Algebraic & Geometric Topology 19 (2019) 2795–2836

Local cut points and splittings of relatively hyperbolic groups

MATTHEW HAULMARK

We show that the existence of a nonparabolic local cut point in the Bowditch boundary $\partial(G,\mathbb{P})$ of a relatively hyperbolic group $(G,\mathbb{P})$ implies that G splits over a 2–ended subgroup. This theorem generalizes a theorem of Bowditch from the setting of hyperbolic groups to relatively hyperbolic groups. As a consequence we are able to generalize a theorem of Kapovich and Kleiner by classifying the homeomorphism type of 1–dimensional Bowditch boundaries of relatively hyperbolic groups which satisfy certain properties, such as no splittings over 2–ended subgroups and no peripheral splittings.

In order to prove the boundary classification result we require a notion of ends of a group which is more general than the standard notion. We show that if a finitely generated discrete group acts properly and cocompactly on two generalized Peano continua X and Y, then $\mathrm{Ends}(X)$ is homeomorphic to $\mathrm{Ends}(Y)$. Thus we propose an alternative definition of $\mathrm{Ends}(G)$ which increases the class of spaces on which G can act.

20F65, 20F67

1 Introduction

The notion of a group G being hyperbolic relative to a class of subgroups $\mathbb{P}$ was introduced by Gromov [22] to generalize both word hyperbolic and geometrically finite Kleinian groups. The subgroups in the class $\mathbb{P}$ are called *peripheral subgroups*, and when G is hyperbolic relative to $\mathbb{P}$ we often say $(G,\mathbb{P})$ is relatively hyperbolic. Introduced by Bowditch [11] there is a boundary for relatively hyperbolic groups. The Bowditch boundary $\partial(G,\mathbb{P})$ generalizes the Gromov boundary of a word hyperbolic group and the limit set of a geometrically finite Kleinian group. The homeomorphism type of the Bowditch boundary is known to be a quasi-isometry invariant of the group — see Groff [21] — under modest hypotheses on the peripheral subgroups. Consequently, it is desirable to describe the topological features of the Bowditch boundary. Topological features of the boundary are closely related to algebraic properties of the group; in particular they are often related to splittings of the group as a fundamental group of a graph of groups.

Published: 20 October 2019 DOI: 10.2140/agt.2019.19.2795

A point p in $\partial(G,\mathbb{P})$ is a *local cut point* if $\partial(G,\mathbb{P})\backslash\{p\}$ is disconnected or $\partial(G,\mathbb{P})\backslash\{p\}$ is connected and has more than one end. For 1–ended hyperbolic groups, Bowditch [4] shows that the existence of a splitting over a 2–ended subgroup (see Section 2.3) is equivalent to the existence of a local cut point in the Gromov boundary. As evidenced by the work of Kapovich and Kleiner [32], this result has proved useful in classifying the homeomorphism type of 1–dimensional boundaries of hyperbolic groups. Kapovich and Kleiner's result relies on the topological characterization of the Menger curve of R D Anderson [1; 2], which requires that the boundary has no local cut points. Because the existence or nonexistence of 2–ended splittings can be verified directly in many natural settings, Kapovich and Kleiner's results provide techniques for constructing examples of hyperbolic groups with Menger curve or Sierpinski carpet boundary. Obstructions to 2–ended splittings are well understood for hyperbolic 3–manifold groups — see Myers [34] — Coxeter groups — see Mihalik and Tschantz [33] — and random groups; see Dahmani, Guirardel and Przytycki [15].

Papasoglu and Swenson [36; 37] and Groff [21] have extended Bowditch's results [4] from hyperbolic groups to CAT(0) and relatively hyperbolic groups, respectively. Their results describe the relationship between 2–ended splittings and cut pairs in the boundary. In particular, their results make no mention of local cut points. Guralnik [27] has observed that many of Bowditch's local cut point results extend to relatively hyperbolic groups provided that the Bowditch boundary has no global cut points. However, that assumption is quite restrictive. Bowditch has shown [10] that the Bowditch boundary often has many global cut points. Thus a general theorem relating local cut points in the Bowditch boundary to 2–ended splittings was still missing from the literature. The primary result of this paper addresses the general setting with the following theorem, which makes no assumption about the existence or nonexistence of global cut points in the Bowditch boundary.

Theorem 1.1 (splitting theorem) *Let $(G,\mathbb{P})$ be a relatively hyperbolic group with tame peripherals. Assume that $\partial(G,\mathbb{P})$ is connected and not homeomorphic to a circle. If G does not split over a 2–ended subgroup, then $\partial(G,\mathbb{P})$ does not contain a nonparabolic local cut point. Moreover, if G splits over a nonparabolic 2–ended subgroup relative to $\mathbb{P}$, then $\partial(G,\mathbb{P})$ contains a nonparabolic local cut point.*

A relatively hyperbolic group $(G,\mathbb{P})$ has *tame peripherals* if every $P\in\mathbb{P}$ is finitely presented, one- or two-ended, and contains no infinite torsion subgroup. Bowditch has shown [10] that if $(G,\mathbb{P})$ has tame peripherals and the Bowditch boundary $\partial(G,\mathbb{P})$

is connected, then $\partial(G,\mathbb{P})$ is locally connected. In this paper we will always assume that $\partial(G,\mathbb{P})$ is connected and that $(G,\mathbb{P})$ has tame peripherals. Other terms used in the statement of Theorem 1.1 will be defined in Section 2.

Kapovich and Kleiner's classification result for 1–dimensional boundaries of hyperbolic groups [32] shows that under certain group-theoretic conditions if the Gromov boundary is 1–dimensional, then it must be a circle, a Sierpinski carpet, or a Menger curve. Theorem 1.1 is used by the author in [28] to generalize the Kapovich–Kleiner result to 1–dimensional visual boundaries of CAT(0) groups with isolated flats which do not split over 2–ended subgroups. (We point out that visual boundary and the Bowditch boundary are not the same in general; see Tran [41].) The application of Theorem 1.1 is a critical step in the proof in Theorem 1.2 of [28] and requires an understanding of the general case where the Bowditch boundary has global cut points.

In the present paper we use Theorem 1.1 to obtain an alternative generalization of the Kapovich–Kleiner theorem for 1–dimensional Bowditch boundaries of relatively hyperbolic groups with 1–ended and tame peripherals.

Theorem 1.2 (classification theorem) *Let $(G,\mathbb{P})$ be a 1–ended relatively hyperbolic group with tame peripherals, and let $\mathcal{P}$ be the set of all subgroups of elements of $\mathbb{P}$. Assume that G does not split over a virtually cyclic subgroup and does not split over any subgroup in $\mathcal{P}$. If every $P\in\mathbb{P}$ is one-ended and $\partial(G,\mathbb{P})$ is 1–dimensional, then one of the following holds*:

(1) $\partial(G,\mathbb{P})$ *is a circle.*

(2) $\partial(G,\mathbb{P})$ *is a Sierpinski carpet.*

(3) $\partial(G,\mathbb{P})$ *is a Menger curve.*

A CAT(0) group with isolated flats is relatively hyperbolic with respect virtually abelian subgroups — see Hruska and Kleiner [30] — and thus is relatively hyperbolic with respect to a collection of tame 1–ended peripherals. However, there are distinctions between Theorem 1.2 of this paper and Theorem 1.2 of [28] worth mentioning here. First and foremost, the Bowditch boundary $\partial(G,\mathbb{P})$ is generally a quotient space of the visual boundary [41]. Second, peripheral splittings (see Section 2.3) play a role in both theorems. Peripheral splittings are not allowed for Theorem 1.2, whereas in the isolated flats setting only certain types of peripheral splittings are excluded. Lastly, in the CAT(0) setting the boundary has no global cut points [37], but the Bowditch boundary of a relatively hyperbolic group may have many global cut points in general.

1.1 Method of proof

The proof of Theorem 1.1 utilizes arguments of Bowditch [4] developed for hyperbolic groups; however, because we are interested in the relatively hyperbolic setting and Bowditch's results depend on hyperbolicity in an essential way additional techniques are required. In particular, Lemmas 5.2 and 5.17 of [4] are key steps in which Bowditch explicitly uses hyperbolicity. Using results of Tukia [44], Guralnik [27] proved a relatively hyperbolic version of Bowditch's Lemma 5.2, which may be found as Lemma 4.1 in this exposition. In Section 4.1 we provide a simple self-contained proof of Lemma 4.1 using techniques different from those of [27]. Guralnik also observed that given Lemma 4.1 and given the Bowditch boundary has no global cut points, some of Bowditch's local cut points carry over to the relatively hyperbolic setting using Bowditch's exact arguments. In [4, Lemma 5.17] Bowditch shows that the stabilizer of a necklace (see Section 2.4 for the definition of a necklace) in the boundary of a relatively hyperbolic group is quasiconvex. Proposition 4.5 provides a relatively hyperbolic version of this result. Namely, we show that the stabilizer of a necklace in the Bowditch boundary of a relatively hyperbolic group is relatively quasiconvex. The importance of Lemma 4.1 and Proposition 4.5 is that they allow us to use Bowditch's arguments verbatim to determine the local cut point structure of the Bowditch boundary in the special case that the Bowditch boundary does not contain any global cut points.

If the boundary of a relatively hyperbolic group $(G,\mathbb{P})$ is connected, then it is a Peano continuum. However, $\partial(G,\mathbb{P})$ may have many global cut points [10]. Our strategy involves demonstrating that it suffices to consider only the case when $\partial(G,\mathbb{P})$ has no global cut points. In particular, using the theory of peripheral splittings [10] and basic decomposition theory we are able to restrict our attention to "blocks" of $\partial(G,\mathbb{P})$, where a *block* of $\partial(G,\mathbb{P})$ is a maximal subcontinuum consisting of points which cannot be separated from one another by global cut points. Blocks have two key features. The first is that a block of $\partial(G,\mathbb{P})$ is the limit set of a relatively hyperbolic subgroup $(H,\mathbb{Q})$ of $(G,\mathbb{P})$ (see Theorem 3.1(4)). The second is that there is a retraction of $\partial(G,\mathbb{P})$ onto any given block; moreover, the retraction map has nice decomposition-theoretic properties. This combination of Bowditch's theory of peripheral splittings with decomposition theory techniques is one of the major contributions of this paper, and it is the focus of Section 3. Using these techniques allows us to reduce the proof of Theorem 1.1 to proving Theorem 4.20, which describes nonparabolic local cut points in a boundary without global cut points. The proof of Theorem 4.20 can be found in

Section 4.3 and relies on the observation from the above paragraph that Lemma 4.1 and Proposition 4.5 allow us to directly use the arguments of [4].

The other main result of this paper is Theorem 1.2. Two key tools used in the proof of Theorem 1.2 are the topological characterization of the Menger curve due to Anderson [1; 2] and the topological characterization of the Sierpinski carpet due to Whyburn [45]. Anderson's theorem states that a compact metric space M is a Menger curve provided M is 1–dimensional, M is connected, M is locally connected, M has no local cut points, and no nonempty open subset of M is planar. We note that if the last condition is replaced with "M is planar", then we have the topological characterization of the Sierpinski carpet (see Whyburn [45]).

In order to apply Anderson's and Whyburn's theorems we must rule out the existence of local cut points. Theorem 1.1 can be used to rule out nonparabolic local cut points, but we also need to rule out the existence of parabolic local cut points. In Theorem 1.2 we are in a setting where $\partial(G,\mathbb{P})$ contains no global cut points, so $\partial(G,\mathbb{P})\setminus\{p\}$ is connected. Thus we need only know that $\partial(G,\mathbb{P})\setminus\{p\}$ is 1–ended. Because the group $P=\mathrm{Stab}(p)$ is 1–ended, and Bowditch [11] has shown that P acts properly and cocompactly on $\partial(G,\mathbb{P})\setminus\{p\}$, a reader familiar with geometric group theory may think that we are done. However, the author was unable to find sufficiently general results in the literature. To define ends of a group one must make a choice of a space on which the group acts, and it is well known that any two CW–complexes on which G acts properly and cocompactly have the same number of ends; see Geoghegan [18] and Guilbault [25].

In this paper we require an understanding of the ends of a connected open subset of a Peano continuum on which a group acts properly and cocompactly. The study of ends as introduced by Freudenthal [17] can be described as inverse limits in the setting of generalized continua (ie locally compact, locally connected, σ–compact, connected Hausdorff spaces) as explained by Baues and Quintero [3]. Given a finitely generated discrete group G acting properly and cocompactly on two "nice" topological spaces X and Y a natural question to ask is: What topological properties are required to guarantee that $\mathrm{Ends}(X)$ homeomorphic to $\mathrm{Ends}(Y)$? In other words, what is the natural setting in which $\mathrm{Ends}(G)$ is well defined?

A *generalized Peano continuum* is a generalized continuum which is metrizable. This general class of spaces includes open connected subspaces of Peano continua, proper geodesic metric spaces and locally finite CW–complexes. Theorem 1.3 extends known ends results to generalized Peano continua. For groups acting properly and compactly on metric spaces with a proper equivariant geodesic metric, Theorem 1.3 follows from Proposition I.8.29 of Bridson and Haefliger [12]. However, it is unknown whether a generalized Peano continuum with a proper, cocompact action admits an equivariant proper geodesic metric (see Section 6 of Guilbault and Moran [26]). Thus Theorem 1.3 is a new contribution to the literature. Theorem 1.3 has already proved useful outside of this paper. Groves and Manning (see Section 7 of [24]) use Theorem 1.3 to prove a result similar to Theorem 1.1 for the restricted case where $\partial(G, \mathbb{P})$ has no global cut points, and the peripheral subgroups are 1–ended and tame.

Theorem 1.3 *Let X and Y be two generalized Peano continua, and assume that G is a finitely generated group acting properly and cocompactly by homeomorphisms on X and Y. Then* $\mathrm{Ends}(X)$ *is homeomorphic to* $\mathrm{Ends}(Y)$.

A generalized Peano continuum X is locally path connected (see for example Willard [46, Exercise 31C.1]). The proof of Theorem 1.3 depends heavily on this fact. In particular, the existence of proper rays in X plays an important role in the proof of Theorem 1.3. The author would be interested to know if there is an alternative argument that could be extended from the metric setting to the more general, nonmetric setting of generalized continua.

Problem 1.4 *Let X and Y be two generalized continua, and G be a finitely generated discrete group acting properly and cocompactly on X and Y. Is* $\mathrm{Ends}(X)$ *homeomorphic to* $\mathrm{Ends}(Y)$?

Theorem 1.3 is also closely related to the recent work of Guilbault and Moran [26]. Their work implies Theorem 1.3 for geometric actions on proper metric ARs.

Acknowledgements First I would like to thank my advisor Chris Hruska for providing comments and guidance throughout this project. Second I would like to give special thanks to Genevieve Walsh for encouraging me to write up the details of these results. Lastly, I would like to thank Craig Guilbault for valuable conversations regarding Freudenthal ends.

2 Preliminaries

In this section we review basic facts and definitions required by the exposition of this paper. The topics in this section include convergence groups, relatively hyperbolic groups, splittings, cut point structures in metric spaces, and ends of generalized continua.

2.1 Convergence group actions

A detailed account of convergence group actions may be found in [8]. Let M be a compact metrizable space. Let G be a group acting by homeomorphisms on M. The group G is called a *convergence group* if for every sequence of distinct group elements (g_k) there exist points $\alpha, \beta \in M$ (not necessarily distinct) and a subsequence $(g_{k_i}) \subset (g_k)$ such that $g_{k_i}(x) \to \alpha$ locally uniformly on $M \setminus \{\beta\}$ and $g_{k_i}^{-1}(x) \to \beta$ converges locally uniformly on $M \setminus \{\alpha\}$. By *locally uniformly* we mean that if C is a compact subset of $M \setminus \{\beta\}$ and U is any open neighborhood of α, then there is an $N \in \mathbb{N}$ such that $g_{k_i} C \subset U$ for all $i > N$.

Elements of convergence groups can be classified into three types: elliptic, loxodromic and parabolic. A group element is *elliptic* if it has finite order. An element g of G is *loxodromic* if has infinite order and fixes exactly two points of M. A subgroup of G is loxodromic if it is virtually infinite cyclic. If $g \in G$ has infinite order and fixes a single point of M then g is *parabolic*. An infinite subgroup P of G is *parabolic* if it contains no loxodromic elements and stabilizes a single point p of M. The point p is uniquely determined by P, and the point p is called a *parabolic* point. We call p a *bounded parabolic* point if P acts properly and cocompactly on $M \setminus \{p\}$.

A point $x \in \partial(G, \mathbb{P})$ is a *conical limit* point if there exists a sequence of group elements $(g_n) \in G$ and distinct points $\alpha, \beta \in M$ such that $g_n x \to \alpha$ and $g_n y \to \beta$ for every $y \in M \setminus \{x\}$. Tukia has shown (see [44]) that:

Proposition 2.1 *A conical limit point cannot be a parabolic point.*

A convergence group G acting on M is called *uniform* if every point of M is a conical limit point, or equivalently the action on space of distinct triples of M is proper and cocompact (see [8]). Bowditch [5] has shown G acts as a uniform convergence group on a perfect compact metric space if and only if it is hyperbolic. G is called *geometrically finite* if every point of M is a conical limit point or a bounded parabolic point.

2.2 Relatively hyperbolic groups and their boundaries

We refer the reader to [11] for a more thorough introduction to relatively hyperbolic groups. Let G be a group acting properly and isometrically on a δ–hyperbolic space X. Tukia [43] has shown that G acts on the Gromov boundary of X as a convergence group. Let $\mathbb{P}$ be a collection of infinite subgroups of G that is closed under conjugation, called *peripheral subgroups*.

Definition We say that G is *hyperbolic relative to* $\mathbb{P}$ if:

(1) $\mathbb{P}$ is the set of all maximal parabolic subgroups of G.

(2) There exists a G–invariant system of disjoint open horoballs based at the parabolic points of G such that if $\mathcal{B}$ is the union of these horoballs, then G acts cocompactly on $X \setminus \mathcal{B}$.

Any action of a group G on a proper δ–hyperbolic space satisfying the above definition is called *cusp uniform*. In [11] Bowditch shows:

Theorem 2.2 *If G is hyperbolic relative to $\mathbb{P}$, then $\mathbb{P}$ consists of only finitely many conjugacy classes.*

The Bowditch boundary $\partial(G,\mathbb{P})$ is defined to be the Gromov boundary of X, ie the set of equivalence classes of geodesic rays of X, where two geodesic rays are equivalent if their Hausdorff distance is finite. It is a result of Bowditch [11] that $\partial(G,\mathbb{P})$ does not depend on the choice of X.

We say that a relatively hyperbolic group $(G,\mathbb{P})$ has *tame peripherals* if every $P \in \mathbb{P}$ is finitely presented, one- or two-ended, and contains no infinite torsion subgroup. Under the assumption of tame peripherals Bowditch has shown the following two results in [7] and [10], respectively.

Theorem 2.3 *Suppose that $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals and that $\partial(G,\mathbb{P})$ is connected. Then every global cut point of $\partial(G,\mathbb{P})$ is a parabolic point.*

Assume that $\partial(G,\mathbb{P})$ is connected. A *global cut point* is a point whose removal disconnects $\partial(G,\mathbb{P})$.

Theorem 2.4 *If $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals and $\partial(G,\mathbb{P})$ is connected, then $\partial(G,\mathbb{P})$ is locally connected.*

In this paper we are interested in the case where $\partial(G,\mathbb{P})$ is locally connected, so we will generally assume that $(G,\mathbb{P})$ has tame peripherals and that $\partial(G,\mathbb{P})$ is connected.

Remark (convergence groups and relatively hyperbolic groups) Recall from Section 2.1 that a group G acts as a uniform convergence group on a perfect compact metric space if and only if G is hyperbolic. A generalization of this result was completed by Bowditch [11] and Yaman [47]. Bowditch [11] shows that a relatively hyperbolic group with finitely generated peripheral subgroups acts on its Bowditch boundary as a geometrically finite convergence group (see Proposition 2.5), and Yaman [47] proves a strong converse. In general, geometrically finite convergence group actions are not uniform (see Proposition 2.1).

Proposition 2.5 (Bowditch [11, Proposition 6.12]) *Assume G acts properly and isometrically on a proper δ–hyperbolic space X, and let $\mathbb{P}$ be a collection of infinite subgroups of G. If the action of $(G,\mathbb{P})$ on X is cusp uniform, then the action on ∂X is geometrically finite.*

2.3 Splittings

A graph of groups is called *trivial* [39; 38] if there exists a vertex group equal to G. A *splitting* of a group G over a given class of subgroups is a nontrivial finite graph-of-groups representation of G, where each edge group belongs to the given class. We say that G *splits* over a subgroup A if G splits over the class $\{A\}$. The group G is said to split *relative* to another class of subgroups $\mathbb{P}$ if each element of $\mathbb{P}$ is conjugate into one of the vertex groups. Assume that G is hyperbolic relative to a collection $\mathbb{P}$. A *peripheral splitting* of $(G,\mathbb{P})$ is a finite bipartite graph-of-groups representation of G, where $\mathbb{P}$ is the set of conjugacy classes of vertex groups of one color of the partition, called peripheral vertices. Nonperipheral vertex groups will be referred to as *components*. This terminology stems from the tame peripheral setting, where there is a correspondence between the cut point tree of $\partial(G,\mathbb{P})$ and the peripheral splitting of $(G,\mathbb{P})$. In this correspondence elements of $\mathbb{P}$ correspond to stabilizers of cut point vertices and the components correspond to stabilizers of blocks in the boundary (see Theorem 3.1).

A peripheral splitting $\mathcal{G}$ is a refinement of another peripheral splitting $\mathcal{G}'$ if $\mathcal{G}'$ can be obtained from $\mathcal{G}$ via a finite sequence of foldings that preserve the vertex coloring. In [10] Bowditch proves the following accessibility result:

Theorem 2.6 *Suppose that $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals and connected boundary. Then $(G,\mathbb{P})$ admits a (possibly trivial) peripheral splitting which is maximal in the sense that it is not a refinement of any other peripheral splitting.*

Combining Proposition 5.1 and Theorem 1.2 of [6], Bowditch also shows:

Theorem 2.7 *If $(G, \mathbb{P})$ is a relatively hyperbolic with tame peripherals, $\partial(G, \mathbb{P})$ is connected, and $\partial(G, \mathbb{P})$ has a global cut point, then there exists a nontrivial peripheral splitting of $(G, \mathbb{P})$.*

2.4 Local cut point structures in Peano continua

We refer the reader to [4] for a more detailed account of local cut point structures in Peano continua. Recall that a *Peano continuum* is a compact, connected and locally connected metric space. Let M be a Peano continuum. A *global cut point* of M is a point $x \in M$ such that $M \setminus \{x\}$ is disconnected. A *cut pair* is a set of two distinct points $\{a, b\} \subset M$ which contains no global cut points and is such that $M \setminus \{a, b\}$ is disconnected. The set of components of $M \setminus \{a, b\}$ will be denoted by $\mathcal{U}(a, b)$, and $\mathcal{N}(a, b)$ will denote the cardinality of $\mathcal{U}(a, b)$. We leave it as an exercise to show if x is a global cut point and $\{a, b\}$ is a cut pair, then a and b cannot lie in different components of $M \setminus \{x\}$. Two cut pairs $\{a, b\}$ and $\{c, d\}$ are said to *mutually separate* M if c and d lie in different components of $M \setminus \{a, b\}$ and vice versa. A cut pair $\{a, b\}$ is called *inseparable* if there does not exist any other cut pair $\{c, d\}$ such that a and b lie in distinct components of $M \setminus \{c, d\}$. Let G be a group acting on M by homeomorphisms. A cut pair $\{a, b\}$ will be called *translate inseparable* (or *G–translate inseparable*) if there does not exist a cut pair $\{c, d\}$ in $\mathrm{Orb}_G(\{a, b\})$ such that a and b lie in distinct components of $M \setminus \{c, d\}$. If $(G, \mathbb{P})$ is relatively hyperbolic and $M = \partial(G, \mathbb{P})$, then a cut pair $\{a, b\}$ will be called *loxodromic* if it is stabilized by a loxodromic element $g \in G$.

Let Δ be a subset of M. A *cyclic order* on Δ is a quaternary relation $\sigma(a, b, c, d)$, such that the following holds: if F is any finite subset of Δ, then there is an embedding $i\colon F \to S^1$ such that given any four $a, b, c, d \in F$ the relation $\sigma(a, b, c, d)$ holds if and only if the pairs $\{i(a), i(c)\}$ and $\{i(b), i(d)\}$ mutually separate in S^1. The set Δ is called a *cyclically separating set* if it has a cyclic order σ such that for any $a, b, c, d \in \Delta$ we have $\sigma(a, b, c, d)$ if and only if $\{a, c\}$ separates $\{b, d\}$ in M. Two points a and b in a cyclically separating set are called *adjacent* if $\{i(a), i(b)\}$ cannot be mutually separated by $\{i(c), i(d)\}$ for any $c, d \in \Delta$. An unordered pair of adjacent points will be referred to as a *jump*. Notice that if two distinct jumps of a cyclically separating set Δ intersect, they must intersect in an isolated point of Δ.

A point $x \in M$ is a *local cut point* if $M \setminus \{x\}$ is disconnected or $M \setminus \{x\}$ is connected and has more than one end. If $M \setminus \{x\}$ is connected, the *valence*, $\mathrm{val}(x)$, of a local cut point is the number of ends of $M \setminus \{x\}$. A detailed discussion of ends of spaces can be found in Section 2.5, but we remark that saying a point $x \in M$ is a local cut point is equivalent to saying that there exists a neighborhood U of x such that for every neighborhood V of x with $V \subset U$, there exist points $z, y \in V \setminus \{x\}$ such that there does not exist a connected subset of $U \setminus \{x\}$ containing z and y. Alternatively, to check that x is not a local cut point it suffices to show that given a neighborhood U of x there exists a neighborhood $V \ni x$ with $V \subset U$ and $V \setminus \{x\}$ connected. We wish to "collect" all the local cut points, so we introduce notation similar to that of Bowditch [4] to describe the various "local cut point structures" in M. Let

$$M(n) = \{x \in M \mid \mathrm{val}(x) = n \text{ and } x \text{ is not a global cut point}\},$$
$$M(n+) = \{x \in M \mid \mathrm{val}(x) \geq n \text{ and } x \text{ is not a global cut point}\}.$$

Now assume that a group G acts on M with a geometrically finite convergence group action. Then G is relatively hyperbolic and M is homeomorphic to $\partial(G, \mathbb{P})$ [5; 47]. Additionally, if $(G, \mathbb{P})$ has tame peripherals, then global cut points in M correspond to parabolic points (see Section 2.1). Because parabolic points cannot be conical limit points (Proposition 2.1), the goal is to understand local cut points which are conical limit points to ensure that the points we are considering do not separate M globally. Define $\mathcal{C}$ to be the collection of conical limit points in M. We will denote by $M^*(n)$ and $M^*(n+)$ the intersections of $M(n)$ and $M(n+)$ with $\mathcal{C}$. We define relations on $M^*(2)$ and $M^*(3+)$. Let $x, y \in M^*(2)$. We write $x \sim y$ if and only if $x = y$ or $\mathcal{N}(x, y) = 2$. For two elements $a, b \in M^*(3+)$ we write $a \approx b$ if $a \neq b$ and $\mathcal{N}(a, b) = \mathrm{val}(a) = \mathrm{val}(b) \geq 3$. From the definitions above we immediately obtain a partition of the set of conical limit points which are local cut points. In other words:

Lemma 2.8 *Let $x \in M$ be a conical limit point which is a local cut point. Then $x \in M^*(2) \cup M^*(3+)$.*

The following three results are proved using arguments analogous to those found in Lemmas 3.1, 3.3 and 3.8 of [4].

Lemma 2.9 *The collection of $\approx$–classes in $M^*(3+)$ is partitioned into pairs which do not mutually separate.*

Lemma 2.10 *The relation $\sim$ is an equivalence relation on $M^*(2)$.*

We say that a cut pair $\{c, d\}$ in M *separates* a subset $C \subset M$ if C is contained in at least two distinct components of $M \setminus \{c, d\}$.

Lemma 2.11 *Let $a, b, c, d \in M^*(2)$. If $a \sim b$ and $\{c, d\}$ separates $\{a, b\}$, then $c \sim d \sim a \sim b$, and the pairs $\{a, b\}$ and $\{c, d\}$ mutually separate.*

An argument similar to that of Bowditch [4] shows that there are no singleton $\sim$–classes in $M^*(2)$; consequently, a $\sim$–class in $M^*(2)$ consists of either a cut pair or a cyclically separating collection of cut pairs. The closure of a $\sim$–class ν containing at least three elements will be called a *necklace*. Notice that if ν is infinite, then $\overline{\nu}$ may contain parabolic points. Lastly note that because cut pairs cannot be separated by global cut points, neither can $\sim$–classes or their closures.

2.5 Ends of generalized continua

In this section we review ends of spaces. Roughly speaking the number of ends of a connected space X counts the number of components at infinity in X. A more detailed discussion about ends of spaces may be found in [25, Section 3; 3, Section I.9].

A *continuum* is a compact, connected, locally connected Hausdorff space. A *generalized continuum* is a connected, locally compact, locally connected, σ–compact, Hausdorff space. A *generalized Peano continuum* is a metrizable generalized continuum. A nested sequence $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ of compact sets in a space X is called an *exhaustion* of X if $X = \bigcup_{i=1}^{\infty} C_i$ and $C_i \subset \operatorname{Int}(C_{i+1})$ for every i. Note that σ–compactness implies that a generalized continuum can always be covered by a sequence of compact sets, and by local compactness we may always assume that $C_i \subset \operatorname{Int}(C_{i+1})$. Also note that for generalized Peano continua the components of the complement of a compact set are path components of the complement. The context of this paper makes it worth noting that a connected open subset of a Peano continuum is a generalized Peano continuum.

Let X be a generalized continuum and let $\{C_i\}_{i=1}^{\infty}$ be an exhaustion of X. Define $\mathcal{U}(C_i)$ to be the set of components of $X \setminus C_i$. Because the sequence $\{C_i\}_{i=1}^{\infty}$ is nested, the sets $\mathcal{U}(C_i)$ form an inverse sequence,

$$\mathcal{U}(C_1) \leftarrow \mathcal{U}(C_2) \leftarrow \mathcal{U}(C_3) \leftarrow \cdots.$$

The set $\operatorname{Ends}(X)$ is defined to be $\varprojlim\{\mathcal{U}(C_i) \mid i \geq 1\}$. The cardinality of $\operatorname{Ends}(X)$ is the *number of ends* of the space X. The set $\operatorname{Ends}(X)$ is independent of choice of $\{C_i\}$ (see Remark I.9.2(a) of [3]). Let G be a finitely generated discrete group acting properly and cocompactly on a generalized Peano continuum X. We define $\operatorname{Ends}(G)$ to be $\operatorname{Ends}(X)$. We show in Theorem 1.3 that $\operatorname{Ends}(G)$ is independent of the choice of X and agrees with the more traditional notion of ends of a Cayley graph for any finite generating set.

The *Freudenthal compactification* of X is $X \cup \mathrm{Ends}(X)$ with the topology generated by the basis consisting of all open subsets of X and all sets $\bar{E}_i$ where $E_i \in \mathcal{U}(C_i)$ for some $i \geq 1$ and

$$\bar{E}_i = E_i \cup \{(F_1, F_2, F_3, \dots) \in \mathrm{Ends}(X) \mid F_i = E_i\}.$$

It is well known that the Freudenthal compactification is compact and metrizable. The space $\mathrm{Ends}(X)$ is given the subspace topology.

Recall that a map between two spaces $f\colon X \to Y$ is called *proper* if for every compact subset C of Y we have $f^{-1}(C)$ is compact. The following well-known result can be found as Proposition I.9.11 of [3].

Proposition 2.12 *Let $f\colon X \to Y$ be a proper map between generalized Peano continua, then f can be uniquely extended to a continuous map $\hat{f}$ from $X \cup \mathrm{Ends}(X)$ to $Y \cup \mathrm{Ends}(Y)$.*

The restriction of $\hat{f}$ to $\mathrm{Ends}(X)$ will be denoted by f^*, and we say that f^* is the *ends map induced* by f.

A useful and more geometric way to describe the ends of a generalized Peano continuum X is by proper rays. By *proper ray* we mean any proper map $\alpha\colon [0,\infty) \to X$. Two rays α and β are *ladder equivalent* if there is a proper map h of the *infinite ladder* (or simply *ladder*)

$$L_{[0,\infty)} = ([0,\infty) \times \{0,1\}) \cup (\mathbb{N} \times [0,1])$$

such that α and β are the *sides*, ie $\alpha = h|_{[0,\infty)\times\{0\}}$ and $\beta = h|_{[0,\infty)\times\{1\}}$. We will write $\alpha \simeq \beta$ to denote that α is ladder equivalent to β. The image under h of $n \times [0,1]$ is called a *rung*. Let $\mathcal{L}(X)$ be the collection of ladder classes of proper rays in X.

Assume α is a proper ray in X. By Proposition 2.12 there is a continuous extension $\hat{\alpha}\colon [0,\infty) \cup \{\infty\} \to X \cup \mathrm{Ends}(X)$ of α such that $\alpha(\infty)$ is an element of $\mathrm{Ends}(X)$. By Proposition I.9.20 of [3] we have:

Proposition 2.13 *Let X be a generalized Peano continuum. The map $\varphi\colon \mathcal{L}(X) \to \mathrm{Ends}(X)$ given by setting $\varphi([\alpha]_{\simeq})$ equal to $E = \hat{\alpha}(\infty)$ defines a one-to-one correspondence between $\mathrm{Ends}(X)$ and $\mathcal{L}(X)$.*

Lemma 2.14 *Let $\hat{f}\colon X \cup \mathrm{Ends}(X) \to Y \cup \mathrm{Ends}(Y)$ be the continuous extension of a proper map f between two generalized Peano continua X and Y, and let f^**

denote the restriction of f to $\mathrm{Ends}(X)$*. Assume that* $\varphi_1\colon \mathcal{L}(X) \to \mathrm{Ends}(X)$ *and* $\varphi_2\colon \mathcal{L}(Y) \to \mathrm{Ends}(Y)$ *are the bijections as given in Proposition 2.13. Let the map* $g\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ *be given by* $g([\alpha]) = [f\alpha]$ *for every* $[\alpha] \in \mathcal{L}(X)$*. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{L}(X) \arrow[r, "g"] \arrow[d, "\varphi_1"'] & \mathcal{L}(Y) \arrow[d, "\varphi_2"] \\ \mathrm{Ends}(X) \arrow[r, "f^*"] & \mathrm{Ends}(Y) \end{tikzcd}$$

Proof Assume the hypotheses and let $\alpha\colon [0,\infty) \to X$ be a proper ray. By Proposition 2.12 there is a continuous extension $\hat{\alpha}\colon [0,\infty] \to X \cup \mathrm{Ends}(X)$ of α, and there is a continuous extension $\widehat{f\alpha}\colon [0,\infty] \to Y \cup \mathrm{Ends}(Y)$ of $f\alpha$. The composition $\hat{f}\hat{\alpha}$ is also a continuous extension of α to $[0,\infty]$ with range space Y. The subspace $[0,\infty)$ is dense in $[0,\infty]$, and extensions of continuous maps into Hausdorff spaces from dense subspaces to their closures are unique. So, $\hat{f}\hat{\alpha}(\infty)$ must equal $\widehat{f\alpha}(\infty)$. Thus $\varphi_2 g([\alpha]) = \varphi_2(f\alpha) = \widehat{f\alpha}(\infty) = \hat{f}\hat{\alpha}(\infty) = f^*(\hat{\alpha}(\infty)) = f^*\varphi_1([\alpha])$. □

2.6 Limit sets, joins and relative quasiconvexity

In this section let X be a δ–hyperbolic space, let $(G,\mathbb{P})$ be a group acting on X with a cusp uniform action, and let H be any subgroup of $(G,\mathbb{P})$. For a sequence $(h_n) \subset H$ we write $h_n \to \xi \in \partial X$ if $h_n x \to \xi$ for some $x \in X$. Note that if $h_n x \to \xi$ for some x, then $h_n x' \to \xi$ for any $x' \in X$. The *limit set* $\Lambda(H)$ of H is the subset of ∂X consisting of all such limits. The set $\Lambda(H)$ is closed and H–invariant.

Given a subset Ω of ∂X containing at least two points, we will denote by $\mathrm{Join}(\Omega)$ the union of all geodesic lines joining points of Ω. If Ω is closed, then it follows from a standard diagonal argument that $\mathrm{Join}(\Omega)$ is closed. The space $\mathrm{Join}(\Omega)$ is quasi-isometric to a geodesic Gromov hyperbolic space $\mathrm{Join}^+(\Omega)$ (see [22, Section 7.5.A]). In fact, $\mathrm{Join}^+(\Omega)$ is finite neighborhood of $\mathrm{Join}^+(\Omega)$ in the space X endowed with the length metric. An infinite subgroup H of $(G,\mathbb{P})$ is *relatively quasiconvex* if H is parabolic, or $(H,\mathbb{Q})$ has a cusp uniform action on $\mathrm{Join}^+(\Lambda(H))$ where $\mathbb{Q} = \{Q \mid Q = H \cap P \text{ with } Q \text{ infinite and } P \in \mathbb{P}\}$.

The following may be found as Proposition 7.1 of [29]:

Proposition 2.15 *A subgroup H of $(G,\mathbb{P})$ is relatively quasiconvex if and only if the induced convergence action of H on $\Lambda(H)$ is geometrically finite.*

We will implicitly be using the following proposition throughout Sections 3 and 4. Proposition 2.16 is distillate from the proof of Theorem 9.1 of [29].

Proposition 2.16 *Let H be a relatively quasiconvex subgroup of G, and let $x \in \Lambda(H)$; then the following hold*:

(1) *x is a conical limit point of the induced action of H on $\Lambda(H)$ if and only if it is a conical limit point of the action of G on $\partial(G,\mathbb{P})$.*

(2) *x is a bounded parabolic point of the induced action of H on $\Lambda(H)$ if and only if it is a bounded parabolic point of the action of G on $\partial(G,\mathbb{P})$.*

3 Reduction

Let $(G,\mathbb{P})$ be a relatively hyperbolic group with tame peripherals. The results in this section can be considered the first step in the proof of Theorem 1.1. In particular, we show that the proof of Theorem 1.1 can be reduced to the case where the Bowditch boundary $\partial(G,\mathbb{P})$ has no global cut points.

3.1 Blocks and branches

In this subsection we look at cut point decompositions of $\partial(G,\mathbb{P})$. For a more in-depth overview, see [9; 40].

Let M be a Peano continuum, and let Π be the set of global cut points of M. We define a relation R on M by xRy if x and y cannot be separated by an element of Π. In other words, xRy means there does not exist a $z \in \Pi$ such that x and y lie in different components of $M \setminus \{z\}$. Assume x is not a global cut point; then the *block* containing x is the collection of points $y \in M$ such that xRy, and will be denoted by $[x]$. We make the exception that any singleton set satisfying these conditions will not be considered a block. If two blocks $[u]$ and $[v]$ intersect, then they intersect in an element of Π or $[u]=[v]$ (see [40]).

If M is the boundary of a relatively hyperbolic group with tame peripherals, then M is a Peano continuum and the relation R naturally associates to M a simplicial bipartite tree T [10]. The vertices of T correspond to elements of Π and the set of blocks B. Additionally, two vertices $b \in B$ and $p \in \Pi$ are adjacent if $p \in b$.

Now, let T be the Bass–Serre tree for the maximal peripheral splitting $\mathcal{G}$ of G (see Theorem 2.6), and assume that $\mathcal{R}$ and $\mathcal{P}$ are the collections of component and peripheral vertices respectively. Let $v \in \mathcal{P}$. A subtree S of T is a *branch rooted at v* if it is the closure of a component of $T \setminus \{p\}$. The following is a partial summary of results due to Bowditch [10]. In particular, we refer the reader to Sections 7 and 8 of [10] for details.

Theorem 3.1 *Let $(G, \mathbb{P})$ be relatively hyperbolic with tame peripherals and connected Bowditch boundary. Assume that T, $\mathcal{R}$ and $\mathcal{P}$ are as above. There exists an injective map $\beta\colon \mathcal{P} \cup \partial T \to \partial(G, \mathbb{P})$ and for every $v \in \mathcal{R}$ there exists a unique set $B(v) \subset \partial(G, \mathbb{P})$ satisfying the following:*

(1) *$B(v)$ is a subcontinuum of $\partial(G, \mathbb{P})$ for every $v \in \mathcal{R}$ which contains a point not in the image of β. If the maximal peripheral splitting is nontrivial then $B(v)$ is a proper subcontinuum of $\partial(G, \mathbb{P})$. Additionally, if $u, v \in \mathcal{R}$ are distinct and $B(u) \cap B(v) \neq \varnothing$, then $B(u) \cap B(v) = \{\beta(p)\}$ for some $p \in \mathcal{P}$ adjacent to both* u and v.

(2) *If $x \in \mathcal{P}$ then $\beta(x)$ is a parabolic point.*

(3) *If $(x_n) \subset \mathcal{P}$ is a sequence of points converging to $i \in \partial T$, then the sequence $\beta(x_n)$ converges to a point $\iota = \beta(i)$ in $\partial(G, \mathbb{P})$. Such a point will be referred to as an **ideal** point.*

(4) *If v is a vertex in $\mathcal{R}$ then $B(v)$ is block which cannot be separated by a cut point. If $H = \mathrm{Stab}_G(R)$, then the action of H on $B(v)$ is geometrically finite with maximal parabolic subgroups*

$$\mathbb{Q} = \{Q \mid Q = \mathrm{Stab}_G(v) \cap P \text{ with } Q \text{ infinite and } P \in \mathbb{P}\}.$$

Consequently, $(H, \mathbb{Q})$ is relatively hyperbolic with $\partial(H, \mathbb{Q}) = B(v)$. Additionally, $\partial(H, \mathbb{Q})$ is locally connected (see [6]).

(5) *Given a subtree S in T, let $\mathcal{P}(S)$ and $\mathcal{R}(S)$ be $\mathcal{P} \cap S$ and $\mathcal{R} \cap S$, respectively. Then the set $\Psi^0(S) = \beta(\mathcal{P}(S)) \cup \bigcup_{v \in \mathcal{R}(S)} B(v)$ is connected and its closure is the set $\Psi(S) = \beta(\mathcal{P}(S) \cup \partial S) \cup \bigcup_{v \in \mathcal{R}(S)} B(v)$. If S is a branch in T rooted at $v \in \mathcal{P}$, then $\Psi(S)$ is called a **branch** of $\partial(G, \mathbb{P})$ **rooted at** $\beta(v)$.*

(6) $\Psi(T) = \partial(G, \mathbb{P})$.

(7) *If v is a vertex in $\mathcal{R}$, then $B(v)$ does not contain any **ideal** points.*

(8) *Every ideal point ι has a neighborhood base consisting of branches, and any branch containing ι is a neighborhood of ι.*

(9) *Let $\beta^*\colon (\mathcal{P} \cup \partial T) \cup \mathcal{R} \to \partial(G, \mathbb{P})$ be the multivalued map defined by $\beta^*(v) = \beta(v)$ for every $v \in \mathcal{P} \cup \partial T$ and $\beta^*(v) = B(v)$ for any $v \in \mathcal{R}$. Then β^* is G–equivariant.*

Corollary 3.2 *A local cut point in $\partial(G,\mathbb{P})$ must be in a block, ie ideal points are not local cut points.*

Proof Let ι be an ideal point in $\partial(G,\mathbb{P})$. Then ι is contained in some branch, $\Psi(B)$. We first show that $\Psi(B)\setminus\{i\}$ connected. We have that $\Psi^0(B)\subset\Psi(B)\setminus\{\iota\}\subset\Psi(B)$, the set $\Psi^0(B)$ is connected, and $\Psi(B)$ is the closure of $\Psi^0(B)$. So, $\Psi(B)\setminus\{\iota\}$ must be connected. Thus $\partial(G,\mathbb{P})\setminus\{\iota\}$ is connected.

Now if U is any neighborhood of ι, we have from Theorem 3.1(8) that there is branch $B\subset U$ containing ι. By the argument in the preceding paragraph $B\setminus\{\iota\}$ is connected and ι cannot be a local cut point (see Section 2.4). □

The following theorem was communicated to the author by Chris Hruska and relies on Theorem 3.1(4) and known results about the action of the G on $\partial(G,\mathbb{P})$. In particular, Bowditch [11] has shown that the action of G on $\partial(G,\mathbb{P})$ is *minimal*, ie $\partial(G,\mathbb{P})$ does not properly contain a closed G–invariant subset. (We refer the reader to Theorem 9.4 and the subsequent discussion in [11] for details regarding this claim.) Because it will be of use in Section 7, it is worth noting that the action of G on $\partial(G,\mathbb{P})$ is minimal if and only if $\mathrm{Orb}_G(m)$ is dense for every $m\in\partial(G,\mathbb{P})$.

Theorem 3.3 *Assume $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals, $\partial(G,\mathbb{P})$ is connected, and $\partial(G,\mathbb{P})$ contains a global cut point. Then $(G,\mathbb{P})$ splits nontrivially over each edge group in the maximal peripheral splitting of $(G,\mathbb{P})$ that corresponds to an edge connecting a component vertex to a peripheral cut point vertex.*

Proof Assume that T is the Bass–Serre tree for the maximal peripheral splitting of G. Assume there exists an edge e in T connecting a component vertex c to a peripheral cut point vertex p; also assume that G does not split over the edge group G_e nontrivially. Then there is a G–invariant subtree B in T which does not contain e (see [31, Lemma 12.8]). As a cut point vertex, p is adjacent to at least two component vertices. Because $e\not\subset B$, there is at least one component vertex u which is not in B. Then by Theorem 3.1(1) there is a block $B(u)$ which is not entirely contained in $\Psi(B)$. Thus $\Psi(B)\neq\Psi(T)$. By Theorem 3.1(9) the map β^* is G–equivariant, so $\Psi(B)$ is a closed G–invariant proper subspace $\partial(G,\mathbb{P})$. This implies that the action of G on $\partial(G,\mathbb{P})$ is not minimal, a contradiction. □

As a corollary we have:

Corollary 3.4 *Assume $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals, $\partial(G,\mathbb{P})$ is connected, and let $T=\mathcal{R}\sqcup\mathcal{P}$ be the Bass–Serre tree for the maximal peripheral splitting of $(G,\mathbb{P})$. Suppose that $p\in\mathcal{P}$ is a cut point vertex of T, and set $P=\mathrm{Stab}_G(p)$. If $H=\mathrm{Stab}_G(v)$ for some $v\in\mathcal{R}$ which is adjacent to p, then G splits nontrivially relative to $\mathbb{P}$ over an infinite subgroup G_e of $P\cap H$.*

Proof Theorem 3.1(2) gives that $\beta(p)$ is a parabolic point of $\partial(G,\mathbb{P})$. By hypothesis p separates T into at least two components, Theorem 3.1(5) gives that $\beta(p)$ is a cut point of $\partial(G,\mathbb{P})$. The result follows from Theorem 3.3 and [11, Proposition 10.1], which says that $\partial(G,\mathbb{P})$ is connected if and only if G does not split nontrivially over any finite subgroup relative to $\mathbb{P}$. □

3.2 Decompositions and reduction

A *decomposition* $\mathcal{D}$ of a topological space X is a partition of X. Associated to $\mathcal{D}$ is the *decomposition space* whose underlying point set is $\mathcal{D}$, but denoted by $X/\mathcal{D}$. The topology of $X/\mathcal{D}$ is given by the *decomposition map* $\pi\colon X\to X/\mathcal{D}$, with $x\mapsto D$, and where $D\in\mathcal{D}$ is the unique element of the decomposition containing x. A set U in $X/\mathcal{D}$ is deemed open if and only if $\pi^{-1}(U)$ is open in X. A subset A of X is called *saturated* (or $\mathcal{D}$–saturated) if $\pi^{-1}(\pi(A))=A$. The *saturation* $\mathrm{Sat}(A)$ of A is the union of A with all $D\in\mathcal{D}$ that intersect A. The decomposition $\mathcal{D}$ is said to be *upper semicontinuous* if every $D\in\mathcal{D}$ is closed and compact, and for every open set U containing D there exists an open set $V\subset U$ such that $D\subset V$ and $\mathrm{Sat}(V)$ is contained in U. In Proposition I.3.1 of [16], Daverman shows that the decomposition map of an upper semicontinuous decomposition is proper. It is then an easy corollary that in an upper semicontinuous decomposition the saturation of a compact set is compact. An upper semicontinuous decomposition $\mathcal{D}$ is called *monotone* if the elements of $\mathcal{D}$ are connected. The following is a key characteristic of monotone decompositions and may be found as Proposition I.4.1 of [16]:

Proposition 3.5 *Let $\mathcal{D}$ be a decomposition of a space X. Then $\mathcal{D}$ is monotone if and only if $\pi^{-1}(A)$ is connected for every connected subset A of $X\setminus\mathcal{D}$.*

A collection of subsets $\mathcal{S}$ of a metric space is called a *null family* if for every $\epsilon>0$ there are only finitely $S\in\mathcal{S}$ with $\mathrm{diam}(S)>\epsilon$. The following proposition can be found as Proposition I.2.3 in [16]:

Proposition 3.6 *Let $\mathcal{S}$ be a null family of closed disjoint subsets of a compact metric space X. Then the associated decomposition of X is upper semicontinuous.*

Lemma 3.7 *If $\mathcal{D}$ is an upper semicontinuous decomposition of a space X, then the saturation of a closed set is closed.*

Lemma 3.7 follows from Proposition I.1.1 of [16].

Lemma 3.8 *If $\mathcal{D}$ is an upper semicontinuous decomposition of a generalized Peano continuum X, then $X/\mathcal{D}$ is a generalized Peano continuum.*

Proof Let $Y = X/\mathcal{D}$. We want that Y is connected, locally connected, locally compact, σ–compact and metrizable. Clearly, Y is connected. By Theorem 27.12 of [46] the quotient of a locally connected space is locally connected.

To prove local compactness note that Lemma 3.7 implies the quotient map $f\colon X \to Y$ is closed. The image of a locally compact space under a closed map is locally compact provided the preimage of each point is compact (see [46, Exercise 18C.2]). Thus Y is locally compact, because the elements of $\mathcal{D}$ are compact.

We still require that Y is σ–compact and metrizable. The continuous image of a σ–compact space is σ–compact, so Y is σ–compact. Proposition I.2.2 of [16] gives that the image of a metric space under an upper semicontinuous decomposition is metrizable. Thus Y is a generalized Peano continuum. □

Lemma 3.9 *Let X be a generalized continuum. Assume that $\mathcal{D}$ is a monotone upper semicontinuous decomposition, and let $f\colon X \to X/\mathcal{D}$ be the decomposition map. If $Q = X/\mathcal{D}$, then f induces a homeomorphism between* $\mathrm{Ends}(X)$ *and* $\mathrm{Ends}(Q)$*.*

Proof By Proposition I.3.1 of [16] the decomposition map of an upper semicontinuous decomposition is proper. So, by Proposition 2.12, we have that f can be continuously extended to a map $\hat{f}\colon X \cup \mathrm{Ends}(X) \to Q \cup \mathrm{Ends}(Q)$. We only need that the restriction $f^*\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Q)$ is a bijection.

Let $(C_1, C_2, C_3, \dots)$ be an exhaustion of X. The elements of $\mathcal{D}$ are compact and we have that the saturation of a compact set is compact. So, we may assume that each C_i is saturated. We first show that f^* is surjective. The sequence $(f(C_i))_{i=1}^{\infty} = (f(C_1), f(C_2), f(C_3), \dots)$ is an exhaustion of Q and $\mathrm{Ends}(Q)$ is independent of choice of exhaustion, so let $(A_1, A_2, A_3, \dots) \in \mathrm{Ends}(Q)$ be defined using the exhaustion $(f(C_i))$. For each i the preimage $f^{-1}(A_i)$ is contained in $f^{-1}(Q \setminus f(C_i)) = X \setminus C_i$, and by monotonicity $f^{-1}(A_i)$ is connected by Proposition 3.5. Since $f^{-1}(A_1) \supset f^{-1}(A_2) \supset f^{-1}(A_3) \supset \cdots$, we have that $(f^{-1}(A_i))_{i=1}^{\infty}$ is an end of $\mathrm{Ends}(X)$.

Now, let $(E_i)_{i=1}^{\infty}$ and $(F_i)_{i=1}^{\infty}$ be distinct elements of $\mathrm{Ends}(X)$. Then there exists an $i \in \mathbb{N}$ such that $E_j \neq F_j$ for all $j \geq i$. Because the C_i are saturated, monotonicity implies $f(E_j) \neq f(F_j)$ for all $j \geq i$. Thus $(f(E_n))_{n=1}^{\infty}$ and $(f(F_n))_{n=1}^{\infty}$ are distinct. □

Corollary 3.10 *Assume that $\mathcal{D}$ is a monotone upper semicontinuous decomposition, let $f\colon X \to X/\mathcal{D}$ be the decomposition, and let x be a point of X such that $\{x\} \in \mathcal{D}$. If x is a local cut point which is not a global cut point, then $f(x)$ is a local cut point and $\mathrm{val}(x) = \mathrm{val}(f(x))$.*

Returning to the setting of Bowditch boundaries we will use the notation introduced in Section 3.1. Let r be an element of $\mathcal{R}$. Any branch not containing the block $B(r)$ but rooted at a point in the block $B(r)$ is said to be *attached to* $B(r)$. The union of all branches attached to $B(r)$ with common root will be called a *full branch attached to* $B(r)$.

Lemma 3.11 *Let $R = B(v)$ for some $v \in \mathcal{R}$. The collection of full branches attached to R forms a null family of disjoint connected closed sets.*

Proof Let F be a full branch attached to R with root $\beta(p)$ for some $p \in \mathcal{P}$. Then F is the subcontinuum associated by Ψ to the subtree S of T consisting of all branches in T rooted at p which do not contain the vertex v. By Theorem 3.1, F is connected and closed.

Let F' be another full branch attached to R, and assume that F' is rooted at $\beta(q)$ for some $q \neq p$. If S' is the subtree of T of all branches in T rooted q and not containing v, then $\Psi(S') = F'$. As $S \cap S' = \varnothing$, we have that $F \cap F' = \varnothing$, because the map β is injective and blocks associated to component vertices are unique.

Now, let $\epsilon > 0$. Bowditch has shown in Section 8 of [10] that the set of all branches attached to a component of $\partial(G, \mathbb{P})$ forms a null family, so there are only finitely many branches of diameter $\frac{\epsilon}{2}$. Let x_1 and x_2 be two points in a full branch D, and let $B_1 \subset D$ and $B_2 \subset D$ be branches containing x_1 and x_2, respectively. The distance between x_1 and x_2 is at most $\mathrm{diam}(B_1) + \mathrm{diam}(B_2)$. If there were infinitely many full branches of diameter greater than ϵ, then there would be infinitely many branches of diameter greater than $\frac{\epsilon}{2}$, a contradiction. □

It follows from Lemma 3.11 that:

Lemma 3.12 *Let $R = B(v)$ for some $v \in \mathcal{R}$, and define $f\colon \partial(G,\mathbb{P}) \to R$ to be the quotient map obtained by identifying all full branches attached to R with their roots. Then f is an upper semicontinuous monotone retraction onto R.*

Lemma 3.13 *Let x be a point contained in a block R. The point x is a local cut point of $\partial(G,\mathbb{P})$ and a conical limit point of the action of G on $\partial(G,\mathbb{P})$ if and only if $f(x)$ is a local cut point of R and a conical limit point of the action of $\mathrm{Stab}_G(R)$ on R.*

Proof Notice that x is not contained in a full branch attached to R, so $\{x\}$ is an element of the decomposition which is not a global cut point. Lemma 3.12 gives that f is an upper semicontinuous decomposition. Then Corollary 3.10 implies that $f(x)$ is a local cut point of R. By Proposition 2.16, $f(x)$ is a conical limit point of the action of $\mathrm{Stab}_G(R)$ on R. The reverse direction follows from Lemma 3.12, Proposition 2.16 and the observation that if $f(x)$ is a local cut point, then $|\mathrm{Ends}(R \setminus \{f(x)\})| \geq 2$ and Lemma 3.9 implies $|\mathrm{Ends}(\partial(G,\mathbb{P}) \setminus \{x\})| \geq 2$. □

Lemma 3.14 *Let $\{x,y\} \subset R$ and assume that x and y are not parabolic points. Then $\{x,y\}$ is a cut pair in $\partial(G,\mathbb{P})$ if and only if $\{f(x), f(y)\}$ is a cut pair in R. Moreover, f induces a bijection between components of $\partial(G,\mathbb{P}) \setminus \{x,y\}$ and components of $R \setminus \{f(x), f(y)\}$.*

Proof This result follows from Proposition 3.5. We first show that there is a bijection between components of $\partial(G,\mathbb{P}) \setminus \{x,y\}$ and components of $R \setminus \{f(x), f(y)\}$. Assume $\{f(x), f(y)\}$ is a cut pair in R. Because $\{x\}, \{y\} \in \mathcal{D}$ the preimage of each component of $R \setminus \{f(x), f(y)\}$ is a saturated connected set in $\partial(G,\mathbb{P}) \setminus \{x,y\}$. Because $\mathcal{D}$ is monotone, the connected components of $\partial(G,\mathbb{P}) \setminus \{x,y\}$ are saturated, and therefore not identified under f. Thus we have a bijection between components of $\partial(G,\mathbb{P}) \setminus \{x,y\}$ and components of $R \setminus \{f(x), f(y)\}$.

Now, assume $\{x,y\}$ is a cut pair. Then $\partial(G,\mathbb{P}) \setminus \{x,y\}$ has at least two components and the above implies that $R \setminus \{f(x), f(y)\}$ has at least two components. If $\{f(x), f(y)\}$ is a cut pair, then again the result follows from the above. □

By Proposition 2.16 a conical limit point of the action of $\mathrm{Stab}_G(R)$ on R is a conical limit point of the action of G on $\partial(G,\mathbb{P})$, so as a corollary of Lemma 3.14 we obtain:

Corollary 3.15 *Let $x, y \in R$ be conical limit points of the action of $\mathrm{Stab}_G(R)$ on R. Then $x \sim y$ in $\partial(G,\mathbb{P})$ if and only if $f(x) \sim f(y)$ in R.*

Lemma 3.16 *Assume $R = B(v)$ for some $v \in \mathcal{R} \subset \partial(G,\mathbb{P})$ and let $H = \mathrm{Stab}_G(R)$. A cut pair $\{a,b\}$ in R is H–translate inseparable if and only if it is G–translate inseparable.*

Proof Notice that, by Lemma 3.14, $\{a,b\}$ is a cut pair of $\partial(G,\mathbb{P})$. If $h \in H$, then by Lemma 3.14 a translate $\{ha,hb\}$ separates $\{a,b\}$ in R if and only if $\{ha,hb\}$ separates $\{a,b\}$ in $\partial(G,\mathbb{P})$. So, if $\{a,b\}$ is G–translate inseparable, then $\{a,b\}$ is H–translate inseparable.

Now assume that $\{a,b\}$ is H–translate inseparable and let $g \in G$. Because cut pairs cannot be separated by cut points, the pair $\{ga,gb\}$ must be in R or $\partial(G,\mathbb{P}) \setminus R$. If $\{ga,gb\}$ is in $\partial(G,\mathbb{P}) \setminus R$ then $\{ga,gb\}$ cannot separate $\{a,b\}$. If $\{ga,gb\}$ is in R then g must be in H. That $g \in H$ follows from the definition of a block as a maximal set of points of $\partial(G,\mathbb{P})$ which cannot be separated by cut points and the fact that g is a homeomorphism. For if g sent a point $x \in R \setminus \{a,b\}$ to a point of $\partial(G,\mathbb{P}) \setminus R$, then by the definition of R there must exist a cut point of $\partial(G,\mathbb{P})$ which separates either the pair $\{gx,ga\}$ or the pair $\{gx,gb\}$. Thus gR contains points which can be separated by a cut point, but g is a homeomorphism. So, R contains points which can be separated by a cut point, a contradiction. Thus $\{a,b\}$ is H–translate inseparable, so $\{a,b\}$ is G–translate inseparable. □

Corollary 3.17 *Let $R = B(v)$ for some $v \in \mathcal{R}$. If $\{a,b\}$ is a cut pair of R which is $\mathrm{Stab}_G(R)$–translate inseparable and does not contain any parabolic points, then $\{a,b\}$ is G–translate inseparable cut pair of $\partial(G,\mathbb{P})$ which does not contain any parabolic points. Additionally, if R contains a necklace ν and $i\colon R \hookrightarrow \partial(G,\mathbb{P})$ is the inclusion map, then $i(\nu)$ is a necklace in $\partial(G,\mathbb{P})$.*

Proof The first result follows from Lemma 3.16.

Let ν be a necklace in R. Then ν is the closure of an $\sim$–class which is contained in R. Call this class C. Because cut pairs in $\partial(G,\mathbb{P})$ cannot be separated by global cut points and the elements of C are conical limit points, Corollary 3.15 implies that $i(C)$ is a $\sim$–class in $\partial(G,\mathbb{P})$. R is closed, so the inclusion map is closed. Thus, $i(\nu) = i(\overline{C}) = \overline{i(C)}$, which is a necklace in $\partial(G,\mathbb{P})$. □

4 Local cut points in $\partial(G,\mathbb{P})$

The goal of this section is to describe the ways local cut points occur in $\partial(G,\mathbb{P})$ (see Theorem 4.22). In the hyperbolic setting, Bowditch [4] showed that a local cut point must be contained in a translate inseparable loxodromic cut pair or a necklace. We wish to adapt Bowditch's result to the relatively hyperbolic setting (see Theorem 4.20). As first observed by Guralnik [27], a careful examination of [4] reveals that much of Bowditch's argument in [4] could directly translate to the Bowditch boundary $\partial(G,\mathbb{P})$ if one only considers local cut points which are conical limit points and assumes that $\partial(G,\mathbb{P})$ has no global cut points. However, there are two key steps [4, Lemmas 5.25 and 5.17] in the proof where Bowditch depends heavily on hyperbolicity. Namely, in Section 5 of [4] Bowditch requires that G act as a uniform convergence group on its boundary, ie that the action on the triple space is proper and cocompact. As mentioned in Section 2, in the relatively hyperbolic setting the action of G on $\partial(G,\mathbb{P})$ is not uniform in general. Lemma 4.1 and Proposition 4.5 generalize the critical steps [4, Lemmas 5.2 and 5.17] of Bowditch's argument to the relatively hyperbolic setting and allow us to use Bowditch's results concerning local cut points to prove the main result of this section (Theorem 4.22). We remark that Lemma 4.1 is also proved in [27], but for completeness we include an alternative more self-contained proof, which uses different techniques.

4.1 A key lemma

In this section we prove the following technical lemma:

Lemma 4.1 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group. There exist finite collections $(U_i)_{i=1}^p$ and $(V_i)_{i=1}^p$ of open connected sets of $\partial(G,\mathbb{P})$ with $\overline{U}_i \cap \overline{V}_i = \varnothing$, and such that if $K \subseteq \partial(G,\mathbb{P})$ is closed and $x \in \partial(G,\mathbb{P})\backslash K$ is a conical limit point then there exists $g \in G$ and $i \in \{1,\dots,p\}$ such that $gx \in U_i$ and $gK \subseteq V_i$.*

We postpone the proof of Lemma 4.1, as it will require a few lemmas. Let X be the proper δ–hyperbolic space on which G acts as given by the definition of relatively hyperbolic (see Section 2). We know from Theorem 2.2 that there are finitely many orbits of horoballs in $\mathcal{B}$. Let $B_1, B_2, \dots, B_n$ be representatives from each orbit and $p_1, p_2, \dots, p_n$ the associated parabolic points for each representative horoball. In Lemma 6.4 of [11] it is shown that $C_i = \mathrm{Fr}(B_i)/\mathrm{Stab}_G(p_i)$ is compact for every $i \in \{1,2,\dots,n\}$ and from the definition of relatively hyperbolic we know that G acts cocompactly on $(X \setminus \mathcal{B})$. Let A be the fundamental domain of the action of G on $(X \setminus \mathcal{B})$, and define

$$C = A \cup C_1 \cup C_2 \cup \cdots \cup C_n.$$

Then C is a compact subset of X and $\mathrm{Orb}_G(C) \supseteq X \setminus \mathcal{B}$.

Let $\Theta_2\partial(G,\mathbb{P})$ the space of distinct pairs in $\partial(G,\mathbb{P})$ and define $E(C)\subseteq\Theta_2\partial X$ to be the collection of pairs (x,y) such that $x=c(\infty)$ and $y=c(-\infty)$ for some line $c\colon\mathbb{R}\to X$ with $\operatorname{im}(c)\cap C\neq\varnothing$.

Lemma 4.2 *The set $E(C)$ is compact in $\Theta_2\partial(G,\mathbb{P})$.*

Lemma 4.2 follows from the fact that for any pair $(x,y)\in\Theta_2\partial(G,\mathbb{P})$ we may find a line whose ends are x and y (see [12, Chapter III]). Then sequential compactness and a standard diagonal argument show that a sequence of lines each meeting C converges to a line meeting C. We leave the details as an exercise.

The action of G is by isometries, the translates of C cover $X\setminus\mathcal{B}$, and a line ℓ cannot be completely contained in a horoball. So, there must be a $g\in G$ and a line ℓ' such that $gC\cap\ell\neq\varnothing$, $\ell'\cap C\neq\varnothing$ and $g\ell'=\ell$. Consequently, we obtain:

Corollary 4.3 (Tukia, Gerasimov) *G acts cocompactly on $\Theta_2\partial(G,\mathbb{P})$.*

Corollary 4.3 was first observed by Gerasimov [19], following results of Tukia [44].

Lemma 4.4 *There exist finite collections $(U_i)_{i=1}^p$ and $(V_i)_{i=1}^p$ such that $\overline{U}_i\cap\overline{V}_i=\varnothing$ for every $i\in\{1,\dots,p\}$, and such that if $x,y\in\partial(G,\mathbb{P})$ then there exists $g\in G$ and $i\in\{1,\dots,p\}$ with $gx\in U_i$ and $gy\in V_i$.*

Proof Let d be a metric on $\partial(G,\mathbb{P})$, and let K be a compact set whose G–translates cover $\Theta_2\partial(G,\mathbb{P})$. Clearly the intersection of K with the diagonal of $\partial(G,\mathbb{P})\times\partial(G,\mathbb{P})$ is empty. For every $(x,y)\in K$ define $r(x,y)=\frac{1}{4}d(x,y)$ and define $U_x=B(x,r(x,y))$ and $V_y=B(y,r(x,y))$. Then $\bigcup_{(x,y)\in C}(U_x\times U_y)$ covers K. By compactness there exist finitely many $(x_i,y_i)\in K$ such that $U_{x_i}\times V_{x_i}$ cover K. Notice that by construction $\overline{U}_{x_i}\cap\overline{V}_{y_i}=\varnothing$. Thus by the cocompactness of the action we are done. □

Proof of Lemma 4.1 Let x be a conical limit point. By the definition of conical limit point there exists $(g_n)\in G$ and distinct points $\alpha,\beta\in\partial(G,\mathbb{P})$ such that $g_nx\to\alpha$ and $g_ny\to\beta$ for every $y\in\partial(G,\mathbb{P})\setminus\{x\}$; moreover, by passing to a subsequence we may assume that the members of the sequence (g_n) are distinct.

Because G acts on $\partial(G,\mathbb{P})$ as a convergence group, every sequence (g_n) of distinct group elements has a subsequence (g_i) such that if $K\subset\partial(G,\mathbb{P})\setminus\{x\}$ then for any neighborhood $V\ni\beta$ there exists $g_{i_0}\in(g_i)$ such that $g_{i_0}\in V$.

Let $(U_i')_{i=1}^p$ and $(V_i')_{i=1}^p$ be the neighborhoods found in Lemma 4.4. As $(\alpha,\beta)\in\Theta_2\partial(G,\mathbb{P})$ there exists $g\in G$ and $i\in\{1,\dots,p\}$ such that $g\alpha\in U_i'$ and $g\beta\in V_i'$. Set $U_i=g^{-1}U_i'$ and $V_i=g^{-1}V_i'$. Then for large enough n we have $g_nx\in U_i$ and $g_nK\subseteq V_i$. □

4.2 The stabilizer of a necklace is relatively quasiconvex

The goal of this section is to prove the following proposition:

Proposition 4.5 *Suppose $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals, and set $M=\partial(G,\mathbb{P})$. Assume M is connected, has no global cut points and is not homeomorphic to S^1, and ν is a necklace in M. Then* $\mathrm{Stab}_G(\nu)$ *is relatively quasiconvex, acts minimally on ν, and any jump in ν is a loxodromic cut pair and translate inseparable.*

The proof of Proposition 4.5 is divided into the following series of lemmas and corollaries.

A collection of subspaces $\mathcal{A}$ of a metric space Z is called *locally finite* if only finitely many members of $\mathcal{A}$ intersect any compact set $K\subset Z$. To prove Proposition 4.5 we will use the following proposition twice, which may be found as Proposition 7.2 of [31].

Proposition 4.6 *Let G be a group acting properly and cocompactly on a metric space Z, and let $\mathcal{A}$ be a locally finite collection of closed subspaces of Z. Then* $\mathrm{Stab}(A)$ *acts cocompactly on A for every $A\in\mathcal{A}$, and the elements of $\mathcal{A}$ lie in finitely many G–orbits.*

Until otherwise stated we will assume the following hypotheses in this section. Let $(G,\mathbb{P})$, M and ν be as in the statement of Proposition 4.5. Let X be a δ–hyperbolic space on which G acts with a cusp uniform action (see Section 2.2). Let $\mathcal{B}$ be a G–equivariant family of open horoballs based at the parabolic points of ∂X as given by the definition of relatively hyperbolic, and let K be a compact set such that $X\setminus\mathcal{B}\subset\mathrm{Orb}_G(K)$.

The following lemma was observed by Gromov in Section 7.5.A of [22]. The following proof is based on an argument due to Dahmani (see Proposition 1.8 of [14]).

Lemma 4.7 *The limit set* $\Lambda(\mathrm{Join}(\nu))$ *is equal to ν.*

Before we prove Lemma 4.7, we recall some basic definitions from the theory of hyperbolic groups (see for example [22] or [20]). Let z be a basepoint of X. The *Gromov product* of $x, y \in X$ with respect to z is defined to be

$$(x \mid y)_z = \tfrac{1}{2}(d(x,z) + d(y,z) - d(x,y)).$$

The Gromov product is extended to $X \cup \partial X$ by setting

$$(x \mid y)_z = \sup \liminf_{i,j\to 0} (x_i \mid x_j),$$

where the supremum is taken over all sequences $(x_i) \to x$ and $(y_j) \to y$. The Gromov product measures the distance from the point z to the geodesic between x and y up to finite error (see for example [12, Definition III.H.1.19 and Exercise III.H.3.18(3)]); in other words, we have:

Lemma 4.8 *There is a constant ρ such that $|(x \mid y)_z - d(z,[x,y])| < \rho$ for all $z \in X$ and all $x, y \in X \cup \partial X$.*

Also, by [12, Remark III.H.3.17(6)] we have:

Lemma 4.9 *If (a_i) is a sequence of points in $X \cap \partial X$ and $a \in \partial X$, then $(a_i) \to a$ if and only if $(a_i \mid a) \to \infty$.*

Proof of Lemma 4.7 Clearly $\nu \subset \Lambda(\mathrm{Join}(\nu))$. Let (x_n) be a sequence of points from distinct lines in $\mathrm{Join}(\nu)$, and converging to a point x in ∂X. For each i let c_i be the line containing x_i, and set $a_i = c_i(\infty)$ and $b_i = c_i(-\infty)$. The pairs $\{a_i, b_i\}$ form a sequence in $\Theta_2\nu$. As $\Theta_2\nu$ is a compact subset of $\Theta_2\partial(G,\mathbb{P})$, by passing to a subsequence if necessary we may assume $(\{a_i, b_i\})$ converges to some pair $\{a, b\} \in \Theta_2\nu$. We claim that $x = a$ or $x = b$.

Let z be a basepoint for X. For each i let y_i denote the point of c_i nearest z. The points y_i divide the lines c_i into two sides A_i and B_i with $A_i \cap B_i = \{y_i\}$, $a_i \in A_i$ and $b_i \in B_i$. The infinitely many members of the sequence (x_i) must be in one of the collections $\{A_i\}$ or $\{B_i\}$. We may assume without loss of generality that infinitely many members of the sequence (x_i) are in $\{B_i\}$. Let $t_i \in [x_i, b_i] \subset B_i$. Then $d(z,t_i) \geq d(z,x_i) - 2\delta$; taking the minimum over all t_i we have that $d(z,[x_i,b_i]) \geq d(z,x_i) - 2\delta$. Thus, by Lemma 4.8, $(x_i \mid b_i)_z \geq d(z,x_i) - 2\delta - \rho$.

Now, as $(b_i) \to b$ we have that $(b_i \mid b) \to \infty$. By δ–hyperbolicity of X we have $(x \mid b) \geq \min\{(x_i \mid b_i)_z, (b_i \mid b)_z\} - \delta'$, where δ' is some multiple of δ. Thus $(x \mid b)_z \to \infty$, which implies that $(x_i) \to b$. □

Lemma 4.10 *Let* $\mathcal{J} = \{\mathrm{Join}(\nu) \mid \nu$ *is a necklace in* $\partial X\}$. $\mathcal{J}$ *is locally finite in* X.

Proof Let C be a compact subset in X. Lemma 4.2 implies that endpoints of the set of all lines intersecting C gives us a compact subset $E(C)$ of $\Theta_2\partial(G,\mathbb{P})$. If $J = \mathrm{Join}(\nu) \in \mathcal{J}$ and $J \cap C$ is nonempty, then there exists a line ℓ contained in J such that $\{\ell(\infty), \ell(-\infty)\} \subset \nu$ and $\{\ell(\infty), \ell(-\infty)\} \in E(C)$. Lemma 3.16 of [4] shows that a compact set of $\Theta_2\partial(G,\mathbb{P})$ can only intersect finitely many $\sim$–classes, thus only finitely many members of $\mathcal{J}$ can intersect C. □

For the remainder of this section define $H = \mathrm{Stab}_G(\mathrm{Join}(\nu))$ for some fixed necklace ν.

Lemma 4.11 *The subgroup* H *acts properly and cocompactly on* $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$.

Proof Lemma 4.10 implies that $\mathrm{Orb}_G(\mathrm{Join}(\nu))$ is locally finite in X. Thus only finitely many members of $\mathrm{Orb}_G(\mathrm{Join}(\nu))$ intersect any compact set $C \subset \mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$, which implies that only finitely many members of the collection

$$\mathcal{A} = \mathrm{Orb}_G(\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B}))$$

intersect C. Setting $Z = (X \setminus \mathcal{B})$, relative hyperbolicity of G implies that G acts cocompactly on Z. Applying Proposition 4.6 we get that H acts cocompactly on $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$. □

Lemma 4.12 *Let* $B \in \mathcal{B}$ *be a horoball intersecting* $\mathrm{Join}^+(\nu)$. *Then* $S = \mathrm{Stab}_H(B)$ *acts cocompactly on the horosphere* $\mathrm{Fr}(B) \cap \mathrm{Join}^+(\nu)$.

Proof The family $\mathcal{B}$ of G–equivariant horoballs based at the parabolic points is locally finite. As H acts cocompactly on $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$, we have that H acts cocompactly on $\mathrm{Join}^+(\nu) \cap (X \setminus \mathcal{B})$. By Proposition 4.6 the group $S = \mathrm{Stab}_H(B)$ acts cocompactly on $\mathrm{Fr}(B) \cap \mathrm{Join}^+(\nu)$. □

Lemma 4.13 *Let* $p \in \nu$ *be a parabolic point of the action of* G *on* X. *A horoball* $B_p \in \mathcal{B}$ *based at* p *has unbounded intersection with* $\mathrm{Join}^+(\nu)$, *the stabilizer* $\mathrm{Stab}_H(B_p)$ *is an infinite subgroup of* H, *and* $p \in \Lambda(H)$.

Proof Assume $p \in \nu$ is a bounded parabolic point, and let B_p be the open horoball in $\mathcal{B}$ based at p. Because a necklace is the closure of a $\sim$–class, p must be an accumulation point. So, we may find a sequence of conical limit points (x_n) which converge to p. As $\mathrm{Join}^+(\nu)$ is a visibility space we find a sequence of geodesic lines $([p, x_n])$ connecting

p to the elements of the sequence (x_n). Define $y_n = [p, x_n] \cap \mathrm{Fr}(B_p)$. Then (y_n) must be unbounded, because the sequence of endpoints $(\{p, x_n\})$ converges to the pair $\{p, p\}$. Thus, by Lemma 4.9, the Gromov product $(p \mid x_i)_z \to \infty$ as $i \to \infty$.

By Lemma 4.12, $S = \mathrm{Stab}_H(B_p)$ acts cocompactly on $\mathrm{Fr}(B_p) \cap \mathrm{Join}^+(\nu)$. Let A be the fundamental domain for this action. By covering the sequence (y_n) with the S–translates of A we may find a sequence of group elements (h'_n) in H such that $h_n \to p$. □

Lemma 4.14 *Let $\mathcal{B}^*$ be the collection of all horoballs for the action of G which are based at parabolic points outside of ν. The elements of $\mathcal{B}^*$ have uniformly bounded intersections with* $\mathrm{Join}(\nu)$.

Proof The family of G–equivariant horoballs is locally finite, and H acts properly and cocompactly on $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$. So, there are only finitely many H–orbits of horoballs in $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$. In particular, there are only finitely many H–orbits of horoballs based outside of ν which intersect $\mathrm{Join}(\nu)$.

By Lemma 4.12, if B_p is a horoball based at a parabolic point $p \notin \nu$ which intersects $\mathrm{Join}(\nu)$, then $S = \mathrm{Stab}_H(B_p)$ acts cocompactly on $\mathrm{Fr}(B_p) \cap \mathrm{Join}^+(\nu)$. This implies that if the intersection of B_p with $\mathrm{Join}(\nu)$ is infinite, then $p \in \Lambda(H)$. The limit set $\Lambda(H)$ is the minimal closed H–invariant subset of ∂X. Thus, $p \in \Lambda(H)$ implies $p \in \nu$, a contradiction. Therefore any horoball based outside of ν must have bounded intersection with $\mathrm{Join}(\nu)$. □

Let $\mathcal{B}' \subset \mathcal{B}$ be the collection of horoballs based at parabolic points of ν.

Corollary 4.15 *Let $H = \mathrm{Stab}_G(\nu)$. Then H acts properly and cocompactly on $Y = \mathrm{Join}(\nu) \cap (X \setminus \mathcal{B}')$.*

Proof Lemma 4.11 gives us that H acts cocompactly on $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$. Let C be a fundamental domain for the action of H on $\mathrm{Join}(\nu) \cap (X \setminus \mathcal{B})$. Since there are only finitely many orbits of horoballs meeting C, there are only finitely many orbits of horoballs from $B \setminus \mathcal{B}'$ which meet C, so Lemma 4.14 implies that we may increase C to a larger compact set C' such that H acts cocompactly on Y. □

Let $\mathbb{P}' \subset \mathbb{P}$ be $\{P \in \mathbb{P} \mid P = \mathrm{Stab}_G(B') \text{ for some } B' \in \mathcal{B}'\}$.

Corollary 4.16 *Let $\mathbb{Q} = \{H \cap P \mid P \in \mathbb{P}'\}$. The action of $(H, \mathbb{Q})$ on $\mathrm{Join}^+(\nu)$ is cusp uniform.*

Proof By definition each group in $\mathbb{Q}$ stabilizes a horoball in $\mathcal{B}'$, and Lemma 4.13 gives that every $Q \in \mathbb{Q}$ is infinite. The collection $\mathcal{B}'$ is H–equivariant and Corollary 4.15 gives us that the action is cocompact on $Y = \mathrm{Join}(\nu) \cap (X \setminus \mathcal{B}')$. Therefore the action of $(H, \mathbb{Q})$ on $\mathrm{Join}^+(\nu)$ is cusp uniform. □

Corollary 4.17 *H acts geometrically finitely on ν.*

Proof Corollary 4.16 shows that the action on Y is cusp uniform, and Lemma 4.7 shows that $\nu = \partial\,\mathrm{Join}^+(\nu)$. Then, by Proposition 2.5, the action on $\partial\,\mathrm{Join}^+(\nu)$ is geometrically finite. □

Thus, $\nu = \Lambda(H)$. In other words, we have shown:

Corollary 4.18 *H acts minimally on ν*

To complete the proof of Proposition 4.5 it remains to show:

Lemma 4.19 *The stabilizer of a jump in ν is a loxodromic and translate inseparable.*

Proof We first show that he stabilizer of a jump is loxodromic. Let $\mathrm{Jump}(\nu)$ be the set of jumps in ν and let $K' \subset Y$ be a compact set whose H–translates cover Y. In the hyperbolic setting Bowditch showed that $\mathrm{Jump}(\nu)/H$ is finite (see Lemma 5.19 of [4]). Bowditch's argument only used the fact that G acts cocompactly on the space of unordered pairs in the boundary of G and the fact that $\nu = \Lambda(H)$. Thus, using Corollary 4.3 and Lemma 4.7, and the argument of Lemma 5.19 of [4], we may conclude that $\mathrm{Jump}(\nu)/H$ is finite. Thus $\{\mathrm{Join}(J) \cap Y \mid J \in \mathrm{Jump}(\nu)\}$ is locally finite in Y, and we may apply Proposition 4.6 to show that $\mathrm{Stab}_H(J)$ acts cocompactly on $\mathrm{Join}(J) \cap Y$ for any jump $J \in \mathrm{Jump}(\nu)$. Notice that if we knew that parabolic points did not participate in jumps, then we would know that for every jump J there is a line in $\mathrm{Join}(J)$ having bi-infinite intersection with Y. Then the action of $\mathrm{Stab}_H(J)$ on $\mathrm{Join}(J) \cap Y$ is cocompact and by isometries we may extend the action of $\mathrm{Stab}_H(J)$ to $\mathbb{R}$, which implies that $\mathrm{Stab}_H(J)$ is loxodromic.

Let $S = \mathrm{Stab}_H(J)$. To see that we may extend the action of S to $\mathbb{R}$, first let ℓ be a line in $\mathrm{Join}(J)$ and notice that if C is a fundamental domain for the action of S on ℓ then the convex hull $\mathrm{Hull}(C)$ of C is a connected subset of ℓ. Because the action of S on ℓ is cocompact, there is a bound on the diameter of components of $\ell \setminus \{\mathrm{Orb}_S(C)\}$. Additionally, there are at most two components of $\ell \setminus \{\mathrm{Orb}_S(C)\}$ which are adjacent to

$\mathrm{Hull}(C)$. Let $\mathcal{V}$ be the collection of components of $\ell \setminus \{\mathrm{Orb}_S(C)\}$ whose intersection with $\mathrm{Hull}(C)$ is nonempty and define C' to be the closure of $\mathrm{Hull}(C) \cap \mathcal{V}$. Then C' is compact and $\ell \subset \mathrm{Orb}_S(C')$. As ℓ is isometric to $\mathbb{R}$, we are done.

We now show that parabolic points cannot participate in jumps. Let $J = \{x, y\}$ be a jump of ν and assume that x was parabolic. Then $\mathrm{Stab}_H(x)$ cannot fix y. Let $h \in \mathrm{Stab}_H(x)$ be nontrivial. Then h is a homeomorphism of ∂X, so $\{x, h(y)\}$ is also a jump. Thus x is a point participating in two jumps and must be isolated (see Section 2.4), but by definition the parabolic points in a necklace cannot be isolated. Therefore x could not have been parabolic.

Let $J = \{x, y\}$ be a jump in ν. If $\mathcal{N}(x, y) \geq 3$, then $\{x, y\}$ is inseparable and thus G–translate inseparable. So, assume that $x \sim y$. To see that $\{x, y\}$ cannot be separated by a pair $\{a, b\}$ with $\mathrm{val}(a) = \mathrm{val}(b) = \mathcal{N}(a, b) = 2$, notice that given such an $\{a, b\}$ Lemma 2.11 would imply that $x \sim y \sim a \sim b$, contradicting the fact that $\{x, y\}$ is a jump.

Assume $\{a, b\}$ is a cut pair with $\mathcal{N}(a, b) \geq 3$. Arguments of Bowditch [4] show that such a pair is an inseparable $M(3+)$ class (see specifically [4, Lemma 3.8 and Proposition 5.13]). As $\{a, b\}$ is inseparable, $\{a, b\}$ must lie in a single component of $M \setminus \{x, y\}$. Because $\{x, y\}$ is a cut pair, there exists at least one component U of $M \setminus \{x, y\}$ which does not contain $\{a, b\}$. Then $\overline{U}$ is a connected set contained in $M \setminus \{a, b\}$ such that $\{x, y\} \subset \overline{U}$. Thus $\{a, b\}$ cannot separate $\{x, y\}$.

As every loxodromic cut pair has two or more components in its complement, we have shown that $\{x, y\}$ is translate inseparable. □

4.3 Collecting local cut points

Now that we have proved Lemma 4.1 and Proposition 4.5, we may plug into the arguments of Bowditch [4] in the case when $\partial(G, \mathbb{P})$ has no global cut points to describe the ways local cut points occur in $\partial(G, \mathbb{P})$.

Theorem 4.20 *Let $(G, \mathbb{P})$ be relatively hyperbolic and set $M = \partial(G, \mathbb{P})$. If M is connected, has no global cut points and is not homeomorphic to S^1, then we have the following:*

(1) *A point $m \in M^*(2)$ is either in a necklace or a translate inseparable loxodromic cut pair.*

(2) *$M^*(3+)$ consists of equivalence classes of translate inseparable loxodromic cut pairs.*

(3) *A necklace ν in M is homeomorphic to a Cantor set. Moreover, if ν is a Cantor set the jumps are loxodromic cut pairs which are translate inseparable.*

As first observed by Guralnik [27], Lemma 4.1 allows us to apply Bowditch's arguments verbatim when considering only conical limit points to obtain Theorem 4.20(1)–(2). Part (3) also follows from the arguments of Bowditch, by substituting Proposition 4.5 for Bowditch's Lemma 5.17. We refer the reader to arguments of Section 5 of [4] for details.

As an immediate corollary we have:

Corollary 4.21 *Assume $\partial(G,\mathbb{P})$ is connected with no global cut points and not homeomorphic to S^1. If $\partial(G,\mathbb{P})$ has a nonparabolic local cut point, then $\partial(G,\mathbb{P})$ contains a G–translate inseparable loxodromic cut pair.*

We remark that Theorem 4.20(3) is related to the work of Groff (see Proposition 7.2 and the definition of relatively QH in [21]). Also note that cut pairs are not separated by global cut points. Thus a necklace ν will be contained in some block of the form $\partial(H,\mathbb{Q})$. This means we may now invoke the results of Section 3 to remove the hypothesis that $\partial(G,\mathbb{P})$ has global cut points and show:

Theorem 4.22 *Let $(G,\mathbb{P})$ be a relatively hyperbolic group with tame peripherals and assume $\partial(G,\mathbb{P})$ is connected. If $p\in\partial(G,\mathbb{P})$ is a local cut point, then one of the following holds:*

(1) *p is parabolic point.*

(2) *p is contained in a G–translate inseparable loxodromic cut pair.*

(3) *p is in a necklace.*

Proof Let p be a local cut point. By Corollary 3.2, p must be either a parabolic point or a conical limit point contained in a block. If p is parabolic, we are done, so assume that p is a conical limit point. By Theorem 3.1 the block is stabilized by a subgroup H, and H is hyperbolic relative to $\mathbb{Q}$. Theorem 3.1 also implies that $\partial(H,\mathbb{Q})$ is connected and has no global cut points; if $\partial(H,\mathbb{Q})$ is not a circle, we may apply Theorem 4.20 to $\partial(H,\mathbb{Q})$. Thus $\partial(H,\mathbb{Q})$ contains a necklace or an inseparable loxodromic cut pair which contains p. Corollary 3.17 implies that inseparable loxodromic cut pairs and necklaces in $\partial(H,\mathbb{Q})$ correspond to inseparable loxodromic cut pairs and necklaces in $\partial(G,\mathbb{P})$. If $\partial(H,\mathbb{Q})$ is a circle, then $\partial(H,\mathbb{Q})$ is a necklace containing p, and we are done by Corollary 3.17. □

5 Splitting theorem

Throughout this section we will assume that $(G,\mathbb{P})$ is relatively hyperbolic with tame peripherals and that $\partial(G,\mathbb{P})$ is connected. Having developed the appropriate tools in Sections 3 and 4, we now wish to prove Theorem 1.1. We start with a few lemmas.

Lemma 5.1 *Assume that $\partial(G,\mathbb{P})$ is not homeomorphic to a circle. If H is the stabilizer of a block and $\partial(H,\mathbb{Q})$ is homeomorphic to a circle, then there exists a nontrivial peripheral splitting of G over a 2–ended subgroup.*

Proof If $\partial(H,\mathbb{Q})$ is a circle, then Theorem 3.1(4) and a result of Tukia (see [42, Theorem 6B]) imply that H is virtually a surface group, and the peripheral subgroups are the boundary subgroups of that surface. Because $\partial(G,\mathbb{P})$ is not a circle, there is a global cut point p of $\partial(G,\mathbb{P})$ contained in $\partial(H,\mathbb{Q})$ such that $\mathrm{Stab}_H(p)$ is a 2–ended subgroup. As the boundary is connected, Corollary 3.4 implies that G must split over an infinite subgroup of $\mathrm{Stab}_H(p)$. □

Lemma 5.2 *Let $\{a,b\}$ be a translate inseparable cut pair in $\partial(G,\mathbb{P})$ and Q the quotient space obtained by identifying ga to gb for every $g\in G$. Then Q contains a cut point for each pair in $\mathrm{Orb}_G(\{a,b\})$.*

Proof Let $M=\partial(G,\mathbb{P})$ and assume $\{a,b\}$ is a translate inseparable cut pair. As the action of G is by homeomorphisms, $\{ga,gb\}$ is inseparable for every $g\in G$.

Define $q\colon \partial(G,\mathbb{P})\to Q$ to be the quotient map described in the statement of the lemma. Let $\mathcal{C}$ be the collection of components of $\partial(G,\mathbb{P})\setminus\{c,d\}$ for some pair $\{c,d\}$ in $\mathrm{Orb}_G(\{a,b\})$. Because every pair in $\mathrm{Orb}_G(\{a,b\})$ is inseparable, there does not exist a pair $\{x,y\}\in\mathrm{Orb}_G(\{a,b\})$ which meets two elements of $\mathcal{C}$. Thus, if C_1 and C_2 are distinct components of $\partial(\Gamma,\mathbb{P})\setminus\{c,d\}$, we have that $q(C_1)$ and $q(C_2)$ are disjoint connected components of $Q\setminus\{q(c)=q(d)\}$. □

By Corollary 1.7 of [35] we have:

Lemma 5.3 *Let $(G,\mathbb{P})$ be relatively hyperbolic. Assume that g is a loxodromic element contained in a maximal 2–ended subgroup H. By adding H and all of its conjugates to $\mathbb{P}$, we may extend $\mathbb{P}$ to a new peripheral structure $\mathbb{P}'$ such that $(G,\mathbb{P}')$ is relatively hyperbolic.*

Let $(G,\mathbb{P})$ and $(G,\mathbb{P}')$ be as in Lemma 5.3. We say that $(G,\mathbb{P}')$ is the *loxodromic extension of $(G,\mathbb{P})$ by g*.

Lemma 5.4 *Assume $\partial(G,\mathbb{P})$ contains an inseparable loxodromic cut pair $\{a,b\}$ stabilized by a loxodromic element g, and let Q the quotient space obtained by identifying $g'a$ to $g'b$ for every $g'\in G$. If $(G,\mathbb{P}')$ is the loxodromic extension of $(G,\mathbb{P})$ by g, then Q is equivariantly homeomorphic to $\partial(G,\mathbb{P}')$.*

Lemma 5.4 was proved by Dahmani [14] in the case where $\langle g\rangle$ is a maximal 2–ended subgroup and follows from Lemma 4.16 of [48] in the general case.

Lemma 5.5 *Assume $\partial(G,\mathbb{P})$ contains a translate inseparable loxodromic cut pair. Then G splits relative to $\mathbb{P}$ over a two-ended group.*

Proof Assume the hypothesis. Then there is a loxodromic group element $g\in G$ which stabilizes the loxodromic cut pair and is such that $\langle g\rangle$ is contained in a maximal 2–ended subgroup H. Let $(G,\mathbb{P}')$ be the loxodromic extension of $(G,\mathbb{P})$ by g. By Lemmas 5.2 and 5.4 there is a cut point $\partial(G,\mathbb{P}')$ stabilized by H, which by Corollary 3.4 implies that $(G,\mathbb{P}')$ has a nontrivial peripheral splitting over a subgroup of H. As $\partial(G,\mathbb{P}')$ is connected, G does not split over a finite group relative to $\mathbb{P}$. Since every infinite subgroup of H is 2–ended, we are done. □

Lastly, to prove Theorem 1.1 we will use the following lemma, taken from the first paragraph in the proof of Theorem 7.8 of [24]. Lemma 5.6 below is more general than what is stated in [24], but follows directly from Groves and Manning's proof, which we include for completeness. The proof of Lemma 5.6 uses the cusped space for $(G,\mathbb{P})$, and we refer the reader to [23, Section 3] for the construction of the cusped space.

Lemma 5.6 *Let $(G,\mathbb{P})$ be relatively hyperbolic with tame peripherals. Assume that $\partial(\Gamma,\mathbb{P})$ is connected and not homeomorphic to a circle. If G splits over a nonparabolic 2–ended subgroup relative to $\mathbb{P}$, then $\partial(G,\mathbb{P})$ contains a nonparabolic local cut point.*

Proof Assume the hypotheses, and let H be a nonparabolic 2–ended subgroup over which G splits relative to $\mathbb{P}$. Because H is nonparabolic, H quasi-isometrically embeds in the cusped space $X(G,\mathbb{P})$. Since this splitting is relative to $\mathbb{P}$, the cusped space $X(G,\mathbb{P})$ can be realized as a tree of cusped spaces glued together in the pattern of the Bass–Serre tree for the splitting over H. Thus H coarsely separates $X(G,\mathbb{P})$ into at least two components, and the limit set of H is a pair of nonparabolic local cut points which separate $\partial(G,\mathbb{P})$. □

Proof of Theorem 1.1 By Lemma 5.6, if G splits over a nonparabolic 2–ended subgroup relative to $\mathbb{P}$, then $\partial(\Gamma,\mathbb{P})$ contains a nonparabolic local cut point.

Now, assume that $x \in \partial(G,\mathbb{P})$ is a nonparabolic local cut point. By Theorem 4.22 we know that x is contained in either a translate inseparable loxodromic cut pair or a necklace. If x is in a translate inseparable loxodromic cut pair, we are done by Lemma 5.5.

Assume x is in a necklace ν. Then ν is either a circle or it is not. If ν is homeomorphic to S^1, we are done by Lemma 5.1. If ν is not a circle, then ν contains a translate inseparable loxodromic cut pair by Theorem 4.20, and again we are done by Lemma 5.5 and Corollary 3.17. □

6 Ends of generalized Peano continua admitting proper and cocompact group actions

Let G be a group with finite generating set S, and let $\Upsilon(G,S)$ denote the Cayley graph of (G,S). We define $\mathrm{Ends}(G)$ to be $\mathrm{Ends}(X)$ for any generalized Peano continuum X on which G acts properly and cocompactly. The goal of this section is to prove that $\mathrm{Ends}(G)$ is well defined (see Theorem 1.3). In particular, we show that $\mathrm{Ends}(\Upsilon(G,S))$ is homeomorphic to $\mathrm{Ends}(X)$ for all generalized Peano continua X. A special case of this result is well known for groups acting on CW–complexes (see for example [18]). Theorem 1.3 provides a generalization to generalized Peano continua, a class of spaces which need not be CW–complexes. We remark that the techniques used to prove Theorem 1.3 differ from those found in [18].

One consequence of Theorem 1.3 for the boundary $\partial(G,\mathbb{P})$ of a relatively hyperbolic group G is that if the peripherals are one-ended then a parabolic point can be a local cut point if and only if it is a global cut point (see Corollary 6.6). This particular fact will be required for the proof of Theorem 1.2.

Let G be a finitely generated discrete group acting properly and cocompactly on a generalized Peano continuum X. We will use Proposition 2.12 to prove Theorem 1.3.

To begin the proof first construct a proper map $\Phi\colon \Upsilon(G,S) \to X$ from the Cayley graph of G to S in the following way:

Let S be a finite generating set for the group G. Fix a basepoint x_0 in the fundamental domain of the action of G on X, and for every vertex v_g in $\Upsilon(G,S)$ define $\Phi(v_g) = g.x_0$. For every $s \in S \cup S^{-1}$ fix a path p_s in X with $p_s(0) = x_0$ and $p_s(1) = s.x_0$. We will denote by $P(S)$ the collection of paths found in this way, ie $P(S) = \{p_s \mid s \in S\}$. Now, for any edge $e_s \in \Upsilon(G,S)$ with endpoints v_g and v_{gs}, define $\Phi(e_s)$ to be gp_s. Notice that Φ is well defined because gp_s is a path with endpoints gx_0 and gsx_0 for every g and s. Also, note that, by the pasting lemma, Φ is continuous.

Lemma 6.1 *The map $\Phi\colon \Upsilon(G,S) \to X$ is proper for all S.*

Proof Let $A \subseteq X$ be compact. As X is Hausdorff, A is closed, therefore $\Phi^{-1}(A)$ is closed. We show that $\Phi^{-1}(A)$ intersects only finitely many vertices and edges. Assume that $\Phi^{-1}(A)$ meets infinitely vertices. This implies that A contains $g_n x_0$ for infinitely many $g_n \in G$, contradicting properness of the action of G on X.

Now assume that infinitely many edges meet $\Phi^{-1}(A)$. As there are finitely many orbits of edges, there must be infinitely many edges with the same label, say s, meeting $\Phi^{-1}(A)$. Thus we may find an infinite sequence of group elements, $(g_i)_{i=1}^{\infty}$, such that $g_i p_s \cap A \neq \varnothing$ for every i. Set $C = p_s \cup A$; then C is compact and $C \cap g_i C \neq \varnothing$ for every i, again a contradiction. $\square$

Define $\Phi^*\colon \mathrm{Ends}(\Upsilon(G,S)) \to \mathrm{Ends}(X)$ to be the ends map induced by Φ.

Lemma 6.2 *Φ^* is a surjection for all S.*

Proof Let $K \subset X$ be a compact connected set whose G–translates cover X, let $\{C_i\}_{i=1}^{\infty}$ be an exhaustion of X, and let $E = (E_1, E_2, E_3, \dots) \in \mathrm{Ends}(X)$.

Let $x_i \in E_i$ for some i. The translates of K cover X, so there exists some $g_i \in G$ such that $x_i \in g_i K$. As $g_i K$ is compact, there exists some $j \in \mathbb{N}$ such that $gK \subseteq C_j$. Let $x_j \in E_j \subset X \backslash C_j$; as before, there exists some $g_j \in G$ such that $x_j \in g_j K$ and some C_k containing $g_j K$. So we may pass to a subsequence $(E_{i_1}, E_{i_2}, E_{i_3}, \dots)$ of E corresponding to a sequence of distinct group elements $(g_{i_1}, g_{i_2}, g_{i_3}, \dots)$ of G found in the manner just described.

The sequence $(g_{i_1}, g_{i_2}, g_{i_3}, \dots)$ corresponds to an infinite sequence, $(v_{g_{i_j}})_{j=1}^{\infty}$, of distinct vertices in $\Upsilon(G,S)$. Because the map Φ is proper, by compactness of $\Upsilon(G,S) \cup \mathrm{Ends}(\Upsilon(G,S))$ we have that some subsequence $(v_{g_{i_{j_k}}})_{k=1}^{\infty}$ of $(v_{g_{i_j}})_{j=1}^{\infty}$ must converge to an end of $\Upsilon(G,S)$. Using the path-connectedness of E_i we may find a proper ray, r, in $\Upsilon(G,S)$ containing the vertices $(v_{g_{i_{j_k}}})_{k=1}^{\infty}$. The ray r determines an end of $\Upsilon(G,S)$, which, by construction, Φ maps to the end E under Φ^*. Thus Φ^* is surjective. □

To complete the proof of Theorem 1.3 we will need the following well-known result about ends of Cayley graphs [38], which is a special case of Theorem 1.3:

Theorem 6.3 *Assume G is finitely generated, and let S and T be two finite generating sets for G. Then $\mathrm{Ends}(\Upsilon(G,S))$ is homeomorphic to $\mathrm{Ends}(\Upsilon(G,T))$.*

We will also require the following lemma:

Lemma 6.4 *Let G act properly and cocompactly on a generalized Peano continuum X. Then there exists a connected compact set K whose G–translates cover X.*

Proof The proof is similar to that of Lemma 9.6 of [31]. Let C be a compact set whose G–translates cover X. Let $x \in C$. By local compactness we have that x has a compact neighborhood U, and local connectedness implies that the interior $\mathrm{Int}(U)$ contains a connected neighborhood V of x. The closure $\overline{V}$ of V is a compact connected neighborhood of x. As C is compact we may cover C by finitely many such neighborhoods. The union of these neighborhoods K' is compact and consists of finitely many components. As X is arcwise connected, we may attach finitely many arcs to K' to find a compact connected set K such that $C \subset K' \subset K$. □

Proof of Theorem 1.3 By Lemma 6.4 there exists a connected compact set K whose G–translates cover X. We will also assume that K contains the basepoint x_0. Define S to be $\{s \in G \mid K \cap sK \neq \varnothing\}$. It is a standard result that S generates G. By Theorem 6.3, $\mathrm{Ends}(\Upsilon(G,S))$ is independent of the choice of generating set.

Let $\Phi\colon \Upsilon(G,S) \to X$ be the map defined at the beginning of this section with $\Phi(1_G) = x_0$. By Lemma 6.2 we need only show that Φ^* is injective. To do this we will make use of Lemma 2.14.

Let α and β be proper rays in $\Upsilon(G,S)$ and (a_i) and (b_i) the corresponding sequences of vertices. Note that, if necessary, α and β may be homotoped to combinatorial proper rays, so we may assume that no vertex in (a_i) or (b_i) occurs infinitely many times. Assume that $\Phi(\alpha)$ and $\Phi(\beta)$ are in the same ladder equivalence class. Then we may find a proper map of the infinite ladder into X such that $\Phi(\alpha)$ and $\Phi(\beta)$ form the sides; moreover, by concatenating paths if necessary we may assume that the rungs, r_i, of the ladder have endpoints $\Phi(a_i)$ and $\Phi(b_i)$. Call this ladder L. Note that the rungs r_i of L may not pull back to paths in $\Upsilon(G,S)$ under Φ^{-1}. We show that we can find an alternative sequence of rungs ρ_i connecting $\Phi(a_i)$ to $\Phi(b_i)$ and such that each ρ_i pulls back to an edge path in $\Upsilon(G,S)$.

For any rung r_i we may find a finite number of translates of K that cover r_i. Let $\{g_1,g_2,\dots,g_n\}$ be such that $\operatorname{im}(r_i)\subset\bigcup_{j=1}^n g_jK$. Notice that by connectedness of the rung r_i we may assume that $\{g_1,g_2,\dots,g_n\}$ is enumerated in such a way that $g_jK\cap g_{j+1}K\neq\varnothing$. Consequently, the g_jK form a chain of connected compact neighborhoods such that the points g_ix_0 in the translates of K can be connected by paths which are translates of paths in $P(S)$ (see the construction of Φ); in other words, because of the specific choice of generating set they are the images of edges in $\Upsilon(G,S)$. By concatenating paths in $\operatorname{Orb}_G(P(S))$ we may find a path ρ_i which pulls back to an edge path in $\Upsilon(G,S)$ connecting (a_i) and (b_i).

Lastly, we need to check that some subladder of the ladder L pulls back to a ladder in $\Upsilon(G,S)$ under Φ. Let $C\subset\Upsilon(G,S)$ be a compact. We find a ρ_i such that $\Phi^{-1}(\rho_i)$ is in $\Upsilon(G,S)\setminus C$.

Set $C'=\Phi(C)$ and $K'=\left(\bigcup_{s\in S}sK\right)\cup P(S)$. Assume that there does not exist a subsequence of rungs $\{\rho_i\}$ entirely outside of C'. Then we may find a compact set $N=\bigcup_{g\in I}gK'$, where $I=\{g\in G\mid K'\cap gK'\neq\varnothing\}$, such that every rung r_i of L meets N. As the ladder L was proper this is a contradiction. Thus there must exist a ρ_i outside of $\Phi(C)$, which implies that $\Phi^{-1}(\rho_i)\subset\Upsilon(G,S)\setminus C$. Therefore, as C was chosen to be arbitrary, α and β represent the same end of $\Upsilon(G,S)$. □

As an immediate corollary we obtain:

Corollary 6.5 *Let G be a one-ended finitely generated group acting properly and cocompactly on a generalized Peano continuum X. Then X is 1–ended.*

In particular, we have:

Corollary 6.6 *Let $(G,\mathbb{P})$ be relatively hyperbolic with tame peripherals and every $P\in\mathbb{P}$ 1–ended. If p is parabolic point in $\partial(G,\mathbb{P})$ which is not a global cut point, then p cannot be a local cut point.*

Proof Assume the hypotheses and let P be the maximal parabolic subgroup which stabilizes p. Bowditch [11] has shown that P acts properly and cocompactly on $\partial(G,\mathbb{P})\setminus\{p\}$. Because p is not a global cut point, we know that $\partial(G,\mathbb{P})\setminus\{p\}$ is connected. We are assuming that $(G,\mathbb{P})$ has tame peripherals, so $\partial(G,\mathbb{P})$ is locally connected. Thus, $\partial(G,\mathbb{P})\setminus\{p\}$ is an open connected subset of a Peano continuum; consequently, $\partial(G,\mathbb{P})\setminus\{p\}$ is a generalized Peano continuum and we may apply Corollary 6.5. □

7 Classification theorem

In this section we prove Theorem 1.2. This theorem is a generalization of a theorem due to Kapovich and Kleiner [32] concerning the boundaries of hyperbolic groups. Kapovich and Kleiner's proof used the topological characterizations of the Menger curve [1; 2] and Sierpinski carpet [45]. A compact metric space M is a Menger curve provided M is 1–dimensional, M is connected, M is locally connected, M has no local cut points and no nonempty open subset of M is planar. If the last condition is replaced with "M is planar", then we have the topological characterization of the Sierpinski carpet. Having proved Theorems 1.1 and 1.3, the only remaining step in the proof of Theorem 1.2 is the following argument, inspired by Kapovich and Kleiner [32]:

Proof of Theorem 1.2 Assume the hypotheses and assume that $\partial(G,\mathbb{P})$ is not homeomorphic to a circle. Then $\partial(G,\mathbb{P})$ is a compact and 1–dimensional metric space. Because G is one-ended, $\partial(G,\mathbb{P})$ is connected. Since we are assuming $(G,\mathbb{P})$ has tame peripherals, connectedness of $\partial(G,\mathbb{P})$ implies that it must also be locally connected (see Theorem 2.4).

There are two types of local cut points: those that separate $\partial(G,\mathbb{P})$ globally and those that do not. By Theorem 2.7 the "no peripheral splitting" hypothesis implies that $\partial(G,\mathbb{P})$ is without global cut points. Additionally, the peripheral subgroups are assumed to be 1–ended, so by Theorem 1.3 there are no parabolic local cut points. Thus any local cut point must be a conical limit point. If there were a conical limit local cut point, then Theorem 1.1 would imply that G splits over a 2–ended subgroup, a contradiction.

Now, $\partial(G,\mathbb{P})$ is planar or it is not. If it is planar then it is a Sierpinski carpet. Assume $\partial(G,\mathbb{P})$ is not planar; then, by the Claytor embedding theorem [13], it must contain a topological embedding of a nonplanar graph K. It suffices to find a homeomorphic copy of K inside any open neighborhood V in $\partial(G,\mathbb{P})$.

As conical limit points are dense, let x be a conical limit point in $\partial(G,\mathbb{P})\setminus\{K\}$. By definition of conical limit point there exist $a,b\in\partial(G,\mathbb{P})$ and a sequence of group elements $(g_i)\subset G$ such that $g_i x\to a$ and $g_i z\to b\neq a$ for every $z\in\partial(G,\mathbb{P})\setminus\{x\}$. Now, G acts on $\partial(G,\mathbb{P})$ as a convergence group. Thus the sequence (g_i) restricted to $\partial(G,\mathbb{P})\setminus\{x\}$ converges locally uniformly to b, so we may find a homeomorphic copy of K inside any neighborhood U of b.

Let V be any neighborhood in $\partial(G,\mathbb{P})$. The action of G on $\partial(G,\mathbb{P})$ is minimal (see [11]), so there exists some group element g such that $gb\in V$. Let W be a neighborhood of gb inside V and set U from the previous paragraph equal to $g^{-1}(W)$. Then we may find a homeomorphic copy of K inside of V. $\square$

References

[1] **R D Anderson**, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. 67 (1958) 313–324 MR

[2] **R D Anderson**, *One-dimensional continuous curves and a homogeneity theorem*, Ann. of Math. 68 (1958) 1–16 MR

[3] **H-J Baues**, **A Quintero**, *Infinite homotopy theory*, K–Monographs in Mathematics 6, Kluwer, Dordrecht (2001) MR

[4] **B H Bowditch**, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998) 145–186 MR

[5] **B H Bowditch**, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998) 643–667 MR

[6] **B H Bowditch**, *Boundaries of geometrically finite groups*, Math. Z. 230 (1999) 509–527 MR

[7] **B H Bowditch**, *Connectedness properties of limit sets*, Trans. Amer. Math. Soc. 351 (1999) 3673–3686 MR

[8] **B H Bowditch**, *Convergence groups and configuration spaces*, from "Geometric group theory down under" (J Cossey, W D Neumann, M Shapiro, editors), de Gruyter, Berlin (1999) 23–54 MR

[9] **B H Bowditch**, *Treelike structures arising from continua and convergence groups*, Mem. Amer. Math. Soc. 662, Amer. Math. Soc., Providence, RI (1999) MR

[10] **B H Bowditch**, *Peripheral splittings of groups*, Trans. Amer. Math. Soc. 353 (2001) 4057–4082 MR

[11] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250016 MR

[12] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR

[13] **S Claytor**, *Topological immersion of Peanian continua in a spherical surface*, Ann. of Math. 35 (1934) 809–835 MR

[14] **F Dahmani**, *Combination of convergence groups*, Geom. Topol. 7 (2003) 933–963 MR

[15] **F Dahmani**, **V Guirardel**, **P Przytycki**, *Random groups do not split*, Math. Ann. 349 (2011) 657–673 MR

[16] **R J Daverman**, *Decompositions of manifolds*, Pure and Applied Mathematics 124, Academic, Orlando, FL (1986) MR

[17] **H Freudenthal**, *Über die Enden topologischer Räume und Gruppen*, Math. Z. 33 (1931) 692–713 MR

[18] **R Geoghegan**, *Topological methods in group theory*, Graduate Texts in Mathematics 243, Springer (2008) MR

[19] **V Gerasimov**, *Expansive convergence groups are relatively hyperbolic*, Geom. Funct. Anal. 19 (2009) 137–169 MR

[20] **E Ghys**, **P de la Harpe** (editors), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics 83, Birkhäuser, Boston, MA (1990) MR

[21] **B W Groff**, *Quasi-isometries, boundaries and JSJ-decompositions of relatively hyperbolic groups*, J. Topol. Anal. 5 (2013) 451–475 MR

[22] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR

[23] **D Groves**, **J F Manning**, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. 168 (2008) 317–429 MR

[24] **D Groves**, **J F Manning**, *Dehn fillings and elementary splittings*, Trans. Amer. Math. Soc. 370 (2018) 3017–3051 MR

[25] **C R Guilbault**, *Ends, shapes, and boundaries in manifold topology and geometric group theory*, from "Topology and geometric group theory" (M W Davis, J Fowler, J-F Lafont, I J Leary, editors), Springer Proc. Math. Stat. 184, Springer (2016) 45–125 MR

[26] **C R Guilbault**, **M A Moran**, *Proper homotopy types and $\mathcal{Z}$–boundaries of spaces admitting geometric group actions*, Expo. Math. 37 (2019) 292–313 MR

[27] **D P Guralnik**, *Ends of cusp-uniform groups of locally connected continua, I*, Internat. J. Algebra Comput. 15 (2005) 765–798 MR

[28] **M Haulmark**, *Boundary classification and two-ended splittings of groups with isolated flats*, J. Topol. 11 (2018) 645–665 MR

[29] **G C Hruska**, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. 10 (2010) 1807–1856 MR

[30] **G C Hruska**, **B Kleiner**, *Hadamard spaces with isolated flats*, Geom. Topol. 9 (2005) 1501–1538 MR

[31] **G C Hruska**, **K Ruane**, *Connectedness properties and splittings of groups with isolated flats*, preprint (2017) arXiv

[32] **M Kapovich**, **B Kleiner**, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. École Norm. Sup. 33 (2000) 647–669 MR

[33] **M Mihalik**, **S Tschantz**, *Visual decompositions of Coxeter groups*, Groups Geom. Dyn. 3 (2009) 173–198 MR

[34] **R Myers**, *Excellent* 1*–manifolds in compact* 3*–manifolds*, Topology Appl. 49 (1993) 115–127 MR

[35] **D V Osin**, *Elementary subgroups of relatively hyperbolic groups and bounded generation*, Internat. J. Algebra Comput. 16 (2006) 99–118 MR

[36] **P Papasoglu**, **E Swenson**, *From continua to* $\mathbb{R}$*–trees*, Algebr. Geom. Topol. 6 (2006) 1759–1784 MR

[37] **P Papasoglu**, **E Swenson**, *Boundaries and JSJ decompositions of* CAT(0)*–groups*, Geom. Funct. Anal. 19 (2009) 559–590 MR

[38] **P Scott**, **T Wall**, *Topological methods in group theory*, from "Homological group theory" (C T C Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 137–203 MR

[39] **J-P Serre**, *Trees*, Springer (2003) MR

[40] **E L Swenson**, *A cutpoint tree for a continuum*, from "Computational and geometric aspects of modern algebra" (M Atkinson, N Gilbert, J Howie, S Linton, E Robertson, editors), London Math. Soc. Lecture Note Ser. 275, Cambridge Univ. Press (2000) 254–265 MR

[41] **H C Tran**, *Relations between various boundaries of relatively hyperbolic groups*, Internat. J. Algebra Comput. 23 (2013) 1551–1572 MR

[42] **P Tukia**, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988) 1–54 MR

[43] **P Tukia**, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. 23 (1994) 157–187 MR

[44] **P Tukia**, *Conical limit points and uniform convergence groups*, J. Reine Angew. Math. 501 (1998) 71–98 MR

[45] **G T Whyburn**, *Topological characterization of the Sierpiński curve*, Fund. Math. 45 (1958) 320–324 MR

[46] **S Willard**, *General topology*, Addison-Wesley, Reading, MA (1970) MR

[47] **A Yaman**, *A topological characterisation of relatively hyperbolic groups*, J. Reine Angew. Math. 566 (2004) 41–89 MR

[48] **W-y Yang**, *Peripheral structures of relatively hyperbolic groups*, J. Reine Angew. Math. 689 (2014) 101–135 MR

Department of Mathematics, Vanderbilt University
Nashville, TN, United States

m.haulmark@vanderbilt.edu

Received: 18 August 2017 Revised: 3 October 2018