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PERRON'S METHOD FOR NONLOCAL FULLY NONLINEAR EQUATIONS

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This paper is concerned with the existence of viscosity solutions of nonlocal fully nonlinear equations that are not translation-invariant. We construct a discontinuous viscosity solution of such a nonlocal equation by Perron's method. If the equation is uniformly elliptic, we prove the discontinuous viscosity solution is Hölder continuous and thus it is a viscosity solution.

1. Introduction

We investigate the existence of a viscosity solution of

$$\begin{cases} I(x,u(x),u(\cdot))=0 & \text{in } \Omega, \\ u=g & \text{in } \Omega^c, \end{cases} \tag{1-1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, I is a nonlocal operator that is not translation-invariant and g is a bounded continuous function in $\mathbb{R}^n$.

An important example of (1-1) is the Dirichlet problem for nonlocal Bellman–Isaacs equations, i.e.,

$$\begin{cases} \sup_{a\in\mathcal{A}}\inf_{b\in\mathcal{B}}\{-I_{ab}[x,u]+b_{ab}(x)\cdot\nabla u(x)+c_{ab}(x)u(x)+f_{ab}(x)\}=0 & \text{in } \Omega, \\ u=g & \text{in } \Omega^c, \end{cases} \tag{1-2}$$

where $\mathcal{A},\mathcal{B}$ are two index sets, $b_{ab}:\mathbb{R}^n\to\mathbb{R}^n$, $c_{ab}:\mathbb{R}^n\to\mathbb{R}^+$, $f_{ab}:\mathbb{R}^n\to\mathbb{R}$ are uniformly continuous functions and I_{ab} is a Lévy operator. If the Lévy measures are symmetric and absolutely continuous with respect to the Lebesgue measure, then they can be represented as

$$I_{ab}[x,u]:=\int_{\mathbb{R}^n}[u(x+z)-u(x)]K_{ab}(x,z)\,dz, \tag{1-3}$$

where $\{K_{ab}(x,\cdot):x\in\Omega,\ a\in\mathcal{A},\ b\in\mathcal{B}\}$ are kernels of Lévy measures satisfying

$$\int_{\mathbb{R}^n}\min\{|z|^2,1\}K_{ab}(x,z)\,dz<+\infty \quad \text{for all } x\in\Omega. \tag{1-4}$$

In fact, we will not assume our Lévy measures to be symmetric in the following sections.

Existence of viscosity solutions has been well established for the Dirichlet problem for integro-differential equations by Perron's method when the equations satisfy the comparison principle. G. Barles and C. Imbert [Barles and Imbert 2008] studied the comparison principle for degenerate second-order

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integro-differential equations assuming the nonlocal operators are of Lévy–Itô type and the equations satisfy the coercive assumption. Then G. Barles, E. Chasseigne and C. Imbert [Barles et al. 2008] obtained the existence of viscosity solutions for such integro-differential equations by Perron's method. L. A. Caffarelli and L. Silvestre [2009, Section 5] proved the comparison principle for uniformly elliptic translation-invariant integro-differential equations where the nonlocal operators are of Lévy type. Then existence of viscosity solutions follows, if suitable barriers can be constructed, by Perron's method. Later H. Chang-Lara and G. Davila [2014a, Section 3; 2016b] extended the comparison and existence results of [Caffarelli and Silvestre 2009] to parabolic equations. The existence for (1-1) when I is a nonlocal operator that is not translation-invariant is much more difficult to tackle since we do not have a good comparison principle; see [Mou and Święch 2015], where the authors proved comparison assuming that either a viscosity subsolution or a supersolution is more regular. To our knowledge, the only available results for the existence of solutions for equations that are not translation-invariant are the following. D. Kriventsov [2013, Section 5] studied the existence of viscosity solutions of some uniformly elliptic nonlocal equations. J. Serra [2015b, Section 4] proved the existence of viscosity solutions of uniformly elliptic nonlocal Bellman equations. H. Chang-Lara and D. Kriventsov [2017, Section 5] extended existence results in [Kriventsov 2013] to a class of uniformly parabolic nonlocal equations. In all these proofs, the authors used fixed-point arguments. O. Alvarez and A. Tourin [1996] obtained the existence of viscosity solutions of degenerate parabolic nonlocal equations by Perron's method with a restrictive assumption that the Lévy measures are bounded. The boundedness of Lévy measures allowed them to obtain the comparison principle. The reader can consult [Crandall et al. 1992; Ishii 1987; 1989; Koike 2005] for Perron's method for viscosity solutions of fully nonlinear partial differential equations.

The probability literature on the existence of viscosity solutions of nonlocal Bellman–Isaacs equations is enormous. It is well known that Bellman–Isaacs equations arise when people study differential games, where the equations carry information about the value and strategies of the games. Probabilists represent viscosity solutions of nonlocal Bellman–Isaacs equations as value functions of certain stochastic differential games with jump diffusion via the dynamic programming principle. However, mostly in the probability literature, the nonlocal terms of nonlocal Bellman–Isaacs equations are of Lévy–Itô type and Ω is the whole space $\mathbb{R}^n$. We refer the reader to [Barles et al. 1997; Biswas 2012; Biswas et al. 2010; Buckdahn et al. 2011; Ishikawa 2004; Kharroubi and Pham 2015; Koike and Święch 2013; Øksendal and Sulem 2007; Pham 1998; Soner 1986; 1988; Święch and Zabczyk 2016] for stochastic representation formulas for viscosity solutions of nonlocal Bellman–Isaacs equations.

In Section 3, we adapt to the nonlocal case the approach from [Ishii 1987; 1989; Koike 2005] for obtaining existence of a discontinuous viscosity solution u of (1-1) without using the comparison principle. For applying Perron's method, we need to assume that there exist a continuous viscosity subsolution and a continuous supersolution of (1-1) and both satisfy the boundary condition. Since (1-1) involves the nonlocal term, the proof of the existence is more delicate than the PDE case.

In Section 4, we obtain a Hölder estimate for the discontinuous viscosity solution of (1-1) constructed by Perron's method assuming the equation is uniformly elliptic. In most of the literature, the nonlocal operator I is assumed to be uniformly elliptic with respect to a class of linear nonlocal operators of form

(1-3) with kernels K satisfying

$$(2-\sigma)\frac{\lambda}{|z|^{n+\sigma}} \leq K(x,z) \leq (2-\sigma)\frac{\Lambda}{|z|^{n+\sigma}}, \tag{1-5}$$

where $0 < \lambda \leq \Lambda$. Various regularity results were obtained in recent years under the above uniform ellipticity, such as [Caffarelli and Silvestre 2009; 2011a; 2011b; Chang-Lara and Dávila 2014a; 2014b; 2016a; 2016b; Chang-Lara and Kriventsov 2017; Dong and Kim 2013; Jin and Xiong 2015; 2016; Kriventsov 2013; Serra 2015a; 2015b; Silvestre 2006; 2011; Dong and Zhang 2016] for both elliptic and parabolic integro-differential equations. In this paper, we follow [Schwab and Silvestre 2016] to assume a much weaker uniform ellipticity. Roughly speaking, we let I be uniformly elliptic with respect to a larger class of linear nonlocal operators where the kernels K satisfy the right-hand side of (1-5) in an integral sense and the left-hand side of that in a symmetric subset of each annulus domain with positive measure. The main tool we use is the weak Harnack inequality obtained in [Schwab and Silvestre 2016]. With the weak Harnack inequality, we are able to prove the oscillation between the upper- and lower-semicontinuous envelopes of the discontinuous viscosity solution u in the ball B_r is of order r^α for some $\alpha > 0$ and any small $r > 0$. This proves that u is Hölder continuous and thus it is a viscosity solution of (1-1). Recently, L. Silvestre [2016] applied the regularity for nonlocal equations under this weak ellipticity to obtain the regularity for the homogeneous Boltzmann equation without cut-off. We also want to mention that M. Kassmann, M. Rang and R. Schwab [Kassmann et al. 2014] studied Hölder regularity for a class of integro-differential operators with kernels which are positive along some given rays or cone-like sets.

To complete the existence results, we construct continuous sub/supersolutions in both uniformly elliptic and degenerate cases in Section 5. In the uniformly elliptic case, we follow the idea of [Ros-Oton and Serra 2016] to construct appropriate barrier functions. We then use them to construct a subsolution and a supersolution which satisfy the boundary condition. The weak uniform ellipticity and the lower-order terms of I make the proofs more involved. With all these ingredients in hand, we can conclude one of the main results in this manuscript, that (1-1) admits a viscosity solution if I is uniformly elliptic; see Theorem 5.6 in Section 5A. This main result generalizes nearly all the previous existence results for uniformly elliptic integro-differential equations. In the degenerate case, it is natural to construct a sub/supersolution only for (1-2) since we have little information about the nonlocal operator I. Moreover, we need to assume the nonlocal Bellman–Isaacs equation in (1-2) satisfies the coercive assumption, i.e., $c_{ab} \geq \gamma$ for some $\gamma > 0$. The coercive assumption is often made to study uniqueness, existence and regularity of viscosity solutions of degenerate elliptic PDEs and integro-PDEs; see [Barles et al. 2008; Barles and Imbert 2008; Crandall et al. 1992; Ishii 1987; 1989; Ishii and Lions 1990; Jakobsen and Karlsen 2006; Mou 2016; Mou and Święch 2015]. In Section 5B, we obtain a subsolution and a supersolution which satisfy the boundary condition in the degenerate case. The difficulty here lies in giving a degenerate assumption on the kernels which allows us to construct barrier functions. Roughly speaking, we only need to assume that the kernels $K_{ab}(x,\cdot)$ are nondegenerate in the outer-pointing normal direction of the boundary for the points x which are sufficiently close to the boundary. That means we allow our kernels K_{ab} to be degenerate in the whole domain. Then we can conclude the second main result, the existence of a discontinuous viscosity solution of (1-2), given in Theorem 5.13. If the comparison principle holds for (1-2), we obtain that the discontinuous viscosity solution is a viscosity solution. Finally, we notice that our method could be adapted to the nonlocal parabolic equations for obtaining the corresponding existence results.

2. Notation and definitions

We write B_δ for the open ball centered at the origin with radius $\delta > 0$ and $B_\delta(x) := B_\delta + x$. We set $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ for $\delta > 0$. For each nonnegative integer r and $0 < \alpha \le 1$, we denote by $C^{r,\alpha}(\Omega)$ ($C^{r,\alpha}(\bar{\Omega})$) the subspace of $C^{r,0}(\Omega)$ ($C^{r,0}(\bar{\Omega})$) consisting of functions whose r-th partial derivatives are locally (uniformly) α-Hölder continuous in Ω. For any $u \in C^{r,\alpha}(\bar{\Omega})$, where r is a nonnegative integer and $0 \le \alpha \le 1$, define

$$[u]_{r,\alpha;\Omega} := \begin{cases} \sup_{x\in\Omega,|j|=r} |\partial^j u(x)| & \text{if } \alpha = 0, \\ \sup_{x,y\in\Omega,x\neq y,|j|=r} |\partial^j u(x) - \partial^j u(y)|/|x-y|^\alpha & \text{if } \alpha > 0, \end{cases}$$

and

$$\|u\|_{C^{r,\alpha}(\bar{\Omega})} := \begin{cases} \sum_{j=0}^r [u]_{j,0,\Omega} & \text{if } \alpha = 0, \\ \|u\|_{C^{r,0}(\bar{\Omega})} + [u]_{r,\alpha;\Omega} & \text{if } \alpha > 0. \end{cases}$$

For simplicity, we use the notation $C^\beta(\Omega)$ ($C^\beta(\bar{\Omega})$), where $\beta > 0$, to denote the space $C^{r,\alpha}(\Omega)$ ($C^{r,\alpha}(\bar{\Omega})$), where r is the largest integer smaller than β and $\alpha = \beta - r$. The set $C_b^\beta(\Omega)$ consist of functions from $C^\beta(\Omega)$ which are bounded. We write $\mathrm{USC}(\mathbb{R}^n)$ for the space of upper-semicontinuous functions in $\mathbb{R}^n$ and $\mathrm{LSC}(\mathbb{R}^n)$ for the space of lower-semicontinuous functions in $\mathbb{R}^n$.

We will give a definition of viscosity solutions of (1-1). We first state the general assumptions on the nonlocal operator I in (1-1). For any $\delta > 0$, $r, s \in \mathbb{R}$, $x, x_k \in \Omega$, $\varphi, \varphi_k, \psi \in C^2(B_\delta(x)) \cap L^\infty(\mathbb{R}^n)$, we assume:

(A0) The function $(x, r) \to I(x, r, \varphi(\cdot))$ is continuous in $B_\delta(x) \times \mathbb{R}$.

(A1) If $x_k \to x$ in Ω, $\varphi_k \to \varphi$ a.e. in $\mathbb{R}^n$, $\varphi_k \to \varphi$ in $C^2(B_\delta(x))$ and $\{\varphi_k\}_k$ is uniformly bounded in $\mathbb{R}^n$, then

$$I(x_k, r, \varphi_k(\cdot)) \to I(x, r, \varphi(\cdot)).$$

(A2) If $r \le s$, then $I(x, r, \varphi(\cdot)) \le I(x, s, \varphi(\cdot))$.

(A3) For any constant C, we have $I(x, r, \varphi(\cdot) + C) = I(x, r, \varphi(\cdot))$.

(A4) If φ touches ψ from above at x, then $I(x, r, \varphi(\cdot)) \le I(x, r, \psi(\cdot))$.

Remark 2.1. If I is uniformly elliptic and satisfies (A0), (A2), then (A0)–(A4) hold for I. See Lemma 4.2.

Remark 2.2. The nonlocal operator I in [Schwab and Silvestre 2016] has only two components, i.e., $(x, \varphi) \to I(x, \varphi(\cdot))$. Here we let our nonlocal operator I have three components and assume (A2)–(A3) hold. This is because we want to let I include the left-hand side of the nonlocal Bellman–Isaacs equation in (1-2) and, moreover, we want to describe the two properties

$$-I_{ab}[x, \varphi + C] + b_{ab}(x) \cdot \nabla(\varphi + C)(x) = -I_{ab}[x, \varphi] + b_{ab}(x) \cdot \nabla\varphi(x),$$
$$c_{ab}(x) r \le c_{ab}(x) s \quad \text{if } r \le s$$

in abstract forms.

Remark 2.3. The left-hand side of the nonlocal Bellman–Isaacs equation in (1-2) satisfies (A0)–(A4) if (1-4) holds and its coefficients K_{ab}, b_{ab}, c_{ab} and f_{ab} are uniformly continuous with respect to x in Ω, uniformly in $a \in \mathcal{A}$, $b \in \mathcal{B}$. See [Guillen and Schwab 2016] for when the nonlocal operator I has a min-max structure.

Throughout the paper, we always assume the nonlocal operator I satisfies (A0)–(A4).

Definition 2.4. A bounded function $u \in \mathrm{USC}(\mathbb{R}^n)$ is a viscosity subsolution of $I = 0$ in Ω if whenever $u - \varphi$ has a maximum over $\mathbb{R}^n$ at $x \in \Omega$ for $\varphi \in C_b^2(\mathbb{R}^n)$, then

$$I(x, u(x), \varphi(\cdot)) \leq 0.$$

A bounded function $u \in \mathrm{LSC}(\mathbb{R}^n)$ is a viscosity supersolution of $I = 0$ in Ω if whenever $u - \varphi$ has a minimum over $\mathbb{R}^n$ at $x \in \Omega$ for $\varphi \in C_b^2(\mathbb{R}^n)$, then

$$I(x, u(x), \varphi(\cdot)) \geq 0.$$

A bounded function u is a viscosity solution of $I = 0$ in Ω if it is both a viscosity subsolution and viscosity supersolution of $I = 0$ in Ω.

Remark 2.5. In Definition 2.4, all the maximums and minimums can be replaced by strict ones.

Definition 2.6. A bounded function u is a viscosity subsolution of (1-1) if u is a viscosity subsolution of $I = 0$ in Ω and $u \leq g$ in Ω^c. A bounded function u is a viscosity supersolution of (1-1) if u is a viscosity supersolution of $I = 0$ in Ω and $u \geq g$ in Ω^c. A bounded function u is a viscosity solution of (1-1) if u is a viscosity subsolution and supersolution of (1-1).

We will use the following notations: if u is a function on Ω, then, for any $x \in \Omega$,

$$u^*(x) = \lim_{r \to 0} \sup\{u(y) : y \in \Omega \text{ and } |y - x| \leq r\},$$
$$u_*(x) = \lim_{r \to 0} \inf\{u(y) : y \in \Omega \text{ and } |y - x| \leq r\}.$$

One calls u^* the upper-semicontinuous envelope of u and u_* the lower semicontinuous envelope of u.

We then give a definition of discontinuous viscosity solutions of (1-1).

Definition 2.7. A bounded function u is a discontinuous viscosity subsolution of (1-1) if u^* is a viscosity subsolution of (1-1). A bounded function u is a discontinuous viscosity supersolution of (1-1) if u_* is a viscosity supersolution of (1-1). A function u is a discontinuous viscosity solution of (1-1) if it is both a discontinuous viscosity subsolution and a discontinuous viscosity supersolution of (1-1).

Remark 2.8. If u is a discontinuous viscosity solution of (1-1) and u is continuous in $\mathbb{R}^n$, then u is a viscosity solution of (1-1).

3. Perron's method

In this section, we obtain the existence of a discontinuous viscosity solution of (1-1) by Perron's method. We remind you that I satisfies (A0)–(A4).

Lemma 3.1. *Let $\mathcal{F}$ be a family of viscosity subsolutions of $I = 0$ in Ω. Let $w(x) = \sup\{u(x) : u \in \mathcal{F}\}$ in $\mathbb{R}^n$ and assume that $w^*(x) < \infty$ for all $x \in \mathbb{R}^n$. Then w is a discontinuous viscosity subsolution of $I = 0$ in Ω.*

Proof. Suppose that φ is a $C_b^2(\mathbb{R}^n)$ function such that $w^* - \varphi$ has a strict maximum (equal to 0) at $x_0 \in \Omega$ over $\mathbb{R}^n$. We can construct a uniformly bounded sequence of $C^2(\mathbb{R}^n)$ functions $\{\varphi_m\}_m$ such that $\varphi_m = \varphi$ in $B_1(x_0)$, $\varphi \le \varphi_m$ in $\mathbb{R}^n$, $\sup_{x\in B_2^c(x_0)}\{w^*(x) - \varphi_m(x)\} \le -\frac{1}{m}$ and $\varphi_m \to \varphi$ pointwise. Thus, for any positive integer m, we know $w^* - \varphi_m$ has a strict maximum (equal to 0) at x_0 over $\mathbb{R}^n$. Therefore, $\sup_{x\in B_1^c(x_0)}\{w^*(x) - \varphi_m(x)\} = \epsilon_m < 0$. By the definition of w^*, we have, for any $u \in \mathcal{F}$, $\sup_{x\in B_1^c(x_0)}\{u(x) - \varphi_m(x)\} \le \epsilon_m < 0$. Again, by the definition of w^*, we have, for any $\epsilon_m < \epsilon < 0$, there exist $u_\epsilon \in \mathcal{F}$ and $\bar{x}_\epsilon \in B_1(x_0)$ such that $u_\epsilon(\bar{x}_\epsilon) - \varphi(\bar{x}_\epsilon) > \epsilon$. Since $u_\epsilon \in \mathrm{USC}(\mathbb{R}^n)$ and $\varphi_m \in C_b^2(\mathbb{R}^n)$, there exists $x_\epsilon \in B_1(x_0)$ such that $u_\epsilon(x_\epsilon) - \varphi_m(x_\epsilon) = \sup_{x\in\mathbb{R}^n}\{u_\epsilon(x) - \varphi(x)\} \ge u_\epsilon(\bar{x}_\epsilon) - \varphi_m(\bar{x}_\epsilon) > \epsilon$. Since $w^* - \varphi_m$ attains a strict maximum (equal to 0) at x_0 over $\mathbb{R}^n$ and $u \le w^*$ for any $u \in \mathcal{F}$, we have $u_\epsilon(x_\epsilon) \to w^*(x_0)$ and $x_\epsilon \to x_0$ as $\epsilon \to 0^-$. Since u_ϵ is a viscosity subsolution of $I = 0$ in Ω, we have

$$I(x_\epsilon, u_\epsilon(x_\epsilon), \varphi_m(\cdot)) \le 0. \tag{3-1}$$

Since $x_\epsilon \to x_0$, $u_\epsilon(x_\epsilon) \to w^*(x_0)$ as $\epsilon \to 0^-$, $\varphi_m = \varphi$ in $B_1(x_0)$, $\varphi_m \to \varphi$ pointwise, $\{\varphi_m\}_m$ is uniformly bounded, $\varphi \in C_b^2(\mathbb{R}^n)$, (A0) and (A1) hold, we have, letting $\epsilon \to 0^-$ and $m \to +\infty$ in (3-1),

$$I(x_0, w^*(x_0), \varphi(\cdot)) \le 0.$$

Therefore, w is a discontinuous viscosity subsolution of $I = 0$. □

Theorem 3.2. *Let $\underline{u}$, $\bar{u}$ be bounded continuous functions and be respectively a viscosity subsolution and a viscosity supersolution of $I = 0$ in Ω. Assume moreover that $\bar{u} = \underline{u} = g$ in Ω^c for some bounded continuous function g and $\underline{u} \le \bar{u}$ in $\mathbb{R}^n$. Then*

$$w(x) = \sup_{u\in\mathcal{F}} u(x),$$

where

$$\mathcal{F} = \big\{u \in C^0(\mathbb{R}^n) : \underline{u} \le u \le \bar{u} \text{ in } \mathbb{R}^n \text{ and } u \text{ is a viscosity subsolution of } I = 0 \text{ in } \Omega\big\},$$

is a discontinuous viscosity solution of (1-1).

Proof. Since $\underline{u} \in \mathcal{F}$, we know $\mathcal{F} \ne \varnothing$. Thus, w is well defined, $\underline{u} \le w \le \bar{u}$ in $\mathbb{R}^n$ and $w = \bar{u} = \underline{u}$ in Ω^c. By Lemma 3.1, w is a discontinuous viscosity subsolution of $G = 0$ in Ω. We claim that w is a discontinuous viscosity supersolution of $G = 0$ in Ω. If not, there exist a point $x_0 \in \Omega$ and a function $\varphi \in C_b^2(\mathbb{R}^n)$ such that $w_* - \varphi$ has a strict minimum (equal to 0) at the point x_0 over $\mathbb{R}^n$ and

$$I(x_0, w_*(x_0), \varphi(\cdot)) < -\epsilon_0,$$

where ϵ_0 is a positive constant. Thus, we can find sufficiently small constants $\epsilon_1 > 0$ and $\delta_0 > 0$ such that $B_{\delta_0}(x_0) \subset \Omega$ and there exists a $C_b^2(\mathbb{R}^n)$ function φ_{ϵ_1} satisfying that $\varphi_{\epsilon_1} = \varphi$ in $B_{\delta_0}(x_0)$, $\varphi_{\epsilon_1} \le \varphi$ in $\mathbb{R}^n$, $\inf_{x\in B_{2\delta_0}^c(x_0)}\{w_*(x) - \varphi_{\epsilon_1}(x)\} \ge \epsilon_1 > 0$ and

$$I(x_0, \varphi_{\epsilon_1}(x_0), \varphi_{\epsilon_1}(\cdot)) < -\tfrac{1}{2}\epsilon_0. \tag{3-2}$$

Thus, by (A0), there exists $\delta_1 < \delta_0$ such that, for any $x \in B_{\delta_1}(x_0)$,

$$I(x, \varphi_{\epsilon_1}(x), \varphi_{\epsilon_1}(\cdot)) < -\tfrac{1}{4}\epsilon_0. \tag{3-3}$$

By the definition of w, we have $\varphi_{\epsilon_1} \le w_* \le \bar{u}$ in $\mathbb{R}^n$. If $\varphi_{\epsilon_1}(x_0) = w_*(x_0) = \bar{u}(x_0)$, then $\bar{u} - \varphi_{\epsilon_1}$ has a strict minimum at the point x_0 over $\mathbb{R}^n$. Since $\bar{u}$ is a viscosity supersolution of $I = 0$ in Ω, we have

$$I(x_0, \varphi_{\epsilon_1}(x_0), \varphi_{\epsilon_1}(\cdot)) \ge 0,$$

which contradicts (3-2). Thus, we have $\varphi_{\epsilon_1}(x_0) < \bar{u}(x_0)$. Since $\bar{u}$ and φ_{ϵ_1} are continuous functions in $\mathbb{R}^n$, we have $\varphi_{\epsilon_1}(x) < \bar{u}(x) - \epsilon_2$ in $B_{\delta_2}(x_0)$ for some $0 < \delta_2 < \delta_1$ and $\epsilon_2 > 0$. We define

$$\Delta_r = \sup_{x \in B_r^c(x_0)} \{\varphi_{\epsilon_1}(x) - w_*(x)\}.$$

Since $\inf_{x \in B^c_{2\delta_0}(x_0)}\{w_*(x) - \varphi_{\epsilon_1}(x)\} \ge \epsilon_1 > 0$, $w_* - \varphi_{\epsilon_1}$ has a strict minimum (equal to 0) at the point x_0 and $-w_* \in \mathrm{USC}(\mathbb{R}^n)$, we have $\Delta_r < 0$ for each $r > 0$. For any $y \in \bar{\Omega} \setminus B_r(x_0)$, there exists a function $v_y \in \mathcal{F}$ such that $v_y(y) - \varphi_{\epsilon_1}(y) \ge -\frac{3}{4}\Delta_r$. Since v_y and φ_{ϵ_1} are continuous in $\mathbb{R}^n$, there exists a positive constant δ_y such that $\inf_{x \in B_{\delta_y}(y)}\{v_y(x) - \varphi_{\epsilon_1}(x)\} \ge -\frac{1}{2}\Delta_r$. Since $\bar{\Omega} \setminus B_r(x_0)$ is a compact set in $\mathbb{R}^n$, there exists a finite set $\{y_i\}_{i=1}^{n_r} \subset \bar{\Omega} \setminus B_r(x_0)$ such that $\bar{\Omega} \setminus B_r(x_0) \subset \bigcup_{i=1}^{n_r} B_{\delta_{y_i}}(y_i)$. Thus, we define

$$v_r(x) = \sup_{1 \le i \le n_r} \{v_{y_i}(x)\}, \quad x \in \mathbb{R}^n.$$

By Lemma 3.1 and the definition of v_r, we have $v_r \in \mathcal{F}$ and $\inf_{x \in \bar{\Omega} \setminus B_r(x_0)}\{v_r(x) - \varphi_{\epsilon_1}(x)\} \ge -\frac{1}{2}\Delta_r$. Let α_r be a constant such that $0 < \alpha_r < \frac{1}{2}$ and $-\alpha_r\Delta_r < \epsilon_2$. Thus, we define

$$U(x) = \begin{cases} \max\{\varphi_{\epsilon_1}(x) - \alpha\Delta_r, v_r(x)\}, & x \in B_r(x_0), \\ v_r(x), & x \in B_r^c(x_0), \end{cases}$$

where $0 < r < \delta_2$ and $0 < \alpha < \alpha_r$. By the definition of U, we obtain $U \in C^0(\mathbb{R}^n)$, $\underline{u} \le U \le \bar{u}$ in $\mathbb{R}^n$, and there exists a sequence $\{x_n\}_n \subset B_r(x_0)$ such that $x_n \to x_0$ as $n \to +\infty$ and $U(x_n) > w(x_n)$.

We claim that U is a viscosity subsolution of $I = 0$ in Ω. For any $y \in \Omega$, suppose that there is a function $\psi \in C_b^2(\mathbb{R}^n)$ such that $U - \psi$ has a maximum (equal to 0) at y over $\mathbb{R}^n$. We then divide the proof into two cases.

<u>Case 1</u>: $U(y) = v_r(y)$. Since $v_r \le U \le \psi$ in $\mathbb{R}^n$, we know $v_r - \psi$ has a maximum (equal to 0) at y over $\mathbb{R}^n$. We recall that v_r is a viscosity subsolution of $I = 0$ in Ω. Therefore, we have

$$I(y, U(y), \psi(\cdot)) \le 0.$$

<u>Case 2</u>: $U(y) = \varphi_{\epsilon_1}(y) - \alpha\Delta_r$. We first notice that $y \in B_r(x_0)$. Since $\varphi_{\epsilon_1} - \alpha\Delta_r \le U \le \psi$ in $B_r(x_0)$, then $\varphi_{\epsilon_1} - \alpha\Delta_r - \psi \le 0$ in $B_r(x_0)$. By the definition of U, we have $\psi \ge U = v_r$ in $B_r^c(x_0)$. Thus, $\varphi_{\epsilon_1} - \alpha\Delta_r - \psi \le \varphi_{\epsilon_1} - \alpha\Delta_r - v_r \le \frac{1}{2}\Delta_r - \alpha\Delta_r \le 0$ in $B_r^c(x_0)$. Therefore, we have $\varphi_{\epsilon_1} - \alpha\Delta_r - \psi$ has a maximum (equal to 0) at $y \in B_r(x_0) \subset B_{\delta_1}(x_0)$ over $\mathbb{R}^n$. Since (3-3), (A0), (A3)–(A4) hold, we can choose α independent of ψ and sufficiently small that

$$I(y, \psi(y), \psi(\cdot)) \le I(y, \varphi_{\epsilon_1}(y) - \alpha\Delta_r, \varphi_{\epsilon_1}(\cdot)) \le 0.$$

Based on the two cases, we have that U is a viscosity subsolution of $I=0$ in Ω. Therefore, $U\in\mathcal{F}$, which contradicts with the definition of w. Thus, w is a discontinuous viscosity supersolution of $I=0$ in Ω. Therefore, w is a discontinuous viscosity solution of $I=0$ in Ω. Since $w=g$ in Ω^c, we know w is a discontinuous viscosity solution of (1-1). □

Remark 3.3. Under the assumptions of Theorem 3.2, if the comparison principle holds for (1-1), the discontinuous viscosity solution w is the unique viscosity solution of (1-1). For example, if I is a translation-invariant nonlocal operator, (1-1) admits a unique viscosity solution.

Before applying Theorem 3.2 to (1-2), we now give the precise assumptions on its equation. For any $0<\lambda\le\Lambda$ and $0<\sigma<2$, we consider the family of kernels $K:\mathbb{R}^n\to\mathbb{R}$ satisfying the following assumptions:

(H0) $K(z)\ge 0$ for any $z\in\mathbb{R}^n$.

(H1) For any $\delta>0$,

$$\int_{B_{2\delta}\setminus B_\delta} K(z)\,dz\le(2-\sigma)\Lambda\delta^{-\sigma}.$$

(H2) For any $\delta>0$,

$$\left|\int_{B_{2\delta}\setminus B_\delta} zK(z)\,dz\right|\le\Lambda|1-\sigma|\delta^{1-\sigma}.$$

We define our nonlocal operator

$$I_{ab}[x,u]:=\int_{\mathbb{R}^n}\delta_z u(x)K_{ab}(x,z)\,dz, \tag{3-4}$$

where

$$\delta_z u(x):=\begin{cases}u(x+z)-u(x) & \text{if } \sigma<1,\\ u(x+z)-u(x)-\mathbb{1}_{B_1}(z)\nabla u(x)\cdot z & \text{if } \sigma=1,\\ u(x+z)-u(x)-\nabla u(x)\cdot z & \text{if } \sigma>1.\end{cases}$$

We consider the following nonlocal Bellman–Isaacs equation

$$\sup_{a\in\mathcal{A}}\inf_{b\in\mathcal{B}}\big\{-I_{ab}[x,u]+b_{ab}(x)\cdot\nabla u(x)+c_{ab}(x)u(x)+f_{ab}(x)\big\}=0\quad\text{in }\Omega. \tag{3-5}$$

Corollary 3.4. *Assume that $0<\sigma<2$, $b_{ab}\equiv 0$ in Ω if $\sigma<1$ and $c_{ab}\ge 0$ in Ω. Let $\underline{u}$, $\bar{u}$ be bounded continuous functions and be respectively a viscosity subsolution and a viscosity supersolution of (3-5), where $\{K_{ab}(\cdot,z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$ and $\{f_{ab}\}_{a,b}$ are sets of uniformly continuous functions in Ω, uniformly in $a\in\mathcal{A}$, $b\in\mathcal{B}$, and $\{K_{ab}(x,\cdot):x\in\Omega,\ a\in\mathcal{A},\ b\in\mathcal{B}\}$ are kernels satisfying (H0)–(H2). Assume moreover that $\bar{u}=\underline{u}=g$ in Ω^c for some bounded continuous function g and $\underline{u}\le\bar{u}$ in $\mathbb{R}^n$. Then*

$$w(x)=\sup_{u\in\mathcal{F}}u(x),$$

where

$$\mathcal{F}=\big\{u\in C^0(\mathbb{R}^n):\underline{u}\le u\le\bar{u}\text{ in }\mathbb{R}^n\text{ and }u\text{ is a viscosity subsolution of (3-5)}\big\},$$

is a discontinuous viscosity solution of (1-2).

Proof. We define

$$I(x,r,u(\cdot)) := \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} \{-I_{ab}[x,u] + b_{ab}(x)\cdot\nabla u(x) + c_{ab}(x)r + f_{ab}(x)\}.$$

It follows from (H1) and (H2) that I_{ab} satisfies (1-4); see Lemma 2.3 in [Schwab and Silvestre 2016]. Then, by (1-4) and uniform continuity of the coefficients, (A0) and (A1) hold. Since $c_{ab} \geq 0$ in Ω, (A2) holds. By (H0) and the structure of I_{ab}, (A3) and (A4) hold. □

4. Hölder estimates

In this section we give Hölder estimates of the discontinuous viscosity solution constructed by Perron's method in the previous section. To obtain Hölder estimates, we will assume that the nonlocal operator I is uniformly elliptic.

We define $\mathcal{L} := \mathcal{L}(\sigma,\lambda,\Lambda)$ to be the class of all the nonlocal operators of form

$$Lu(x) := \int_{\mathbb{R}^n} \delta_z u(x) K(z)\,dz,$$

where K is a kernel satisfying the assumptions (H0)–(H2) given above and the following assumption:

(H3) There exist positive constants λ and μ such that, for any $\delta > 0$, there is a set A_δ satisfying

- (i) $A_\delta \subset B_{2\delta} \setminus B_\delta$;
- (ii) $A_\delta = -A_\delta$;
- (iii) $|A_\delta| \geq \mu |B_{2\delta} \setminus B_\delta|$;
- (iv) $K(z) \geq (2-\sigma)\lambda\delta^{-n-\sigma}$ for any $z \in A_\delta$.

We note that we will also write $K \in \mathcal{L}$ if the corresponding nonlocal operator $L \in \mathcal{L}$. We then define the extremal operators

$$M^+_{\mathcal{L}} u(x) := \sup_{L\in\mathcal{L}} Lu(x), \quad M^-_{\mathcal{L}} u(x) := \inf_{L\in\mathcal{L}} Lu(x).$$

We denote by $m : [0,+\infty) \to [0,+\infty)$ a modulus of continuity. We say that the nonlocal operator I is uniformly elliptic if for every $r, s \in \mathbb{R},\ x \in \Omega,\ \delta > 0,\ \varphi,\psi \in C^2(B_\delta(x)) \cap L^\infty(\mathbb{R}^n)$,

$$\begin{aligned} M^-_{\mathcal{L}}(\varphi-\psi)(x) - C_0|\nabla(\psi-\varphi)(x)| - m(|r-s|) &\leq I(x,r,\psi(\cdot)) - I(x,s,\varphi(\cdot)) \\ &\leq M^+_{\mathcal{L}}(\varphi-\psi)(x) + C_0|\nabla(\psi-\varphi)(x)| + m(|r-s|), \end{aligned}$$

where C_0 is a nonnegative constant such that $C_0 = 0$ if $\sigma < 1$.

Remark 4.1. The definition of uniform ellipticity is different from that in [Schwab and Silvestre 2016] since the nonlocal operator I contains the second component r.

Lemma 4.2. *If the nonlocal operator I is uniformly elliptic and satisfies (A0), (A2), then I satisfies (A0)–(A4).*

Proof. Suppose that $\delta > 0$, $x_k \to x$ in Ω, $\varphi_k \to \varphi$ a.e. in $\mathbb{R}^n$, $\varphi_k \to \varphi$ in $C^2(B_\delta(x))$ and $\{\varphi_k\}_k$ is uniformly bounded in $\mathbb{R}^n$. Since I is uniformly elliptic, we have, for any $r \in \mathbb{R}$,

$$M^-_{\mathcal{L}}(\varphi-\varphi_k)(x_k) - C_0|\nabla(\varphi_k-\varphi)(x_k)| \le I(x_k, r, \varphi_k(\cdot)) - I(x_k, r, \varphi(\cdot)) \le M^+_{\mathcal{L}}(\varphi-\varphi_k)(x_k) + C_0|\nabla(\varphi_k-\varphi)(x_k)|. \tag{4-1}$$

Since $K \in \mathcal{L}$, we know, by Lemma 2.3 in [Schwab and Silvestre 2016], that K satisfies (1-4). Letting $k \to +\infty$ in (4-1), we have, by (A0),

$$\lim_{k\to+\infty} I(x_k, r, \varphi_k(\cdot)) = I(x, r, \varphi(\cdot)).$$

Therefore, (A1) holds. For any constant C, we have

$$0 = M^-_{\mathcal{L}}(-C) - C_0|\nabla C| \le I(x, r, \varphi(\cdot)+C) - I(x, r, \varphi(\cdot)) \le M^+_{\mathcal{L}}(-C) + C_0|\nabla C| = 0.$$

Thus, (A3) holds. If φ touches a $C^2(B_\delta(x)) \cap L^\infty(\mathbb{R}^n)$ function ψ from above at x, then

$$I(x, r, \varphi) - I(x, r, \psi) \le M^+_{\mathcal{L}}(\psi-\varphi)(x) \le 0.$$

Therefore, (A4) holds. □

The following lemma is an elliptic version of Theorem 6.1 in [Schwab and Silvestre 2016].

Lemma 4.3. *Assume $0 < \sigma_0 \le \sigma < 2$, $C_0, C_1 \ge 0$, and further assume $C_0 = 0$ if $\sigma < 1$. Let u be a viscosity supersolution of*

$$M^-_{\mathcal{L}} u - C_0|\nabla u| = C_1 \quad \text{in } B_2$$

and $u \ge 0$ in $\mathbb{R}^n$. Then there exist constants C and ϵ_3 such that

$$\left(\int_{B_1} u^{\epsilon_3}\, dx\right)^{\frac{1}{\epsilon_3}} \le C(\inf_{B_1} u + C_1),$$

where ϵ_3 and C depend on σ_0, λ, Λ, C_0, n and μ.

The following lemma is a direct corollary of Lemma 4.3.

Corollary 4.4. *Assume $0 < \sigma_0 \le \sigma < 2$, $0 < r < 1$, $C_0, C_1 \ge 0$, and further assume $C_0 = 0$ if $\sigma < 1$. Let u be a viscosity supersolution of*

$$M^-_{\mathcal{L}} u - C_0|\nabla u| = C_1 \quad \text{in } B_{2r}$$

and $u \ge 0$ in $\mathbb{R}^n$. Then there exist constants C and ϵ_3 such that

$$\left(\left|\{u > t\} \cap B_r\right|\right) \le C r^n (u(0) + C_1 r^\sigma)^{\epsilon_3} t^{-\epsilon_3} \quad \text{for any } t \ge 0, \tag{4-2}$$

where ϵ_3 and C depend on σ_0, λ, Λ, C_0, n and μ.

Proof. Now let $v(x) = u(rx)$. By Lemma 2.2 in [Schwab and Silvestre 2016], we have

$$M^-_{\mathcal{L}} v - C_0 r^{\sigma-1}|\nabla v| \le C_1 r^\sigma \quad \text{in } B_2. \tag{4-3}$$

Now we apply Lemma 4.3 to (4-3). Thus, for any $t \geq 0$, we have

$$t\big|\{v>t\}\cap B_1\big|^{\frac{1}{\epsilon_3}} \leq \left(\int_{B_1} v^{\epsilon_3}\,dx\right)^{\frac{1}{\epsilon_3}} \leq C(\inf_{B_1} v + C_1 r^\sigma) \leq C(v(0)+C_1 r^\sigma).$$

Then

$$r^{-n}\big|\{u>t\}\cap B_r\big| \leq \big|\{v>t\}\cap B_1\big| \leq C(v(0)+C_1 r^\sigma)^{\epsilon_3} t^{-\epsilon_3} = C(u(0)+C_1 r^\sigma)^{\epsilon_3} t^{-\epsilon_3}.$$

Therefore, (4-2) holds. □

Then we follow the idea in [Caffarelli and Silvestre 2009] to obtain a Hölder estimate.

Theorem 4.5. *Assume $0<\sigma_0\leq\sigma<2$, $C_0\geq 0$, and further assume $C_0=0$ if $\sigma<1$. For any $\epsilon>0$, let $\mathcal{F}$ be a class of bounded continuous functions u in $\mathbb{R}^n$ such that $-\frac{1}{2}\leq u\leq\frac{1}{2}$ in $\mathbb{R}^n$, u is a viscosity subsolution of $M_{\mathcal{L}}^+ u + C_0|\nabla u| = -\frac{1}{2}\epsilon$ in B_1 and $w=\sup_{u\in\mathcal{F}} u$ is a discontinuous viscosity supersolution of $M_{\mathcal{L}}^- w - C_0|\nabla w| = \frac{1}{2}\epsilon$ in B_1. Then there exist constants ϵ_4, α and C such that, if $\epsilon<\epsilon_4$,*

$$-C|x|^\alpha \leq w_*(x)-w^*(0) \leq w^*(x)-w_*(0) \leq C|x|^\alpha,$$

where ϵ_4, α and C depend on σ_0, λ, Λ, C_0, n and μ.

Proof. We claim that there exist an increasing sequence $\{m_k\}_k$ and a decreasing sequence $\{M_k\}_k$ such that $M_k-m_k=8^{-\alpha k}$ and $m_k\leq\inf_{B_{8^{-k}}} w_* \leq \sup_{B_{8^{-k}}} w^* \leq M_k$. We will prove this claim by induction.

For $k=0$, we choose $m_0=-\frac{1}{2}$ and $M_0=\frac{1}{2}$ since $-\frac{1}{2}\leq u\leq\frac{1}{2}$ for any $u\in\mathcal{F}$. Assume that we have the sequences up to m_k and M_k. In $B_{8^{-k-1}}$, we have either

$$\Big|\Big\{w_* \geq \tfrac{1}{2}M_k+m_k\Big\}\cap B_{8^{-k-1}}\Big| \geq \tfrac{1}{2}|B_{8^{-k-1}}| \tag{4-4}$$

or

$$\Big|\Big\{w_* \leq \tfrac{1}{2}M_k+m_k\Big\}\cap B_{8^{-k-1}}\Big| \geq \tfrac{1}{2}|B_{8^{-k-1}}|. \tag{4-5}$$

Case 1: (4-4) holds. We define

$$v(x) := \frac{w_*(8^{-k}x)-m_k}{\frac{1}{2}(M_k-m_k)}.$$

Thus, $v\geq 0$ in B_1 and

$$\big|\{v\geq 1\}\cap B_{\frac{1}{8}}\big| \geq \tfrac{1}{2}\big|B_{\frac{1}{8}}\big|.$$

Since w is a discontinuous viscosity supersolution of $M_{\mathcal{L}}^- w - C_0|\nabla w| = \frac{1}{2}\epsilon$ in B_1, we know v is a viscosity supersolution of

$$M_{\mathcal{L}}^- v - C_0 8^{k(1-\sigma)}|\nabla v| = 8^{k(\alpha-\sigma)}\epsilon \quad \text{in } B_{8^k}.$$

We notice that $C_0=0$ if $\sigma<1$ and choose $\alpha<\sigma_0$. Thus, for any $0<\sigma<2$, v is a viscosity supersolution of

$$M_{\mathcal{L}}^- v - C_0|\nabla v| = \epsilon \quad \text{in } B_{8^k}.$$

By the inductive assumption, we have, for any $k \ge j \ge 0$,

$$v \ge \frac{m_{k-j} - m_k}{\frac{1}{2}(M_k - m_k)} \ge \frac{m_{k-j} - M_{k-j} + M_k - m_k}{\frac{1}{2}(M_k - m_k)} = 2(1 - 8^{\alpha j}) \quad \text{in } B_{8^j}. \tag{4-6}$$

Moreover, we have

$$v \ge 2 \cdot 8^{\alpha k}\left[-\tfrac{1}{2} - \left(\tfrac{1}{2} - 8^{-\alpha k}\right)\right] = 2(1 - 8^{\alpha k}) \quad \text{in } B^c_{8^k}. \tag{4-7}$$

By (4-6) and (4-7), we have

$$v(x) \ge -2(|8x|^\alpha - 1) \quad \text{for any } x \in B_1^c.$$

We define

$$v^+(x) := \max\{v(x), 0\} \quad \text{and} \quad v^-(x) := -\min\{v(x), 0\}.$$

Since $v \ge 0$ in B_1, we have $v^-(x) = 0$ and $\nabla v^-(x) = 0$ for any $x \in B_1$. By (H1), we can choose α independent of σ and sufficiently small that, for any $x \in B_{\frac{3}{4}}$ and $\sigma_0 \le \sigma < 2$,

$$\begin{aligned}
M^-_{\mathcal{L}} v^+(x) &\le M^-_{\mathcal{L}} v(x) + M^+_{\mathcal{L}} v^-(x) \\
&\le M^-_{\mathcal{L}} v(x) + \sup_{K \in \mathcal{L}} \int_{\mathbb{R}^n} \delta_z v^-(x) K(z)\, dz \\
&\le M^-_{\mathcal{L}} v(x) + \sup_{K \in \mathcal{L}} \int_{B^c_{\frac{1}{4}} \cap \{v(x+z) < 0\}} v^-(x+z) K(z)\, dz \\
&\le M^-_{\mathcal{L}} v(x) + \sup_{K \in \mathcal{L}} \int_{B^c_{\frac{1}{4}}} \max\bigl\{2(|8(x+z)|^\alpha - 1), 0\bigr\} K(z)\, dz \\
&\le M^-_{\mathcal{L}} v(x) + 2(2-\sigma)\Lambda \sum_{l=0}^{+\infty} \left(\frac{2^l}{4}\right)^{-\sigma} (2^{(l+4)\alpha} - 1) \\
&\le M^-_{\mathcal{L}} v(x) + 2^{13}(2-\sigma_0)\Lambda \left(\frac{2^{4(\alpha - \sigma_0)}}{1 - 2^{\alpha - \sigma_0}} - \frac{2^{-4\sigma_0}}{1 - 2^{-\sigma_0}}\right) \le M^-_{\mathcal{L}} v(x) + \epsilon.
\end{aligned}$$

Therefore, we have

$$M^-_{\mathcal{L}} v^+ - C_0 |\nabla v^+| \le 2\epsilon \quad \text{in } B_{\frac{3}{4}}.$$

Given any point $x \in B_{1/8}$, we can apply Corollary 4.4 in $B_{1/4}(x)$ to obtain

$$C(v^+(x) + 2\epsilon)^{\epsilon_3} \ge \left|\{v^+ > 1\} \cap B_{\frac{1}{4}}(x)\right| \ge \left|\{v^+ > 1\} \cap B_{\frac{1}{8}}\right| \ge \tfrac{1}{2}\left|B_{\frac{1}{8}}\right|.$$

Thus, we can choose ϵ_4 sufficiently small that $v^+ \ge \epsilon_4$ in $B_{1/8}$ if $\epsilon < \epsilon_4$. Therefore,

$$v(x) = \frac{w_*(8^{-k}x) - m_k}{\frac{1}{2}(M_k - m_k)} \ge \epsilon_4 \quad \text{in } B_{\frac{1}{8}}.$$

If we set $m_{k+1} = m_k + \frac{1}{2}\epsilon_4(M_k - m_k)$ and $M_{k+1} = M_k$, we must have

$$m_{k+1} \le \inf_{B_{8^{-k-1}}} w_* \le \sup_{B_{8^{-k-1}}} w^* \le M_{k+1}.$$

Case 2: (4-5) holds. For any $u \in \mathcal{F}$, we obtain that $u \in C^0(\mathbb{R}^n)$ is a viscosity subsolution of $M^+_{\mathcal{L}} u + C_0|\nabla u| = -\frac{1}{2}\epsilon$ in B_1 and $u \le w_*$ in $\mathbb{R}^n$. Thus, we have

$$\left|\left\{u \le \tfrac{1}{2}(M_k + m_k)\right\} \cap B_{8^{-k-1}}\right| \ge \tfrac{1}{2}|B_{8^{-k-1}}|.$$

We define

$$v_u(x) := \frac{M_k - u(8^{-k}x)}{\frac{1}{2}(M_k - m_k)}.$$

Thus, $v_u \ge 0$ in B_1 and

$$\left|\{v_u \ge 1\} \cap B_{\frac{1}{8}}\right| \ge \tfrac{1}{2}\left|B_{\frac{1}{8}}\right|.$$

Since u is a viscosity subsolution of $M^+_{\mathcal{L}} u + C_0|\nabla u| = -\frac{1}{2}\epsilon$ in B_1, then v_u is a viscosity supersolution of

$$M^-_{\mathcal{L}} v_u - C_0|\nabla v_u| = \epsilon \quad \text{in } B_{8^k}.$$

Similar to Case 1, we have, if $\epsilon < \epsilon_4$,

$$v_u(x) = \frac{M_k - u(8^{-k}x)}{\frac{1}{2}(M_k - m_k)} \ge \epsilon_4 \quad \text{in } B_{\frac{1}{8}},$$

which implies

$$u(8^{-k}x) \le M_k - \tfrac{1}{2}\epsilon_4(M_k - m_k) \quad \text{in } B_{\frac{1}{8}}.$$

By the definition of w, we have

$$w^*(8^{-k}x) \le M_k - \tfrac{1}{2}\epsilon_4(M_k - m_k) \quad \text{in } B_{\frac{1}{8}}.$$

If we set $m_{k+1} = m_k$ and $M_{k+1} = M_k - \frac{1}{2}\epsilon_4(M_k - m_k)$, we must have

$$m_{k+1} \le \inf_{B_{8^{-k-1}}} w_* \le \sup_{B_{8^{-k-1}}} w^* \le M_{k+1}.$$

Therefore, in both of the cases, we have $M_{k+1} - m_{k+1} = \left(1 - \frac{1}{2}\epsilon_4\right)8^{-\alpha k}$. We then choose α and ϵ_4 sufficiently small that $\left(1 - \frac{1}{2}\epsilon_4\right) = 8^{-\alpha}$. Thus we have $M_{k+1} - m_{k+1} = 8^{-\alpha(k+1)}$. □

Theorem 4.6. *Assume that $0 < \sigma_0 \le \sigma < 2$ and $I(x,0,0)$ is bounded in Ω. Assume that I is uniformly elliptic and satisfies (A0), (A2). Let w be the bounded discontinuous viscosity solution of (1-1) constructed in Theorem 3.2. Then, for any sufficiently small $\tilde{\delta} > 0$, there exists a constant C such that $w \in C^\alpha(\Omega)$ and*

$$\|w\|_{C^\alpha(\bar{\Omega}_{\tilde{\delta}})} \le C\left(C_2 + m(C_2) + \|I(\cdot,0,0)\|_{L^\infty(\Omega)}\right),$$

where α is given in Theorem 4.5, $C_2 := \max\{\|\underline{u}\|_{L^\infty(\mathbb{R}^n)}, \|\bar{u}\|_{L^\infty(\mathbb{R}^n)}\}$ and C depends on σ_0, $\tilde{\delta}$, λ, Λ, C_0, n, μ.

Proof. It is obvious that $\|u\|_{L^\infty(\mathbb{R}^n)} \le C_2$ if $u \in \mathcal{F}$. Since I is uniformly elliptic, we have

$$I(x,0,0) - I(x,u(x),u(\cdot)) \le M^+_{\mathcal{L}} u(x) + C_0|\nabla u(x)| + m(C_2) \quad \text{in } \Omega.$$

Since u is a viscosity subsolution of $I = 0$ in Ω, we have

$$-m(C_2) - \|I(\cdot,0,0)\|_{L^\infty(\Omega)} \le M^+_{\mathcal{L}} u + C_0|\nabla u| \quad \text{in } \Omega.$$

Similarly, we have

$$M^-_{\mathcal{L}} w_* - C_0|\nabla w_*| \le m(C_2) + \|I(\cdot,0,0)\|_{L^\infty(\Omega)} \quad \text{in } \Omega.$$

By normalization, the result follows from Theorem 4.5. □

By applying Theorem 4.6 to Bellman–Isaacs equation, we have the following corollary.

Corollary 4.7. *Assume that* $0 < \sigma_0 \le \sigma < 2$, $b_{ab} \equiv 0$ *in* Ω *if* $\sigma < 1$ *and* $c_{ab} \ge 0$ *in* Ω. *Assume that* $\{K_{ab}(\cdot,z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$, $\{f_{ab}\}_{a,b}$ *are sets of uniformly bounded and continuous functions in* Ω, *uniformly in* $a \in \mathcal{A}$, $b \in \mathcal{B}$, *and* $\{K_{ab}(x,\cdot) : x \in \Omega,\ a \in \mathcal{A},\ b \in \mathcal{B}\}$ *are kernels satisfying (H0)–(H3). Let* w *be the bounded discontinuous viscosity solution of (1-2) constructed in Corollary 3.4. Then, for any sufficiently small* $\tilde{\delta} > 0$, *there exists a constant* C *such that* $w \in C^\alpha(\Omega)$ *and*

$$\|w\|_{C^\alpha(\overline{\Omega}_{\tilde{\delta}})} \le C\big(C_2 + \sup_{a\in\mathcal{A},b\in\mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)}\big),$$

where α *and* C_2 *are given in Theorem 4.6 and* C *depends on* σ_0, $\tilde{\delta}$, λ, Λ, $\sup_{a\in\mathcal{A},b\in\mathcal{B}} \|b_{ab}\|_{L^\infty(\Omega)}$, $\sup_{a\in\mathcal{A},b\in\mathcal{B}} \|c_{ab}\|_{L^\infty(\Omega)}$, n, μ.

Remark 4.8. In this section we assume our nonlocal equations satisfy the weak uniform ellipticity introduced in [Schwab and Silvestre 2016] mainly because, to our knowledge, this is the weakest assumption to get the weak Harnack inequality. In fact, our approach to get Hölder continuity of the discontinuous viscosity solution constructed by Perron's method could be applied to more general nonlocal equations as long as the weak Harnack inequality holds for such an equation.

5. Continuous sub/supersolutions

In this section we construct continuous sub/supersolutions in both uniformly elliptic and degenerate cases.

5A. ***Uniformly elliptic case.*** In the uniformly elliptic case, we follow the idea in [Ros-Oton and Serra 2016] to establish barrier functions. We define $v_\alpha(x) = ((x_1 - 1)^+)^\alpha$, where $0 < \alpha < 1$ and $x = (x_1, x_2, \dots, x_n)$.

Lemma 5.1. *Assume that* $0 < \sigma < 2$. *Then there exists a sufficiently small* $\alpha > 0$ *such that*

$$M^+_{\mathcal{L}} v_\alpha((1+r)e_1) \le -\epsilon_5 r^{\alpha-\sigma}$$

for any $r > 0$, *where* $e_1 = (1,0,\dots,0)$ *and* ϵ_5 *is some positive constant.*

Proof. Case 1: $0<\sigma<1$. By Lemma 2.2 in [Schwab and Silvestre 2016], we have, for any $r>0$ and $\alpha>0$,

$$\begin{aligned}M_{\mathcal{L}}^{+}v_\alpha((1+r)e_1) &= \sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(v_\alpha((1+r)e_1+z)-v_\alpha((1+r)e_1)\big)K(z)\,dz\\ &= \sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((r+z_1)^+)^\alpha-r^\alpha\big)K(z)\,dz\\ &= r^{\alpha-\sigma}\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((1+z_1)^+)^\alpha-1\big)r^{n+\sigma}K(rz)\,dz\\ &= r^{\alpha-\sigma}\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((1+z_1)^+)^\alpha-1\big)K(z)\,dz\\ &\le r^{\alpha-\sigma}\bigg(\sup_{K\in\mathcal{L}}\int_{z_1>-1}\big((1+z_1)^\alpha-1\big)K(z)\,dz-\inf_{K\in\mathcal{L}}\int_{z_1\le-1}K(z)\,dz\bigg).\end{aligned}$$

By (H3), we have, for any $K\in\mathcal{L}$ and any $\delta>0$, there is a set A_δ satisfying $A_\delta\subset B_{2\delta}\setminus B_\delta$, $A_\delta=-A_\delta$, $|A_\delta|\ge\mu|B_{2\delta}\setminus B_\delta|$ and $K(z)\ge(2-\sigma)\lambda\delta^{-n-\sigma}$ in A_δ. It is obvious that

$$\mu_\delta:=\frac{\big|(B_{2\delta}\setminus B_\delta)\cap\{z:|z_1|<1\}\big|}{|B_{2\delta}\setminus B_\delta|}\to0\quad\text{as }\delta\to+\infty.$$

Thus, there exists $\delta_3>0$ such that $\mu_\delta<\frac{1}{2}\mu$ if $\delta\ge\delta_3$. Then

$$\frac{\big|\{z:|z_1|\ge1\}\cap A_{\delta_3}\big|}{|B_{2\delta_3}\setminus B_{\delta_3}|}\ge\frac{|A_{\delta_3}|-\big|(B_{2\delta_3}\setminus B_{\delta_3})\cap\{z:|z_1|<1\}\big|}{|B_{2\delta_3}\setminus B_{\delta_3}|}\ge\frac{\mu}{2}.$$

By the symmetry of A_{δ_3}, we have

$$\frac{\big|\{z:z_1\le-1\}\cap A_{\delta_3}\big|}{|B_{2\delta_3}\setminus B_{\delta_3}|}\ge\frac{\mu}{4}.$$

Therefore, we have, for any $K\in\mathcal{L}$,

$$\int_{z_1\le-1}K(z)\,dz\ge\int_{\{z:z_1\le-1\}\cap A_{\delta_3}}K(z)\,dz\ge\frac{(2-\sigma)\lambda\mu}{4}\delta_3^{-n-\sigma}|B_{2\delta_3}\setminus B_{\delta_3}|=:2\epsilon_5. \tag{5-1}$$

By (H1) and (H2), we have, for any $K\in\mathcal{L}$,

$$\begin{aligned}\int_{z_1>-1}((1+z_1)^\alpha-1)K(z)\,dz &= \int_{\{z:z_1>-1\}\cap B_{\frac12}}+\int_{\{z:z_1>-1\}\cap B^c_{\frac12}}\\ &\le\alpha2^{1-\alpha}\bigg|\int_{B_{\frac12}}zK(z)\,dz\bigg|+\int_{\{z:z_1>-1\}\cap B^c_{\frac12}}((1+z_1)^\alpha-1)K(z)\,dz\\ &\le\alpha2^{1-\alpha}(1-\sigma)\Lambda\sum_{l=0}^{+\infty}\bigg(\frac{1}{2^{l+2}}\bigg)^{1-\sigma}+(2-\sigma)\Lambda\sum_{l=0}^{+\infty}(2^{l-1})^{-\sigma}((1+2^l)^\alpha-1)\\ &\le2\alpha\Lambda\frac{1-\sigma}{1-2^{\sigma-1}}+8\Lambda\bigg(\frac{2^{\alpha-\sigma}}{1-2^{\alpha-\sigma}}-\frac{2^{-\sigma}}{1-2^{-\sigma}}\bigg).\end{aligned} \tag{5-2}$$

Thus, we have

$$\lim_{\alpha\to 0^+}\sup_{K\in\mathcal{L}}\int_{z_1>-1}((1+z_1)^\alpha-1)K(z)\,dz-\inf_{K\in\mathcal{L}}\int_{z_1\le -1}K(z)\,dz\le -2\epsilon_5.$$

Then there exists a sufficiently small α such that

$$M_{\mathcal{L}}^+v_\alpha((1+r)e_1)\le -\epsilon_5 r^{\alpha-\sigma}.$$

Case 2: $\sigma=1$. Using (H2), we have, for any $r>0$ and $\alpha>0$,

$$\begin{aligned}M_{\mathcal{L}}^+v_\alpha((1+r)e_1)&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(v_\alpha((1+r)e_1+z)-v_\alpha((1+r)e_1)-\mathbb{1}_{B_1}(z)\nabla v_\alpha((1+r)e_1)\cdot z\big)K(z)\,dz\\&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((r+z_1)^+)^\alpha-r^\alpha-\mathbb{1}_{B_1}(z)\alpha r^{\alpha-1}z_1\big)K(z)\,dz\\&=r^{\alpha-1}\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((1+z_1)^+)^\alpha-1-\mathbb{1}_{B_{\frac{1}{r}}}(z)\alpha z_1\big)r^{n+1}K(rz)\,dz\\&=r^{\alpha-1}\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((1+z_1)^+)^\alpha-1-\mathbb{1}_{B_{\frac{1}{2}}}(z)\alpha z_1\big)K(z)\,dz\\&\le r^{\alpha-1}\bigg(\sup_{K\in\mathcal{L}}\int_{z_1>-1}\big((1+z_1)^\alpha-1-\mathbb{1}_{B_{\frac{1}{2}}}(z)\alpha z_1\big)K(z)\,dz-\inf_{K\in\mathcal{L}}\int_{z_1\le -1}K(z)\,dz\bigg).\end{aligned}$$

By (H1), we have, for any $K\in\mathcal{L}$,

$$\begin{aligned}&\int_{z_1>-1}\big((1+z_1)^\alpha-1-\mathbb{1}_{B_{\frac{1}{2}}}(z)\alpha z_1\big)K(z)\,dz\\&\quad=\int_{\{z:z_1>-1\}\cap B_{\frac{1}{2}}}((1+z_1)^\alpha-1-\alpha z_1)K(z)\,dz+\int_{\{z:z_1>-1\}\cap B^c_{\frac{1}{2}}}((1+z_1)^\alpha-1)K(z)\,dz\\&\quad\le\alpha(1-\alpha)2^{2-\alpha}\int_{B_{\frac{1}{2}}}|z|^2K(z)\,dz+\int_{\{z:z_1>-1\}\cap B^c_{\frac{1}{2}}}((1+z_1)^\alpha-1)K(z)\,dz\\&\quad\le\alpha(1-\alpha)2^{2-\alpha}\Lambda\sum_{l=0}^{+\infty}\left(\frac{1}{2^{l+2}}\right)^{-1}\left(\frac{1}{2^{l+1}}\right)^2+\Lambda\sum_{l=0}^{+\infty}(2^{l-1})^{-1}((1+2^l)^\alpha-1)\\&\quad\le 8\alpha\Lambda+4\Lambda\left(\frac{2^{\alpha-1}}{1-2^{\alpha-1}}-\frac{2^{-1}}{1-2^{-1}}\right).\end{aligned}$$

Then the rest of proof is similar to Case 1.

Case 3: $1<\sigma<2$. For any $r>0$ and $\alpha>0$, we have

$$\begin{aligned}M_{\mathcal{L}}^+v_\alpha((1+r)e_1)&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(v_\alpha((1+r)e_1+z)-v_\alpha((1+r)e_1)-\nabla v_\alpha((1+r)e_1)\cdot z\big)K(z)\,dz\\&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((r+z_1)^+)^\alpha-r^\alpha-\alpha r^{\alpha-1}z_1\big)K(z)\,dz\end{aligned}$$

$$= r^{\alpha-\sigma}\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}\big(((1+z_1)^+)^\alpha-1-\alpha z_1\big)K(z)\,dz$$
$$\le r^{\alpha-\sigma}\bigg(\sup_{K\in\mathcal{L}}\int_{z_1>-1}\big(((1+z_1)^+)^\alpha-1-\alpha z_1\big)K(z)\,dz-\inf_{K\in\mathcal{L}}\int_{z_1\le -1}(1+\alpha z_1)K(z)\,dz\bigg).$$

Using (5-1) and (H2), we have

$$\inf_{K\in\mathcal{L}}\int_{z_1\le -1}(1+\alpha z_1)K(z)\,dz \ge \inf_{K\in\mathcal{L}}\int_{z_1\le -1}K(z)\,dz-\alpha\sup_{K\in\mathcal{L}}\bigg|\int_{B_1^c} zK(z)\,dz\bigg| \ge 2\epsilon_5-\frac{\alpha\Lambda(\sigma-1)}{1-2^{1-\sigma}}.$$

By (H1) and (H2), we have, for any $K\in\mathcal{L}$,

$$\begin{aligned}\int_{z_1>-1}\big((1+z_1)^\alpha-1-\alpha z_1\big)K(z)\,dz &= \int_{\{z:z_1>-1\}\cap B_{\frac12}}+\int_{\{z:z_1>-1\}\cap B_{\frac12}^c}\\ &\le \alpha(1-\alpha)2^{2-\alpha}\int_{B_{\frac12}}|z|^2K(z)\,dz+\alpha\bigg|\int_{\{z:z_1>-1\}\cap B_{\frac12}^c} zK(z)\,dz\bigg|\\ &\qquad+\int_{\{z:z_1>-1\}\cap B_{\frac12}^c}((1+z_1)^\alpha-1)K(z)\,dz\\ &\le \frac{16\alpha(2-\sigma)\Lambda}{1-2^{\sigma-2}}+\frac{2\alpha\Lambda(\sigma-1)}{1-2^{1-\sigma}}+16(2-\sigma)\Lambda\bigg(\frac{2^{\alpha-\sigma}}{1-2^{\alpha-\sigma}}-\frac{2^{-\sigma}}{1-2^{-\sigma}}\bigg).\end{aligned}$$

Then we have

$$\begin{aligned}&\lim_{\alpha\to0^+}\sup_{K\in\mathcal{L}}\int_{z_1>-1}\big(((1+z_1)^+)^\alpha-1-\alpha z_1\big)K(z)\,dz-\inf_{K\in\mathcal{L}}\int_{z_1\le-1}(1+\alpha z_1)K(z)\,dz\\ &\quad\le \lim_{\alpha\to0^+}\frac{16\alpha(2-\sigma)\Lambda}{1-2^{\sigma-2}}+\frac{2\alpha\Lambda(\sigma-1)}{1-2^{1-\sigma}}+16(2-\sigma)\Lambda\bigg(\frac{2^{\alpha-\sigma}}{1-2^{\alpha-\sigma}}-\frac{2^{-\sigma}}{1-2^{-\sigma}}\bigg)-2\epsilon_5+\frac{\alpha\Lambda(\sigma-1)}{1-2^{1-\sigma}}\\ &\quad=-2\epsilon_5.\end{aligned}$$

Similar to Case 1, there exists a sufficiently small α such that

$$M_{\mathcal{L}}^+v_\alpha((1+r)e_1)\le-\epsilon_5 r^{\alpha-\sigma}. \qquad \square$$

Lemma 5.2. *Assume that $0<\sigma<2$, $C_0\ge0$ and further assume $C_0=0$ if $\sigma<1$. Then there are $\alpha>0$ and $0<r_0<1$ sufficiently small so that the function $u_\alpha(x):=((|x|-1)^+)^\alpha$ satisfies $M_{\mathcal{L}}^+u_\alpha+C_0|\nabla u_\alpha|\le-1$ in $\bar B_{1+r_0}\setminus\bar B_1$.*

Proof. We notice that u_α and $|\nabla|$ are rotation invariant. By Lemma 2.2 in [Schwab and Silvestre 2016], $M_{\mathcal{L}}^+$ is also rotation invariant. Then we only need to prove that $M_{\mathcal{L}}^+u_\alpha((1+r)e_1)+C_0|\nabla u_\alpha((1+r)e_1)|\le-1$ for any $r\in(0,r_0]$, where r_0 and α are sufficiently small positive constants. Note that, for all $r>0$, $u_\alpha((1+r)e_1)=v_\alpha((1+r)e_1)$, $\nabla u_\alpha((1+r)e_1)=\nabla v_\alpha((1+r)e_1)$ and

$$\big|\big(|(1+r)e_1+z|-1\big)^+-(r+z_1)^+\big|\le C|z'|^2 \quad \text{for any } z\in B_1,$$

where $z=(z_1,z')$. Therefore, we have

$$0\le (u_\alpha-v_\alpha)((1+r)e_1+z)\le \begin{cases} Cr^{\alpha-1}|z'|^2, & z\in B_{\frac{r}{2}},\\ C|z'|^{2\alpha}, & z\in B_1\setminus B_{\frac{r}{2}},\\ C|z|^{\alpha}, & z\in \mathbb{R}^n\setminus B_1.\end{cases}$$

Using (H1), we have, for any $0<\sigma<2$ and $L\in\mathcal{L}$,

$$\begin{aligned} 0&\le L(u_\alpha-v_\alpha)((1+r)e_1)\\ &=\int_{\mathbb{R}^n}(u_\alpha-v_\alpha)((1+r)e_1+z)K(z)\,dz\\ &\le C\biggl(\int_{B_{\frac{r}{2}}} r^{\alpha-1}|z'|^2K(z)\,dz+\int_{B_1\setminus B_{\frac{r}{2}}}|z'|^{2\alpha}K(z)\,dz+\int_{\mathbb{R}^n\setminus B_1}|z|^{\alpha}K(z)\,dz\biggr)\\ &\le C\biggl(\int_{B_{\frac{r}{2}}} r^{\alpha-1}|z|^2K(z)\,dz+\int_{B^c_{\frac{r}{2}}}|z|^{2\alpha}K(z)\,dz\biggr)\le C\Lambda(r^{\alpha-\sigma+1}+r^{2\alpha-\sigma}). \end{aligned}$$

Thus, we have $M^+_{\mathcal{L}}(u_\alpha-v_\alpha)((1+r)e_1)\le C\Lambda(r^{\alpha-\sigma+1}+r^{2\alpha-\sigma})$. Therefore, by Lemma 5.1, there exists a sufficiently small $\alpha>0$ such that

$$\begin{aligned} &M^+_{\mathcal{L}}u_\alpha((1+r)e_1)+C_0|\nabla u_\alpha((1+r)e_1)|\\ &\qquad\le M^+_{\mathcal{L}}(u_\alpha-v_\alpha)((1+r)e_1)+M^+_{\mathcal{L}}v_\alpha((1+r)e_1)+C_0|\nabla u_\alpha((1+r)e_1)|\\ &\qquad\le C\Lambda(r^{\alpha-\sigma+1}+r^{2\alpha-\sigma})-\epsilon_5 r^{\alpha-\sigma}+\alpha C_0r^{\alpha-1}. \end{aligned}$$

We notice that $\alpha-\sigma+1>\alpha-\sigma,\ 2\alpha-\sigma>\alpha-\sigma$ and

(i) if $0<\sigma<1$, then $C_0=0$;

(ii) if $\sigma=1$, then $\alpha C_0\to 0$ as $\alpha\to 0$;

(iii) if $1<\sigma<2$, then $\alpha-1>\alpha-\sigma$.

Thus, there exist sufficiently small $0<r_0<1$ such that we have, for any $r\in(0,r_0]$,

$$M^+_{\mathcal{L}}u_\alpha((1+r)e_1)+C_0|\nabla u_\alpha((1+r)e_1)|\le -1. \tag{5-3}$$

This completes the proof. □

In the rest of this section, we assume that Ω satisfies the uniform exterior ball condition, i.e., there is a constant $r_\Omega>0$ such that, for any $x\in\partial\Omega$ and $0<r\le r_\Omega$, there exists $y^r_x\in\Omega^c$ satisfying $\overline{B}_r(y^r_x)\cap\overline{\Omega}=\{x\}$. Without loss of generality, we can assume that $r_\Omega<1$. Since Ω is a bounded domain, there exists a sufficiently large constant $R_0>0$ such that $\Omega\subset\{y:|y_1|<R_0\}$.

Remark 5.3. At this stage, we are not sure about whether the exterior ball condition is necessary for the construction of sub/supersolutions. In future work, we plan to construct sub/supersolutions under a weaker assumption on Ω, such as the cone condition.

Lemma 5.4. *Assume that* $0<\sigma<2$, $C_0\geq 0$ *and further assume* $C_0=0$ *if* $\sigma<1$. *There exists an* $\epsilon_7>0$ *such that, for any* $x\in\partial\Omega$ *and* $0<r<r_\Omega$, *there is a continuous function* $\varphi_{x,r}$ *satisfying*

$$\begin{cases}\varphi_{x,r}\equiv 0 & \text{in } \bar{B}_r(y_x^r),\\ \varphi_{x,r}>0 & \text{in } \bar{B}_r^c(y_x^r),\\ \varphi_{x,r}\geq 1 & \text{in } B_{2r}^c(y_x^r),\\ M_{\mathcal{L}}^+\varphi_{x,r}+C_0|\nabla\varphi_{x,r}|\leq-\epsilon_7 & \text{in } \Omega.\end{cases}$$

Proof. We define a uniformly continuous function φ in $\mathbb{R}^n$ such that $1\leq\varphi\leq 2$ and

$$\varphi(y)=1\quad\text{in } y_1>R_0+1,\qquad \varphi(y)=2\quad\text{in } y_1\leq R_0.$$

We pick some sufficiently large $C_3>2/r_0^\alpha$ and we define

$$\varphi_{x,r}(y)=\min\left\{\varphi(y),C_3u_\alpha\left(\frac{y-y_x^r}{r}\right)\right\},$$

where α and r_0 are defined in Lemma 5.2. It is easy to verify that $\varphi_{x,r}\equiv 0$ in $\bar{B}_r(y_x^r)$, $\varphi_{x,r}>0$ in $\bar{B}_r^c(y_x^r)$, and $\varphi_{x,r}\geq 1$ in $B_{2r}^c(y_x^r)$. By Lemma 5.2, we have $M_{\mathcal{L}}^+u_\alpha+C_0|\nabla u_\alpha|\leq -1$ in $\bar{B}_{1+r_0}\setminus\bar{B}_1$. It is obvious that, for any $y\in\bar{B}_{(1+r_0)r}(y_x^r)\setminus\bar{B}_r(y_x^r)$, we have

$$\left(M_{\mathcal{L}}^+u_\alpha\left(\frac{\cdot-y_x^r}{r}\right)\right)(y)+C_0r^{1-\sigma}\left|\left(\nabla u_\alpha\left(\frac{\cdot-y_x^r}{r}\right)\right)(y)\right|\leq -r^{-\sigma}\quad\text{for any } 0<r<r_\Omega.$$

Since $C_0=0$ if $0<\sigma<1$, and $0<r<1$, we have

$$\left(M_{\mathcal{L}}^+u_\alpha\left(\frac{\cdot-y_x^r}{r}\right)\right)(y)+C_0\left|\left(\nabla u_\alpha\left(\frac{\cdot-y_x^r}{r}\right)\right)(y)\right|\leq -1\quad\text{for any } 0<r<r_\Omega.$$

For any $y\in\bar{B}_{(1+(2/C_3)^{1/\alpha})r}(y_x^r)\setminus\bar{B}_r(y_x^r)$, we have $\varphi_{x,r}(y)=C_3u_\alpha((y-y_x^r)/r)$. Suppose that there exists a test function $\psi\in C_b^2(\mathbb{R}^n)$ that touches $\varphi_{x,r}$ from below at y. Thus, ψ/C_3 touches $u_\alpha((\cdot-y_x^r)/r)$ from below at y. Hence, $M_{\mathcal{L}}^+\psi(y)+C_0|\nabla\psi(y)|\leq -C_3$. For any $y\in\Omega\cap\bar{B}^c_{(1+(2/C_3)^{1/\alpha})r}(y_x^r)$, we have $\varphi_{x,r}(y)=\varphi(y)=\max_{\mathbb{R}^n}\varphi_{x,r}=2$. Therefore, for any $0<\sigma<2$, we have

$$\begin{aligned}(M_{\mathcal{L}}^+\varphi_{x,r})(y)+C_0|\nabla\varphi_{x,r}(y)|&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}(\varphi_{x,r}(y+z)-\varphi_{x,r}(y))K(z)\,dz\\&=\sup_{K\in\mathcal{L}}\int_{\mathbb{R}^n}(\varphi_{x,r}(y+z)-2)K(z)\,dz\\&\leq-\inf_{K\in\mathcal{L}}\int_{\{z|z_1>-y_1+R_0+1\}}K(z)\,dz\\&\leq-\inf_{K\in\mathcal{L}}\int_{\{z|z_1>2R_0+1\}}K(z)\,dz.\end{aligned}$$

By a similar estimate to (5-1), there exists a positive constant ϵ_6 such that, for any $K\in\mathcal{L}$, we have

$$\int_{\{z|z_1>2R_0+1\}}K(z)\,dz\geq\epsilon_6.$$

Then, for any $y \in \Omega \cap \overline{B}^c_{(1+(2/C_3)^{1/\alpha})r}(y^r_x)$, we have

$$M^+_{\mathcal{L}}\varphi_{x,r}(y) + C_0|\nabla\varphi_{x,r}(y)| \le -\epsilon_6. \tag{5-4}$$

Based on the above estimates, if we set $\epsilon_7 = \min\{C_3, \epsilon_6\}$, we have

$$M^+_{\mathcal{L}}\varphi_{x,r} + C_0|\nabla\varphi_{x,r}| \le -\epsilon_7 \quad \text{in } \Omega. \qquad \square$$

Theorem 5.5. *Assume that* $0 < \sigma < 2$, $I(x, 0, 0)$ *is bounded in* Ω *and* g *is a bounded continuous function in* $\mathbb{R}^n$. *Assume that* I *is uniformly elliptic and satisfies* (A0), (A2). *Then* (1-1) *admits a continuous viscosity supersolution* $\bar{u}$ *and a continuous viscosity subsolution* $\underline{u}$ *and* $\bar{u} = \underline{u} = g$ *in* Ω^c.

Proof. We only prove (1-1) admits a viscosity supersolution $\bar{u}$ and $\bar{u} = g$ in Ω^c. For a viscosity subsolution, the construction is similar. Since I is uniformly elliptic, we have, for any $x \in \Omega$,

$$-m(\|g\|_{L^\infty(\mathbb{R}^n)}) \le I(x, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0) - I(x, 0, 0) \le m(\|g\|_{L^\infty(\mathbb{R}^n)}).$$

Thus, we have $\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\|_{L^\infty(\Omega)} < +\infty$. Since g is a continuous function, let ρ_R be a modulus of continuity of g in B_R. Let R_1 be a sufficiently large constant such that $\Omega \subset B_{R_1-1}$. For any $x \in \partial\Omega$, we let

$$u_{x,r} = \rho_{R_1}(3r) + g(x) + \max\left\{2\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\left\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\right\|_{L^\infty(\Omega)}}{\epsilon_7}\right\}\varphi_{x,r},$$

where $\varphi_{x,r}$ and ϵ_7 are given in Lemma 5.4. It is obvious that $u_{x,r}(x) = \rho_{R_1}(3r) + g(x)$, $u_{x,r} \ge g$ in $\mathbb{R}^n$ and

$$M^+_{\mathcal{L}}u_{x,r} + C_0|\nabla u_{x,r}| \le -\left\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\right\|_{L^\infty(\Omega)} \quad \text{in } \Omega.$$

Now we define $\tilde{u} = \inf_{x\in\partial\Omega, 0<r<r_\Omega}\{u_{x,r}\}$. Therefore, $\tilde{u} = g$ in $\partial\Omega$ and $\tilde{u} \ge g$ in $\mathbb{R}^n$. For any $x \in \partial\Omega$ and $y \in \mathbb{R}^n$, we have

$$\begin{aligned} g(y) - g(x) &\le \tilde{u}(y) - \tilde{u}(x) = \tilde{u}(y) - g(x) \\ &\le \rho_{R_1}(3r) + \max\left\{2\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\left\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\right\|_{L^\infty(\Omega)}}{\epsilon_7}\right\}\varphi_{x,r}(y) \end{aligned}$$

for any $0 < r < r_\Omega$. Therefore, $\tilde{u}$ is continuous on $\partial\Omega$. For any $y \in \Omega$, we define $d_y = \operatorname{dist}(y, \partial\Omega) > 0$. If $r < \frac{1}{2}d_y$, then we have, for any $z \in B_{d_y/2}(y)$,

$$u_{x,r}(z) = \rho_{R_1}(3r) + g(x) + 2\max\left\{2\|g\|_{L^\infty(\mathbb{R}^n)}, \frac{\left\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\right\|_{L^\infty(\Omega)}}{\epsilon_7}\right\}, \quad \text{for any } x \in \partial\Omega.$$

Thus, we have, for any $z \in B_{d_y/2}(y)$,

$$\inf_{x\in\partial\Omega, \frac{d_y}{2}<r<r_\Omega}\{u_{x,r}(z) - u_{x,r}(y), 0\} \le \tilde{u}(z) - \tilde{u}(y) \le \sup_{x\in\partial\Omega, \frac{d_y}{2}<r<r_\Omega}\{u_{x,r}(z) - u_{x,r}(y), 0\}.$$

Since $\{u_{x,r}\}_{x\in\partial\Omega, d_y/2<r<r_\Omega}$ has a uniform modulus of continuity, $\tilde{u}$ is continuous in Ω. Therefore, $\tilde{u}$ is a bounded continuous function in $\overline{\Omega}$. By Lemma 3.1, in Ω we have

$$M^+_{\mathcal{L}}\tilde{u} + C_0|\nabla\tilde{u}| \le -\|I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\|_{L^\infty(\Omega)}.$$

Now we define

$$\bar{u} := \begin{cases} \tilde{u} & \text{in } \Omega, \\ g & \text{in } \Omega^c. \end{cases}$$

By the properties of $\tilde{u}$, we have $\bar{u}$ is a bounded continuous function in $\mathbb{R}^n$, $\bar{u} = g$ in Ω^c and

$$M_{\mathcal{L}}^{+}\bar{u} + C_0|\nabla \bar{u}| \leq -\big\| I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\big\|_{L^\infty(\Omega)}$$

in Ω. Using (A2) and uniform ellipticity, we have, for any $x \in \Omega$,

$$\begin{aligned} I(x, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0) - I(x, \bar{u}(x), \bar{u}(\cdot)) &\leq I(x, \bar{u}(x), 0) - I(x, \bar{u}(x), \bar{u}(\cdot)) \\ &\leq M_{\mathcal{L}}^{+}\bar{u}(x) + C_0|\nabla \bar{u}(x)| \leq -\big\| I(\cdot, -\|g\|_{L^\infty(\mathbb{R}^n)}, 0)\big\|_{L^\infty(\Omega)}. \end{aligned}$$

Thus, $I(x, \bar{u}(x), \bar{u}(\cdot)) \geq 0$ in Ω. ☐

Now we have enough ingredients to conclude:

Theorem 5.6. *Let Ω be a bounded domain satisfying the uniform exterior ball condition. Assume that $0 < \sigma < 2$, $I(x, 0, 0)$ is bounded in Ω and g is a bounded continuous function. Assume that I is uniformly elliptic and satisfies (A0), (A2). Then (1-1) admits a viscosity solution u.*

Proof. The result follows from Theorems 3.2, 4.6 and 5.5. ☐

Corollary 5.7. *Let Ω be a bounded domain satisfying the uniform exterior ball condition. Assume that $0 < \sigma < 2$, $b_{ab} \equiv 0$ in Ω if $\sigma < 1$ and $c_{ab} \geq 0$ in Ω. Assume that g is a bounded continuous function in $\mathbb{R}^n$, $\{K_{ab}(\cdot, z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$, $\{f_{ab}\}_{a,b}$ are sets of uniformly bounded and continuous functions in Ω, uniformly in $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\{K_{ab}(x, \cdot) : x \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}\}$ are kernels satisfying (H0)–(H3). Then (1-2) admits a viscosity solution u.*

5B. ***Degenerate case.*** In the degenerate case, it is natural to construct a sub/supersolution only for (1-2) when $c_{ab} \geq \gamma$ for some $\gamma > 0$. Recall that Ω is a bounded domain satisfying the uniform exterior ball condition with a uniform radius r_Ω and, for any $x \in \partial\Omega$ and $0 < r \leq r_\Omega$, we have y_x^r is a point satisfying $\bar{B}_r(y_x^r) \cap \bar{\Omega} = \{x\}$. From now on, we will hide the dependence on x for all variables and functions to make the notation simpler. For example, we will let $y^r := y_x^r$. For any $x \in \partial\Omega$, $y \in \Omega$ and $0 < r \leq r_\Omega$, we let

$$n := \frac{x - y^r}{|x - y^r|}, \quad n_y^r := \frac{y - y^r}{|y - y^r|}, \quad \text{and} \quad v_\alpha^r(y) := \left(\left(\frac{(y - y^r)\cdot n}{r} - 1\right)^{+}\right)^{\alpha}$$

(see Figure 1).

Instead of letting $\{K_{ab}(x, \cdot) : x \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}\}$ satisfy (H3), we let the set of kernels satisfy the following weaker assumption:

($\overline{\text{H3}}$) There exist $C_4 > 0$, $0 < r_1 < r_\Omega$, $\lambda > 0$ and $\mu > 0$ such that, for any $x \in \partial\Omega$, $0 < r < r_1$ and $y \in \Omega \cap B_{2r}(y^r)$, there is a set A_y^r satisfying
- (i) $A_y^r \subset \{z : z_{n_y^r} < -r s_y^r\} \cap (B_{C_4 r s_y^r} \setminus B_{r s_y^r})$, where $z_{n_y^r} := z \cdot n_y^r$ and $s_y^r := |y - y^r|/r - 1$;
- (ii) $|A_y^r| \geq \mu |B_{r s_y^r}|$;
- (iii) $K(y, z) \geq (2 - \sigma)\lambda (r s_y^r)^{-n-\sigma}$ for any $z \in A_y^r$.

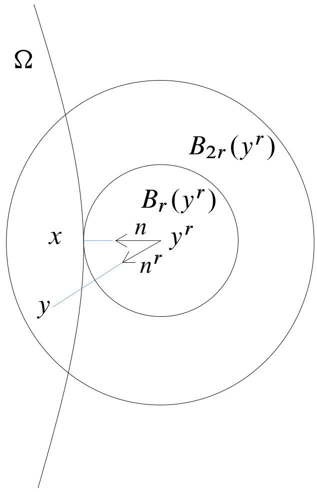

Figure 1. The exterior ball centered at y^r.

Lemma 5.8. *Suppose that* $\{K_{ab}(x,\cdot) : a \in \mathcal{A},\, b \in \mathcal{B},\, x \in \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) < r_1\}\}$ *satisfies* (H3) *for some* $r_1 \in (0, r_\Omega)$. *Then* $(\overline{\text{H3}})$ *holds for the set of kernels.*

Proof. For any $x \in \partial\Omega,\ 0 < r < r_1$ and $y \in \Omega \cap B_{2r}(y^r)$, we define

$$\mu_{C_4} := \frac{\left|(B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}) \cap \{z : |z_{n_y^r}| \le r s_y^r\}\right|}{|B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}|}. \tag{5-5}$$

We notice that the right-hand side of (5-5) depends only on C_4. It is obvious that

$$\lim_{C_4 \to +\infty} \mu_{C_4} = 0.$$

By (H3), there exists a set A satisfying

$$A \subset B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}, \quad A = -A, \quad |A| \ge \mu \left|B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}\right|,$$

and, for any $z \in A$,

$$K(y,z) \ge (2-\sigma)\lambda\left(\tfrac{1}{2} C_4 r s_y^r\right)^{-n-\sigma} = (2-\sigma)\lambda\left(\tfrac{1}{2} C_4\right)^{-n-\sigma} (r s_y^r)^{-n-\sigma} := (2-\sigma)\bar{\lambda}(r s_y^r)^{-n-\sigma}.$$

There exists a sufficiently large constant $C_4 (\ge 2)$ such that $\mu_{C_4} < \frac{1}{2}\mu$. Then

$$\frac{\left|\{z : |z_{n_y^r}| > r s_y^r\} \cap A\right|}{|B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}|} \ge \frac{|A| - \left|(B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}) \cap \{z : |z_{n_y^r}| \le r s_y^r\}\right|}{\left|B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}\right|} \ge \frac{\mu}{2}.$$

Let $A_y^r := A \cap \{z : z_{n_y^r} < -r s_y^r\}$. By the symmetry of A, we have

$$|A_y^r| \ge \tfrac{1}{4}\mu \left|B_{C_4 r s_y^r} \setminus B_{\frac{C_4 r s_y^r}{2}}\right| \ge \tfrac{1}{4}\mu |B_{r s_y^r}| := \bar{\mu} |B_{r s_y^r}|.$$

Therefore, $(\overline{\text{H3}})$ holds for the set of kernels with C_4, r_1, $\bar{\lambda}$ and $\bar{\mu}$. ☐

Lemma 5.9. *Assume that* $0<\sigma<2$ *and* $\{K_{ab}(x,\cdot):x\in\Omega, a\in\mathcal{A}, b\in\mathcal{B}\}$ *are kernels satisfying* (H0)–(H2), $(\overline{\text{H3}})$. *Then there exists a sufficiently small* $\alpha>0$ *such that, for any* $x\in\partial\Omega$, $0<r<r_1$ *and* $s\in\{l\in(0,1):y^r+(1+l)rn\in\Omega\}$, *we have* $I_{ab}[y^r+(1+s)rn, v^r_\alpha]\le-\epsilon_8 r^{-\sigma}s^{\alpha-\sigma}$, *where* ϵ_8 *is some positive constant.*

Proof. We only prove the result for the case $0<\sigma<1$. For the rest of cases, the proofs are similar to those in Lemma 5.1. For any $x\in\partial\Omega$, $0<r<r_1$ and $s\in\{l\in(0,1):y^r+(1+l)rn\in\Omega\}$, we have

$$\begin{aligned}
I_{ab}[y^r+(1+s)rn, v^r_\alpha] &= \int_{\mathbb{R}^n}\big(v^r_\alpha(y^r+(1+s)rn+z)-v^r_\alpha(y^r+(1+s)rn)\big)K_{ab}(y^r+(1+s)rn,z)\,dz\\
&= \int_{\mathbb{R}^n}\left[\left(\left(s+\frac{\tilde z_n}{r}\right)^+\right)^\alpha - s^\alpha\right]K_{ab}(y^r+(1+s)rn,z)\,dz\\
&= r^{-\sigma}s^{\alpha-\sigma}\int_{\mathbb{R}^n}\big[((1+\tilde z_n)^+)^\alpha-1\big](rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn,rsz)\,dz\\
&= r^{-\sigma}s^{\alpha-\sigma}\Bigg\{\int_{\tilde z_n>-1}[(1+\tilde z_n)^\alpha-1](rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn,rsz)\,dz\\
&\qquad\qquad - \int_{\tilde z_n\le -1}(rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn,rsz)\,dz\Bigg\},
\end{aligned}$$

where $\tilde z_n := z\cdot n$. Using $(\overline{\text{H3}})$, we have

$$\begin{aligned}
\int_{\tilde z_n\le -1}(rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn,rsz)\,dz &= (rs)^\sigma\int_{\tilde z_n\le -rs}K_{ab}(y^r+(1+s)rn,z)\,dz\\
&\ge (rs)^\sigma\int_{A^r_{y^r+(1+s)rn}}K_{ab}(y^r+(1+s)rn,z)\,dz\\
&\ge (2-\sigma)\lambda\mu(rs)^{-n}|B_{rs}| := 2\epsilon_8.
\end{aligned}$$

We notice that the kernel $(rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn, rs\,\cdot)$ still satisfies (H1) and (H2). By a similar calculation to (5-2), we have

$$\int_{\tilde z_n>-1}[(1+\tilde z_n)^\alpha-1](rs)^{n+\sigma}K_{ab}(y^r+(1+s)rn,rsz)\,dz\le\epsilon(\alpha),$$

where $\epsilon(\alpha)$ is a positive constant satisfying that $\epsilon(\alpha)\to0$ as $\alpha\to0$. Then there exists a sufficiently small α such that

$$I_{ab}[y^r+(1+s)rn, v^r_\alpha]\le-\epsilon_8 r^{-\sigma}s^{\alpha-\sigma}.$$ □

Lemma 5.10. *Assume that* $0<\sigma<2$, *and* $b_{ab}\equiv0$ *in* Ω *if* $\sigma<1$. *Assume that* $\{b_{ab}\}_{a,b}$ *are sets of uniformly bounded functions in* Ω *and* $\{K_{ab}(x,\cdot):x\in\Omega, a\in\mathcal{A}, b\in\mathcal{B}\}$ *are kernels satisfying* (H0)–(H2), $(\overline{\text{H3}})$. *Then there are* $\alpha>0$ *and* $0<s_0<1$ *sufficiently small so that, for any* $x\in\partial\Omega$ *and* $0<r<r_1$, *the function*

$$u^r_\alpha(y) := \left(\left(\frac{|y-y^r|}{r}-1\right)^+\right)^\alpha$$

satisfies, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$,

$$-I_{ab}[y, u_\alpha^r] + b_{ab}(y) \cdot \nabla u_\alpha^r(y) \geq 1 \quad in\ \Omega \cap \big(\bar{B}_{(1+s_0)r}(y^r) \setminus \bar{B}_r(y^r)\big).$$

Proof. Note that, for all $s > 0$, we have $u_\alpha^r(y^r + (1+s)rn) = v_\alpha^r(y^r + (1+s)rn)$, $\nabla u_\alpha^r(y^r + (1+s)rn) = \nabla v_\alpha^r(y^r + (1+s)rn)$ and

$$\left| \left(\frac{|(1+s)rn + z|}{r} - 1 \right)^+ - \left(s + \frac{\tilde{z}_n}{r} \right)^+ \right| \leq C \frac{|z - \tilde{z}_n|^2}{r^2} \quad \text{for any } z \in B_r.$$

Thus, we have

$$0 \leq (u_\alpha^r - v_\alpha^r)(y^r + (1+s)rn + z) \leq \begin{cases} C s^{\alpha-1} |z - \tilde{z}_n|^2 / r^2, & z \in B_{\frac{rs}{2}}, \\ C|z - \tilde{z}_n|^{2\alpha} / r^{2\alpha}, & z \in B_r \setminus B_{\frac{rs}{2}}, \\ C|z|^\alpha / r^\alpha, & z \in \mathbb{R}^n \setminus B_r. \end{cases}$$

Using (H1), we have, for any $0 < \sigma < 2$, $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $s \in \{l \in (0,1) : y^r + (1+l)rn \in \Omega\}$,

$$\begin{aligned} 0 &\leq I_{ab}[y^r + (1+s)rn, u_\alpha^r - v_\alpha^r] \\ &\leq \int_{\mathbb{R}^n} (u_\alpha^r - v_\alpha^r)(y^r + (1+s)rn + z) K_{ab}(y^r + (1+s)rn, z)\, dz \\ &\leq C \bigg(\int_{B_{\frac{rs}{2}}} s^{\alpha-1} \frac{|z - \tilde{z}_n|^2}{r^2} K_{ab}(y^r + (1+s)rn, z)\, dz + \int_{B_{\frac{r}{2}} \setminus B_{\frac{rs}{2}}} \frac{|z - \tilde{z}_n|^{2\alpha}}{r^{2\alpha}} K_{ab}(y^r + (1+s)rn, z)\, dz \\ &\qquad + \int_{\mathbb{R}^n \setminus B_r} \frac{|z|^\alpha}{r^\alpha} K_{ab}(y^r + (1+s)rn, z)\, dz \bigg) \\ &\leq C \bigg(\int_{B_{\frac{rs}{2}}} s^{\alpha-1} \frac{|z|^2}{r^2} K_{ab}(y^r + (1+s)rn, z)\, dz + \int_{\mathbb{R}^n \setminus B_{\frac{rs}{2}}} \frac{|z|^{2\alpha}}{r^{2\alpha}} K_{ab}(y^r + (1+s)rn, z)\, dz \bigg) \\ &\leq C\Lambda r^{-\sigma} (s^{\alpha-\sigma+1} + s^{2\alpha-\sigma}). \end{aligned}$$

By Lemma 5.9, we have

$$\begin{aligned} -I_{ab}[y^r + (1+s)rn, u_\alpha^r] &\geq -I_{ab}[y^r + (1+s)rn, v_\alpha^r] - I_{ab}[y^r + (1+s)rn, u_\alpha^r - v_\alpha^r] \\ &\geq r^{-\sigma} [\epsilon_8 s^{\alpha-\sigma} - C\Lambda(s^{\alpha-\sigma+1} + s^{2\alpha-\sigma})]. \end{aligned} \tag{5-6}$$

For any $y \in \Omega \cap (B_{2r}(y^r) \setminus \bar{B}_r(y^r))$, we have

$$\begin{aligned} -I_{ab}[y, u_\alpha^r] &= -\int_{\mathbb{R}^n} \delta_z u_\alpha^r(y) K_{ab}(y, z)\, dz \\ &= -\int_{\mathbb{R}^n} \delta_z u_\alpha^r(y^r + (1+s_y^r)rn_y^r) K_{ab}(y, z)\, dz \\ &= -\int_{\mathbb{R}^n} \delta_z u_\alpha^r(y^r + (1+s_y^r)rn) K_{ab}\Big(y, \Big(\frac{z}{|z|} + n_y^r - n\Big)|z|\Big)\, dz. \end{aligned}$$

Using $(\overline{\text{H3}})$ and a similar estimate to (5-6), we have

$$-I_{ab}[y, u_\alpha^r] \geq r^{-\sigma} \big[\epsilon_8 (s_y^r)^{\alpha-\sigma} - C\Lambda((s_y^r)^{\alpha-\sigma+1} + (s_y^r)^{2\alpha-\sigma})\big].$$

By a similar estimate to (5-3), there exists a sufficiently small constant $0 < s_0 < 1$ such that we have, for any $y \in \Omega \cap (\bar{B}_{(1+s_0)r}(y^r) \setminus \bar{B}_r(y^r))$,

$$-I_{ab}[y, u_\alpha^r] + b_{ab}(y) \cdot \nabla u_\alpha^r(y) \geq 1. \qquad \square$$

Lemma 5.11. *Assume that $0 < \sigma < 2$, $b_{ab} \equiv 0$ in Ω if $\sigma < 1$ and $c_{ab} \geq \gamma$ in Ω for some $\gamma > 0$. Assume that $\{K_{ab}(\cdot, z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$, $\{f_{ab}\}_{a,b}$ are sets of uniformly bounded and continuous functions in Ω, uniformly in $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\{K_{ab}(x, \cdot) : x \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}\}$ are kernels satisfying (H0)–(H2), ($\overline{\text{H3}}$). Then, for any $x \in \partial\Omega$ and $0 < r < r_1$, there is a continuous viscosity supersolution ψ_r of (3-5) such that $\psi_r \equiv 0$ in $\bar{B}_r(y^r)$, $\psi_r > 0$ in $\bar{B}_r^c(y^r)$ and*

$$\psi_r \equiv \frac{\sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1}{\gamma} \quad \text{in } B^c_{(1+s_0)r}(y^r), \tag{5-7}$$

where s_0 is given by Lemma 5.10.

Proof. Without loss of generality, we assume that $0 < \gamma < 1$. We pick a sufficiently large $C_5 > 0$ that

$$C_5 > \frac{\sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1}{s_0^\alpha \gamma}. \tag{5-8}$$

We then define, for any $x \in \partial\Omega$ and $0 < r < r_1$,

$$\psi_r(y) = \min\left\{ \frac{\sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1}{\gamma}, C_5 u_\alpha^r(y) \right\}.$$

It is easy to verify that $\psi_r \equiv 0$ in $\bar{B}_r(y^r)$, $\psi_r > 0$ in $\bar{B}_r^c(y^r)$ and ψ_r is a continuous function in $\mathbb{R}^n$. Using (5-8), we know that

$$C_5 u_\alpha^r \geq C_5 s_0^\alpha \geq \frac{\sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1}{\gamma} \quad \text{in } B^c_{(1+s_0)r}(y^r).$$

Therefore, (5-7) holds. Since $c_{ab} \geq \gamma > 0$ in Ω, $(\sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1)/\gamma$ is a viscosity supersolution of (3-5) in Ω. By Lemma 5.10 and (5-7), we have, for any $y \in \Omega \cap (\bar{B}_{(1+s_0)r}(y^r) \setminus \bar{B}_r(y^r))$,

$$\begin{aligned} \sup_{a \in \mathcal{A}} \inf_{b \in \mathcal{B}} &\{-I_{ab}[y, C_5 u_\alpha^r] + C_5 b_{ab}(x) \cdot \nabla u_\alpha^r(y) + C_5 c_{ab}(x) u_\alpha^r(y) + f_{ab}(y)\} \\ &\geq \sup_{a \in \mathcal{A}, b \in \mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1 + f_{ab}(y) \geq 0. \end{aligned} \tag{5-9}$$

Therefore, ψ_r is a continuous viscosity supersolution of (3-5) in Ω. $\square$

Theorem 5.12. *Assume that $0 < \sigma < 2$, $b_{ab} \equiv 0$ in Ω if $\sigma < 1$ and $c_{ab} \geq \gamma$ in Ω for some $\gamma > 0$. Assume that g is a bounded continuous function in $\mathbb{R}^n$, $\{K_{ab}(\cdot, z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$, $\{f_{ab}\}_{a,b}$ are sets of uniformly bounded and continuous functions in Ω, uniformly in $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\{K_{ab}(x, \cdot) : x \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}\}$ are kernels satisfying (H0)–(H2), ($\overline{\text{H3}}$). Then (1-2) admits a continuous viscosity supersolution $\bar{u}$ and a continuous viscosity subsolution $\underline{u}$ and $\bar{u} = \underline{u} = g$ in Ω^c.*

Proof. We only prove (1-2) admits a viscosity supersolution $\bar{u}$ such that $\bar{u} = g$ in Ω^c. Since g is a continuous function, let ρ_R be a modulus of continuity of g in B_R. Let R_1 be a sufficiently large constant such that $\Omega \subset B_{R_1-1}$. For any $x \in \partial\Omega$, we let

$$u_r = \rho_{R_1}(3r) + g(x) + \left(2\|g\|_{L^\infty(\mathbb{R}^n)} \frac{\gamma}{\sup_{a\in\mathcal{A},b\in\mathcal{B}} \|f_{ab}\|_{L^\infty(\Omega)} + 1} + 1\right)\psi_r,$$

where ψ_r is given in Lemma 5.11. Using Lemma 5.11, $u_r(x) = \rho_{R_1}(3r) + g(x)$, $u_r \geq g$ in $\mathbb{R}^n$ and u_r is a continuous viscosity supersolution of (3-5) in Ω. Then the rest of the proof is similar to Theorem 5.5. □

Theorem 5.13. *Let Ω be a bounded domain satisfying the uniform exterior ball condition. Assume that $0 < \sigma < 2$, $b_{ab} \equiv 0$ in Ω if $\sigma < 1$ and $c_{ab} \geq \gamma$ in Ω for some $\gamma > 0$. Assume that g is a bounded continuous function in $\mathbb{R}^n$, $\{K_{ab}(\cdot, z)\}_{a,b,z}$, $\{b_{ab}\}_{a,b}$, $\{c_{ab}\}_{a,b}$, $\{f_{ab}\}_{a,b}$ are sets of uniformly bounded and continuous functions in Ω, uniformly in $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\{K_{ab}(x, \cdot) : x \in \Omega, a \in \mathcal{A}, b \in \mathcal{B}\}$ are kernels satisfying (H0)–(H2), $(\overline{\text{H3}})$. Then (1-2) admits a discontinuous viscosity solution u.*

Proof. The result follows from Corollary 3.4 and Theorem 5.12. □

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