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THE A_∞-PROPERTY OF THE KOLMOGOROV MEASURE

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We consider the Kolmogorov–Fokker–Planck operator

$$\mathcal{K} := \sum_{i=1}^{m} \partial_{x_i x_i} + \sum_{i=1}^{m} x_i \partial_{y_i} - \partial_t$$

in unbounded domains of the form

$$\Omega = \{(x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} \mid x_m > \psi(x, y, t)\}.$$

Concerning ψ and Ω, we assume that Ω is what we call an (unbounded) admissible Lip_K-domain: ψ satisfies a uniform Lipschitz condition, adapted to the dilation structure and the (non-Euclidean) Lie group underlying the operator $\mathcal{K}$, as well as an additional regularity condition formulated in terms of a Carleson measure. We prove that in admissible Lip_K-domains the associated parabolic measure is absolutely continuous with respect to a surface measure and that the associated Radon–Nikodym derivative defines an A_∞ weight with respect to this surface measure. Our result is sharp.

1. Introduction

A major breakthrough in the study of boundary value problems for the heat operator

$$\mathcal{H} := \sum_{i=1}^{m} \partial_{x_i x_i} - \partial_t, \tag{1-1}$$

in $\mathbb{R}^{m+1}$, $m \geq 1$, in (unbounded) Lipschitz-type domains

$$\Omega = \{(x, x_m, t) \in \mathbb{R}^{m+1} \mid x_m > \psi(x, t)\}, \tag{1-2}$$

was achieved in [Lewis and Silver 1988; Lewis and Murray 1995; Hofmann and Lewis 1996; Hofmann 1997]; see also [Hofmann and Lewis 2001b]. In these papers the correct notion of time-dependent Lipschitz-type cylinders, correct from the perspective of parabolic measure, parabolic singular integral operators, parabolic layer potentials, as well as from the perspective of the Dirichlet, Neumann and regularity problems with data in L^p for the heat operator, was found. In particular, in [Lewis and Silver 1988; Lewis and Murray 1995] the mutual absolute continuity of the parabolic measure, with respect to surface measure, and the A_∞-property was studied/established and in [Hofmann and Lewis 1996]

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the authors solved the Dirichlet, Neumann and regularity problems with data in L^2. The Neumann and regularity problems with data in L^p were considered in [Hofmann and Lewis 1999; 2005]. For further and related results concerning the fine properties of parabolic measures we refer to the impressive and influential work [Hofmann and Lewis 2001a].

The assumptions on the time-dependent function ψ underlying the analysis in all of the papers mentioned can be formulated as follows: there exist constants $0 < M_1, M_2 < \infty$ such that

$$|\psi(x,t) - \psi(\tilde{x},\tilde{t})| \le M_1(|x-\tilde{x}| + |t-\tilde{t}|^{1/2}) \tag{1-3}$$

whenever $(x,t), (\tilde{x},\tilde{t}) \in \mathbb{R}^m$ and such that

$$\sup_{(x,t)\in\mathbb{R}^m,\, r>0} r^{-(m+1)} \int_0^r \int_{B_\lambda(x,t)} (\gamma_\psi(\tilde{x},\tilde{t},\lambda))^2 \,\frac{d\tilde{x}\, d\tilde{t}\, d\lambda}{\lambda} \le M_2. \tag{1-4}$$

In (1-4), $B_\lambda(x,t)$ is the parabolic ball centered at $(x,t)\in\mathbb{R}^m$, with radius λ, and

$$\gamma_\psi(\tilde{x},\tilde{t},\lambda) := \left(\lambda^{-(m+1)} \int_{B_\lambda(\tilde{x},\tilde{t})} \left| \frac{\psi(\bar{x},\bar{t}) - \psi(\tilde{x},\tilde{t}) - \mathcal{P}_\lambda(\nabla_x\psi)(\tilde{x},\tilde{t})(\bar{x}-\tilde{x})}{\lambda}\right|^2 d\bar{x}\, d\bar{t}\right)^{1/2}, \tag{1-5}$$

where $\mathcal{P} \in C_0^\infty(B_1(0,0))$ is a standard approximation of the identity, $\mathcal{P}_\lambda(x,t) = \lambda^{-(m+1)}\mathcal{P}(\lambda^{-1}x, \lambda^{-2}t)$, for $\lambda > 0$, and $\mathcal{P}_\lambda(\nabla_x\psi)$ denotes the convolution of $\nabla_x\psi$ with $\mathcal{P}_\lambda$. Inequality (1-3) is sufficient for the validity of the doubling property of the caloric/parabolic measure, while the additional regularity imposed through (1-4) is necessary and sufficient for the A_∞-property of caloric measure, with respect to the surface measure $d\sigma_t\, dt$, to hold: this is a consequence of [Lewis and Silver 1988; Lewis and Murray 1995; Hofmann 1997; Hofmann et al. 2003; 2004].

In this paper we initiate the corresponding developments for the Kolmogorov–Fokker–Planck operator

$$\mathcal{K} := \sum_{i=1}^m \partial_{x_i x_i} + \sum_{i=1}^m x_i \partial_{y_i} - \partial_t \tag{1-6}$$

in $\mathbb{R}^{N+1}$, $N = 2m$, $m \ge 1$, equipped with coordinates $(X,Y,t) := (x_1,\dots,x_m,y_1,\dots,y_m,t) \in \mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}$, in unbounded domains of the form

$$\Omega = \{(x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} \mid x_m > \psi(x,y,t)\}, \tag{1-7}$$

The function $\psi : \mathbb{R}^{m-1}\times\mathbb{R}^{m-1}\times\mathbb{R} \to \mathbb{R}$ is, for reasons to be explained, assumed to be independent of the variable y_m.

The operator $\mathcal{K}$, referred to as the Kolmogorov or Kolmogorov–Fokker–Planck operator plays a central role in many application in analysis, physics and finance. $\mathcal{K}$ was introduced and studied by Kolmogorov [1934] as an example of a degenerate parabolic operator having strong regularity properties. Kolmogorov proved that $\mathcal{K}$ has a fundamental solution $\Gamma = \Gamma(X,Y,t,\widetilde{X},\widetilde{Y},\tilde{t})$ which is smooth on the set $\{(X,Y,t) \ne (\widetilde{X},\widetilde{Y},\tilde{t})\}$. As a consequence,

$$\mathcal{K}u := f \in C^\infty \quad\Rightarrow\quad u \in C^\infty \tag{1-8}$$

for every distributional solution of $\mathcal{K}u = f$. The property in (1-8) can also be stated as

$$\mathcal{K} \text{ is hypoelliptic;} \tag{1-9}$$

see (2-3) below.

Kolmogorov was originally motivated by statistical physics and he studied $\mathcal{K}$ in the context of stochastic processes. Indeed, the fundamental solution $\Gamma(\,\cdot\,,\cdot\,,\cdot,\widetilde{X},\widetilde{Y},\tilde{t})$ describes the density of the stochastic process (X_t, Y_t), which solves the Langevin system

$$\begin{cases} dX_t = \sqrt{2}dW_t, & X_{\tilde{t}} = \widetilde{X}, \\ dY_t = X_t dt, & Y_{\tilde{t}} = \widetilde{Y}, \end{cases} \tag{1-10}$$

where W_t is an m-dimensional Wiener process. The system in (1-10) describes the density of a system with $2m$ degrees of freedom. Given $Z := (X, Y) \in \mathbb{R}^{2m}$, the variables $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_m)$ are, respectively, the velocity and the position of the system. The model in (1-10) and the equation in (1-6) are of fundamental importance in kinetic theory as they form the basis for Langevin-type models for particle dispersion, see [Bernardin et al. 2009; 2010; Chauvin et al. 2010; Bossy et al. 2011; Pope 2000], but they also appear in many other applied areas including finance [Barucci et al. 2001; Pascucci 2011], and vision [Citti and Sarti 2006; 2014].

In this paper we are concerned with the solvability of the Dirichlet problem for the operator $\mathcal{K}$ in unbounded domains of the form (1-7), and throughout the paper we will assume that Ω is a Lip_K-domain in the sense of Definition 1.1 below. Given $\varphi \in C(\partial\Omega)$ with compact support, we consider the boundary value problem

$$\begin{cases} \mathcal{K}u = 0 & \text{in } \Omega, \\ \quad u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1-11}$$

Using the Perron–Wiener–Brelot method one can prove the existence of a solution to this problem and, in the sequel, $u = u_\varphi$ will denote this solution to (1-11). Using the results of [Manfredini 1997], and assuming that Ω is a Lip_K-domain in the sense of Definition 1.1, it follows that all points on $\partial\Omega$ are regular for the Dirichlet problem for $\mathcal{K}$, i.e.,

$$\lim_{\substack{(Z,t)\to(Z_0,t_0)\\ (Z,t)\in\Omega}} u_\varphi(Z,t) = \varphi(Z_0,t_0) \quad \text{for any } \varphi \in C(\partial\Omega) \tag{1-12}$$

whenever $(Z_0, t_0) \in \partial\Omega$. Moreover, there exists, for every $(Z,t) \in \Omega$, a unique probability measure $\omega(Z,t,\cdot\,)$ on $\partial\Omega$ such that

$$u(Z,t) = \int_{\partial\Omega} \varphi(\widetilde{Z},\tilde{t})\, d\omega(Z,t,\widetilde{Z},\tilde{t}). \tag{1-13}$$

We refer to $\omega(Z,t,\cdot\,)$ as the Kolmogorov measure, or parabolic measure, associated to $\mathcal{K}$, relative to (Z,t) and Ω. In this paper we are particularly interested in scale- and translation-invariant estimates of $\omega(Z,t,\cdot\,)$ in terms of a (physical) surface measure, $d\sigma$, on $\partial\Omega$. In particular, assuming that Ω is an admissible Lip_K-domain in the sense of Definition 1.1 below, we establish a scale-invariant form of mutual absolute continuity of $\omega(Z,t,\cdot\,)$ with respect to $d\sigma$ on $\partial\Omega$.

Despite the relevance of the operator $\mathcal{K}$ to analysis, stochastics, physics, and in the applied sciences, the analysis of its properties is in several respects fundamentally underdeveloped. Indeed, geometry, the fine properties of the Dirichlet problem in (1-11) and the Kolmogorov measure, the boundary behavior of nonnegative solutions and the Green function, are currently only modestly studied and explored in the literature. One reason for this may be the intrinsic and intricate complexity built into the operator $\mathcal{K}$ through the lack of diffusion in the coordinates $(y_1, \dots, y_m)$ and through the presence of the lower-order drift term $\sum_{i=1}^m x_i \partial_{y_i} - \partial_t$. These two features of $\mathcal{K}$, which make this operator decisively different from the heat operator $\mathcal{H}$, have the consequence that the Lie group of translations $(\mathbb{R}^{N+1}, \circ)$ underlying $\mathcal{K}$ is different from the standard group of Euclidean translations and that already fundamental principles like the Harnack inequality and the construction of appropriate Harnack chains under geometrical restrictions become issues; see [Nyström and Polidoro 2016].

To briefly outline the current state of the literature, in our context, it is fair to mention that the first proof of the scale-invariant Harnack inequality, which constitutes one of the building blocks for our paper, can be found in [Garofalo and Lanconelli 1990]. In that paper the Harnack inequality is expressed in terms of level sets of the fundamental solution; hence it depends implicitly on the underlying Lie group structure. This fact was used in [Lanconelli and Polidoro 1994], where the group law, see (1-15) below, was used explicitly and the Harnack inequality, in the form we use it, was proved for the first time. The Perron–Wiener–Brelot method in the context of the Dirichlet problem in (1-11), as well as criteria based on which boundary points can be proved to be regular, were developed in the important work [Manfredini 1997]. In [Cinti et al. 2010; 2012; 2013], the author, together with Chiara Cinti and Sergio Polidoro, developed a number of important preliminary estimates concerning the boundary behavior of nonnegative solutions to equations of Kolmogorov–Fokker–Planck type in Lipschitz-type domains. These papers were the result of our ambition to establish scale- and translation-invariant boundary comparison principles, boundary Harnack inequalities and doubling properties of associated parabolic measures, results previously established for uniformly parabolic equations with bounded measurable coefficients in Lipschitz-type domains, see [Fabes and Safonov 1997; Fabes et al. 1986; 1999; Safonov and Yuan 1999; Nyström 1997; Salsa 1981], for nonnegative solutions to the equation $\mathcal{K}u = 0$ and for more general equations of Kolmogorov–Fokker–Planck type. In [Nyström and Polidoro 2016], the author together with Sergio Polidoro took the program started in [Cinti et al. 2010; 2012; 2013] a large step forward by establishing several new results concerning the boundary behavior of nonnegative solutions to the equation $\mathcal{K}u = 0$ near the noncharacteristic part of the boundary of local versions of the Lip_K-domains defined in Definition 1.1 below. Generalizations to more general operators of Kolmogorov–Fokker–Planck type were also discussed. In particular, in [Nyström and Polidoro 2016] scale- and translation-invariant quantitative estimates concerning the behavior, at the boundary, for nonnegative solutions vanishing on a portion of the boundary were proved as well as a scale- and translation-invariant doubling property of the Kolmogorov measure. The results in that paper are developed under the regularity condition stated below in (1-25) in Definition 1.1; in particular, for reasons that they explain in detail, the results, including the translation-invariant doubling property of the Kolmogorov measure, were derived using the assumption that the defining function for Ω in (1-7), ψ, was assumed to be independent of the variable y_m. This assumption gave us a crucial additional degree of freedom at our disposal when building Harnack chains to connect points: we could freely connect points in the x_m-variable, taking geometric restriction into account, accepting that the path in the y_m-variable will most likely not end up in "the right spot". This is one reason why we also in this paper consider domains which are constant in the y_m-direction.

The main achievement of this paper is that we take the analysis in [Nyström and Polidoro 2016] one step further by proving, see Theorem 1.6 below, that if Ω is an admissible Lip_K-domain with constants (M_1, M_2) in the sense of Definition 1.1 below, then ω is mutually absolutely continuous with respect to a (physical) surface measure σ on $\partial\Omega$ and $\omega \in A_\infty(\partial\Omega, d\sigma)$ with constants depending only on N, M_1 and M_2. This gives a generalization of [Lewis and Silver 1988; Lewis and Murray 1995] to the operator $\mathcal{K}$ and in the case of graphs which are independent of all y-variables, our assumptions coincide with the geometrical conditions underlying [Lewis and Silver 1988; Lewis and Murray 1995; Hofmann and Lewis 1996; Hofmann 1997]; see (1-3) and (1-4) above.

1A. *Notation.* The natural family of dilations for $\mathcal{K}$, denoted by $(\delta_r)_{r>0}$, on $\mathbb{R}^{N+1}$ is defined by

$$\delta_r(X, Y, t) = (rX, r^3Y, r^2t) \tag{1-14}$$

for $(X, Y, t) \in \mathbb{R}^{N+1}$, $r > 0$. Due to the presence of nonconstant coefficients in the drift term of $\mathcal{K}$, the usual Euclidean change of variable does not preserve the Kolmogorov equation. Instead the Lie group on $\mathbb{R}^{N+1}$ preserving $\mathcal{K}u = 0$ is defined by the group law

$$(\widetilde{Z}, \tilde{t}) \circ (Z, t) = (\widetilde{X}, \widetilde{Y}, \tilde{t}) \circ (X, Y, t) = (\widetilde{X} + X, \widetilde{Y} + Y - t\widetilde{X}, \tilde{t} + t) \tag{1-15}$$

whenever $(Z, t), (\widetilde{Z}, \tilde{t}) \in \mathbb{R}^{N+1}$. Note that

$$(Z, t)^{-1} = (X, Y, t)^{-1} = (-X, -Y - tX, -t), \tag{1-16}$$

and hence

$$(\widetilde{Z}, \tilde{t})^{-1} \circ (Z, t) = (\widetilde{X}, \widetilde{Y}, \tilde{t})^{-1} \circ (X, Y, t) = (X - \widetilde{X}, Y - \widetilde{Y} + (t - \tilde{t})\widetilde{X}, t - \tilde{t}) \tag{1-17}$$

whenever $(Z, t), (\widetilde{Z}, \tilde{t}) \in \mathbb{R}^{N+1}$. Given $(Z, t) = (X, Y, t) \in \mathbb{R}^{N+1}$ we let

$$\|(Z, t)\| = \|(X, Y, t)\| := |(X, Y)| + |t|^{1/2}, \quad |(X, Y)| = |X| + |Y|^{1/3}. \tag{1-18}$$

Note that $\|\delta_r(X, Y, t)\| = r\|(X, Y, t)\|$ when $(X, Y, t) \in \mathbb{R}^{N+1}$, $r > 0$. We define

$$d((Z, t), (\widetilde{Z}, \tilde{t})) := \tfrac{1}{2}\big(\|(\widetilde{Z}, \tilde{t})^{-1} \circ (Z, t)\| + \|(Z, t)^{-1} \circ (\widetilde{Z}, \tilde{t})\|\big). \tag{1-19}$$

Then, as discussed in the bulk of the paper, d is a symmetric quasidistance on $\mathbb{R}^{N+1}$. Based on d we introduce the balls

$$\mathcal{B}_r(Z, t) := \{(\widetilde{Z}, \tilde{t}) \in \mathbb{R}^{N+1} \mid d((\widetilde{Z}, \tilde{t}), (Z, t)) < r\} \tag{1-20}$$

for $(Z, t) \in \mathbb{R}^{N+1}$ and $r > 0$. The measure of the ball $\mathcal{B}_r(Z, t)$, denoted by $|\mathcal{B}_r(Z, t)|$, is approximately $r^{\boldsymbol{q}}$, where $\boldsymbol{q} := 4m + 2$, independent of (Z, t). Similarly, given $(z, t) = (x, y, t) \in \mathbb{R}^{N-1} = \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \times \mathbb{R}$ we let

$$\mathcal{B}_r(z, t) := \{(\tilde{z}, \tilde{t}) \in \mathbb{R}^{N-1} \mid d((\tilde{x}, 0, \tilde{y}, 0, \tilde{t}), (x, 0, y, 0, t)) < r\}. \tag{1-21}$$

The measure of the ball $\mathcal{B}_r(z,t)$, denoted by $|\mathcal{B}_r(z,t)|$, is approximately $r^{\boldsymbol{q}-4}$, independent of (z,t). With a slight abuse of notation we will by $\mathcal{B}_r(Z,t)$, note the capital Z, always denote a ball in $\mathbb{R}^{N+1}$, and by $\mathcal{B}_r(z,t)$, note the lowercase z, we will always denote a ball in $\mathbb{R}^{N-1}$.

1B. *Geometry.* Our geometrical setting is that of unbounded domains of the form

$$\Omega = \{(x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} \mid x_m > \psi(x,y,t)\}, \tag{1-22}$$

and here we define the restrictions that we impose on the function $\psi : \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}$. Let $\mathcal{P} \in C_0^\infty(\mathcal{B}_1(0,0))$, where $\mathcal{B}_1(0,0) \subset \mathbb{R}^{N-1}$, be a standard approximation of the identity. Let $\mathcal{P}_\lambda(x,y,t) = \lambda^{-(\boldsymbol{q}-4)}\mathcal{P}(\lambda^{-1}x, \lambda^{-3}y, \lambda^{-2}t)$ for $\lambda > 0$. Given a function f defined on $\mathbb{R}^{N-1}$ we let

$$\begin{aligned}\mathcal{P}_\lambda f(x,y,t) &:= \int_{\mathbb{R}^{N-1}} f(\bar{x},\bar{y},\bar{t})\,\mathcal{P}_\lambda\big((\bar{x},\bar{y},\bar{t})^{-1} \circ (x,y,t)\big)\,d\bar{x}\,d\bar{y}\,d\bar{t} \\ &= \int_{\mathbb{R}^{N-1}} f(\bar{x},\bar{y},\bar{t})\,\mathcal{P}_\lambda\big(x-\bar{x},\, y-\bar{y}+(t-\bar{t})\bar{x},\, t-\bar{t}\big)\,d\bar{x}\,d\bar{y}\,d\bar{t}.\end{aligned} \tag{1-23}$$

$\mathcal{P}_\lambda f$ represents a regularization of f. Given $(\tilde{z},\tilde{t}) \in \mathbb{R}^{N-1}$, $\lambda > 0$, we introduce

$$\gamma_\psi(\tilde{z},\tilde{t},\lambda) := \left(\lambda^{-(\boldsymbol{q}-4)} \int_{\mathcal{B}_\lambda(\tilde{z},\tilde{t})} \left|\frac{\psi(\bar{x},\bar{y},\bar{t}) - \psi(\tilde{x},\tilde{y},\tilde{t}) - \mathcal{P}_\lambda(\nabla_x\psi)(\tilde{x},\tilde{y},\tilde{t})(\bar{x}-\tilde{x})}{\lambda}\right|^2 d\bar{x}\,d\bar{y}\,d\bar{t}\right)^{1/2}. \tag{1-24}$$

We are now ready to formulate our conditions on $\psi : \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}$ and Ω.

Definition 1.1. Assume that there exist constants $0 < M_1, M_2 < \infty$ such that

$$|\psi(z,t) - \psi(\tilde{z},\tilde{t})| \le M_1 \|(\tilde{z},\tilde{t})^{-1} \circ (z,t)\| \tag{1-25}$$

whenever $(z,t), (\tilde{z},\tilde{t}) \in \mathbb{R}^{N-1}$ and such that

$$\sup_{(z,t)\in\mathbb{R}^{N-1},\, r>0} r^{-(\boldsymbol{q}-4)} \int_0^r \int_{\mathcal{B}_\lambda(z,t)} (\gamma_\psi(\tilde{z},\tilde{t},\lambda))^2 \frac{d\tilde{z}\,d\tilde{t}\,d\lambda}{\lambda} \le M_2. \tag{1-26}$$

Let $\Omega = \Omega_\psi$ be defined as in (1-22). We say that Ω, defined by a function ψ satisfying (1-25), is a Lip_K-domain with constant M_1. We say that Ω, defined by a function ψ satisfying (1-25) and (1-26), is an admissible Lip_K-domain with constants (M_1, M_2).

Remark 1.2. Inequality (1-25) implies

$$\begin{aligned}|\psi(x,y,t) - \psi(\tilde{x},y,t)| &\le M_1\|(x-\tilde{x},0,0)\| = M_1|x-\tilde{x}|, \\ |\psi(x,y,t) - \psi(x,\tilde{y},t)| &\le M_1\|(0,y-\tilde{y},0)\| = M_1|y-\tilde{y}|^{1/3}, \\ |\psi(x,y,t) - \psi(x,y,\tilde{t})| &\le M_1\|(0,(t-\tilde{t})x,(t-\tilde{t}))\| = M_1(|(t-\tilde{t})x|^{1/3} + |t-\tilde{t}|^{1/2})\end{aligned} \tag{1-27}$$

uniformly with respect to the remaining variables. From the perspective of dilations and translations, Lip_K-domains are, assuming y_m-independence, the natural replacement in the context of the operator $\mathcal{K}$ of the $\mathrm{Lip}\big(1, \frac{1}{2}\big)$-domains considered in the context of the heat operator.

Remark 1.3. Inequality (1-26) states that the measure

$$(\gamma_\psi(\tilde z, \tilde t, \lambda))^2 \, \frac{d\tilde z \, d\tilde t \, d\lambda}{\lambda}$$

is a Carleson measure on $\mathbb{R}^{N-1} \times \mathbb{R}_+$. In this paper we prove, from the perspective of the finer properties of the Kolmogorov measure, that admissible Lip_K-domains are, assuming y_m-independence, the natural replacements in the context of the operator $\mathcal{K}$ of the admissible time-varying domains discovered and explored in [Lewis and Murray 1995; Hofmann 1997; Hofmann and Lewis 1996; 2001b] in the context of the heat operator.

Remark 1.4. Assume that $\Omega = \Omega_\psi \subset \mathbb{R}^{N+1}$ is a Lip_K-domain, with constant M_1. We define a (physical) measure σ on $\partial\Omega$ as

$$d\sigma(X, Y, t) := \sqrt{1 + |\nabla_x \psi(x, y, t)|^2} \, dx \, dY \, dt, \quad (X, Y, t) \in \partial\Omega. \tag{1-28}$$

We will refer to σ as the surface measure on $\partial\Omega$.

1C. ***Statement of the main result.*** Given $\varrho > 0$ and $\Lambda > 0$, we let

$$A^+_{\varrho,\Lambda} = \big(0, \Lambda\varrho, 0, -\tfrac{2}{3}\Lambda\varrho^3, \varrho^2\big) \in \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}. \tag{1-29}$$

We let

$$A^+_{\varrho,\Lambda}(Z_0, t_0) = (Z_0, t_0) \circ A^+_{\varrho,\Lambda} \tag{1-30}$$

whenever $(Z_0, t_0) \in \mathbb{R}^{N+1}$. Using the main result of [Nyström and Polidoro 2016], see Lemma 4.12 below, one can prove the following theorem.

Theorem 1.5. *Assume that $\Omega = \Omega_\psi \subset \mathbb{R}^{N+1}$ is a (unbounded) Lip_K-domain with constant M_1. Then there exist $\Lambda = \Lambda(N, M_1)$, $1 \le \Lambda < \infty$, and $c = c(N, M_1)$, $1 \le c < \infty$, such that the following is true. Let $(Z_0, t_0) \in \partial\Omega$, $0 < \varrho_0 < \infty$. Then*

$$\omega\big(A^+_{c\varrho_0,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{2\varrho}(\widetilde Z_0, \tilde t_0)\big) \le c\,\omega\big(A^+_{c\varrho_0,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\varrho}(\widetilde Z_0, \tilde t_0)\big)$$

for all balls $\mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0)$, $(\widetilde Z_0, \tilde t_0) \in \partial\Omega$ such that $\mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0) \subset \mathcal{B}_{4\varrho_0}(Z_0, t_0)$.

The following is the main new result proved in this paper.

Theorem 1.6. *Assume that $\Omega \subset \mathbb{R}^{N+1}$ is an (unbounded) admissible Lip_K-domain with constants (M_1, M_2) in the sense of Definition 1.1. Then there exist $\Lambda = \Lambda(N, M_1)$, $1 \le \Lambda < \infty$, and $c = c(N, M_1)$, $1 \le c < \infty$, and $\tilde c = \tilde c(N, M_1, M_2)$, $1 \le \tilde c < \infty$, and $\eta = \eta(N, M_1, M_2)$, $0 < \eta < 1$, such that the following is true. Let $(Z_0, t_0) \in \partial\Omega$, $0 < \varrho_0 < \infty$. Then*

$$\tilde c^{-1}\bigg(\frac{\sigma(E)}{\sigma(\partial\Omega \cap \mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0))}\bigg)^{1/\eta} \le \frac{\omega(A^+_{c\varrho_0,\Lambda}(Z_0, t_0), E)}{\omega\big(A^+_{c\varrho_0,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0)\big)} \le \tilde c\bigg(\frac{\sigma(E)}{\sigma(\partial\Omega \cap \mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0))}\bigg)^{\eta}$$

whenever $E \subset \partial\Omega \cap \mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0)$ for some ball $\mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0)$, $(\widetilde Z_0, \tilde t_0) \in \partial\Omega$ such that $\mathcal{B}_\varrho(\widetilde Z_0, \tilde t_0) \subset \mathcal{B}_{\varrho_0}(Z_0, t_0)$.

Remark 1.7. A short formulation of the conclusion of Theorem 1.6 is that

$$\omega(A^+_{c\varrho_0,\Lambda}(Z_0,t_0),\cdot)\in A_\infty(\partial\Omega\cap\mathcal{B}_{\varrho_0}(Z_0,t_0),d\sigma)$$

for all $(Z_0,t_0)\in\partial\Omega,\ 0<\varrho_0<\infty$, and with constants independent of (Z_0,t_0) and ϱ_0.

Remark 1.8. Theorem 1.6 states that a sufficient condition for the conclusion that $\omega(A^+_{c\varrho_0,\Lambda}(Z_0,t_0),\cdot)\in A_\infty(\partial\Omega\cap\mathcal{B}_{\varrho_0}(Z_0,t_0),d\sigma)$ uniformly is that $\Omega\subset\mathbb{R}^{N+1}$ is an (unbounded) admissible Lip_K-domain with constants (M_1,M_2) in the sense of Definition 1.1. In fact, the condition in (1-26) in Definition 1.1 is also necessary in the following sense. Using [Lewis and Silver 1988; Hofmann et al. 2003] one can conclude that there exists a function $\psi:\mathbb{R}^{m-1}\times\mathbb{R}\to\mathbb{R}$ which satisfies (1-3) for some M_1, but violates (1-4) for all $M_2<\infty$, and such that the parabolic measure associated to the heat operator in

$$\{(x,x_m,t)\in\mathbb{R}^{m+1}\mid x_m>\psi(x,t)\},\tag{1-31}$$

denoted by $\omega_{\mathcal{H}}$, is singular with respect to the surface measure $d\sigma_t\,dt$. Obviously this ψ also satisfies (1-25) with constant M_1, but violates (1-26) for all $M_2<\infty$. Consider now the domain

$$\Omega:=\{(x,x_m,y,y_m,t)\in\mathbb{R}^{2m+1}\mid x_m>\psi(x,t)\},\tag{1-32}$$

which is constant as a function of (y,y_m). Using that solutions to $\mathcal{H}u=0$ also satisfy $\mathcal{K}u=0$, estimates for nonnegative solutions to $\mathcal{H}u=0$, see [Hofmann et al. 2004] for example, and Lemma 4.11, Theorem 4.8 and Theorem 4.9 stated below, it can then be proved that the Kolmogorov measure in Ω must be singular with respect to the surface measure $d\sigma$ defined in Remark 1.4.

1D. *Discussion of the proof.* To prove Theorem 1.6 it suffices to prove Theorem 5.1 below. To prove Theorem 5.1 we use, and expand on, results from [Nyström and Polidoro 2016] and we implement ideas similar to the ideas in the recent paper [Kenig et al. 2016], where similar types of results are established but in the context of elliptic measure and second-order elliptic operators in divergence form. The final part of our proof of Theorem 1.6 is based on a crucial square function estimate, Lemma 5.3 below. The lemma states that if $u(Z,t):=\omega(Z,t,S)$, where $S\subset\partial\Omega$ is a Borel set, and if $c=c(N,M_1)\ge 1$, then there exists $\tilde c=\tilde c(N,M_1,M_2),\ 1\le\tilde c<\infty$, such that

$$\iint_{T_{cQ_0}}\bigl(|\nabla_X u|^2\delta+|\nabla_Y u|^2\delta^5+|(X\cdot\nabla_Y-\partial_t)(u)|^2\delta^3\bigr)\,dZ\,dt\le\tilde c\sigma(Q_0),$$

where T_{cQ_0} is a Carleson box associated with cQ_0, where $Q_0\subset\Omega$ is a (dyadic) surface cube, see Section 5A and (5-13), and where $\delta=\delta(Z,t)$ is the relevant distance from $(Z,t)\in\Omega$ to $\partial\Omega$. To prove Lemma 5.3 and to enable partial integration, we use a Dahlberg–Kenig–Stein-type of mapping adapted to the underlying group law,

$$(w,w_m,y,y_m,t)\in U\to(w,w_m+P_{\gamma w_m}\psi(w,y,t),y,y_m,t),\tag{1-33}$$

where

$$U=\bigl\{(W,Y,t)=(w,w_m,y,y_m,t)\in\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}\bigm| w_m>0\bigr\}.\tag{1-34}$$

Then u satisfies $\mathcal{K}u = 0$ in Ω if and only if $v(W, Y, t) = u(w, w_m + P_{\gamma w_m}\psi(w, y, t), y, y_m, t)$ satisfies

$$\nabla_W \cdot (A\nabla_W v) + B \cdot \nabla_W v + \big((w, w_m + P_{\gamma w_m}\psi(w, y, t)) \cdot \nabla_Y - \partial_t\big)v = 0 \quad \text{in } U. \tag{1-35}$$

Using this change of coordinates, it turns out to be sufficient to prove Lemma 5.3 below for solutions to the equation in (1-35) and our proof explores, as a consequence of our assumptions on ψ and as discussed in Section 2, that the coefficients A and B are independent of the variable y_m and that A and B define certain Carleson measures on U; see (6-10) below.

1E. *Organization of the paper.* In Section 2 we give additional preliminaries and we discuss implications of the assumptions in (1-25) and (1-26). In particular, considering a Dahlberg–Kenig–Stein-type of mapping as in (1-33), (1-34), we prove, as a consequence of the assumptions on ψ, that certain measures defined based on ψ are Carleson measures; see Lemma 2.2. In Section 3 we discuss the Dirichlet problem (1-11). In Section 4 we state, and elaborate on, some crucial estimates from [Nyström and Polidoro 2016]. In Section 5 we prove Theorem 1.6, assuming the square function estimate referred to above. The proof of the square function estimate is then given in Section 6.

2. Preliminaries

As discussed in Section 1A, see (1-14)–(1-17), the natural family of dilations for $\mathcal{K}$, denoted by $(\delta_r)_{r>0}$, on $\mathbb{R}^{N+1}$, and the Lie group on $\mathbb{R}^{N+1}$ preserving $\mathcal{K}u = 0$ are different from standard parabolic dilations and Euclidean translations applicable in the context of the heat operator. Using the notation of Section 1A, the operator $\mathcal{K}$ is δ_r-homogeneous of degree two, i.e., $\mathcal{K} \circ \delta_r = r^2(\delta_r \circ \mathcal{K})$, for all $r > 0$, and the operator $\mathcal{K}$ can be expressed as

$$\mathcal{K} = \sum_{i=1}^{m} X_i^2 + X_0,$$

where

$$X_i := \partial_{x_i}, \quad i = 1, \dots, m, \qquad X_0 := \sum_{i=1}^{m} x_i \partial_{y_i} - \partial_t, \tag{2-1}$$

and the vector fields $X_1, \dots, X_m$ and X_0 are left-invariant with respect to the group law (1-15) in the sense that

$$X_i(u((\widetilde{Z}, \tilde{t}) \circ \cdot)) = (X_i u)((\widetilde{Z}, \tilde{t}) \circ \cdot), \quad i = 0, \dots, m, \tag{2-2}$$

for every $(\widetilde{Z}, \tilde{t}) \in \mathbb{R}^{N+1}$. Consequently,

$$\mathcal{K}(u((\widetilde{Z}, \tilde{t}) \circ \cdot)) = (\mathcal{K}u)((\widetilde{Z}, \tilde{t}) \circ \cdot).$$

Taking commutators we see that $[X_i, X_0] = \partial_{y_i}$ for $i \in \{1, \dots, m\}$ and that the vector fields $\{X_1, \dots, X_m, X_0\}$ generate the Lie algebra associated to the Lie group $(\mathbb{R}^{N+1}, \circ)$. In particular, (1-9) is equivalent to the Hörmander condition,

$$\operatorname{rank} \operatorname{Lie}(X_1, \dots, X_m, X_0)(Z, t) = N + 1 \quad \text{for all } (Z, t) \in \mathbb{R}^{N+1}; \tag{2-3}$$

see [Hörmander 1967]. Furthermore, while X_i represents a differential operator of order one, ∂_{y_i} acts as a third-order operator. This fact is also reflected in the dilations group $(\delta_r)_{r>0}$ defined above.

2A. ***A symmetric quasidistance.*** Recall the notation $\|(Z,t)\| = \|(X,Y,t)\|$ for $(Z,t) = (X,Y,t) \in \mathbb{R}^{N+1}$, introduced in (1-18). We recall the following pseudotriangular inequality: there exists a positive constant c such that

$$\|(Z,t)^{-1}\| \le c\|(Z,t)\|, \quad \|(Z,t)\circ(\widetilde{Z},\tilde{t})\| \le c(\|(Z,t)\| + \|(\widetilde{Z},\tilde{t})\|) \tag{2-4}$$

whenever $(Z,t), (\widetilde{Z},\tilde{t}) \in \mathbb{R}^{N+1}$. Using (2-4) it follows directly that

$$\|(\widetilde{Z},\tilde{t})^{-1}\circ(Z,t)\| \le c\,\|(Z,t)^{-1}\circ(\widetilde{Z},\tilde{t})\| \tag{2-5}$$

whenever $(Z,t), (\widetilde{Z},\tilde{t}) \in \mathbb{R}^{N+1}$. Furthermore, defining $d((Z,t),(\widetilde{Z},\tilde{t}))$ as in (1-19), and using (2-5), it follows that

$$\|(\widetilde{Z},\tilde{t})^{-1}\circ(Z,t)\| \sim d((Z,t),(\widetilde{Z},\tilde{t})) \sim \|(Z,t)^{-1}\circ(\widetilde{Z},\tilde{t})\| \tag{2-6}$$

with constants of comparison independent of $(Z,t), (\widetilde{Z},\tilde{t}) \in \mathbb{R}^{N+1}$. Again using (2-4) we also see that

$$d((Z,t),(\widetilde{Z},\tilde{t})) \le c\big(d((Z,t),(\widehat{Z},\hat{t})) + d((\widehat{Z},\hat{t}),(\widetilde{Z},\tilde{t}))\big) \tag{2-7}$$

whenever $(Z,t), (\widehat{Z},\hat{t}), (\widetilde{Z},\tilde{t}) \in \mathbb{R}^{N+1}$, and hence that d is a symmetric quasidistance. Based on d, in (1-20) we introduced the balls $\mathcal{B}_r(Z,t)$ for $(Z,t) \in \mathbb{R}^{N+1}$ and $r > 0$, and in (1-21) we introduced the balls $\mathcal{B}_r(z,t)$ for $(z,t) \in \mathbb{R}^{N-1}$ and $r > 0$. Note that

$$\begin{aligned} \mathcal{B}_r(Z,t) &= (Z,t)\circ\big\{(\widetilde{Z},\tilde{t}) \in \mathbb{R}^{N+1} \;\big|\; \|(\widetilde{Z},\tilde{t})\| + \|(\widetilde{Z},\tilde{t})^{-1}\| < r\big\}, \\ \mathcal{B}_r(z,t) &= (z,t)\circ\big\{(\tilde{z},\tilde{t}) \in \mathbb{R}^{N-1} \;\big|\; \|(\tilde{z},\tilde{t})\| + \|(\tilde{z},\tilde{t})^{-1}\| < r\big\}. \end{aligned} \tag{2-8}$$

We emphasize that throughout the paper we will stick to the convention that $\mathcal{B}_r(Z,t)$, with a capital Z, always denotes a ball in $\mathbb{R}^{N+1}$, and that $\mathcal{B}_r(z,t)$, with a lowercase z, always denotes a ball in $\mathbb{R}^{N-1}$.

2B. ***Geometry and Carleson measures.*** Assume ψ satisfies (1-25) and (1-26) for some constants $0 < M_1, M_2 < \infty$. Let $\gamma \in (0,1)$ and consider the change of coordinates/mapping

$$(W,Y,t) = (w, w_m, y, y_m, t) \in U \to (w, w_m + P_{\gamma w_m}\psi(w,y,t), y, y_m, t)$$

defined in (1-33), where $P_{\gamma w_m}\psi(x,y,t)$ is defined in (1-23), and where U is defined in (1-34). This mapping is a version of the Dahlberg–Kenig–Stein mapping used in elliptic and parabolic problems. The purpose of this section is to prove properties of this change of coordinates assuming that ψ satisfies (1-25) and (1-26). In particular, we prove that if ψ satisfies (1-25) and (1-26), then certain measures, naturally associated to $P_{\gamma w_m}\psi$, are Carleson measures. Throughout the section and the paper $\mathcal{P}$ will denote a parabolic approximation of the identity chosen based on a finite stock of functions and fixed throughout the paper. Let $\mathcal{P} \in C_0^\infty(\mathcal{B}_1(0,0))$, where $\mathcal{B}_1(0,0) \subset \mathbb{R}^{N-1}$, $\mathcal{P} \ge 0$ be real-valued, and $\int \mathcal{P}\,dz\,dt = 1$. We will assume, as we may by imposing a product structure on $\mathcal{P}$, that $\mathcal{P}$ is even in the sense that

$$\int x_i\mathcal{P}(z,t)\,dz\,dt = \int y_i\mathcal{P}(z,t)\,dz\,dt = \int t\mathcal{P}(z,t)\,dz\,dt = 0 \tag{2-9}$$

for $i \in \{1, \dots, m-1\}$. We set $\mathcal{P}_\lambda(z,t) = \mathcal{P}_\lambda(x,y,t) = \lambda^{-(q-4)}\mathcal{P}(\lambda^{-1}x, \lambda^{-3}y, \lambda^{-2}t)$ whenever $\lambda > 0$. Given $\mathcal{P}$, we let $\mathcal{P}_\lambda$ define a convolution operator as introduced in (1-23). Similarly, we will by $\mathcal{Q}_\lambda$ denote a generic approximation to the zero operator, not necessarily the same at each instance, but chosen from a finite set of such operators depending only on our original choice of $\mathcal{P}_\lambda$. In particular, $\mathcal{Q}_\lambda(z,t) = \mathcal{Q}_\lambda(x,y,t) = \lambda^{-(q-4)}\mathcal{Q}(\lambda^{-1}x, \lambda^{-3}y, \lambda^{-2}t)$, where $\mathcal{Q} \in C_0^\infty(\mathcal{B}_1(0,0))$, $\int \mathcal{Q}\,dz\,dt = 0$. We first prove the following lemma.

Lemma 2.1. *Let ψ be a function satisfying (1-25) for some constant $0 < M_1 < \infty$, let $\gamma \in (0,1)$ and let $P_{\gamma w_m}\psi$ be defined as above for $w_m > 0$. Let $\theta, \tilde\theta \ge 0$ be integers and let $(\phi_1, \dots, \phi_{m-1})$ and $(\tilde\phi_1, \dots, \tilde\phi_{m-1})$ denote multi-indices. Let $\ell := (\theta + |\phi| + 3|\tilde\phi| + 2\tilde\theta)$. Then*

$$\left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\big((w\cdot\nabla_y - \partial_t)^{\tilde\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\big)\right| \le c(m,l)\gamma^{1-(l-\theta)} w_m^{1-l} M_1 \tag{2-10}$$

whenever $(W, Y, t) \in U$.

Proof. We first consider the case $\theta = 1$, $\phi = 0$, $\tilde\phi = 0$, $\tilde\theta = 0$. In this case, simply using that $\mathcal{P}_{\gamma w_m}$ is an approximation of the identity operator, we see that (1-25) immediately implies

$$\left|\frac{\partial}{\partial w_m}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right| \le c(m)\gamma M_1. \tag{2-11}$$

By similar considerations we have

$$\begin{aligned} w_m^{\theta-1}\left|\frac{\partial^\theta}{\partial w_m^\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right| &\le c(m,l)\gamma M_1,\\ w_m^{|\phi|-1}\left|\frac{\partial^{|\phi|}}{\partial w^\phi}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right| &\le c(m,l)\gamma^{1-|\phi|} M_1,\\ w_m^{3|\tilde\phi|-1}\left|\frac{\partial^{|\tilde\phi|}}{\partial y^{\tilde\phi}}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right| &\le c(m,l)\gamma^{1-3|\tilde\phi|} M_1 \end{aligned} \tag{2-12}$$

whenever $\theta \ge 1$, $|\phi| \ge 1$, $|\tilde\phi| \ge 1$. Furthermore,

$$(\gamma w_m)(w\cdot\nabla_y - \partial_t)(\mathcal{P}_{\gamma w_m}\psi(w,y,t)) = (\gamma w_m)^{-1}(w\cdot\nabla_y\mathcal{P} - \partial_t\mathcal{P})_{\gamma w_m}\psi(w,y,t), \tag{2-13}$$

and hence, again arguing as above, we can conclude that

$$w_m^{2|\tilde\theta|-1}\left|(w\cdot\nabla_y - \partial_t)^{\tilde\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right| \le c(m,l)\gamma^{1-2\tilde\theta} M_1. \tag{2-14}$$

Combining the above, the lemma follows. □

Lemma 2.2. *Let ψ be a function satisfying (1-25) and (1-26) for some constants $0 < M_1, M_2 < \infty$, let $\gamma \in (0,1)$ and let $P_{\gamma w_m}\psi$ be defined as above for $w_m > 0$. Let $\theta, \tilde\theta \ge 0$ be integers and let $(\phi_1, \dots, \phi_{m-1})$ and $(\tilde\phi_1, \dots, \tilde\phi_{m-1})$ denote multi-indices. Let $\ell := (\theta + |\phi| + 3|\tilde\phi| + 2\tilde\theta)$. Let*

$$d\mu = d\mu(W,Y,t) := \left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\big((w\cdot\nabla_y - \partial_t)^{\tilde\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\big)\right|^2 w_m^{2l-3}\,dW\,dy\,dt, \tag{2-15}$$

defined on U. Then

$$\mu(U\cap\mathcal{B}_r)\le c(m,l,M_1,M_2)\gamma^{2-2(l-\theta)}r^{\mathbf{q}-1}$$

for all balls $\mathcal{B}_r=\mathcal{B}_r(Z_0,t_0)\subset\mathbb{R}^{N+1}$ centered on ∂U, $r>0$.

Proof. As in the proof of Lemma 2.1, we first consider the case $\theta=1,\ \phi=0,\ \tilde\phi=0,\ \tilde\theta=0$. Then

$$\frac{\partial}{\partial w_m}(\mathcal{P}_{\gamma w_m})(w,y,t)=\frac{1}{w_m}(\mathcal{Q}_{\gamma w_m})(w,y,t), \tag{2-16}$$

where $\mathcal{Q}$ is such that $\int w_i\mathcal{Q}_{\gamma w_m}(w,y,t)\,dw\,dy\,dt=0$ for all $i\in\{1,\dots,m-1\}$. Let

$$l^{\psi}_{(w,y,t)}(\bar w,\bar y,\bar t)=\psi(\bar w,\bar y,\bar t)-\psi(w,y,t)-\mathcal{P}_{w_m}(\nabla_w\psi)(w,y,t)(\bar w-w). \tag{2-17}$$

Then,

$$\begin{aligned}\left|\frac{\partial}{\partial w_m}(\mathcal{P}_{\gamma w_m}\psi)(w,y,t)\right|&=\frac{1}{w_m}\left|\int_{\mathbb{R}^{N-1}}\psi(\bar w,\bar y,\bar t)\,\mathcal{Q}_{\gamma w_m}((\bar w,\bar y,\bar t)^{-1}\circ(w,y,t))\,d\bar w\,d\bar y\,d\bar t\right|\\&\le\frac{1}{w_m}\left|\int_{\mathbb{R}^{N-1}}(l^{\psi}_{(w,y,t)}(\bar w,\bar y,\bar t))\,\mathcal{Q}_{\gamma w_m}((\bar w,\bar y,\bar t)^{-1}\circ(w,y,t))\,d\bar w\,d\bar y\,d\bar t\right|\\&\le c\gamma\gamma_\psi(z,t,cw_m)\end{aligned} \tag{2-18}$$

for some $c=c(m)$, $1\le c<\infty$. Hence, using (1-26) we have

$$\begin{aligned}\iint_{U\cap\mathcal{B}_r}\left|\frac{\partial}{\partial w_m}(\mathcal{P}_{\gamma w_m}\psi)(w,y,t)\right|^2w_m^{-1}\,dW\,dy\,dt&\le c\gamma^2\iint_{U\cap\mathcal{B}_r}(\gamma_\psi(w,y,t,cw_m))^2w_m^{-1}\,dW\,dy\,dt\\&\le cM_2\gamma^2r^{\mathbf{q}-1}\end{aligned} \tag{2-19}$$

for all balls $\mathcal{B}_r\subset\mathbb{R}^{N+1}$ centered on ∂U, $r>0$. By similar considerations, using (1-26), we have

$$\begin{aligned}&\iint_{U\cap\mathcal{B}_r}\left|\frac{\partial^\theta}{\partial w_m^\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right|^2w_m^{2\theta-3}\,dW\,dy\,dt\le c(m,l)\gamma^2r^{\mathbf{q}-1},\\&\iint_{U\cap\mathcal{B}_r}\left|\frac{\partial^{|\phi|}}{\partial w^\phi}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right|^2w_m^{2|\phi|-3}\,dW\,dy\,dt\le c(m,l)\gamma^{2-2|\phi|}r^{\mathbf{q}-1},\\&\iint_{U\cap\mathcal{B}_r}\left|\frac{\partial^{|\tilde\phi|}}{\partial y^{\tilde\phi}}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right|^2w_m^{6|\tilde\phi|-3}\,dW\,dy\,dt\le c(m,l)\gamma^{2-6|\tilde\phi|}r^{\mathbf{q}-1},\\&\iint_{U\cap\mathcal{B}_r}\left|(w\cdot\nabla_y-\partial_t)^{\tilde\theta}(\mathcal{P}_{\gamma w_m}\psi(w,y,t))\right|^2w_m^{4|\tilde\theta|-3}\,dW\,dy\,dt\le c(m,l)\gamma^{2-4\tilde\theta}r^{\mathbf{q}-1}\end{aligned} \tag{2-20}$$

for all balls $\mathcal{B}_r\subset\mathbb{R}^{N+1}$ centered on ∂U, $r>0$, whenever $\theta\ge1$, $|\phi|\ge2$, $|\tilde\phi|\ge1$, $\tilde\theta\ge1$. Combining the above, the lemma follows. □

Remark 2.3. Using Lemma 2.1 we see that there exists $\hat\gamma=\hat\gamma(m,M_1)\in(0,1)$ such that if $\gamma\in(0,\hat\gamma)$ then

$$\tfrac12\le1+\frac{\partial}{\partial w_m}(\mathcal{P}_{\gamma w_m}\psi)(w,y,t)\le\tfrac32$$

whenever $(w, w_m, y, y_m, t) \in U$. This implies, in particular, that the map $(w, w_m, y, y_m, t) \in U \to (w, w_m + P_{\gamma w_m}\psi(w, y, t), y, y_m, t)$ is one-to-one.

2C. *A Poincaré inequality.* We introduce the open cube

$$\mathcal{Q}_r(0,0) = \big\{(Z,t) = (X,Y,t) \in \mathbb{R}^{N+1} \;\big|\; |x_i| < r,\ |y_i| < r^3,\ |t| < r^2\big\}, \tag{2-21}$$

where $i \in \{1, \dots, m\}$. Given $(Z_0, t_0) \in \mathbb{R}^{N+1}$, we let $\mathcal{Q}_r(Z_0, t_0) = (Z_0, t_0) \circ \mathcal{Q}_r(0, 0)$. We will need the following Poincaré inequality.

Lemma 2.4. *Consider $\mathcal{Q}_r := \mathcal{Q}_r(Z_0, t_0) \subset \mathbb{R}^{N+1}$ and let p, $1 < p < \infty$, be given. Let u be a (smooth) function defined on $\mathcal{Q}_r$ and let E denote the mean value of u on $\mathcal{Q}_r$. Then there exists a constant $c = c(N, p)$, $1 \le c < \infty$, such that*

$$\iint_{\mathcal{Q}_r} |u - E|^p \, dZ\, dt \le c \iint_{\mathcal{Q}_r} \big(r^p |\nabla_X u|^p + r^{3p} |\nabla_Y u|^p + r^{2p} |X_0(u)|^p\big)\, dZ\, dt.$$

Proof. Assume first that $(Z_0, t_0) = (0, 0)$ and let $\tilde{u}$ be a (smooth) function defined on $\mathcal{Q}_r(0, 0)$. Then, using the mean value theorem and arguing, for example, as in the proof of Lemma 6.12 in [Lieberman 1996], we see that

$$\iint_{\mathcal{Q}_r(0,0)} |\tilde{u} - E(\tilde{u}, \mathcal{Q}_r(0,0))|^p \, dZ\, dt \le c(N,p) \iint_{\mathcal{Q}_r(0,0)} \big(r^p |\nabla_X \tilde{u}|^p + r^{3p} |\nabla_Y \tilde{u}|^p + r^{2p} |\partial_t \tilde{u}|^p\big)\, dZ\, dt, \tag{2-22}$$

where $E(\tilde{u}, \mathcal{Q}_r(0,0))$ denotes the mean value of $\tilde{u}$ on $\mathcal{Q}_r(0,0)$. Next, consider a function u defined on $\mathcal{Q}_r(Z_0, t_0)$ for some $(Z_0, t_0) \ne (0, 0)$. Let $\tilde{u}(Z, t) = u((Z_0, t_0) \circ (Z, t))$. Then $\tilde{u}$ is a function defined on $\mathcal{Q}_r(Z_0, t_0)$, $E(\tilde{u}, \mathcal{Q}_r(0,0)) = E(u, \mathcal{Q}_r(Z_0, t_0))$ and (2-22) applies to $\tilde{u}$. Applying (2-22) to $\tilde{u}$ and expressing the result in terms of u the conclusion of the lemma follows. □

2D. *Interior regularity.*

Lemma 2.5. *Assume that $\mathcal{K}u = 0$ in $\mathcal{B}_{2r} = \mathcal{B}_{2r}(Z_0, t_0) \subset \mathbb{R}^{N+1}$. Then there exists a constant $c = c(N)$, $1 \le c < \infty$, such that*

(i) $r^{\mathbf{q}} (\sup_{\mathcal{B}_r} |u|)^2 \le c \iint_{\mathcal{B}_{2r}} |u|^2 \, dZ\, dt$,

(ii) $\iint_{\mathcal{B}_r} |\nabla_X u|^2 \, dZ\, dt \le \dfrac{c}{r^2} \iint_{\mathcal{B}_{2r}} u^2 \, dZ\, dt$,

(iii) $\sup_{\mathcal{B}_r} \big(r|\nabla_X u| + r^3 |\nabla_Y u| + r^2 |X_0(u)|\big) \le c \sup_{\mathcal{B}_{2r}} |u|$.

Proof. For (i) and (iii) we refer to [Lanconelli and Polidoro 1994]; (ii) is an energy estimate which can be proved by standard arguments. □

Lemma 2.6. *Assume that $\mathcal{K}u = 0$ in $\mathcal{B}_{2r} = \mathcal{B}_{2r}(Z_0, t_0) \subset \mathbb{R}^{N+1}$. Let $\zeta \in C_0^\infty(\mathcal{B}_{2r})$ be such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ on $\mathcal{B}_r$, and such that $r|\nabla_X \zeta| + r^3 |\nabla_Y \zeta| + r^2 |X_0(\zeta)| \le c(N)$. Let $i \in \{1, \dots, m\}$. Then there exists a constant $c = c(N)$, $1 \le c < \infty$, such that*

(i) $\displaystyle\iint_{\mathcal{B}_{2r}} |\nabla_X(\partial_{y_i} u)|^2 \zeta^6 \, dZ\, dt \le \frac{c}{r^2} \iint_{\mathcal{B}_{2r}} |\partial_{y_i} u|^2 \zeta^4 \, dZ\, dt,$

(ii) $\displaystyle\iint_{\mathcal{B}_{2r}} |\partial_{y_i} u|^2 \zeta^4 \, dZ\, dt \le \frac{c}{r^2} \iint_{\mathcal{B}_{2r}} |X_0(u)|^2 \zeta^2 \, dZ\, dt + \frac{c}{r^4} \iint_{\mathcal{B}_{2r}} |\nabla_X u|^2 \, dZ\, dt,$

(iii) $\displaystyle\iint_{\mathcal{B}_{2r}} |X_0(u)|^2 \zeta^2 \, dZ\, dt \le \frac{c}{r^2} \iint_{\mathcal{B}_{2r}} |\nabla_X u|^2 \, dZ\, dt.$

Proof. Let

$$A_1 = \iint_{\mathcal{B}_{2r}} |\nabla_X(\partial_{y_i} u)|^2 \zeta^6 \, dZ\, dt, \quad A_2 = \iint_{\mathcal{B}_{2r}} |\partial_{y_i} u|^2 \zeta^4 \, dZ\, dt, \quad A_3 = \iint_{\mathcal{B}_{2r}} |X_0(u)|^2 \zeta^2 \, dZ\, dt. \tag{2-23}$$

As $\tilde u := \partial_{y_i} u$ solves $\mathcal{K}\tilde u = 0$, we see that (i) follows immediately from Lemma 2.5(ii) and its proof. To prove (ii) we first note, integrating by parts,

$$\begin{aligned} A_2 &= \iint_{\mathcal{B}_{2r}} (\partial_{y_i} u)\big(X_0(\partial_{x_i} u) - \partial_{x_i}(X_0(u)\big)\zeta^4 \, dZ\, dt \\ &= -\iint_{\mathcal{B}_{2r}} X_0(\partial_{y_i} u)(\partial_{x_i} u)\zeta^4 \, dZ\, dt - 4\iint_{\mathcal{B}_{2r}} (\partial_{y_i} u)(\partial_{x_i} u)\zeta^3 X_0(\zeta)\, dZ\, dt \\ &\qquad + \iint_{\mathcal{B}_{2r}} (\partial_{x_i y_i} u) X_0(u)\zeta^4 \, dZ\, dt + 4\iint_{\mathcal{B}_{2r}} (\partial_{y_i} u) X_0(u)\zeta^3 \partial_{x_i}(\zeta)\, dZ\, dt. \end{aligned} \tag{2-24}$$

Next, writing $X_0(\partial_{y_i} u) = \partial_{y_i}(X_0(u))$ and integrating by parts in the first term, we see that

$$\begin{aligned} A_2 &= \iint_{\mathcal{B}_{2r}} X_0(u)\partial_{y_i x_i} u \zeta^4 \, dZ\, dt + 4\iint_{\mathcal{B}_{2r}} X_0(u)\partial_{x_i} u \zeta^3 \partial_{y_i}\zeta \, dZ\, dt - 4\iint_{\mathcal{B}_{2r}} (\partial_{y_i} u)(\partial_{x_i} u)\zeta^3 X_0(\zeta)\, dZ\, dt \\ &\qquad + \iint_{\mathcal{B}_{2r}} (\partial_{x_i y_i} u) X_0(u)\zeta^4 \, dZ\, dt + 4\iint_{\mathcal{B}_{2r}} (\partial_{y_i} u) X_0(u)\zeta^3 \partial_{x_i}(\zeta)\, dZ\, dt. \end{aligned} \tag{2-25}$$

Using this we see that

$$\begin{aligned} A_2 &\le \iint_{\mathcal{B}_{2r}} |\nabla_X(\partial_{y_i} u)||X_0(u)|\zeta^4 \, dZ\, dt + \frac{c}{r^3}\iint_{\mathcal{B}_{2r}} |X_0(u)||\nabla_X u|\zeta^3 \, dZ\, dt \\ &\qquad + \frac{c}{r^2}\iint_{\mathcal{B}_{2r}} |\partial_{y_i} u||\nabla_X u|\zeta^3 \, dZ\, dt + \frac{c}{r}\iint_{\mathcal{B}_{2r}} |\partial_{y_i} u||X_0(u)|\zeta^3 \, dZ\, dt. \end{aligned} \tag{2-26}$$

Hence, using Cauchy–Schwarz we see that

$$A_2 \le \epsilon r^2 A_1 + \tilde\epsilon A_2 + c(\epsilon, \tilde\epsilon, n)\bigg(\frac{1}{r^2} A_3 + \frac{1}{r^4}\iint_{\mathcal{B}_{2r}} |\nabla_X u|^2 \, dZ\, dt\bigg), \tag{2-27}$$

where $\epsilon > 0$ and $\tilde\epsilon > 0$ are degrees of freedom. Furthermore, using the conclusion established in (i) we see that

$$A_2 \le c\epsilon A_2 + \tilde\epsilon A_2 + \tilde c(\epsilon, \tilde\epsilon, n)\bigg(\frac{1}{r^2} A_3 + \frac{1}{r^4}\iint_{\mathcal{B}_{2r}} |\nabla_X u|^2 \, dZ\, dt\bigg). \tag{2-28}$$

Part (ii) now follows by elementary manipulations. To prove (iii) we use the equation $\mathcal{K}u = 0$ and write

$$A_3 = -\sum_{i=1}^m \iint_{\mathcal{B}_{2r}} X_0(u)(\partial_{x_i x_i} u)\zeta^2 \, dZ\, dt = A_{31} + A_{32} + A_{33}, \tag{2-29}$$

where

$$A_{31} := 2\sum_{i=1}^{m}\iint_{\mathcal{B}_{2r}} X_0(u)(\partial_{x_i}u)\zeta\,\partial_{x_i}(\zeta)\,dZ\,dt,$$
$$A_{32} := \sum_{i=1}^{m}\iint_{\mathcal{B}_{2r}} X_0(\partial_{x_i}u)(\partial_{x_i}u)\zeta^2\,dZ\,dt, \tag{2-30}$$
$$A_{33} := \sum_{i=1}^{m}\iint_{\mathcal{B}_{2r}} (\partial_{y_i}u)(\partial_{x_i}u)\zeta^2\,dZ\,dt.$$

Then

$$|A_{31}|+|A_{33}| \le \epsilon A_3 + \tilde{\epsilon}r^2 A_1 + \frac{c(\epsilon,\tilde{\epsilon})}{r^2}\iint_{\mathcal{B}_{2r}} |\nabla_X u|^2\,dZ\,dt, \tag{2-31}$$

where $\epsilon > 0$ and $\tilde{\epsilon} > 0$ are degrees of freedom. To handle A_{32} we simply note, lifting the vector field X_0 by partial integration, that

$$2A_{32} = -2\sum_{i=1}^{m}\iint_{\mathcal{B}_{2r}} |\partial_{x_i}u|^2\zeta X_0(\zeta)\,dZ\,dt. \tag{2-32}$$

Hence,

$$A_3 \le \epsilon A_3 + \tilde{\epsilon}r^2 A_1 + \frac{c(\epsilon,\tilde{\epsilon})}{r^2}\iint_{\mathcal{B}_{2r}} |\nabla_X u|^2\,dZ\,dt. \tag{2-33}$$

Combining (2-33) and (i), (ii) of the lemma, we see that (iii) follows. □

Remark 2.7. To construct ζ as in the statement of Lemma 2.6, simply choose $\zeta(Z,t) := \tilde{\zeta}((Z_0,t_0)\circ(Z,t))$, where $\tilde{\zeta} \in C_0^\infty(\mathcal{B}_{2r}(0,0))$ is such that $0 \le \tilde{\zeta} \le 1$, $\tilde{\zeta} \equiv 1$ on $\mathcal{B}_r(0,0)$, and such that

$$r|\nabla_X\tilde{\zeta}| + r^3|\nabla_Y\tilde{\zeta}| + r^2|\partial_t\tilde{\zeta}| \le c(N).$$

We can construct $\tilde{\zeta}$ in a standard manner by smoothing out the indicator function of say $\mathcal{B}_{3r/2}(0,0)$.

Lemma 2.8. *Assume that $\mathcal{K}u = 0$ in $\mathcal{B}_{2r} = \mathcal{B}_{2r}(Z_0,t_0) \subset \mathbb{R}^{N+1}$. Let $i \in \{1,\dots,m\}$. Then there exists a constant $c = c(N)$, $1 \le c < \infty$, such that*

$$\iint_{\mathcal{B}_r} \big(r^4|\nabla_X(\partial_{y_i}u)|^2 + r^2|\nabla_Y u|^2 + |X_0(u)|^2\big)\,dZ\,dt \le \frac{c}{r^2}\iint_{\mathcal{B}_{2r}} |\nabla_X u|^2\,dZ\,dt.$$

Proof. The lemma is an immediate consequence of Lemma 2.6. □

3. The Dirichlet problem

Let $\Omega = \Omega_\psi \subset \mathbb{R}^{N+1}$ be an unbounded Lip_K-domain in the sense of Definition 1.1. We consider here the well-posedness of the boundary value problem

$$\begin{cases} \mathcal{K}u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

Note that we can without loss of generality assume that $\psi(0,0,0)=0$ and hence that $(0,0,0,0,0)\in\partial\Omega$. To conform with the notation used in [Nyström and Polidoro 2016] we let

$$\Omega_r := \Omega_{\psi,r} := \left\{(X,Y,t) \,\middle|\, |x_i|<r^2,\ |y_i|<r^3,\ |t|<2r^2,\ |y_m|<r^3,\ \psi(x,y,t)<x_m<4M_1 r\right\} \tag{3-2}$$

for $r>0$ and where $i\in\{1,\dots,m-1\}$. As outlined in Subsection 2.4 of that paper, using the Perron–Wiener–Brelot method, the existence of a solution to the problem in (3-1) with Ω replaced by Ω_r can be established. In Definition 3 of the same paper, we introduced what we here refer to as the Kolmogorov boundary of Ω_r, denoted by $\partial_K\Omega_r$. The notion of the Kolmogorov boundary replaces the notion of the parabolic boundary used in the context of uniformly parabolic equations and by definition $\partial_K\Omega_r\subset\partial\Omega_r$ is the set of all points on the topological boundary of Ω_r, which is contained in the closure of the propagation of at least one interior point in Ω_r. The importance of the Kolmogorov boundary of Ω_r is highlighted in the following lemma; see Lemma 2.2 in [Nyström and Polidoro 2016].

Lemma 3.1. *Consider the Dirichlet problem in (3-1), with Ω replaced by Ω_r, with boundary data $\varphi\in C(\partial\Omega_r)$ and let $u=u_\varphi$ be the corresponding Perron–Wiener–Brelot solution. Then*

$$\sup_{\Omega_r}|u|\le\sup_{\partial_K\Omega_r}|\varphi|.$$

In particular, if $\varphi\equiv 0$ on $\partial_K\Omega_r$ then $u\equiv 0$ in Ω_r.

The set $\partial_K\Omega_r$ is the largest subset of the topological boundary of Ω_r on which we can attempt to impose boundary data if we want to construct nontrivial solutions to the Dirichlet problem in (3-1), with Ω replaced by Ω_r. The notion of regular points on $\partial\Omega_r$ for the Dirichlet problem only makes sense for points on the Kolmogorov boundary and we let $\partial_R\Omega_r$ be the set of all $(z_0,t_0)\in\partial_K\Omega_r$ such that

$$\lim_{(Z,t)\to(Z_0,t_0)} u_\varphi(Z,t)=\varphi(Z_0,t_0)\quad\text{for any } \varphi\in C(\partial\Omega_r). \tag{3-3}$$

We refer to $\partial_R\Omega_r$ as the regular boundary of Ω_r with respect to the operator $\mathcal{K}$. By definition, $\partial_R\Omega_r\subseteq\partial_K\Omega_r$.

Lemma 3.2 [Nyström and Polidoro 2016, Lemma 2.2]. *Let $\Omega\subset\mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1 and let Ω_r be as defined in (3-2). Then*

$$\partial_R\Omega_r=\partial_K\Omega_r;$$

i.e., all points on the Kolmogorov boundary of Ω_r are regular for the operator $\mathcal{K}$.

Lemma 3.3. *Let $\Omega\subset\mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1, consider the Dirichlet problem in (3-1) and assume that $\varphi\in C(\partial\Omega)\cap L^\infty(\partial\Omega)$ is such that $\varphi(Z,t)\to 0$ as $\|(Z,t)\|\to\infty$. Then there exists a unique solution to the Dirichlet problem in (3-1) in Ω such that $u\in C(\bar\Omega)$, $u=\varphi$ on $\partial\Omega$. Furthermore, $\|u\|_{L^\infty(\Omega)}\le\|\varphi\|_{L^\infty(\Omega)}$.*

Proof. This can be proved by exhausting Ω with the bounded domains Ω_{r_j}, $j\in\mathbb{Z}_+$, $r_j=j$, for example, and by constructing u as the limit of $\{u_j\}$, where $\mathcal{K}u_j=0$ in Ω_{r_j} and with u_j having appropriate data on $\partial\Omega_{r_j}$. We here omit the routine details. □

Remark 3.4. The operator adjoint to $\mathcal{K}$ is

$$\mathcal{K}^* = \sum_{i=1}^{m} \partial_{x_i x_i} - \sum_{i=1}^{m} x_i \partial_{y_i} + \partial_t. \tag{3-4}$$

In the case of the adjoint operator $\mathcal{K}^*$ we denote the associated Kolmogorov boundary of Ω_r by $\partial_K^* \Omega_r$. The above discussion, lemmas and Lemma 3.2 then apply to $\mathcal{K}^*$ subject to the natural modifications.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\varphi \in C(\partial\Omega) \cap L^\infty(\partial\Omega)$ be such that $\varphi(Z,t) \to 0$ as $\|(Z,t)\| \to \infty$. Then there exist unique solutions $u = u_\varphi$, $u \in C(\bar{\Omega})$, and $u^* = u_{\varphi^*}$, $u^* \in C(\bar{\Omega})$, to the Dirichlet problem in (3-1) and to the corresponding Dirichlet problem for $\mathcal{K}^*$, respectively. Moreover, there exist, for every $(Z,t) \in \Omega$, unique probability measures $\omega(Z,t,\cdot)$ and $\omega^*(Z,t,\cdot)$ on $\partial\Omega$ such that*

$$u(Z,t) = \int_{\partial\Omega} \varphi(\tilde{Z},\tilde{t})\, d\omega(Z,t,\tilde{Z},\tilde{t}), \quad u^*(Z,t) = \int_{\partial\Omega} \varphi^*(\tilde{Z},\tilde{t})\, d\omega^*(Z,t,\tilde{Z},\tilde{t}). \tag{3-5}$$

Proof. The lemma is an immediate consequence of Lemma 3.2. □

Definition 3.6. Let $(Z,t) \in \Omega$. Then $\omega(Z,t,\cdot)$ is referred to as the Kolmogorov measure relative to (Z,t) and Ω, and $\omega^*(Z,t,\cdot)$ is referred to as the adjoint Kolmogorov measure relative to (Z,t) and Ω.

3A. *The fundamental solution and the Green function.* Following [Kolmogorov 1934] and [Lanconelli and Polidoro 1994], it is well known that an explicit fundamental solution, Γ, associated to $\mathcal{K}$ can be constructed. Indeed, let

$$B := \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}, \qquad E(s) = \exp(-sB^*)$$

for $s \in \mathbb{R}$, where I_m, 0, represent the identity matrix and the zero matrix in $\mathbb{R}^m$, respectively. Here * denotes the transpose. Furthermore, let

$$\mathcal{C}(t) := \int_0^t E(s) \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} E^*(s)\, ds = \begin{pmatrix} tI_m & -\frac{1}{2}t^2 I_m \\ -\frac{1}{2}t^2 I_m & \frac{1}{3}t^3 I_m \end{pmatrix}$$

whenever $t \in \mathbb{R}$. Note that $\det \mathcal{C}(t) = \frac{1}{12} t^{4m}$ and that

$$(\mathcal{C}(t))^{-1} = 12 \begin{pmatrix} \frac{1}{3}t^{-1} I_m & \frac{1}{2}t^{-2} I_m \\ \frac{1}{2}t^{-2} I_m & t^{-3} I_m \end{pmatrix}.$$

Using this notation, a fundamental solution, with pole at $(\tilde{Z},\tilde{t})$, $\Gamma(\cdot,\cdot,\tilde{Z},\tilde{t})$, can be defined by

$$\Gamma(Z,t,\tilde{Z},\tilde{t}) = \Gamma\big(Z - E(t-\tilde{t})\tilde{Z},\, t-\tilde{t}, 0, 0\big), \tag{3-6}$$

where $\Gamma(Z,t,0,0) = 0$ if $t \le 0$, $Z \ne 0$, and

$$\Gamma(Z,t,0,0) = \frac{(4\pi)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} \exp\big(-\tfrac{1}{4}\langle \mathcal{C}(t)^{-1} Z, Z\rangle\big) \quad \text{if } t > 0. \tag{3-7}$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard inner product on $\mathbb{R}^N$. We also note that

$$\Gamma(Z,t,\widetilde{Z},\tilde{t}) \le \frac{c(N)}{\|(\widetilde{Z},\tilde{t})^{-1}\circ(Z,t)\|^{\boldsymbol{q}-2}} \quad \text{for all } (Z,t),\,(\widetilde{Z},\tilde{t})\in\mathbb{R}^{N+1},\ t>\tilde{t}. \tag{3-8}$$

We define the Green function for Ω, with pole at $(\widehat{Z},\hat{t})\in\Omega$, as

$$G(Z,t,\widehat{Z},\hat{t}) = \Gamma(Z,t,\widehat{Z},\hat{t}) - \int_{\partial\Omega}\Gamma(\widetilde{Z},\tilde{t},\widehat{Z},\hat{t})\,d\omega(Z,t,\widetilde{Z},\tilde{t}), \tag{3-9}$$

where Γ is the fundamental solution to the operator $\mathcal{K}$ introduced in (3-6). If we instead consider $(Z,t)\in\Omega$ as fixed, then, for $(\widehat{Z},\hat{t})\in\Omega$,

$$G(Z,t,\widehat{Z},\hat{t}) = \Gamma(Z,t,\widehat{Z},\hat{t}) - \int_{\partial\Omega}\Gamma(Z,t,\widetilde{Z},\tilde{t})\,d\omega^*(\widehat{Z},\hat{t},\widetilde{Z},\tilde{t}), \tag{3-10}$$

where $\omega^*(\widehat{Z},\hat{t},\cdot)$ is the associated adjoint Kolmogorov measure relative to $(\widehat{Z},\hat{t})$ and Ω. Given $\theta\in C_0^\infty(\mathbb{R}^{N+1})$, we have the representation formulas

$$\begin{aligned}\theta(Z,t) &= \int_{\partial\Omega}\theta(\widetilde{Z},\tilde{t})\,d\omega(Z,t,\widetilde{Z},\tilde{t}) + \int G(Z,t,\widehat{Z},\hat{t})\,\mathcal{K}\theta(\widehat{Z},\hat{t})\,d\widehat{Z}\,d\hat{t},\\ \theta(\widehat{Z},\hat{t}) &= \int_{\partial\Omega}\theta(\widetilde{Z},\tilde{t})\,d\omega^*(\widehat{Z},\hat{t},\widetilde{Z},\tilde{t}) + \int G(Z,t,\widehat{Z},\hat{t})\,\mathcal{K}^*\theta(Z,t)\,dZ\,dt\end{aligned} \tag{3-11}$$

whenever $(Z,t),\,(\widehat{Z},\hat{t})\in\Omega$. In particular,

$$\begin{aligned}\int G(Z,t,\widehat{Z},\hat{t})\,\mathcal{K}\theta(\widehat{Z},\hat{t})\,d\widehat{Z}\,d\hat{t} &= -\int\theta(\widetilde{Z},\tilde{t})\,d\omega(Z,t,\widetilde{Z},\tilde{t}),\\ \int G(Z,t,\widehat{Z},\hat{t})\,\mathcal{K}^*\theta(Z,t)\,dZ\,dt &= -\int\theta(\widetilde{Z},\tilde{t})\,d\omega^*(\hat{x},\hat{t},\widetilde{Z},\tilde{t})\end{aligned} \tag{3-12}$$

whenever $\theta\in C_0^\infty(\mathbb{R}^{N+1}\setminus\{(Z,t)\})$ and $\theta\in C_0^\infty(\mathbb{R}^{N+1}\setminus\{(\widehat{Z},\hat{t})\})$, respectively.

Remark 3.7. Recall that $\boldsymbol{q}=4m+2$. However, we note that in [Nyström and Polidoro 2016] a different definition of $\boldsymbol{q}$ ($\boldsymbol{q}=4m$) was used. Hence, in this paper some statements containing $\boldsymbol{q}$ differ slightly compared to the corresponding statements in that paper.

4. Estimates for nonnegative solutions

In this section we develop and state a number of estimates concerning nonnegative solutions, the Kolmogorov measure as well as the kernel function. Throughout this section we assume that $\Omega=\Omega_\psi\subset\mathbb{R}^{N+1}$ is a Lip_K-domain, with constant M_1, in the sense of Definition 1.1. Given $\varrho>0$ and $\Lambda>0$ we let

$$\begin{aligned}A^+_{\varrho,\Lambda} &= \big(\Lambda\varrho,0,-\tfrac{2}{3}\Lambda\varrho^3,0,\varrho^2\big)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R},\\ A_{\varrho,\Lambda} &= (\Lambda\varrho,0,0,0,0)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R},\\ A^-_{\varrho,\Lambda} &= \big(\Lambda\varrho,0,\tfrac{2}{3}\Lambda\varrho^3,0,-\varrho^2\big)\in\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}.\end{aligned} \tag{4-1}$$

Given $(Z_0, t_0) \in \mathbb{R}^{N+1}$ we let

$$A^{\pm}_{\varrho,\Lambda}(Z_0, t_0) = (Z_0, t_0) \circ A^{\pm}_{\varrho,\Lambda}, \quad A_{\varrho,\Lambda}(Z_0, t_0) = (Z_0, t_0) \circ A_{\varrho,\Lambda}.$$

Furthermore, given $(Z_0, t_0) = (X_0, Y_0, t_0) = (x_0, \psi(x_0, y_0, t_0), Y_0, t_0) \in \partial\Omega_\psi$ and $r > 0$ we let $\Omega_r(Z_0, t_0) = \Omega_{\psi,r}(Z_0, t_0)$ be the set of all points $(X, Y, t) = (x, x_m, y, y_m, t)$ which satisfy the conditions

$$\begin{gathered} |x_i - x_{0,i}| < r, \quad |y_i - y_{0,i} + (t - t_0)x_{0,i}| < r^3 \quad \text{for } i \in \{1, \dots, m-1\}, \\ |t - t_0| < 2r^2, \quad \big|y_m - y_{0,m} + (t - t_0)\psi(x_0, y_0, t_0)\big| < r^3, \\ \psi(x, y, t) < x_m < 4M_1 r + \psi(x_0, y_0, t_0). \end{gathered} \tag{4-2}$$

Note that if we let $(\widetilde{X}, \widetilde{Y}, \tilde{t}) := (X_0, Y_0, t_0)^{-1} \circ (X, Y, t)$, and if we define

$$\tilde{\psi}(\tilde{x}, \tilde{y}, \tilde{t}) := \psi((x_0, y_0, t_0) \circ (\tilde{x}, \tilde{y}, \tilde{t})) - \psi(x_0, y_0, t_0),$$

then

$$\Omega_r(Z_0, t_0) = \Omega_{\psi,r}(Z_0, t_0) = \Omega_{\tilde{\psi},r}, \tag{4-3}$$

with $\Omega_{\tilde{\psi},r}$ defined as in (3-2). To be consistent with the notation used and the estimates proved in [Nyström and Polidoro 2016], we here simply note that there exists $c = c(N)$, $1 \le c < \infty$, such that

$$\Omega \cap \mathcal{B}_{r/c}(Z_0, t_0) \subset \Omega_r(Z_0, t_0) \subset \Omega \cap \mathcal{B}_{cr}(Z_0, t_0) \tag{4-4}$$

for all $(Z_0, t_0) \in \partial\Omega_\psi$, $r > 0$.

4A. ***The Harnack inequality.*** To formulate the Harnack inequality we first need to introduce some additional notation. We let, for $r > 0$ and $(Z_0, t_0) \in \mathbb{R}^{N+1}$,

$$Q^- = \big(B\big(\tfrac{1}{2}e_1, 1\big) \cap B\big(-\tfrac{1}{2}e_1, 1\big)\big) \times [-1, 0], \quad Q^-_r(Z_0, t_0) = (Z_0, t_0) \circ \delta_r(Q^-), \tag{4-5}$$

where e_1 is the unit vector pointing in the direction of x_m and $B\big(\frac{1}{2}e_1, 1\big)$ and $B\big(-\frac{1}{2}e_1, 1\big)$ are standard Euclidean balls of radius 1 in $\mathbb{R}^N$, centered at $\frac{1}{2}e_1$ and $-\frac{1}{2}e_1$, respectively. Similarly, we let

$$Q = \big(B\big(\tfrac{1}{2}e_1, 1\big) \cap B\big(-\tfrac{1}{2}e_1, 1\big)\big) \times [-1, 1], \quad Q_r(Z_0, t_0) = (Z_0, t_0) \circ \delta_r(Q). \tag{4-6}$$

Given $\alpha, \beta, \gamma, \theta \in \mathbb{R}$ such that $0 < \alpha < \beta < \gamma < \theta^2$, we set

$$\begin{aligned} \widetilde{Q}^+_r(Z_0, t_0) &= \big\{(x, t) \in Q^-_{\theta r}(Z_0, t_0) \mid t_0 - \alpha r^2 \le t \le t_0\big\}, \\ \widetilde{Q}^-_r(Z_0, t_0) &= \big\{(x, t) \in Q^-_{\theta r}(Z_0, t_0) \mid t_0 - \gamma r^2 \le t \le t_0 - \beta r^2\big\}. \end{aligned}$$

In the following we will formulate two versions of the Harnack inequality. The first version reads as follows and we refer to [Lanconelli and Polidoro 1994] for details and proofs.

Lemma 4.1. *There exist constants $c > 1$ and $\alpha, \beta, \gamma, \theta \in (0, 1)$, with $0 < \alpha < \beta < \gamma < \theta^2$, such that the following is true. Assume u is a nonnegative solution to $\mathcal{K}u = 0$ in $Q^-_r(Z_0, t_0)$ for some $r > 0$, $(Z_0, t_0) \in \mathbb{R}^{N+1}$. Then,*

$$\sup_{\widetilde{Q}^-_r(Z_0,t_0)} u \le c \inf_{\widetilde{Q}^+_r(Z_0,t_0)} u.$$

To formulate another version of the Harnack inequality we recall that the tool used to build Harnack chains is that of $\mathcal{K}$-admissible paths. A path $\gamma : [0, T] \to \mathbb{R}^{N+1}$ is called $\mathcal{K}$-admissible if it is absolutely continuous and satisfies

$$\frac{d}{d\tau}\gamma(\tau) = \sum_{j=1}^{m} \omega_j(\tau) X_j(\gamma(\tau)) + \lambda(\tau) X_0(\gamma(\tau)) \quad \text{for a.e. } \tau \in [0, T], \tag{4-7}$$

where $\omega_j \in L^2([0, T])$ for $j = 1, \dots, m$, and λ are nonnegative measurable functions. We say that γ connects $(Z, t) = (X, Y, t) \in \mathbb{R}^{N+1}$ to $(\tilde{Z}, \tilde{t}) = (\tilde{X}, \tilde{Y}, \tilde{t}) \in \mathbb{R}^{N+1}$, $\tilde{t} < t$, if $\gamma(0) = (Z, t)$ and $\gamma(T) = (\tilde{Z}, \tilde{t})$. When considering Kolmogorov operators in the domain $\mathbb{R}^N \times (T_0, T_1)$, it is well known that (2-3) implies the existence of a $\mathcal{K}$-admissible path γ for any points $(Z, t), (\tilde{Z}, \tilde{t}) \in \mathbb{R}^{N+1}$ with $T_0 < \tilde{t} < t < T_1$. Given a domain $\Omega \subset \mathbb{R}^{N+1}$, and a point $(Z, t) \in \Omega$, we let $A_{(Z,t)} = A_{(Z,t)}(\Omega)$ denote the set

$$\big\{(\tilde{Z}, \tilde{t}) \in \Omega \,\big|\, \exists \text{ a } \mathcal{K}\text{-admissible } \gamma : [0, T] \to \Omega \text{ connecting } (Z, t) \text{ to } (\tilde{Z}, \tilde{t})\big\},$$

and we define $\mathcal{A}_{(Z,t)} = \mathcal{A}_{(Z,t)}(\Omega) = \overline{A_{(Z,t)}(\Omega)}$. Here and in the sequel, $\mathcal{A}_{(Z,t)}(\Omega)$ is referred to as the propagation set of the point (Z, t) with respect to Ω. The presence of the drift term in $\mathcal{K}$ considerably changes the geometric structure of $A_{(Z,t)}(\Omega)$ and $\mathcal{A}_{(Z,t)}(\Omega)$ compared to the case of uniformly parabolic equations. The second version of the Harnack inequality reads as follows and we refer to [Cinti et al. 2010] for details and proofs.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a domain and let $(Z_0, t_0) \in \Omega$. Let K be a compact set contained in the interior of $A_{(Z_0,t_0)}(\Omega)$. Then there exists a positive constant c_K, depending only on Ω and K, such that*

$$\sup_K u \le c_K\, u(Z_0, t_0)$$

for every nonnegative solution u of $\mathcal{K}u = 0$ in Ω.

Remark 4.3. We emphasize, and this is different compared to the case of uniform parabolic equations, that the constants $\alpha, \beta, \gamma, \theta$ in Lemma 4.1 cannot be arbitrarily chosen. In particular, according to Lemma 4.2, the cylinder $\tilde{Q}_r^-(Z_0, t_0)$ has to be contained in the interior of the propagation set $\mathcal{A}_{(Z_0,t_0)}(Q_r^-(Z_0, t_0))$.

Several arguments in [Nyström and Polidoro 2016] involving the Harnack inequality explore that, by construction,

$$\gamma^+(\tau) = A^+_{(1-\tau)\varrho,\Lambda}(Z_0, t_0), \quad \gamma^-(\tau) = A^-_{(1-\tau)\varrho,\Lambda}(Z_0, t_0), \quad \tau \in [0, 1], \tag{4-8}$$

are $\mathcal{K}$-admissible paths; see Lemma 3.5–Lemma 3.8 in [Nyström and Polidoro 2016]. Here we state one of the results established in the same paper, which will be used in the forthcoming sections.

Lemma 4.4 [Nyström and Polidoro 2016, Lemma 3.9]. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Then there exist $\Lambda = \Lambda(N, M_1)$, $1 \le \Lambda < \infty$, and $c = c(N, M_1)$, $1 \le c < \infty$, and $\gamma = \gamma(N, M_1)$, $0 < \gamma < \infty$, such that the following is true. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Assume that u is a nonnegative solution to $\mathcal{K}u = 0$ in $\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$ and consider $\varrho, \tilde{\varrho}$, with $0 < \tilde{\varrho} \le \varrho < r/c$. Then*

$$\begin{aligned} u(A^+_{\tilde{\varrho},\Lambda}(Z_0, t_0)) &\le c(\varrho/\tilde{\varrho})^{\gamma} u(A^+_{\varrho,\Lambda}(Z_0, t_0)),\\ u(A^-_{\tilde{\varrho},\Lambda}(Z_0, t_0)) &\ge c^{-1}(\tilde{\varrho}/\varrho)^{\gamma} u(A^-_{\varrho,\Lambda}(Z_0, t_0)). \end{aligned} \tag{4-9}$$

Proof. Note that the lemma follows from the construction of Harnack chains along the paths in (4-8) and from Lemma 3.8 in [Nyström and Polidoro 2016]. For the details we refer to Lemma 3.9 in that paper and to Lemma 4.3 in [Cinti et al. 2013]. □

Remark 4.5. Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. The constants $\Lambda = \Lambda(N, M_1)$, $1 \le \Lambda < \infty$, and $c = c(N, M_1)$, $1 \le c < \infty$, referred to in Lemma 4.4, are fixed in Remark 3.7 in [Nyström and Polidoro 2016]. In particular, these constants are fixed so that the validity of Lemmas 3.5–Lemma 3.7 in that paper are ensured. In the following we also let Λ and c be determined accordingly.

4B. *Hölder continuity estimates and boundary comparison principles.*

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Let $\epsilon \in (0, 1)$ be given. Then there exists $c = c(N, M_1, \epsilon)$, $1 \le c < \infty$, such that following holds. Assume that u is a nonnegative solution to $\mathcal{K}u = 0$ in $\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$, vanishing continuously on $\partial\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$. Then*

$$\sup_{\Omega\cap\mathcal{B}_{r/c}(Z_0,t_0)} u \le \epsilon \sup_{\Omega\cap\mathcal{B}_{2r}(Z_0,t_0)} u. \tag{4-10}$$

Proof. This follows from Lemma 3.11 in [Nyström and Polidoro 2016]. □

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\Lambda = \Lambda(N, M_1)$ and $c = c(N, M_1)$ be in accordance with Remark 4.5. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Assume that u is a nonnegative solution to $\mathcal{K}u = 0$ in $\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$, vanishing continuously on $\partial\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$. Then*

$$u(Z, t) \le c u(A^+_{\varrho,\Lambda}(Z_0, t_0))$$

whenever $(Z, t) \in \Omega \cap \mathcal{B}_{2\varrho/c}(Z_0, t_0)$, $0 < \varrho < r/c$.

Proof. This is essentially Theorem 1.1 in [Cinti et al. 2013]. □

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\Lambda = \Lambda(N, M_1)$ and $c = c(N, M_1)$ be in accordance with Remark 4.5. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Assume that u is a nonnegative solution to $\mathcal{K}u = 0$ in $\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$, vanishing continuously on $\partial\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$. Let $\varrho_0 = r/c$,*

$$m^+ = u(A^+_{\varrho_0,\Lambda}(Z_0, t_0)), \quad m^- = u(A^-_{\varrho_0,\Lambda}(Z_0, t_0)), \tag{4-11}$$

and assume $m^- > 0$. Then there exist constants $c_1 = c_1(N, M_1)$, $1 \le c_1 < \infty$, and $c_2 = c_2(N, M_1, m^+/m^-)$, $1 \le c_2 < \infty$, such that if we let $\varrho_1 = \varrho_0/c_1$, then

$$u(Z, t) \le c_2 u(A_{\varrho,\Lambda}(\tilde{Z}_0, \tilde{t}_0)),$$

whenever $(Z, t) \in \Omega \cap \mathcal{B}_{\varrho/c_1}(\tilde{Z}_0, \tilde{t}_0)$, for some $0 < \varrho < \varrho_1$ and $(\tilde{Z}_0, \tilde{t}_0) \in \partial\Omega \cap \mathcal{B}_{\varrho_1}(Z_0, t_0)$.

Proof. Using (4-3) and (4-4), it is easily seen that the theorem is a consequence of Theorem 1.1 in [Nyström and Polidoro 2016]. □

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\Lambda = \Lambda(N, M_1)$ and $c = c(N, M_1)$ be in accordance with Remark 4.5. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Assume that u and v are nonnegative* *solutions to* $\mathcal{K}u = 0$ *in* Ω*, vanishing continuously on* $\partial\Omega \cap \mathcal{B}_{2r}(Z_0, t_0)$*. Let* $\varrho_0 = r/c$*,*

$$\begin{aligned} m_1^+ &= v(A^+_{\varrho_0,\Lambda}(Z_0, t_0)), \quad m_1^- = v(A^-_{\varrho_0,\Lambda}(Z_0, t_0)),\\ m_2^+ &= u(A^+_{\varrho_0,\Lambda}(Z_0, t_0)), \quad m_2^- = u(A^-_{\varrho_0,\Lambda}(Z_0, t_0)), \end{aligned} \tag{4-12}$$

and assume $m_1^-, m_2^- > 0$*. Then there exist constants* $c_1 = c_1(N, M_1)$*,* $c_2 = c_2(N, M_1, m_1^+/m_1^-, m_2^+/m_2^-)$*,* $1 \le c_1, c_2 < \infty$*, and* $\sigma = \sigma(N, M_1, m_1^+/m_1^-, m_2^+/m_2^-)$*,* $\sigma \in (0, 1)$*, such that if we let* $\varrho_1 = \varrho_0/c_1$*, then*

$$\left|\frac{v(Z,t)}{u(Z,t)} - \frac{v(\widetilde{Z},\tilde{t})}{u(\widetilde{Z},\tilde{t})}\right| \le c_2\left(\frac{d((Z,t),(\widetilde{Z},\tilde{t}))}{\varrho}\right)^{\sigma} \frac{v(A_{\varrho,\Lambda}(\widetilde{Z}_0,\tilde{t}_0))}{u(A_{\varrho,\Lambda}(\widetilde{Z}_0,\tilde{t}_0))},$$

whenever $(Z,t), (\widetilde{Z},\tilde{t}) \in \Omega \cap \mathcal{B}_{\varrho/c_1}(\widetilde{Z}_0,\tilde{t}_0)$*, for some* $0 < \varrho < \varrho_1$ *and* $(\widetilde{Z}_0,\tilde{t}_0) \in \partial\Omega \cap \mathcal{B}_{\varrho_1}(Z_0, t_0)$*.*

Proof. Again using (4-3) and (4-4) we see that the theorem is a special case of Theorem 1.2 in [Nyström and Polidoro 2016]. □

4C. *Doubling of parabolic measure and estimates of the kernel function.*

Lemma 4.10. *Let* $\Omega \subset \mathbb{R}^{N+1}$ *be a* Lip_K*-domain with constant* M_1*. Let* $\Lambda = \Lambda(N, M_1)$ *be in accordance with Remark 4.5. Let* $(Z_0, t_0) \in \partial\Omega$ *and* $r > 0$*. Then*

$$\omega\big(A^+_{r/c,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_r(Z_0, t_0)\big) \ge c^{-1}.$$

Proof. This is an immediate consequence of Lemma 4.6. □

Lemma 4.11. *Let* $\Omega \subset \mathbb{R}^{N+1}$ *be a* Lip_K*-domain with constant* M_1*. Let* $\Lambda = \Lambda(N, M_1)$ *be in accordance with Remark 4.5. Let* $(Z_0, t_0) \in \partial\Omega$ *and* $r > 0$*. Let* $\omega(Z, t, \cdot)$ *be the Kolmogorov measure relative to* $(Z, t) \in \Omega$ *and* Ω *and let* $G(Z, t, \cdot, \cdot)$ *be the adjoint Green function for* Ω *with pole at* (Z, t)*. Then there exists* $c = c(N, M_1)$*,* $1 \le c < \infty$*, such that*

(1) $c^{-1}r^{\boldsymbol{q}-2}G(Z, t, A^+_{r,\Lambda}(Z_0, t_0)) \le \omega(Z, t, \partial\Omega \cap \mathcal{B}_r(Z_0, t_0))$*,*

(2) $\omega(Z, t, \partial\Omega \cap \mathcal{B}_{r/c}(Z_0, t_0)) \le cr^{\boldsymbol{q}-2}G(Z, t, A^-_{r,\Lambda}(Z_0, t_0))$

whenever $(Z, t) \in \Omega$*,* $t - t_0 \ge cr^2$*.*

Proof. This is a consequence of Lemma 4.1 in [Nyström and Polidoro 2016]. However, we emphasize that in that paper the definition of $\boldsymbol{q}$ is different compared to the definition used in this paper; see Remark 3.7. Based on the $\boldsymbol{q}$ used in this paper, $\boldsymbol{q} = 4m + 2$, (i) and (ii) are the correct formulation of the corresponding inequalities in Lemma 4.1 in [Nyström and Polidoro 2016]. □

Lemma 4.12. *Let* $\Omega \subset \mathbb{R}^{N+1}$ *be a* Lip_K*-domain with constant* M_1*. Let* $\Lambda = \Lambda(N, M_1)$ *be in accordance with Remark 4.5. Let* $(Z_0, t_0) \in \partial\Omega$ *and* $r > 0$*. Then there exists* $c = c(N, M_1)$*,* $1 \le c < \infty$*, such that*

$$\omega\big(A^+_{r,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{2\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)\big) \le c\,\omega\big(A^+_{r,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)\big)$$

whenever $(\widetilde{Z}_0, \tilde{t}_0) \in \partial\Omega$*,* $\mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0) \subset \mathcal{B}_{r/c}(Z_0, t_0)$*.*

Proof. This is a consequence of one of the main results, Theorem 1.3, proved in [Nyström and Polidoro 2016]. However, for the convenience of the reader we include here a proof using the results stated above. Consider $(\widetilde{Z}_0, \tilde{t}_0) \in \partial\Omega,\ \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0) \subset \mathcal{B}_{r/C}(Z_0, t_0)$, where $C = C(N, M_1) \gg 1$ is a degree of freedom. Choosing $\mathcal{C}$ large enough and using Lemma 4.11(ii) we see that

$$\omega\big(A^+_{r,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{2\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)\big) \le \tilde{c}\tilde{r}^{\boldsymbol{q}-2} G\big(A^+_{r,\Lambda}(Z_0, t_0), A^-_{2c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0)\big)$$

for some $\tilde{c} = \tilde{c}(N, M_1),\ 1 \le \tilde{c} < \infty$. Let

$$m^+ = G(A^+_{r,\Lambda}(Z_0, t_0), A^+_{r/1000,\Lambda}(Z_0, t_0)), \quad m^- = G(A^+_{r,\Lambda}(Z_0, t_0), A^-_{r/1000,\Lambda}(Z_0, t_0)). \tag{4-13}$$

By elementary estimates and the Harnack inequality, see Lemma 4.4, we see that

$$\bar{c}^{-1} \le r^{\boldsymbol{q}-2} m^+ \le \bar{c}, \qquad r^{\boldsymbol{q}-2} m^- \le \bar{c} \tag{4-14}$$

for some $\bar{c} = \bar{c}(N, M_1),\ 1 \le \bar{c} < \infty$. To prove the lemma we intend to use the adjoint version of Theorem 4.8 and hence we need to establish a lower bound on $r^{\boldsymbol{q}-2} m^-$. To establish this lower bound we first use the adjoint version of Lemma 4.7 to conclude that there exists $c = c(N, M_1),\ 1 \le c < \infty$, such that

$$\sup\nolimits_{(Z,t)\in\Omega\cap\mathcal{B}_{r/C}(Z_0,t_0)} G\big(A^+_{r,\Lambda}(Z_0, t_0), (Z, t)\big) \le cm^-. \tag{4-15}$$

However,

$$\sup\nolimits_{(Z,t)\in\Omega\cap\mathcal{B}_{r/C}(Z_0,t_0)} G\big(A^+_{r,\Lambda}(Z_0, t_0), (Z, t)\big) \ge G\big(A^+_{r,\Lambda}(Z_0, t_0), A^+_{r/(100C),\Lambda}(Z_0, t_0)\big) \ge cr^{2-\boldsymbol{q}} \tag{4-16}$$

by elementary estimates. In particular, (4-14)–(4-16) imply that

$$c^{-1} \le m^+/m^- \le c \quad \text{for some } c = c(N, M_1),\ 1 \le c < \infty.$$

Using this, the adjoint version of Theorem 4.8, and the scale invariance of Theorem 4.8, we deduce that there exist $\tilde{c} = \tilde{c}(N, M_1),\ 1 \le \tilde{c} < \infty$, such that

$$G\big(A^+_{r,\Lambda}(Z_0, t_0), A^-_{2c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0)\big) \le \tilde{c} G\big(A^+_{r,\Lambda}(Z_0, t_0), A^+_{2c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0)\big), \tag{4-17}$$

provided $(\widetilde{Z}_0, \tilde{t}_0) \in \partial\Omega,\ \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0) \subset \mathcal{B}_{r/C}(Z_0, t_0)$. Finally, using the (adjoint) Harnack inequality, we here use the adjoint version of Lemma 4.4, and Lemma 4.11, we see that

$$\begin{aligned}\tilde{r}^{\boldsymbol{q}-2} G\big(A^+_{r,\Lambda}(Z_0, t_0), A^+_{2c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0)\big) &\le c\tilde{r}^{\boldsymbol{q}-2} G\big(A^+_{r,\Lambda}(Z_0, t_0), A^+_{\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0)\big) \\ &\le c^2 \omega\big(A^+_{r,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)\big)\end{aligned} \tag{4-18}$$

for some $c = c(N, M_1)$, $1 \le c < \infty$. Combining these, we can conclude the proof of Lemma 4.12. □

Lemma 4.13. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\Lambda = \Lambda(N, M_1)$ be in accordance with Remark 4.5. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Let $(\widetilde{Z}_0, \tilde{t}_0) \in \partial\Omega$ and $\tilde{r} > 0$ be such that $\mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0) \subset \mathcal{B}_r(Z_0, t_0)$. Then there exists $c = c(N, M_1),\ 1 \le c < \infty$, such that*

$$K(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), \bar{Z}, \bar{t}) := \lim_{\bar{r}\to 0} \frac{\omega(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), \partial\Omega \cap \mathcal{B}_{\bar{r}}(\bar{Z}, \bar{t}))}{\omega(A^+_{cr,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\bar{r}}(\bar{Z}, \bar{t}))} \tag{4-19}$$

exists for $\omega(A^+_{cr,\Lambda}(Z_0,t_0),\cdot)$ *a.e.* $(\bar Z,\bar t)\in\partial\Omega\cap\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)$, *and*

$$c^{-1}\le\omega\big(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)\big)K\left(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),\bar Z,\bar t\right)\le c \tag{4-20}$$

whenever $(\bar Z,\bar t)\in\partial\Omega\cap\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)$.

Proof. Using the Harnack inequality, see Lemma 4.4, we see that the only thing we have to prove is (4-20). To prove (4-20), consider $(\bar Z,\bar t)\in\partial\Omega\cap\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)$ and $\bar r\ll\tilde r$. Using Lemma 4.11 we see that there exists $c=c(N,M_1)$, $1\le c<\infty$, such that

$$\begin{aligned}c^{-1}\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A^+_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A^-_{\bar r,\Lambda}(\bar Z,\bar t))}&\le\frac{\omega(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}{\omega(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}\\&\le c\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A^-_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A^+_{\bar r,\Lambda}(\bar Z,\bar t))}.\end{aligned}\tag{4-21}$$

Furthermore, using the adjoint version of Theorem 4.8 and by arguing as in the proof of Lemma 4.12, we see that

$$\begin{aligned}c^{-1}\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}&\le\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A^+_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A^-_{\bar r,\Lambda}(\bar Z,\bar t))},\\ \frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A^-_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A^+_{\bar r,\Lambda}(\bar Z,\bar t))}&\le c\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}.\end{aligned}\tag{4-22}$$

Hence we can conclude that

$$\frac{\omega(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}{\omega(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}\sim\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))},\tag{4-23}$$

where $\sim$ means that the quotient between the expression on the left-hand side and the expression on the right-hand side is bounded from above and below by constants depending only on N, M_1. Next, using the boundary Harnack inequality for solutions to the adjoint equation, which is a consequence of the adjoint version of Theorem 4.9, we deduce that

$$\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\bar r,\Lambda}(\bar Z,\bar t))}\sim\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\tilde r/\tilde c,\Lambda}(\widetilde Z_0,\tilde t_0))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\tilde r/\tilde c,\Lambda}(\widetilde Z_0,\tilde t_0))}\tag{4-24}$$

for some $\tilde c=\tilde c(N,M_1)\gg 1$. Combining the inequalities in the last two displays we see that

$$\frac{\omega(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}{\omega(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}\sim\frac{G(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),A_{\tilde r/\tilde c,\Lambda}(\widetilde Z_0,\tilde t_0))}{G(A^+_{cr,\Lambda}(Z_0,t_0),A_{\tilde r/\tilde c,\Lambda}(\widetilde Z_0,\tilde t_0))}\tag{4-25}$$

for some $\tilde c=\tilde c(N,M_1)\gg 1$. Finally, using this and arguing by the same principles, using the doubling properties of ω and related estimates, we can conclude that

$$\frac{\omega(A^+_{c\tilde r,\Lambda}(\widetilde Z_0,\tilde t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}{\omega(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\bar r}(\bar Z,\bar t))}\sim\frac{1}{\omega(A^+_{cr,\Lambda}(Z_0,t_0),\partial\Omega\cap\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)))}\tag{4-26}$$

whenever $(\bar Z,\bar t)\in\mathcal{B}_{\tilde r}(\widetilde Z_0,\tilde t_0)$. Using this and (4-19), we deduce (4-20) by letting $\bar r\to 0$. □

Lemma 4.14. *Let $\Omega \subset \mathbb{R}^{N+1}$ be a Lip_K-domain with constant M_1. Let $\Lambda = \Lambda(N, M_1)$ be in accordance with Remark 4.5. Let $(Z_0, t_0) \in \partial\Omega$ and $r > 0$. Let $(\widetilde{Z}_0, \tilde{t}_0) \in \partial\Omega$ and $\tilde{r} > 0$ be such that $\mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0) \subset \mathcal{B}_r(Z_0, t_0)$. Then there exist $c = c(N, M_1)$, $1 \le c < \infty$, and $\tilde{c} = \tilde{c}(N, M_1)$, $1 \le \tilde{c} < \infty$, such that*

$$\tilde{c}^{-1}\omega(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), E) \le \frac{\omega(A^+_{cr,\Lambda}(Z_0, t_0), E)}{\omega(A^+_{cr,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0))} \le \tilde{c}\,\omega(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), E)$$

whenever $E \subset \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)$.

Proof. Consider $E \subset \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)$. Then, by definition

$$\omega(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), E) = \int_E K(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), \bar{Z}, \bar{t})\, d\omega(A^+_{cr,\Lambda}(Z_0, t_0), \bar{Z}, \bar{t}).$$

Hence, using Lemma 4.13 we see that

$$\omega\big(A^+_{cr,\Lambda}(Z_0, t_0), \partial\Omega \cap \mathcal{B}_{\tilde{r}}(\widetilde{Z}_0, \tilde{t}_0)\big)\, \omega(A^+_{c\tilde{r},\Lambda}(\widetilde{Z}_0, \tilde{t}_0), E) \sim \omega(A^+_{cr,\Lambda}(Z_0, t_0), E),$$

which is the statement to be proved. □

5. Proof of Theorem 1.6

In order to introduce some efficient notation, we will use the terminology of spaces of homogeneous type in the sense of [Coifman and Weiss 1971]. Indeed, assuming that $\Omega = \Omega_\psi \subset \mathbb{R}^{N+1}$ is a Lip_K-domain, with constant M_1, in the sense of Definition 1.1, we let

$$\Sigma := \partial\Omega = \big\{(x, x_m, y, y_m, t) \in \mathbb{R}^{N+1} \;\big|\; x_m = \psi(x, y, t)\big\}.$$

Then $(\Sigma, d, d\sigma)$ is a space of homogeneous type, with homogeneous dimension $\boldsymbol{q} - 1$. Furthermore, $(\mathbb{R}^{N+1}, d, dZ\,dt)$ is also a space of homogeneous type, but with homogeneous dimension $\boldsymbol{q}$.

5A. ***Dyadic grids, Whitney cubes and Carleson boxes.*** By the results in [Christ 1990] there exists what we here will refer to as a dyadic grid on Σ having a number of important properties in relation to d. To formulate this we introduce, for any $(Z, t) = (X, Y, t) \in \Sigma$ and $E \subset \Sigma$,

$$\mathrm{dist}((Z, t), E) := \inf\big\{d((Z, t), (\widetilde{Z}, \tilde{t})) \;\big|\; (\widetilde{Z}, \tilde{t}) \in E\big\}, \tag{5-1}$$

and we let

$$\mathrm{diam}(E) := \sup\big\{d((Z, t), (\widetilde{Z}, \tilde{t})) \;\big|\; (Z, t), (\widetilde{Z}, \tilde{t}) \in E\big\}. \tag{5-2}$$

Using [Christ 1990] we can conclude that there exist constants $\alpha > 0$, $\beta > 0$ and $c_* < \infty$ such that for each $k \in \mathbb{Z}$ there exists a collection of Borel sets, $\mathbb{D}_k$, which we will call cubes, such that

$$\mathbb{D}_k := \{Q^k_j \subset \Sigma \mid j \in \mathfrak{I}_k\}, \tag{5-3}$$

where $\mathfrak{I}_k$ denotes some index set depending on k, satisfying:

(i) $\Sigma = \bigcup_j Q^k_j$ for each $k \in \mathbb{Z}$.

(ii) If $m \ge k$ then either $Q^m_i \subset Q^k_j$ or $Q^m_i \cap Q^k_j = \varnothing$.

(iii) For each (j,k) and each $m<k$, there is a unique i such that $Q_j^k \subset Q_i^m$.

(iv) $\operatorname{diam}(Q_j^k) \le c_* 2^{-k}$.

(v) Each Q_j^k contains $\Sigma \cap \mathcal{B}_{\alpha 2^{-k}}(Z_j^k, t_j^k)$ for some $(Z_j^k, t_j^k) \in \Sigma$.

(vi) $\sigma\big(\big\{(Z,t) \in Q_j^k \,\big|\, \operatorname{dist}((Z,t), \Sigma \setminus Q_j^k) \le \varrho\, 2^{-k}\big\}\big) \le c_* \varrho^\beta \sigma(Q_j^k)$ for all k, j and for all $\varrho \in (0,\alpha)$.

Let us make a few remarks concerning this result and discuss some related notation and terminology. First, in the setting of a general space of homogeneous type, this result is due to Christ [1990], with the dyadic parameter $\frac{1}{2}$ replaced by some constant $\delta \in (0,1)$. In fact, one may always take $\delta = \frac{1}{2}$; see [Hofmann et al. 2017, proof of Proposition 2.12]. We shall denote by $\mathbb{D} = \mathbb{D}(\Sigma)$ the collection of all Q_j^k; i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k.$$

Note that (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $(Z_Q, t_Q) = (X_Q, Y_Q, t_Q) \in \Sigma$ and a ball $\mathcal{B}_r(Z_Q, t_Q)$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

$$\Sigma \cap \mathcal{B}_r(Z_Q, t_Q) \subset Q \subset \Sigma \cap \mathcal{B}_{cr}(Z_Q, t_Q) \tag{5-4}$$

for some uniform constant c. We will denote the associated surface ball by

$$\Delta_Q := \Sigma \cap \mathcal{B}_r(Z_Q, t_Q), \tag{5-5}$$

and we shall refer to the point (Z_Q, t_Q) as the center of Q. Given a dyadic cube $Q \subset \Sigma$, we define its γ dilate by

$$\gamma Q := \Sigma \cap \mathcal{B}_{\gamma \operatorname{diam}(Q)}(Z_Q, t_Q). \tag{5-6}$$

For a dyadic cube $Q \in \mathbb{D}_k$, we let $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the length of Q. Clearly, $\ell(Q) \approx \operatorname{diam}(Q)$. For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the dyadic generation to which Q belongs; i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$, thus, $\ell(Q) = 2^{-k(Q)}$. For any $Q \in \mathbb{D}(\Sigma)$, we set $\mathbb{D}_Q := \{Q' \in \mathbb{D} \mid Q' \subset Q\}$.

Using that also $(\mathbb{R}^{N+1}, d, dZ\,dt)$ is a space of homogeneous type, we see that we can partition Ω into a collection of (closed) dyadic Whitney cubes $\{I\}$, in the following denoted $\mathcal{W} = \mathcal{W}(\Omega)$, such that the cubes in $\mathcal{W}$ form a covering of Ω with nonoverlapping interiors, and which satisfy

$$4 \operatorname{diam}(I) \le \operatorname{dist}(4I, \Sigma) \le \operatorname{dist}(I, \Sigma) \le 40 \operatorname{diam}(I) \tag{5-7}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2) \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.} \tag{5-8}$$

Given $I \in \mathcal{W}$, we let $\ell(I)$ denote its size. Given $Q \in \mathbb{D}(\Sigma)$, we set

$$\mathcal{W}_Q := \big\{I \in \mathcal{W} \,\big|\, 100^{-1}\ell(Q) \le \ell(I) \le 100\,\ell(Q) \ \text{ and } \ \operatorname{dist}(I, Q) \le 100\,\ell(Q)\big\}. \tag{5-9}$$

We fix a small, positive parameter τ, and given $I \in \mathcal{W}$, we let

$$I^* = I^*(\tau) := (1+\tau)I \tag{5-10}$$

denote the corresponding "fattened" Whitney cube. Choosing τ small, we see that the cubes I^* will retain the usual properties of Whitney cubes, in particular that

$$\operatorname{diam}(I) \approx \operatorname{diam}(I^*) \approx \operatorname{dist}(I^*, \Sigma) \approx \operatorname{dist}(I, \Sigma).$$

We then define a Whitney region with respect to Q by setting

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*. \tag{5-11}$$

Finally, given $Q \in \mathbb{D}(\Sigma)$, $\gamma \geq 0$, we let

$$T_Q := \operatorname{int}\Big(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'} \Big) \tag{5-12}$$

denote the Carleson box associated to Q. Furthermore, given $\gamma \geq 1$ we let

$$T_{\gamma Q} := \operatorname{int}\Big(\bigcup_{Q' : Q' \cap (\gamma Q) \neq \varnothing} U_{Q'} \Big) \tag{5-13}$$

denote the Carleson set associated to the γ dilate of Q.

5B. ***Reduction of Theorem 1.6 to two key lemmas.*** Using Lemma 4.12, we see that to prove Theorem 1.6 it suffices to prove the following version of Theorem 1.6.

Theorem 5.1. *Assume that $\Omega \subset \mathbb{R}^{N+1}$ is an (unbounded) admissible Lip_K-domain with constants (M_1, M_2) in the sense of Definition 1.1. Then there exist $\Lambda = \Lambda(N, M_1)$, $1 \leq \Lambda < \infty$, and $c = c(N, M_1)$, $1 \leq c < \infty$, and $\tilde{c} = \tilde{c}(N, M_1, M_2)$, $1 \leq \tilde{c} < \infty$, and $\eta = \eta(N, M_1, M_2)$, $0 < \eta < 1$, such that the following is true. Let $Q_0 \in \mathbb{D}$, $\varrho_0 := l(Q_0)$ and let $\omega(\,\cdot\,) := \omega(A^+_{c\varrho_0,\Lambda}(Z_{Q_0}, t_{Q_0}), \cdot\,)$. Then*

$$\tilde{c}^{-1}\Big(\frac{\sigma(E)}{\sigma(Q)}\Big)^{1/\eta} \leq \frac{\omega(E)}{\omega(Q)} \leq \tilde{c}\Big(\frac{\sigma(E)}{\sigma(Q)}\Big)^{\eta}$$

whenever $E \subset Q$, $Q \in \mathbb{D}$, $Q \subseteq Q_0$.

The proof of Theorem 5.1 is based on the following lemmas.

Lemma 5.2. *Let $Q_0 \in \mathbb{D}$ and let $\omega(\,\cdot\,)$ be as in the statement of Theorem 1.6. Let $\kappa \gg 1$ be given and consider $\delta_0 \in (0, 1)$. Assume that $E \subset Q_0$ with $\omega(E) \leq \delta_0$. If $\delta_0 = \delta_0(N, M_1, \kappa)$ is chosen sufficiently small, then there exist a Borel set $S \subset \partial\Omega$, and a constant $c = c(N, M_1)$, $1 \leq c < \infty$, such that if we let $u(Z, t) := \omega(Z, t, S)$, then*

$$\kappa^2 \sigma(E) \leq c \iint_{T_{cQ_0}} \big(|\nabla_X u|^2 \delta + |\nabla_Y u|^2 \delta^5 + |X_0(u)|^2 \delta^3\big)\, dZ\, dt.$$

Here $\delta = \delta(Z, t)$ is the distance from $(Z, t) \in \Omega$ to Σ and T_{cQ_0} is the Carleson set associated to cQ_0 as defined in (5-13).

Lemma 5.3. *Let $Q_0 \in \mathbb{D}$ and let $\omega(\cdot)$ be as in the statement of Theorem 1.6. Let $u(Z,t) := \omega(Z,t,S)$ and c be as stated in Lemma 5.2. Then there exists $\tilde{c} = \tilde{c}(N, M_1, M_2)$, $1 \le \tilde{c} < \infty$, such that*

$$\iint_{T_{cQ_0}} \big(|\nabla_X u|^2\delta + |\nabla_Y u|^2\delta^5 + |X_0(u)|^2\delta^3\big)\, dZ\, dt \le \tilde{c}\sigma(Q_0).$$

The proof of Lemma 5.2 is given below. The proof of Lemma 5.3 is given in the next section. We here prove Theorem 5.1, hence completing the proof of Theorem 1.6, assuming Lemmas 5.2 and 5.3. Indeed, first using Lemmas 4.14 and 4.12 we see that it suffices to prove Theorem 5.1 with $Q = Q_0$. Then, using Lemmas 5.2 and 5.3 we see that we can, for $\Gamma \gg 1$ given, choose $\delta_0 = \delta_0(N, M_1, \Gamma)$ so that if $E \subset Q_0$ with $\omega(E) \le \delta_0$, then

$$\Gamma^2\sigma(E) \le \hat{c}\bar{\sigma}(Q_0) \tag{5-14}$$

for some $\hat{c} = \hat{c}(N, M_1, M_2)$, $1 \le \hat{c} < \infty$. In particular, we can conclude that there exists, for every $\epsilon > 0$, a positive $\delta_0 = \delta_0(N, M_1, M_2, \epsilon)$ such that

$$\omega(E) \le \delta_0 \le c\delta_0\omega(Q_0) \implies \sigma(E) \le \epsilon\sigma(Q_0), \tag{5-15}$$

where we have also applied Lemma 4.10. Theorem 1.6 now follows from the doubling property of ω, see Lemma 4.12, and the classical result in [Coifman and Fefferman 1974].

5C. *Good ϵ_0 covers.* Recall that in the following $\omega(\cdot) := \omega(A^+_{c\varrho_0,\Lambda}(Z_{Q_0}, t_{Q_0}), \cdot)$, $Q_0 \in \mathbb{D}$, $\varrho_0 := l(Q_0)$.

Definition 5.4. Let $E \subset Q_0$ be given, let $\epsilon_0 \in (0,1)$ and let k be an integer. A good ϵ_0 cover of E, of length k, is a collection $\{\mathcal{O}_l\}_{l=1}^k$ of nested (relatively) open subsets of Q_0, together with collections $\mathcal{F}_l = \{\Delta_i^l\}_i \subset Q_0$, $\Delta_i^l \in \mathbb{D}$, such that

$$E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset \cdots \subset \mathcal{O}_1 \subset Q_0, \tag{5-16}$$

$$\mathcal{O}_l = \bigcup_{\mathcal{F}_l} \Delta_i^l, \tag{5-17}$$

$$\omega(\mathcal{O}_l \cap \Delta_i^{l-1}) \le \epsilon_0\omega(\Delta_i^{l-1}) \quad \text{for all } \Delta_i^{l-1} \in \mathcal{F}_{l-1}. \tag{5-18}$$

Lemma 5.5. *Let $E \subset Q_0$ be given and consider $\epsilon_0 \in (0,1)$. There exists $\gamma = \gamma(N, M_1)$, $0 < \gamma \ll 1$, and $\Gamma = \Gamma(N, M_1)$, $1 \ll \Gamma$, such that if we let $\delta_0 = \gamma(\epsilon_0/\Gamma)^k$, and if $\omega(E) \le \delta_0$, then E has a good ϵ_0 cover of length k.*

Proof. Let $k \in \mathbb{Z}_+$ be given. Let γ, $0 < \gamma \ll 1$, and Γ, $1 \ll \Gamma$, be degrees of freedom to be chosen depending only on N and M_1. Let $\delta_0 = \gamma(\epsilon_0/\Gamma)^k$. Suppose that $\omega(E) \le \delta_0$. Using that ω is a regular Borel measure, we see that there exists a (relatively) open subset of Q_0, containing E, which we denote by $\mathcal{O}_{k+1}$, satisfying $\omega(\mathcal{O}_{k+1}) \le 2\omega(E)$. Using Lemma 4.10 and the Harnack inequality, see Lemma 4.4, we see that there exists $c = c(N, M_1)$, $1 \le c < \infty$, such that

$$\omega(\mathcal{O}_{k+1}) \le 2\delta_0 \le c\delta_0\omega(Q_0) \le \frac{1}{2}\Big(\frac{\epsilon_0}{\Gamma}\Big)^k\omega(Q_0) \tag{5-19}$$

if we let $\gamma := 2/c$. Let $f \in L^1_{\text{loc}}(\Sigma, d\omega)$, and let

$$M_\omega(f)(Z,t) := \sup_{\mathfrak{B}} \frac{1}{\omega(\mathcal{B}_r(\widetilde{Z},\tilde{t}))} \int_{\mathcal{B}_r(\widetilde{Z},\tilde{t})} f\, d\omega,$$

where $\mathfrak{B} = \{\mathcal{B}_r(\widetilde{Z},\tilde{t}) \mid (\widetilde{Z},\tilde{t}) \in \partial\Omega,\ (Z,t) \in \mathcal{B}_r(\widetilde{Z},\tilde{t})\}$, denote the Hardy–Littlewood maximal function of f, with respect to ω, and where the supremum is taken over all balls $\mathcal{B}_r(\widetilde{Z},\tilde{t})$, $(\widetilde{Z},\tilde{t}) \in \partial\Omega$, containing (Z,t). Set

$$\mathcal{O}_k := \{(Z,t) \in Q_0 \mid M_\omega(1_{\mathcal{O}_{k+1}}) \geq \epsilon_0/\bar{c}\},$$

where we let $\bar{c} = \bar{c}(N, M_1)$, $1 \leq \bar{c} < \infty$, denote the constant appearing in Lemma 4.12. Then, by construction, $\mathcal{O}_{k+1} \subset \mathcal{O}_k$, $\mathcal{O}_k$ is relatively open in Q_0 and $\mathcal{O}_k$ is properly contained in Q_0. As ω is doubling, see Lemma 4.12, $(2Q_0, d, \omega)$ is a space of homogeneous type, weak L^1 estimates for the Hardy–Littlewood maximal function apply and hence

$$\omega(\mathcal{O}_k) \leq \tilde{c}\,\frac{\bar{c}}{\epsilon_0}\,\omega(\mathcal{O}_{k+1}) \leq \frac{1}{2}\left(\frac{\epsilon_0}{\bar{c}}\right)^{k-1} \omega(Q_0), \tag{5-20}$$

if we let $\Gamma = \tilde{c}\bar{c}$ and where $\tilde{c} = \tilde{c}(N, M_1)$, $1 \leq \tilde{c} < \infty$. By definition and by the construction, see (i)–(iii) on page 1733, Q_0 can be dyadically subdivided, and we can select a collection $\mathcal{F}_k = \{\Delta_i^k\}_i \subset Q_0$, comprised of the cubes that are maximal with respect to containment in $\mathcal{O}_k$, and thus $\mathcal{O}_k := \bigcup_i \Delta_i^k$. Then, by the maximality of the cubes in $\mathcal{F}_k$, and by the doubling property of ω, we find that

$$\omega(\mathcal{O}_{k+1} \cap \Delta_i^k) \leq \epsilon_0 \omega(\Delta_i^k) \quad \text{for all } \Delta_i^k \in \mathcal{F}_k. \tag{5-21}$$

We now iterate this argument, to construct $\mathcal{O}_{j-1}$ from $\mathcal{O}_j$ for $2 \leq j \leq k$, just as we constructed $\mathcal{O}_k$ from $\mathcal{O}_{k+1}$. It is then a routine matter to verify that the sets $\mathcal{O}_1, \dots, \mathcal{O}_k$, form a good ϵ_0 cover of E. We omit further details. □

Remark 5.6. From now on we fix a small dyadic number $\eta = 2^{-k_0}$, where k_0 is to be chosen. Given $Q \in \mathbb{D}$, we consider the k_0-grandchildren of Q, i.e., the subcubes $Q' \subset Q$, $Q' \in \mathbb{D}$, with length $l(Q') = \eta l(Q)$. We let $\widetilde{Q}$ denote the particular such grandchild which contains the center of Q, (Z_Q, t_Q).

Remark 5.7. Given $Q \in \mathbb{D}$ we let $A_Q^+ = A^+_{cl(Q),\Lambda}(Z_Q, t_Q)$ and $A^+_{\widetilde{Q}} = A^+_{cl(\widetilde{Q}),\Lambda}(Z_{\widetilde{Q}}, t_{\widetilde{Q}})$, where $\widetilde{Q}$ was defined in Remark 5.6.

Remark 5.8. Consider the special case $\Delta := \Delta_i^l \in \mathcal{F}_l$; i.e., Δ is a cube arising in some good ϵ_0 cover. We then set $\tilde{\Delta}_i^l := \tilde{\Delta}$, where $\tilde{\Delta}$ is defined as in Remark 5.6, and we define

$$\widetilde{\mathcal{O}}_l := \bigcup_{\Delta_i^l \in \mathcal{F}_l} \tilde{\Delta}_i^l. \tag{5-22}$$

Remark 5.9. Let $E \subset Q$ and consider the setup of Lemma 5.5. We note that for every $(Z_0, t_0) \in E$ we have $(Z_0, t_0) \in \mathcal{O}_l$ for all $l = 1, 2, \dots, k$, and therefore there exists, for each l, a cube $\Delta_i^l = \Delta_i^l(Z_0, t_0) \in \mathcal{F}_l$ containing (Z_0, t_0). With (Z_0, t_0) fixed, we let $\hat{\Delta}_i^l = \hat{\Delta}_i^l(Z_0, t_0)$ denote the particular k_0-grandchild, as defined in Remark 5.6, of Δ_i^l that contains (Z_0, t_0).

5D. ***Proof of Lemma 5.2.*** To prove Lemma 5.2, let $\epsilon_0 > 0$ be a degree of freedom to be specified below and depending only on N, M_1, let $\delta_0 = \gamma(\epsilon_0/\Gamma)^k$ be as specified in Lemma 5.5, where k is to be chosen depending only on N, M_1 and κ. Consider $E \subset Q_0$ with $\omega(E) \le \delta_0$. Using Lemma 5.5, we see that E has a good ϵ_0 cover of length k, $\{\mathcal{O}_l\}_{l=1}^k$ with corresponding collections $\mathcal{F}_l = \{\Delta_i^l\}_i \subset Q_0$. Let $\{\widetilde{\mathcal{O}}_l\}_{l=1}^k$ be defined as in (5-22). Using this good ϵ_0 cover of E we let

$$F(Z,t) := \sum_{j=2}^{k} \chi_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}(Z,t),$$

where $\chi_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}$ is the indicator function for the set $\chi_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}$. Then F equals the indicator function of some Borel set $S \subset \Sigma$ and we let $u(Z,t) := \omega(Z,t,S)$. Consider

$$(Z_0,t_0) \in E \quad \text{and} \quad \text{an index } l \in \{1,\dots,k\}.$$

In the following let

$$\Delta_i^l \in \mathcal{F}_l \text{ be a cube in the collection } \mathcal{F}_l \text{ which contains } (Z_0,t_0).$$

Given $k_0 \in \mathbb{Z}_+$ we let

$$\tilde{\Delta}_i^l \text{ be the } k_0\text{-grandchild of } \Delta_i^l \text{ which contains } (Z_{\Delta_i^l}, t_{\Delta_i^l}).$$

With (Z_0,t_0) and Δ_i^l fixed, we let $\hat{\Delta}_i^l$ be defined as in Remark 5.9; i.e., we let

$$\hat{\Delta}_i^l \text{ be the } k_0\text{-grandchild of } \Delta_i^l \text{ which contains } (Z_0,t_0).$$

Finally, we let

$$\bar{\Delta}_i^l \text{ be the } k_0\text{-grandchild of } \hat{\Delta}_i^l \text{ which contains } (Z_{\hat{\Delta}_i^l}, t_{\hat{\Delta}_i^l}).$$

Hence, based on $(Z_0,t_0) \in E$ and an index $l \in \{1,\dots,k\}$, we have specified Δ_i^l, $\tilde{\Delta}_i^l$, $\hat{\Delta}_i^l$ and $\bar{\Delta}_i^l$ satisfying

$$\tilde{\Delta}_i^l \subset \Delta_i^l, \quad \bar{\Delta}_i^l \subset \hat{\Delta}_i^l \subset \Delta_i^l.$$

We let

$$A^+_{\tilde{\Delta}_i^l} = A^+_{cl(\tilde{\Delta}_i^l),\Lambda}(Z_{\tilde{\Delta}_i^l}, t_{\tilde{\Delta}_i^l}), \quad A^+_{\bar{\Delta}_i^l} = A^+_{cl(\bar{\Delta}_i^l),\Lambda}(Z_{\bar{\Delta}_i^l}, t_{\bar{\Delta}_i^l}). \tag{5-23}$$

We first intend to prove that there exists $\beta > 0$, depending only on N, M_1, κ, such that if ϵ_0 and $\eta = 2^{-k_0}$ are chosen sufficiently small, then

$$|u(A^+_{\tilde{\Delta}_i^l}) - u(A^+_{\bar{\Delta}_i^l})| \ge \beta. \tag{5-24}$$

To estimate $u(A^+_{\tilde{\Delta}_i^l})$ we write

$$u(A^+_{\tilde{\Delta}_i^l}) = \int_{Q_0\setminus\Delta_i^l} F(\bar{Z},\bar{t})\, d\omega(A^+_{\tilde{\Delta}_i^l}, \bar{Z},\bar{t}) + \int_{\Delta_i^l} F(\bar{Z},\bar{t})\, d\omega(A^+_{\tilde{\Delta}_i^l}, \bar{Z},\bar{t}) =: I + II. \tag{5-25}$$

Using Lemma 4.6 and the definition of $A^+_{\tilde{\Delta}_i^l}$ we see that

$$|I| \le \omega(A^+_{\tilde{\Delta}_i^l}, Q_0 \setminus \Delta_i^l) \le c\eta^\sigma \tag{5-26}$$

for some $c = c(N, M_1)$, $\sigma = \sigma(N, M_1) \in (0, 1)$. Furthermore, by the definition of F we see that

$$II = II_1 + II_2 + II_3, \tag{5-27}$$

where

$$\begin{aligned} II_1 &:= \sum_{j=2}^{l} \int_{\Delta_i^l} 1_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}\, d\omega(A^+_{\tilde{\Delta}_i^l}, \bar{Z}, \bar{t}),\\ II_2 &:= \sum_{j=l+2}^{k} \int_{\Delta_i^l} 1_{\widetilde{\mathcal{O}}_{j-1}\setminus\mathcal{O}_j}\, d\omega(A^+_{\tilde{\Delta}_i^l}, \bar{Z}, \bar{t}),\\ III_3 &:= \int_{\Delta_i^l} 1_{\widetilde{\mathcal{O}}_{l}\setminus\mathcal{O}_{l+1}}\, d\omega(A^+_{\tilde{\Delta}_i^l}, \bar{Z}, \bar{t}). \end{aligned} \tag{5-28}$$

Note that if $j \le l$, then $\Delta_i^l \subset \mathcal{O}_l \subset \mathcal{O}_j$ and $(\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j) \cap \Delta_i^l = \varnothing$. Hence $II_1 = 0$. Obviously,

$$|II_2| \le \sum_{j=l+2}^{k} \omega\big(A^+_{\tilde{\Delta}_i^l}, (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j) \cap \Delta_i^l\big) \le c_\eta \sum_{j=l+2}^{k} \omega\big(A^+_{\Delta_i^l}, (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j) \cap \Delta_i^l\big), \tag{5-29}$$

where we in the second estimate have used the Harnack inequality; see Lemma 4.4. Consider $(\bar{Z}, \bar{t}) \in (\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j) \cap \Delta_i^l$. Then, using Lemma 4.13 we have

$$K(A^+_{\Delta_i^l}, \bar{Z}, \bar{t}) := \lim_{\varrho \to 0} \frac{\omega(A^+_{\Delta_i^l}, \partial\Omega \cap \mathcal{B}_\varrho(\bar{Z}, \bar{t}))}{\omega(\partial\Omega \cap \mathcal{B}_\varrho(\bar{Z}, \bar{t}))} \tag{5-30}$$

exists for ω a.e. $(\bar{Z}, \bar{t}) \in \Delta_i^l$, and

$$K(A^+_{\Delta_i^l}, \bar{Z}, \bar{t}) \le \frac{c}{\omega(\Delta_i^l)} \quad \text{whenever } (\bar{Z}, \bar{t}) \in \Delta_i^l. \tag{5-31}$$

In the last conclusion we have also used Lemma 4.12. Using these facts, and using the definition of the good ϵ_0 cover, we see that

$$\begin{aligned} |II_2| &\le \frac{c_\eta}{\omega(\Delta_i^l)} \sum_{j=l+2}^{k} \omega((\widetilde{\mathcal{O}}_{j-1} \setminus \mathcal{O}_j) \cap \Delta_i^l)\\ &\le \frac{c_\eta}{\omega(\Delta_i^l)} \sum_{j=l+2}^{k} \omega(\mathcal{O}_{j-1} \cap \Delta_i^l) \le \frac{c_\eta}{\omega(\Delta_i^l)} \sum_{j=l+2}^{k} \epsilon_0^{j-1-l} \omega(\Delta_i^l) \le c_\eta \epsilon_0. \end{aligned} \tag{5-32}$$

To estimate the term II_3 we first observe that $\Delta_i^l \cap \widetilde{\mathcal{O}}_l = \tilde{\Delta}_i^l$ by the definition of $\widetilde{\mathcal{O}}_l$. Hence,

$$\begin{aligned} II_3 &= \omega(A^+_{\tilde{\Delta}_i^l}, \Delta_i^l \cap \widetilde{\mathcal{O}}_l) - \omega(A^+_{\tilde{\Delta}_i^l}, \Delta_i^l \cap \mathcal{O}_{l+1})\\ &= \omega(A^+_{\tilde{\Delta}_i^l}, \tilde{\Delta}_i^l) - \omega(A^+_{\tilde{\Delta}_i^l}, \Delta_i^l \cap \mathcal{O}_{l+1}) =: II_{31} + II_{32}. \end{aligned} \tag{5-33}$$

Arguing as in (5-32), we see that

$$|II_{32}| \le \frac{c_\eta}{\omega(\Delta_i^l)} \omega(\Delta_i^l \cap \mathcal{O}_{l+1}) \le C_\eta \epsilon_0, \tag{5-34}$$

by the construction. Putting these together we can conclude that so far we have proved that

$$|u(A^+_{\tilde\Delta^l_i}) - II_{31}| \le c\eta^\sigma + c_\eta\epsilon_0, \tag{5-35}$$

and it remains to analyze II_{31}. However, using Lemma 4.10, and elementary estimates, we see that there exists $\tilde c = \tilde c(N, M_1)$, $1 \le \tilde c < \infty$, such that

$$\tilde c^{-1} \le II_{31} \le 1 - \tilde c^{-1}.$$

Combining the last result and (5-35) we can conclude, by first choosing $\eta = \eta(N, M_1)$ small and then $\epsilon_0 = \epsilon_0(N, M_1, \eta)$ small, that

$$\tfrac{3}{4}\tilde c^{-1} \le u(A^+_{\tilde\Delta^l_i}) \le 1 - \tfrac{3}{4}\tilde c^{-1}. \tag{5-36}$$

To estimate $u(A^+_{\hat\Delta^l_i})$ we write

$$u(A^+_{\hat\Delta^l_i}) = \int_{Q_0\setminus\hat\Delta^l_i} F(\bar Z,\bar t)\, d\omega(A^+_{\hat\Delta^l_i}, \bar Z, \bar t) + \int_{\hat\Delta^l_i} F(\bar Z,\bar t)\, d\omega(A^+_{\hat\Delta^l_i}, \bar Z, \bar t) =: \hat I + \widehat{II}. \tag{5-37}$$

We split $\widehat{II}$ as

$$\widehat{II} = \widehat{II}_1 + \widehat{II}_2 + \widehat{II}_3, \tag{5-38}$$

where

$$\begin{aligned} \widehat{II}_1 &:= \sum_{j=2}^{l} \int_{\hat\Delta^l_i} 1_{\tilde{\mathcal O}_{j-1}\setminus\mathcal O_j}\, d\omega(A^+_{\hat\Delta^l_i}, \bar Z, \bar t,), \\ \widehat{II}_2 &:= \sum_{j=l+2}^{k} \int_{\hat\Delta^l_i} 1_{\tilde{\mathcal O}_{j-1}\setminus\mathcal O_j}\, d\omega(A^+_{\hat\Delta^l_i}, \bar Z, \bar t), \\ \widehat{II}_3 &:= \int_{\hat\Delta^l_i} 1_{\tilde{\mathcal O}_{l}\setminus\mathcal O_{l+1}}\, d\omega(A^+_{\hat\Delta^l_i}, \bar Z, \bar t). \end{aligned} \tag{5-39}$$

We can now conclude, by essentially repeating the estimates in the corresponding estimate for $u(A^+_{\tilde\Delta^l_i})$ above, that

$$|\hat I| + |\widehat{II}_1| + |\widehat{II}_2| \le c\eta^\sigma + c_\eta\epsilon_0. \tag{5-40}$$

Note that the key estimate is now the kernel estimate

$$K(A^+_{\hat\Delta^l_i}, \bar Z, \bar t) \le \frac{c}{\omega(\Delta^l_i)} \quad \text{whenever } (\bar Z, \bar t) \in \Delta^l_i, \tag{5-41}$$

which again follows from Lemma 4.13. We now focus on $\widehat{II}_3$ and we observe that

$$\widehat{II}_3 = \omega(A^+_{\hat\Delta^l_i}, \hat\Delta^l_i \cap (\tilde{\mathcal O}_l \setminus \mathcal O_{l+1})) = \omega(A^+_{\hat\Delta^l_i}, (\hat\Delta^l_i \cap \tilde\Delta^l_i) \setminus \mathcal O_{l+1}), \tag{5-42}$$

and that either $(\hat\Delta^l_i \cap \tilde\Delta^l_i) = \varnothing$ or $\hat\Delta^l_i = \tilde\Delta^l_i$. If $(\hat\Delta^l_i \cap \tilde\Delta^l_i) = \varnothing$, then $\widehat{II}_3 = 0$. If $\hat\Delta^l_i = \tilde\Delta^l_i$, then

$$\widehat{II}_3 = \omega(A^+_{\hat\Delta^l_i}, \hat\Delta^l_i) - \omega(A^+_{\hat\Delta^l_i}, \hat\Delta^l_i \setminus \mathcal O_{l+1}) =: \widehat{II}_{31} + \widehat{II}_{32}. \tag{5-43}$$

Also in this case

$$\widehat{II}_{32} \le c_\eta\epsilon_0,$$

and we are left with $\widehat{II}_{31}$. However, by Lemma 4.6 we deduce that

$$\widehat{II}_{31} = \omega(A^+_{\tilde\Delta^l_i}, \hat\Delta^l_i) \geq 1 - c\eta^\sigma,$$

where $c = c(N, M_1)$, $\sigma = \sigma(N, M_1) \in (0, 1)$. Combining our estimates we find that either

$$0 \leq u(A^+_{\tilde\Delta^l_i}) \leq c\eta^\sigma + c_\eta\epsilon_0 \quad \text{or} \quad u(A^+_{\tilde\Delta^l_i}) \geq 1 - (c\eta^\sigma + c_\eta\epsilon_0). \tag{5-44}$$

Combining (5-36) and (5-44) we can conclude, in either case, that

$$|u(A^+_{\tilde\Delta^l_i}) - u(A^+_{\tilde\Delta^l_i})| \geq \tfrac{3}{4}\tilde c^{-1} - c\eta^\sigma + c_\eta\epsilon_0 \geq \tfrac{1}{2}\tilde c^{-1} \tag{5-45}$$

by first choosing $\eta = \eta(N, M_1)$ small and then choosing $\epsilon_0 = \epsilon_0(N, M_1, \eta)$ small. Hence the proof of (5-24) is complete.

Next, using (5-24), Lemma 2.5(i), an elementary connectivity/covering argument and the Poincaré inequality, see Lemma 2.4, we see that

$$c^{-1}\beta^2 \leq \iint_{\widetilde W_{\Delta^l_i}} \big(|\nabla_X u|^2\delta^{2-\boldsymbol{q}} + |\nabla_Y u|^2\delta^{6-\boldsymbol{q}} + |X_0(u)|^2\delta^{4-\boldsymbol{q}}\big)\, dZ\, dt,$$

where $\widetilde W_{\Delta^l_i}$ is a natural Whitney-type region associated to Δ^l_i, $\delta = \delta(Z, t)$ is the distance from (Z, t) to Σ, and $c = c(N, M_1, \eta)$, $1 \leq c < \infty$. Consequently, for $(Z_0, t_0) \in E$ fixed we find, by summing over all indices i, l, such that $(Z_0, t_0) \in \Delta^l_i$, that

$$c^{-1}\beta^2 k \leq \sum_{i,l:(Z_0,t_0)\in\Delta^l_i} \bigg(\iint_{\widetilde W_{\Delta^l_i}} \big(|\nabla_X u|^2\delta^{2-\boldsymbol{q}} + |\nabla_Y u|^2\delta^{6-\boldsymbol{q}} + |X_0(u)|^2\delta^{4-\boldsymbol{q}}\big)\, dZ\, dt\bigg). \tag{5-46}$$

The construction can be made so that the Whitney-type regions $\{\widetilde W_{\Delta^l_i}\}$ have bounded overlaps measured by a constant depending only on N, M_1, and such that $W_{Q^l_i} \subset T_{cQ_0}$ for some $c = c(N, M_1)$, $1 \leq c < \infty$, where T_{cQ_0} is defined in (5-13). Hence, integrating with respect to $d\sigma$, we deduce that

$$c^{-1}\beta^2 k\sigma(E) \leq \bigg(\iint_{T_{cQ_0}} \big(|\nabla_X u|^2\delta + |\nabla_Y u|^2\delta^5 + |X_0(u)|^2\delta^3\big)\, dZ\, dt\bigg), \tag{5-47}$$

where, resolving the dependencies, $c = c(N, M_1)$, $1 \leq c < \infty$. Furthermore,

$$k \approx \frac{\log(\delta_0)}{\log(\epsilon_0)},$$

where η and ϵ_0 now have been fixed, and δ_0 is at our disposal. Given κ we obtain the conclusion of the lemma by specifying $\delta_0 = \delta_0(N, M_1, \kappa)$ sufficiently small. This completes the proof of Lemma 5.2.

6. The square function estimate: proof of Lemma 5.3

The purpose of the section is to prove Lemma 5.3. Hence we consider $Q_0 \in \mathbb{D}(\Sigma)$, we let $\varrho_0 = l(Q_0)$, $u(Z, t) := \omega(Z, t, S)$ and we let c be as stated in Lemma 5.2. We want to prove that there exists

$\tilde c = \tilde c(N, M_1, M_2),\ 1 \le \tilde c < \infty$, such that

$$\iint_{T_c Q_0} \big(|\nabla_X u|^2\delta + |\nabla_Y u|^2\delta^5 + |X_0(u)|^2\delta^3\big)\, dZ\, dt \le \tilde c \varrho_0^{\boldsymbol{q}-1}. \tag{6-1}$$

However, using Lemma 2.8, and a simple covering argument, we first note that to prove (6-1) it suffices to prove that

$$\iint_{T_c Q_0} |\nabla_X u|^2\delta\, dZ\, dt \le \tilde c \varrho_0^{\boldsymbol{q}-1} \tag{6-2}$$

for potentially new constants c, $\tilde c$ having the same dependence as the original constants c, $\tilde c$. Inequality (6-2) will be proved using partial integration. To enable partial integration, we perform the change of variables

$$(w, w_m, y, y_m, t) \in U \to (w, w_m + P_{\gamma w_m}\psi(w, y, t), y, y_m, t) \tag{6-3}$$

where

$$U = \big\{(W, Y, t) = (w, w_m, y, y_m, t) \in \mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R}^{m-1}\times\mathbb{R}\times\mathbb{R} \;\big|\; w_m > 0\big\}. \tag{6-4}$$

Then, by a straightforward calculation we see that u satisfies $\mathcal{K}u = 0$ in Ω if and only if $v(W, Y, t) := u(w, w_m + P_{\gamma w_m}\psi(w, y, t), y, y_m, t)$ satisfies

$$\nabla_W\cdot(A\nabla_W v) + B\cdot\nabla_W v + \big((w, w_m + P_{\gamma w_m}\psi(w, y, t))\cdot\nabla_Y - \partial_t\big)v = 0 \quad \text{in } U. \tag{6-5}$$

Here A is an $m\times m$-matrix-valued function, $B : U \to \mathbb{R}^m$ and

$$\begin{aligned} a_{m,m} &= \frac{1 + |\nabla_w P_{\gamma w_m}\psi|^2}{(1 + \partial_{w_m} P_{\gamma w_m}\psi)^2},\\ a_{j,m} = a_{m,j} &= -\frac{\partial_{w_j} P_{\gamma w_m}\psi}{(1 + \partial_{w_m} P_{\gamma w_m}\psi)}, \qquad j = 1, \dots, m-1,\\ a_{i,j} &= \delta_{i,j}, \qquad i, j \in \{1, \dots, m-1\}, \end{aligned} \tag{6-6}$$

and

$$\begin{aligned} b_m &= \frac{\partial_{w_m w_m} P_{\gamma w_m}\psi}{(1 + \partial_{w_m} P_{\gamma w_m}\psi)} + \frac{\big((w, w_m + P_{\gamma w_m}\psi(w, y, t))\cdot\nabla_Y - \partial_t\big)(P_{\gamma w_m}\psi)}{(1 + \partial_{w_m} P_{\gamma w_m}\psi)},\\ b_j &= \frac{\partial_{w_m w_j} P_{\gamma w_m}\psi}{(1 + \partial_{w_m} P_{\gamma w_m}\psi)}, \qquad j = 1, \dots, m-1. \end{aligned} \tag{6-7}$$

Choosing $\gamma = \gamma(N, M_1),\ \gamma > 0$, small enough, see Remark 2.3, we have

$$(c(N, M_1))^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{i,j}(W, Y, t)\xi_i\xi_j \le c(N, M_1)|\xi|^2, \quad c(N, M_1) \ge 1, \tag{6-8}$$

for all $(W, Y, t) \in \mathbb{R}^{N+1}$ and for all $\xi \in \mathbb{R}^m$. Let $\theta, \tilde\theta \ge 0$ be integers, let $(\phi_1, \dots, \phi_{m-1})$ and $(\tilde\phi_1, \dots, \tilde\phi_{m-1})$ denote multi-indices and let $\ell = (\theta + |\phi| + 3|\tilde\phi| + 2\tilde\theta)$. Then, using Lemma 2.1 and the fact that ψ is independent of y_m, we deduce that

$$\begin{aligned}\left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\Big(\big((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)^{\tilde\theta}(A(W,Y,t))\Big)\right| &\le c(m,l,\gamma)w_m^{-l}M_1,\\ \left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\Big(\big((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)^{\tilde\theta}(B(W,Y,t))\Big)\right| &\le c(m,l,\gamma)w_m^{-1-l}M_1\end{aligned} \tag{6-9}$$

whenever $(W,Y,t)\in U$. Similarly, using Lemma 2.2 we see that if we let $d\mu_i=d\mu_i(W,Y,t),\ i\in\{1,2\}$,

$$d\mu_1:=\left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\Big(\big((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)^{\tilde\theta}(A(W,Y,t))\Big)\right|^2 w_m^{2l-1}\,dW\,dy\,dt,$$

$$d\mu_2:=\left|\frac{\partial^{\theta+|\phi|+|\tilde\phi|}}{\partial w_m^\theta \partial w^\phi \partial y^{\tilde\phi}}\Big(\big((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)^{\tilde\theta}(B(W,Y,t))\Big)\right|^2 w_m^{2l+1}\,dW\,dy\,dt,$$

be defined on U, then

$$\mu_1(U\cap\mathcal{B}_\varrho(w_0,0,Y_0,t_0))+\mu_2(U\cap\mathcal{B}_\varrho(w_0,0,Y_0,t_0))\le c(m,l,\gamma,M_1,M_2)\varrho^{q-1} \tag{6-10}$$

whenever $(w_0,0,Y_0,t_0)\in\partial U,\ \varrho>0$, and $\mathcal{B}_\varrho(w_0,0,Y_0,t_0)\subset\mathbb{R}^{N+1}$. We emphasize that

$$A \text{ and } B \text{ are independent of } y_m. \tag{6-11}$$

As the equation $\mathcal{K}u=0$ and the statements in (6-9)–(6-11) are invariant under left translation defined by $\circ$, we can in the following without loss of generality assume that

$$(Z_{Q_0},t_{Q_0})=\big(x_{Q_0},\psi(x_{Q_0},y_{Q_0},t_{Q_0}),y_{Q_0},y_{m,Q_0},t_{Q_0}\big)=(0,0,0,0,0). \tag{6-12}$$

Furthermore, given $c\ge 1$ we claim that there exist $\alpha=\alpha(N,c)\ge 1$ and $\beta=\beta(N,M_1,c)\ge 1$ such that if we define

$$\square_0=\big\{(W,Y,t)=(w,w_m,y,y_m,t)\ \big|\ |w|<\alpha\varrho_0,\ 0<w_m<\beta\varrho_0,\ |Y|<\alpha^3\varrho_0^3,\ |t|<\alpha^2\varrho_0^2\big\}, \tag{6-13}$$

then

$$T_{cQ_0}\subset\big\{\big(w,w_m+P_{\gamma w_m}\psi(w,y,t),y,y_m,t\big)\ \big|\ (w,w_m,y,y_m,t)\in\square_0\big\}. \tag{6-14}$$

In the following we let $2\square_0$ be defined as in (6-13) but with ϱ_0 replaced by $2\varrho_0$. Letting $(\square_0)^*$ and $(2\square_0)^*$ denote the sets we get if we reflect $\square_0$ and $2\square_0$, respectively, in the boundary ∂U, in the following we let $\zeta\in C_0^\infty(2\square_0\cup(2\square_0)^*)$ be such that $0\le\zeta\le 1,\ \zeta\equiv 1$ on $\square_0\cup\square_0^*$, and such that

$$\varrho_0|\nabla_W\zeta|+\varrho_0^3|\nabla_Y\zeta|+\varrho_0^2|(W\cdot\nabla_Y-\partial_t)(\zeta)|\le c(N,M_1). \tag{6-15}$$

Letting $v(W,Y,t)=u\big(w,w_m+P_{\gamma w_m}\psi(w,y,t),y,y_m,t\big)$, where $u(Z,t)=\omega(Z,t,S)$, we see that

$$|v(W,Y,t)|\le 1 \quad \text{whenever } (W,Y,t)\in U. \tag{6-16}$$

Using (6-10) we see that

$$\iint_{2\square_0} |v(W,Y,t)|^2 \, d\mu_i(W,Y,t) \le c(m,l,\gamma,M_1,M_2,i)\varrho_0^{q-1}, \quad i \in \{1,2\}. \tag{6-17}$$

To prove (6-2), and hence to complete the proof of Lemma 5.3, it suffices to prove that

$$\iint_{2\square_0} |\nabla_W v(W,Y,t)|^2 \zeta^2 w_m \, dW\, dy\, dt \le c(N,M_1,M_2)\varrho_0^{q-1}. \tag{6-18}$$

The rest of the proof is devoted to the proof of (6-18) and in the proof of (6-18) we will use the notation

$$\begin{aligned} T_1 &:= \iint_{2\square_0} |\nabla_W v|^2 \zeta^2 w_m \, dW\, dy\, dt, \\ T_2 &:= \iint_{2\square_0} \big|\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)\big|^2 \zeta^4 w_m^3 \, dW\, dy\, dt. \end{aligned} \tag{6-19}$$

Inequality (6-18) is a consequence of the following two lemmas.

Lemma 6.1. *Let $\square_0$, ζ and v be as above. Then there exists, for $\epsilon > 0$ given, $c = c(N, M_1, M_2, \epsilon)$, $1 \le c < \infty$, such that*

$$T_1 \le c\varrho_0^{q-1} + \epsilon T_2.$$

Lemma 6.2. *Let $\square_0$, ζ and v be as above. Then there exists $c = c(N, M_1, M_2)$, $1 \le c < \infty$, such that*

$$T_2 \le c(T_1 + \varrho_0^{q-1}).$$

6A. ***Proof of Lemma 6.1.*** Using (6-8) we see that

$$T_1 \le cI, \quad I := \sum_{i,j=1}^{m} I_{i,j},$$

where

$$I_{i,j} := 2\iint_{2\square_0} \left(\frac{1}{a_{m,m}}\right) a_{i,j}(\partial_{w_i} v)(\partial_{w_j} v) w_m \zeta^2 \, dW\, dY\, dt.$$

Assume first that $i \neq m$. Then, integrating by parts in $I_{i,j}$ with respect to w_i we see that

$$\begin{aligned} I_{i,j} = &-2\iint_{2\square_0} \left(\frac{1}{a_{m,m}}\right) v\,\partial_{w_i}(a_{i,j}\partial_{w_j} v) w_m \zeta^2 \, dW\, dY\, dt \\ &-2\iint_{2\square_0} \partial_{w_i}\left(\frac{1}{a_{m,m}}\right) a_{i,j} v(\partial_{w_j} v) w_m \zeta^2 \, dW\, dY\, dt \\ &-4\iint_{2\square_0} \left(\frac{a_{i,j}}{a_{m,m}}\right) v(\partial_{w_j} v) w_m \zeta\,\partial_{w_i}\zeta \, dW\, dY\, dt. \end{aligned}$$

Similarly we see that

$$\begin{aligned}
I_{m,j} = &-\lim_{\delta\to 0} 2\int_{2\square_0\cap\{w_m=\delta\}}\left(\frac{a_{m,j}}{a_{m,m}}\right)(w,\delta,Y,t)v(w,\delta,Y,t)(\partial_{w_j}v(w,\delta,Y,t))\delta\zeta^2\,dw\,dY\,dt\\
&-2\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v\partial_{w_m}(a_{m,j}\partial_{w_j}v)w_m\zeta^2\,dW\,dY\,dt\\
&-2\iint_{2\square_0}\partial_{w_m}\left(\frac{1}{a_{m,m}}\right)a_{m,j}v(\partial_{w_j}v)w_m\zeta^2\,dW\,dY\,dt\\
&-2\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v(\partial_{w_j}v)\zeta^2\,dW\,dY\,dt\\
&-4\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v(\partial_{w_j}v)w_m\zeta\partial_{w_m}\zeta\,dW\,dY\,dt.
\end{aligned}$$

Combining the above,

$$I=\lim_{\delta\to 0} I_1^\delta + I_2+I_3+I_4+I_5,$$

where

$$\begin{aligned}
I_1^\delta &:= -2\sum_j\int_{2\square_0\cap\{w_m=\delta\}}\left(\frac{a_{m,j}}{a_{m,m}}\right)(w,\delta,Y,t)v(w,\delta,Y,t)(\partial_{w_j}v(w,\delta,Y,t))\delta\zeta^2\,dw\,dY\,dt,\\
I_2 &:= -2\sum_{i,j}\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v\partial_{w_i}(a_{i,j}\partial_{w_j}v)w_m\zeta^2\,dW\,dY\,dt,\\
I_3 &:= -2\sum_{i,j}\iint_{2\square_0}\partial_{w_i}\left(\frac{1}{a_{m,m}}\right)a_{i,j}v(\partial_{w_j}v)w_m\zeta^2\,dW\,dY\,dt,\\
I_4 &:= -4\sum_{i,j}\iint_{2\square_0}\left(\frac{a_{i,j}}{a_{m,m}}\right)v(\partial_{w_j}v)w_m\zeta\partial_{w_i}\zeta\,dW\,dy\,dt,\\
I_5 &:= -2\sum_{j}\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v(\partial_{w_j}v)\zeta^2\,dW\,dy\,dt.
\end{aligned}$$

Using (6-9), Lemma 2.5(iii) and (6-15) we see that

$$|I_1^\delta|\le c\varrho_0^{q-1}. \tag{6-20}$$

We next analyze I_2. Using the equation

$$\begin{aligned}
I_2 = 2\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v\big(((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t)v\big)w_m\zeta^2\,dW\,dy\,dt&\\
+2\sum_i\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)vb_i\partial_{w_i}vw_m\zeta^2\,dW\,dy\,dt&\\
=: I_{21}+I_{22},&
\end{aligned}$$

we have

$$I_{22}\le c\left(\iint_{2\square_0}v^2|B|^2w_m\zeta^2\,dW\,dy\,dt\right)^{1/2}\left(\iint_{2\square_0}|\nabla_W v|^2w_m\zeta^2\,dW\,dy\,dt\right)^{1/2}\le c(\epsilon)\varrho_0^{q-1}+\epsilon T_1$$

by (6-10), (6-17) and where ϵ is a degree of freedom. Furthermore, integrating by parts with respect to w_m we see that

$$I_{21} = \lim_{\delta\to 0} I_{211}^{\delta} + I_{212} + I_{213} + I_{214} + I_{215},$$

where

$$\begin{aligned}
I_{211}^{\delta} &= \int_{2\square_0\cap\{w_m=\delta\}} \Big(\frac{1}{a_{m,m}}\Big) v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)\delta^2\zeta^2\, dw\, dY\, dt,\\
I_{212} &= -\iint_{2\square_0} \partial_{w_m}\Big(\frac{1}{a_{m,m}}\Big) v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)w_m^2\zeta^2\, dW\, dY\, dt,\\
I_{213} &= -\iint_{2\square_0} \Big(\frac{1}{a_{m,m}}\Big)\partial_{w_m} v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)w_m^2\zeta^2\, dW\, dY\, dt,\\
I_{214} &= -\iint_{2\square_0} \Big(\frac{1}{a_{m,m}}\Big) v\partial_{w_m}\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)w_m^2\zeta^2\, dW\, dY\, dt,\\
I_{215} &= -2\iint_{2\square_0} \Big(\frac{1}{a_{m,m}}\Big) v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)w_m^2\zeta\,\partial_{w_m}\zeta\, dW\, dY\, dt.
\end{aligned}$$

To estimate I_{211}^{δ} we again have to use Lemma 2.5. Indeed, using that

$$v(W,Y,t) = u\big(w, w_m + P_{\gamma w_m}\psi(w,y,t), y, y_m, t\big) = u(x, x_m, y, y_m, t)$$

we see that

$$\begin{aligned}
&\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\\
&\qquad = (X\cdot\nabla_Y - \partial_t)u(X,Y,t) + \partial_{x_m}u(X,Y,t)\big((w\cdot\nabla_y - \partial_t)(\mathcal{P}_{\gamma w_m}\psi(x,y,t))\big).
\end{aligned}$$

Hence, using (6-9), Lemma 2.5(iii) and Lemma 2.1 we first see that

$$\big|\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\big| \le c\delta^{-2}$$

whenever $(W,Y,t)\in 2\square_0\cap\{w_m=\delta\}$ and then that $|I_{211}^{\delta}| \le c\varrho_0^{q-1}$. Focusing on I_{212} we see that

$$\begin{aligned}
I_{212} &= -\iint_{2\square_0}\Big(\frac{\partial_{w_m}a_{m,m}}{a_{m,m}^2}\Big) v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)v\big)w_m^2\zeta^2\, dW\, dy\, dt\\
&\le c\Big(\iint_{2\square_0} |\partial_{w_m}a_{m,m}|^2 v^2 w_m\, dW\, dy\, dt\Big)T_2^{1/2}\\
&\le c(\epsilon)\varrho_0^{q-1} + \epsilon T_2,
\end{aligned}$$

by (6-10), (6-17), and where ϵ is a degree of freedom. To continue we see that

$$\begin{aligned}
I_{214} = -\iint_{2\square_0}\Big(\frac{1}{a_{m,m}}\Big) v\big(((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t)\partial_{w_m}v\big)w_m^2\zeta^2\, dW\, dy\, dt\\
-\iint_{2\square_0}\Big(\frac{1}{a_{m,m}}\Big) v(1 + \partial_{w_m}P_{\gamma w_m}\psi(w,y,t))(\partial_{y_m}v)w_m^2\zeta^2\, dW\, dy\, dt\\
=: I_{2141} + I_{2142}.
\end{aligned}$$

To estimate I_{2142} we write

$$\begin{aligned} I_{2142} &= -\frac{1}{2}\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)(1+\partial_{w_m}P_{\gamma w_m}\psi(w,y,t))(\partial_{y_m}v^2)w_m^2\zeta^2\,dW\,dy\,dt \\ &= \iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)(1+\partial_{w_m}P_{\gamma w_m}\psi(w,y,t))v^2w_m^2\zeta\,\partial_{y_m}\zeta\,dW\,dy\,dt, \end{aligned}$$

where we have used that ψ is independent of y_m. In particular, $|I_{2142}|\le c\varrho_0^{\boldsymbol{q}-1}$. Focusing on I_{2141},

$$\begin{aligned} I_{2141} &= \iint_{2\square_0}\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\left(\frac{1}{a_{m,m}}\right)v(\partial_{w_m}v)w_m^2\zeta^2\,dW\,dy\,dt \\ &\quad+\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)\Big(\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)v\Big)(\partial_{w_m}v)w_m^2\zeta^2\,dW\,dy\,dt, \\ &\quad+2\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v(\partial_{w_m}v)w_m^2\zeta\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\zeta\,dW\,dy\,dt \\ &=: I_{21411}+I_{21412}+I_{21413}. \end{aligned}$$

Again using (6-10), (6-17), (6-9), and elementary estimates we see that

$$|I_{21411}|+|I_{21413}|\le c(\epsilon)\varrho_0^{\boldsymbol{q}-1}+\epsilon T_1,$$

where ϵ is a degree of freedom. Furthermore,

$$I_{21412}=-I_{213}.$$

Finally,

$$\begin{aligned} I_{215} &= -\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)\Big(\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)v^2\Big)w_m^2\zeta\,\partial_{w_m}\zeta\,dW\,dy\,dt \\ &= \iint_{2\square_0}\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\left(\frac{1}{a_{m,m}}\right)v^2w_m^2\zeta\,\partial_{w_m}\zeta\,dW\,dy\,dt \\ &\quad+\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v^2w_m^2\Big(\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\zeta\Big)\partial_{w_m}\zeta\,dW\,dy\,dt \\ &\quad+\iint_{2\square_0}\left(\frac{1}{a_{m,m}}\right)v^2w_m^2\zeta\big((w,w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\partial_{w_m}\zeta\,dW\,dy\,dt \\ &=: I_{2151}+I_{2152}+I_{2153}. \end{aligned}$$

Using (6-9), (6-10), (6-17), (6-12), (6-15), and by now familiar arguments, we see that $|I_{215}|\le c\varrho_0^{\boldsymbol{q}-1}$. Combining the above, we can conclude that

$$\begin{aligned} |I_2| &\le |I_{211}^\delta|+|I_{212}|+|I_{213}|+|I_{214}|+|I_{215}|+|I_{22}| \\ &\le c(\epsilon,\tilde{\epsilon})\varrho_0^{\boldsymbol{q}-1}+\epsilon T_1+\tilde{\epsilon}T_2, \end{aligned}$$

where ϵ and $\tilde{\epsilon}$ are degrees of freedom. Similarly,

$$|I_3|+|I_4|\le c(\epsilon)\varrho_0^{\boldsymbol{q}-1}+\epsilon T_1,$$

and we can conclude that

$$|I_1^\delta|+|I_2|+|I_3|+|I_4|\le c(\epsilon)\varrho_0^{\boldsymbol{q}-1}+\epsilon T_1.$$

Finally we consider I_5,

$$I_5=-2\sum_j\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v(\partial_{w_j}v)\zeta^2\,dW\,dy\,dt.$$

First we consider the term in the definition of I_5 which corresponds to $j=m$. Then

$$\begin{aligned}-\iint_{2\square_0}&\partial_{w_m}(v^2)\zeta^2\,dW\,dy\,dt\\&=-\lim_{\delta\to 0}\int_{2\square_0\cap\{w_m=\delta\}}(v^2)(w,\delta,Y,t)\zeta^2\,dw\,dY\,dt+2\iint_{2\square_0}v^2\zeta\,\partial_{w_m}\zeta\,dW\,dY\,dt,\end{aligned}$$

and obviously the absolute value of the terms on the right-hand side is bounded by $c\varrho_0^{\boldsymbol{q}-1}$. Next we consider the terms in the definition of I_5 which correspond to $j\neq m$. By integration by parts we see that

$$\begin{aligned}-2\iint_{2\square_0}&\left(\frac{a_{m,j}}{a_{m,m}}\right)v(\partial_{w_j}v)\partial_{w_m}(w_m)\zeta^2\,dW\,dY\,dt\\
&=\lim_{\delta\to 0}2\int_{2\square_0\cap\{w_m=\delta\}}\left(\frac{a_{m,j}}{a_{m,m}}\right)(w,\delta,Y,t)v(w,\delta,Y,t)\partial_{w_j}v(w,\delta,Y,t)\delta\zeta^2\,dw\,dY\,dt\\
&\quad+2\iint_{2\square_0}\partial_{w_m}\left(\frac{a_{m,j}}{a_{m,m}}\right)v\partial_{w_j}vw_m\zeta^2\,dW\,dY\,dt\\
&\quad+2\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)\partial_{w_m}v\partial_{w_j}vw_m\zeta^2\,dW\,dY\,dt\\
&\quad+2\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v\partial_{w_mw_j}vw_m\zeta^2\,dW\,dY\,dt\\
&\quad+4\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v\partial_{w_j}vw_m\zeta\,\partial_{w_m}\zeta\,dW\,dY\,dt.\end{aligned}$$

Let

$$\begin{aligned}I_{51}&:=2\sum_{j\neq m}\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)\partial_{w_m}v\partial_{w_j}vw_m\zeta^2\,dW\,dY\,dt,\\
I_{52}&:=2\sum_{j\neq m}\iint_{2\square_0}\left(\frac{a_{m,j}}{a_{m,m}}\right)v\partial_{w_mw_j}vw_m\zeta^2\,dW\,dY\,dt.\end{aligned}$$

By the above deductions, and using by now familiar arguments, we can conclude that

$$|I_5-I_{51}-I_{52}|\le c(\epsilon)\varrho_0^{\boldsymbol{q}-1}+\epsilon T_1.$$

To estimate I_{52} we use that $j \neq m$. Integrating by parts

$$\begin{aligned} I_{52} = &-2\sum_{j\neq m}\iint_{2\square_0} \partial_{w_j}\left(\frac{a_{m,j}}{a_{m,m}}\right) v\partial_{w_m} v w_m \zeta^2\, dW\, dY\, dt \\ &-2\sum_{j\neq m}\iint_{2\square_0} \left(\frac{a_{m,j}}{a_{m,m}}\right)\partial_{w_j} v\partial_{w_m} v w_m \zeta^2\, dW\, dY\, dt \\ &-4\sum_{j\neq m}\iint_{2\square_0} \left(\frac{a_{m,j}}{a_{m,m}}\right) v\partial_{w_m} v w_m \zeta\partial_{w_j}\zeta\, dW\, dY\, dt \\ := &\; I_{521}+I_{522}+I_{523}. \end{aligned}$$

Note that

$$I_{522} = -I_{51},$$

and that

$$|I_{521}|+|I_{523}| \leq c(\epsilon)\varrho_0^{\boldsymbol{q}-1} + \epsilon T_1$$

by familiar arguments. Summarizing we can conclude that

$$c^{-1}T_1 \leq I \leq |I_1^\delta| + |I_2| + |I_3| + |I_4| + |I_5| \leq c(\epsilon,\tilde{\epsilon})\varrho_0^{\boldsymbol{q}-1} + \epsilon T_1 + \tilde{\epsilon} T_2,$$

where ϵ, $\tilde{\epsilon}$ are degrees of freedom. This completes the proof of the lemma.

6B. ***Additional technical estimates.*** In this subsection we prove some additional technical estimates that will be used in the proof of Lemma 6.2. Let

$$\begin{aligned} T_3 &= \sum_{i=1}^m \iint_{2\square_0} |\nabla_W(\partial_{w_i} v)|^2 w_m^3 \zeta^4\, dW\, dY\, dt, \\ T_4 &= \sum_{i=1}^m \iint_{2\square_0} |\nabla_W(\partial_{y_i} v)|^2 w_m^7 \zeta^8\, dW\, dY\, dt, \\ T_5 &= \iint_{2\square_0} |\nabla_Y v|^2 w_m^5 \zeta^6\, dW\, dY\, dt. \end{aligned} \tag{6-21}$$

Lemma 6.3. *Let $\square_0$, ζ and v be as in Lemma 6.1. Then there exists, for positive $\epsilon_1 - \epsilon_4$ given, $c = c(N, M_1, M_2, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, $1 \leq c < \infty$, such that*

(i) $T_3 \leq c\varrho_0^{\boldsymbol{q}-1} + cT_1 + \epsilon_1 T_5$,

(ii) $T_4 \leq c\varrho_0^{\boldsymbol{q}-1} + cT_5 + \epsilon_2 T_1 + \epsilon_3 T_3$,

(iii) $T_5 \leq c\varrho_0^{\boldsymbol{q}-1} + cT_1 + cT_2 + \epsilon_4 T_4$.

Proof. To prove (i) we introduce $\tilde{v} = \partial_{w_i} v$. Using (6-5) we see that $\tilde{v}$ solves

$$\begin{aligned} &\nabla_W\cdot(A\nabla_W\tilde{v}) + B\cdot\nabla_W\tilde{v} + \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)\tilde{v} \\ &\qquad = -\nabla_W\cdot(\partial_{w_i}A\nabla_W v) - \partial_{w_i}B\cdot\nabla_W v - \partial_{y_i} v - \partial_{w_i}P_{\gamma w_m}\psi(w,y,t)\partial_{y_m} v \end{aligned} \tag{6-22}$$

in U. Multiplying the equation in (6-22) with $\tilde v w_m^3\zeta^4$ and integrating we see that

$$J := -\iint_{2\Box_0} (\nabla_W\cdot(A\nabla_W\tilde v))\tilde v w_m^3\zeta^4\,dW\,dY\,dt = J_1+J_2+J_3+J_4, \tag{6-23}$$

where

$$\begin{aligned} J_1 &:= \iint_{2\Box_0} \big(((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)\tilde v\big)\tilde v w_m^3\zeta^4\,dW\,dy\,dt,\\ J_2 &:= \iint_{2\Box_0} \big(\nabla_W\cdot((\partial_{w_i}A)\nabla_W v)\big)\tilde v w_m^3\zeta^4\,dW\,dy\,dt,\\ J_3 &:= \iint_{2\Box_0} (\partial_{w_i}(B\cdot\nabla_W v))\tilde v w_m^3\zeta^4\,dW\,dy\,dt,\\ J_4 &:= \iint_{2\Box_0} \big(\partial_{y_i}v+\partial_{w_i}P_{\gamma w_m}\psi(w,y,t)\partial_{y_m}v\big)\tilde v w_m^3\zeta^4\,dW\,dy\,dt. \end{aligned} \tag{6-24}$$

Using (6-9) we immediately see that

$$|J_2|+|J_3|+|J_4| \le cT_1+cT_1^{1/2}T_3^{1/2}+cT_1^{1/2}T_5^{1/2}. \tag{6-25}$$

Furthermore, using (6-12), (6-15) we see that

$$2|J_1| \le 4\left|\iint_{2\Box_0} \tilde v^2\big((w, w_m+P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y-\partial_t\big)(\zeta)w_m^3\zeta^3\,dW\,dy\,dt\right| \le cT_1, \tag{6-26}$$

and we can conclude that

$$|J_1|+|J_2|+|J_3|+|J_4| \le c(\epsilon,\tilde\epsilon)T_1+\epsilon T_3+\tilde\epsilon T_5, \tag{6-27}$$

where ϵ and $\tilde\epsilon$ are positive degrees of freedom. Next, integrating by parts in J we see that

$$\begin{aligned} J &= -\sum_j \int_{2\Box_0\cap\{w_m=\delta\}} a_{m,j}\partial_{w_j}\tilde v\tilde v\delta^3\zeta^4\,dw\,dY\,dt\\ &\quad+\sum_j \iint_{2\Box_0} a_{m,j}(\partial_{w_j}\tilde v)\partial_{w_m}(\tilde v w_m^3\zeta^4)\,dW\,dy\,dt\\ &\quad+\sum_{i\neq m}\sum_j \iint_{2\Box_0} a_{i,j}(\partial_{w_j}\tilde v)\partial_{w_i}(\tilde v w_m^3\zeta^4)\,dW\,dy\,dt\\ &= \sum_i\sum_j \iint_{2\Box_0} a_{i,j}(\partial_{w_j}\tilde v)(\partial_{w_i}\tilde v)(w_m^3\zeta^4)\,dW\,dy\,dt+\tilde J, \end{aligned}$$

where

$$\begin{aligned} \tilde J &:= -\sum_j \int_{2\Box_0\cap\{w_m=\delta\}} a_{m,j}(\partial_{w_j}\tilde v)\tilde v\delta^3\zeta^4\,dw\,dY\,dt\\ &\quad+4\sum_i\sum_j \iint_{2\Box_0} a_{i,j}(\partial_{w_j}\tilde v)\tilde v\zeta^3\partial_{w_i}\zeta\, w_m^3\zeta^3\,dW\,dy\,dt\\ &\quad+3\sum_j \iint_{2\Box_0} a_{m,j}(\partial_{w_j}\tilde v)\tilde v w_m^2\zeta^4\,dW\,dy\,dt. \end{aligned}$$

Using this notation we see that

$$c^{-1}T_3 \le |\tilde J| + |J_1| + |J_2| + |J_3| + |J_4|. \tag{6-28}$$

Furthermore, using (6-9) and Lemma 2.5 it is easy to see that

$$|\tilde J| \le c\varrho_0^{\boldsymbol{q}-1} + cT_1 + \epsilon T_3, \tag{6-29}$$

where ϵ is a degree of freedom. Combining (6-27) and (6-29), (i) follows. To prove (ii) we introduce $\tilde v = \partial_{y_i} v$. Again using (6-5) we see that $\tilde v$ solves

$$\begin{aligned}\nabla_W\cdot(A\nabla_W\tilde v) + B\cdot\nabla_W\tilde v + \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)\tilde v \\ = -\nabla_W\cdot(\partial_{y_i}A\nabla_W v) - \partial_{y_i}B\cdot\nabla_W v + \partial_{y_i}P_{\gamma w_m}\psi(w,y,t)\partial_{y_m}v\end{aligned} \tag{6-30}$$

in U. Arguing similarly to the proof of (i) we derive that

$$T_4 \le c\varrho_0^{\boldsymbol{q}-1} + cT_5 + cT_4^{1/2}T_5^{1/2} + cT_1^{1/2}T_5^{1/2} + cT_4^{1/2}T_5^{1/2} + cT_3^{1/2}T_5^{1/2}. \tag{6-31}$$

Hence,

$$T_4 \le c\varrho_0^{\boldsymbol{q}-1} + c(\epsilon,\tilde\epsilon)T_5 + \epsilon T_1 + \tilde\epsilon T_3, \tag{6-32}$$

where ϵ and $\tilde\epsilon$ are positive degrees of freedom. To prove (iii) we have to estimate

$$T_5 = \sum_{i=1}^m T_{5,i},\quad T_{5,i} := \iint_{2\square_0} (\partial_{y_i}v)(\partial_{y_i}v)w_m^5\zeta^6\,dW\,dy\,dt. \tag{6-33}$$

Note that

$$\begin{aligned}\partial_{y_i}v = -\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(\partial_{w_i}v) \\ +\partial_{w_i}\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v) - (\partial_{w_i}P_{\gamma w_m}\psi(w,y,t))\partial_{y_m}v.\end{aligned} \tag{6-34}$$

Hence,

$$\begin{aligned}T_{5,i} = &-\iint_{2\square_0}(\partial_{y_i}v)\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(\partial_{w_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\ &+\iint_{2\square_0}(\partial_{y_i}v)\partial_{w_i}\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)w_m^5\zeta^6\,dW\,dy\,dt \\ &-\iint_{2\square_0}(\partial_{y_i}v)\big((\partial_{w_i}P_{\gamma w_m}\psi(w,y,t))\partial_{y_m}v\big)w_m^5\zeta^6\,dW\,dy\,dt \\ =: &\; T_{5,i,1} + T_{5,i,2} + T_{5,i,3}.\end{aligned} \tag{6-35}$$

Using partial integration we immediately see that

$$|T_{5,i,2}| \le c\varrho_0^{\boldsymbol{q}-1} + cT_4^{1/2}T_2^{1/2} + cT_{5,i}^{1/2}T_2^{1/2}. \tag{6-36}$$

Furthermore,

$$
\begin{aligned}
T_{5,i,1} &= \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(\partial_{y_i}v)(\partial_{w_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\
&\quad + 6\iint_{2\Box_0} (\partial_{y_i}v)(\partial_{w_i}v)w_m^5\zeta^5\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)\zeta\,dW\,dy\,dt \\
&= \iint_{2\Box_0} \partial_{y_i}\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)(\partial_{w_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\
&\quad - \iint_{2\Box_0} (\partial_{y_i}P_{\gamma w_m}\psi(w,y,t))(\partial_{y_m}v)(\partial_{w_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\
&\quad + 6\iint_{2\Box_0} (\partial_{y_i}v)(\partial_{w_i}v)w_m^5\zeta^5\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)\zeta\,dW\,dy\,dt.
\end{aligned}
$$

Integrating by parts we have

$$
\begin{aligned}
T_{5,i,1} = &-\iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)(\partial_{w_i y_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\
&- 6\iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)(\partial_{w_i}v)w_m^5\zeta^5\partial_{y_i}\zeta\,dW\,dy\,dt \\
&- \iint_{2\Box_0} (\partial_{y_i}P_{\gamma w_m}\psi(w,y,t))(\partial_{y_m}v)(\partial_{w_i}v)w_m^5\zeta^6\,dW\,dy\,dt \\
&+ 6\iint_{2\Box_0} (\partial_{y_i}v)(\partial_{w_i}v)w_m^5\zeta^5\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)\zeta\,dW\,dy\,dt.
\end{aligned}
$$

Hence,

$$|T_{5,i,1}| \le cT_2^{1/2}T_4^{1/2} + cT_1^{1/2}T_2^{1/2} + cT_1^{1/2}(T_{5,i}^{1/2} + T_{5,m}^{1/2}) \tag{6-37}$$

and

$$|T_{5,i}| \le c\varrho_0^{\boldsymbol{q}-1} + cT_2^{1/2}T_4^{1/2} + cT_1^{1/2}T_2^{1/2} + cT_1^{1/2}(T_{5,i}^{1/2} + T_{5,m}^{1/2}) + cT_2^{1/2}T_{5,i}^{1/2} + |T_{5,i,3}|. \tag{6-38}$$

We now first consider the case $i = m$. Using Remark 2.3 and (6-38) we immediately see that

$$T_{5,m} \le c\varrho_0^{\boldsymbol{q}-1} + cT_1 + cT_2 + \epsilon T_4, \tag{6-39}$$

where ϵ is a positive degree of freedom. Consider now $i \ne m$. Then, using (6-38) we have

$$|T_{5,i}| \le c\varrho_0^{\boldsymbol{q}-1} + cT_2^{1/2}T_4^{1/2} + cT_1^{1/2}T_2^{1/2} + cT_1^{1/2}(T_{5,i}^{1/2} + T_{5,m}^{1/2}) + cT_2^{1/2}T_{5,i}^{1/2} + cT_{5,i}^{1/2}T_{5,m}^{1/2}, \tag{6-40}$$

and hence

$$T_{5,i} \le c\varrho_0^{\boldsymbol{q}-1} + cT_1 + cT_2 + \epsilon T_4 + cT_{5,m}. \tag{6-41}$$

Using (6-39) and (6-41) we can now complete the proof of (iii) and the lemma. □

6C. *Proof of Lemma 6.2*. To start the proof of Lemma 6.2 we first use the equation in (6-5) and write

$$T_2 = -\iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(v)\big(\nabla_W\cdot(A\nabla_W v) + B\cdot\nabla_W v\big)\zeta^4 w_m^3\, dW\, dy\, dt$$

and

$$T_2 = T_{21} + T_{22} + T_{23} + T_{24}, \tag{6-42}$$

where

$$\begin{aligned}
T_{21} &:= -\sum_j \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\,\partial_{w_m}(a_{m,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt,\\
T_{22} &:= -\sum_{i\neq m} \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\,\partial_{w_i}(a_{i,m}\partial_{w_m}v)w_m^3\zeta^4\, dW\, dy\, dt,\\
T_{23} &:= -\sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\,\partial_{w_i}(a_{i,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt,\\
T_{24} &:= -\sum_i \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v\,b_i\partial_{w_i}v\,w_m^3\zeta^4\, dW\, dy\, dt.
\end{aligned}$$

Using (6-9) we immediately see that

$$|T_{21}| + |T_{22}| + |T_{24}| \le cT_1^{1/2}T_2^{1/2} + cT_3^{1/2}T_2^{1/2}. \tag{6-43}$$

Next, focusing on T_{23}, and integrating by parts with respect to w_i, we see that

$$\begin{aligned}
T_{23} &= \sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} \partial_{w_i}\big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v(a_{i,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt\\
&\quad + \sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)v(a_{i,j}\partial_{w_j}v)\partial_{w_i}(\zeta)w_m^3\zeta^3\, dW\, dy\, dt\\
&=: T_{231} + T_{232},
\end{aligned}$$

and that $|T_{232}| \le T_1^{1/2}T_2^{1/2}$. Furthermore

$$\begin{aligned}
T_{231} &= \sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} (\partial_{y_i}v)(a_{i,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt\\
&\quad + \sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} \big((w, w_m + P_{\gamma w_m}\psi(w,y,t))\cdot\nabla_Y - \partial_t\big)(\partial_{w_i}v)(a_{i,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt\\
&\quad + \sum_{i\neq m}\sum_{j\neq m} \iint_{2\Box_0} (\partial_{w_i}P_{\gamma w_m}\psi(w,y,t))(\partial_{y_m}v)(a_{i,j}\partial_{w_j}v)w_m^3\zeta^4\, dW\, dy\, dt\\
&=: T_{2311} + T_{2312} + T_{2313}.
\end{aligned}$$

Then

$$|T_{2311}| + |T_{2313}| \le cT_5^{1/2}T_1^{1/2}. \tag{6-44}$$

To estimate T_{2312} we lift the vector field $\big((w, w_m + P_{\gamma w_m}\psi(w, y, t)) \cdot \nabla_Y - \partial_t\big)$ through partial integration and use the symmetry of the matrix $\{a_{i,j}\}$ to see that

$$\begin{aligned}2T_{2312} = &-\sum_{i\neq m}\sum_{j\neq m}\iint_{2\square_0} (\partial_{w_i}v)\Big(\big((w, w_m+P_{\gamma w_m}\psi(w, y, t))\cdot\nabla_Y-\partial_t\big)(a_{i,j})\partial_{w_j}v\Big)w_m^3\zeta^4\,dW\,dy\,dt\\ &-4\sum_{i\neq m}\sum_{j\neq m}\iint_{2\square_0} (\partial_{w_i}v)(a_{i,j}\partial_{w_j}v)\big((w, w_m+P_{\gamma w_m}\psi(w, y, t))\cdot\nabla_Y-\partial_t\big)(\zeta)w_m^3\zeta^3\,dW\,dy\,dt\\ =:&\ T_{23121}+T_{23122}.\end{aligned}$$

Then, by familiar arguments,

$$|T_{23121}| + |T_{23122}| \le cT_1. \tag{6-45}$$

Putting all estimates together we can conclude that

$$T_2 \le |T_{21}| + |T_{22}| + |T_{23}| + |T_{24}| \le cT_1 + cT_1^{1/2}T_2^{1/2} + cT_2^{1/2}T_3^{1/2} + cT_1^{1/2}T_5^{1/2}. \tag{6-46}$$

Hence

$$T_2 \le c(T_1 + T_3) + \epsilon_1 T_5, \tag{6-47}$$

where ϵ_1 is a positive degree of freedom. Now, using Lemma 6.3 we see, given positive degrees of freedom $\epsilon_2 - \epsilon_5$, that there exists $c = c(N, M_1, M_2, \epsilon_2 - \epsilon_5)$, $1 \le c < \infty$, such that

$$\begin{aligned}T_3 &\le c\varrho_0^{q-1} + cT_1 + \epsilon_2 T_5,\\ T_4 &\le c\varrho_0^{q-1} + cT_5 + \epsilon_3 T_1 + \epsilon_4 T_3,\\ T_5 &\le c\varrho_0^{q-1} + cT_1 + cT_2 + \epsilon_5 T_4.\end{aligned} \tag{6-48}$$

Using the estimates on the last two lines in (6-48) we see that

$$T_5 \le c\varrho_0^{q-1} + cT_1 + cT_2 + \epsilon_6 T_3, \tag{6-49}$$

where again ϵ_6 is a degree of freedom. Using (6-49) in the first estimate in (6-48) we deduce that

$$T_3 \le c\varrho_0^{q-1} + cT_1 + c\epsilon_2 T_2 + \epsilon_2\epsilon_6 T_3, \tag{6-50}$$

and hence, consuming ϵ_6,

$$\begin{aligned}T_3 &\le c\varrho_0^{q-1} + cT_1 + \epsilon_7 T_2,\\ T_5 &\le c\varrho_0^{q-1} + cT_1 + cT_2,\end{aligned} \tag{6-51}$$

for yet another degree of freedom ϵ_7. Putting the estimates from (6-51) into (6-47), we deduce that

$$T_2 \le c_1\varrho_0^{q-1} + c_1 T_1 + c_2 T_2(\epsilon_1 + \epsilon_7), \tag{6-52}$$

where $c_1 = c_1(N, M_1, M_2, \epsilon_1, \epsilon_7)$ and $c_2 = c_2(N, M_1, M_2)$. Elementary manipulations now complete the proof of the lemma. □

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