$$\max\{\|\Delta\chi\|_\infty, \||\nabla\chi|\|_\infty\} \leq \frac{\widetilde{\Theta}_1}{\delta^4} =: \Theta_1,$$

where $\widetilde{\Theta}_1 = \widetilde{\Theta}_1(d)$ depends only on the dimension. This is due to the fact that the distance of S_2 and $\mathbb{R}^{d+1}_+ \setminus S_3$ is bounded from below by $\delta^2/16$. Let φ be a nonnegative function in $C_c^\infty(\mathbb{R}^d)$ with the properties that $\|\varphi\|_1 = 1$ and $\operatorname{supp}\varphi \subset B(1)$. For $\varepsilon > 0$ we define $\varphi_\varepsilon : \mathbb{R}^d \to \mathbb{R}_0^+$ by $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$. The function φ_ε belongs to $C_c^\infty(\mathbb{R}^d)$ and satisfies $\operatorname{supp}\varphi_\varepsilon \subset (\varepsilon)$. Now we continuously extend the eigenfunctions $\phi_k : \Lambda_{RL} \to \mathbb{R}$ to the set $\mathbb{R}^d$ by zero and define for $\varepsilon > 0$ the function $F_\varepsilon : \mathbb{R}^d \times \mathbb{R}$ by

$$F_\varepsilon(x, x_{d+1}) = \sum_{\substack{k\in\mathbb{N}\\ E_k \leq E}} \alpha_k(\varphi_\varepsilon * \phi_k)(x)\, \mathrm{s}_k(x_{d+1}).$$

By construction, the function $g = \chi F_\varepsilon$ is an element of $C^\infty_{c,0}(B_\delta^+)$. Hence, we can apply Proposition 3.1 with $g = \chi F_\varepsilon$ and $\rho = \frac{1}{2}$ and obtain for all $\beta \geq \beta_0 \geq 1$

$$\int_{S_3} \mathrm{e}^{2\beta\psi}\left(\beta|\nabla(\chi F_\varepsilon)|^2 + \beta^3|\chi F_\varepsilon|^2\right) \leq C_1\int_{S_3} \mathrm{e}^{2\beta\psi}|\Delta(\chi F_\varepsilon)|^2 + \beta C_1\int_{B(\delta)} \mathrm{e}^{2\beta\psi_0}|(\partial_{d+1}(\chi F_\varepsilon))_0|^2. \tag{15}$$

Note that β_0 and C_1 only depend on the dimension. By [Ziemer 1989, Theorem 1.6.1(iii)] we have $\varphi_\varepsilon * \phi_k \to \phi_k$, $\nabla(\varphi_\varepsilon * \phi_k) \to \nabla\phi_k$ and $\Delta(\varphi_\varepsilon * \phi_k) \to \Delta\phi_k$ in $L^2(S_3)$ as ε tends to zero. Consequently, the same holds for F_ε, ∇F_ε and ΔF_ε and thus we obtain (15) with F_ε replaced by F. For the first summand on the right-hand side we have the upper bound

$$\begin{aligned}\int_{S_3} \mathrm{e}^{2\beta\psi}|\Delta(\chi F)|^2 &\leq 3\int_{S_3} \mathrm{e}^{2\beta\psi}\left(4|\nabla\chi|^2|\nabla F|^2 + |\Delta\chi|^2|F|^2 + |\Delta F|^2|\chi|^2\right)\\ &\leq 3\mathrm{e}^{2\beta\psi_2}\int_{S_3\setminus S_2}\left(4\Theta_1^2|\nabla F|^2 + \Theta_1^2|F|^2\right) + \int_{S_3} 3\mathrm{e}^{2\beta\psi}|V_L F\chi|^2\\ &\leq 12\Theta_1^2\mathrm{e}^{2\beta\psi_2}\|F\|^2_{H^1(S_3)} + 3\|V\|^2_\infty\int_{S_3} \mathrm{e}^{2\beta\psi}|\chi F|^2.\end{aligned}$$

The second summand is bounded from above by $\beta C_1 \int_{B(\delta)} |(\partial_{d+1} F)_0|^2$, since $F = 0$ and $\psi \leq 0$ on $\{x_{d+1} = 0\}$. Hence,

$$\beta \int_{S_3} \mathrm{e}^{2\beta\psi} |\nabla(\chi F)|^2 + (\beta^3 - 3\|V\|_\infty^2 C_1) \int_{S_3} \mathrm{e}^{2\beta\psi} |\chi F|^2 \leq 12 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + C_1 \beta \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

Additionally to $\beta \geq \beta_0$ we choose $\beta \geq (6\|V\|_\infty^2 C_1)^{1/3} =: \tilde{\beta}_0$. This ensures that for all

$$\beta \geq \beta_1 := \max\{\beta_0, \tilde{\beta}_0\} \tag{16}$$

we have

$$\frac{1}{2} \int_{S_3} \mathrm{e}^{2\beta\psi} \big(\beta |\nabla(\chi F)|^2 + \beta^3 |\chi F|^2\big) \leq 12 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + C_1 \beta \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

Since $\beta \geq 1,\ S_3 \supset S_1,\ \chi = 1$ and $\mathrm{e}^{2\beta\psi} \geq \mathrm{e}^{2\beta\psi_1}$ on S_1, we obtain

$$\mathrm{e}^{2\beta\psi_1} \|F\|^2_{H^1(S_1)} \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \|F\|^2_{H^1(S_3)} + 2 C_1 \|(\partial_{d+1} F)_0\|^2_{L^2(B(\delta))}.$$

We apply this inequality for translates $S_i(z_j)$ and obtain by summing over $j \in Q_L = \mathbb{Z}^d \cap \Lambda_L$

$$\mathrm{e}^{2\beta\psi_1} \sum_{j \in Q_L} \|F\|^2_{H^1(S_1(z_j))} \leq 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta\psi_2} \sum_{j \in Q_L} \|F\|^2_{H^1(S_3(z_j))} + 2 C_1 \sum_{j \in Q_L} \|(\partial_{d+1} F)_0\|^2_{L^2(B(z_j, \delta))}.$$

Recall that $U_i(L) = \bigcup_{j \in Q_L} S_i(z_j)$ and $W_\delta(L) = \bigcup_{j \in Q_L} B(z_j, \delta)$. Hence, for all $\beta \geq \beta_1$ we have

$$\|F\|^2_{H^1(U_1(L))} \leq \widetilde{D}_1 \|F\|^2_{H^1(U_3(L))} + \widehat{D}_1 \|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L))},$$

where

$$\widetilde{D}_1(\beta) = 24 C_1 \Theta_1^2 \mathrm{e}^{2\beta(\psi_2 - \psi_1)} \quad \text{and} \quad \widehat{D}_1(\beta) = 2 C_1 \mathrm{e}^{-2\beta\psi_1}. \tag{17}$$

We choose β such that

$$\mathrm{e}^{\beta} = \left[\frac{1}{12\Theta_1^2} \frac{\|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L)}}{\|F\|^2_{H^1(U_3(L))}} \right]^{1/(2\psi_2)}. \tag{18}$$

Now we distinguish two cases. If $\beta \geq \beta_1$ we obtain by using $\psi_1 = 2\psi_2$

$$\|F\|^2_{H^1(U_1(L))} \leq 8\sqrt{3} C_1 \Theta_1 \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \tag{19}$$

If $\beta < \beta_1$ we use Lemma 5.2 of [Le Rousseau and Lebeau 2012]. In particular, one concludes from (18) that

$$\|F\|^2_{H^1(U_3(L))} < \frac{1}{12\Theta_1^2} \mathrm{e}^{-2\beta_1\psi_2} \|(\partial_{d+1} F)_0\|^2_{L^2(W_\delta(L))}.$$

This gives us in the case $\beta < \beta_1$

$$\|F\|^2_{H^1(U_1(L))} \leq \|F\|^2_{H^1(U_3(L))} < \frac{\mathrm{e}^{-\beta_1\psi_2}}{\sqrt{12}\Theta_1} \|F\|_{H^1(U_3(L))} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}. \tag{20}$$

If we set

$$D_1^2 = \max\left\{ 8\sqrt{3} C_1 \Theta_1, \frac{\mathrm{e}^{-\beta_1\psi_2}}{\Theta_1 \sqrt{12}} \right\}, \tag{21}$$

we conclude the statement of the proposition from inequalities(19) and (20). □

Now we deduce from the second Carleman estimate, Proposition 3.2, another interpolation inequality.

Proposition 3.5. *For all $\delta \in \left(0, \frac{1}{2}\right)$, all $(1, \delta)$-equidistributed sequences z_j, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \geq 0$ and all $\phi \in \mathrm{Ran}(\chi_{(-\infty,E]}(H_L))$:*

(a) *There is $\alpha_1 = \alpha_1(d, \|V\|_\infty) \geq 1$ such that for all $\alpha \geq \alpha_1$ we have*

$$\|F\|^2_{H^1(X_1)} \leq \widetilde{D}_2(\alpha)\|F\|^2_{H^1(U_1(L))} + \widehat{D}_2(\alpha)\|F\|_{H^1(\widetilde{X}_{R_3})},$$

where α_1 is given in (23), and $\widetilde{D}_2(\alpha)$ and $\widehat{D}_2(\alpha)$ are given in (27).

(b) *We have*

$$\|F\|_{H^1(X_1)} \leq D_2\|F\|^{\gamma}_{H^1(U_1(L))}\|F\|^{1-\gamma}_{H^1(\widetilde{X}_{R_3})},$$

where γ and D_2 are given in (32) and (33).

Proof. We choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}^{d+1}; [0, 1])$ with $\operatorname{supp}\chi \subset B(R_3) \setminus \overline{B(r_1)}$, $\chi(x) = 1$ if $x \in B(r_3) \setminus \overline{B(R_1)}$,

$$\max\{\|\Delta\chi\|_{\infty,V_1}, \||\nabla\chi|\|_{\infty,V_1}\} \leq \frac{\widetilde{\Theta}_2}{\delta^4} =: \Theta_2,$$
$$\max\{\|\Delta\chi\|_{\infty,V_3}, \||\nabla\chi|\|_{\infty,V_3}\} \leq \Theta_3,$$

where $\widetilde{\Theta}_2$ depends only on the dimension and Θ_3 is an absolute constant. We set $u = \chi F$. We apply Proposition 3.2 with $\rho = R_3$ to the function u and obtain for all $\alpha \geq \alpha_0 \geq 1$

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{1-2\alpha}|\nabla u|^2 + \alpha^3 w^{-1-2\alpha}|u|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}|\Delta u|^2 \,\mathrm{d}x.$$

Since $w \leq 1$ on $B(R_3)$ we can replace the exponent of the weight function w at all three places by $2 - 2\alpha$; i.e.,

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha}|\nabla u|^2 + \alpha^3 w^{2-2\alpha}|u|^2\right) \mathrm{d}x \leq C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}|\Delta u|^2 \,\mathrm{d}x =: I. \tag{22}$$

For the right-hand side we use

$$\Delta u = 2(\nabla\chi)(\nabla F) + (\Delta\chi)F + (\Delta F)\chi,$$

and $\Delta F = V_L F$, and obtain

$$I \leq 3C_2 R_3^4 \int_{B(R_3)} w^{2-2\alpha}\left(4|(\nabla\chi)(\nabla F)|^2 + |(\Delta\chi)F|^2 + \|V\|^2_\infty|\chi F|^2\right) \mathrm{d}x =: I_1 + I_2 + I_3.$$

If we choose α sufficiently large, i.e.,

$$\alpha \geq (6C_2 R_3^4\|V\|^2_\infty)^{1/3} =: \tilde{\alpha}_0,$$

we can subsume the term I_3 into the left-hand side of (22). We obtain for all

$$\alpha \geq \alpha_1 := \max\{\alpha_0, \tilde{\alpha}_0\} \tag{23}$$

the estimate

$$\int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha}|\nabla u|^2 + \tfrac{1}{2}\alpha^3 w^{2-2\alpha}|u|^2\right) \mathrm{d}x \leq I_1 + I_2.$$

For the "new" left-hand side we have the lower bound

$$I_1 + I_2 \geq \int_{B(R_3)} \left(\alpha R_3^2 w^{2-2\alpha} |\nabla u|^2 + \tfrac{1}{2}\alpha^3 w^{2-2\alpha}|u|^2\right) \mathrm{d}x \geq \frac{1}{2}\left(\frac{R_3}{R_2}\right)^{2\alpha-2} \|F\|_{H^1(V_2)}^2.$$

For I_1 and I_2 we have the estimates

$$I_1 \leq 3C_2 R_3^4 \left[4\Theta_2^2 \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \int_{V_1} |\nabla F|^2 + 4\Theta_3^2 \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \int_{V_3} |\nabla F|^2 \right],$$

$$I_2 \leq 3C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_3}{r_1}\right)^{2\alpha-2} \int_{V_1} |F|^2 + \Theta_3^2 \left(\frac{\mathrm{e}R_3}{r_3}\right)^{2\alpha-2} \int_{V_3} |F|^2 \right].$$

Putting everything together, the Carleman estimate from Proposition 3.2 implies for $\alpha \geq \alpha_1$

$$\|F\|_{H^1(V_2)}^2 \leq 24 C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \|F\|_{H^1(V_1)}^2 + \Theta_3^2 \left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2} \|F\|_{H^1(V_3)}^2 \right]. \tag{24}$$

By translation, (24) is still true if we replace V_1, V_2 and V_3 by its translates $V_1(z_j)$, $V_2(z_j)$ and $V_3(z_j)$ for all $j \in Q_L$. Hence,

$$\sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j))}^2 \leq 24 C_2 R_3^4 \left[\Theta_2^2 \left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \sum_{j \in Q_L} \|F\|_{H^1(V_1(z_j))}^2 + \Theta_3^2 \left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2} \sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \right]. \tag{25}$$

For all $L \in \mathbb{N}_{\mathrm{odd}}$ Lemma 3.3 tells us that $\bigcup_{k \in Q_5} \bigcup_{j \in Q_L} V_2(z_j + kL) \supset X_1 = \Lambda_L \times [-1, 1]$ and the left-hand side is bounded from below by

$$\sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j))}^2 = \frac{1}{5^d} \sum_{k \in Q_5} \sum_{j \in Q_L} \|F\|_{H^1(V_2(z_j + kL))}^2 \geq \frac{1}{5^d} \|F\|_{H^1(X_1)}^2.$$

Since $V_1(z_j) \cap \mathbb{R}_+^{d+1} \subset S_1(z_j)$, $S_1(z_i) \cap S_1(z_j) = \varnothing$ for $i \neq j$, and since F is antisymmetric with respect to its last coordinate, we have

$$\sum_{j \in Q_L} \|F\|_{H^1(V_1(z_j))}^2 \leq 2 \sum_{j \in Q_L} \|F\|_{H^1(S_1(z_j))}^2 = 2 \|F\|_{H^1(U_1(L))}^2.$$

For the second summand on the right-hand side of (25), we find by Lemma 3.3(iii) that there exists a constant K_d such that

$$\sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \leq K_d \|F\|_{H^1(\bigcup_{j \in Q_L} V_3(z_j))}^2.$$

Moreover, since $\bigcup_{j \in Q_L} V_3(z_j) \subset \widetilde{X}_{R_3} = \Lambda_{L+R_3} \times [-R_3, R_3]$, we have

$$\sum_{j \in Q_L} \|F\|_{H^1(V_3(z_j))}^2 \leq K_d \|F\|_{H^1(\widetilde{X}_{R_3})}.$$

Putting everything together we obtain for all $\alpha \geq \alpha_1$

$$\frac{1}{5^d} \|F\|_{H^1(X_1)}^2 \leq \widetilde{D}_2(\alpha) \|F\|_{H^1(U_1(L))}^2 + \widehat{D}_2(\alpha) \|F\|_{H^1(\widetilde{X}_{R_3})}, \tag{26}$$

where

$$\widetilde{D}_2(\alpha) = 48C_2R_3^4\Theta_2^2\left(\frac{\mathrm{e}R_2}{r_1}\right)^{2\alpha-2} \quad \text{and} \quad \widehat{D}_2(\alpha) = 24C_2R_3^4\Theta_3^2K_d\left(\frac{\mathrm{e}R_2}{r_3}\right)^{2\alpha-2}. \tag{27}$$

If we let $c_1 = 48C_2\Theta_2^2R_3^4r_1^2/(\mathrm{e}R_2)^2$, $c_2 = 24C_2\Theta_3^2K_dR_3^4r_3^2/(\mathrm{e}R_2)^2$,

$$p^+ = 2\ln\left(\frac{\mathrm{e}R_2}{r_1}\right) > 0 \quad \text{and} \quad p^- = 2\ln\left(\frac{\mathrm{e}R_2}{r_3}\right) < 0,$$

then (26) reads as

$$\frac{1}{5^d}\|F\|^2_{H^1(X_1)} \le c_1\mathrm{e}^{p^+\alpha}\|F\|^2_{H^1(U_1(L))} + c_2\mathrm{e}^{p^-\alpha}\|F\|^2_{H^1(\widetilde{X}_{R_3})}. \tag{28}$$

We choose α such that

$$\mathrm{e}^\alpha = \left(\frac{c_2}{c_1}\frac{\|F\|^2_{H^1(\widetilde{X}_{R_3})}}{\|F\|^2_{H^1(U_1(L))}}\right)^{1/(p^+-p^-)}. \tag{29}$$

If $\alpha \ge \alpha_1$ we obtain from (28) that

$$\frac{1}{5^d}\|F\|^2_{H^1(X_1)} \le 2c_1^\gamma c_2^{1-\gamma}\|F\|^{2\gamma}_{H^1(U_1(L))}\|F\|^{2-2\gamma}_{H^1(\widetilde{X}_{R_3})}, \quad \text{where} \quad \gamma = \frac{-p^-}{p^+-p^-}. \tag{30}$$

If $\alpha < \alpha_1$, we proceed as in the last part of the proof of Proposition 3.4; i.e., we conclude from (29) that

$$\|F\|^2_{H^1(\widetilde{X}_{R_3})} < \frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\|F\|^2_{H^1(U_1(L))},$$

and thus

$$\|F\|^2_{H^1(X_1)} \le \|F\|^{2(p^+-p^-)/(p^+-p^-)}_{H^1(\widetilde{X}_{R_3})} < \|F\|^{2(1-\gamma)}_{H^1(\widetilde{X}_{R_3})}\left(\frac{c_1}{c_2}\mathrm{e}^{\alpha_1(p^+-p^-)}\right)^\gamma\|F\|^{2\gamma}_{H^1(U_1(L))}. \tag{31}$$

We calculate

$$\gamma = \frac{\ln 2}{\ln(r_3/r_1)}, \tag{32}$$

set

$$D_2^2 = \max\left\{5^d192\cdot 9^4C_2\Theta_3^2K_d\mathrm{e}^4d^2\left(\frac{2\Theta_2^2r_1^2}{\Theta_3^2K_dr_3^2}\right)^\gamma, \left(\frac{2\Theta_2^2}{\Theta_3^2K_d}\left(\frac{r_3}{r_1}\right)^{2(\alpha_1-1)}\right)^\gamma\right\} \tag{33}$$

and conclude the statement of the proposition from (30) and (31). □

Proofs of Theorem 2.2 and Corollary 2.6.

Proposition 3.6. *For all $T > 0$, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}_{\mathrm{odd}}$, all $E \ge 0$ and all $\phi \in \mathrm{Ran}(\chi_{(-\infty,E]}(H_L))$ we have*

$$\frac{T}{2}\sum_{\substack{k\in\mathbb{N}\\E_k\le E}}|\alpha_k|^2 \le \frac{\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])}}{R^d} \le 2T(1+(1+\|V\|_\infty)T^2)\sum_{\substack{k\in\mathbb{N}\\E_k\le E}}\beta_k(T)|\alpha_k|^2,$$

where

$$\beta_k(T) = \begin{cases} 1 & \text{if } E_k \le 0, \\ \mathrm{e}^{2T\sqrt{E_k}} & \text{if } E_k > 0. \end{cases}$$

Proof. For the function $F : \Lambda_{RL} \times \mathbb{R} \to \mathbb{C}$ we have for $T > 0$

$$\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} = \int_{-T}^{T} \int_{\Lambda_{RL}} \left(|\partial_{d+1} F|^2 + |\nabla' F|^2 + |F|^2\right) \mathrm{d}x.$$

Note that $\|\phi_k\|_{L^2(\Lambda_{RL})} = R^d$. By Green's theorem we have

$$\int_{\Lambda_{RL}} |\nabla' F|^2 \,\mathrm{d}x' = \int_{\Lambda_{RL}} \left(-\sum_{i=1}^{d} \partial_i^2 F \right) \bar{F} \,\mathrm{d}x' = -\int_{\Lambda_{RL}} V|F|^2 \,\mathrm{d}x' + \int_{\Lambda_{RL}} (\partial_{d+1}^2 F)\bar{F} \,\mathrm{d}x'$$

for all $x_{d+1} \in \mathbb{R}$. First we estimate

$$\begin{aligned}\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} &= \int_{-T}^{T} \int_{\Lambda_{RL}} \left(|\partial_{d+1} F|^2 - V|F|^2 + (\partial_{d+1}^2 F)\bar{F} + |F|^2\right) \mathrm{d}x \\ &\leq \int_{-T}^{T} \int_{\Lambda_{RL}} \left(|\partial_{d+1} F|^2 + (\partial_{d+1}^2 F)\bar{F} + (1+\|V\|_\infty)|F|^2\right) \mathrm{d}x = 2R^d \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} |\alpha_k|^2 I_k,\end{aligned}$$

where

$$\begin{aligned} I_k &:= \int_0^T \left((1+\|V\|_\infty)\,\mathrm{s}_k(x_{d+1})^2 + \mathrm{s}_k'(x_{d+1})^2 + \mathrm{s}_k''(x_{d+1})\,\mathrm{s}_k(x_{d+1})\right) \mathrm{d}x_{d+1} \\ &= (1+\|V\|_\infty) \int_0^T \mathrm{s}_k(x_{d+1})^2 \,\mathrm{d}x_{d+1} + \mathrm{s}_k'(T)\,\mathrm{s}_k(T). \end{aligned}$$

If $E_k \leq 0$, we estimate using $s_k(t) \leq t$ and $s_k'(t)s_k(t) \leq t$ for $t > 0$

$$I_k \leq \tfrac{1}{3}(1+\|V\|_\infty)T^3 + T \leq ((1+\|V\|_\infty)T^3 + T)\beta_k(T).$$

For $E_k > 0$ we use $\sinh(\omega_k t)/\omega_k \leq t\cosh(\omega_k t)$ for $t > 0$ and $\cosh(\omega_k T)^2 \leq \mathrm{e}^{2\omega_k T}$ to obtain

$$\begin{aligned} I_k &= (1+\|V\|_\infty) \int_0^T \frac{\sinh^2(\omega_k x_{d+1})}{\omega_k^2} \,\mathrm{d}x_{d+1} + \sinh(\omega_k T)\cosh(\omega_k T)/\omega_k \\ &\leq ((1+\|V\|_\infty)T^3 \cosh^2(\omega_k T) + T\cosh^2(\omega_k T)) \leq ((1+\|V\|_\infty)T^3 + T)\beta_k(T). \end{aligned}$$

This shows the upper bound. For the lower bound we drop the gradient term and obtain

$$\|F\|^2_{H^1(\Lambda_{RL}\times[-T,T])} \geq \int_{-T}^{T} \int_{\Lambda_{RL}} (|\partial_{d+1} F|^2 + |F|^2) \,\mathrm{d}x = 2 \cdot R^d \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} |\alpha_k|^2 \tilde{I}_k,$$

where

$$\tilde{I}_k := \int_0^T [\mathrm{s}_k(x_{d+1})^2 + \mathrm{s}_k'(x_{d+1})^2] \,\mathrm{d}x_{d+1}.$$

If $E_k = 0$, the lower bound $\tilde{I}_k \geq T$ follows immediately. Else, we have $\mathrm{s}_k(t)^2 \geq \sin^2(\omega_k t)/\omega_k$ and $\mathrm{s}_k'(t)^2 \geq \cos(\omega_k t)$, whence

$$\tilde{I}_k \geq \int_0^T \frac{\sin^2(\omega_k x_{d+1})}{\omega_k^2} + \cos^2(\omega_k x_{d+1}) \,\mathrm{d}x_{d+1} \geq \int_0^T \cos^2(\omega_k x_{d+1}) \,\mathrm{d}x_{d+1} = \frac{T}{2} + \frac{\sin(2\omega_k T)}{4\omega_k}.$$

Now, if $2\omega_k T < \pi$, the sinus term is positive and we drop it to find $\tilde{I}_k \geq T/2$. If $2\omega_k T \geq \pi$, we have $\sin(2\omega_k T) \geq -1$ and estimate

$$\tilde{I}_k \geq \frac{T}{2} - \frac{1}{4\omega_k} = \frac{T}{2} - \frac{\pi}{4\pi\omega_k} \geq \frac{T}{2} - \frac{T}{2\pi} \geq \frac{T}{4}. \qquad \square$$

Proof of Theorem 2.2. First we consider the case $L \in \mathbb{N}_{\text{odd}}$. We note that Proposition 3.6 remains true if we replace Λ_{RL} by Λ_L and R^d by 1; i.e., for all $T > 0$ and $L \in \mathbb{N}_{\text{odd}}$ we have

$$\frac{T}{2} \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} |\alpha_k|^2 \leq \|F\|^2_{H^1(\Lambda_L \times [-T,T])} \leq 2T(1 + (1 + \|V\|_\infty)T^2) \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} \beta_k(T)|\alpha_k|^2. \tag{34}$$

We have $\widetilde{X}_{R_3} \subset \Lambda_{RL} \times [-R_3, R_3]$. By (34) and Proposition 3.6 we have

$$\frac{\|F\|^2_{H^1(\widetilde{X}_{R_3})}}{\|F\|^2_{H^1(X_1)}} \leq \frac{\|F\|^2_{H^1(\Lambda_{RL} \times [-R_3, R_3])}}{\|F\|^2_{H^1(X_1)}} \leq \widetilde{D}_3^2 D_4^2$$

with

$$\widetilde{D}_3^2 = \frac{\sum_{E_k \leq E} \theta_k |\alpha_k|^2}{\sum_{E_k \leq E} |\alpha_k|^2} \quad \text{and} \quad D_4^2 = 4 \cdot R^d R_3 (1 + (1 + \|V\|_\infty) R_3^2),$$

where $\theta_k = \beta_k(R_3)$. We use Propositions 3.4 and 3.5 and obtain

$$\|F\|_{H^1(\widetilde{X}_{R_3})} \leq \widetilde{D}_3 D_4 \|F\|_{H^1(X_1)} \leq D_1^\gamma D_2 \widetilde{D}_3 D_4 \|F\|^{1-\gamma}_{H^1(\widetilde{X}_{R_3})} \|(\partial_{d+1} F)_0\|^{\gamma/2}_{L^2(W_\delta(L))} \|F\|^{\gamma/2}_{H^1(U_3(L))}.$$

Since $U_3(L) \subset \widetilde{X}_{R_3}$ we have

$$\|F\|_{H^1(\widetilde{X}_{R_3})} \leq D_1^2 D_2^{2/\gamma} \widetilde{D}_3^{2/\gamma} D_4^{2/\gamma} \|(\partial_{d+1} F)_0\|_{L^2(W_\delta(L))}.$$

By (34), the square of the left-hand side is bounded from below by

$$\|F\|^2_{H^1(\widetilde{X}_{R_3})} \geq \|F\|^2_{H^1(\Lambda_L \times [-R_3, R_3])} \geq \tfrac{1}{2} R_3 \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} |\alpha_k|^2.$$

Putting everything together we obtain by using $(\partial_{d+1} F)_0 = \phi$

$$\tfrac{1}{2} R_3 \sum_{\substack{k\in\mathbb{N} \\ E_k \leq E}} |\alpha_k|^2 \leq D_1^4 (D_2 \widetilde{D}_3 D_4)^{4/\gamma} \|\phi\|^2_{L^2(W_\delta(L))}.$$

In order to end the proof we will give an upper bound on $\widetilde{D}_3$ which is independent of α_k, $k \in \mathbb{N}$. For this purpose, we recall that $\theta_k = \beta_k(R_3)$. Since $\theta_k \leq \mathrm{e}^{2R_3\sqrt{E}}$ for all $k \in \mathbb{N}$ with $E_k \leq E$, we have

$$\widetilde{D}_3^4 \leq D_3^4 := \mathrm{e}^{4R_3\sqrt{E}}.$$

Hence, using $\sum_{E_k \leq E} |\alpha_k|^2 = \|\phi\|^2_{L^2(\Lambda_L)}$, we obtain for all $L \in \mathbb{N}_{\text{odd}}$ the estimate

$$\widetilde{C}_{\text{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)} \leq \|\phi\|^2_{L^2(W_\delta(L))},$$

where $\widetilde{C}_{\mathrm{sfuc}} = \widetilde{C}_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) = D_1^{-4}(D_2 D_3 D_4)^{-4/\gamma}$. From the definitions of $D_i,\ i \in \{1, 2, 3, 4\}$, and γ one calculates that

$$\widetilde{C}_{\mathrm{sfuc}} \geq \delta^{\widetilde{N}\left(1+\|V\|_\infty^{2/3}+\sqrt{E}\right)}$$

with some constant $\widetilde{N} = \widetilde{N}(d)$. Now we treat the case of $L \in \mathbb{N}_{\mathrm{even}} = \{2, 4, 6, \ldots\}$. By a scaling argument as in Corollary 2.2 of [Rojas-Molina and Veselić 2013], we immediately obtain that for all $G > 0$, $\delta \in (0, G/2),\ L/G \in \mathbb{N}_{\mathrm{odd}}$ and all (G, δ)-equidistributed sequences q_j we have

$$\|\phi\|^2_{L^2(W_\delta^q(L))} \geq \widetilde{C}^G_{\mathrm{sfuc}} \|\phi\|^2_{L^2(\Lambda_L)} \tag{35}$$

and $\widetilde{C}^G_{\mathrm{sfuc}}(d, \delta, E, \|V\|_\infty) = \widetilde{C}_{\mathrm{sfuc}}(d, \delta/G, EG^2, \|V\|_\infty G^2)$. Here $W_\delta^q(L)$ denotes the set $W_\delta(L)$ corresponding to the sequence q_j. Now we define

$$G = \begin{cases} L/(L/2-1) & \text{if } L \in 4\mathbb{N}, \\ 2 & \text{otherwise,} \end{cases}$$

which satisfies $G \in [2, 4]$ and $L/G \in \mathbb{N}_{\mathrm{odd}}$. Since $G \geq 2$, every elementary cell $\Lambda_G + j,\ j \in (G\mathbb{Z})^d$, contains at least one elementary cell $\Lambda_1 + j,\ j \in \mathbb{Z}^d$. Hence we can choose a (G, δ)-equidistributed subsequence q_j of z_j. We apply (35) to this subsequence and obtain

$$\|\phi\|^2_{L^2(W_\delta(L))} \geq \|\phi\|^2_{L^2(W_\delta^q(L))} \geq \widetilde{C}^G_{\mathrm{sfuc}} \|\phi\|_{L^2(\Lambda_L)}.$$

Note that $W_\delta(L)$ corresponds to the sequence z_j. Putting everything together we obtain the statement of the theorem with

$$\min\{\widetilde{C}_{\mathrm{sfuc}}, \inf_{G\in[2,4]} \widetilde{C}^G_{\mathrm{sfuc}}\} \geq \delta^{N\left(1+\|V\|_\infty^{2/3}+\sqrt{E}\right)} =: C_{\mathrm{sfuc}}$$

and some constant $N = N(d)$. For the last inequality we use that $\left(\frac{1}{4}\right)^{\widetilde{N}} \geq \delta^{2\widetilde{N}}$. □

Proof of Corollary 2.6. We denote the normalized eigenfunctions of $-\Delta_L + A + B$ corresponding to the eigenvalues $\lambda_i(-\Delta_L + A + B)$ by ϕ_i. Then we have

$$\begin{aligned}
\lambda_i(-\Delta_L + A + B) &= \langle \phi_i, (-\Delta_L + A + B)\phi_i \rangle \\
&= \max_{\substack{\phi \in \mathrm{Span}\{\phi_1,\ldots,\phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle + \langle \phi, B\phi \rangle \\
&\geq \max_{\substack{\phi \in \mathrm{Span}\{\phi_1,\ldots,\phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle + \alpha \langle \phi, \chi_{W_\delta(L)}\phi \rangle.
\end{aligned}$$

By Corollary 2.4, we conclude that for all $\phi \in \mathrm{Span}\{\phi_1, \ldots, \phi_i\},\ \|\phi\| = 1$, we have

$$\langle \phi, \chi_{W_\delta(L)}\phi \rangle \geq C^{G,1}_{\mathrm{sfuc}}(d, \delta, E, \|A + B\|_\infty)$$

and furthermore, by the variational characterization of eigenvalues, we find

$$\max_{\substack{\phi \in \mathrm{Span}\{\phi_1,\ldots,\phi_i\} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle \geq \inf_{\dim \mathcal{D}=i} \max_{\substack{\phi \in \mathcal{D} \\ \|\phi\|=1}} \langle \phi, (-\Delta_L + A)\phi \rangle = \lambda_i(-\Delta_L + A).$$

Thus, we obtain the statement of the corollary. □

4. Proof of Wegner and initial scale estimates

Recall that $0 < G_1 < G_2$ are the numbers from the Delone property such that $\sharp\{\mathcal{D} \cap (\Lambda_{G_1} + x)\} \le 1$, $\sharp\{\mathcal{D} \cap (\Lambda_{G_2} + x)\} \ge 1$ for any $x \in \mathbb{R}^d$, and that for all $t \in [0, 1]$ we have $\operatorname{supp} u_t \subset \Lambda_{G_u}$. Let $\delta_{\max} := 1 - \omega_+$ and $K_u := u_{\max} \lceil G_u/G_1 \rceil^d$. For $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$ and $\delta \le \delta_{\max}$, we use the notation $V_{\omega+\delta}$ for the potential V_ω, where every ω_j, $j \in \mathcal{D}$, has been replaced by $\omega_j + \delta$. The following lemma is a consequence of the properties of a Delone set, in particular $\sharp\Lambda_L \cap \mathcal{D} \le \lceil L/G_1 \rceil^d$, and our assumption (5).

Lemma 4.1. (i) *For all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $0 < \delta \le \delta_{\max}$ and all $L \in (G_2 + G_u)\mathbb{N}$, the difference $V_{\omega+\delta} - V_\omega$ is on Λ_L bounded from below by $\alpha_1 \delta^{\alpha_2}$ times the characteristic function of $W_{\beta_1 \delta^{\beta_2}}(L)$ which corresponds to a $(G_2 + G_u, \beta_1 \delta^{\beta_2})$-equidistributed sequence.*

(ii) *For all $\omega \in [0, 1]^{\mathcal{D}}$ we have $\|V_\omega\|_\infty \le K_u$.*

(iii) *For all $L \in (G_2 + G_u)\mathbb{N}$, we have*

$$\sharp\big\{j \in \mathcal{D} : \exists t \in [0, 1],\ \operatorname{supp} u_t(\cdot - j) \cap \Lambda_L \neq \varnothing\big\} \le \lceil (L + G_u)/G_1 \rceil^d \le (2L/G_1)^d.$$

Proof of Theorem 2.10. Note that for all $E_0 \in \mathbb{R}$, $\lambda_i(H_{\omega,L}) \le E_0$ implies, by Lemma 4.1(ii), that $\lambda_i(H_{\omega+\delta,L}) \le E_0 + \|V_{\omega+\delta} - V_\omega\| \le E_0 + 2K_u$. Now we apply Corollary 2.6 with $A = V_\omega$ and $B = V_{\omega+\delta} - V_\omega$ (both restricted to Λ_L). Together with Lemma 4.1(i), we obtain for all $E_0 \in \mathbb{R}$, all $L \in G_u\mathbb{N}$, all $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, all $\delta \le \delta_{\max}$ and all $i \in \mathbb{N}$ with $\lambda_i(H_{\omega,L}) \le E_0$ the inequality

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + \alpha_1 \delta^{\alpha_2} C_{\mathrm{sfuc}}^{G_2+G_u,1}(d, \beta_1\delta^{\beta_2}, E_0 + 2K_u, K_u).$$

In particular, there is $\kappa = \kappa(d, \omega_+, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u, E_0) > 0$ such that

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + \delta^\kappa. \tag{36}$$

Now let $\varepsilon > 0$, satisfying $\varepsilon \le \varepsilon_{\max} := \delta_{\max}^\kappa/4$. We choose $\delta := (4\varepsilon)^{1/\kappa}$, whence

$$\lambda_i(H_{\omega+\delta,L}) \ge \lambda_i(H_{\omega,L}) + 4\varepsilon. \tag{37}$$

Let $\rho \in C^\infty(\mathbb{R}, [-1, 0])$ be smooth, nondecreasing such that $\rho = -1$ on $(-\infty; -\varepsilon]$ and $\rho = 0$ on $[\varepsilon; \infty)$. We can assume $\|\rho'\|_\infty \le 1/\varepsilon$. It holds that

$$\chi_{[E-\varepsilon; E+\varepsilon]}(x) \le \rho(x - E + 2\varepsilon) - \rho(x - E - 2\varepsilon) = \rho(x - E - 2\varepsilon + 4\varepsilon) - \rho(x - E - 2\varepsilon)$$

for all $x \in \mathbb{R}$ and together with (37) this implies

$$\begin{aligned}\mathbb{E}\big[\operatorname{Tr}[\chi_{[E-\varepsilon; E+\varepsilon]}(H_{\omega,L})]\big] &\le \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon + 4\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]\big] \\ &\le \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon) - \rho(H_{\omega,L} - E - 2\varepsilon)]\big].\end{aligned} \tag{38}$$

Now let $\tilde{\Lambda}_L := \{j \in \mathcal{D} : \exists t \in [0, 1],\ \operatorname{supp} u_t(\cdot - j) \cap \Lambda_L \neq \varnothing\}$ be the set of lattice sites which can influence the potential within Λ_L. Note that $\sharp\tilde{\Lambda}_L \le (2L/G_1)^d$. We enumerate the points in $\tilde{\Lambda}_L$ by $k : \{1, \dots, \sharp\tilde{\Lambda}_L\} \to \mathcal{D}$, $n \mapsto k(n)$. The upper bound in (38) will be expanded in a telescopic sum by changing the $|\tilde{\Lambda}_L|$ indices from ω_j to $\omega_j + \delta$ successively. In order to do that some notation is needed.

Given $\omega \in [\omega_-, \omega_+]^{\mathcal{D}}$, $n \in \{1, \dots, |\tilde{\Lambda}_L|\}$, $\delta \in [0, \delta_{\max}]$ and $t \in [\omega_-, \omega_+]$, we define $\tilde{\omega}^{(n,\delta)}(t) \in [\omega_-, 1]^{\mathcal{D}}$ inductively via

$$(\tilde{\omega}^{(1,\delta)}(t))_j := \begin{cases} t & \text{if } j = k(1), \\ \omega_j & \text{else} \end{cases} \quad \text{and} \quad (\tilde{\omega}^{(n,\delta)}(t))_j := \begin{cases} t & \text{if } j = k(n), \\ (\tilde{\omega}^{(n-1,\delta)}(\omega_j + \delta))_j & \text{else.} \end{cases}$$

The function $\tilde{\omega}^{(n,\delta)} : [\omega_-, 1] \to [\omega_-, 1]^{\mathcal{D}}$ is the rank-1 perturbation of ω in the $k(n)$-th coordinate with the additional requirement that all sites $k(1), \dots, k(n-1)$ have already been blown up by δ. We define

$$\Theta_n(t) := \operatorname{Tr}[\rho(H_{\tilde{\omega}^{(n,\delta)}(t),L} - E - 2\varepsilon)] \quad \text{for } n = 1, \dots, |\tilde{\Lambda}_L|.$$

Note that

$$\begin{aligned} \Theta_1(\omega_{k(1)}) &= \operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon)], \\ \Theta_n(\omega_{k(n)}) &= \Theta_{n-1}(\omega_{k(n-1)} + \delta) \quad \text{for } n = 2, \dots, |\tilde{\Lambda}_L| \quad \text{and} \\ \Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) &= \operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon)]. \end{aligned}$$

Hence the upper bound in (38) is

$$\begin{aligned} \mathbb{E}\big[\operatorname{Tr}[\rho(H_{\omega+\delta,L} - E - 2\varepsilon)] - \operatorname{Tr}[\rho(H_{\omega,L} - E - 2\varepsilon]\big] &= \mathbb{E}\big[\Theta_{|\tilde{\Lambda}_L|}(\omega_{k(|\tilde{\Lambda}_L|)} + \delta) - \Theta_1(\omega_{k(1)})\big] \\ &= \sum_{n=1}^{|\tilde{\Lambda}_L|} \mathbb{E}\big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\big]. \end{aligned}$$

Due to the product structure of the probability space, we can apply Fubini's theorem to each summand and obtain

$$\mathbb{E}\big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\big] = \mathbb{E}\bigg[\int_{\omega_-}^{\omega_+} \Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)}) \, \mathrm{d}\mu(\omega_{k(n)})\bigg].$$

Note that $\Theta_n : [\omega_-, 1] \to \mathbb{R}$ is monotone and bounded. We will use the following lemma.

Lemma 4.2. *Let $-\infty < \omega_- < \omega_+ \le +\infty$. Assume that μ is a probability distribution with bounded density ν_μ and support in the interval $[\omega_-, \omega_+]$ and let Θ be a nondecreasing, bounded function. Then for all $\delta > 0$*

$$\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \, \mathrm{d}\mu(\lambda) \le \|\nu_\mu\|_\infty \cdot \delta[\Theta(\omega_+ + \delta) - \Theta(\omega_-)].$$

Proof of Lemma 4.2. We calculate

$$\begin{aligned} &\int_{\mathbb{R}} [\Theta(\lambda + \delta) - \Theta(\lambda)] \, \mathrm{d}\mu(\lambda) \\ &\quad \le \|\nu_\mu\|_\infty \int_{\omega_-}^{\omega_+} [\Theta(\lambda + \delta) - \Theta(\lambda)] \, \mathrm{d}\lambda = \|\nu_\mu\|_\infty \bigg[\int_{\omega_- + \delta}^{\omega_+ + \delta} \Theta(\lambda) \, \mathrm{d}\lambda - \int_{\omega_-}^{\omega_+} \Theta(\lambda) \, \mathrm{d}\lambda\bigg] \\ &\quad = \|\nu_\mu\|_\infty \bigg[\int_{\omega_+}^{\omega_+ + \delta} \Theta(\lambda) \, \mathrm{d}\lambda - \int_{\omega_-}^{\omega_- + \delta} \Theta(\lambda) \, \mathrm{d}\lambda\bigg] \le \|\nu_\mu\|_\infty \cdot \delta[\Theta(\omega_+ + \delta) - \Theta(\omega_-)]. \qquad \square \end{aligned}$$

Thus, we find for all $n = 1, \dots, |\tilde{\Lambda}_L|$

$$\int_{\omega_-}^{\omega_+} \big[\Theta_n(\omega_{k(n)} + \delta) - \Theta_n(\omega_{k(n)})\, \mathrm{d}\mu(\omega_{k(n)})\big] \le \|\nu_\mu\|_\infty \cdot \delta[\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-)].$$

We will also need the following result; see, e.g., Theorem 2 in [Hundertmark et al. 2006].

Proposition 4.3. *Let $H_0 := -\Delta + A$ be a Schrödinger operator with a bounded potential $A \ge 0$, and let $H_1 := H_0 + B$ for some bounded $B \ge 0$ with compact support. Denote the corresponding Dirichlet restrictions to Λ by H_0^Λ and H_1^Λ, respectively. There are constants K_1, K_2 depending only on d and monotonously on* $\operatorname{diam}\operatorname{supp} B$ *such that for any smooth, bounded function $g : \mathbb{R} \to \mathbb{R}$ with compact support in $(-\infty, E_0]$ and the property that $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class we have*

$$\operatorname{Tr}[g(H_1^\Lambda) - g(H_0^\Lambda)] \le K_1 \mathrm{e}^{E_0} + K_2(\ln(1 + \|g'\|_\infty)^d)\|g'\|_1.$$

Proposition 4.3 implies:

Lemma 4.4. *Let $0 < \varepsilon \le \varepsilon_{\max}$. Then $\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) \le (K_1 \mathrm{e}^{E_0} + 2^d K_2)|\ln \varepsilon|^d$, where K_1, K_2 are as in Proposition 4.3 and thus only depend on d and on G_u.*

Proof of Lemma 4.4. Let $g(\cdot) := \rho(\cdot - E - 2\varepsilon))$. By our choice of ρ, we know g has support in $(-\infty, E_0]$, $\|g'\|_\infty \le 1/\varepsilon$ and $\|g'\|_1 = 1$. We define the operators

$$H_0^\Lambda := H(\tilde{\omega}^{(n,\delta)}(\omega_-), L) \quad \text{and} \quad H_1^\Lambda := H(\tilde{\omega}^{(n,\delta)}(\omega_+ + \delta), L).$$

They are lower semibounded operators with purely discrete spectrum and since g has support in $(-\infty, E_0]$, the difference $g(H_1^\Lambda) - g(H_0^\Lambda)$ is trace class. By the previous proposition

$$\Theta_n(\omega_+ + \delta) - \Theta_n(\omega_-) = \operatorname{Tr}[g(H_1^\Lambda) - g(H_0^\Lambda)] \le K_1 \mathrm{e}^{E_0} + K_2(\ln(1 + 1/\varepsilon))^d.$$

To conclude, note that $\varepsilon \le \varepsilon_{\max} < \frac{1}{2}$ and thus $\ln(1 + 1/\varepsilon) \le 2|\ln \varepsilon|$ and $1 \le |\ln \varepsilon| \le |\ln \varepsilon|^d$. □

Putting everything together and recalling $\delta = (4\varepsilon)^{1/\kappa}$ we find

$$\begin{aligned}\mathbb{E}\big[\operatorname{Tr}[\chi_{[E-\varepsilon, E+\varepsilon]}(H_{\omega,L})]\big] &\le (K_1 \mathrm{e}^{E_0} + 2^d K_2)\|\nu_\mu\|_\infty \cdot \delta |\ln \varepsilon|^d |\tilde{\Lambda}_L| \\ &\le (K_1 \mathrm{e}^{E_0} + 2^d K_2)\|\nu_\mu\|_\infty \cdot (4\varepsilon)^{1/\kappa} |\ln \varepsilon|^d (2/G_1)^d L^d.\end{aligned}$$ □

Proof of Theorem 2.11. We follow the ideas developed in [Barbaroux et al. 1997; Kirsch et al. 1998]. Let $t \le \delta_{\max}$, $V_{t,L}$ be the restriction of V_ω to Λ_L obtained by setting all random variables to t, and $H_{t,L} = -\Delta_{\Lambda_L} + V_{L,t}$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions. Note that $H_{0,L} = -\Delta_{\Lambda_L} + V_{0,L}$ and that the first eigenvalue of $H_{t,L}$ is bounded from above by $d(\pi/L)^2 + K_u$. Inequality (36) with $E_0 = d\pi^2 + K_u$, $\omega_k = 0$, $k \in \mathcal{D}$, and $\delta = t$ yields that there is $\kappa = \kappa(d, \delta_{\max}, \alpha_1, \alpha_2, \beta_1, \beta_2, G_2, G_u, K_u)$ such that for all $t \le \delta_{\max}$

$$\lambda_1(H_{t,L}) \ge \lambda_1(H_{0,L}) + t^\kappa.$$

We choose $t = L^{-7/(4\kappa)}$ and L sufficiently large such that $t < \min\{\delta_{\max}, t_0\}$. Then,

$$\lambda_1(H_{t,L}) - \lambda_1(H_{0,L}) \ge L^{-7/4}.$$

Let $\Omega_0 := \{\omega \in \Omega : \lambda_1(H_{\omega,L}) \geq \lambda_1(H_{t,L})\}$. Since the potential values in Λ_L only depend on ω_k, $k \in \Lambda_{L+G_u} \cap \mathcal{D}$, we calculate using $\sharp\Lambda_{L+G_u} \cap \mathcal{D} \leq \lceil (L+G_u)/G_1 \rceil^d$ and our assumption on the measure μ that

$$\mathbb{P}(\Omega_0) \geq 1 - \mathbb{P}(\exists \gamma \in \Lambda_{L+G_u} \cap \mathcal{D},\ \omega_\gamma \leq t) \geq 1 - \left\lceil \frac{L+G_u}{G_1} \right\rceil^d \mu([0,t]) \geq 1 - \left\lceil \frac{L+G_u}{G_1} \right\rceil^d \frac{C}{L^{7d/4}}.$$

Since $\lceil (L+G_u)/G_1 \rceil^d \leq L^{5d/4}$ for L sufficiently large, we obtain the statement of the theorem. □

5. Proof of the observability estimate

We want to apply [Miller 2010, Theorem 2.2] where we choose $A = \Delta_L - V_L$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions, $C = \chi_{W_\delta(L)}$ and $C_0 = \mathrm{Id}$. Note that A is self-adjoint with spectrum contained in $(-\infty, \|V\|_\infty]$. For $\lambda > 0$ we define the increasing sequence of spectral subspaces $\mathcal{E}_\lambda := \operatorname{Ran} \chi_{[-\lambda,\infty)}(\Delta_L - V_L)$.

We need to check [Miller 2010, (5),(6),(7)]. By spectral calculus, we have for all $\lambda > 0$

$$\|\mathrm{e}^{(\Delta_L - V_L)t} u\|_{\Lambda_L} \leq \mathrm{e}^{-\lambda t} \|u\|_{\Lambda_L}, \quad u \in \mathcal{E}_\lambda^\perp = \operatorname{Ran} \chi_{(-\infty,-\lambda)}(\Delta_L - V_L), \quad t > 0.$$

Furthermore, Corollary 2.4 implies for all $\lambda > 0$ and $u \in \mathcal{E}_\lambda$

$$\|u\|_{\Lambda_L}^2 \leq a_0 \mathrm{e}^{-N \ln(\delta/G) G \sqrt{\lambda}} \|u\|_{W_\delta(L)}^2.$$

For $T \leq 1$ we have $\mathrm{e}^{2T\|V\|_\infty}/T \leq \mathrm{e}^{2\|V\|_\infty} \mathrm{e}^{2/T}$, whence

$$\|\mathrm{e}^{T(\Delta - V)} u\|_{\Lambda_L}^2 \leq \frac{\mathrm{e}^{2T\|V\|_\infty}}{T} \int_0^T \|\mathrm{e}^{t(\Delta - V)} u\|_{\Lambda_L}^2 \, \mathrm{d}t \leq \mathrm{e}^{2\|V\|_\infty} \mathrm{e}^{2/T} \int_0^T \|\mathrm{e}^{t(\Delta - V)} u\|_{\Lambda_L}^2 \, \mathrm{d}t.$$

Thus we found [Miller 2010, (5),(6),(7)] with $m_0 = 1$, $m = 0$, $\alpha = \nu = \frac{1}{2}$, a_0 and b_0 as in the theorem, $a = -(N/2)\ln(\delta/G)G > 0$, $b = 1$ and $\beta = 1$. By [Miller 2010, Theorem 2.2 and Corollary 1(i)], there exists $T' > 0$ such that for all $T \leq T'$

$$\kappa_T \leq 4 a_0 b_0 \mathrm{e}^{2c_*/T}, \quad \text{where } c_* = 4(\sqrt{a+2} - \sqrt{a})^{-4}.$$

From the proof in [Miller 2010], it can be inferred that T' only depends on m_0, α, β, a, b, a_0, b_0 and on our choice of $T \leq 1$. Thus, in our case, T' only depends on G, δ and $\|V\|_\infty$.

Using $\sqrt{a+2} - \sqrt{a} = \int_a^{a+2} (2\sqrt{x})^{-1} \, \mathrm{d}x \geq (a+2)^{-1/2}$ and the fact that from $\delta \leq G/2$, it follows that $2 \leq 2a/a_{\min}$, where $a_{\min} := (N/2)\ln(2)G$, and we obtain

$$c_* \leq 4(a+2)^2 \leq 4a^2(1 + 2/a_{\min})^2 = \ln(G/\delta)^2 (NG + 4/\ln 2)^2.$$

Appendix: On single-site potentials for the breather model

Our assumptions. In this section we discuss our conditions on the single-site potential in the random breather model. Recall that the ω_j were supported in $[\omega_-, \omega_+] \subset [0, 1)$, whence we consider $t \in [\omega_-, \omega_+]$ and $\delta \in [0, 1 - \omega_+]$.

Definition A.1. We say that a family $\{u_t\}_{t\in[0,1]}$ of measurable functions $u_t:\mathbb{R}^d\to\mathbb{R}$ satisfies:

- condition (A) if the u_t are uniformly bounded, have uniform compact support and if there are $\alpha_1,\beta_1>0$ and $\alpha_2,\beta_2\geq 0$ such that for all $t\in[\omega_-,\omega_+]$, $\delta\leq 1-\omega_+$ there is $x_0=x_0(t,\delta)\in\mathbb{R}^d$ with
$$u_{t+\delta}-u_t\geq \alpha_1\delta^{\alpha_2}\chi_{B(x_0,\beta_1\delta^{\beta_2})}, \tag{39}$$
- condition (B) if u_t is the dilation of a function u by t, defined as $u_t(x):=u(x/t)$ for $t>0$ and $u_0\equiv 0$, where u is the characteristic function of a bounded, convex, open set K with $0\in\overline{K}$,
- condition (C) if u_t is the dilation of a measurable function u which is positive, radially symmetric, compactly supported, bounded with monotonously decreasing radial part $r_u:[0,\infty)\to[0,\infty)$ and such there is a point $\tilde{x}>0$ where r_u is differentiable, $r_u'(\tilde{x})<0$ and $r_u(\tilde{x})>0$,
- condition (D) if u_t is the dilation of a measurable function u which is positive, radially symmetric, radially decreasing, compactly supported, bounded and which has a discontinuity away from 0,
- condition (E) if u_{1-t} is the dilation of a measurable function which is nonpositive, radially symmetric, radially increasing, compactly supported, bounded, and such there is a point $\tilde{x}>0$ where the radial part r_u is differentiable, $r_u'(\tilde{x})>0$ and $r_u(\tilde{x})<0$.

Remark A.2. Condition (A) is the abstract assumption we used in the proof of the Wegner estimate for the random breather model. Conditions (B) to (E) are relatively easy to verify for specific examples of single-site potentials. In particular, (C) holds for many natural choices of single-site potentials such as the smooth function $\chi_{|x|<1}\exp(1/(|x|^2-1))$ or the hat-potential $\chi_{|x|<1}(1-|x|)$. Furthermore, we note that if we have families $\{u_t\}_{t\in[0,1]}$ and $\{v_t\}_{t\in[0,1]}$ where u_t satisfies (A) and $v_{t+\delta}-v_t\geq 0$ for all $t\in[\omega_-,\omega_+]$ and $\delta\in(0,1-\omega_+]$, then the family $\{u_t+v_t\}_{t\in[0,1]}$ also satisfies (A).

Lemma A.3. *We have that each of the assumptions* (*B*) *to* (*E*) *implies* (*A*).

Proof. Assume (B). We will show (A) with $\alpha_1=1$, $\alpha_2=0$, $\beta_2=1$ and $\beta_1=c$, and hence it is enough to show the existence of a $c\delta$-ball in $K_{t+\delta}\backslash K_t$.

For $K\subset\mathbb{R}^d$ and $t>0$ we define $K_t:=\{x\in\mathbb{R}^d : x/t\in K\}$ and $K_0:=\varnothing$. Without loss of generality let $x:=(1,0,\dots,0)$ be a point in $\overline{K}$ which maximizes $|x|$ over $\overline{K}$. For $\lambda\in\mathbb{R}$ define the half-space $H_\lambda:=\{x\in\mathbb{R}^d : x_1\leq\lambda\}$, where x_1 stands for the first coordinate of x. By scaling, the existence of a $c\delta$-ball in $K_{t+\delta}\backslash K_t$ is equivalent to the existence of a $c\delta/(t+\delta)$-ball in $K\backslash K_{t/(t+\delta)}$. By maximality of $(1,0,\dots,0)$, we have $K\subset H_1$ and hence $K_{t/(t+\delta)}\subset H_{t/(t+\delta)}$. Thus, it is sufficient to find a $c\delta/(t+\delta)$-ball in $K\backslash H_{t/(t+\delta)}$. By convexity of K, the set $\left\{z\in K : z_1=\frac{1}{2}\right\}$ is nonempty and since K is open, we find $z_0\in K$ with $z_1=\frac{1}{2}$ and $0<c<\frac{1}{2}$ such that $B(z_0,c)\subset K$. We define for $\lambda\in[0,1)$ the set $X(\lambda)\subset\mathbb{R}^d$ as $X(\lambda):=B(z_0+\lambda((1,0,\dots,0)-z_0),c\cdot(1-\lambda))$. By convexity and the fact that $(1,0,\dots,0)\in\overline{K}$, we have $X(\lambda)\subset K$. In fact, let $\{x_n\}_{n\in\mathbb{N}}\subset K$ be a sequence with $x_n\to(1,0,\dots,0)$. We define open sets $X_n(\lambda)$ by replacing $(1,0,\dots,0)$ by x_n in the definition of $X(\lambda)$. By convexity of K, every X_n is a subset of K, whence $\bigcup_{n\in\mathbb{N}}X_n(\lambda)\subset K$. Furthermore we have $X(\lambda)\subset\bigcup_{n\in\mathbb{N}}X_n(\lambda)$. Thus $X(\lambda)\subset K$. We now choose $\lambda:=t/(t+\delta)$. Then $X(\lambda)\cap H_\lambda=\varnothing$. Noting that $c(1-\lambda)=c\delta/(t+\delta)$, we see that $X(\lambda)$ is the desired $c\delta/(t+\delta)$-ball.

Now we assume (C). Let $r_u'(\tilde{x}) = -C_1$. Then there is $\tilde{\varepsilon} > 0$ such that

$$r_u(\tilde{x}+\varepsilon) - r_u(\tilde{x}) \in \Big[-2\varepsilon C_1, -\frac{\varepsilon}{2}C_1\Big] \quad \text{for all } |\varepsilon| < \tilde{\varepsilon}. \tag{40}$$

It is sufficient to prove the following: there are $C_2, C_3 > 0$ such that for every $0 \le t \le \omega_+$ and every $0 < \delta \le 1 - \omega_+$ there is $\hat{x} = \hat{x}(t, \delta)$ such that

$$r_u\Big(\frac{\hat{x} + C_2\delta}{t+\delta}\Big) - r_u\Big(\frac{\hat{x}}{t}\Big) \ge C_3\delta. \tag{41}$$

Indeed, by monotonicity of r_u, (41) implies that for every $x \in [\hat{x}, \hat{x} + C_2\delta]$ we have

$$r_u\Big(\frac{x}{t+\delta}\Big) - r_u\Big(\frac{x}{t}\Big) \ge r_u\Big(\frac{\hat{x} + C_2\delta}{t+\delta}\Big) - r_u\Big(\frac{\hat{x}}{t}\Big) \ge C_3\delta,$$

whence (A) holds with $x_0 := (\hat{x} + C_2\delta/2)e_1$, $\alpha_1 = C_3$, $\beta_1 = C_2/2$, $\alpha_2 = \beta_2 = 1$.

In order to see (41), let $\hat{x} = (t+\delta)\tilde{x}$. We choose $\kappa \in \big(0, \frac{1}{4}\big)$ and assume that $\tilde{x} - 4\kappa\tilde{\varepsilon} > 0$ (this is no restriction since (40) also holds for smaller $\tilde{\varepsilon}$). Furthermore, we define $C_2 := \kappa\tilde{\varepsilon}$. Now we distinguish two cases. If $\tilde{x}\delta/t \le \tilde{\varepsilon}$, then (40) implies

$$\begin{aligned} r_u\Big(\frac{\hat{x} + C_2\delta}{t+\delta}\Big) - r_u\Big(\frac{\hat{x}}{t}\Big) &= r_u\Big(\tilde{x} + \kappa\frac{\tilde{\varepsilon}\delta}{t+\delta}\Big) - r_u(\tilde{x}) + r_u(\tilde{x}) - r_u\Big(\tilde{x} + \tilde{x}\frac{\delta}{t}\Big) \\ &\ge -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t+\delta} + C_1\frac{\tilde{x}\delta}{2t} \ge \delta\frac{C_1}{2}\frac{\tilde{x} - 4\kappa\tilde{\varepsilon}}{t+\delta}. \end{aligned}$$

If $\tilde{x}\delta/t > \tilde{\varepsilon}$, we use $r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{x}\delta/t) \ge r_u(\tilde{x}) - r_u(\tilde{x} + \tilde{\varepsilon})$ and (40) to obtain

$$r_u\Big(\frac{\hat{x} + C_2\delta}{t+\delta}\Big) - r_u\Big(\frac{\hat{x}}{t}\Big) \ge -2\kappa C_1\frac{\tilde{\varepsilon}\delta}{t+\delta} + C_1\frac{\tilde{\varepsilon}}{2} = \frac{C_1\tilde{\varepsilon}}{2}\Big(1 - \frac{4\kappa\delta}{t+\delta}\Big) \ge \frac{C_1\tilde{\varepsilon}}{2}(1-4\kappa).$$

Hence

$$r_u\Big(\frac{\hat{x} + C_2\delta}{t+\delta}\Big) - r_u\Big(\frac{\hat{x}}{t}\Big) \ge C_3\delta, \qquad \text{where } C_3 := \min\Big\{\frac{C_1(\tilde{x} - 4\kappa\tilde{\varepsilon})}{2}, \frac{C_1\tilde{\varepsilon}(1-4\kappa)}{2(1-\omega_+)}\Big\} > 0.$$

The fact that (D) implies (A) is a consequence of (B). In fact, a function u as in (D) can be decomposed as $u = v + w$, where v is (a multiple of) a characteristic function of a ball, centered at the origin, and w is positive, radially symmetric and decreasing. Indeed, let x_0 be the point of discontinuity with the smallest norm. Then we can take $v = (u(x_0-) - u(x_0+))\chi_{B(0,|x_0|)}$, where χ_A denotes the characteristic function of the set A.

The function v satisfies (A) by (B) (since balls are convex) and we have $w_{t+\delta} - w_t \ge 0$. By Remark A.2, the family $\{u_t\}_{t\in[0,1]} = \{v_t + w_t\}_{t\in[0,1]}$ also satisfies (A). The case (E) is an adaptation of (C). □

Earlier assumptions. For certain types of random breather potentials, Wegner estimates have been given before; see [Combes et al. 1996; 2001]. As we will show below, none of these results covers the *standard breather model*. The methods of [Combes et al. 1996; 2001] seem to be motivated by reducing, thanks to linearization, the random breather model to a model of alloy type and then applying methods designed for the latter one. They are not focused to take advantage of the inherent, albeit nonlinear, monotonicity of the random breather model. The following assumptions on the single site potential are considered in [Combes et al. 1996] and [Combes et al. 2001], respectively.

Definition A.4. We say that a measurable function $u : \mathbb{R}^d \to [0, \infty)$ satisfies:

- condition (F) if u is compactly supported, in $C^2(\mathbb{R}^d)$, nonzero in a neighborhood of the origin and for some $c_0 > 0$ we have the inequalities

$$-x \cdot \nabla u \geq 0 \quad \text{for all } x \in \mathbb{R}^d \qquad \text{and} \qquad \left| \frac{(x, \operatorname{Hess}[u]x)}{x \cdot \nabla u} \right| \leq c_0 < \infty \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\}, \tag{42}$$

- condition (G) if $u \not\equiv 0$ is compactly supported, in $C^1(B_1 \setminus \{0\})$, and there is $\varepsilon_0 > 0$ such that

$$-x \cdot \nabla u - \varepsilon_0 u \geq 0 \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\}. \tag{43}$$

We have the following lemma.

Lemma A.5. *We have that*

- (F) *never holds*,
- (G) *implies that u has a singularity at the origin.*

Proof. We first show the statements in dimension one. Assume (F) and let $x_0 := \min \operatorname{supp} u$. Note that $x_0 < 0$. By the first inequality in (42) we have that $u' \geq 0$ for $x \in (x_0, 0)$. The second inequality in (42) implies

$$|u''(x)| \leq \frac{c_0 u'(x)}{|x|} \leq \frac{2c_0 u'(x)}{|x_0|} \quad \text{for all } x \in (x_0, x_0/2),$$

whence we have

$$u'(x) = \int_{x_0}^{x} u''(y)\,\mathrm{d}y \leq \int_{x_0}^{x} |u''(y)|\,\mathrm{d}y \leq \frac{2c_0}{|x_0|} \int_{x_0}^{x} u'(y)\,\mathrm{d}y,$$

and iteratively

$$\begin{aligned} u'(x) &\leq \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^{x} \int_{x_0}^{x^{(1)}} \cdots \int_{x_0}^{x^{(n-1)}} u'(x^{(n)})\,\mathrm{d}x^{(n)} \cdots \mathrm{d}x^{(1)} \\ &\leq \|u'\|_\infty \cdot \frac{(2c_0)^n}{|x_0|^n} \int_{x_0}^{x} \int_{x_0}^{x^{(1)}} \cdots \int_{x_0}^{x^{(n-1)}} \mathrm{d}x^{(n)} \cdots \mathrm{d}x^{(1)} \\ &= \|u'\|_\infty \cdot \left(\frac{2c_0(x - x_0)}{|x_0|} \right)^n (n!)^{-1} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

for all $x \in (x_0, x_0/2)$. We found $u' \equiv 0$ on $(x_0, x_0/2)$, which is a contradiction.

Now we assume (G). The function u cannot have its supremum at a point of differentiability for else it would have to be zero at its maximum, which would imply $u \equiv 0$. Condition (43) implies that u is increasing on the negative half-axis and decreasing on the positive half-axis. We conclude that the supremum has to be the limit at the only possible nondifferentiable point $x = 0$ and we will show that this limit is ∞. By monotonicity of u and the assumption $u \not\equiv 0$, there is $\delta_0 > 0$ such that

$$u(x) \geq u(\delta_0) > 0 \quad \text{on } (0, \delta_0) \qquad \text{or} \qquad u(x) \geq u(-\delta_0) > 0 \quad \text{on } (-\delta_0, 0).$$

Without loss of generality, we assume $u(x) \geq u(\delta_0) > 0$ on $(0, \delta_0)$. Furthermore, from (43) it follows that

$$-u'(x) \geq \varepsilon_0 \frac{u(x)}{x} \quad \text{for } x > 0.$$

Using this inequality we estimate for $0 < x < \delta_0$:

$$u(x) \geq u(x) - u(\delta_0) = -\int_x^{\delta_0} u'(s)\,\mathrm{d}s \geq \varepsilon_0 \int_x^{\delta_0} \frac{u(s)}{s}\,\mathrm{d}s$$
$$\geq \varepsilon_0 u(\delta_0) \int_x^{\delta_0} s^{-1}\mathrm{d}s = \varepsilon_0 u(\delta_0)[\ln(\delta_0) - \ln(x)] \to \infty \text{ as } x \to 0.$$

Now we show the claim in higher dimensions. If the single site potential $U : \mathbb{R}^d \to [0, \infty)$ does not vanish identically there is a point y such that $U(y) > 0$. Assume without loss of generality that y lies on the x_1-axis and define $u : \mathbb{R} \to [0, \infty)$ by $u(x_1) = U(x_1, 0, \ldots, 0)$. Note that if U satisfies the assumption (F) or (G), respectively, then u satisfies (F) or (G) as well and the one-dimensional argument can be applied to u. Hence, the statement of the lemma also holds for U. □

In the light of the comments made at the beginning of this section, the occurrence of a singularity is not surprising since in the case of a single-site potential with a polynomial singularity, $u(x) = |x|^{-\alpha}$, we have

$$u(x/\omega_j) = |x/\omega_j|^{-\alpha} = \omega_j^\alpha |x|^{-\alpha} = \omega_j^\alpha u(x),$$

and thus the random breathing would correspond to a multiplication, which would allow to reduce the breather model to the well-understood alloy-type model $V_\omega(x) = \sum_j \omega_j u(x - j)$.

Acknowledgements

This work has been partially supported by the DFG under grant *Unique continuation principles and equidistribution properties of eigenfunctions*, and by the binational German-Croatian DAAD-MZOS projects *Scale-uniform controllability of partial differential equations* and *The cost of controlling the heat flow in a multiscale setting*. Nakić was partially supported by HRZZ project grant 9345. Tautenhahn thanks Constanza Rojas-Molina for pointing out that the initial length scale estimate follows from the unique continuation principle.

References

[Bakri 2013] L. Bakri, "Carleman estimates for the Schrödinger operator: applications to quantitative uniqueness", *Comm. Partial Differential Equations* **38**:1 (2013), 69–91. MR Zbl

[Barbaroux et al. 1997] J. M. Barbaroux, J. M. Combes, and P. D. Hislop, "Localization near band edges for random Schrödinger operators", *Helv. Phys. Acta* **70**:1-2 (1997), 16–43. MR Zbl

[Bourgain and Kenig 2005] J. Bourgain and C. E. Kenig, "On localization in the continuous Anderson–Bernoulli model in higher dimension", *Invent. Math.* **161**:2 (2005), 389–426. MR Zbl

[Combes et al. 1996] J. M. Combes, P. D. Hislop, and E. Mourre, "Spectral averaging, perturbation of singular spectra, and localization", *Trans. Amer. Math. Soc.* **348**:12 (1996), 4883–4894. MR Zbl

[Combes et al. 2001] J. M. Combes, P. D. Hislop, and S. Nakamura, "The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators", *Comm. Math. Phys.* **218**:1 (2001), 113–130. MR Zbl

[Combes et al. 2003] J.-M. Combes, P. D. Hislop, and F. Klopp, "Hölder continuity of the integrated density of states for some random operators at all energies", *Int. Math. Res. Not.* **2003**:4 (2003), 179–209. MR Zbl

[Combes et al. 2007] J.-M. Combes, P. D. Hislop, and F. Klopp, "An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators", *Duke Math. J.* **140**:3 (2007), 469–498. MR Zbl

[Davey 2014] B. Davey, "Some quantitative unique continuation results for eigenfunctions of the magnetic Schrödinger operator", *Comm. Partial Differential Equations* **39**:5 (2014), 876–945. MR Zbl

[Dietlein et al. 2017] A. Dietlein, M. Gebert, and P. Müller, "Bounds on the effect of perturbations of continuum random Schrödinger operators and applications", preprint, 2017. To appear in *J. Spectr. Theory*. arXiv

[Egidi and Veselić 2016] M. Egidi and I. Veselić, "Scale-free unique continuation estimates and Logvinenko–Sereda theorems on the torus", preprint, 2016. arXiv

[Ervedoza and Zuazua 2011] S. Ervedoza and E. Zuazua, "Sharp observability estimates for heat equations", *Arch. Ration. Mech. Anal.* **202**:3 (2011), 975–1017. MR Zbl

[Escauriaza and Vessella 2003] L. Escauriaza and S. Vessella, "Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients", pp. 79–87 in *Inverse problems: theory and applications* (Cortona/Pisa, 2002), edited by G. Alessandrini and G. Uhlmann, Contemp. Math. **333**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Fernández-Cara and Zuazua 2000] E. Fernández-Cara and E. Zuazua, "The cost of approximate controllability for heat equations: the linear case", *Adv. Differential Equations* **5**:4-6 (2000), 465–514. MR Zbl

[Fursikov and Imanuvilov 1996] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series **34**, Seoul National University, Seoul, 1996. MR Zbl

[Germinet 2008] F. Germinet, "Recent advances about localization in continuum random Schrödinger operators with an extension to underlying Delone sets", pp. 79–96 in *Mathematical results in quantum mechanics*, edited by I. Beltita et al., World Scientific, Hackensack, NJ, 2008. MR Zbl

[Germinet and Klein 2013] F. Germinet and A. Klein, "A comprehensive proof of localization for continuous Anderson models with singular random potentials", *J. Eur. Math. Soc.* (*JEMS*) **15**:1 (2013), 53–143. MR Zbl

[Germinet et al. 2007] F. Germinet, P. D. Hislop, and A. Klein, "Localization for Schrödinger operators with Poisson random potential", *J. Eur. Math. Soc.* (*JEMS*) **9**:3 (2007), 577–607. MR Zbl

[Germinet et al. 2015] F. Germinet, P. Müller, and C. Rojas-Molina, "Ergodicity and dynamical localization for Delone–Anderson operators", *Rev. Math. Phys.* **27**:9 (2015), art. id. 1550020. MR Zbl

[Güichal 1985] E. N. Güichal, "A lower bound of the norm of the control operator for the heat equation", *J. Math. Anal. Appl.* **110**:2 (1985), 519–527. MR Zbl

[Helm and Veselić 2007] M. Helm and I. Veselić, "Linear Wegner estimate for alloy-type Schrödinger operators on metric graphs", *J. Math. Phys.* **48**:9 (2007), art. id. 092107. MR Zbl

[Hundertmark et al. 2006] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, and I. Veselić, "Bounds on the spectral shift function and the density of states", *Comm. Math. Phys.* **262**:2 (2006), 489–503. MR Zbl

[Jerison and Lebeau 1999] D. Jerison and G. Lebeau, "Nodal sets of sums of eigenfunctions", pp. 223–239 in *Harmonic analysis and partial differential equations* (Chicago, IL, 1996), edited by M. Christ et al., Univ. Chicago Press, 1999. MR Zbl

[Kellendonk et al. 2015] J. Kellendonk, D. Lenz, and J. Savinien (editors), *Mathematics of aperiodic order*, Progress in Mathematics **309**, Springer, 2015. MR Zbl

[Kenig et al. 2011] C. E. Kenig, M. Salo, and G. Uhlmann, "Inverse problems for the anisotropic Maxwell equations", *Duke Math. J.* **157**:2 (2011), 369–419. MR Zbl

[Kirsch 1996] W. Kirsch, "Wegner estimates and Anderson localization for alloy-type potentials", *Math. Z.* **221**:3 (1996), 507–512. MR Zbl

[Kirsch and Veselić 2002] W. Kirsch and I. Veselić, "Existence of the density of states for one-dimensional alloy-type potentials with small support", pp. 171–176 in *Mathematical results in quantum mechanics* (Taxco, 2001), edited by R. Weder and P. Exner, Contemp. Math. **307**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[Kirsch and Veselić 2010] W. Kirsch and I. Veselić, "Lifshitz tails for a class of Schrödinger operators with random breather-type potential", *Lett. Math. Phys.* **94**:1 (2010), 27–39. MR Zbl

[Kirsch et al. 1998] W. Kirsch, P. Stollmann, and G. Stolz, "Localization for random perturbations of periodic Schrödinger operators", *Random Oper. Stochastic Equations* **6**:3 (1998), 241–268. MR Zbl

[Klein 2013] A. Klein, "Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators", *Comm. Math. Phys.* **323**:3 (2013), 1229–1246. MR Zbl

[Klein and Tsang 2016] A. Klein and C. S. S. Tsang, "Quantitative unique continuation principle for Schrödinger operators with singular potentials", *Proc. Amer. Math. Soc.* **144**:2 (2016), 665–679. MR Zbl

[Kukavica 1998] I. Kukavica, "Quantitative uniqueness for second-order elliptic operators", *Duke Math. J.* **91**:2 (1998), 225–240. MR Zbl

[Le Rousseau and Lebeau 2012] J. Le Rousseau and G. Lebeau, "On Carleman estimates for elliptic and parabolic operators: applications to unique continuation and control of parabolic equations", *ESAIM Control Optim. Calc. Var.* **18**:3 (2012), 712–747. MR Zbl

[Lebeau and Robbiano 1995] G. Lebeau and L. Robbiano, "Contrôle exact de l'équation de la chaleur", *Comm. Partial Differential Equations* **20**:1-2 (1995), 335–356. MR Zbl

[Lissy 2012] P. Lissy, "A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation", *C. R. Math. Acad. Sci. Paris* **350**:11-12 (2012), 591–595. MR Zbl

[Miller 2004] L. Miller, "Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time", *J. Differential Equations* **204**:1 (2004), 202–226. MR Zbl

[Miller 2006] L. Miller, "The control transmutation method and the cost of fast controls", *SIAM J. Control Optim.* **45**:2 (2006), 762–772. MR

[Miller 2010] L. Miller, "A direct Lebeau–Robbiano strategy for the observability of heat-like semigroups", *Discrete Contin. Dyn. Syst. Ser. B* **14**:4 (2010), 1465–1485. MR Zbl

[Boutet de Monvel et al. 2006] A. Boutet de Monvel, S. Naboko, P. Stollmann, and G. Stolz, "Localization near fluctuation boundaries via fractional moments and applications", *J. Anal. Math.* **100** (2006), 83–116. MR Zbl

[Boutet de Monvel et al. 2011] A. Boutet de Monvel, D. Lenz, and P. Stollmann, "An uncertainty principle, Wegner estimates and localization near fluctuation boundaries", *Math. Z.* **269**:3-4 (2011), 663–670. MR Zbl

[Nakić et al. 2015a] I. Nakić, C. Rose, and M. Tautenhahn, "A quantitative Carleman estimate for second order elliptic operators", preprint, 2015. to appear in *Proc. Roy. Soc. Edinburgh Sect. A*. arXiv

[Nakić et al. 2015b] I. Nakić, M. Täufer, M. Tautenhahn, and I. Veselić, "Scale-free uncertainty principles and Wegner estimates for random breather potentials", *C. R. Math. Acad. Sci. Paris* **353**:10 (2015), 919–923. MR Zbl

[Phung 2004] K.-D. Phung, "Note on the cost of the approximate controllability for the heat equation with potential", *J. Math. Anal. Appl.* **295**:2 (2004), 527–538. MR Zbl

[Privat et al. 2015a] Y. Privat, E. Trélat, and E. Zuazua, "Complexity and regularity of maximal energy domains for the wave equation with fixed initial data", *Discrete Contin. Dyn. Syst.* **35**:12 (2015), 6133–6153. MR Zbl

[Privat et al. 2015b] Y. Privat, E. Trélat, and E. Zuazua, "Optimal shape and location of sensors for parabolic equations with random initial data", *Arch. Ration. Mech. Anal.* **216**:3 (2015), 921–981. MR Zbl

[Rojas-Molina and Veselić 2013] C. Rojas-Molina and I. Veselić, "Scale-free unique continuation estimates and applications to random Schrödinger operators", *Comm. Math. Phys.* **320**:1 (2013), 245–274. MR Zbl

[Schumacher and Veselić 2017] C. Schumacher and I. Veselić, "Lifshitz tails for Schrödinger operators with random breather potential", preprint, 2017. To appear in *C. R. Math. Acad. Sci. Paris*.

[Shirley 2015] C. Shirley, "Decorrelation estimates for some continuous and discrete random Schrödinger operators in dimension one, without covering condition", preprint, 2015. arXiv

[Stollmann 2001] P. Stollmann, *Caught by disorder: bound states in random media*, Progress in Mathematical Physics **20**, Birkhäuser, Boston, 2001. MR Zbl

[Täufer and Tautenhahn 2017] M. Täufer and M. Tautenhahn, "Scale-free and quantitative unique continuation for infinite dimensional spectral subspaces of Schrödinger operators", *Commun. Pure Appl. Anal.* **16**:5 (2017), 1719–1730. MR Zbl

[Täufer and Veselić 2015] M. Täufer and I. Veselić, "Conditional Wegner estimate for the standard random breather potential", *J. Stat. Phys.* **161**:4 (2015), 902–914. MR Zbl

[Täufer and Veselić 2016] M. Täufer and I. Veselić, "Wegner estimate for Landau-breather Hamiltonians", *J. Math. Phys.* **57**:7 (2016), art. id. 072102. MR Zbl

[Täufer et al. 2016] M. Täufer, M. Tautenhahn, and I. Veselić, "Harmonic analysis and random Schrödinger operators", pp. 223–255 in *Spectral theory and mathematical physics*, edited by M. Mantoiu et al., Oper. Theory Adv. Appl. **254**, Springer, 2016. MR

[Tenenbaum and Tucsnak 2007] G. Tenenbaum and M. Tucsnak, "New blow-up rates for fast controls of Schrödinger and heat equations", *J. Differential Equations* **243**:1 (2007), 70–100. MR Zbl

[Tucsnak and Weiss 2009] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser, Basel, 2009. MR Zbl

[Veselić 1996] I. Veselić, *Lokalisierung bei zufällig gestörten periodischen Schrödingeroperatoren in Dimension Eins*, diploma thesis, Ruhr-Universität Bochum, 1996, http://www.ruhr-uni-bochum.de/mathphys/ivan/diplomski-www-abstract.htm.

[Veselić 2007] I. Veselić, "Lifshitz asymptotics for Hamiltonians monotone in the randomness", *Oberwolfach Rep.* **4**:1 (2007), 378–381.

[Veselić 2008] I. Veselić, *Existence and regularity properties of the integrated density of states of random Schrödinger operators*, Lecture Notes in Mathematics **1917**, Springer, 2008. MR Zbl

[Ziemer 1989] W. P. Ziemer, *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics **120**, Springer, 1989. MR Zbl

[Zuazua 2007] E. Zuazua, "Controllability and observability of partial differential equations: some results and open problems", pp. 527–621 in *Handbook of differential equations: evolutionary equations, III*, edited by C. M. Dafermos and E. Feireisl, North-Holland, Amsterdam, 2007. MR Zbl

Received 26 May 2017. Revised 11 Aug 2017. Accepted 16 Oct 2017.

IVICA NAKIĆ: nakic@math.hr
Department of Mathematics, University of Zagreb, Zagreb, Croatia

MATTHIAS TÄUFER: matthias.taeufer@mathematik.tu-dortmund.de
Fakultät für Mathematik, Technische Universität Dortmund, Dortmund, Germany

MARTIN TAUTENHAHN: martin.tautenhahn@mathematik.tu-chemnitz.de
Fakultät für Mathematik, Technische Universität Chemnitz, Chemnitz, Germany

and

Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, Jena, Germany

IVAN VESELIĆ: ivan.veselic@udo.edu
Fakultät für Mathematik, Technische Universität Dortmund, Dortmund, Germany

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2018 is US $275/year for the electronic version, and $480/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 11 No. 4 2018