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# HIGH-FREQUENCY APPROXIMATION OF THE INTERIOR DIRICHLET-TO-NEUMANN MAP AND APPLICATIONS TO THE TRANSMISSION EIGENVALUES

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We study the high-frequency behaviour of the Dirichlet-to-Neumann map for an arbitrary compact Riemannian manifold with a nonempty smooth boundary. We show that far from the real axis it can be approximated by a simpler operator. We use this fact to get new results concerning the location of the transmission eigenvalues on the complex plane. In some cases we obtain optimal transmission eigenvalue-free regions.

## 1. Introduction and statement of results

Let $(X, \mathcal{G})$ be a compact Riemannian manifold of dimension $d = \dim X \geq 2$ with a nonempty smooth boundary $\partial X$ and let $\Delta_X$ denote the negative Laplace–Beltrami operator on $(X, \mathcal{G})$. Denote also by $\Delta_{\partial X}$ the negative Laplace–Beltrami operator on $(\partial X, \mathcal{G}_0)$, which is a Riemannian manifold without boundary of dimension $d-1$, where $\mathcal{G}_0$ is the Riemannian metric on $\partial X$ induced by the metric $\mathcal{G}$. Given a function $f \in H^{m+1}(\partial X)$, let $u$ solve

$$\begin{cases} (\Delta_X + \lambda^2 n(x))u = 0 & \text{in } X, \\ u = f & \text{on } \partial X, \end{cases} \tag{1-1}$$

where $\lambda \in \mathbb{C}$, $1 \ll |\operatorname{Im} \lambda| \ll \operatorname{Re} \lambda$ and $n \in C^\infty(\bar{X})$ is a strictly positive function. Then the Dirichlet-to-Neumann (DN) map

$$\mathcal{N}(\lambda; n) : H^{m+1}(\partial X) \to H^m(\partial X)$$

is defined by

$$\mathcal{N}(\lambda; n) f := \partial_\nu u|_{\partial X},$$

where $\nu$ is the unit inner normal to $\partial X$. One of our goals in the present paper is to approximate the operator $\mathcal{N}(\lambda; n)$ when $n(x) \equiv 1$ in $X$ by a simpler one of the form $p(-\Delta_{\partial X})$ with a suitable complex-valued function $p(\sigma)$, $\sigma \geq 0$. More precisely, the function $p$ is defined as

$$p(\sigma) = \sqrt{\sigma - \lambda^2}, \quad \operatorname{Re} p < 0.$$

Our first result is the following:

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*Keywords:* Dirichlet-to-Neumann map, transmission eigenvalues.

**Theorem 1.1.** *Let $0 < \epsilon < 1$ be arbitrary. Then, for every $0 < \delta \ll 1$ there are constants $C_\delta, C_{\epsilon,\delta} > 1$ such that we have*

$$\|\mathcal{N}(\lambda; 1) - p(-\Delta_{\partial X})\|_{L^2(\partial X)\to L^2(\partial X)} \le \delta|\lambda| \tag{1-2}$$

*for $C_\delta \le |\operatorname{Im}\lambda| \le (\operatorname{Re}\lambda)^{1-\epsilon}$, $\operatorname{Re}\lambda \ge C_{\epsilon,\delta}$.*

Note that this result has been previously proved in [Petkov and Vodev 2017b] in the case when $X$ is a ball in $\mathbb{R}^d$ and the metric is the Euclidean one. In fact, in this case we have a better approximation of the operator $\mathcal{N}(\lambda; 1)$. In the general case when the function $n$ is arbitrary, the DN map can be approximated by $h$-ΨDOs, where $0 < h \ll 1$ is a semiclassical parameter such that $\operatorname{Re}(h\lambda)^2 = 1$. To describe this more precisely let us introduce the class of symbols $S^k_\delta(\partial X)$, $0 \le \delta < \frac{1}{2}$, as being the set of all functions $a(x', \xi') \in C^\infty(T^*\partial X)$ satisfying the bounds

$$\left|\partial^\alpha_{x'}\partial^\beta_{\xi'} a(x', \xi')\right| \le C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle\xi'\rangle^{k-|\beta|}$$

for all multi-indices $\alpha$ and $\beta$ with constants $C_{\alpha,\beta}$ independent of $h$. We let $\mathrm{OP}S^k_\delta(\partial X)$ denote the set of all $h$-ΨDOs, $\mathrm{Op}_h(a)$, with symbol $a \in S^k_\delta(\partial X)$, defined by

$$(\mathrm{Op}_h(a)f)(x') = (2\pi h)^{-d+1} \int_{T^*\partial X} e^{-(i/h)\langle x'-y', \xi'\rangle} a(x', \xi') f(y')\, dy'\, d\xi'.$$

It is well known that for this class of symbols we have a very nice pseudodifferential calculus; e.g., see [Dimassi and Sjöstrand 1999]. It was proved in [Vodev 2015] that for $|\operatorname{Im}\lambda| \ge |\lambda|^{1/2+\epsilon}$, $0 < \epsilon \ll 1$, the operator $h\mathcal{N}(\lambda; n)$ is an $h$-ΨDO of class $\mathrm{OP}S^1_{1/2-\epsilon}(\partial X)$ with a principal symbol

$$\rho(x', \xi') = \sqrt{r_0(x', \xi') - (h\lambda)^2 n_0(x')}, \qquad \operatorname{Re}\rho < 0,\ n_0 := n|_{\partial X},$$

$r_0 \ge 0$ being the principal symbol of $-\Delta_{\partial X}$. Note that it is still possible to construct a semiclassical parametrix for the operator $h\mathcal{N}(\lambda; n)$ when $|\operatorname{Im}\lambda| \ge |\lambda|^\epsilon$, $0 < \epsilon \ll 1$, if one supposes that the boundary $\partial X$ is strictly concave; see [Vodev 2016]. This construction, however, is much more complex and one has to work with symbols belonging to much worse classes near the glancing region $\Sigma = \{(x', \xi') \in T^*\partial X : r_\sharp(x', \xi') = 1\}$, where $r_\sharp = n_0^{-1} r_0$. On the other hand, it seems that no parametrix construction near $\Sigma$ is possible in the important region $1 \ll \text{const.} \le |\operatorname{Im}\lambda| \le |\lambda|^\epsilon$. Therefore, in the present paper we follow a different approach which consists of showing that, for arbitrary manifold $X$, the norm of the operator $h\mathcal{N}(\lambda; n)\mathrm{Op}_h(\chi^0_\delta)$ is $\mathcal{O}(\delta)$ for every $0 < \delta \ll 1$ independent of $\lambda$, provided $|\operatorname{Im}\lambda|$ and $\operatorname{Re}\lambda$ are taken big enough (see Proposition 3.3 below). Here the function $\chi^0_\delta \in C^\infty_0(T^*\partial X)$ is supported in $\{(x', \xi') \in T^*\partial X : |r_\sharp(x', \xi') - 1| \le 2\delta^2\}$ and $\chi^0_\delta = 1$ in $\{(x', \xi') \in T^*\partial X : |r_\sharp(x', \xi') - 1| \le \delta^2\}$ (see Section 3 for the precise definition of $\chi^0_\delta$). Theorem 1.1 is an easy consequence of the following semiclassical version.

**Theorem 1.2.** *Let $0 < \epsilon < 1$ be arbitrary. Then, for every $0 < \delta \ll 1$ there are constants $C_\delta, C_{\epsilon,\delta} > 1$ such that we have*

$$\left\|h\mathcal{N}(\lambda; n) - \mathrm{Op}_h(\rho(1 - \chi^0_\delta) + hb)\right\|_{L^2(\partial X)\to H^1_h(\partial X)} \le C\delta \tag{1-3}$$

*for $C_\delta \le |\operatorname{Im}\lambda| \le (\operatorname{Re}\lambda)^{1-\epsilon}$, $\operatorname{Re}\lambda \ge C_{\epsilon,\delta}$, where $C > 0$ is a constant independent of $\lambda$ and $\delta$, and $b \in S^0_0(\partial X)$ is independent of $\lambda$ and the function $n$.*

Here $H_h^1(\partial X)$ denotes the Sobolev space equipped with the semiclassical norm (see Section 3 for the precise definition). Thus, to prove (1-3), as well as (1-2), it suffices to construct a semiclassical parametrix outside a $\delta^2$-neighbourhood of $\Sigma$, which turns out to be much easier and can be done for an arbitrary $X$. In the elliptic region $\{(x',\xi') \in T^*\partial X : r_\sharp(x',\xi') \geq 1+\delta^2\}$ we use the same parametrix construction as in [Vodev 2015] with slight modifications. In the hyperbolic region $\{(x',\xi') \in T^*\partial X : r_\sharp(x',\xi') \leq 1-\delta^2\}$, however, we need to improve the parametrix construction of that paper. We do this in Section 4 for $1 \ll \text{const.} \leq |\operatorname{Im}\lambda| \leq |\lambda|^{1-\epsilon}$. Then we show that the difference between the operator $h\mathcal{N}(\lambda; n)$ microlocalized in the hyperbolic region and its parametrix is $\mathcal{O}(e^{-\beta|\operatorname{Im}\lambda|}) + \mathcal{O}_{\epsilon,M}(|\lambda|^{-M})$, where $\beta > 0$ is some constant and $M \geq 1$ is arbitrary. So, we can make it small by taking $|\operatorname{Im}\lambda|$ and $|\lambda|$ big enough.

These kinds of approximations of the DN map are important for the study of the location of the complex eigenvalues associated to boundary-value problems with dissipative boundary conditions; e.g., see [Petkov 2016]. In particular, Theorem 1.2 leads to significant improvements of the eigenvalue-free regions in that paper. In the present paper we use Theorem 1.2 to study the location of the interior transmission eigenvalues (see Section 2). We improve most of the results in [Vodev 2015], as well as those in [Petkov and Vodev 2017b; Vodev 2016], and provide simpler proofs. In some cases we get optimal transmission eigenvalue-free regions (see Theorem 2.1). Note that for the applications in the anisotropic case it suffices to have a weaker analogue of the estimate (1-3) with the space $H_h^1$ replaced by $L^2$, in which case the operator $\operatorname{Op}_h(hb)$ becomes negligible. In the isotropic case, however, it is essential to have in (1-3) the space $H_h^1$ and that the function $b$ does not depend on the refraction index $n$.

Note finally that Theorem 1.2 can be also used to study the location of the resonances for the exterior transmission problems considered in [Cardoso et al. 2001; Galkowski 2015]. For example, it allows us to simplify the proof of the resonance-free regions in [Cardoso et al. 2001] and to extend it to more general boundary conditions.

## 2. Applications to the transmission eigenvalues

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded, connected domain with a $C^\infty$ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geq 0$, will be said to be a transmission eigenvalue if the following problem has a nontrivial solution:

$$\begin{cases} (\nabla c_1(x)\nabla + \lambda^2 n_1(x))u_1 = 0 & \text{in } \Omega, \\ (\nabla c_2(x)\nabla + \lambda^2 n_2(x))u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad c_1\,\partial_\nu u_1 = c_2\,\partial_\nu u_2 & \text{on } \Gamma, \end{cases} \tag{2-1}$$

where $\nu$ denotes the Euclidean unit inner normal to $\Gamma$, $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$, are strictly positive real-valued functions. We will consider two cases:

$$c_1(x) \equiv c_2(x) \equiv 1 \quad \text{in } \Omega, \qquad n_1(x) \neq n_2(x) \quad \text{on } \Gamma \quad \text{(isotropic case)}, \tag{2-2}$$

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) \neq 0 \quad \text{on } \Gamma \quad \text{(anisotropic case)}. \tag{2-3}$$

In Section 6 we will prove the following:

**Theorem 2.1.** *Assume either the condition (2-2) or the condition*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) < 0 \quad \text{on } \Gamma. \tag{2-4}$$

*Then there exists a constant $C > 0$ such that there are no transmission eigenvalues in the region*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C\}. \tag{2-5}$$

**Remark.** It is proven in [Vodev 2015] that under the condition (2-2) (as well as the condition (2-6) below) there exists a constant $\widetilde{C} > 0$ such that there are no transmission eigenvalues in the region

$$\{\lambda \in \mathbb{C} : 0 \leq \operatorname{Re}\lambda \leq 1,\ |\operatorname{Im}\lambda| \geq \widetilde{C}\}.$$

This is no longer true under the condition (2-4), in which case there exist infinitely many transmission eigenvalues very close to the imaginary axis.

Note that the eigenvalue-free region (2-5) is optimal and cannot be improved in general. Indeed, it follows from the analysis in [Leung and Colton 2012] (see Section 4) that in the isotropic case when the domain $\Omega$ is a ball and the refraction indices $n_1$ and $n_2$ are constant, there may exist infinitely many transmission eigenvalues whose imaginary parts are bounded from below by a positive constant. Note also that the above result has been previously proved in [Petkov and Vodev 2017b] in the case when the domain $\Omega$ is a ball and the coefficients are constant. In the isotropic case, the eigenvalue-free region (2-5) has been also obtained in [Sylvester 2013] when the dimension is 1. In the general case of arbitrary domains, the existence of transmission eigenvalue-free regions has been previously proved in [Hitrik et al. 2011; Lakshtanov and Vainberg 2013; Robbiano 2013] in the isotropic case, and [Vodev 2015, 2016] in both cases. For example, it has been proved in [Vodev 2015] that, under the conditions (2-2) and (2-4), there are no transmission eigenvalues in

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C_\epsilon (\operatorname{Re}\lambda)^{1/2+\epsilon}\}, \quad C_\epsilon > 0,$$

for every $0 < \epsilon \ll 1$. This eigenvalue-free region has been improved in [Vodev 2016] under an additional strict concavity condition on the boundary $\Gamma$ to

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C_\epsilon (\operatorname{Re}\lambda)^{\epsilon}\}, \quad C_\epsilon > 0,$$

for every $0 < \epsilon \ll 1$. When the function in the left-hand side of (2-3) is strictly positive, the existence of parabolic eigenvalue-free regions has been proved in [Vodev 2015] for arbitrary domains, which however are worse than the eigenvalue-free regions we have under the conditions (2-2) and (2-4). In Section 7 we will prove:

**Theorem 2.2.** *Assume the conditions*

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0 \quad \text{on } \Gamma \tag{2-6}$$

*and*

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)} \quad \text{on } \Gamma. \tag{2-7}$$

*Then there exists a constant $C > 0$ such that there are no transmission eigenvalues in the region*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C \log(\operatorname{Re}\lambda + 1)\}. \tag{2-8}$$

Note that in the case when (2-6) is fulfilled but (2-7) is not, the method developed in the present paper does not work and it is not clear if improvements are possible compared with the results in [Vodev 2015]. To the best of our knowledge, no results exist in the degenerate case when the function in the left-hand side of (2-3) vanishes without being identically zero.

It has been proved in [Petkov and Vodev 2017a] that the counting function

$$N(r) = \#\{\lambda - \text{trans. eig.} : |\lambda| \leq r\}, \quad r > 1,$$

satisfies the asymptotics

$$N(r) = (\tau_1 + \tau_2) r^d + \mathcal{O}_\epsilon(r^{d-\kappa+\epsilon}) \quad \forall\, 0 < \epsilon \ll 1,$$

where $0 < \kappa \leq 1$ is such that there are no transmission eigenvalues in the region

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1,\ |\operatorname{Im}\lambda| \geq C(\operatorname{Re}\lambda)^{1-\kappa}\}, \quad C > 0,$$

and

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_\Omega \left(\frac{n_j(x)}{c_j(x)}\right)^{d/2} dx,$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. Using this we obtain from the above theorems the following:

**Corollary 2.3.** *Under the conditions of Theorems 2.1 and 2.2, the counting function of the transmission eigenvalues satisfies the asymptotics*

$$N(r) = (\tau_1 + \tau_2) r^d + \mathcal{O}_\epsilon(r^{d-1+\epsilon}) \quad \forall\, 0 < \epsilon \ll 1. \tag{2-9}$$

This result has been previously proved in [Vodev 2016] under an additional strict concavity condition on the boundary $\Gamma$. In the present paper we remove this additional condition to conclude that in fact the asymptotics (2-9) holds true for an arbitrary domain. We also expect that (2-9) holds with $\epsilon = 0$, but this remains an interesting open problem. In the isotropic case asymptotics for the counting function $N(r)$ with remainder $o(r^d)$ have been previously obtained in [Faierman 2014; Pham and Stefanov 2014; Robbiano 2016].

## 3. A priori estimates in the glancing region

Let $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 1$, $1 < |\operatorname{Im}\lambda| \leq \theta_0 \operatorname{Re}\lambda$, where $0 < \theta_0 < 1$ is a fixed constant, and set $h = \mu^{-1}$, where

$$\mu = \operatorname{Re}\lambda \sqrt{1 - \left(\frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right)^2} \sim \operatorname{Re}\lambda \sim |\lambda|.$$

Clearly, we have $\operatorname{Re}(h\lambda)^2 = 1$ and

$$\lambda^2 = \mu^2(1 + izh), \quad z = 2\mu^{-1} \operatorname{Im}\lambda \operatorname{Re}\lambda \sim 2 \operatorname{Im}\lambda.$$

Given an integer $m \geq 0$, denote by $H_h^m(X)$ the Sobolev space equipped with the semiclassical norm

$$\|v\|_{H_h^m(X)} = \sum_{|\alpha|\leq m} h^{|\alpha|} \|\partial_x^\alpha v\|_{L^2(X)}.$$

We define similarly the Sobolev space $H_h^m(\partial X)$. It is well known that

$$\|v\|_{H_h^m(\partial X)} \sim \|\mathrm{Op}_h(\langle \xi' \rangle^m) v\|_{L^2(\partial X)} \sim \|v\|_{L^2(\partial X)} + \|\mathrm{Op}_h((1-\eta)|\xi'|^m) v\|_{L^2(\partial X)}$$

for any function $\eta \in C_0^\infty(T^*\partial X)$ independent of $h$. Hereafter, $\langle \xi' \rangle = (1+|\xi'|^2)^{1/2}$.

Given functions $V \in L^2(X)$ and $f \in L^2(\partial X)$, we let the function $u$ solve

$$\begin{cases} (\Delta_X + \lambda^2 n(x)) u = \lambda V & \text{in } X, \\ u = f & \text{on } \partial X, \end{cases} \tag{3-1}$$

and set $g = h\, \partial_\nu u|_{\partial X}$. We will first prove:

**Lemma 3.1.** *There is a constant $C > 0$ such that the following estimate holds*:

$$\|u\|_{H_h^1(X)} \leq C |\mathrm{Im}\,\lambda|^{-1} \|V\|_{L^2(X)} + C |\mathrm{Im}\,\lambda|^{-1/2} \|f\|_{L^2(\partial X)}^{1/2} \|g\|_{L^2(\partial X)}^{1/2}. \tag{3-2}$$

*Proof.* By Green's formula we have

$$\mathrm{Im}(\lambda^2) \|n^{1/2} u\|_{L^2(X)}^2 = \mathrm{Im}\langle \lambda V, u\rangle_{L^2(X)} + \mathrm{Im}\langle \partial_\nu u|_{\partial X}, f\rangle_{L^2(\partial X)}$$

which implies

$$|\mathrm{Im}\,\lambda|\, \|u\|_{L^2(X)}^2 \lesssim \|V\|_{L^2(X)} \|u\|_{L^2(X)} + \|f\|_{L^2(\partial X)} \|g\|_{L^2(\partial X)}. \tag{3-3}$$

On the other hand, we have

$$\|\nabla_X u\|_{L^2(X)}^2 - \mathrm{Re}(\lambda^2) \|n^{1/2} u\|_{L^2(X)}^2 = -\mathrm{Re}\langle \lambda V, u\rangle_{L^2(X)} - \mathrm{Re}\langle \partial_\nu u|_{\partial X}, f\rangle_{L^2(\partial X)},$$

which yields

$$\|h \nabla_X u\|_{L^2(X)}^2 \lesssim \|u\|_{L^2(X)}^2 + \mathcal{O}(h^2) \|V\|_{L^2(X)}^2 + \mathcal{O}(h) \|f\|_{L^2(\partial X)} \|g\|_{L^2(\partial X)}. \tag{3-4}$$

Since $h \lesssim |\mathrm{Im}\,\lambda|^{-1}$, the estimate (3-2) follows from (3-3) and (3-4). □

We now equip $X$ with the Riemannian metric $n\mathcal{G}$. We will write the operator $n^{-1}\Delta_X$ in the normal coordinates $(x_1, x')$ with respect to the metric $n\mathcal{G}$ near the boundary $\partial X$, where $0 < x_1 \ll 1$ denotes the distance to the boundary and $x'$ are coordinates on $\partial X$. Set $\Gamma(x_1) = \{x \in X : \mathrm{dist}(x, \partial X) = x_1\}$, $\Gamma(0) = \partial X$. Then $\Gamma(x_1)$ is a Riemannian manifold without boundary of dimension $d-1$ with a Riemannian metric induced by the metric $n\mathcal{G}$, which depends smoothly in $x_1$. It is well known that the operator $n^{-1}\Delta_X$ can be written as

$$n^{-1}\Delta_X = \partial_{x_1}^2 + Q(x_1) + R,$$

where $Q(x_1) = \Delta_{\Gamma(x_1)}$ is the negative Laplace–Beltrami operator on $\Gamma(x_1)$ and $R$ is a first-order differential operator. Clearly, $Q(x_1)$ is a second-order differential operator with smooth coefficients and $Q(0) = \Delta_{\partial X}^{(n)}$ is the negative Laplace–Beltrami operator on $\partial X$ equipped with the Riemannian metric induced by the metric $n\mathcal{G}$.

Let $\chi \in C_0^\infty(\mathbb{R})$, $0 \le \chi(t) \le 1$, $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$. Given a parameter $0 < \delta_1 \ll 1$ independent of $\lambda$ and an integer $k \ge 0$, set $\phi_k(x_1) = \chi(2^{-k}x_1/\delta_1)$. Given integers $0 \le s_1 \le s_2$, we define the norm $\|u\|_{s_1,s_2,k}$ by

$$\|u\|_{s_1,s_2,k}^2 = \|u\|_{H_h^{s_1}(X)}^2 + \sum_{\ell_1=0}^{s_1} \sum_{\ell_2=0}^{s_2-\ell_1} \int_0^\infty \|(h\,\partial_{x_1})^{\ell_1}(\phi_k u)(x_1,\cdot)\|_{H_h^{\ell_2}(\partial X)}^2\,dx_1.$$

Clearly, we have

$$\|u\|_{H_h^{s_1}(X)} \le \|u\|_{s_1,s_2,k} \lesssim \|u\|_{H_h^{s_2}(X)}.$$

Throughout this paper $\eta \in C_0^\infty(T^*\partial X)$, $0 \le \eta \le 1$, $\eta = 1$ in $|\xi'| \le A$, $\eta = 0$ in $|\xi'| \ge A+1$, will be a function independent of $\lambda$, where $A > 1$ is a parameter we may take as large as we want. We will now prove:

**Lemma 3.2.** *Let $u$ solve (3-1) with $V \in H^{s-1}(X)$ and $f \in H^{2s}(\partial X)$ for some integer $s \ge 1$. Then the following estimate holds*:

$$\|u\|_{1,s+1,k} \lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,s-1,k+s-1} + \|\mathrm{Op}_h(1-\eta)f\|_{H_h^{2s}(\partial X)}^{1/2} \|g\|_{L^2(\partial X)}^{1/2}. \tag{3-5}$$

*Proof.* Note that

$$\|u\|_{1,s+1,k} \lesssim \|u\|_{H_h^1(X)} + \|u_{s,k}\|_{H_h^1(X)},$$

where the function $u_{s,k} = \mathrm{Op}_h((1-\eta)|\xi'|^s)(\phi_k u)$ satisfies the equation

$$(h^2\partial_{x_1}^2 + h^2 Q(x_1) + 1 + ihz)u_{s,k} = U_{s,k}$$

with

$$U_{s,k} = \big[h^2 Q(x_1), \mathrm{Op}_h((1-\eta)|\xi'|^s)\big](\phi_k u) + \mathrm{Op}_h((1-\eta)|\xi'|^s)[h^2\partial_{x_1}^2, \phi_k]\phi_{k+1}u \\ - h^2\mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_k R\phi_{k+1}u + h^2\lambda\mathrm{Op}_h((1-\eta)|\xi'|^s)(\phi_k V).$$

We also have

$$f_s := u_{s,k}|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)f,$$
$$g_s := h\,\partial_{x_1}u_{s,k}|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)g_\flat,$$

where $g_\flat := h\,\partial_{x_1}u|_{x_1=0}$. Integrating by parts the above equation and taking the real part, we get

$$\|h\,\partial_{x_1}u_{s,k}\|_{L^2(X)}^2 - \langle (h^2Q(x_1)+1)u_{s,k}, u_{s,k}\rangle_{L^2(X)} \\ \le |\langle U_{s,k}, u_{s,k}\rangle_{L^2(X)}| + h|\langle f_s, g_s\rangle_{L^2(\partial X)}| \\ \lesssim \|u_{s,k}\|_{H_h^1(X)}(\|V\|_{0,s-1,k} + \|u\|_{1,s,k+1}) \\ + \big\|\mathrm{Op}_h((1-\eta)|\xi'|^s)^*\mathrm{Op}_h((1-\eta)|\xi'|^s)f\big\|_{L^2(\partial X)}\|g_\flat\|_{L^2(\partial X)}. \tag{3-6}$$

The principal symbol $r$ of the operator $-Q(x_1)$ satisfies $r(x,\xi') \ge C'|\xi'|^2$, $C' > 0$, on $\operatorname{supp}\phi_k$, provided $\delta_1$ is taken small enough. Therefore, we can arrange by taking the parameter $A$ big enough that $r - 1 \ge C\langle\xi'\rangle$ on $\operatorname{supp}(1-\eta)\phi_k$, where $C > 0$ is some constant. Hence, by Gårding's inequality we have

$$-\langle (h^2Q(x_1)+1)u_{s,k}, u_{s,k}\rangle_{L^2(X)} \ge C\|\mathrm{Op}_h(\langle\xi'\rangle)u_{s,k}\|_{L^2(X)}^2 \tag{3-7}$$

with possibly a new constant $C > 0$. Since the norms of $g$ and $g_\flat$ are equivalent, by (3-6) and (3-7) we get

$$\|u_{s,k}\|_{H_h^1(X)} \lesssim \|V\|_{0,s-1,k} + \|u\|_{H_h^1(X)} + \|u_{s-1,k+1}\|_{H_h^1(X)} + \|\mathrm{Op}_h(1-\eta)f\|^{1/2}_{H_h^{2s}(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3-8}$$

We may now apply the same argument to $u_{s-1,k+1}$. Thus, repeating this argument a finite number of times we can eliminate the term involving $u_{s-1,k+1}$ in the right-hand side of (3-8) and obtain the estimate (3-5). □

Let the functions $\chi_j \in C^\infty(\mathbb{R})$, $0 \le \chi_j(t) \le 1$, $j = 1, 2, 3$, be such that $\chi_1 + \chi_2 + \chi_3 \equiv 1$, $\chi_2 = \chi$, $\chi_1(t) = 1$ for $t \le -2$, $\chi_1(t) = 0$ for $t \ge -1$, $\chi_3(t) = 0$ for $t \le 1$, $\chi_3(t) = 1$ for $t \ge 2$. Given a parameter $0 < \delta \ll 1$ independent of $\lambda$, set

$$\chi_\delta^-(x', \xi') = \chi_1\big((r_\sharp(x', \xi') - 1)/\delta^2\big),$$
$$\chi_\delta^0(x', \xi') = \chi_2\big((r_\sharp(x', \xi') - 1)/\delta^2\big),$$
$$\chi_\delta^+(x', \xi') = \chi_3\big((r_\sharp(x', \xi') - 1)/\delta^2\big),$$

where $r_\sharp = n_0^{-1} r_0$ is the principal symbol of the operator $-\Delta_{\partial X}^{(n)}$. Since $(r_\sharp - 1)^k \chi_\delta^0 = \mathcal{O}(\delta^{2k})$, we have

$$(h^2 \Delta_{\partial X}^{(n)} + 1)^k \mathrm{Op}_h(\chi_\delta^0) = \mathcal{O}(\delta^{2k}) : L^2(\partial X) \to L^2(\partial X) \tag{3-9}$$

for every integer $k \ge 0$. Clearly, we also have

$$\mathrm{Op}_h(\chi_\delta^0) = \mathcal{O}(1) : L^2(\partial X) \to H_h^m(\partial X) \quad \forall\, m \ge 0,$$

uniformly in $\delta$. Using (3-9) we will prove:

**Proposition 3.3.** *Let $u$ solve (3-1) with $f \equiv 0$ and $V \in H^s(X)$ for some integer $s \ge 0$. Then the function $g = h\,\partial_\nu u|_{\partial X}$ satisfies the estimate*

$$\|g\|_{H_h^s(\partial X)} \le C' |\mathrm{Im}\,\lambda|^{-1/2} \|V\|_{0,s,s} \tag{3-10}$$

*with a constant $C' > 0$ independent of $\lambda$.*

*Let $u$ solve (3-1) with $f$ replaced by $\mathrm{Op}_h(\chi_\delta^0) f$ and $V \in H^{s+2}(X)$ for some integer $s \ge 0$. Then the function $g = h\,\partial_\nu u|_{\partial X}$ satisfies the estimate*

$$\|g\|_{H_h^s(\partial X)} \le C(\delta + |\mathrm{Im}\,\lambda|^{-1/4}) \|f\|_{L^2(\partial X)} + C(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8}) \|V\|_{0,s+2,s+2} \tag{3-11}$$

*for $1 < |\mathrm{Im}\,\lambda| \le \delta^2\,\mathrm{Re}\,\lambda$, $\mathrm{Re}\,\lambda \ge C_\delta \gg 1$, with a constant $C > 0$ independent of $\lambda$ and $\delta$.*

*Proof.* Set $w = \phi_0(x_1) u$. We will first show that the estimates (3-10) and (3-11) with $s \ge 1$ follow from (3-10) and (3-11) with $s = 0$, respectively. This follows from the estimate

$$\|g\|_{H_h^s(\partial X)} \lesssim \|g\|_{L^2(\partial X)} + \big\| h\,\partial_{x_1} v_s|_{x_1=0} \big\|_{L^2(\partial X)}, \tag{3-12}$$

where the function $v_s = \mathrm{Op}_h((1-\eta)|\xi'|^s) w$ satisfies (3-1) with $V$ replaced by

$$V_s = n \mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_0 n^{-1} V + \lambda^{-1} n \big[ n^{-1} \Delta_X, \mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_0 \big] u.$$

We can write the commutator as

$$[\partial_{x_1}^2 + R, \phi_0(x_1)] \mathrm{Op}_h((1-\eta)|\xi'|^s)\phi_1(x_1) + \phi_0 \big[ Q(x_1) + R, \mathrm{Op}_h((1-\eta)|\xi'|^s) \big] \phi_1(x_1).$$

Therefore, if $f \equiv 0$, in view of Lemmas 3.1 and 3.2, the function $V_s$ satisfies the bound

$$\|V_s\|_{0,0,0} \lesssim \|V\|_{0,s,0} + \|u\|_{1,s+1,1} \lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,s,s} \lesssim \|V\|_{0,s,s}. \tag{3-13}$$

Clearly, the assertion concerning (3-10) follows from (3-12) and (3-13). The estimate (3-11) can be treated similarly. Indeed, in view of Lemma 3.2, the function $V_s$ satisfies the bound

$$\begin{aligned}\|V_s\|_{0,2,2} &\lesssim \|V\|_{0,s+2,0} + \|u\|_{1,s+3,1}\\ &\lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,s+2,s+2} + \|\mathrm{Op}_h(1-\eta)\mathrm{Op}_h(\chi_\delta^0) f\|^{1/2}_{H_h^{2s+4}(\partial X)} \|g\|^{1/2}_{L^2(\partial X)}.\end{aligned} \tag{3-14}$$

Taking the parameter $A$ big enough we can arrange that $\operatorname{supp}\chi_\delta^0 \cap \operatorname{supp}(1-\eta) = \varnothing$. Hence

$$\mathrm{Op}_h(1-\eta)\mathrm{Op}_h(\chi_\delta^0) = \mathcal{O}(h^\infty) : L^2(\partial X) \to H_h^m(\partial X) \quad \forall m \geq 0. \tag{3-15}$$

By (3-14) and (3-15) together with Lemma 3.1 we conclude

$$\begin{aligned}\|V_s\|_{0,2,2} &\lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,s+2,s+2} + \mathcal{O}(h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}\\ &\lesssim \|V\|_{0,s+2,s+2} + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2} + h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}.\end{aligned}$$

We now apply (3-11) with $s=0$ to the function $v_s$ and note that

$$v_s|_{x_1=0} = \mathrm{Op}_h((1-\eta)|\xi'|^s)\mathrm{Op}_h(\chi_\delta^0) f = \mathcal{O}(h^\infty) f.$$

Hence

$$\begin{aligned}\big\|h\,\partial_{x_1} v_s|_{x_1=0}\big\|_{L^2(\partial X)} &\leq \mathcal{O}(h^\infty)\|f\|_{L^2(\partial X)} + \mathcal{O}(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8})\|V_s\|_{0,2,2}\\ &\leq \mathcal{O}(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8})\|V\|_{0,s+2,s+2} + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2} + h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}.\end{aligned} \tag{3-16}$$

Therefore, the assertion concerning (3-11) follows from (3-12) and (3-16).

We now turn to the proofs of (3-10) and (3-11) with $s=0$. In view of Lemma 3.1, the function

$$U := h(n^{-1}\Delta_X + \lambda^2)w = h[n^{-1}\Delta_X, \phi_0(x_1)]u + h\lambda n^{-1}\phi_0 V$$

satisfies the bound

$$\|U\|_{L^2(X)} \lesssim \|u\|_{H_h^1(X)} + \|V\|_{L^2(X)} \lesssim \|V\|_{L^2(X)} + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2})\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}. \tag{3-17}$$

Observe now that the derivative of the function

$$E(x_1) = \|h\,\partial_{x_1} w\|^2 + \langle (h^2 Q(x_1) + 1)w, w\rangle,$$

where $\|\cdot\|$ and $\langle\cdot\,,\cdot\rangle$ are the norm and the scalar product in $L^2(\partial X)$, satisfies

$$\begin{aligned}E'(x_1) &= 2\,\mathrm{Re}\big\langle (h^2\partial_{x_1}^2 + h^2 Q(x_1) + 1)w, \partial_{x_1} w\big\rangle + \langle h^2 Q'(x_1) w, w\rangle\\ &= 2\,\mathrm{Re}\big\langle (U - izw - hRw), h\,\partial_{x_1} w\big\rangle + \langle h^2 Q'(x_1) w, w\rangle.\end{aligned}$$

If we put $g_\flat := h\,\partial_{x_1} u|_{x_1=0}$, we have

$$\begin{aligned}\|g_\flat\|^2 + \big\langle (h^2\Delta_{\partial X}^{(n)}+1)\mathrm{Op}_h(\chi_\delta^0) f, \mathrm{Op}_h(\chi_\delta^0) f\big\rangle &= E(0) = -\int_0^\infty E'(x_1)\,dx_1 \\ &\lesssim \big(\|U\|_{L^2(X)} + |z|\,\|w\|_{L^2(X)} + \|hRw\|_{L^2(X)}\big)\|h\,\partial_{x_1} w\|_{L^2(X)} + \|w\|^2_{H_h^1(X)} \\ &\le \mathcal{O}(|z|)\|h\,\partial_{x_1} w\|_{L^2(X)}\|w\|_{L^2(X)} + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1})F^2, \qquad (3\text{-}18)\end{aligned}$$

where we have used Lemma 3.1 together with (3-17) and we have put

$$F = \|f\|^{1/2}\|g\|^{1/2} + \|V\|_{L^2(X)}.$$

Clearly, (3-10) with $s=0$ follows from (3-18) applied with $f \equiv 0$ and Lemma 3.1. To prove (3-11) with $s=0$, observe that (3-9) and (3-18) lead to

$$\|g\| \le \mathcal{O}(\delta)\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2})F + \mathcal{O}(|\mathrm{Im}\,\lambda|^{1/2})\|h\,\partial_{x_1} w\|^{1/2}_{L^2(X)}\|w\|^{1/2}_{L^2(X)}. \qquad (3\text{-}19)$$

We now need a better bound on the norm $\|h\,\partial_{x_1} w\|_{L^2(X)}$ in the right-hand side of (3-19) than what the estimate (3-2) gives. To this end, observe that integrating by parts yields

$$\begin{aligned}\|h\,\partial_{x_1} w\|^2_{L^2(X)} - \langle (h^2 Q(x_1)+1)w, w\rangle_{L^2(X)} &= -h\,\mathrm{Re}\langle (U-hRw), w\rangle_{L^2(X)} - h\,\mathrm{Re}\langle f, g_\flat\rangle \\ &\le \mathcal{O}(h)\|w\|^2_{H_h^1(X)} + \mathcal{O}(h)\|U\|^2_{L^2(X)} + \mathcal{O}(h)\|f\|\,\|g\| \\ &\le \mathcal{O}(h)F^2. \qquad (3\text{-}20)\end{aligned}$$

By (3-19) and (3-20), together with Lemma 3.1, we get

$$\begin{aligned}\|g\| &\le \mathcal{O}(\delta)\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{1/2})\|w_1\|^{1/4}_{L^2(X)}\|w\|^{3/4}_{L^2(X)} + \mathcal{O}(h^{1/4}|\mathrm{Im}\,\lambda|^{1/2})F^{1/2}\|w\|^{1/2}_{L^2(X)} + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2})F \\ &\le \mathcal{O}(\delta)\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{1/8})\|w_1\|^{1/4}_{L^2(X)}F^{3/4} + \mathcal{O}\big(|\mathrm{Im}\,\lambda|^{-1/2} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\big)F, \qquad (3\text{-}21)\end{aligned}$$

where we have put $w_1 := (h^2 Q(x_1)+1)w$. We need now the following:

**Lemma 3.4.** *The function $w_1$ satisfies the estimate*

$$|\mathrm{Im}\,\lambda|^{1/2}\|w_1\|_{L^2(X)} \le \mathcal{O}(\delta^2 + |\mathrm{Im}\,\lambda|^{-1} + h^\infty)\|f\|^{1/2}\|g\|^{1/2} + \mathcal{O}(h^{1/2})\|f\| + \mathcal{O}(|\mathrm{Im}\,\lambda|^{-1/2})\|V\|_{0,2,2}. \qquad (3\text{-}22)$$

Let us show that this lemma implies the estimate (3-11) with $s=0$. Set

$$\widetilde{F} = \|f\|^{1/2}\|g\|^{1/2} + \|V\|_{0,2,2} \ge F.$$

By (3-21) and (3-22),

$$\begin{aligned}\|g\| &\le \mathcal{O}(\delta)\|f\| + \mathcal{O}(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8} + h^\infty)\widetilde{F} + \mathcal{O}(h^{1/8})(\|f\| + F) + \mathcal{O}\big(|\mathrm{Im}\,\lambda|^{-1/2} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\big)F \\ &\le \mathcal{O}(\delta + h^{1/8})\|f\| + \mathcal{O}\big(\delta^{1/2} + |\mathrm{Im}\,\lambda|^{-1/8} + h^{1/8} + h^{1/4}|\mathrm{Im}\,\lambda|^{1/4}\big)\widetilde{F}. \qquad (3\text{-}23)\end{aligned}$$

Since by assumption $h^{1/4}|\mathrm{Im}\,\lambda|^{1/4} = \mathcal{O}(\delta^{1/2})$, one can easily see that (3-11) with $s=0$ follows from (3-23). ∎

*Proof of Lemma 3.4.* Observe that the function $w_1$ satisfies the equation

$$(h^2\partial_{x_1}^2 + h^2 Q(x_1) + 1 + ihz)w_1 = hU_1,$$

where

$$U_1 := (h^2 Q(x_1) + 1)(U - hRw) + 2h^3 Q'(x_1)\partial_{x_1} w + h^3 Q''(x_1)w.$$

We also have

$$f_1 := w_1|_{x_1=0} = (h^2 Q(0) + 1)\mathrm{Op}_h(\chi_\delta^0) f,$$
$$g_1 := h\,\partial_{x_1} w_1|_{x_1=0} = (h^2 Q(0) + 1)g_\flat + h^3 Q'(0)\mathrm{Op}_h(\chi_\delta^0) f.$$

Integrating by parts the above equation and taking the imaginary part, we get

$$\begin{aligned}|z|\,\|w_1\|^2_{L^2(X)} &\le |\langle U_1, w_1\rangle_{L^2(X)}| + |\langle f_1, g_1\rangle| \\ &\le \|U_1\|_{L^2(X)}\|w_1\|_{L^2(X)} + \mathcal{O}(1)\|(h^2 Q(0)+1)^2\mathrm{Op}_h(\chi_\delta^0) f\|\,\|g\| \\ &\qquad + \mathcal{O}(h)\|\mathrm{Op}_h(\chi_\delta^0) f\|_{H_h^2(\partial X)}\|(h^2 Q(0)+1)\mathrm{Op}_h(\chi_\delta^0) f\| \\ &\le \|U_1\|_{L^2(X)}\|w_1\|_{L^2(X)} + \mathcal{O}(\delta^4)\|f\|\,\|g\| + \mathcal{O}(h)\|f\|^2,\end{aligned}$$

where we have used (3-9). Hence

$$|z|\|w_1\|^2_{L^2(X)} \le \mathcal{O}(|z|^{-1})\|U_1\|^2_{L^2(X)} + \mathcal{O}(\delta^4)\|f\|\|g\| + \mathcal{O}(h)\|f\|^2. \tag{3-24}$$

Recall that the function $U$ is of the form $(2h\,\partial_{x_1} + a(x))\phi_1(x_1)u + h\lambda n^{-1}\phi_0 V$, where $a$ is some smooth function. Hence the function $U_1$ satisfies the estimate

$$\|U_1\|_{L^2(X)} \lesssim \|u\|_{1,3,1} + \|V\|_{0,2,0} \lesssim \|u\|_{H_h^1(X)} + \|V\|_{0,2,2} + \mathcal{O}(h^\infty)\|f\|^{1/2}_{L^2(\partial X)}\|g\|^{1/2}_{L^2(\partial X)}, \tag{3-25}$$

where we have used Lemma 3.2 together with (3-15). By (3-24) and (3-25),

$$|z|\,\|w_1\|^2_{L^2(X)} \le \mathcal{O}(|z|^{-1})\|u\|^2_{H_h^1(X)} + \mathcal{O}(|z|^{-1})\|V\|^2_{0,2,2} + \mathcal{O}(\delta^4 + h^\infty)\|f\|\,\|g\| + \mathcal{O}(h)\|f\|^2. \tag{3-26}$$

Clearly, (3-22) follows from (3-26) and Lemma 3.1. □

## 4. Parametrix construction in the hyperbolic region

Let $\lambda$ be as in Theorems 1.1 and 1.2, and let $h$, $z$, $\delta$, $r_0$, $n_0$, $r_\sharp$, $\chi$ and $\chi_\delta^-$ be as in the previous sections. Set $\theta = \mathrm{Im}(h\lambda)^2 = hz = \mathcal{O}(h^\epsilon)$, $|\theta| \gg h$, and

$$\rho(x', \xi') = \sqrt{r_0(x', \xi') - (1 + i\theta)n_0(x')}, \qquad \mathrm{Re}\,\rho < 0.$$

It is easy to see that $\rho\chi_\delta^- \in S_0^0(\partial X)$. In this section we will prove:

**Proposition 4.1.** *There are constants $C, C_1 > 0$ depending on $\delta$ but independent of $\lambda$ such that*

$$\left\|h\mathcal{N}(\lambda; n)\mathrm{Op}_h(\chi_\delta^-) - \mathrm{Op}_h(\rho\chi_\delta^-)\right\|_{L^2(\partial X)\to H_h^1(\partial X)} \le C_1(h + e^{-C|\mathrm{Im}\,\lambda|}). \tag{4-1}$$

*Proof.* To prove (4-1) we will build a parametrix near the boundary of the solution to (1-1) with $f$ replaced by $\mathrm{Op}_h(\chi_\delta^-) f$. Let $x = (x_1, x')$, $x_1 > 0$, be the normal coordinates with respect to the metric $\mathcal{G}$, which of course are different from those introduced in the previous section. In these coordinates the operator $\Delta_X$ is given by

$$\Delta_X = \partial_{x_1}^2 + \widetilde{Q} + \widetilde{R},$$

where $\widetilde{Q} \le 0$ is a second-order differential operator with respect to the variable $x'$ and $\widetilde{R}$ is a first-order differential operator with respect to the variable $x$, both with coefficients depending smoothly on $x$. Let $(x^0, \xi^0) \in \operatorname{supp} \chi_\delta^-$ and let $\mathcal{U} \subset T^*\partial X$ be a small open neighbourhood of $(x^0, \xi^0)$ contained in $\{r_\sharp \le 1 - \delta^2/2\}$. Take a function $\psi \in C_0^\infty(\mathcal{U})$. We will construct a parametrix $\tilde{u}_\psi^-$ of the solution of (1-1) with $\tilde{u}_\psi^-|_{x_1=0} = \mathrm{Op}_h(\psi) f$ in the form $\tilde{u}_\psi^- = \phi(x_1)\mathcal{K}^- f$, where $\phi(x_1) = \chi(x_1/\delta_1)$, $0 < \delta_1 \ll 1$, is a parameter independent of $\lambda$ to be fixed later on depending on $\delta$, and

$$(\mathcal{K}^- f)(x) = (2\pi h)^{-d+1} \iint e^{(i/h)(\langle y', \xi'\rangle + \varphi(x, \xi', \theta))} a(x, \xi', \lambda) f(y')\, d\xi'\, dy'.$$

The phase $\varphi$ is complex-valued such that $\varphi|_{x_1=0} = -\langle x', \xi'\rangle$ and satisfies the eikonal equation mod $\mathcal{O}(\theta^M)$:

$$(\partial_{x_1}\varphi)^2 + \langle B(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi\rangle - (1 + i\theta)n(x) = \theta^M \mathcal{R}_M, \tag{4-2}$$

where $M \gg 1$ is an arbitrary integer, the function $\mathcal{R}_M$ is bounded uniformly in $\theta$, and $B$ is a matrix-valued function such that $r(x, \xi') = \langle B(x)\xi', \xi'\rangle$, $r(x, \xi') \ge 0$, is the principal symbol of the operator $-\widetilde{Q}$. We clearly have $r_0(x', \xi') = r(0, x', \xi')$. Let us see that for $(x', \xi') \in \mathcal{U}$, $0 \le x_1 \le 3\delta_1$, (4-2) has a smooth solution satisfying

$$\partial_{x_1}\varphi|_{x_1=0} = -i\rho + \mathcal{O}(\theta^{M/2}) \tag{4-3}$$

provided $\delta_1$ and $\mathcal{U}$ are small enough. We will be looking for $\varphi$ in the form

$$\varphi = \sum_{j=0}^{M-1} (i\theta)^j \varphi_j(x, \xi'),$$

where $\varphi_j$ are real-valued functions depending only on the sign of $\theta$ and satisfying the equations

$$(\partial_{x_1}\varphi_0)^2 + \langle B(x)\nabla_{x'}\varphi_0, \nabla_{x'}\varphi_0\rangle = n(x), \tag{4-4}$$

$$\sum_{j=0}^{k} \partial_{x_1}\varphi_j\, \partial_{x_1}\varphi_{k-j} + \sum_{j=0}^{k} \langle B(x)\nabla_{x'}\varphi_j, \nabla_{x'}\varphi_{k-j}\rangle = \epsilon_k n(x), \quad 1 \le k \le M-1, \tag{4-5}$$

$\varphi_0|_{x_1=0} = -\langle x', \xi'\rangle$, $\varphi_j|_{x_1=0} = 0$ for $j \ge 1$, where $\epsilon_1 = 1$, $\epsilon_k = 0$ for $k \ge 2$. It is easy to check that with this choice the function $\varphi$ satisfies (4-2) with $\mathcal{R}_M$ being polynomial in $\theta$.

Clearly, if $\varphi_0$ is a solution to (4-4), then we have $(\partial_{x_1}\varphi_0|_{x_1=0})^2 = n_0(x') - r_0(x', \xi') \ge C'$ with some constant $C' > 0$ depending on $\delta$. It is well known that (4-4) has a local (that is, for $\delta_1$ and $\mathcal{U}$ small enough) real-valued solution $\varphi_0^\pm$ such that $\partial_{x_1}\varphi_0^\pm|_{x_1=0} = \pm\sqrt{n_0 - r_0}$. We now define the function $\varphi_0$ by $\varphi_0 = \varphi_0^+$ if $\theta > 0$ and $\varphi_0 = \varphi_0^-$ if $\theta < 0$. Hence $|\partial_{x_1}\varphi_0(x, \xi')| \ge \text{const.} > 0$ for $x_1$ small enough. Therefore, the equations (4-5) can be solved locally. Taking $x_1 = 0$ in (4-5) with $k = 1$, we find

$$\theta\partial_{x_1}\varphi_1|_{x_1=0} = \theta n_0 (2\partial_{x_1}\varphi_0|_{x_1=0})^{-1} = \tfrac{1}{2}|\theta| n_0 (n_0 - r_0)^{-1/2} \ge \tfrac{1}{2}C|\theta| \tag{4-6}$$

on $\mathcal{U}$, where $C = \min \sqrt{n_0(x')}$. Hence

$$\operatorname{Im} \partial_{x_1}\varphi|_{x_1=0} = \theta \partial_{x_1}\varphi_1|_{x_1=0} + \mathcal{O}(\theta^2) \geq \tfrac{1}{3}C|\theta| \tag{4-7}$$

if $|\theta|$ is taken small enough. On the other hand, taking $x_1 = 0$ in (4-2) we find

$$(\partial_{x_1}\varphi|_{x_1=0})^2 = (i\rho)^2 + \mathcal{O}(\theta^M) = (i\rho)^2(1+\mathcal{O}(\theta^M)), \tag{4-8}$$

where we have used that $|\rho| \geq \text{const.} > 0$ on $\mathcal{U}$. Since $\operatorname{Re}\rho < 0$, we get (4-3) from (4-7) and (4-8). By (4-6) we also get

$$\theta\varphi_1(x_1, x', \xi') = \theta x_1 \partial_{x_1}\varphi_1(0, x', \xi') + \mathcal{O}(\theta x_1^2) \geq \tfrac{1}{2}Cx_1|\theta| - \mathcal{O}(|\theta| x_1^2) \geq \tfrac{1}{3}Cx_1|\theta|$$

provided $x_1$ is taken small enough. This implies

$$\operatorname{Im}\varphi(x, \xi', \theta) = \theta\varphi_1(x_1, x', \xi') + \mathcal{O}(\theta^2 x_1) \geq \tfrac{1}{4}Cx_1|\theta|. \tag{4-9}$$

The amplitude $a$ is of the form

$$a = \sum_{k=0}^{m} h^k a_k(x, \xi', \theta),$$

where $m \gg 1$ is an arbitrary integer and the functions $a_k$ satisfy the transport equations mod $\mathcal{O}(\theta^M)$:

$$2i\partial_{x_1}\varphi\partial_{x_1}a_k + 2i\langle B(x)\nabla_{x'}\varphi, \nabla_{x'}a_k\rangle + i(\Delta_X\varphi)a_k + \Delta_X a_{k-1} = \theta^M \mathcal{Q}_M^{(k)}, \quad 0 \leq k \leq m, \tag{4-10}$$

$a_0|_{x_1=0} = \psi,\ a_k|_{x_1=0} = 0$ for $k \geq 1$, where $a_{-1} = 0$. Let us see that the transport equations have smooth solutions for $(x', \xi') \in \mathcal{U},\ 0 \leq x_1 \leq 3\delta_1$, provided $\delta_1$ and $\mathcal{U}$ are taken small enough. As above, we will be looking for $a_k$ in the form

$$a_k = \sum_{j=0}^{M-1} (i\theta)^j a_{k,j}(x, \xi').$$

We let $a_{k,j}$ satisfy the equations

$$2i\sum_{\nu=0}^{j} \partial_{x_1}\varphi_\nu\, \partial_{x_1}a_{k,j-\nu} + 2i\sum_{\nu=0}^{j}\langle B(x)\nabla_{x'}\varphi_\nu, \nabla_{x'}a_{k,j-\nu}\rangle + i(\Delta_X\varphi_j)a_{k,j} + \Delta_X a_{k-1,j} = 0, \tag{4-11}$$

$0 \leq j \leq M-1,\ a_{0,0}|_{x_1=0} = \psi,\ a_{k,j}|_{x_1=0} = 0$ for $k + j \geq 1$. Then the functions $a_k$ satisfy (4-10) with $\mathcal{Q}_M^{(k)}$ polynomial in $\theta$. As in the case of (4-5) one can solve (4-11) locally. Then we can write

$$V_- := h^{-1}\big(h^2\Delta_X + (1+i\theta)n(x)\big)\tilde{u}_\psi^- = \mathcal{K}_1^- f + \mathcal{K}_2^- f,$$

where

$$\begin{aligned}\mathcal{K}_1^- f &= h[\Delta_X, \phi]\mathcal{K}^- f = h\big(2\phi'(x_1)\partial_{x_1} + c(x)\phi''(x_1)\big)\mathcal{K}^- f\\ &= (2\pi h)^{-d+1}\iint e^{(i/h)(\langle y', \xi'\rangle + \varphi(x, \xi', \theta))} A_1^-(x, \xi', \lambda) f(y')\, d\xi'\, dy',\end{aligned}$$

$c$ being some smooth function and

$$A_1^- = 2i\phi' a\, \partial_{x_1}\varphi + hc\phi''\, \partial_{x_1}a,$$

and

$$(\mathcal{K}_2^- f)(x) = (2\pi h)^{-d+1} \iint e^{(i/h)(\langle y',\xi'\rangle+\varphi(x,\xi',\theta))} A_2^-(x,\xi',\lambda) f(y')\, d\xi'\, dy',$$

where

$$A_2^- = \phi(x_1)\Big(-h^{-1}\theta^M \mathcal{R}_M a + \theta^M \sum_{k=0}^{m} h^k \mathcal{Q}_M^{(k)} + h^{m+1}\Delta_X a_m\Big).$$

We claim that Proposition 4.1 follows from:

**Lemma 4.2.** *The function $V_-$ satisfies the estimate*

$$\|V_-\|_{H_h^1(X)} \lesssim e^{-C|\operatorname{Im}\lambda|}\|f\| + \mathcal{O}_m(h^{m-d})\|f\| + \mathcal{O}_M(h^{\epsilon M-d})\|f\| \tag{4-12}$$

*with some constant $C > 0$.*

Indeed, if $u_\psi^-$ denotes the solution to (1-1) with $f$ replaced by $\mathrm{Op}_h(\psi)f$ and $\tilde u_\psi^-$ is the parametrix built above, then the function $v = u_\psi^- - \tilde u_\psi^-$ satisfies (3-1) with $f \equiv 0$. Therefore, by the estimates (3-10) and (4-12) we have

$$\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\psi) - T_\psi^-\|_{L^2(\partial X)\to H_h^1(\partial X)} \lesssim e^{-C|\operatorname{Im}\lambda|} + \mathcal{O}_m(h^{m-d}) + \mathcal{O}_M(h^{\epsilon M-d}), \tag{4-13}$$

where the operator $T_\psi^-$ is defined by

$$T_\psi^- f = h\,\partial_{x_1}\mathcal{K}^- f|_{x_1=0}.$$

Hence, in view of (4-3),

$$\begin{aligned}(T_\psi^- f)(x') &= (2\pi h)^{-d+1} \iint e^{(i/h)\langle y'-x',\xi'\rangle}\big(i\psi\,\partial_{x_1}\varphi(0,x',\xi',\theta) + h\,\partial_{x_1}a(0,x',\xi',\lambda)\big) f(y')\, d\xi'\, dy' \\ &= \mathrm{Op}_h(\rho\psi + \mathcal{O}(\theta^{M/2}))f + \sum_{k=0}^{m} h^{k+1}\mathrm{Op}_h(\partial_{x_1}a_k(0,x',\xi',\theta))f.\end{aligned}$$

Since

$$\mathrm{Op}_h(\partial_{x_1}a_k(0,x',\xi',\theta)) = \mathcal{O}(1) : L^2(\partial X) \to H_h^1(\partial X)$$

uniformly in $\theta$, it follows from (4-13) that

$$\big\|h\mathcal{N}(\lambda;n)\mathrm{Op}_h(\psi) - \mathrm{Op}_h(\rho\psi)\big\|_{L^2(\partial X)\to H_h^1(\partial X)} \lesssim e^{-C|\operatorname{Im}\lambda|} + \mathcal{O}(h). \tag{4-14}$$

On the other hand, using a suitable partition of the unity we can write the function $\chi_\delta^-$ as $\sum_{j=1}^J \psi_j$, where each function $\psi_j$ has the same properties as the function $\psi$ above. In other words, we have (4-14) with $\psi$ replaced by each $\psi_j$, which after summing up leads to (4-1). □

*Proof of Lemma 4.2.* Let $\alpha$ be a multi-index such that $|\alpha| \le 1$. Since

$$i|\alpha| A_2^- \partial_x^\alpha \varphi + (h\,\partial_x)^\alpha A_2^- = \mathcal{O}_m(h^{m+1}) + \mathcal{O}_M(h^{\epsilon M-1})$$

and $\operatorname{Im}\varphi \ge 0$, the kernel of the operator $(h\,\partial_x)^\alpha \mathcal{K}_2^- : L^2(\partial X) \to L^2(X)$ is $\mathcal{O}_m(h^{m-d}) + \mathcal{O}_M(h^{\epsilon M-d})$, and hence so is its norm. Since the function $A_1^-$ is supported in the interval $[\delta_1/2, 3\delta_1]$ with respect to the variable $x_1$, to bound the norm of the operator $\mathcal{K}^-_{1,\alpha} := (h\,\partial_x)^\alpha \mathcal{K}^-_1 : L^2(\partial X) \to L^2(X)$ it suffices to show that

$$\|\mathcal{K}^-_{1,\alpha}\|_{L^2(\partial X)\to L^2(\partial X)} \lesssim e^{-C|\theta|/h} + \mathcal{O}(h^\infty) \tag{4-15}$$

uniformly in $x_1 \in [\delta_1/2, 3\delta_1]$. Since $|\theta|/h \sim |\operatorname{Im}\lambda|$, (4-15) will imply (4-12). We would like to consider $\mathcal{K}^-_{1,\alpha}$ as an $h$-FIO with phase $\operatorname{Re}\varphi$ and amplitude

$$A_\alpha = e^{-\operatorname{Im}\varphi/h}(i|\alpha|A^-_1\partial_x^\alpha\varphi + (h\,\partial_x)^\alpha A^-_1).$$

To do so, we need to have that the phase satisfies the condition

$$\left|\det\left(\frac{\partial^2 \operatorname{Re}\varphi}{\partial x'\partial\xi'}\right)\right| \geq \widetilde{C} > 0 \tag{4-16}$$

for $|\theta|$ small enough, where $\widetilde{C}$ is a constant independent of $\theta$. Since $\operatorname{Re}\varphi = \varphi_0 + \mathcal{O}(|\theta|)$, it suffices to show (4-16) for the phase $\varphi_0$. This, however, is easy to arrange by taking $x_1$ small enough because $\varphi_0 = -\langle x', \xi'\rangle + \mathcal{O}(x_1)$ and (4-16) is trivially fulfilled for the phase $-\langle x', \xi'\rangle$. On the other hand, using that $\operatorname{Im}\varphi = \mathcal{O}(|\theta|)$ together with (4-9) we get the following bounds for the amplitude:

$$|\partial_{x'}^{\beta_1}\partial_{\xi'}^{\beta_2} A_\alpha| \leq C_{\beta_1,\beta_2} \sum_{0\leq k\leq|\beta_1|+|\beta_2|} \left(\frac{|\theta|}{h}\right)^k e^{-C\delta_1|\theta|/(8h)} \leq \widetilde{C}_{\beta_1,\beta_2} e^{-C\delta_1|\theta|/(9h)} \tag{4-17}$$

for all multi-indices $\beta_1$ and $\beta_2$. It follows from (4-16) and (4-17) that, mod $\mathcal{O}(h^\infty)$, the operator $(\mathcal{K}^-_{1,\alpha})^*\mathcal{K}^-_{1,\alpha}$ is an $h$-ΨDO in the class $\mathrm{OP}S^0_0(\partial X)$ uniformly in $\theta$ with a symbol which is $\mathcal{O}(e^{-2C|\theta|/h})$ together with all derivatives, where $C > 0$ is a new constant. Therefore, its norm is also $\mathcal{O}(e^{-2C|\theta|/h})$, which clearly implies (4-15). □

## 5. Parametrix construction in the elliptic region

We keep the notations from the previous sections and note that $\rho\chi^+_\delta \in S^1_0(\partial X)$. It is easy also to see that $0 < C_1\langle\xi'\rangle \leq |\rho| \leq C_2\langle\xi'\rangle$ on supp $\chi^+_\delta$, where $C_1$ and $C_2$ are constants depending on $\delta$. In this section we will prove:

**Proposition 5.1.** *There is a constant $C > 0$ depending on $\delta$ but independent of $\lambda$ such that*

$$\left\| h\mathcal{N}(\lambda; n)\mathrm{Op}_h(\chi^+_\delta) - \mathrm{Op}_h(\rho\chi^+_\delta + hb)\right\|_{L^2(\partial X)\to H^1_h(\partial X)} \leq Ch, \tag{5-1}$$

*where $b \in S^0_0(\partial X)$ does not depend on $\lambda$ or the function $n$.*

*Proof.* The estimate (5-1) is a consequence of the parametrix built in [Vodev 2015]. In what follows we will recall this construction. We will first proceed locally and then we will use partition of the unity to get the global parametrix. Fix a point $x^0 \in \partial X$ and let $\mathcal{U}_0 \subset \partial X$ be a small open neighbourhood of $x^0$. Let $(x_1, x')$, $x_1 > 0$, $x' \in \mathcal{U}_0$, be the normal coordinates used in the previous section. Take a function $\psi^0 \in C_0^\infty(\mathcal{U}_0)$ and set $\psi = \psi^0\chi^+_\delta$. As in the previous section, we will construct a parametrix $\tilde{u}^+_\psi$ of the solution of (1-1) with $\tilde{u}^+_\psi|_{x_1=0} = \mathrm{Op}_h(\psi)f$ in the form $\tilde{u}^+_\psi = \phi(x_1)\mathcal{K}^+ f$, where $\phi(x_1) = \chi(x_1/\delta_1)$, $0 < \delta_1 \ll 1$, is a parameter independent of $\lambda$ to be fixed later on, and

$$(\mathcal{K}^+ f)(x) = (2\pi h)^{-d+1}\iint e^{(i/h)(\langle y',\xi'\rangle + \varphi(x,\xi',\theta))} a(x, \xi', \lambda) f(y')\, d\xi'\, dy'.$$

The phase $\varphi$ is complex-valued such that $\varphi|_{x_1=0} = -\langle x', \xi' \rangle$ and satisfies the eikonal equation mod $\mathcal{O}(x_1^M)$:

$$(\partial_{x_1}\varphi)^2 + \langle B(x)\nabla_{x'}\varphi, \nabla_{x'}\varphi \rangle - (1+i\theta)n(x) = x_1^M \widetilde{\mathcal{R}}_M, \tag{5-2}$$

where $M \gg 1$ is an arbitrary integer, and the function $\widetilde{\mathcal{R}}_M$ is smooth up to the boundary $x_1 = 0$. It is shown in [Vodev 2015, Section 4] that for $(x', \xi') \in \operatorname{supp} \psi$, (5-2) has a smooth solution of the form

$$\varphi = \sum_{k=0}^{M-1} x_1^k \varphi_k(x', \xi', \theta), \quad \varphi_0 = -\langle x', \xi' \rangle,$$

satisfying

$$\partial_{x_1}\varphi|_{x_1=0} = \varphi_1 = -i\rho. \tag{5-3}$$

Moreover, taking $\delta_1$ small enough we can arrange that

$$\operatorname{Im} \varphi \geq -\tfrac{1}{2} x_1 \operatorname{Re} \rho \geq C x_1 \langle \xi' \rangle, \quad C > 0, \tag{5-4}$$

for $0 \leq x_1 \leq 3\delta_1$, $(x', \xi') \in \operatorname{supp} \psi$. The amplitude $a$ is of the form

$$a = \sum_{j=0}^{m} h^j a_j(x, \xi', \theta),$$

where $m \gg 1$ is an arbitrary integer and the functions $a_j$ satisfy the transport equations mod $\mathcal{O}(x_1^M)$:

$$2i\partial_{x_1}\varphi \partial_{x_1} a_j + 2i\langle B(x)\nabla_{x'}\varphi, \nabla_{x'} a_j \rangle + i(\Delta_X \varphi)a_j + \Delta_X a_{j-1} = x_1^M \widetilde{\mathcal{Q}}_M^{(j)}, \quad 0 \leq j \leq m, \tag{5-5}$$

$a_0|_{x_1=0} = \psi$, $a_j|_{x_1=0} = 0$ for $j \geq 1$, where $a_{-1} = 0$ and the functions $\widetilde{\mathcal{Q}}_M^{(j)}$ are smooth up to the boundary $x_1 = 0$. It is shown in [Vodev 2015, Section 4] that the equations (5-5) have unique smooth solutions of the form

$$a_j = \sum_{k=0}^{M-1} x_1^k a_{k,j}(x', \xi', \theta)$$

with functions $a_{k,j} \in S_0^{-j}(\partial X)$ uniformly in $\theta$. We can write

$$V_+ := h^{-1}\big(h^2\Delta_X + (1+i\theta)n(x)\big)\tilde{u}_\psi^+ = \mathcal{K}_1^+ f + \mathcal{K}_2^+ f,$$

where

$$\begin{aligned}\mathcal{K}_1^+ f &= h[\Delta_X, \phi]\mathcal{K}^+ f = h\big(2\phi'(x_1)\partial_{x_1} + c(x)\phi''(x_1)\big)\mathcal{K}^+ f \\ &= (2\pi h)^{-d+1} \iint e^{(i/h)(\langle y', \xi' \rangle + \varphi(x, \xi', \theta))} A_1^+(x, \xi', \lambda) f(y')\, d\xi'\, dy',\end{aligned}$$

with

$$A_1^+ = 2i\phi' a \partial_{x_1}\varphi + hc\phi''\partial_{x_1} a,$$

and

$$(\mathcal{K}_2^+ f)(x) = (2\pi h)^{-d+1} \iint e^{(i/h)(\langle y', \xi' \rangle + \varphi(x, \xi', \theta))} A_2^+(x, \xi', \lambda) f(y')\, d\xi'\, dy',$$

where

$$A_2^+ = \phi(x_1)\bigg(-h^{-1}x_1^M \widetilde{\mathcal{R}}_M a + x_1^M \sum_{j=0}^{m} h^j \widetilde{\mathcal{Q}}_M^{(j)} + h^{m+1}\Delta_X a_m\bigg).$$

As in the previous section, we will derive Proposition 5.1 from (5-3) and the following:

**Lemma 5.2.** *The function $V_+$ satisfies the estimate*

$$\|V_+\|_{H_h^1(X)} \le \mathcal{O}_m(h^{m-d})\|f\| + \mathcal{O}_M(h^{M-d})\|f\|. \tag{5-6}$$

*Proof.* Let $\alpha$ be a multi-index such that $|\alpha| \le 1$. In view of (5-4) we have

$$\left|e^{i\varphi/h}(i|\alpha|A_1^+\partial_x^\alpha\varphi + (h\,\partial_x)^\alpha A_1^+)\right| \lesssim \sup_{\delta_1/2\le x_1\le 3\delta_1} e^{-\operatorname{Im}\varphi/h} \lesssim e^{-C\langle\xi'\rangle/h} = \mathcal{O}_M((h/\langle\xi'\rangle)^M)$$

for every integer $M \gg 1$. Therefore, the kernel of the operator $(h\,\partial_x)^\alpha\mathcal{K}_1^+ : L^2(\partial X) \to L^2(X)$ is $\mathcal{O}_M(h^{M-d+1})$, and hence so is its norm. By (5-4) we also have

$$x_1^M e^{-\operatorname{Im}\varphi/h} \le x_1^M e^{-Cx_1\langle\xi'\rangle/h} = \mathcal{O}_M((h/\langle\xi'\rangle)^M).$$

This implies

$$e^{i\varphi/h}\left(i|\alpha|A_2^+\partial_x^\alpha\varphi + (h\,\partial_x)^\alpha A_2^+\right) = \mathcal{O}_M((h/\langle\xi'\rangle)^{M-1}) + \mathcal{O}_m((h/\langle\xi'\rangle)^m),$$

which again implies the desired bound for the norm of the operator $(h\,\partial_x)^\alpha\mathcal{K}_2^+$. □

By the estimates (3-10) and (5-6) we have

$$\left\|h\mathcal{N}(\lambda;n)\operatorname{Op}_h(\psi) - T_\psi^+\right\|_{L^2(\partial X)\to H_h^1(\partial X)} \le \mathcal{O}_m(h^{m-d}) + \mathcal{O}_M(h^{M-d}), \tag{5-7}$$

where the operator $T_\psi^+$ is defined by

$$T_\psi^+ f = h\,\partial_{x_1}\mathcal{K}^+ f|_{x_1=0}.$$

In view of (5-3), we have

$$\begin{aligned}(T_\psi^+ f)(x') &= (2\pi h)^{-d+1}\iint e^{(i/h)\langle y'-x',\xi'\rangle}\left(i\psi\,\partial_{x_1}\varphi(0,x',\xi',\theta) + h\,\partial_{x_1}a(0,x',\xi',\lambda)\right)f(y')\,d\xi'\,dy' \\ &= \operatorname{Op}_h(\rho\psi)f + \sum_{j=0}^{m} h^{j+1}\operatorname{Op}_h(a_{1,j}(x',\xi',\theta))f,\end{aligned}$$

where $a_{1,j} \in S_0^{-j}(\partial X)$. Hence

$$\operatorname{Op}_h(a_{1,j}) = \mathcal{O}(1) : L^2(\partial X) \to H_h^j(\partial X).$$

Therefore it follows from (5-7) that

$$\left\|h\mathcal{N}(\lambda;n)\operatorname{Op}_h(\psi) - \operatorname{Op}_h(\rho\psi + ha_{1,0})\right\|_{L^2(\partial X)\to H_h^1(\partial X)} \le \mathcal{O}(h). \tag{5-8}$$

We need now the following:

**Lemma 5.3.** *There exists a function $b^0 \in S_0^0(\partial X)$, independent of $\lambda$ and $n$, such that*

$$a_{1,0} - b^0 \in S_0^{-1}(\partial X). \tag{5-9}$$

*Proof.* We will calculate the function $a_{1,0}$ explicitly. Note that this lemma (as well as Proposition 5.1) is also used in [Vodev 2015], but the proof therein is not correct since $a_{1,0}$ is calculated incorrectly. Therefore we will give here a new proof. Clearly, it suffices to prove (5-9) with $a_{1,0}$ replaced by $(1-\eta)a_{1,0}$ with some function $\eta \in C_0^\infty(T^*\partial X)$ independent of $h$. Since $\rho = -\sqrt{r_0}(1+\mathcal{O}(r_0^{-1}))$ as $r_0 \to \infty$, it is easy to see that

$$(1-\eta)\rho^{-k} - (1-\eta)(-\sqrt{r_0})^{-k} \in S_0^{-k-1}(\partial X) \tag{5-10}$$

for every integer $k \geq 0$, provided $\eta$ is taken such that $\eta = 1$ for $|\xi'| \leq A$ with some $A > 1$ big enough. We will now calculate the function $\varphi_2$ from the eikonal equation. To this end, write

$$B(x) = B_0(x') + x_1 B_1(x') + \mathcal{O}(x_1^2), \quad n(x) = n_0(x') + x_1 n_1(x') + \mathcal{O}(x_1^2)$$

and observe that the left-hand side of (5-2) is equal to

$$x_1\big(4\varphi_1\varphi_2 + 2\langle B_0\nabla_{x'}\varphi_0, \nabla_{x'}\varphi_1\rangle + \langle B_1\nabla_{x'}\varphi_0, \nabla_{x'}\varphi_0\rangle - (1+i\theta)n_1\big) + \mathcal{O}(x_1^2).$$

Hence, taking into account that $\varphi_0 = -\langle x', \xi'\rangle$ and $\varphi_1 = -i\rho$, we get

$$\varphi_2 = (2\rho)^{-1}\langle B_0\xi', \nabla_{x'}\rho\rangle + (4i\rho)^{-1}\langle B_1\xi', \xi'\rangle - (1+i\theta)(4i\rho)^{-1}n_1.$$

Using the identity

$$2\rho\nabla_{x'}\rho = \nabla_{x'}r_0 - (1+i\theta)\nabla_{x'}n_0$$

we can write $\varphi_2$ in the form

$$\varphi_2 = (2\rho)^{-2}\langle B_0\xi', \nabla_{x'}r_0\rangle + (4i\rho)^{-1}\langle B_1\xi', \xi'\rangle - (1+i\theta)(2\rho)^{-2}\langle B_0\xi', \nabla_{x'}n_0\rangle - (1+i\theta)(4i\rho)^{-1}n_1.$$

By (5-10) we conclude that, mod $S_0^{-1}(\partial X)$,

$$(1-\eta)\frac{\varphi_2}{\varphi_1} = -i4^{-1}(1-\eta)r_0^{-3/2}\langle B_0\xi', \nabla_{x'}r_0\rangle + (1-\eta)(4r_0)^{-1}\langle B_1\xi', \xi'\rangle. \tag{5-11}$$

Write now the operator $\Delta_X$ in the form

$$\Delta_X = \partial_{x_1}^2 + \langle B_0\nabla_{x'}, \nabla_{x'}\rangle + q_1(x')\partial_{x_1} + \langle q_2(x'), \nabla_{x'}\rangle + \mathcal{O}(x_1)$$

and observe that

$$\Delta_X\varphi = 2\varphi_2 + q_1\varphi_1 - \langle q_2(x'), \xi'\rangle + \mathcal{O}(x_1).$$

We now calculate the left-hand side of (5-5) with $j = 0$ modulo $\mathcal{O}(x_1)$. Recall that $a_{0,0} = \psi$. We obtain

$$2i\varphi_1 a_{1,0} + 2i\langle B_0\nabla_{x'}\varphi_0, \nabla_{x'}a_{0,0}\rangle + i(\Delta_X\varphi)a_{0,0} = 2i\varphi_1 a_{1,0} + 2i\langle B_0\xi', \nabla_{x'}\psi\rangle + i\big(2\varphi_2 + q_1\varphi_1 - \langle q_2(x'), \xi'\rangle\big)\psi.$$

Since the right-hand side is $\mathcal{O}(x_1^M)$, the above function must be identically zero. Thus we get the following expression for the function $a_{1,0}$:

$$a_{1,0} = -\varphi_1^{-1}\langle B_0\xi', \nabla_{x'}\psi\rangle - \big(\varphi_1^{-1}\varphi_2 + 2^{-1}q_1 - (2\varphi_1)^{-1}\langle q_2(x'), \xi'\rangle\big)\psi. \tag{5-12}$$

Taking into account that $\psi = \psi^0$ on $\operatorname{supp}(1-\eta)$, we find from (5-10)–(5-12) that (5-9) holds with

$$b^0 = i(1-\eta)r_0^{-1/2}\langle B_0\xi', \nabla_{x'}\psi^0\rangle$$
$$-4^{-1}(1-\eta)\psi^0\big(-ir_0^{-3/2}\langle B_0\xi', \nabla_{x'}r_0\rangle + r_0^{-1}\langle B_1\xi', \xi'\rangle + 2q_1 + 2r_0^{-1/2}\langle q_2(x'), \xi'\rangle\big). \tag{5-13}$$

Clearly, $b^0 \in S_0^0(\partial X)$ is independent of $\lambda$ and $n$, as desired. □

Lemma 5.3 implies that

$$\operatorname{Op}_h(a_{1,0} - b^0) = \mathcal{O}(1) : L^2(\partial X) \to H_h^1(\partial X). \tag{5-14}$$

Now, using a suitable partition of the unity on $\partial X$ we can write $1 = \sum_{j=1}^J \psi_j^0$. Hence, we can write the function $\chi_\delta^+$ as $\sum_{j=1}^J \psi_j$, where $\psi_j = \psi_j^0\chi_\delta^+$. Since we have (5-8) and (5-14) with $\psi$ replaced by each $\psi_j$, we get (5-1) by summing up all the estimates. □

It follows from the estimate (3-11) applied with $V \equiv 0$ that

$$h\mathcal{N}(\lambda; n)\operatorname{Op}_h(\chi_\delta^0) = \mathcal{O}(\delta) : L^2(\partial X) \to H_h^1(\partial X) \tag{5-15}$$

provided $|\operatorname{Im}\lambda| \geq \delta^{-4}$ and $\operatorname{Re}\lambda \geq C_\delta \gg 1$. Now Theorem 1.2 follows from (5-15) and Propositions 4.1 and 5.1. Let us now see that Theorem 1.1 follows from Theorem 1.2. Since the operator $-h^2\Delta_{\partial X} \geq 0$ is self-adjoint, we have the bound

$$\begin{aligned}\big\|hp(-\Delta_{\partial X})\chi_2((-h^2\Delta_{\partial X}-1)\delta^{-2})\big\| &= \big\|\sqrt{-h^2\Delta_{\partial X}-1-i\theta}\,\chi((-h^2\Delta_{\partial X}-1)\delta^{-2})\big\| \\ &\leq \sup_{\sigma\geq 0}\big|\sqrt{\sigma-1-i\theta}\,\chi((\sigma-1)\delta^{-2})\big| \\ &\leq \sup_{\delta^2\leq|\sigma-1|\leq 2\delta^2}\sqrt{|\sigma-1|+|\theta|} \leq \mathcal{O}(\delta+|\theta|^{1/2}) = \mathcal{O}(\delta+h^{\epsilon/2}).\end{aligned} \tag{5-16}$$

On the other hand, it is well known that the operator $hp(-\Delta_{\partial X})(1-\chi_2)((-h^2\Delta_{\partial X}-1)\delta^{-2})$ is an $h$-ΨDO in the class $\mathrm{OP}S_0^1(\partial X)$ with principal symbol $\rho(1-\chi_\delta^0)$. This implies the bound

$$hp(-\Delta_{\partial X})(1-\chi_2)((-h^2\Delta_{\partial X}-1)\delta^{-2}) - \operatorname{Op}_h(\rho(1-\chi_\delta^0)) = \mathcal{O}(h) : L^2(\partial X) \to L^2(\partial X). \tag{5-17}$$

It is easy to see that Theorem 1.1 follows from (1-3) together with (5-16) and (5-17). □

## 6. Proof of Theorem 2.1

Define the DN maps $\mathcal{N}_j(\lambda)$, $j = 1, 2$, by

$$\mathcal{N}_j(\lambda)f = \partial_\nu u_j|_\Gamma,$$

where $\nu$ is the Euclidean unit normal to $\Gamma$ and $u_j$ is the solution to the equation

$$\begin{cases}(\nabla c_j(x)\nabla + \lambda^2 n_j(x))u_j = 0 & \text{in } \Omega,\\ u_j = f & \text{on } \Gamma,\end{cases} \tag{6-1}$$

and consider the operator

$$T(\lambda) = c_1\mathcal{N}_1(\lambda) - c_2\mathcal{N}_2(\lambda).$$

Clearly, $\lambda$ is a transmission eigenvalue if there exists a nontrivial function $f$ such that $T(\lambda)f = 0$. Therefore Theorem 2.1 is a consequence of the following:

**Theorem 6.1.** *Under the conditions of Theorem 2.1, the operator $T(\lambda)$ sends $H^{(1+k)/2}(\Gamma)$ into $H^{(1-k)/2}(\Gamma)$, where $k = -1$ if (2-2) holds and $k = 1$ if (2-4) holds. Moreover, there exists a constant $C > 0$ such that $T(\lambda)$ is invertible for $\operatorname{Re}\lambda \geq 1$ and $|\operatorname{Im}\lambda| \geq C$ with an inverse satisfying in this region the bound*

$$\|T(\lambda)^{-1}\|_{H^{(1-k)/2}(\Gamma)\to H^{(1+k)/2}(\Gamma)} \lesssim |\lambda|^{(k-1)/2}, \tag{6-2}$$

*where the Sobolev spaces are equipped with the classical norms.*

*Proof.* We may suppose that $\lambda \in \Lambda_\epsilon = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq C_\epsilon \gg 1,\ |\operatorname{Im}\lambda| \leq |\lambda|^\epsilon\},\ 0 < \epsilon \ll 1$, since the case when $\lambda \in \{\operatorname{Re}\lambda \geq 1\} \setminus \Lambda_\epsilon$ follows from the analysis in [Vodev 2015]. We will equip the boundary $\Gamma$ with the Riemannian metric induced by the Euclidean metric $g_E$ in $\Omega$ and will denote by $r_0$ the principal symbol of the Laplace–Beltrami operator $-\Delta_\Gamma$. We would like to apply Theorem 1.2 to the operators $\mathcal{N}_j(\lambda)$. However, some modifications must be done coming from the presence of the function $c_j$ in (6-1). Indeed, in the definition of the operator $\mathcal{N}(\lambda; n)$ in Section 1, the normal derivative is taken with respect to the Riemannian metric $g_j = c_j^{-1} g_E$, while in the definition of the operator $\mathcal{N}_j(\lambda)$ it is taken with respect to the metric $g_E$. The first observation to be done is that the glancing region corresponding to the problem (6-1) is defined by $\Sigma_j := \{(x', \xi') \in T^*\Gamma : r_j(x', \xi') = 1\}$, where $r_j := m_j^{-1} r_0,\ m_j := (n_j/c_j)|_\Gamma$. We define now the cut-off functions $\chi^0_{\delta,j}$ by replacing in the definition of $\chi^0_\delta$ the function $r_\sharp$ by $r_j$. Secondly, the function $\rho$ must be replaced by

$$\rho_j(x', \xi') = \sqrt{r_0(x', \xi') - (1 + i\theta) m_j(x')}, \quad \operatorname{Re}\rho_j < 0.$$

With these changes, the operator $\mathcal{N}_j(\lambda)$ satisfies the estimate (1-3). Set

$$\tau_\delta = c_1\rho_1(1 - \chi^0_{\delta,1}) - c_2\rho_2(1 - \chi^0_{\delta,2}) = \tau - c_1\rho_1\chi^0_{\delta,1} + c_2\rho_2\chi^0_{\delta,2},$$

where

$$\tau = c_1\rho_1 - c_2\rho_2 = \frac{\tilde{c}(x')\big(c_0(x') r_0(x', \xi') - 1 - i\theta\big)}{c_1\rho_1 + c_2\rho_2}, \tag{6-3}$$

where $\tilde{c}$ and $c_0$ are the restrictions on $\Gamma$ of the functions

$$c_1 n_1 - c_2 n_2 \quad \text{and} \quad \frac{c_1^2 - c_2^2}{c_1 n_1 - c_2 n_2}$$

respectively. Clearly, under the conditions of Theorem 2.1, we have $\tilde{c}(x') \neq 0$ for all $x' \in \Gamma$. Moreover, (2-2) implies $c_0 \equiv 0$, while (2-4) implies $c_0(x') < 0$ for all $x' \in \Gamma$. Hence,

$$0 < C_1 \leq |c_0 r_0 - 1 - i\theta| \leq C_2$$

if (2-2) holds, and

$$0 < C_1\langle r_0\rangle \leq |c_0 r_0 - 1 - i\theta| \leq C_2\langle r_0\rangle$$

if (2-4) holds. Using this, together with (6-3), and the fact that $\rho_j \sim -\sqrt{r_0}$ as $r_0 \to \infty$, we get

$$0 < C_1'\langle\xi'\rangle^k \leq C_1\langle r_0\rangle^{k/2} \leq |\tau| \leq C_2\langle r_0\rangle^{k/2} \leq C_2'\langle\xi'\rangle^k, \tag{6-4}$$

where $k=-1$ if (2-2) holds and $k=1$ if (2-4) holds. Let $\eta\in C_0^\infty(T^*\Gamma)$ be such that $\eta=1$ on $|\xi'|\le A$ and $\eta=0$ on $|\xi'|\ge A+1$, where $A\gg 1$ is a big parameter independent of $\lambda$ and $\delta$. Taking $A$ big enough we can arrange that $(1-\eta)\tau_\delta=(1-\eta)\tau$. On the other hand, we have $\eta\tau_\delta=\eta\tau+\mathcal{O}(\delta+|\theta|^{1/2})$. Therefore, taking $\delta$ and $|\theta|$ small enough, we get from (6-4) that the function $\tau_\delta$ satisfies the bounds

$$\widetilde{C}_1\langle\xi'\rangle^k\le|\tau_\delta|\le\widetilde{C}_2\langle\xi'\rangle^k \tag{6-5}$$

with positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ independent of $\delta$ and $\theta$. Furthermore, one can easily check that $(1-\eta)\tau\in S_0^k(\Gamma)$ and $\eta\tau_\delta\in S_0^{-2}(\Gamma)$. Hence, $\tau_\delta\in S_0^k(\Gamma)$, which in turn implies that the operator $\mathrm{Op}_h(\tau_\delta)$ sends $H^{(1+k)/2}(\Gamma)$ into $H^{(1-k)/2}(\Gamma)$. Moreover, it follows from (6-5) that the operator $\mathrm{Op}_h(\tau_\delta):H_h^{(1+k)/2}(\Gamma)\to H_h^{(1-k)/2}(\Gamma)$ is invertible with an inverse satisfying the bound

$$\|\mathrm{Op}_h(\tau_\delta)^{-1}\|_{H_h^{(1-k)/2}(\Gamma)\to H_h^{(1+k)/2}(\Gamma)}\le\widetilde{C} \tag{6-6}$$

with a constant $\widetilde{C}>0$ independent of $\lambda$ and $\delta$. We now apply Theorem 2.1 to the operators $\mathcal{N}_j(\lambda)$. We get, for $\lambda\in\Lambda_\epsilon$, $|\mathrm{Im}\,\lambda|\ge C_\delta\gg 1$, $\mathrm{Re}\,\lambda\ge C_{\epsilon,\delta}\gg 1$, that

$$\|hT(\lambda)-\mathrm{Op}_h(\tau_\delta)\|_{L^2(\Gamma)\to L^2(\Gamma)}\le C\delta \tag{6-7}$$

in the anisotropic case, and

$$\|hT(\lambda)-\mathrm{Op}_h(\tau_\delta)\|_{L^2(\Gamma)\to H_h^1(\Gamma)}\le C\delta \tag{6-8}$$

in the isotropic case, where $C>0$ is a constant independent of $\lambda$ and $\delta$. We introduce the operators

$$\mathcal{A}_1(\lambda)=(hT(\lambda)-\mathrm{Op}_h(\tau_\delta))\mathrm{Op}_h(\tau_\delta)^{-1},$$
$$\mathcal{A}_2(\lambda)=\mathrm{Op}_h(\tau_\delta)^{-1}(hT(\lambda)-\mathrm{Op}_h(\tau_\delta)).$$

It follows from (6-6)–(6-8) that in the anisotropic case we have the bound

$$\|\mathcal{A}_1(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)}\le C'\delta, \tag{6-9}$$

while in the isotropic case we have the bound

$$\|\mathcal{A}_2(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)}\le C'\delta, \tag{6-10}$$

where $C'>0$ is a constant independent of $\lambda$ and $\delta$. Hence, taking $\delta$ small enough we can arrange that the operators $1+\mathcal{A}_j(\lambda)$ are invertible on $L^2(\Gamma)$ with inverses whose norms are bounded by 2. We now write the operator $hT(\lambda)$ as

$$hT(\lambda)=(1+\mathcal{A}_1(\lambda))\mathrm{Op}_h(\tau_\delta)$$

in the anisotropic case, and as

$$hT(\lambda)=\mathrm{Op}_h(\tau_\delta)(1+\mathcal{A}_2(\lambda))$$

in the isotropic case. Therefore, the operator $hT(\lambda)$ is invertible in the desired region and by (6-6) we get the bound

$$\|(hT(\lambda))^{-1}\|_{H_h^{(1-k)/2}(\Gamma)\to H_h^{(1+k)/2}(\Gamma)}\le 2\widetilde{C}. \tag{6-11}$$

Passing from semiclassical to classical Sobolev norms, one can easily see that (6-11) implies (6-2). □

## 7. Proof of Theorem 2.2

We keep the notations from the previous section. Theorem 2.2 is a consequence of the following:

**Theorem 7.1.** *Under the conditions of Theorem 2.2, there exists a constant $C>0$ such that the operator $T(\lambda): H^1(\Gamma)\to L^2(\Gamma)$ is invertible for $\operatorname{Re}\lambda\ge 1$ and $|\operatorname{Im}\lambda|\ge C\log(\operatorname{Re}\lambda+1)$ with an inverse satisfying in this region the bound*

$$\|T(\lambda)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1. \tag{7-1}$$

*Proof.* As in the previous section we may suppose that $\lambda\in\Lambda_\epsilon$. We will again make use of the identity (6-3) with the difference that under the condition (2-6) we have $c_0(x')>0$ for all $x'\in\Gamma$. This means that $|\tau|$ can get small near the characteristic variety $\Sigma=\{(x',\xi')\in T^*\Gamma: r(x',\xi')=1\}$, where $r:=c_0r_0$. Clearly, the assumption (2-7) implies that $\Sigma_1\cap\Sigma_2=\varnothing$. This in turn implies that $\Sigma\cap\Sigma_j=\varnothing,\ j=1,2$. Indeed, if we suppose that there is a $\zeta^0\in\Sigma\cap\Sigma_j$ for $j=1$ or $j=2$, then it is easy to see that $\zeta^0\in\Sigma_1\cap\Sigma_2$, which however is impossible in view of (2-7). Therefore, we can choose a cut-off function $\chi^0\in C^\infty(T^*\Gamma)$ such that $\chi^0=1$ in a small neighbourhood of $\Sigma$, $\chi^0=0$ outside another small neighbourhood of $\Sigma$, and $\operatorname{supp}\chi^0\cap\Sigma_j=\varnothing,\ j=1,2$. This means that $\operatorname{supp}\chi^0$ belongs either to the hyperbolic region $\{r_j\le 1-\delta^2\}$ or to the elliptic region $\{r_j\ge 1+\delta^2\}$, provided $\delta>0$ is taken small enough. Therefore, we can use Propositions 4.1 and 5.1 to get the estimate

$$\left\|h\mathcal{N}_j(\lambda)\mathrm{Op}_h(\chi^0)-\mathrm{Op}_h(\rho_j\chi^0)\right\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim h+e^{-C|\operatorname{Im}\lambda|},$$

which implies

$$\left\|hT(\lambda)\mathrm{Op}_h(\chi^0)-\mathrm{Op}_h(\tau\chi^0)\right\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim h+e^{-C|\operatorname{Im}\lambda|}. \tag{7-2}$$

It follows from (6-3) that near $\Sigma$ the function $\tau$ is of the form $\tau=\tau_0(r-1-i\theta)$ with some smooth function $\tau_0\neq 0$. We now extend $\tau_0$ globally on $T^*\Gamma$ to a function $\tilde\tau_0\in S_0^0(\Gamma)$ such that $\tilde\tau_0=\tau_0$ on $\operatorname{supp}\chi^0$ and $|\tilde\tau_0|\ge\text{const.}>0$ on $T^*\Gamma$. Hence, we can write the operator $\mathrm{Op}_h(\tau\chi^0)$ as

$$\mathrm{Op}_h(\tau\chi^0)=\mathrm{Op}_h(\chi^0)\mathrm{Op}_h(\tilde\tau_0)(\mathcal{B}-i\theta)+\mathcal{O}(h),$$

where $\mathcal{B}=\frac12\mathrm{Op}_h(r-1)+\frac12\mathrm{Op}_h(r-1)^*$ is a self-adjoint operator. Hence

$$(\mathcal{B}-i\theta)^{-1}=\mathcal{O}(|\theta|^{-1}):L^2(\Gamma)\to L^2(\Gamma).$$

Since $\tilde\tau_0$ is globally elliptic, we also have

$$\mathrm{Op}_h(\tilde\tau_0)^{-1}=\mathcal{O}(1):L^2(\Gamma)\to L^2(\Gamma).$$

This implies

$$K_1:=\mathrm{Op}_h(\chi^0)(\mathcal{B}-i\theta)^{-1}\mathrm{Op}_h(\tilde\tau_0)^{-1}=\mathcal{O}(|\theta|^{-1}):L^2(\Gamma)\to L^2(\Gamma)$$

and (7-2) leads to the estimate

$$\|hT(\lambda)K_1-\mathrm{Op}_h(\chi^0)\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim|\theta|^{-1}(h+e^{-C|\operatorname{Im}\lambda|})\lesssim|\operatorname{Im}\lambda|^{-1}+\operatorname{Re}\lambda\, e^{-C|\operatorname{Im}\lambda|}\le\delta \tag{7-3}$$

for any $0 < \delta \ll 1$, provided $|\mathrm{Im}\,\lambda| \ge C_\delta \log(\mathrm{Re}\,\lambda)$, $\mathrm{Re}\,\lambda \ge \widetilde{C}_\delta$ with some constants $C_\delta, \widetilde{C}_\delta > 0$. On the other hand, by Theorem 1.2 we have, for $\lambda \in \Lambda_\epsilon$, $|\mathrm{Im}\,\lambda| \ge C_\delta \gg 1$, $\mathrm{Re}\,\lambda \ge C_{\epsilon,\delta} \gg 1$,

$$\left\| hT(\lambda)\mathrm{Op}_h(1-\chi^0) - \mathrm{Op}_h(\tau_\delta(1-\chi^0)) \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C\delta. \tag{7-4}$$

As in the proof of (6-5), one can see that the function $\tau_\delta$ satisfies

$$\widetilde{C}_1\langle\xi'\rangle \le |\tau_\delta| \le \widetilde{C}_2\langle\xi'\rangle \quad \text{on } \mathrm{supp}(1-\chi^0) \tag{7-5}$$

with positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ independent of $\delta$ and $\theta$. Moreover, $\tau_\delta \in S_0^1(\Gamma)$. We extend the function $\tau_\delta$ on the whole of $T^*\Gamma$ to a function $\tilde{\tau}_\delta \in S_0^1(\Gamma)$ such that $\tilde{\tau}_\delta(1-\chi^0) = \tau_\delta(1-\chi^0)$ and

$$\widetilde{C}_1'\langle\xi'\rangle \le |\tilde{\tau}_\delta| \le \widetilde{C}_2'\langle\xi'\rangle \quad \text{on } T^*\Gamma. \tag{7-6}$$

Hence

$$\|\mathrm{Op}_h(\tilde{\tau}_\delta)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \widetilde{C} \tag{7-7}$$

with a constant $\widetilde{C} > 0$ independent of $\lambda$ and $\delta$. By (7-4) and (7-7) we obtain

$$\left\| hT(\lambda)K_2 - \mathrm{Op}_h(1-\chi^0) \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \le C\delta \tag{7-8}$$

with a new constant $C > 0$ independent of $\lambda$ and $\delta$, where

$$K_2 := \mathrm{Op}_h(1-\chi^0)\mathrm{Op}_h(\tilde{\tau}_\delta)^{-1} = \mathcal{O}(1) : L^2(\Gamma) \to L^2(\Gamma).$$

By (7-3) and (7-8),

$$\left\| hT(\lambda)(K_1+K_2) - 1 \right\|_{L^2(\Gamma)\to L^2(\Gamma)} \le (C+1)\delta. \tag{7-9}$$

It follows from (7-9) that if $\delta$ is taken small enough, the operator $hT(\lambda)$ is invertible with an inverse satisfying the bound

$$\|(hT(\lambda))^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \le 2\|K_1\|_{L^2(\Gamma)\to L^2(\Gamma)} + 2\|K_2\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim |\theta|^{-1} + 1. \tag{7-10}$$

It is easy to see that (7-10) implies (7-1). ☐

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