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FOR A CLASS OF KORTEWEG–DE VRIES EQUATIONS
BY MEANS OF TIME-VARYING FEEDBACK LAWS

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We study the exponential stabilization problem for a nonlinear Korteweg-de Vries equation on a bounded interval in cases where the linearized control system is not controllable. The system has Dirichlet boundary conditions at the end-points of the interval and a Neumann nonhomogeneous boundary condition at the right end-point, which is the control. We build a class of time-varying feedback laws for which the solutions of the closed-loop systems with small initial data decay exponentially to 0. We present also results on the well-posedness of the closed-loop systems for general time-varying feedback laws.

1. Introduction

Let $L \in (0, +\infty)$. We consider the stabilization of the controlled Korteweg–de Vries (KdV) system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t, x) \in (s, +\infty) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t \in (s, +\infty), \\ y_x(t, L) = u(t) & \text{for } t \in (s, +\infty), \end{cases} \tag{1-1}$$

where $s \in \mathbb{R}$ and where, at time $t \in [s, +\infty)$, the state is $y(t, \cdot) \in L^2(0, L)$ and the control is $u(t) \in \mathbb{R}$.

Boussinesq [1877] and Korteweg and de Vries [1895] introduced KdV equations for describing the propagation of small-amplitude long water waves. For a better understanding of KdV equations, one can see [Whitham 1974], in which different mathematical models of water waves are deduced. These equations have turned out to be good models, not only for water waves but also to describe other physical phenomena. For mathematical studies on these equations, let us mention [Bona and Smith 1975; Constantin and Saut 1988; Craig et al. 1992; Temam 1969], as well as the discovery of solitons and the inverse scattering method [Gardner et al. 1967; Murray 1978] to solve these equations. We also refer here to [Bona et al. 2003; 2009; Coron and Crépeau 2004; Rivas et al. 2011; Zhang 1999] for well-posedness results of initial-boundary-value problems of our KdV equation (1-1) or for other equations which are similar to (1-1). Finally, let us refer to [Cerpa 2014; Rosier and Zhang 2009] for reviews on recent progresses on the control of various KdV equations.

The controllability research on (1-1) began when Lionel Rosier [1997] showed that the linearized KdV control system (around 0 in $L^2(0, L)$)

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0, T) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (0, L), \\ y_x(t, L) = u(t) & \text{on } (0, T) \end{cases} \tag{1-2}$$

is controllable if and only if $L \notin \mathcal{N}$, where $\mathcal{N}$ is called the set of critical lengths and is defined by

$$\mathcal{N} := \left\{ 2\pi \sqrt{\tfrac{1}{3}(l^2 + lk + k^2)} : l, k \in \mathbb{N}^* \right\}. \tag{1-3}$$

From this controllability result Lionel Rosier, in the same article, deduced that the nonlinear KdV equations (1-1) are locally controllable (around 0 in $L^2(0, L)$) if $L \notin \mathcal{N}$. His work also shows that the $L^2(0, L)$ space can be decomposed as $H \oplus M$, where M is the "uncontrollable" part for the linearized KdV control systems (1-2), and H is the "controllable" part. Moreover, M is of finite dimension, a dimension which strongly depends on some number theory property of the length L. More precisely, the dimension of M is the number of different pairs of positive integers (l_j, k_j) satisfying

$$L = 2\pi \sqrt{\tfrac{1}{3}(l_j^2 + l_j k_j + k_j^2)}. \tag{1-4}$$

For each such pair of (l_j, k_j) with $l_j \geqslant k_j$, we can find two nonzero real-valued functions φ_1^j and φ_2^j such that $\varphi^j := \varphi_1^j + i\varphi_2^j$ is a solution of

$$\begin{cases} -i\omega(l_j, k_j)\varphi^j + (\varphi^j)' + (\varphi^j)''' = 0, \\ \varphi^j(0) = \varphi^j(L) = 0, \\ (\varphi^j)'(0) = (\varphi^j)'(L) = 0, \end{cases} \tag{1-5}$$

where $\varphi_1^j, \varphi_2^j \in C^\infty([0, L])$ and $\omega(l_j, k_j)$ is defined by

$$\omega(l_j, k_j) := \frac{(2l_j + k_j)(l_j - k_j)(2k_j + l_j)}{3\sqrt{3}(l_j^2 + l_j k_j + k_j^2)^{3/2}}. \tag{1-6}$$

When $l_j > k_j$, the functions φ_1^j, φ_2^j are linearly independent, but when $l_j = k_j$, we have $\omega(l_j, k_j) = 0$ and φ_1^j, φ_2^j are linearly dependent. It is also proved in [Rosier 1997] that

$$M = \text{Span}\{\varphi_1^1, \varphi_2^1, \ldots, \varphi_1^n, \varphi_2^n\}. \tag{1-7}$$

Multiplying (1-2) by φ^j, integrating on $(0, L)$, performing integrations by parts and combining with (1-5), we get

$$\frac{d}{dt}\left(\int_0^L y(t, x)\varphi^j(x)\, dx\right) = i\omega(l_j, k_j)\int_0^L y(t, x)\varphi^j(x)\, dx,$$

which shows that M is included in the "uncontrollable" part of (1-2). Let us point out that there exists at most one pair of (l_j, k_j) such that $l_j = k_j$. Hence we can classify $L \in \mathbb{R}^+$ into five different cases and therefore divide $\mathbb{R}^+$ into five disjoint subsets of $(0, +\infty)$, which are defined as follows:

(1) $\mathcal{C} := \mathbb{R}^+ \setminus \mathcal{N}$. Then $M = \{0\}$.

(2) $\mathcal{N}_1 := \{L \in \mathcal{N} : \text{there exists exactly one ordered pair } (l_j, k_j) \text{ satisfying (1-4) and } l_j = k_j\}$. Then the dimension of M is 1.

(3) $\mathcal{N}_2 := \{L \in \mathcal{N} : \text{there exists exactly one ordered pair } (l_j, k_j) \text{ satisfying (1-4) and } l_j > k_j\}$. Then the dimension of M is 2.

(4) $\mathcal{N}_3 := \{L \in \mathcal{N} : \text{there exist } n \geqslant 2 \text{ distinct ordered pairs } (l_j, k_j) \text{ satisfying (1-4) and none satisfy } l_j = k_j\}$. Then the dimension of M is $2n$.

(5) $\mathcal{N}_4 := \{L \in \mathcal{N} : \text{there exist } n \geqslant 2 \text{ distinct ordered pairs } (l_j, k_j) \text{ satisfying (1-4) and one satisfies } l_j = k_j\}$. Then the dimension of M is $2n - 1$.

The five sets $\mathcal{C}$, $\{\mathcal{N}_i\}_{i=1}^4$ are pairwise disjoint and

$$\mathbb{R}^+ = \mathcal{C} \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \cup \mathcal{N}_4,$$
$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \cup \mathcal{N}_4.$$

Additionally, Eduardo Cerpa [2007, Lemma 2.5] proved that each of these five sets has infinite number of elements; see also [Coron 2007, Proposition 8.3] for the case of $\mathcal{N}_1$.

Let us point out that $L \notin \mathcal{N}$ is equivalent to $M = \{0\}$. Hence, Lionel Rosier solved the (local) controllability problem of nonlinear KdV equations for $L \in \mathcal{C}$. Later on Jean-Michel Coron and Emmanuelle Crépeau [2004] proved the small-time local controllability of nonlinear KdV equations for the second case $L \in \mathcal{N}_1$, by a "power series expansion" method; the nonlinear term yy_x gives this controllability. Later on, Eduardo Cerpa [2007] proved the local controllability in large time for the third case $L \in \mathcal{N}_2$, still by using the "power series expansion" method. In this case, an expansion to the order 2 is sufficient but the local controllability in small time remains open. Finally Eduardo Cerpa and Emmanuelle Crépeau [2009a] concluded the study by proving the local controllability in large time of (1-1) for the two remaining critical cases (for which dim $M \geqslant 3$). The proofs of all these results rely on the "power series expansion" method, introduced in [Coron and Crépeau 2004]. This method has also been used to prove controllability results for Schrödinger equations [Beauchard 2005; Beauchard and Coron 2006; Beauchard and Morancey 2014; Morancey 2014] and for rapid asymptotic stability of a Navier-Stokes control system in [Chowdhury and Ervedoza 2017]. In this article we use it to get exponential stabilization of (1-1). For studies on the controllability of other KdV control systems problems, let us refer to [Capistrano-Filho et al. 2015; Gagnon 2016; Glass and Guerrero 2010; Goubet and Shen 2007; Rosier 2004; Zhang 1999].

The asymptotic stability of 0 without control (control term equal to 0) has been studied for years; see, in particular, [Cerpa and Coron 2013; Goubet and Shen 2007; Jia and Zhang 2012; Massarolo et al. 2007; Pazoto 2005; Perla Menzala et al. 2002; Rosier and Zhang 2006; Russell and Zhang 1995; 1996]. For example, the local exponential stability for our KdV equation if $L \notin \mathcal{N}$ was proved in [Perla Menzala et al. 2002]. Let also point out here that in [Doronin and Natali 2014], the authors give the existence of (large) stationary solutions, which ensures that the exponential stability result in [Perla Menzala et al. 2002] is only local.

Concerning the stabilization by means of feedback laws, the locally exponential stabilization with arbitrary decay rate (rapid stabilization) with some linear feedback law was obtained by Eduardo Cerpa

and Emmanuelle Crépeau in [2009b] for the linear KdV equation (1-2). For the nonlinear case, the first rapid stabilization for Korteweg–de Vries equations was obtained by Camille Laurent, Lionel Rosier and Bing-Yu Zhang [Laurent et al. 2010] in the case of localized distributed control on a periodic domain. In that case, the linearized control system, let us write it $\dot{y} = Ay + Bu$, is controllable. These authors used an approach due to Marshall Slemrod [1974] to construct linear feedback laws leading to the rapid stabilization of $\dot{y} = Ay + Bu$ and then proved that the same feedback laws give the rapid stabilization of the nonlinear Korteweg de Vries equation. In the case of distributed control, the operator B is bounded. For boundary control the operator B is unbounded. The Slemrod approach has been modified to handle this case by Vilmos Komornik [1997] and by Jose Urquiza [2005], and [Cerpa and Crépeau 2009b] precisely uses the modification presented in [Urquiza 2005]. However, in contrast with the case of distributed control, it leads to unbounded linear feedback laws and one does not know for the moment if these linear feedback laws lead to asymptotic stabilization for the nonlinear Korteweg de Vries equation. One does not even know if the closed system is well posed for this nonlinear equation. The first rapid stabilization result in the nonlinear case and with boundary controls was obtained by Eduardo Cerpa and Jean-Michel Coron [2013]. Their approach relies on the backstepping method/transformation, a method introduced by Miroslav Krstic and his collaborators (see [Krstic and Smyshlyaev 2008] for an excellent starting point to this method). When $L \notin \mathcal{N}$, by using a more general transformation and the controllability of (1-2), Jean-Michel Coron and Qi Lü [2014] proved the rapid stabilization of our KdV control system. Their method can be applied to many other equations, like Schrödinger equations [Coron et al. 2016] and Kuramoto–Sivashinsky equations [Coron and Lü 2015]. When $L \in \mathcal{N}$, as mentioned above, the linearized control system (1-2) is not controllable, but the control system (1-1) is controllable. Let us recall that for the finite-dimensional case, the controllability doesn't imply the existence of a (continuous) stationary feedback law which stabilizes (asymptotically, exponentially, etc.) the control system; see [Brockett 1983; Coron 1990]. However the controllability in general implies the existence of (continuous) *time-varying* feedback laws which asymptotically (and even in finite time) stabilize the control system; see [Coron 1995]. Hence it is natural to look for time-varying feedback laws $u(t, y(t, \cdot))$ such that 0 is (locally) asymptotically stable for the closed-loop system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t, x) \in (s, +\infty) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t \in (s, +\infty), \\ y_x(t, L) = u(t, y(t, \cdot)) & \text{for } t \in (s, +\infty). \end{cases} \quad (1\text{-}8)$$

Let us also point out that in [Laurent et al. 2010], as in [Coron and Rosier 1994] by Jean-Michel Coron and Lionel Rosier, which dealt with finite-dimensional control systems, time-varying feedback laws were used in order to combine two different feedback laws to get rapid *global* asymptotic stability of the closed loop system. Let us emphasize that $u = 0$ leads to (local) asymptotic stability when $L \in \mathcal{N}_1$ [Chu et al. 2015] and $L \in \mathcal{N}_2$ [Tang et al. 2016]. However, in both cases, the convergence is not exponential. It is then natural to ask if we can get exponential convergence to 0 with the help of some suitable time-varying feedback laws $u(t, y(t, \cdot))$. The aim of this paper is to prove that it is indeed possible in the case where

$$L \text{ is in } \mathcal{N}_2 \text{ or in } \mathcal{N}_3. \quad (1\text{-}9)$$

Let us denote by

$$P_H : L^2(0, L) \to H \quad \text{and} \quad P_M : L^2(0, L) \to M$$

the orthogonal projections (for the L^2-scalar product) on H and M respectively. Our main result is the following one, where the precise definition of a solution of (1-10) is given in Section 2.

Theorem 1. *Assume that (1-9) holds. Then there exists a periodic time-varying feedback law u, $C > 0$, $\lambda > 0$ and $r > 0$ such that, for every $s \in \mathbb{R}$ and for every $\|y_0\|_{L^2_L} < r$, the Cauchy problem*

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t, x) \in (s, +\infty) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t \in (s, +\infty), \\ y_x(t, L) = u(t, y(t, \cdot)) & \text{for } t \in (s, +\infty), \\ y(s, \cdot) = y_0 & \text{for } x \in (0, L) \end{cases} \tag{1-10}$$

has at least one solution in $C^0\big([s, +\infty); L^2(0, L)\big) \cap L^2_{\mathrm{loc}}\big([s, +\infty); H^1(0, L)\big)$ and every solution y of (1-10) is defined on $[s, +\infty)$ and satisfies, for every $t \in [s, +\infty)$,

$$\|P_H(y(t))\|_{L^2_L} + \|P_M(y(t))\|_{L^2_L}^{1/2} \leqslant Ce^{-\lambda(t-s)}\big(\|P_H(y_0)\|_{L^2_L} + \|P_M(y_0)\|_{L^2_L}^{1/2}\big). \tag{1-11}$$

In order to simplify the notations, in this paper we sometimes simply denote $y(t, \cdot)$ by $y(t)$, if there is no misunderstanding; sometimes we also simply denote $L^2(0, L)$ by L^2_L and $L^2(0, T)$ by L^2_T. Let us explain briefly an important ingredient of our proof of Theorem 1. Taking into account the uncontrollability of the linearized system, it is natural to split the KdV system into a coupled system for $(P_H(y), P_M(y))$. Then the finite-dimensional analogue of our KdV control system is

$$\dot{x} = Ax + R_1(x, y) + Bu, \quad \dot{y} = Ly + Q(x, x) + R_2(x, y), \tag{1-12}$$

where A, B, and L are matrices, Q is a quadratic map, R_1, R_2 are polynomials and u is the control. The state variable x plays the role of $P_H(y)$, while y plays the role of $P_M(y)$. The two polynomials R_1 and R_2 are quadratic and $R_2(x, y)$ vanishes for $y = 0$. For this ODE system, in many cases the Brockett condition [1983] and the Coron condition [2007] for the existence of continuous stationary stabilizing feedback laws do not hold. However, as shown in [Coron and Rivas 2016], many physical systems of form (1-12) can be exponentially stabilized by means of time-varying feedback laws. We follow the construction of these time-varying feedback laws given in this article. However, due to the fact that H is of infinite dimension, many parts of the proof have to be modified compared to those given in [Coron and Rivas 2016]; in particular we do not know how to use a Lyapunov approach, in contrast to what is done in that paper.

This article is organized as follows. In Section 2, we recall some classical results and definitions about (1-1) and (1-2). In Section 3, we study the existence and uniqueness of solutions to the closed-loop system (1-10) with time-varying feedback laws u which are not smooth. In Section 4, we construct our time-varying feedback laws. In Section 5, we prove two estimates for solutions to the closed-loop system (1-10) (Propositions 15 and 16) which imply Theorem 1. The article ends with three appendices where proofs of propositions used in the main parts of the article are given.

2. Preliminaries

We first recall some results on KdV equations and give the definition of a solution to the Cauchy problem (1-10). Let us start with the nonhomogeneous linear Cauchy problem

$$
\begin{cases}
y_t + y_{xxx} + y_x = \tilde{h} & \text{in } (T_1, T_2) \times (0, L), \\
y(t, 0) = y(t, L) = 0 & \text{on } (T_1, T_2), \\
y_x(t, L) = h(t) & \text{on } (T_1, T_2), \\
y(T_1, x) = y_0(x) & \text{on } (0, L)
\end{cases}
\tag{2-1}
$$

for

$$
-\infty < T_1 < T_2 < +\infty,
\tag{2-2}
$$

$$
y_0 \in L^2(0, L),
\tag{2-3}
$$

$$
\tilde{h} \in L^1(T_1, T_2; L^2(0, L)),
\tag{2-4}
$$

$$
h \in L^2(T_1, T_2).
\tag{2-5}
$$

Let us now give the definition of a solution to (2-1).

Definition 2. A solution to the Cauchy problem (2-1) is a function $y \in L^1(T_1, T_2; L^2(0, L))$ such that, for almost every $\tau \in [T_1, T_2]$, the following holds: for every $\phi \in C^3([T_1, \tau] \times [0, L])$ such that

$$
\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0 \quad \forall t \in [T_1, \tau],
\tag{2-6}
$$

one has

$$
-\int_{T_1}^{\tau} \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y \, dx \, dt - \int_{T_1}^{\tau} h(t) \phi_x(t, L) \, dt - \int_{T_1}^{\tau} \int_0^L \phi \tilde{h} \, dx \, dt
$$
$$
+ \int_0^L y(\tau, x) \phi(\tau, x) \, dx - \int_0^L y_0 \phi(T_1, x) \, dx = 0. \tag{2-7}
$$

For T_1 and T_2 satisfying (2-2), let us define the linear space $\mathcal{B}_{T_1, T_2}$ by

$$
\mathcal{B}_{T_1, T_2} := C^0([T_1, T_2]; L^2(0, L)) \cap L^2(T_1, T_2; H^1(0, L)).
\tag{2-8}
$$

This linear space $\mathcal{B}_{T_1, T_2}$ is equipped with the norm

$$
\|y\|_{\mathcal{B}_{T_1, T_2}} := \max\{\|y(t)\|_{L_L^2} : t \in [T_1, T_2]\} + \left(\int_{T_1}^{T_2} \|y_x(t)\|_{L_L^2}^2 \, dt \right)^{1/2}.
\tag{2-9}
$$

With this norm, $\mathcal{B}_{T_1, T_2}$ is a Banach space.

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0, L) \to L^2(0, L)$ be the linear operator defined by

$$
\mathcal{D}(\mathcal{A}) := \{\phi \in H^3(0, L) : \phi(0) = \phi(L) = \phi_x(L) = 0\},
\tag{2-10}
$$

$$
\mathcal{A}\phi := -\phi_x - \phi_{xxx} \quad \forall \phi \in \mathcal{D}(\mathcal{A}).
\tag{2-11}
$$

It is known that both $\mathcal{A}$ and $\mathcal{A}^*$ are closed and dissipative (see, e.g., [Coron 2007, page 39]), and therefore $\mathcal{A}$ generates a strongly continuous semigroup of contractions $S(t)$, $t \in [0, +\infty)$ on $L^2(0, L)$.

Rosier [1997], using the above properties of $\mathcal{A}$ together with multiplier techniques, proved the following existence and uniqueness result for the Cauchy problem (2-1).

Lemma 3. *The Cauchy problem (2-1) has one and only one solution. This solution is in $\mathcal{B}_{T_1,T_2}$ and there exists a constant $C_2 > 0$ depending only on $T_2 - T_1$ such that*

$$\|y\|_{\mathcal{B}_{T_1,T_2}} \leqslant C_2\big(\|y_0\|_{L_L^2} + \|h\|_{L^2(T_1,T_2)} + \|\tilde{h}\|_{L^1(T_1,T_2;L^2(0,L))}\big). \tag{2-12}$$

In fact the notion of solution to the Cauchy problem (2-1) considered in [Rosier 1997] is a priori stronger than the one we consider here (it is required to be in $C^0([T_1, T_2]; L^2(0, L))$. However, the uniqueness of the solution in the sense of Definition 2 still follows from classical arguments; see, for example, [Coron 2007, Proof of Theorem 2.37, page 53].

Let us now turn to the nonlinear KdV equation

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = \widetilde{H} & \text{in } (T_1, T_2) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (T_1, T_2), \\ y_x(t, L) = H(t) & \text{on } (T_1, T_2), \\ y(T_1, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{2-13}$$

Inspired by Lemma 3, we adopt the following definition.

Definition 4. A solution to (2-13) is a function $y \in \mathcal{B}_{T_1,T_2}$ which is a solution of (2-1) for $\tilde{h} := \widetilde{H} - yy_x \in L^1(T_1, T_2; L^2(0, L))$ and $h := H$.

Throughout this article we will use similar definitions without giving them precisely, as, for example, in the case for system (3-15).

Coron and Crépeau [2004] proved the following lemma on the well-posedness of the Cauchy problem (2-13) for small initial data.

Lemma 5. *There exist $\eta > 0$ and $C_3 > 0$ depending on L and $T_2 - T_1$ such that, for every $y_0 \in L^2(0, L)$, every $H \in L^2(T_1, T_2)$ and every $\widetilde{H} \in L^1(T_1, T_2; L^2(0, L))$ satisfying*

$$\|y_0\|_{L_L^2} + \|H\|_{L^2(T_1,T_2)} + \|\widetilde{H}\|_{L^1(T_1,T_2;L^2(0,L))} \leqslant \eta, \tag{2-14}$$

the Cauchy problem (2-13) has a unique solution and this solution satisfies

$$\|y\|_{\mathcal{B}_{T_1,T_2}} \leqslant C_3\big(\|y_0\|_{L_L^2} + \|H\|_{L^2(T_1,T_2)} + \|\widetilde{H}\|_{L^1(T_1,T_2;L^2(0,L))}\big). \tag{2-15}$$

3. Time-varying feedback laws and well-posedness of the associated closed-loop system

Throughout this section u denotes a time-varying feedback law; it is a map from $\mathbb{R} \times L^2(0, L)$ with values into $\mathbb{R}$. We assume that this map is a Carathéodory map, i.e., it satisfies the three properties

$$\forall R > 0, \ \exists\, C_B(R) > 0 \text{ such that } \big(\|y\|_{L_L^2} \leqslant R \implies |u(t, y)| \leqslant C_B(R) \quad \forall t \in \mathbb{R}\big), \tag{3-1}$$

$$\forall y \in L^2(0, L), \text{ the function } t \in \mathbb{R} \mapsto u(t, y) \in \mathbb{R} \text{ is measurable}, \tag{3-2}$$

$$\text{for almost every } t \in \mathbb{R}, \text{ the function } y \in L^2(0, L) \mapsto u(t, y) \in \mathbb{R} \text{ is continuous.} \tag{3-3}$$

In this article we always assume that

$$C_B(R) \geq 1 \quad \forall R \in [0, +\infty), \tag{3-4}$$

$$R \in [0, +\infty) \mapsto C_B(R) \in \mathbb{R} \text{ is a nondecreasing function.} \tag{3-5}$$

Let $s \in \mathbb{R}$ and let $y_0 \in L^2(0, L)$. We start by giving the definition of a solution to

$$\begin{cases} y_t + y_{xxx} + y_x + y y_x = 0 & \text{for } t \in \mathbb{R}, \ x \in (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t \in \mathbb{R}, \\ y_x(t, L) = u(t, y(t, \cdot)) & \text{for } t \in \mathbb{R}, \end{cases} \tag{3-6}$$

and to the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x + y y_x = 0 & \text{for } t > s, \ x \in (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{for } t > s, \\ y_x(t, L) = u(t, y(t, \cdot)) & \text{for } t > s, \\ y(s, x) = y_0(x) & \text{for } x \in (0, L), \end{cases} \tag{3-7}$$

where y_0 is a given function in $L^2(0, L)$ and s is a given real number.

Definition 6. Let I be an interval of $\mathbb{R}$ with a nonempty interior. A function y is a solution of (3-6) on I if $y \in C^0(I; L^2(0, L))$ is such that, for every $[T_1, T_2] \subset I$ with $-\infty < T_1 < T_2 < +\infty$, the restriction of y to $[T_1, T_2] \times (0, L)$ is a solution of (2-13) with $\widetilde{H} := 0$, $H(t) := u(t, y(t))$ and $y_0 := y(T_1)$. A function y is a solution to the Cauchy problem (3-7) if there exists an interval I with a nonempty interior satisfying $I \cap (-\infty, s] = \{s\}$ such that $y \in C^0(I; L^2(0, L))$ is a solution of (3-6) on I and satisfies the initial condition $y(s) = y_0$ in $L^2(0, L)$. The interval I is denoted by $D(y)$. We say that a solution y to the Cauchy problem (3-7) is maximal if, for every solution z to the Cauchy problem (3-7) such that

$$D(y) \subset D(z), \tag{3-8}$$

$$y(t) = z(t) \quad \text{for every } t \text{ in } D(y), \tag{3-9}$$

one has

$$D(y) = D(z). \tag{3-10}$$

Let us now state our theorems concerning the Cauchy problem (3-7).

Theorem 7. *Assume that u is a Carathéodory function and that, for every $R > 0$, there exists $K(R) > 0$ such that*

$$\big(\|y\|_{L^2_L} \leqslant R \quad \text{and} \quad \|z\|_{L^2_L} \leqslant R\big) \implies \big(|u(t, y) - u(t, z)| \leqslant K(R)\|y - z\|_{L^2_L} \quad \forall t \in \mathbb{R}\big). \tag{3-11}$$

Then, for every $s \in \mathbb{R}$ and for every $y_0 \in L^2(0, L)$, the Cauchy problem (3-7) has one and only one maximal solution y. If $D(y)$ is not equal to $[s, +\infty)$, there exists $\tau \in \mathbb{R}$ such that $D(y) = [s, \tau)$ and one has

$$\lim_{t \to \tau^-} \|y(t)\|_{L^2_L} = +\infty. \tag{3-12}$$

Moreover, *if* $C_B(R)$ *satisfies*

$$\int_0^{+\infty} \frac{R}{(C_B(R))^2} \, dR = +\infty, \tag{3-13}$$

then

$$D(y) = [s, +\infty). \tag{3-14}$$

Theorem 8. *Assume that u is a Carathéodory function which satisfies condition* (3-13). *Then, for every* $s \in \mathbb{R}$ *and for every* $y_0 \in L^2(0, L)$, *the Cauchy problem* (3-7) *has at least one maximal solution y such that* $D(y) = [s, +\infty)$.

The proofs of Theorems 7 and 8 will be given in Appendix B.

We end this section with the following proposition, which gives the expected connection between the evolution of $P_M(y)$ and $P_H(y)$ and the fact that y is a solution to (3-6).

Proposition 9. *Let* $u : \mathbb{R} \times L^2(0, L) \to \mathbb{R}$ *be a Carathéodory feedback law. Let* $-\infty < s < T < +\infty$, *let* $y \in \mathcal{B}_{s,T}$ *and let* $y_0 \in L^2(0, L)$. *Denote* $P_H(y)$ *by* y_1 *and* $P_M(y)$ *by* y_2. *Then y is a solution to the Cauchy problem* (3-7) *if and only if*

$$\begin{cases} y_{1t} + y_{1x} + y_{1xxx} + P_H\big((y_1 + y_2)(y_1 + y_2)_x\big) = 0, \\ y_1(t, 0) = y_1(t, L) = 0, \\ y_{1x}(t, L) = u(t, y_1 + y_2), \\ y_1(0, \cdot) = P_H(y_0), \\ y_{2t} + y_{2x} + y_{2xxx} + P_M\big((y_1 + y_2)(y_1 + y_2)_x\big) = 0, \\ y_2(t, 0) = y_2(t, L) = 0, \\ y_{2x}(t, L) = 0, \\ y_2(0, \cdot) = P_M(y_0). \end{cases} \tag{3-15}$$

The proof of this proposition is given in Appendix A.

4. Construction of time-varying feedback laws

In this section, we construct feedback laws which will lead to the local exponential stability stated in Theorem 1. Let us denote by M_1 the set of elements in M having an L^2-norm equal to 1:

$$M_1 := \big\{ y \in M : \|y\|_{L^2_L} = 1 \big\}. \tag{4-1}$$

Let M^j be the linear space generated by φ_1^j and φ_2^j for every $j \in \{1, 2, \dots, n\}$:

$$M^j := \mathrm{Span}\{\varphi_1^j, \varphi_2^j\}. \tag{4-2}$$

The construction of our feedback laws relies on the following proposition.

Proposition 10. *There exist* $T > 0$ *and* $v \in L^\infty([0, T] \times M_1; \mathbb{R})$ *such that the following properties hold*:

($\mathcal{P}_1$) *There exists* $\rho_1 \in (0, 1)$ *such that*

$$\|S(T)y_0\|_{L^2(0,L)}^2 \leqslant \rho_1 \|y_0\|_{L^2(0,L)}^2 \quad \text{for every } y_0 \in H.$$

($\mathcal{P}_2$) *For every $y_0 \in M$,*

$$\|S(T)y_0\|^2_{L^2(0,L)} = \|y_0\|^2_{L^2(0,L)}.$$

($\mathcal{P}_3$) *There exists $C_0 > 0$ such that*

$$| v(t, y) - v(t, z) | \leqslant C_0 \|y - z\|_{L^2(0,L)} \quad \forall t \in [0, T], \ \forall y, z \in M_1. \tag{4-3}$$

Moreover, there exists $\delta > 0$ such that, for every $z \in M_1$, the solution (y_1, y_2) to the equation

$$\begin{cases} y_{1t} + y_{1x} + y_{1xxx} = 0, \\ y_1(t, 0) = y_1(t, L) = 0, \\ y_{1x}(t, L) = v(t, z), \\ y_1(0, x) = 0, \\ y_{2t} + y_{2x} + y_{2xxx} + P_M(y_1 y_{1x}) = 0, \\ y_2(t, 0) = y_2(t, L) = 0, \\ y_{2x}(t, L) = 0, \\ y_2(0, x) = 0, \end{cases} \tag{4-4}$$

satisfies

$$y_1(T) = 0 \quad and \quad \langle y_2(T), S(T)z \rangle_{L^2(0,L)} < -2\delta. \tag{4-5}$$

Proof of Proposition 10. Property ($\mathcal{P}_2$) is given in [Rosier 1997]; one can also see (4-14) and (4-44). Property ($\mathcal{P}_1$) follows from the dissipativity of $\mathcal{A}$ and the controllability of (1-2) in H (see also [Perla Menzala et al. 2002]). Indeed, integrations by parts (and simple density arguments) show that, in the distribution sense in $(0, +\infty)$,

$$\frac{d}{dt} \|S(t)y_0\|^2_{L^2_L} = -y_x^2(t, 0). \tag{4-6}$$

Moreover, as Rosier [1997] proved for every $T > 0$, there exists $c > 1$ such that, for every $y_0 \in H$,

$$\|y_0\|^2_{L^2_L} \leqslant c \|y_x(t, 0)\|^2_{L^2(0,T)}. \tag{4-7}$$

Integration of identity (4-6) on $(0, T)$ and the use of (4-7) give

$$\|S(T)y_0\|^2_{L^2_L} \leqslant \frac{c-1}{c} \|y_0\|^2_{L^2_L}. \tag{4-8}$$

Hence $\rho_1 := (c - 1)/c \in (0, 1)$ satisfies the required properties.

Our concern now is to deal with ($\mathcal{P}_3$). Let us first recall a result on the controllability of the linear control system

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0, T) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (0, L), \\ y_x(t, L) = u(t) & \text{on } (0, T), \end{cases} \tag{4-9}$$

where, at time $t \in [0, T]$, the state is $y(t, \cdot) \in L^2(0, L)$. Our goal is to investigate the cases where $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, but in order to explain more clearly our construction of v, we first deal with the case where

$$L = 2\pi \sqrt{\tfrac{1}{3}(1^2 + 1 \times 2 + 2^2)} = 2\pi \sqrt{\tfrac{7}{3}}, \tag{4-10}$$

which corresponds to $l = 1$ and $k = 2$ in (1-3). In that case the uncontrollable subspace M is a two-dimensional vector subspace of $L^2(0, L)$ generated by

$$\varphi_1(x) = C\left(\cos\left(\tfrac{5}{\sqrt{21}}x\right) - 3\cos\left(\tfrac{1}{\sqrt{21}}x\right) + 2\cos\left(\tfrac{4}{\sqrt{21}}x\right)\right),$$

$$\varphi_2(x) = C\left(-\sin\left(\tfrac{5}{\sqrt{21}}x\right) - 3\sin\left(\tfrac{1}{\sqrt{21}}x\right) + 2\sin\left(\tfrac{4}{\sqrt{21}}x\right)\right),$$

where C is a positive constant such that $\|\varphi_1\|_{L_L^2} = \|\varphi_2\|_{L_L^2} = 1$. They satisfy

$$\begin{cases} \varphi_1' + \varphi_1''' = -2\pi\varphi_2/p, \\ \varphi_1(0) = \varphi_1(L) = 0, \\ \varphi_1'(0) = \varphi_1'(L) = 0 \end{cases} \tag{4-11}$$

and

$$\begin{cases} \varphi_2' + \varphi_2''' = 2\pi\varphi_1/p, \\ \varphi_2(0) = \varphi_2(L) = 0, \\ \varphi_2'(0) = \varphi_2'(L) = 0, \end{cases} \tag{4-12}$$

with (see [Cerpa 2007])

$$p := \frac{441\pi}{10\sqrt{21}}. \tag{4-13}$$

For every $t > 0$, one has

$$S(t)M \subset M \quad \text{and} \quad S(t) \text{ restricted to } M \text{ is the rotation of angle } \frac{2\pi t}{p}, \tag{4-14}$$

if the orientation on M is chosen so that (φ_1, φ_2) is a direct basis, a choice which is done from now on. Moreover the control u has no action on M for the linear control system (1-2): for every initial data $y_0 \in M$, whatever $u \in L^2(0, T)$, the solution y of (1-2) with $y(0) = y_0$ satisfies $P_M(y(t)) = S(t)y_0$ for every $t \in [0, +\infty)$. Let us denote by H the orthogonal in $L^2(0, L)$ of M for the L^2-scalar product $H := M^\perp$. This linear space is left invariant by the linear control system (1-2): for every initial data $y_0 \in H$, whatever $u \in L^2(0, T)$, the solution y of (1-2) satisfying $y(0) = y_0$ is such that $y(t) \in H$ for every $t \in [0, +\infty)$. Moreover, as proved by Rosier [1997], the linear control system (1-2) is controllable in H in small time. More precisely, he proved the following lemma.

Lemma 11. *Let $T > 0$. There exists $C > 0$ depending only on T such that, for every $y_0, y_1 \in H$, there exists a control $u \in L^2(0, T)$ satisfying*

$$\|u\|_{L_T^2} \leqslant C(\|y_0\|_{L_L^2} + \|y_1\|_{L_L^2}) \tag{4-15}$$

such that the solution y of the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{in } (0, T) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (0, T), \\ y_x(t, L) = u(t) & \text{on } (0, T), \\ y(0, x) = y_0(x) & \text{on } (0, L) \end{cases}$$

satisfies $y(T, \cdot) = y_1$.

A key ingredient of our construction of v is the following proposition.

Proposition 12. *Let $T > 0$. For every $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, for every $j \in \{1, 2, \ldots, n\}$, there exists $u^j \in H^1(0, T)$ such that*

$$\alpha(T, \cdot) = 0 \quad and \quad P_{M^j}(\beta(T, \cdot)) \neq 0,$$

where (α, β) is the solution of

$$
\begin{cases}
\alpha_t + \alpha_x + \alpha_{xxx} = 0, \\
\alpha(t, 0) = \alpha(t, L) = 0, \\
\alpha_x(t, L) = u^j(t), \\
\alpha(0, x) = 0, \\
\beta_t + \beta_x + \beta_{xxx} + \alpha\alpha_x = 0, \\
\beta(t, 0) = \beta(t, L) = 0, \\
\beta_x(t, L) = 0, \\
\beta(0, x) = 0.
\end{cases}
\tag{4-16}
$$

Proposition 12 is due to Eduardo Cerpa and Emmanuelle Crépeau if one requires only u to be in $L^2(0, T)$ instead of being in $H^1(0, T)$: see [Cerpa 2007, Proposition 3.1] and [Cerpa and Crépeau 2009a, Proposition 3.1]. We explain in Appendix C how to modify the proof of [Cerpa 2007, Proposition 3.1] (as well as [Cerpa and Crépeau 2009a, Proposition 3.1]) in order to get Proposition 12.

We decompose β into $\beta = \beta_1 + \beta_2$, where $\beta_1 := P_H(\beta)$ and $\beta_2 := P_M(\beta)$. Hence, similarly to Proposition 9, we get

$$
\begin{cases}
\beta_{2t} + \beta_{2x} + \beta_{2xxx} + P_M(\alpha\alpha_x) = 0, \\
\beta_2(t, 0) = \beta_2(t, L) = 0, \\
\beta_{2x}(t, L) = 0, \\
\beta_2(0, x) = 0,
\end{cases}
\tag{4-17}
$$

where $\beta_2(T, \cdot) = P_M(\beta(T, \cdot)) \neq 0$. In particular, $P_{M^j}(\beta_2(T, \cdot)) = P_{M^j}(\beta(T, \cdot)) \neq 0$.

Combining (4-16) and (4-17), we get:

Corollary 13. *For every $L \in \mathcal{N}_2 \cup \mathcal{N}_3$, for every $T_0 > 0$, for every $j \in \{1, 2, \ldots, n\}$, there exists $u_0^j \in L^\infty(0, T_0)$ such that the solution (y_1, y_2) to equation (4-4) with $v(t, z) := u_0^j(t)$ satisfies*

$$y_1(T_0) = 0 \quad and \quad P_{M^j}(y_2(T_0)) \neq 0. \tag{4-18}$$

Now we come back to the case when (4-10) holds. Let us fix $T_0 > 0$ such that

$$T_0 < \tfrac{1}{4}p. \tag{4-19}$$

Let

$$q := \tfrac{1}{4}p. \tag{4-20}$$

Let u_0 be as in Corollary 13. We define

$$Y_1(t) := y_1(t), \quad Y_2(t) := y_2(t) \quad \text{for } t \in [0, T_0] \tag{4-21}$$

and

$$\psi_1 := Y_2(T_0) \in M \setminus \{0\}. \tag{4-22}$$

Let

$$\psi_2 = S(q)\psi_1 \in M, \quad \psi_3 = S(2q)\psi_1 \in M, \quad \psi_4 = S(3q)\psi_1 \in M, \tag{4-23}$$

$$T := 3q + T_0, \tag{4-24}$$

$$K_1 := [3q, 3q + T_0], \tag{4-25}$$

$$K_2 := [2q, 2q + T_0], \tag{4-26}$$

$$K_3 := [q, q + T_0], \tag{4-27}$$

$$K_4 := [0, T_0]. \tag{4-28}$$

Note that (4-19) implies

$$K_1, K_2, K_3 \text{ and } K_4 \text{ are pairwise disjoint.} \tag{4-29}$$

Let us define four functions $[0, T] \to \mathbb{R}$: u_1, u_2, u_3 and u_4 by requiring that, for every $i \in \{1, 2, 3, 4\}$,

$$u_i := \begin{cases} 0 & \text{on } [0, T] \setminus K_i, \\ u_0(\cdot - \tau_i) & \text{on } K_i, \end{cases} \tag{4-30}$$

with

$$\tau_1 = 3q, \quad \tau_2 = 2q, \quad \tau_3 = q, \quad \tau_4 = 0. \tag{4-31}$$

One can easily verify that, for every $i \in \{1, 2, 3, 4\}$, the solution of (4-4) for $v = u_i$ is given explicitly by

$$y_{i,1}(t) = \begin{cases} 0 & \text{on } [0, T] \setminus K_i, \\ Y_1(\cdot - \tau_i) & \text{on } K_i \end{cases} \tag{4-32}$$

and

$$y_{i,2}(t) = \begin{cases} 0 & \text{on } [0, \tau_i], \\ Y_2(\cdot - \tau_i) & \text{on } K_i, \\ S(\cdot - \tau_i - T_0)\psi_1 & \text{on } [\tau_i + T_0, T]. \end{cases} \tag{4-33}$$

For $z \in M_1$, let $\alpha_1, \alpha_2, \alpha_3$ and α_4 in $[0, +\infty)$ be such that

$$-S(T)z = \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + \alpha_4\psi_4, \tag{4-34}$$

$$\alpha_1\alpha_3 = 0, \quad \alpha_2\alpha_4 = 0. \tag{4-35}$$

Let us define

$$v(t, z) := \alpha_1 u_1(t) + \alpha_2 u_2(t) + \alpha_3 u_3(t) + \alpha_4 u_4(t). \tag{4-36}$$

We notice that

$$(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2)\|\psi_1\|_{L_L^2}^2 = 1, \tag{4-37}$$

which, together with (4-36), implies that

$$v \in L^\infty([0, T] \times M_1; \mathbb{R}). \tag{4-38}$$

Moreover, using the above construction (and in particular (4-29)), one easily checks that the solution of (4-4) satisfies

$$y_1(t) = \alpha_1 y_{1,1}(t) + \alpha_2 y_{2,1}(t) + \alpha_3 y_{3,1}(t) + \alpha_4 y_{4,1}(t) \quad \text{for } t \in [0, T], \tag{4-39}$$

$$y_2(t) = \alpha_1^2 y_{1,2}(t) + \alpha_2^2 y_{2,2}(t) + \alpha_3^2 y_{3,2}(t) + \alpha_4^2 y_{4,2}(t) \quad \text{for } t \in [0, T]. \tag{4-40}$$

In particular

$$y_1(T) = 0, \tag{4-41}$$

$$y_2(T) = \alpha_1^2 \psi_1 + \alpha_2^2 \psi_2 + \alpha_3^2 \psi_3 + \alpha_4^2 \psi_4. \tag{4-42}$$

From (4-34), (4-37) and (4-42), we can find that (4-5) holds if $\delta > 0$ is small enough. It is easy to check that the Lipschitz condition (4-3) is also satisfied. This completes the construction of $v(t, z)$ such that $(\mathcal{P}_3)$ holds and also the proof of Proposition 10 if (4-10) holds.

For other values of $L \in \mathcal{N}_2$, only the values of φ_1, φ_2 and p have to be modified. For $L \in \mathcal{N}_3$, as mentioned in the Introduction, M is now of dimension $2n$, where n is the number of ordered pairs. It is proved in [Cerpa and Crépeau 2009a] that (compare with (4-11)–(4-14)), by a good choice of order on $\{\varphi^j\}$, one can assume

$$0 < p^1 < p^2 < \cdots < p^n, \tag{4-43}$$

where $p^j := 2\pi/\omega^j$. For every $t > 0$, one has

$$S(t)M^j \subset M^j \quad \text{and} \quad S(t) \text{ restricted to } M^j \text{ is the rotation of angle } \frac{2\pi t}{p^j}. \tag{4-44}$$

From (4-43), (4-44) and Corollary 13, one can get the following corollary (see also [Cerpa and Crépeau 2009a, Proposition 3.3]):

Corollary 14. *For every $L \in \mathcal{N}_3$, there exists $T_L > 0$ such that, for every $j \in \{1, 2, \ldots, n\}$, there exists $u_0^j \in L^\infty(0, T_L)$ such that the solution (y_1, y_2) to equation (4-4) with $v(t, z) := u_0^j(t)$ satisfies*

$$y_1(T_L) = 0 \quad \text{and} \quad y_2(T_L) = \varphi_1^j. \tag{4-45}$$

Let us define

$$\psi_1^j := \varphi_1^j, \quad \psi_2^j := S(q^j)\varphi_1^j, \quad \psi_3^j := S(2q^j)\varphi_1^j, \quad \psi_4^j := S(3q^j)\varphi_1^j, \tag{4-46}$$

where $q^j := p^j/4$.

Comparing with (4-22)–(4-33), we can find $T > T_L$ and closed interval sets $\{K_i^j\}$, where $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, n\}$, such that

$$K_i^j \subset [0, T], \tag{4-47}$$

$$\{K_i^j\} \text{ are pairwise disjoint.} \tag{4-48}$$

We can also find functions $\{u_i^j\} \in L^\infty([0, T]; \mathbb{R})$, with

$$u_i^j(t) \text{ supports on } K_i^j, \tag{4-49}$$

such that when we define the control as u_i^j, we get the solution of (4-4) satisfies

$$y_{i,1}^j(t) \text{ supports on } K_i^j, \tag{4-50}$$

$$y_{i,1}^j(T) = 0, \tag{4-51}$$

$$y_{i,2}^j(T) = \psi_i^j. \tag{4-52}$$

Then for $z \in M_1$, let α_i^j in $[0, +\infty)$ be such that

$$-S(T)z = \sum_{i,j} \alpha_i^j \psi_i^j, \tag{4-53}$$

$$\alpha_1^j \alpha_3^j = 0, \quad \alpha_2^j \alpha_4^j = 0, \quad \sum_{i,j} (\alpha_i^j)^2 = 1, \tag{4-54}$$

where $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, n\}$. Let us define

$$v(t, z) := \sum_{i,j} \alpha_i^j u_i^j(t). \tag{4-55}$$

Then the solution of (4-4) with control defined as $v(t, z)$ satisfies

$$y_1(T) = 0, \tag{4-56}$$

$$y_2(T) = \sum_{i,j} (\alpha_i^j)^2 \psi_i^j. \tag{4-57}$$

One can easily verify that condition (4-5) holds when $\delta > 0$ is small enough, and that Lipschitz condition (4-3) also holds. This completes the construction of $v(t, z)$ and the proof of Proposition 10.   □

We are now able to define the periodic time-varying feedback laws $u_\varepsilon : \mathbb{R} \times L^2(0, L) \to \mathbb{R}$, which will lead to the exponential stabilization of (1-1). For $\varepsilon > 0$, we define u_ε by

$$u_\varepsilon|_{[0,T) \times L_L^2}(t, y) := \begin{cases} 0 & \text{if } \|y^M\|_{L_L^2} = 0, \\ \varepsilon \sqrt{\|y^M\|_{L_L^2}} \, v\big(t, S(-t)y^M/\|y^M\|_{L_L^2}\big) & \text{if } 0 < \|y^M\|_{L_L^2} \leqslant 1, \\ \varepsilon v\big(t, S(-t)y^M/\|y^M\|_{L_L^2}\big) & \text{if } \|y^M\|_{L_L^2} > 1, \end{cases} \tag{4-58}$$

with $y^M := P_M(y)$, and

$$u_\varepsilon(t, y) := u_\varepsilon|_{[0,T) \times L_L^2}(t - [t/T]T, y) \quad \forall t \in \mathbb{R}, \ \forall y \in L^2(0, L). \tag{4-59}$$

5. Proof of Theorem 1

Let us first point out that Theorem 1 is a consequence of the following two propositions.

Proposition 15. *There exist $\varepsilon_1 > 0$, $r_1 > 0$ and C_1 such that, for every Carathéodory feedback law u satisfying*

$$|u(t, z)| \leqslant \varepsilon_1 \min\big\{1, \sqrt{\|P_M(z)\|_{L_L^2}}\big\} \quad \forall t \in \mathbb{R}, \ \forall z \in L^2(0, L), \tag{5-1}$$

for every $s \in \mathbb{R}$ and for every maximal solution y of (3-6) defined at time s and satisfying $\|y(s)\|_{L_L^2} < r_1$, y is well-defined on $[s, s + T]$ and one has

$$\|P_H(y)\|_{\mathcal{B}_{s,s+T}}^2 + \|P_M(y)\|_{\mathcal{B}_{s,s+T}} \leqslant C_1 \big(\|P_H(y(s))\|_{L_L^2}^2 + \|P_M(y(s))\|_{L_L^2} \big). \tag{5-2}$$

Proposition 16. *For ρ_1 as in Proposition 10, let $\rho_2 > \rho_1$. There exists $\varepsilon_0 \in (0, 1)$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists $r_\varepsilon > 0$ such that, for every solution y to (3-6) on $[0, T]$, for the feedback law $u := u_\varepsilon$ defined in (4-58) and (4-59), and satisfying $\|y(0)\|_{L_L^2} < r_\varepsilon$, one has*

$$\|P_H(y(T))\|_{L_L^2}^2 + \varepsilon \|P_M(y(T))\|_{L_L^2} \leqslant \rho_2 \|P_H(y(0))\|_{L_L^2}^2 + \varepsilon(1 - \delta\varepsilon^2) \|P_M(y(0))\|_{L_L^2}. \tag{5-3}$$

Indeed, it suffices to choose $\rho_2 \in (\rho_1, 1)$, $\varepsilon \in (0, \varepsilon_0)$ and $u := u_\varepsilon$ defined in (4-58) and (4-59). Then, using the T-periodicity of u with respect to time, Proposition 15 and Proposition 16, one checks that inequality (1-11) holds with

$$\lambda := \min \left\{ -\frac{\ln(\rho_2)}{2T}, -\frac{\ln(1 - \delta\varepsilon^2)}{2T} \right\}$$

provided that C is large enough and that r is small enough. We now prove Propositions 15 and 16 successively.

Proof of Proposition 15. Performing a time translation if necessary, we may assume without loss of generality that $s = 0$. The fact that the maximal solution y is at least defined on $[0, T]$ follows from Theorem 8 and (5-1). We choose ε_1 and r_1 small enough so that

$$r_1 + \varepsilon_1 T^{1/2} \leqslant \eta, \tag{5-4}$$

where $\eta > 0$ is as in Lemma 5. From (5-1) and (5-4), we have

$$\|y(0)\|_{L_L^2} + \|u(t, y(t))\|_{L_T^2} \leqslant \eta, \tag{5-5}$$

which allows us to apply Lemma 5 with $H(t) := u(t, y(t))$ and $\widetilde{H} := 0$. Then, using (5-1) once more, we get

$$\|y\|_{\mathcal{B}} \leqslant C_3 \big(\|y_0\|_{L_L^2} + \|u(t, y(t))\|_{L_T^2} \big)$$
$$\leqslant C_3 \big(r_1 + \varepsilon_1 \sqrt{T \|P_M(y)\|_{C^0 L_L^2}} \big) \leqslant C_3 \Big(r_1 + \varepsilon_1^2 T C_3 + \frac{1}{4C_3} \|y\|_{\mathcal{B}} \Big),$$

which implies that

$$\|y\|_{\mathcal{B}} \leqslant 2C_3(r_1 + \varepsilon_1^2 T C_3). \tag{5-6}$$

In the above inequalities and until the end of the proof of Proposition 16, $\mathcal{B} := \mathcal{B}_{0,T}$.

We have the following lemma; see the proof of [Rosier 1997, Proposition 4.1 and (4.14)] or [Perla Menzala et al. 2002, page 121].

Lemma 17. *If $y \in L^2(0, T; H^1(0, L))$, then $yy_x \in L^1(0, T; L^2(0, L))$. Moreover, there exists $c_4 > 0$, which is independent of T, such that, for every $T > 0$ and for every $y, z \in L^2(0, T; H^1(0, L))$, we have*

$$\|yy_x - zz_x\|_{L_T^1 L_L^2} \leqslant c_4 T^{1/4} \big(\|y\|_{\mathcal{B}} + \|z\|_{\mathcal{B}} \big) \|y - z\|_{\mathcal{B}}. \tag{5-7}$$

Let us define $C_4 := c_4 T^{1/4}$. To simplify the notation, until the end of this section, we write y_1 and y_2 for $P_H(y)$ and $P_M(y)$ respectively. From (5-1), (5-6), Lemma 3, Lemma 17 and Proposition 9, we get

$$\|y_1\|_{\mathcal{B}} \leqslant C_2\big(\|y_0^H\|_{L_L^2} + \|u(t, y_1 + y_2)\|_{L_T^2} + \big\|P_H\big((y_1 + y_2)(y_1 + y_2)_x\big)\big\|_{L_T^1 L_L^2}\big)$$

$$\leqslant C_2\big(\|y_0^H\|_{L_L^2} + \varepsilon_1\big\|\sqrt{\|y_2\|_{L_L^2}}\big\|_{L_T^2} + \big\|(y_1 + y_2)(y_1 + y_2)_x\big\|_{L_T^1 L_L^2}\big)$$

$$\leqslant C_2\big(\|y_0^H\|_{L_L^2} + \varepsilon_1\|y_2\|_{L_T^1 L_L^2}^{1/2} + C_4\|y_1 + y_2\|_{L_T^2 H_L^1}^2\big) \tag{5-8}$$

and

$$\|y_2\|_{\mathcal{B}} \leqslant C_2\big(\|y_0^M\|_{L_L^2} + \big\|P_M\big((y_1 + y_2)(y_1 + y_2)_x\big)\big\|_{L_T^1 L_L^2}\big)$$

$$\leqslant C_2\big(\|y_0^M\|_{L_L^2} + \big\|(y_1 + y_2)(y_1 + y_2)_x\big\|_{L_T^1 L_L^2}\big)$$

$$\leqslant C_2\big(\|y_0^M\|_{L_L^2} + C_4\|y_1 + y_2\|_{L_T^2 H_L^1}^2\big)$$

$$\leqslant 2C_2\big(\|y_0^M\|_{L_L^2} + C_4\|y_1\|_{\mathcal{B}}^2 + C_4\|y_2\|_{\mathcal{B}}^2\big). \tag{5-9}$$

Since M is a finite-dimensional subspace of $H^1(0, L)$, there exists $C_5 > 0$ such that

$$\|f\|_{H^1(0, L)} \leqslant C_5\|f\|_{L_L^2} \quad \text{for every } f \in M. \tag{5-10}$$

Hence

$$\|y_2\|_{\mathcal{B}} = \|y_2\|_{L_T^\infty L_L^2} + \|y_2\|_{L_T^2 H_L^1} \leqslant \|y_2\|_{L_T^\infty L_L^2} + C_5\sqrt{T}\|y_2\|_{L_T^\infty L_L^2}. \tag{5-11}$$

Since $y_2(t)$ is the L^2-orthogonal projection on M of $y(t)$, we have

$$\|y_2\|_{L_T^\infty L_L^2} \leqslant \|y\|_{L_T^\infty L_L^2} \leqslant \|y\|_{\mathcal{B}},$$

which, together with (5-6) and (5-11), implies

$$\|y_2\|_{\mathcal{B}} \leqslant (1 + C_5\sqrt{T})\|y\|_{\mathcal{B}} \leqslant 2(1 + C_5\sqrt{T})C_3(r_1 + \varepsilon_1^2 T C_3). \tag{5-12}$$

Decreasing if necessary r_1 and ε_1, we may assume

$$4C_2 C_4(1 + C_5\sqrt{T})C_3(r_1 + \varepsilon_1^2 T C_3) < \tfrac{1}{2}. \tag{5-13}$$

From estimation (5-9) and condition (5-13), we get

$$\|y_2\|_{\mathcal{B}} \leqslant 4C_2\big(\|y_0^M\|_{L_L^2} + C_4\|y_1\|_{\mathcal{B}}^2\big). \tag{5-14}$$

From (5-6), (5-8), (5-12) and (5-14), we deduce that

$$\|y_1\|_{\mathcal{B}}^2 \leqslant 3C_2^2\big(\|y_0^H\|_{L_L^2}^2 + \varepsilon_1^2\|y_2\|_{L_T^1 L_L^2} + C_4^2\|y_1 + y_2\|_{L_T^2 H_L^1}^4\big)$$

$$\leqslant 3C_2^2\big(\|y_0^H\|_{L_L^2}^2 + \varepsilon_1^2 T\|y_2\|_{L_T^\infty L_L^2} + 2C_4^2\|y\|_{\mathcal{B}}^2\big(\|y_1\|_{\mathcal{B}}^2 + \|y_2\|_{\mathcal{B}}^2\big)\big)$$

$$\leqslant 3C_2^2\|y_0^H\|_{L_L^2}^2 + 3C_2^2\big(\varepsilon_1^2 T + 16C_4^2(1 + C_5\sqrt{T})C_3^3(r_1 + \varepsilon_1^2 T C_3)^3\big)\|y_2\|_{\mathcal{B}}$$

$$\hspace{6cm} + 24C_2^2 C_4^2 C_3^2(r_1 + \varepsilon_1^2 T C_3)^2\|y_1\|_{\mathcal{B}}^2$$

$$\leqslant 3C_2^2\|y_0^H\|_{L_L^2}^2 + 12C_2^3\big(\varepsilon_1^2 T + 16C_4^2(1 + C_5\sqrt{T})C_3^3(r_1 + \varepsilon_1^2 T C_3)^3\big)\|y_0^M\|_{L_L^2}$$

$$+ \big(12C_2^3 C_4\big(\varepsilon_1^2 T + 16C_4^2(1 + C_5\sqrt{T})C_3^3(r_1 + \varepsilon_1^2 T C_3)^3\big) + 24C_2^2 C_4^2 C_3^2(r_1 + \varepsilon_1^2 T C_3)^2\big)\|y_1\|_{\mathcal{B}}^2. \tag{5-15}$$

Again, decreasing if necessary r_1 and ε_1, we may assume

$$12C_2^3 C_4\big(\varepsilon_1^2 T + 16C_4^2(1 + C_5\sqrt{T})C_3^3(r_1 + \varepsilon_1^2 T C_3)^3\big) + 24C_2^2 C_4^2 C_3^2(r_1 + \varepsilon_1^2 T C_3)^2 < \tfrac{1}{2}. \tag{5-16}$$

From (5-15) and (5-16), we get

$$\|y_1\|_{\mathcal{B}}^2 \leqslant 6C_2^2\|y_0^H\|_{L_L^2}^2 + 24C_2^3\big(\varepsilon_1^2 T + 16C_4^2(1 + C_5\sqrt{T})C_3^3(r_1 + \varepsilon_1^2 T C_3)^3\big)\|y_0^M\|_{L_L^2}$$

$$\leqslant 6C_2^2\|y_0^H\|_{L_L^2}^2 + C_4^{-1}\|y_0^M\|_{L_L^2},$$

which, combined with (5-14), gives the existence of $C_1 > 0$ independent of y such that

$$\|y_1\|_{\mathcal{B}}^2 + \|y_2\|_{\mathcal{B}} \leqslant C_1\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big). \tag{5-17}$$

This completes the proof of Proposition 15. □

Proof of Proposition 16. To simplify the notation, from now on we denote by C various constants which vary from place to place but do not depend on ε and r.

By Lemma 3 applied with $y := y_1(t) - S(t)y_0^H$, $h(t) := u_\varepsilon(t, y(t))$ and $\tilde{h} := (y_1 + y_2)(y_1 + y_2)_x$ and by Proposition 15, we have

$$\|y_1(t) - S(t)y_0^H\|_{\mathcal{B}} \leqslant C\big(\|u_\varepsilon\|_{L_T^2} + \big\|P_H((y_1 + y_2)(y_1 + y_2)_x)\big\|_{L_T^1 L_L^2}\big)$$

$$\leqslant C\big(\varepsilon\|y_2\|_{L_T^1 L_L^2}^{1/2} + \|y_1 + y_2\|_{\mathcal{B}}^2\big)$$

$$\leqslant C\big(\varepsilon\|y_2\|_{\mathcal{B}}^{1/2} + \|y_1\|_{\mathcal{B}}^2 + \|y_2\|_{\mathcal{B}}^2\big)$$

$$\leqslant C(\varepsilon + \sqrt{r})\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big)^{1/2}, \tag{5-18}$$

where $r := \|y_0\|_{L_L^2} < r_\varepsilon < 1$. On r_ε, we impose that

$$r_\varepsilon < \varepsilon^{12}. \tag{5-19}$$

From (5-18) and (5-19), we have

$$\|y_1(t) - S(t)y_0^H\|_{\mathcal{B}} \leqslant C\varepsilon\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big)^{1/2}. \tag{5-20}$$

Notice that, by Lemma 3, we have

$$\|S(t)y_0^M\|_{\mathcal{B}} \leqslant C\|y_0^M\|_{L_L^2}, \tag{5-21}$$

$$\|S(t)y_0^H\|_{\mathcal{B}} \leqslant C\|y_0^H\|_{L_L^2}. \tag{5-22}$$

Proceeding as in the proof of (5-20), we have

$$\|y_2(t) - S(t)y_0^M\|_{\mathcal{B}} \leqslant C\big\|P_M((y_1 + y_2)(y_1 + y_2)_x)\big\|_{L_T^1 L_L^2}$$

$$\leqslant C\|y_1 + y_2\|_{\mathcal{B}}^2$$

$$\leqslant C\big(\|y_2\|_{\mathcal{B}} + \|S(t)y_0^H\|_{\mathcal{B}} + \varepsilon\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big)^{1/2}\big)^2$$

$$\leqslant C\big((r + \varepsilon^2)\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big) + \|y_0^H\|_{L_L^2}^2\big)$$

$$\leqslant C\big(\varepsilon^2\|y_0^M\|_{L_L^2} + \|y_0^H\|_{L_L^2}^2\big). \tag{5-23}$$

Let us now study successively the two cases

$$\|y_0^H\|_{L_L^2} \geqslant \varepsilon^{2/3}\sqrt{\|y_0^M\|_{L_L^2}}, \tag{5-24}$$

$$\|y_0^H\|_{L_L^2} < \varepsilon^{2/3}\sqrt{\|y_0^M\|_{L_L^2}}. \tag{5-25}$$

We start with the case where (5-24) holds. From $(\mathcal{P}_1)$, $(\mathcal{P}_2)$, (5-20), (5-23) and (5-24), we get the existence of $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for every $\varepsilon \in (0, \varepsilon_2)$,

$$
\begin{aligned}
\|y_1(T)\|_{L_L^2}^2 &+ \varepsilon\|y_2(T)\|_{L_L^2} \\
&\leqslant \left(C\varepsilon\left(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\right)^{1/2} + \|S(T)y_0^H\|_{L_L^2}\right)^2 + \varepsilon\left(C\left(\varepsilon^2\|y_0^M\|_{L_L^2} + \|y_0^H\|_{L_L^2}^2\right) + \|S(T)y_0^M\|_{L_L^2}\right) \\
&\leqslant (\rho_1\rho_2)^{1/2}\|y_0^H\|_{L_L^2}^2 + C\varepsilon^2\left(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\right) + C\varepsilon\|y_0^H\|_{L_L^2}^2 + (\varepsilon + C\varepsilon^3)\|y_0^M\|_{L_L^2} \\
&\leqslant \rho_2\|y_0^H\|_{L_L^2}^2 + \varepsilon(1 - \delta\varepsilon^2)\|y_0^M\|_{L_L^2}.
\end{aligned}
\tag{5-26}
$$

Let us now study the case where (5-25) holds. Let us define

$$b := y_0^M. \tag{5-27}$$

Then, from (5-20), (5-22), (5-23) and (5-25), we get

$$\|y_1(t)\|_{\mathcal{B}} \leqslant \|S(t)y_0^H\|_{\mathcal{B}} + C\varepsilon\left(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\right)^{1/2} \leqslant C\varepsilon\sqrt{\|b\|_{L_L^2}} + C\|y_0^H\|_{L_L^2} \leqslant C\varepsilon^{2/3}\sqrt{\|b\|_{L_L^2}} \tag{5-28}$$

and

$$\|y_2(t) - S(t)y_0^M\|_{\mathcal{B}} \leqslant \varepsilon^{4/3}\|b\|_{L_L^2}, \tag{5-29}$$

which shows that $y_2(\cdot)$ is close to $S(\cdot)y_0^M$. Let $z : [0, T] \to L^2(0, L)$ be the solution to the Cauchy problem

$$
\begin{cases}
z_{1t} + z_{1xxx} + z_{1x} = 0 & \text{in } (0, T) \times (0, L), \\
z_1(t, 0) = z_1(t, L) = 0 & \text{on } (0, T), \\
z_{1x}(t, L) = v(t, b/\|b\|_{L_L^2}) & \text{on } (0, T), \\
z_1(0, x) = 0 & \text{on } (0, L).
\end{cases}
\tag{5-30}
$$

From $(\mathcal{P}_3)$, we know that $z_1(T) = 0$. Moreover, Lemma 3 tells us that

$$\|z_1(t)\|_{\mathcal{B}} \leqslant C\left\|v\left(t, \frac{b}{\|b\|_{L_L^2}}\right)\right\|_{L_T^2} \leqslant C. \tag{5-31}$$

Let us define w_1 by

$$w_1 := y_1 - S(t)y_0^H - \varepsilon\|b\|_{L_L^2}^{1/2}z_1. \tag{5-32}$$

Then w_1 is the solution to the Cauchy problem

$$
\begin{cases}
w_{1t} + w_{1xxx} + w_{1x} + P_H\left((y_1 + y_2)(y_1 + y_2)_x\right) = 0, \\
w_1(t, 0) = w_1(t, L) = 0, \\
w_{1x}(t, L) = \varepsilon\left(\|y_2(t)\|_{L_L^2}^{1/2}v\left(t, S(-t)y_2(t)/\|y_2(t)\|_{L_L^2}\right) - \|b\|_{L_L^2}^{1/2}v\left(t, b/\|b\|_{L_L^2}\right)\right), \\
w_1(0, x) = 0.
\end{cases}
\tag{5-33}
$$

By Lemma 3, we get

$$\|w_1\|_{\mathcal{B}} \leqslant C \|P_H((y_1 + y_2)(y_1 + y_2)_x)\|_{L_T^1 L_L^2}$$
$$+\varepsilon C \left\| \left(\|y_2(t)\|_{L_L^2}^{1/2} v\left(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L_L^2}}\right) - \|b\|_{L_L^2}^{1/2} v\left(t, \frac{b}{\|b\|_{L_L^2}}\right) \right) \right\|_{L_T^2}. \quad (5\text{-}34)$$

Note that (5-29) ensures that the right-hand side of (5-34) is of order ε^2. Indeed, for the first term of the right-hand side of inequality (5-34), we have, using (5-19), (5-28) and (5-29),

$$C\|P_H((y_1 + y_2)(y_1 + y_2)_x)\|_{L_T^1 L_L^2} \leqslant C\|y_1 + y_2\|_{\mathcal{B}}^2$$
$$\leqslant C\varepsilon^{4/3}\|b\|_{L_L^2} + C\|b\|_{L_L^2} \leqslant C\|b\|_{L_L^2}^{1/2}\|b\|_{L_L^2}^{1/2} \leqslant C\varepsilon^6\|b\|_{L_L^2}^{1/2}. \quad (5\text{-}35)$$

For the second term of the right-hand side of inequality (5-34), by (4-14), the Lipschitz condition (4-3) on v and (5-29), we get, for every $t \in [0, T]$,

$$\left| \|b\|_{L_L^2}^{1/2}\left(v\left(t, \frac{b}{\|b\|_{L_L^2}}\right) - v\left(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L_L^2}}\right)\right) \right|$$
$$\leqslant C\|b\|_{L_L^2}^{1/2}\left\|\left(\frac{b}{\|b\|_{L_L^2}} - \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L_L^2}}\right)\right\|_{L_L^2}$$
$$\leqslant C\|b\|_{L_L^2}^{-1/2}\|y_2(t)\|_{L_L^2}^{-1}\left(\|y_2(t)\|_{L_L^2}\|b - S(-t)y_2(t)\|_{L_L^2} + \|S(-t)y_2(t)\|_{L_L^2}\big|\|y_2(t)\|_{L_L^2} - \|b\|_{L_L^2}\big|\right)$$
$$\leqslant C\varepsilon^{4/3}\|b\|_{L_L^2}^{1/2} \quad (5\text{-}36)$$

and

$$\left| \left(\|y_2(t)\|_{L_L^2}^{1/2} - \|b\|_{L_L^2}^{1/2}\right) v\left(t, \frac{S(-t)y_2(t)}{\|y_2(t)\|_{L_L^2}}\right) \right| \leqslant C\varepsilon^{4/3}\|b\|_{L_L^2}^{1/2}. \quad (5\text{-}37)$$

Combining (5-35)–(5-37), we obtain the following estimate on w_1:

$$\|w_1\|_{\mathcal{B}} \leqslant C\varepsilon^2\|b\|_{L_L^2}^{1/2}. \quad (5\text{-}38)$$

We fix

$$\rho_3 \in (\rho_1, \rho_2). \quad (5\text{-}39)$$

Then, by (5-32), ($\mathcal{P}_1$) and the fact that $z_1(T) = 0$, we get

$$\|y_1(T)\|_{L_L^2}^2 \leqslant \rho_3\|y_0^H\|_{L_L^2}^2 + C\varepsilon^4\|b\|_{L_L^2}. \quad (5\text{-}40)$$

We then come to the estimate of y_2. Let $\tau_1(t) := S(t)y_0^H$ and let $\tau_2 : [0, T] \to L^2(0, L)$ and $z_2 : [0, T] \to L^2(0, L)$ be the solutions to the Cauchy problems

$$\begin{cases} \tau_{2t} + \tau_{2xxx} + \tau_{2x} + P_M(\tau_1 y_{1x} + \tau_{1x} y_1) - P_M(\tau_1 \tau_{1x}) = 0, \\ \tau_2(t, 0) = \tau_2(t, L) = 0, \\ \tau_{2x}(t, L) = 0, \\ \tau_2(0, x) = 0 \end{cases} \quad (5\text{-}41)$$

and

$$\begin{cases} z_{2t} + z_{2xxx} + z_{2x} + P_M(z_1 z_{1x}) = 0, \\ z_2(t, 0) = z_2(t, L) = 0, \\ z_{2x}(t, L) = 0, \\ z_2(0, x) = 0. \end{cases} \tag{5-42}$$

Lemmas 3 and 17, (5-25) and (5-28) show us that

$$\begin{aligned} \|\tau_2\|_{\mathcal{B}} &\leqslant C \|P_M(\tau_1 y_{1x} + \tau_{1x} y_1 - \tau_1 \tau_{1x})\|_{L_T^1 L_L^2} \\ &\leqslant C \|\tau_1\|_{\mathcal{B}} (\|y_1\|_{\mathcal{B}} + \|\tau_1\|_{\mathcal{B}}) \\ &\leqslant C \varepsilon^{2/3} \|b\|_{L_L^2}^{1/2} \|y_0^H\|_{L_L^2} \end{aligned} \tag{5-43}$$

and

$$\|z_2\|_{\mathcal{B}} \leqslant \|z_1\|_{\mathcal{B}}^2 \leqslant C. \tag{5-44}$$

From $(\mathcal{P}_3)$, (5-30) and (5-42), we get

$$\langle z_2(T), S(T)b \rangle_{(L_L^2, L_L^2)} < -2\delta \|b\|_{L_L^2}. \tag{5-45}$$

Hence

$$\begin{aligned} \left\| S(T)b + \varepsilon^2 \|b\|_{L_L^2} z_2(T) \right\|_{L_L^2} &= \left(\left\langle S(T)b + \varepsilon^2 \|b\|_{L_L^2} z_2(T), S(T)b + \varepsilon^2 \|b\|_{L_L^2} z_2(T) \right\rangle_{(L_L^2, L_L^2)} \right)^{1/2} \\ &\leqslant \left(\|b\|_{L_L^2}^2 + \varepsilon^4 \|b\|_{L_L^2}^2 C - 4\delta \varepsilon^2 \|b\|_{L_L^2}^2 \right)^{1/2} \\ &\leqslant \|b\|_{L_L^2} (1 - 2\delta \varepsilon^2 + C \varepsilon^4). \end{aligned} \tag{5-46}$$

Let us define $w_2 : [0, T] \to L^2(0, L)$ by

$$w_2 := y_2 - \tau_2 - \varepsilon^2 \|b\|_{L_L^2} z_2 - S(t)b. \tag{5-47}$$

Then, from (3-15), (5-41) and (5-42), we get that

$$\begin{aligned} w_{2t} &= y_{2t} - \tau_{2t} - \varepsilon^2 \|b\|_{L_L^2} z_{2t} - (S(t)b)_t \\ &= -w_{2x} - w_{2xxx} - P_M((y_1 + y_2)(y_1 + y_2)_x) + P_M(\tau_1 y_{1x} + \tau_{1x} y_1) - P_M(\tau_1 \tau_{1x}) + \varepsilon^2 \|b\|_{L_L^2} P_M(z_1 z_{1x}) \\ &= -w_{2x} - w_{2xxx} - \varepsilon \|b\|_{L_L^2}^{1/2} P_M(w_1 z_{1x} + w_{1x} z_1) - P_M(w_1 w_{1x}) - P_M(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x}). \end{aligned}$$

Hence, w_2 is the solution to the Cauchy problem

$$\begin{cases} w_{2t} + w_{2xxx} + w_{2x} + \varepsilon \|b\|_{L_L^2}^{1/2} P_M(w_1 z_{1x} + w_{1x} z_1) + P_M(w_1 w_{1x}) + P_M(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x}) = 0, \\ w_2(t, 0) = w_2(t, L) = 0, \\ w_{2x}(t, L) = 0, \\ w_2(0, x) = 0. \end{cases} \tag{5-48}$$

From Lemmas 3 and 17, Proposition 15, (5-19), (5-25) and (5-38), we get

$$\|w_2\|_\mathcal{B} \leqslant C\varepsilon\|b\|_{L_L^2}^{1/2}\big\|P_M(w_1 z_{1x} + w_{1x} z_1)\big\|_{L_T^1 L_L^2} + C\big\|P_M(w_1 w_{1x})\big\|_{L_T^1 L_L^2}$$
$$+ C\big\|P_M(y_1 y_{2x} + y_2 y_{1x} + y_2 y_{2x})\big\|_{L_T^1 L_L^2}$$
$$\leqslant C\varepsilon\|b\|_{L_L^2}^{1/2}\varepsilon^2\|b\|_{L_L^2}^{1/2} + C\varepsilon^4\|b\|_{L_L^2} + C\big(\|y_0^H\|_{L_L^2}^2 + \|y_0^M\|_{L_L^2}\big)^{3/2}$$
$$\leqslant C\varepsilon^3\|b\|_{L_L^2}. \tag{5-49}$$

We can now estimate $y_2(T)$ from (5-43), (5-46), (5-47) and (5-49):

$$\|y_2(T)\|_{L_L^2} = \big\|w_2(T) + \tau_2(T) + \varepsilon^2\|b\|_{L_L^2} z_2(T) + S(T)b\big\|_{L_L^2}$$
$$\leqslant \|b\|_{L_L^2}\big(C\varepsilon^3 + 1 - 2\delta\varepsilon^2 + C\varepsilon^4\big) + C\varepsilon^{2/3}\|b\|_{L_L^2}^{1/2}\|y_0^H\|_{L_L^2}. \tag{5-50}$$

Combining (5-27), (5-39), (5-40) and (5-50), we get the existence of $\varepsilon_3 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_3]$,

$$\|y_1(T)\|_{L_L^2}^2 + \varepsilon\|y_2(T)\|_{L_L^2}$$
$$\leqslant \rho_3\|y_0^H\|_{L_L^2}^2 + C\varepsilon^4\|b\|_{L_L^2} + \varepsilon\big(\|b\|_{L_L^2}\big(C\varepsilon^3 + 1 - 2\delta\varepsilon^2 + C\varepsilon^4\big) + C\varepsilon^{2/3}\|b\|_{L_L^2}^{1/2}\|y_0^H\|_{L_L^2}\big)$$
$$\leqslant \rho_2\|y_0^H\|_{L_L^2}^2 + \varepsilon(1 - \delta\varepsilon^2)\|y_0^M\|_{L_L^2}. \tag{5-51}$$

This concludes the proof of Proposition 16. □

Appendix A: Proof of Proposition 9

Proof of Proposition 9. It is clear that, if (y_1, y_2) is a solution to (3-15), then y is solution to (3-7). Let us assume that y is a solution to the Cauchy problem (3-7). Then, by Definition 4, for every $\tau \in [s, T]$ and for every $\phi \in C^3([s, \tau] \times [0, L])$ satisfying

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0 \quad \forall t \in [s, \tau], \tag{A-1}$$

we have

$$-\int_s^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y \, dx \, dt - \int_s^\tau u(t, y(t, \cdot)) \phi_x(t, L) \, dt + \int_s^\tau \int_0^L \phi y y_x \, dx \, dt$$
$$+ \int_0^L y(\tau, x)\phi(\tau, x) \, dx - \int_0^L y_0\phi(s, x) \, dx = 0. \tag{A-2}$$

Let us denote by ϕ_1 and ϕ_2 the projections of ϕ on H and M respectively: $\phi_1 := P_H(\phi)$, $\phi_2 := P_M(\phi)$. Because M is spanned by φ_1^j and φ_2^j, $j \in \{1, \ldots, n\}$, which are of class C^∞ and satisfy

$$\varphi_1^j(0) = \varphi_1^j(L) = \varphi_{1x}^j(0) = \varphi_{1x}^j(L) = 0,$$
$$\varphi_2^j(0) = \varphi_2^j(L) = \varphi_{2x}^j(0) = \varphi_{2x}^j(L) = 0,$$

the functions $\phi_1, \phi_2 \in C^3([s, \tau] \times [0, L])$ and satisfy

$$\phi_1(t, 0) = \phi_1(t, L) = \phi_{1x}(t, 0) = 0 \qquad \forall t \in [s, \tau], \tag{A-3}$$
$$\phi_2(t, 0) = \phi_2(t, L) = \phi_{2x}(t, 0) = \phi_{2x}(t, L) = 0 \quad \forall t \in [s, \tau]. \tag{A-4}$$

Using (A-2) for $\phi = \phi_2$ in (A-2) together with (A-4), we get

$$-\int_s^\tau \int_0^L (\phi_{2t} + \phi_{2x} + \phi_{2xxx}) y \, dx \, dt + \int_s^\tau \int_0^L \phi_2 y y_x \, dx \, dt$$

$$+ \int_0^L y(\tau, x)\phi_2(\tau, x) \, dx - \int_0^L y_0 \phi_2(s, x) \, dx = 0, \quad \text{(A-5)}$$

which, combined with the fact that $\phi_{2t} + \phi_{2x} + \phi_{2xxx} \in M$, gives

$$-\int_s^\tau \int_0^L (\phi_{2t} + \phi_{2x} + \phi_{2xxx}) y_2 \, dx \, dt + \int_s^\tau \int_0^L \phi_2 P_M(y y_x) \, dx \, dt$$

$$+ \int_0^L y_2(\tau, x)\phi_2(\tau, x) \, dx - \int_0^L P_M(y_0)\phi_2(s, x) \, dx = 0. \quad \text{(A-6)}$$

Simple integrations by parts show that $\phi_{1x} + \phi_{1xxx} \in M^\perp = H$. Since, ϕ_1 and ϕ_{1t} are also in H, we get from (A-6) that

$$-\int_s^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y_2 \, dx \, dt + \int_s^\tau \int_0^L \phi P_M(y y_x) \, dx \, dt$$

$$+ \int_0^L y_2(\tau, x)\phi(\tau, x) \, dx - \int_0^L P_M(y_0)\phi(s, x) \, dx = 0, \quad \text{(A-7)}$$

which is exactly the definition of a solution of the y_2-part of the linear KdV system (3-15). We then combine (A-2) and (A-7) to get

$$-\int_s^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx}) y_1 \, dx \, dt - \int_s^\tau u(t, y(t, \cdot))\phi_x(t, L) \, dt + \int_s^\tau \int_0^L \phi P_H(y y_x) \, dx \, dt$$

$$+ \int_0^L y_1(\tau, x)\phi(\tau, x) \, dx - \int_0^L P_H(y_0)\phi(0, x) \, dx = 0, \quad \text{(A-8)}$$

and we get the definition of a solution to the y_1-part of the linear KdV system (3-15). This concludes the proof of Proposition 9. $\qquad\square$

Appendix B: Proofs of Theorems 7 and 8

Our strategy to prove Theorem 7 is to prove first the existence of a solution for small times and then to use some a priori estimates to control the L_L^2-norm of the solution with which we can extend the solution to a longer time, and to continue until the solution blows up. We start by proving the following lemma.

Lemma 18. *Let $C_2 > 0$ be as in Lemma 3 for $T_2 - T_1 = 1$. Assume that u is a Carathéodory function and that, for every $R > 0$, there exists $K(R) > 0$ such that*

$$\big(\|y\|_{L_L^2} \leqslant R \quad \text{and} \quad \|z\|_{L_L^2} \leqslant R\big) \implies \big(|u(t, y) - u(t, z)| \leqslant K(R)\|y - z\|_{L_L^2} \quad \forall t \in \mathbb{R}\big). \quad \text{(B-1)}$$

Then, for every $R \in (0, +\infty)$, there exists a time $T(R) > 0$ such that, for every $s \in \mathbb{R}$ and for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L_L^2} \leqslant R$, the Cauchy problem (3-7) has one and only one solution y on $[s, s + T(R)]$. Moreover, this solution satisfies

$$\|y\|_{\mathcal{B}_{s,s+T(R)}} \leqslant C_R := 3C_2 R. \quad \text{(B-2)}$$

Proof of Lemma 18. Let us first point out that it follows from our choice of C_2 and Lemma 3 that, for every $-\infty < T_1 < T_2 < +\infty$ such that $T_2 - T_1 \leqslant 1$, for every solution y of problem (2-1), estimation (2-12) holds.

Let $y_0 \in L^2(0, L)$ be such that

$$\|y_0\|_{L_L^2} \leqslant R. \tag{B-3}$$

Let us define $\mathcal{B}_1$ by

$$\mathcal{B}_1 := \{y \in \mathcal{B}_{s,s+T(R)} : \|y\|_{\mathcal{B}_{s,s+T(R)}} \leqslant C_R\}.$$

The set $\mathcal{B}_1$ is a closed subset of $\mathcal{B}_{s,s+T(R)}$. For every $y \in \mathcal{B}_1$, we define $\Psi(y)$ as the solution of (2-1) with $\tilde{h} := -y y_x$, $h(t) := u(t, y(t, \cdot))$ and $y_0 := y_0$. Let us prove that, for $T(R)$ small enough, the smallness being independent of y_0 provided that it satisfies (B-3), we have

$$\Psi(\mathcal{B}_1) \subset \mathcal{B}_1. \tag{B-4}$$

Indeed for $y \in \mathcal{B}_1$, by Lemmas 3 and 17, we have, if $T(R) \leqslant 1$,

$$
\begin{aligned}
\|\Psi(y)\|_{\mathcal{B}} &\leqslant C_2 \big(\|y_0\|_{L_L^2} + \|h\|_{L_T^2} + \|\tilde{h}\|_{L^1(0,T;L^2(0,L))} \big) \\
&\leqslant C_2 \big(\|y_0\|_{L_L^2} + \|u(t, y(t, \cdot))\|_{L_T^2} + \|-y y_x\|_{L^1(s,s+T(R);L^2(0,L))} \big) \\
&\leqslant C_2 \big(R + C_B(C_R) T(R)^{1/2} + c_4 T(R)^{1/4} \|y\|_{\mathcal{B}}^2 \big).
\end{aligned} \tag{B-5}
$$

In (B-5) and until the end of the proof of Lemma 18, for ease of notation, we simply write $\|\cdot\|_{\mathcal{B}}$ for $\|\cdot\|_{\mathcal{B}_{s,s+T(R)}}$. From (B-5), we get that, if

$$T(R) \leqslant \min \left\{ \left(\frac{R}{C_B(C_R)} \right)^2, \left(\frac{1}{9 c_4 C_2^2 R} \right)^4, 1 \right\}, \tag{B-6}$$

then (B-4) holds. From now on, we assume that (B-6) holds.

Note that every $y \in \mathcal{B}_1$ such that $\Psi(y) = y$ is a solution of (3-7). In order to use the Banach fixed point theorem, it remains to estimate $\|\Psi(y) - \Psi(z)\|_{\mathcal{B}}$. We know that $\Psi(y) - \Psi(z)$ is the solution of equation (2-1) with $T_1 := s$, $T_2 = s + T(R)$, $\tilde{h} := -y y_x + z z_x$, $h(t) := u(t, y(t, \cdot)) - u(t, z(t, \cdot))$ and $y_0 := 0$. Hence, from Lemmas 3 and 17 and (B-1), we get

$$
\begin{aligned}
\|\Psi(y) - \Psi(z)\|_{\mathcal{B}} &\leqslant C_2 \big(\|y_0\|_{L_L^2} + \|h\|_{L_T^2} + \|\tilde{h}\|_{L^1(0,T;L^2(0,L))} \big) \\
&\leqslant C_2 \big(0 + T(R)^{1/2} K(C_R) \|y - z\|_{\mathcal{B}} + c_4 T(R)^{1/4} \|y - z\|_{\mathcal{B}} (\|y\|_{\mathcal{B}} + \|z\|_{\mathcal{B}}) \big) \\
&\leqslant C_2 \|y - z\|_{\mathcal{B}} \big(T(R)^{1/2} K(C_R) + 2 c_4 T(R)^{1/4} C_R \big),
\end{aligned}
$$

which shows that, if

$$T(R) \leqslant \min \left\{ \left(\frac{1}{12 c_4 C_2^2 R} \right)^4, \left(\frac{1}{4 C_2 K(3 C_2 R)} \right)^2 \right\}, \tag{B-7}$$

then,

$$\|\Psi(y) - \Psi(z)\|_{\mathcal{B}} \leqslant \tfrac{3}{4} \|y - z\|_{\mathcal{B}}.$$

Hence, by the Banach fixed point theorem, there exists $y \in \mathcal{B}_1$ such that $\Psi(y) = y$, which is the solution that we are looking for. We define $T(R)$ as

$$T(R) := \min\left\{ \left(\frac{R}{C_B(3C_2 R)} \right)^2, \left(\frac{1}{12c_4 C_2^2 R} \right)^4, \left(\frac{1}{4C_2 K(3C_2 R)} \right)^2, 1 \right\}. \tag{B-8}$$

It only remains to prove the uniqueness of the solution to the Cauchy problem (3-7) (the above proof gives only the uniqueness in the set $\mathcal{B}_1$). Clearly it suffices to prove that two solutions to (3-6) which are equal at a time τ are equal in a neighborhood of τ in $[\tau, +\infty)$. This property follows from the above proof and from the fact that, for every solution $y : [\tau, \tau_1] \to L^2(0, L)$ of (3-7), if $T > 0$ is small enough (the smallness depending on y),

$$\|y\|_{\mathcal{B}_{\tau,\tau+T}} \leqslant 3C_2 \|y(\tau)\|_{L^2_L}. \tag{B-9}$$

This concludes the proof of Lemma 18.   □

Proceeding similarly to the proof of Lemma 18, one can get the following lemma concerning the Cauchy problem (2-13).

Lemma 19. *Let $C_2 > 0$ be as in Lemma 3 for $T_2 - T_1 = 1$. Given $R, M > 0$, there exists $T(R, M) > 0$ such that, for every $s \in \mathbb{R}$, for every $y_0 \in L^2(0, L)$ with $\|y_0\|_{L^2_L} \leqslant R$, and for every measurable $H : (s, s + T(R, M)) \to \mathbb{R}$ such that $|H(t)| \leqslant M$ for every $t \in (s, s + T(R, M))$, the Cauchy problem*

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s, s + T(R, M)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s, s + T(R, M)), \\ y_x(t, L) = H(t) & \text{on } (s, s + T(R, M)), \\ y(s, x) = y_0(x) & \text{on } (0, L) \end{cases} \tag{B-10}$$

has one and only one solution y on $[s, s + T(R, M)]$. Moreover, this solution satisfies

$$\|y\|_{\mathcal{B}_{s,s+T(R,M)}} \leqslant 3C_2 R. \tag{B-11}$$

We are now in position to prove Theorem 7.

Proof of Theorem 7. The uniqueness follows from the proof of the uniqueness part of Lemma 18. Let us give the proof of the existence. Let $y_0 \in L^2(0, L)$, let $s \in \mathbb{R}$ and let $T_0 := T(\|y_0\|_{L^2_L})$. By Lemma 18, there exists a solution $y \in \mathcal{B}_{s,s+T_0}$ to the Cauchy problem (3-7). Hence, together with the uniqueness of the solution, we can find a maximal solution $y : D(y) \to L^2(0, L)$ with $[s, s + T_0] \subset D(y)$. By the maximality of the solution y and Lemma 18, there exists $\tau \in [s + T_0, +\infty)$ such that $D(y) = [s, \tau)$. Let us assume that $\tau < +\infty$ and that (3-12) does not hold. Then there exist an increasing sequence $(t_n)_{n \in \mathbb{N}}$ of real numbers in (s, τ) and $R \in (0, +\infty)$ such that

$$\lim_{n \to +\infty} t_n = \tau, \tag{B-12}$$

$$\|y(t_n)\|_{L^2_L} \leq R \quad \forall n \in \mathbb{N}. \tag{B-13}$$

By (B-12), there exists $n_0 \in \mathbb{N}$ such that

$$t_{n_0} \geq \tau - \tfrac{1}{2} T(R). \tag{B-14}$$

From Lemma 18, there is a solution $z : [t_{n_0}, t_{n_0} + T(R)] \to L^2(0, L)$ of (3-7) for the initial time $s := t_{n_0}$ and the initial data $z(t_{n_0}) := y(t_{n_0})$. Let us then define $\tilde{y} : [s, t_{n_0} + T(R)] \to L^2(0, L)$ by

$$\tilde{y}(t) := y(t) \quad \forall t \in [s, t_{n_0}], \tag{B-15}$$

$$\tilde{y}(t) := z(t) \quad \forall t \in [t_{n_0}, t_{n_0} + T(R)]. \tag{B-16}$$

Then $\tilde{y}$ is also a solution to the Cauchy problem (3-7). By the uniqueness of this solution, we have $y = \tilde{y}$ on $D(y) \cap D(\tilde{y})$. However, from (B-14), we have that $D(y) \subsetneq D(\tilde{y})$, in contradiction with the maximality of y.

Finally, we prove that, if $C(R)$ satisfies (3-13), then, for the maximal solution y to (3-7), we have $D(y) = [s, +\infty)$. We argue by contradiction and therefore assume that the maximal solution y is such that $D(y) = [s, \tau)$ with $\tau < +\infty$. Then (3-12) holds. Let us estimate $\|y(t)\|_{L^2_L}$ when t tends to τ^-. We define the energy $E : [s, \tau) \to [0, +\infty)$ by

$$E(t) := \int_0^L |y(t, x)|^2 \, dx. \tag{B-17}$$

Then $E \in C^0([s, \tau))$ and, in the distribution sense, it satisfies

$$\frac{dE}{dt} \leqslant |u(t, y(t, \cdot))|^2 \leqslant C_B^2(\sqrt{E}). \tag{B-18}$$

(We get such an estimate first in the classical sense for regular initial data and regular boundary conditions $y_x(t, L) = \varphi(t)$ with the related compatibility conditions; the general case then follows from this special case by smoothing the initial data and the boundary conditions, by passing to the limit, and by using the uniqueness of the solution.) From (3-12) and (B-18), we get

$$\frac{1}{2} \int_0^{+\infty} \frac{1}{C_B^2(\sqrt{E})} \, dE < +\infty. \tag{B-19}$$

However the left-hand side of (B-19) is equal to the left-hand side of (3-13). Hence (3-13) and (B-19) are in contradiction. This completes the proof of Theorem 7. $\qquad\square$

The proof of Theorem 8 is more difficult. For this proof, we adapt a strategy introduced by Carathéodory to solve ordinary differential equations $\dot{y} = f(t, y)$ when f is not smooth. Roughly speaking it consists in solving $\dot{y} = f(t, y(t - h))$, where h is a positive time-delay, and then letting h tend to 0. Here we do not put the time-delay on y (it does not seem to be possible) but only on the feedback law: $u(t, y(t))$ is replaced by $u(t, y(t - h))$.

Proof of Theorem 8. Let us define $H : [0, +\infty) \to [0, +\infty)$ by

$$H(a) := \int_0^a \frac{1}{(C_B(\sqrt{E}))^2} \, dE = 2 \int_0^{\sqrt{a}} \frac{R}{(C_B(R))^2} \, dR. \tag{B-20}$$

From (3-13), we know that H is a bijection from $[0, +\infty)$ into $[0, +\infty)$. We denote by $H^{-1} : [0, +\infty) \to [0, +\infty)$ the inverse of this map.

For a given $y_0 \in L^2(0, L)$ and $s \in \mathbb{R}$, let us prove that there exists a solution y defined on $[s, +\infty)$ to the Cauchy problem (3-7), which also satisfies

$$\|y(t)\|^2_{L^2(0,L)} \leqslant H^{-1}\big(H(\|y(s)\|^2_{L^2_L}) + (t - s)\big) < +\infty \quad \forall t \in [s, +\infty). \tag{B-21}$$

Let $n \in \mathbb{N}^*$. Let us consider the Cauchy system on $[s, s + 1/n]$

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s, s + (1/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s, s + (1/n)), \\ y_x(t, L) = u(t, y_0) & \text{on } (s, s + (1/n)), \\ y(s, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{B-22}$$

By Theorem 7 applied with the feedback law $(t, y) \mapsto u(t, y_0)$ (a measurable bounded feedback law which now does not depend on y and therefore satisfies (3-11)), the Cauchy problem (B-22) has one and only one solution y. Let us now consider the Cauchy problem on $[s + (1/n), s + (2/n)]$

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s + (1/n), s + (2/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s + (1/n), s + (2/n)), \\ y_x(t, L) = u(t, y(t - (1/n))) & \text{on } (s + (1/n), s + (2/n)), \\ y(s, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{B-23}$$

As for (B-22), this Cauchy problem has one and only one solution, which we still denote by y. We keep going and, by induction on the integer i, define $y \in C^0([s, +\infty); L^2(0, L))$ so that, on $[s + (i/n), s + ((i+1)/n)]$, $i \in \mathbb{N} \setminus \{0\}$, we have y is the solution to the Cauchy problem

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{in } (s + (i/n), s + ((i+1)/n)) \times (0, L), \\ y(t, 0) = y(t, L) = 0 & \text{on } (s + (i/n), s + ((i+1)/n)), \\ y_x(t, L) = u(t, y(t - (1/n))) & \text{on } (s + (i/n), s + ((i+1)/n)), \\ y(s + (i/n)) = y(s + (i/n) - 0) & \text{on } (0, L), \end{cases} \tag{B-24}$$

where, in the last equation, we mean that the initial value, i.e., the value at time $(s + (i/n))$, is the value at time $(s + (i/n))$ of the y defined previously on $[(s + ((i-1)/n)), s + (i/n)]$.

Again, we let, for $t \in [s, +\infty)$,

$$E(t) := \int_0^L |y(t, x)|^2 \, dx. \tag{B-25}$$

Then $E \in C^0([s, +\infty))$ and, in the distribution sense, it satisfies (compare with (B-18))

$$\frac{dE}{dt} \leqslant |u(t, y_0)|^2 \leqslant C_B^2(\sqrt{E(s)}), \qquad\qquad t \in (s, s + (1/n)), \tag{B-26}$$

$$\frac{dE}{dt} \leqslant |u(t, y(t - (1/n)))|^2 \leqslant C_B^2(\sqrt{E(t - (1/n))}), \quad t \in \big(s + (i/n), s + ((i+1)/n)\big), \ i > 0. \tag{B-27}$$

Let $\varphi : [0, +\infty) \to [0, +\infty)$ be the solution of

$$\frac{d\varphi}{dt} = C_B^2\big(\sqrt{\varphi(t)}\big), \quad \varphi(s) = E(s). \tag{B-28}$$

Using (B-26)–(B-28) and simple comparison arguments, one gets

$$E(t) \leqslant \varphi(t) \quad \forall t \in [s, +\infty), \tag{B-29}$$

that is,

$$E(t) \leqslant H^{-1}\big(H(E(s)) + (t-s)\big) \quad \forall t \in [s, +\infty). \tag{B-30}$$

We now want to let $n \to +\infty$. In order to show the dependence on n, we write y^n instead of y. In particular (B-30) becomes

$$\|y^n(t)\|_{L^2(0,L)}^2 \leqslant H^{-1}\big(H(\|y_0(s)\|_{L_L^2}^2) + (t-s)\big) \quad \forall t \in [s, +\infty). \tag{B-31}$$

From Lemma 19, (B-31) and the construction of y^n, we get that, for every $T > s$, there exists $M(T) > 0$ such that

$$\|y^n\|_{\mathcal{B}_{s,T}} \leqslant M(T) \quad \forall n \in \mathbb{N}. \tag{B-32}$$

Hence, upon extracting a subsequence of $(y^n)_n$, which we still denote by $(y^n)_n$, there exists

$$y \in L_{\text{loc}}^\infty\big([s, +\infty); L^2(0, L)\big) \cap L_{\text{loc}}^2\big([s, +\infty); H^1(0, L)\big) \tag{B-33}$$

such that, for every $T > s$,

$$y^n \rightharpoonup y \quad \text{in } L^\infty(s, T; L^2(0, L)) \text{ weak } * \text{ as } n \to +\infty, \tag{B-34}$$

$$y^n \rightharpoonup y \quad \text{in } L^2(s, T; H^1(0, L)) \text{ weak as } n \to +\infty. \tag{B-35}$$

Let us define $z^n : [s, s + \infty) \times (0, L) \to \mathbb{R}$ and $\gamma^n : [s, +\infty) \to \mathbb{R}$ by

$$z^n(t) := y_0 \qquad \forall t \in [s, s + (1/n)], \tag{B-36}$$

$$z^n(t) := y^n(t - (1/n)) \quad \forall t \in (s + (1/n), +\infty), \tag{B-37}$$

$$\gamma^n(t) := u(t, z^n) \qquad \forall t \in [s, +\infty). \tag{B-38}$$

Note that y^n is the solution to the Cauchy problem

$$\begin{cases} y_t^n + y_{xxx}^n + y_x^n + y^n y_x^n = 0 & \text{in } (s, +\infty) \times (0, L), \\ y^n(t, 0) = y^n(t, L) = 0 & \text{on } (s, +\infty), \\ y_x^n(t, L) = \gamma^n(t) & \text{on } (s, +\infty), \\ y^n(s, x) = y_0(x) & \text{on } (0, L). \end{cases} \tag{B-39}$$

From (B-32) and the first line of (B-39), we get that

$$\forall T > 0, \quad \left(\frac{d}{dt} y^n\right)_{n \in \mathbb{N}} \text{ is bounded in } L^2(s, s + T; H^{-2}(0, L)). \tag{B-40}$$

From (B-34), (B-35), (B-40) and the Aubin-Lions lemma [Aubin 1963], we get

$$y^n \to y \quad \text{in } L^2(s, T; L^2(0, L)) \text{ as } n \to +\infty \quad \forall T > s. \tag{B-41}$$

From (B-41) we know that, upon extracting a subsequence if necessary, still denoted by $(y^n)_n$,

$$\lim_{n \to +\infty} \|y^n(t) - y(t)\|_{L_L^2} = 0 \quad \text{for almost every } t \in (s, +\infty). \tag{B-42}$$

Letting $n \to +\infty$ in inequality (B-30) for y^n and using (B-42), we get

$$\|y(t)\|_{L^2(0,L)}^2 \leqslant H^{-1}\big(H(\|y_0\|_{L_L^2}^2) + (t - s)\big) \quad \text{for almost every } t \in (0, +\infty). \tag{B-43}$$

Note that, for every $T > s$,

$$\begin{aligned}
\|z^n - y\|_{L^2((s,T);L_L^2)} &\leq (1/\sqrt{n})\|y_0\|_{L_L^2} + \big\|y^n(\cdot - (1/n)) - y(\cdot - (1/n))\big\|_{L^2(s+(1/n),T;L^2(0,L))} \\
&\quad + \big\|y(\cdot - (1/n)) - y(\cdot)\big\|_{L^2(s+(1/n),T;L^2(0,L))} + \|y\|_{L^2(s,s+(1/n);L^2(0,L))} \\
&\leq (1/\sqrt{n})\|y_0\|_{L_L^2} + \|y^n - y\|_{L^2(s,T;L^2(0,L))} \\
&\quad + \big\|y(\cdot - (1/n)) - y(\cdot)\big\|_{L^2(s+(1/n),T;L^2(0,L))} + \|y(\cdot)\|_{L^2(s,s+(1/n);L^2(0,L))}. \tag{B-44}
\end{aligned}$$

From (B-36), (B-37), (B-41) and (B-44), we get

$$z^n \to y \quad \text{in } L^2(s, T; L^2(0, L)) \text{ as } n \to +\infty \quad \forall T > s. \tag{B-45}$$

Extracting, if necessary, from the sequence $(z^n)_n$ a subsequence, still denoted by $(z^n)_n$, and using (B-45), we have

$$\lim_{n \to +\infty} \|z^n(t) - y(t)\|_{L_L^2} = 0 \quad \text{for almost every } t \in (s, +\infty). \tag{B-46}$$

From (3-1)–(3-3), (B-32), (B-36), (B-37) and (B-46), extracting a subsequence from the sequence $(\gamma^n)_n$ if necessary, still denoted by $(\gamma^n)_n$, we may assume that

$$\gamma^n \rightharpoonup \gamma(t) := u(t, y(t)) \text{ in } L^\infty(s, T) \text{ weak} * \text{ as } n \to +\infty \quad \forall T > s. \tag{B-47}$$

Let us now check that

$$y \text{ is a solution to the Cauchy problem (3-7).} \tag{B-48}$$

Let $\tau \in [s, +\infty)$ and let $\phi \in C^3([s, \tau] \times [0, L])$ be such that

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, 0) = 0 \quad \forall t \in [T_1, \tau]. \tag{B-49}$$

From (B-39), one has, for every $n \in \mathbb{N}$,

$$\begin{aligned}
-\int_{T_1}^\tau \int_0^L (\phi_t + \phi_x + \phi_{xxx})y^n \, dx \, dt &- \int_{T_1}^\tau \gamma^n \phi_x(t, L) \, dt + \int_{T_1}^\tau \int_0^L \phi y^n y_x^n \, dx \, dt \\
&+ \int_0^L y(\tau, x)\phi(\tau, x) \, dx - \int_0^L y_0 \phi(s, x) \, dx = 0. \tag{B-50}
\end{aligned}$$

Let τ be such that

$$\lim_{n \to +\infty} \|y^n(\tau) - y(\tau)\|_{L_L^2} = 0. \tag{B-51}$$

Let us recall that, by (B-42), (B-51) holds for almost every $\tau \in [s, +\infty)$. Using (B-35), (B-41), (B-47), (B-51) and letting $n \to +\infty$ in (B-50), we get

$$-\int_{T_1}^{\tau}\int_0^L (\phi_t + \phi_x + \phi_{xxx})y \, dx \, dt - \int_{T_1}^{\tau} u(t, y(t))\phi_x(t, L) \, dt + \int_{T_1}^{\tau}\int_0^L \phi y y_x \, dx \, dt$$
$$+ \int_0^L y(\tau, x)\phi(\tau, x) \, dx - \int_0^L y_0\phi(s, x) \, dx = 0. \tag{B-52}$$

Thus y is a solution to (2-1), with $T_1 := s$, T_2 arbitrary in $(s, +\infty)$, $\tilde{h} := -yy_x \in L^1([s, T_2]; L^2(0, L))$ and $h = u(\cdot, y(\cdot)) \in L^2(s, T_2)$. Let us emphasize that, by Lemma 3, it also implies that $y \in \mathcal{B}_{s,T}$ for every $T \in (s, +\infty)$. This concludes the proof of (B-48) and of Theorem 8. $\qquad\square$

Appendix C: Proof of Proposition 12

Let us first recall that Proposition 12 is due to Eduardo Cerpa if one requires only u to be in $L^2(0, T)$ instead of being in $H^1(0, T)$; see [Cerpa 2007, Proposition 3.1] and [Cerpa and Crépeau 2009a, Proposition 3.1]. In his proof, he uses Lemma 11, the controllability in H with controls $u \in L^2$. Actually, the only place in his proof where the controllability in H is used is on page 887 of [Cerpa 2007] for the construction of α_1, where, with the notations of that paper $\Re(y_\lambda), \Im(y_\lambda) \in H$. We notice that $\Re(y_\lambda), \Im(y_\lambda)$ share more regularity and better boundary conditions. Indeed, one has

$$\begin{cases} \lambda y_\lambda + y_\lambda' + y_\lambda''' = 0, \\ y_\lambda(0) = y_\lambda(L) = 0, \end{cases}$$

which implies that

$$\Re(y_\lambda), \Im(y_\lambda) \in \mathcal{H}^3,$$

where

$$\mathcal{H}^3 := H \cap \{\omega \in H^3(0, L) : \omega(0) = \omega(L) = 0\}. \tag{C-1}$$

In order to adapt Cerpa's proof in the framework of $u \in H^1(0, T)$, it is sufficient to prove the following controllability result in $\mathcal{H}^3$ with control $u \in H^1(0, T)$.

Proposition 20. *For every $y_0, y_1 \in \mathcal{H}^3$ and for every $T > 0$, there exists a control $u \in H^1(0, T)$ such that the solution $y \in \mathcal{B}$ to the Cauchy problem*

$$\begin{cases} y_t + y_{xxx} + y_x = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = u(t), \\ y(0, \cdot) = y_0 \end{cases}$$

satisfies $y(T, \cdot) = y_1$.

The proof of Proposition 12 is the same as the one of [Cerpa 2007, Proposition 3.1], with the only difference that one uses Proposition 20 instead of Lemma 11.

Proof of Proposition 20. Let us first point out that 0 is not an eigenvalue of the operator $\mathcal{A}$. Indeed this follows from property $(\mathcal{P}_2)$, (1-5) and (1-6). Using Lemma 11 and [Tucsnak and Weiss 2009, Proposition 10.3.4] with $\beta = 0$, it suffices to check that

$$\text{for every } f \in H, \text{ there exists } y \in \mathcal{H}^3 \text{ such that } -y_{xxx} - y_x = f. \tag{C-2}$$

Let $f \in H$. We know that there exists $y \in H^3(0, L)$ such that

$$-y_{xxx} - y_x = f, \tag{C-3}$$

$$y(0) = y(L) = y_x(L) = 0. \tag{C-4}$$

Simple integrations by parts, together with (4-11), (4-12), (C-3) and (C-4), show that, with $\varphi := \varphi_1 + i\varphi_2$,

$$0 = \int_0^L f\varphi \, dx = \int_0^L (-y_{xxx} - y_x)\varphi \, dx = \int_0^L y(\varphi_{xxx} + \varphi_x) \, dx = i\frac{2\pi}{p} \int_0^L y\varphi \, dx, \tag{C-5}$$

which, together with (C-4), implies that $y \in \mathcal{H}^3$. This concludes the proof of (C-2) as well as the proof of Proposition 20 and of Proposition 12. $\qquad\square$

Acknowledgments

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Volume 10   No. 5   2017