

Commensurability invariance for abelian splittings of right-angled Artin groups, braid groups and loop braid groups

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We prove that if a right-angled Artin group A_Γ is abstractly commensurable to a group splitting nontrivially as an amalgam or HNN extension over \mathbb{Z}^n , then A_Γ must itself split nontrivially over \mathbb{Z}^k for some $k \leq n$. Consequently, if two right-angled Artin groups A_Γ and A_Δ are commensurable and Γ has no separating k -cliques for any $k \leq n$, then neither does Δ , so “smallest size of separating clique” is a commensurability invariant. We also discuss some implications for issues of quasi-isometry. Using similar methods we also prove that for $n \geq 4$ the braid group B_n is not abstractly commensurable to any group that splits nontrivially over a “free group-free” subgroup, and the same holds for $n \geq 3$ for the loop braid group LB_n . Our approach makes heavy use of the Bieri–Neumann–Strebel invariant.

20F65; 20F36, 57M07

Introduction

We say two groups are *abstractly commensurable* or for brevity *commensurable* if they contain isomorphic finite-index subgroups. It has been an ongoing problem to understand commensurability for right-angled Artin groups, or RAAGs for short. This can mean either to understand when a group is commensurable to a given RAAG, or to understand when two RAAGs are commensurable to each other. For instance, a RAAG is commensurable to a nonabelian free group if and only if it itself is a nonabelian free group, and on the other hand \mathbb{Z}^n is not commensurable to any RAAG except itself. Related questions include all of the above replacing “commensurable” with “quasi-isometric” everywhere, and the “rigidity” question asking for which RAAGs does quasi-isometry imply commensurability.

Recall that for a finite simplicial graph Γ , the RAAG A_Γ is defined by the presentation with a generator for each vertex of Γ and the relations that two generators commute if and only if their corresponding vertices span an edge in Γ . A great deal of work has been done toward understanding the above questions for RAAGs A_Γ assuming

various restrictions on Γ . For example, in [14] Huang proved that if A_Γ has finite outer automorphism group, which is equivalent to saying that Γ has no separating closed stars and no instances of $\text{lk } v \subseteq \text{st } w$ for vertices $v \neq w$, then a RAAG A_Δ is commensurable to A_Γ if and only if it is quasi-isometric. Moreover, if A_Γ and A_Δ both have finite outer automorphism group then they are quasi-isometric if and only if $\Gamma \cong \Delta$. Other examples of past work include Huang [13; 15], Casals-Ruiz, Kazachkov and Zakharov [6], Casals-Ruiz [5], Behrstock, Januszkiewicz and Neumann [2] and Kim and Koberda [16; 17]. In all of these examples, results are shown assuming the RAAG or RAAGs in question have defining graphs falling into certain classes. For example, there are results if the graph is a tree, or a join, or an atomic graph, or a cyclic graph, or has some other such global structure.

In this paper we do not focus on any particular graph or class of graphs, but rather inspect the commensurability problem in terms of some more local features of the graph, with an eye on separating cliques. These correspond to nontrivial splittings over free abelian groups. Recall that a *nontrivial splitting* of a group G over a subgroup C is a decomposition $G = A *_C B$ with $G \neq A, B$ or $G = A *_C C$ with $G \neq A$. Our main results are:

Theorem 3.5 *Let Γ be a finite simplicial nonclique graph with no separating k -cliques for any $k \leq n$. Then A_Γ is not commensurable to any group splitting nontrivially over \mathbb{Z}^n .*

Corollary 3.6 *If A_Γ and A_Δ are commensurable and Γ has no separating k -cliques for any $k \leq n$, then neither does Δ .*

An equivalent way to phrase **Theorem 3.5** is to say that such an A_Γ does not virtually split nontrivially over \mathbb{Z}^n . Another equivalent formulation is: if a RAAG virtually splits nontrivially over \mathbb{Z}^n then it must actually split nontrivially over \mathbb{Z}^k for some $k \leq n$.

Corollary 3.6 can be phrased informally as “‘smallest size of separating clique’ is a commensurability invariant for RAAGs”. We suspect that the conclusion of **Theorem 3.5** is true even if we only assume Γ has no separating n -cliques, though proving this would require new ideas (for instance, even in the proof of **Proposition 2.3**, concerning when A_Γ itself splits, we cannot precisely control the size of the cliques that arise).

Say that a group is *NF* if it contains no nonabelian free subgroups (so, colloquially, it is a “free group–free group”). It is a fact that RAAGs satisfy a strong Tits alternative, namely every NF subgroup of a RAAG is abelian; even more strongly, every pair of elements in a RAAG either commute or generate a copy of F_2 ; see Baudisch [1],

Carr [4] and Kim and Koberda [18]. This leads to the following corollary in the case when Γ has no separating cliques at all:

Corollary 3.7 *Let Γ be a finite simplicial nonclique graph with no separating cliques. Then A_Γ is not commensurable to any group splitting nontrivially over an NF subgroup.*

The key to proving [Theorem 3.5](#) is understanding the Bieri–Neumann–Strebel (BNS) invariant well enough to produce nontrivial characters of the groups of interest that contain certain prescribed subgroups in their kernels while still lying in the BNS invariant. The BNS invariant of an arbitrary RAAG is known from work of Meier and VanWyk [22]. There has been some other recent interest in using the BNS invariants of RAAGs to distinguish groups; for instance, Koban and Piggott [20] determined precisely when the pure symmetric automorphism group of a RAAG is itself a RAAG, and Day and Wade [11] used a new homology theory to produce similar results for the “outer” version.

Using the BNS invariant to approach questions of commensurability is a natural endeavor, but to the best of our knowledge it has not been exploited in the literature. We expect that our techniques could be used in the future to get similar commensurability results for other groups whose BNS invariants are known. In the interest of providing other explicit examples, we inspect braid groups and loop braid groups, and use similar methods to those used for RAAGs to get the following results:

Theorem 5.1 *For $n \geq 4$ the braid group B_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

Theorem 5.2 *For $n \geq 3$ the loop braid group LB_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

The BNS invariant of the (loop) braid group is known but turns out not to be useful here, since it is too small (characters tend to become trivial as soon as they kill interesting subgroups). Instead we use the BNS invariants of the pure braid group PB_n and pure loop braid group PLB_n , which are known from work of Koban, McCammond and Meier [19] and Orlandi-Korner [23], and are robust enough to use for these purposes. Another relevant comment here is that Clay, Leininger and Margalit proved that for $n \geq 4$ the group B_n is not commensurable to any RAAG [9].

This paper is organized as follows. In [Section 1](#) we recall the BNS invariant and establish some results about kernels of characters. In [Section 2](#) we discuss RAAGs and their BNS invariants, and refine a result of Groves and Hull [12] about which RAAGs

split over which abelian subgroups. In [Section 3](#) we prove our main commensurability results, [Theorem 3.5](#) and [Corollaries 3.6](#) and [3.7](#), about RAAGs, and in [Section 4](#) we discuss the consequences our results have for questions of quasi-isometry. Finally, in [Section 5](#) we prove related commensurability results, [Theorems 5.1](#) and [5.2](#), about braid groups and loop braid groups.

Acknowledgements Thanks are due to Matt Brin, Matt Clay, Thomas Koberda, Ric Wade and Stefan Witzel for helpful discussions, useful comments, clarification of results and general encouragement. I am also grateful to Jingyin Huang for [Lemma 4.1](#). Finally, I thank the anonymous referee for helpful remarks, including pointing out a more concise way to phrase [Theorem 3.5](#).

1 Characters of a group

A *character* of a group G is a homomorphism $G \rightarrow \mathbb{R}$. In this section we recall the definition of the BNS invariant and establish some facts about the behavior of kernels of characters.

1.1 The BNS invariant

The BNS invariant $\Sigma^1(G)$ of a finitely generated group G is a certain subset of the *character sphere*

$$S(G) := \{[\chi] \mid 0 \neq \chi \in \text{Hom}(G, \mathbb{R})\}$$

of G . Here $[\chi]$ is the equivalence class of the character $\chi \in \text{Hom}(G, \mathbb{R})$ under the equivalence relation given by $\chi \sim \chi'$ whenever $\chi = a\chi'$ for some $a \in \mathbb{R}_{>0}$. The character sphere is thus the “sphere at infinity” for the euclidean vector space $\text{Hom}(G, \mathbb{R})$. The invariant $\Sigma^1(G)$ is the subset of $S(G)$ defined as follows:

Definition 1.1 (BNS invariant) Let G be a finitely generated group and let $\text{Cay}(G)$ be its Cayley graph with respect to some finite generating set. For $0 \neq \chi \in \text{Hom}(G, \mathbb{R})$ let $\text{Cay}(G)^{\chi \geq 0}$ be the induced subgraph of $\text{Cay}(G)$ supported on those vertices g with $\chi(g) \geq 0$. The *BNS invariant* $\Sigma^1(G)$ is defined to be

$$\Sigma^1(G) := \{[\chi] \in S(G) \mid \text{Cay}(G)^{\chi \geq 0} \text{ is connected}\}.$$

Denote by $\Sigma^1(G)^c$ the complement $S(G) \setminus \Sigma^1(G)$. For various reasons it will be convenient to adopt the convention that the trivial character 0 lies in $\Sigma^1(G)^c$ (but note that this runs counter to the definition).

In general the BNS invariant can be very difficult to compute. It contains a huge amount of information; for example, it reveals exactly which (normal) subgroups $N \leq G$ containing $[G, G]$ are finitely generated or not, namely N is finitely generated if and only if $[\chi] \in \Sigma^1(G)$ for all $0 \neq \chi$ such that $\chi(N) = 0$.

Even if $\Sigma^1(G)$ is completely known, it can still be very difficult to compute $\Sigma^1(H)$ for H a finite-index subgroup of G . There is a region of $S(H)$ that can be understood based just on knowing $\Sigma^1(G)$, namely the region given by characters of H that are restrictions of characters of G :

Citation 1.2 [24, Proposition B1.11] *Let G be a finitely generated group and H a finite-index subgroup. Let $\chi \in \text{Hom}(G, \mathbb{R})$ and consider the restriction $\chi|_H \in \text{Hom}(H, \mathbb{R})$ of χ to H . We have that $[\chi|_H] \in \Sigma^1(H)$ if and only if $[\chi] \in \Sigma^1(G)$.*

1.2 Kernels of characters

In this subsection we find a way to control which generators of a group must lie in the kernel of a character, given the knowledge that some prescribed subgroup lies in the kernel. The main result is [Proposition 1.4](#).

Fix a finitely generated group G . Let V denote the \mathbb{R} -vector space $(G/[G, G]) \otimes \mathbb{R}$. Let $\phi: G \rightarrow V$ be the “euclideanization” map obtained by composing the abelianization map $G \rightarrow G/[G, G]$ with the map $G/[G, G] \rightarrow (G/[G, G]) \otimes \mathbb{R}$.

Definition 1.3 Given a subset A of G , define the *radical* \sqrt{A} of A to be the set $\{g \in G \mid g^q \in A \text{ for some } q \in \mathbb{Z} \setminus \{0\}\}$. Note that $A \subseteq \sqrt{A} \subseteq G$, and if A is a subgroup of G containing $[G, G]$ then \sqrt{A} is a subgroup of G .

For $J \leq G$, if a character $\chi \in \text{Hom}(G, \mathbb{R})$ contains J in its kernel then it necessarily contains $\sqrt{J[G, G]}$. This next proposition says, first, that χ does not necessarily contain more than this, and, second, that under an addition restriction on G (which will be satisfied by our future groups of interest), the number of generators of J controls the number of generators of G that can lie in $\ker(\chi)$.

Proposition 1.4 (kill J and little else) *Let G be a finitely generated group, and let $J \leq G$. Then there exists $\chi \in \text{Hom}(G, \mathbb{R})$ with $\ker(\chi) = \sqrt{J[G, G]}$. Moreover, if G admits a finite generating set S such that $\dim_{\mathbb{R}}(V) = |S|$, and J is generated by n elements, then at most n elements of S lie in $\ker(\chi)$.*

Proof The quotient $G/\sqrt{J[G, G]}$ is a finitely generated torsion-free abelian group (ie a free abelian group), and hence can be embedded in \mathbb{R} . Composing this embedding

with $G \rightarrow G/\sqrt{J[G, G]}$ yields a character $\chi \in \text{Hom}(G, \mathbb{R})$ with $\ker(\chi) = \sqrt{J[G, G]}$. Now suppose G admits a finite generating set S such that $\dim_{\mathbb{R}}(V) = |S|$, and J is generated by n elements j_1, \dots, j_n . We claim that the image of $\sqrt{J[G, G]}$ in V spans a subspace W of dimension at most n . It suffices to prove that every element of $\sqrt{J[G, G]}$ maps under ϕ to a vector of V in the span of the $\phi(j_i)$. Let $g \in \sqrt{J[G, G]}$, say $g^q = jc$ for $q \neq 0$, $j \in J$ and $c \in [G, G]$. Then $\phi(g) = \frac{1}{q}\phi(g^q) = \frac{1}{q}\phi(jc) = \frac{1}{q}\phi(j)$, which indeed lies in the span of the $\phi(j_i)$. Now, since $\dim_{\mathbb{R}}(V) = |S|$ and $\phi(S)$ spans V , we must have that ϕ is injective on S and $\phi(S)$ is also linearly independent. Hence, at most n elements of S can map into W , and hence at most n elements of S can lie in $\sqrt{J[G, G]} = \ker(\chi)$. \square

2 Right-angled Artin groups

A *right-angled Artin group* or *RAAG* is a group admitting a finite presentation in which each relator is a commutator of two generators. Given a finite simplicial graph Γ , with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, we get a RAAG, denoted by A_Γ , by taking a generator for each vertex and declaring that two vertices commute if and only if they share an edge. For example, if $E(\Gamma) = \emptyset$ then $A_\Gamma \cong F_{|V(\Gamma)|}$, the free group on $|V(\Gamma)|$ generators, and if Γ is a *clique*, ie a graph where every pair of vertices spans an edge, then $A_\Gamma \cong \mathbb{Z}^{|V(\Gamma)|}$.

The BNS invariants of RAAGs were fully computed by Meier and VanWyk [22]. We recall the computation here.

Definition 2.1 (living/dead subgraph) Given a character $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$, define the χ -*living subgraph* Γ_χ^* to be the induced subgraph of Γ supported on those vertices v with $\chi(v) \neq 0$, and the χ -*dead subgraph* Γ_χ^\dagger to be the induced subgraph of Γ supported on those vertices v with $\chi(v) = 0$.

Citation 2.2 (BNS of RAAG [22]) $[\chi] \in \Sigma^1(A_\Gamma)$ if and only if the χ -*living subgraph* Γ_χ^* is connected and dominating in Γ .

Here a subgraph Δ of Γ is called *dominating* (in Γ) if every vertex of $\Gamma \setminus \Delta$ is adjacent to a vertex of Δ .

In [12], Groves and Hull proved that the only way a nonabelian RAAG can split nontrivially over an abelian subgroup is if its defining graph admits a (possibly empty) separating clique. Recall that a subgraph Δ of Γ is called *separating* (for Γ) if $\Gamma \setminus \Delta$ is disconnected.

We now inspect the details of Groves and Hull’s proof of their Theorem A to get the following refined result:

Proposition 2.3 (splittings and cliques) *Let Γ be a finite simplicial graph that is not a clique. The minimal $n \geq 0$ such that A_Γ splits nontrivially over \mathbb{Z}^n equals the minimal $n \geq 0$ such that Γ admits a separating n -clique, with n taken to be ∞ whenever such splittings or cliques do not exist.*

To clarify, by n -clique we mean a clique with n vertices, ie the 1-skeleton of an $(n-1)$ -simplex.

Proof of Proposition 2.3 The $n = \infty$ case is immediate from [12, Theorem A], so assume $n < \infty$. Note that if Γ has a separating n -clique then A_Γ splits nontrivially over \mathbb{Z}^n , so the thing to prove is that if A_Γ splits nontrivially over \mathbb{Z}^n then Γ admits a separating k -clique for some $k \leq n$. The splitting gives us an action of A_Γ on a tree T with edge stabilizers isomorphic to \mathbb{Z}^n , no global fixed points and no edge inversions, and we will inspect this action using the proof of Theorem A in [12] as an outline.

First suppose some $v \in V(\Gamma)$ acts hyperbolically on T . Let e be any edge of the axis of v in T , so $\text{Stab}_{A_\Gamma}(e) \cong \mathbb{Z}^n$. Let u be a vertex in $\text{lk}_\Gamma v$, so u stabilizes the axis of v in T . Hence, there exist $n_u, m_u \in \mathbb{Z}$ with $n_u \neq 0$ such that $u^{n_u} v^{m_u}$ fixes this axis pointwise, and in particular $u^{n_u} v^{m_u} \in \text{Stab}_{A_\Gamma}(e)$. Since this holds for every $u \in \text{lk}_\Gamma v$, and since $\text{Stab}_{A_\Gamma}(e)$ is abelian, we conclude that $[u^{n_u}, w^{n_w}] = 1$ for any $u, w \in \text{lk}_\Gamma v$, which implies that $\text{lk}_\Gamma v$ is a clique (this conclusion is also in [12]), and, even more precisely, since $\text{Stab}_{A_\Gamma}(e) \cong \mathbb{Z}^n$ we conclude that $\text{lk}_\Gamma v$ is a k -clique for some $k \leq n$. Since $\text{lk}_\Gamma v$ separates v from $\Gamma \setminus \text{st}_\Gamma v$ (and the latter is nonempty since Γ is not a clique but $\text{st}_\Gamma v$ is), we have our separating k -clique.

Now assume that every $v \in V(\Gamma)$ acts elliptically on T . Groves and Hull define a map $F: \Gamma \rightarrow T$ that in particular takes each $v \in V(\Gamma)$ to some point of T that it fixes. There is a special point p , at the midpoint of an edge, that is the image under F of every v fixing it. Since the action does not invert edges, all these v even fix the edge containing p . As Groves and Hull show, $F^{-1}(p)$ is a separating clique in Γ , but even more precisely it is a separating k -clique for some $k \leq n$, since the edge stabilizer is isomorphic to \mathbb{Z}^n . □

As a remark, the reason to exclude the case when Γ is a clique is that while cliques have no separating cliques, technically \mathbb{Z}^n does split nontrivially over \mathbb{Z}^{n-1} , as the HNN extension $\mathbb{Z}^n = \mathbb{Z}^{n-1} *_t$ where the stable element t conjugates \mathbb{Z}^{n-1} to itself

via the identity map. Another remark is that the $n = 1$ case was previously proved by Clay [8], and Groves and Hull remarked in [12, Remark 0.1] that their approach could recover Clay's result. Finally, to reiterate a point made in the introduction, the reason that the statement of the proposition is only about the minimal n is that in the proof we cannot control the size of the cliques produced, only an upper bound. However, we suspect that the stronger statement “ A_Γ splits nontrivially over \mathbb{Z}^n if and only if Γ admits a separating n -clique” is true.

3 Commensurability results for RAAGs

In this section we prove our main results about RAAGs, [Theorem 3.5](#) and [Corollaries 3.6](#) and [3.7](#). We first prove a proposition about general finitely generated groups that shows, outside a trivial case, that if a group G is commensurable to a group G' that splits over a subgroup L , then G contains a copy of a finite-index subgroup of L that cannot be killed by any pair of opposite characters $\pm\chi$ in the BNS invariant of G .

Proposition 3.1 *Let L be a group and let G be a finitely generated group that is not virtually of the form $K \rtimes \mathbb{Z}$ for any finite-index subgroup K of L . Suppose G is commensurable to a group G' that splits nontrivially over L . Then there exists $K \leq G$, with K isomorphic to a finite-index subgroup of L , such that for any $\chi \in \text{Hom}(G, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(G)^c$.*

Proof Let H be a finite-index subgroup of G that embeds with finite index into G' . We will abuse notation and write H also for the finite-index image of H in G' . Since G' splits nontrivially over L , we know H decomposes as the fundamental group of a finite reduced graph of groups \mathcal{G} whose edge groups are H intersected with conjugates of L in G' . Since H has finite index in G' , these edge groups are all isomorphic to finite-index subgroups of L . Let $K \leq H$ be one of these edge groups; for example, just take $K := H \cap L$. First suppose \mathcal{G} is a strictly ascending HNN extension, say $H = K *_t$. Then, for any $\psi \in \text{Hom}(H, \mathbb{R})$ such that $\psi(K) = 0$, if moreover $\psi(t) = 0$ then $\psi = 0$ and $[\pm\psi] \in \Sigma^1(H)^c$ by our convention. If $\psi(K) = 0$ and $\psi(t) \neq 0$ then either $[\psi]$ or $[-\psi]$ lies in $\Sigma^1(G)^c$ (see for instance [3, Theorem 2.1]). Next suppose \mathcal{G} is an ascending HNN extension that is not strict, ie $H \cong K \rtimes \mathbb{Z}$. Then G is virtually of the form $K \rtimes \mathbb{Z}$, which we ruled out. Finally suppose \mathcal{G} is not an ascending HNN extension. Then [7, Proposition 2.5] says that for any $\psi \in \text{Hom}(H, \mathbb{R})$, if $\psi(K) = 0$ then $[\psi] \in \Sigma^1(H)^c$. In any case, for any $\chi \in \text{Hom}(G, \mathbb{R})$ with $\chi(K) = 0$, at least one of $[\pm\chi|_H] \in \Sigma^1(H)^c$, so by [Citation 1.2](#) also at least one of $[\pm\chi] \in \Sigma^1(G)^c$. \square

Now we specialize to RAAGs.

Lemma 3.2 *Let Γ be a finite simplicial graph and let K be an abelian subgroup of A_Γ . Let $\Delta_K \subseteq \Gamma$ be the induced subgraph supported on those vertices v such that $v \in \sqrt{K[A_\Gamma, A_\Gamma]}$. Then Δ_K is a clique.*

Proof Suppose v and w are distinct vertices in Δ_K , say with $v^q c, w^r d \in K$ for $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [A_\Gamma, A_\Gamma]$. Since K is abelian, $v^q c$ and $w^r d$ commute. Now suppose v and w are not adjacent, so there is a retract $\pi: A_\Gamma \rightarrow F_2 = \langle v, w \rangle$. We have that $\pi(v^q c) = v^q \pi(c)$ and $\pi(w^r d) = w^r \pi(d)$ commute in F_2 . Since neither is trivial, this means that $(v^q \pi(c))^a = (w^r \pi(d))^b$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. Abelianizing F_2 to $\mathbb{Z}^2 = \langle \bar{v}, \bar{w} \rangle$, this produces $qa\bar{v} = rb\bar{w}$, with $qa, rb \neq 0$, which is absurd. \square

Corollary 3.3 *Let Γ be a finite simplicial graph and let $K \leq A_\Gamma$ with $K \cong \mathbb{Z}^n$. Then there exists $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ and the χ -dead subgraph Γ_χ^\dagger is a k -clique for some $0 \leq k \leq n$.*

Proof Choose χ as in Proposition 1.4 with $G = A_\Gamma$, $S = V(\Gamma)$ and $J = K$. Then $\chi(K) = 0$ and $v \in V(\Gamma)$ satisfies $v \in \Gamma_\chi^\dagger$ if and only if $v \in \sqrt{K[A_\Gamma, A_\Gamma]}$. Since the abelianization of A_Γ is $\mathbb{Z}^{|V(\Gamma)|}$, Proposition 1.4 also says that at most n vertices satisfy this, and Lemma 3.2 says they must span a clique. \square

Proposition 3.1 applied to the $L = \mathbb{Z}^n$ case said that a RAAG commensurable to a group splitting over \mathbb{Z}^n contains a copy of \mathbb{Z}^n that cannot be killed by a pair of opposite characters $\pm\chi$ in the BNS invariant. This next proposition says that for a RAAG that does not obviously split over \mathbb{Z}^n , any copy of \mathbb{Z}^n can be killed by a pair of opposite characters $\pm\chi$ in the BNS invariant.

Proposition 3.4 *Let Γ be a finite simplicial graph with no separating k -cliques for any $k \leq n$. Then, for any proper subgroup $K \cong \mathbb{Z}^n$ of A_Γ , there exists a character $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ but $[\pm\chi] \in \Sigma^1(A_\Gamma)$.*

Proof Choose χ as in Corollary 3.3, so $\chi(K) = 0$ and Γ_χ^\dagger is a k -clique for some $0 \leq k \leq n$. If Γ is a clique, then since K is a proper subgroup of A_Γ we know Γ_χ^\dagger is not all of Γ , so in this case Γ_χ^* is connected and dominating. Now assume Γ is not a clique. Since Γ has no separating k -cliques, Γ_χ^* is connected. Also, it must be dominating since if $\text{st}_\Gamma(v)$ lies in Γ_χ^\dagger then $\text{st}_\Gamma(v)$ is an ℓ -clique for some $\ell \leq k$, and

since Γ is not a clique this means $\text{lk}_\Gamma(v)$ is a separating $(\ell-1)$ -clique, which we have ruled out. In either case [Citation 2.2](#) says $[\chi] \in \Sigma^1(A_\Gamma)$. Since $\Gamma_\chi^* = \Gamma_{-\chi}^*$, we also have $[-\chi] \in \Sigma^1(A_\Gamma)$. □

Now we can prove our main results.

Theorem 3.5 *Let Γ be a finite simplicial nonclique graph with no separating k -cliques for any $k \leq n$. Then A_Γ is not commensurable to any group splitting nontrivially over \mathbb{Z}^n .*

Proof Suppose A_Γ is commensurable to a group splitting nontrivially over \mathbb{Z}^n . By [Proposition 3.1](#) using $L = \mathbb{Z}^n$ (which applies since A_Γ contains F_2 and hence cannot be virtually of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$) there exists a subgroup $K \cong \mathbb{Z}^n$ of A_Γ such that for any $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(A_\Gamma)^c$ (in fact both do since $\Sigma^1(A_\Gamma)$ happens to be closed under inverting characters). But by [Proposition 3.4](#) we know that there exists a character $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ but $[\pm\chi] \in \Sigma^1(A_\Gamma)$, a contradiction. □

We immediately get the following commensurability invariant for RAAGs:

Corollary 3.6 *If A_Γ and A_Δ are commensurable and Γ has no separating k -cliques for any $k \leq n$, then neither does Δ .*

Proof If Γ is itself a clique then we must have $\Gamma = \Delta$. If Γ is not a clique then the result is immediate from [Proposition 2.3](#) and [Theorem 3.5](#). □

We also get the following corollary in the special case where Γ has no separating cliques at all. Recall from the introduction that any NF subgroup (that is, one containing no nonabelian free subgroups) of a RAAG is abelian.

Corollary 3.7 *Let Γ be a finite simplicial nonclique graph with no separating cliques. Then A_Γ is not commensurable to any group splitting nontrivially over an NF subgroup.*

Proof Suppose A_Γ is commensurable to a group that splits nontrivially over an NF subgroup. By [Proposition 3.1](#), which applies since A_Γ is not (virtually) NF, there exists an NF subgroup $K \leq A_\Gamma$ such that for any $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(A_\Gamma)^c$. Since NF subgroups of RAAGs are abelian, in fact K is abelian, so by [Proposition 1.4](#) and [Lemma 3.2](#) we can choose $\chi \in \text{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ and Γ_χ^\dagger is a clique. Since Γ has no separating cliques, this implies $[\pm\chi] \in \Sigma^1(A_\Gamma)$, as explained in the proof of [Proposition 3.4](#), a contradiction. □

As a remark, if A_Γ is commensurable to a group splitting nontrivially over an NF subgroup generated by n elements, then in general we cannot control the number of generators of the subgroup K described in the proof, and hence cannot control the size of the clique Γ_χ^\dagger . Of course, if the NF subgroup is \mathbb{Z}^n then K is also \mathbb{Z}^n , since finite-index subgroups of \mathbb{Z}^n are isomorphic to \mathbb{Z}^n (which is why [Theorem 3.5](#) worked), but in general we do not get a statement like [Corollary 3.7](#) if we merely rule out separating cliques up to some size; we really need to rule out all separating cliques.

4 Quasi-isometry results for RAAGs

This brief section amounts to a collection of examples of results about quasi-isometry, which follow immediately from our results about commensurability together with results by Huang [[14](#); [15](#); [13](#)] tying commensurability to quasi-isometry.

First we need one technical lemma, the proof of which is essentially due to Jingyin Huang.

Lemma 4.1 *Let Γ be a finite simplicial graph. Suppose $\text{Out}(A_\Gamma)$ is finite. Then Γ has no separating cliques.*

Proof (Jingyin Huang) Since $\text{Out}(A_\Gamma)$ is finite we know Γ has no separating closed stars, and no instances of $\text{lk } v \subseteq \text{st } w$ for vertices $v \neq w$. Now suppose Γ has a separating clique K , say the connected components of its complement are C_1, \dots, C_k , so $k \geq 2$. If $K = \emptyset$ (ie it is a 0-clique) then Γ is disconnected and has infinite outer automorphism group, so we know $K \neq \emptyset$. Pick a vertex $v \in K$, so $K \subseteq \text{st } v$. Since $\text{st } v$ is not separating, at most one of the $C_i \setminus \text{st } v$ can be nonempty. Since $k \geq 2$ this means at least one of the $C_i \setminus \text{st } v$ must be empty, say without loss of generality $C_1 \setminus \text{st } v = \emptyset$, ie $C_1 \subseteq \text{st } v$. But now for any vertex w in C_1 , we have $\text{lk } w \subseteq C_1 \cup K \subseteq \text{st } v$, a contradiction. \square

Corollary 4.2 *Suppose A_Γ and A_Δ are quasi-isometric, and that $\text{Out}(A_\Gamma)$ is finite, so by [Lemma 4.1](#) we know Γ has no separating cliques. Then Δ also has no separating cliques.*

Proof This follows from [[14](#), Theorem 1.2] and [Corollary 3.6](#). \square

Corollary 4.3 *Suppose A_Γ and A_Δ are quasi-isometric and Γ is of weak type I or type II as defined in [13]. Then, if Γ has no separating k -cliques for any $k \leq n$, neither does Δ .*

Proof This follows from [13, Theorems 1.3 and 1.6] and Corollary 3.6. □

Corollary 4.4 *Let G be a finitely generated group quasi-isometric to A_Γ . Suppose that every automorphism of Γ fixing a closed star of a vertex pointwise fixes all of Γ , that Γ contains no induced 4-cycles and that $\text{Out}(A_\Gamma)$ is finite. Then G does not split nontrivially over \mathbb{Z}^n for any n .*

Proof Since $\text{Out}(A_\Gamma)$ is finite, Γ has no separating cliques by Lemma 4.1. The result now follows from [15, Theorem 1.2] and Theorem 3.5. □

In general, we would get similar sorts of results anytime there is a graph Γ for which quasi-isometry to A_Γ implies commensurability to A_Γ .

5 Commensurability results for (loop) braid groups

In this section we apply our approach to braid groups and loop braid groups.

5.1 Commensurability results for braid groups

The n -strand braid group is the group presented by

$$B_n = \left\langle s_1, \dots, s_{n-1} \mid \begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } 1 \leq i \leq n-2 \\ \text{and } s_i s_j = s_j s_i \text{ for all } |i-j| > 1 \end{array} \right\rangle.$$

There is a projection $B_n \rightarrow S_n$ given by adding the relations $s_i^2 = 1$ for all i , and the kernel of this map is the n -strand pure braid group PB_n .

We will work with a specific generating set of PB_n , namely $S = \{S_{i,j} \mid 1 \leq i < j \leq n\}$, where $S_{i,j} := s_i s_{i+1} \cdots s_{j-2} s_{j-1}^2 s_{j-2}^{-1} \cdots s_{i+1}^{-1} s_i^{-1}$. Visually, in $S_{i,j}$ the i^{th} strand crosses in front of all the strands between it and the j^{th} strand, spins around the j^{th} strand, and returns to where it came from, again crossing in front of the intermediate strands. An important fact we will use is that $\text{PB}_3 \cong F_2 \times \mathbb{Z}$, with $S_{1,2}$ and $S_{1,3}$ serving as generators of the F_2 factor. We will also make use of the standard projections $\text{PB}_n \rightarrow \text{PB}_m$ for $m < n$, obtained by deleting some collection of $n - m$ strands.

The BNS invariant $\Sigma^1(\text{PB}_n)$ was computed by Koban, McCammond and Meier [19]. We recall the computation here. If $n \geq 5$, then $[\chi] \in \Sigma^1(\text{PB}_n)^c$ if and only if $\chi = \chi' \circ \pi$ for π one of the standard projections $\text{PB}_n \rightarrow \text{PB}_4$ or $\text{PB}_n \rightarrow \text{PB}_3$ given by deleting strands, and $[\chi'] \in \Sigma^1(\text{PB}_3)^c$ or $\Sigma^1(\text{PB}_4)^c$. In particular, to understand $\Sigma^1(\text{PB}_n)^c$ we need only understand $\Sigma^1(\text{PB}_3)^c$ and $\Sigma^1(\text{PB}_4)^c$. For PB_3 , we have $[\chi] \in \Sigma^1(\text{PB}_3)^c$ if and only if $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{2,3}) = 0$. For PB_4 we have $[\chi] \in \Sigma^1(\text{PB}_4)^c$ if and only if either $\chi = \chi' \circ \pi$ for $[\chi'] \in \Sigma^1(\text{PB}_3)^c$ and $\pi: \text{PB}_4 \rightarrow \text{PB}_3$ one of the standard projections, or else χ satisfies the equations $\chi(S_{1,2}) = \chi(S_{3,4})$, $\chi(S_{1,3}) = \chi(S_{2,4})$, $\chi(S_{1,4}) = \chi(S_{2,3})$ and $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{1,4}) = 0$. Note that these characterizations imply that, for any χ , $[\chi] \in \Sigma^1(\text{PB}_n)$ if and only if $[-\chi] \in \Sigma^1(\text{PB}_n)$.

We now use the ideas from the previous sections to prove the following:

Theorem 5.1 *For $n \geq 4$ the braid group B_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

Note that $\text{PB}_3 \cong F_2 \times \mathbb{Z} = \mathbb{Z}^2 *_\mathbb{Z} \mathbb{Z}^2$ and \mathbb{Z} is NF, so the $n \geq 4$ restriction in the theorem is necessary. Also, the NF condition is obviously necessary, since for instance $B_n \cong [B_n, B_n] \rtimes \mathbb{Z}$ is a nontrivial HNN extension.

Proof of Theorem 5.1 We will work with the pure braid group PB_n , which is commensurable to B_n (being a finite-index subgroup). Suppose PB_n is commensurable to a group that splits nontrivially over an NF subgroup. Since PB_n is not NF, Proposition 3.1 implies that PB_n admits an NF subgroup K such that for any $\chi \in \text{Hom}(\text{PB}_n, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(\text{PB}_n)^c$, which means $[\chi] \in \Sigma^1(\text{PB}_n)^c$. By Proposition 1.4, there exists $\chi \in \text{Hom}(\text{PB}_n, \mathbb{R})$ with $\ker(\chi) = \sqrt{K[\text{PB}_n, \text{PB}_n]}$. Since $\chi(K) = 0$ we know $[\chi] \in \Sigma^1(\text{PB}_n)^c$, which implies that either $n = 4$ or else χ is induced from some standard projection onto PB_3 or PB_4 .

In particular, if $n \geq 5$ then there exists j such that $\chi(S_{i,j}) = \chi(S_{j,k}) = 0$ for any $i < j$ or $j < k$ (just choose j to be the label of a strand getting deleted), which implies that $S_{i,j}, S_{j,k} \in \sqrt{K[\text{PB}_n, \text{PB}_n]}$ for any such i or k . Up to automorphisms (note that the BNS invariant is invariant under automorphisms) we can assume $j = 1$, so in particular $S_{1,2}, S_{1,3} \in \sqrt{K[\text{PB}_n, \text{PB}_n]}$. Choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\text{PB}_n, \text{PB}_n]$ such that $S_{1,2}^q c, S_{1,3}^r d \in K$, which, since K is NF, implies that $S_{1,2}^q c$ and $S_{1,3}^r d$ do not generate a copy of F_2 . Now consider the standard projection $\pi: \text{PB}_n \rightarrow \text{PB}_3$ given by deleting all but the first three strands. Then $S_{1,2}^q \pi(c)$ and $S_{1,3}^r \pi(d)$ do not generate a copy of F_2 in PB_3 , and so neither do their images in $\text{PB}_3/Z(\text{PB}_3) \cong F_2$.

Hence, these images commute,¹ and so modulo $Z(\text{PB}_3)$, $S_{1,2}^q \pi(c)$ and $S_{1,3}^r \pi(d)$ have a common power, say $(S_{1,2}^q \pi(c))^a = (S_{1,3}^r \pi(d))^b z$ for $a, b \in \mathbb{Z}$ and $z \in Z(\text{PB}_3)$. But modding out $Z(\text{PB}_3)$ and abelianizing F_2 to $\mathbb{Z}^2 = \langle \bar{S}_{1,2}, \bar{S}_{1,3} \rangle$, this implies that $qa\bar{S}_{1,2} = rb\bar{S}_{1,3}$, which is absurd.

Now suppose $n = 4$. If χ is induced from a standard projection $\text{PB}_4 \rightarrow \text{PB}_3$ then we can use the above argument to get our contradiction, so suppose it is not. Hence, we have $\chi(S_{1,2}) = \chi(S_{3,4})$, $\chi(S_{1,3}) = \chi(S_{2,4})$, $\chi(S_{1,4}) = \chi(S_{2,3})$ and $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{1,4}) = 0$. In particular,

$$S_{1,2}S_{3,4}^{-1}, S_{1,3}S_{2,4}^{-1} \in \ker(\chi) = \sqrt{K[\text{PB}_4, \text{PB}_4]},$$

so we can choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\text{PB}_4, \text{PB}_4]$ such that $(S_{1,2}S_{3,4}^{-1})^q c$ and $(S_{1,3}S_{2,4}^{-1})^r d$ lie in K , and hence do not generate a copy of F_2 . Their images under the standard projection $\pi: \text{PB}_4 \rightarrow \text{PB}_3$ given by deleting all but the first three strands also do not generate a copy of F_2 , so $S_{1,2}^q \pi(c)$ and $S_{1,3}^r \pi(d)$ do not generate a copy of F_2 in PB_3 . We are now in the same situation as in the proof of the $n \geq 5$ case, and as in that proof we reach a contradiction. \square

As a remark, it would not have worked to try and apply this technique to B_n itself, so working with PB_n really was necessary. Indeed, every $[\chi] \in \Sigma^1(B_n)$ satisfies $\ker(\chi) = [B_n, B_n]$, so it is impossible to find such a χ with an arbitrary NF subgroup lying in its kernel.

5.2 Commensurability results for loop braid groups

Much of this subsection proceeds very similarly to [Section 5.1](#).

The *loop braid group* LB_n on n loops is the group of symmetric automorphisms of the free group F_n . Fixing a free generating set $\{x_1, \dots, x_n\}$ for F_n , an automorphism $\alpha \in \text{Aut}(F_n)$ is called *symmetric* if it takes each x_i to a conjugate of some x_j or x_j^{-1} . Sometimes the word symmetric is reserved for those α taking each x_i to a conjugate of some x_j , not allowing x_j^{-1} ; this produces a finite-index subgroup of what we are calling LB_n . This terminological ambiguity will not matter here, since we will actually work with the *pure loop braid group* PLB_n , the group of automorphisms $\alpha \in \text{Aut}(F_n)$ taking each x_i to a conjugate of x_i , which is again a finite-index subgroup of LB_n .

¹Actually $S_{1,2}^q c$ and $S_{1,3}^r d$ already commute in PB_n by [21], but we have to pass to F_2 anyway, so it is not necessary to appeal to the result from [21].

The name loop braid group comes from viewing such automorphisms as pictures of n loops in 3-space moving around and through each other. See [10] for a great deal of background and more details.

The BNS invariant $\Sigma^1(\text{PLB}_n)$ was computed by Orlandi-Korner [23]. We recall here some of her setup. First, PLB_n is generated by $\{\alpha_{i,j} \mid i \neq j\}$, where $\alpha_{i,j}$ is the automorphism of F_n taking x_i to $x_j^{-1}x_ix_j$ and x_k to itself for $k \neq i$. For $m < n$ a standard projection $\text{PLB}_n \rightarrow \text{PLB}_m$ is a map induced from some projection $F_n \rightarrow F_m$ given by sending some choice of $n - m$ generators to the identity and sending the remaining m generators to the generators of F_m . Now $\Sigma^1(\text{PLB}_n)$ is described as follows: For $n \geq 4$, $[\chi] \in \Sigma^1(\text{PLB}_n)^c$ if and only if $\chi = \chi' \circ \pi$ for π a standard projection $\text{PLB}_n \rightarrow \text{PLB}_2$ or $\text{PLB}_n \rightarrow \text{PLB}_3$ and $[\chi']$ in $\Sigma^1(\text{PLB}_2)^c$ or $\Sigma^1(\text{PLB}_3)^c$. For $n = 3$ we have that $[\chi] \in \Sigma^1(\text{PLB}_3)^c$ if and only if it is induced from a standard projection to PLB_2 or else $\chi(\alpha_{2,1}) + \chi(\alpha_{3,1}) = 0$, $\chi(\alpha_{1,2}) + \chi(\alpha_{3,2}) = 0$ and $\chi(\alpha_{1,3}) + \chi(\alpha_{2,3}) = 0$. For $n = 2$ we have $\Sigma^1(\text{PLB}_2) = \emptyset$ (in fact $\text{PLB}_2 \cong F_2$). Note that a consequence of all this is that $[\chi] \in \Sigma^1(\text{PLB}_n)$ if and only if $[-\chi] \in \Sigma^1(\text{PLB}_n)$.

We now use the ideas from the previous sections to prove the following. The proof is very similar to the proof of Theorem 5.1.

Theorem 5.2 *For $n \geq 3$ the loop braid group LB_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

The $n \geq 3$ restriction is necessary since $\text{PLB}_2 \cong F_2$ splits nontrivially over $\{1\}$, and the NF condition is necessary for reasons similar to the braid group case.

Proof of Theorem 5.2 We will work with the pure loop braid group PLB_n , which is commensurable to LB_n (being a finite-index subgroup). Suppose PLB_n is commensurable to a group that splits nontrivially over an NF subgroup. Since PLB_n is not NF, Proposition 3.1 implies that PLB_n admits an NF subgroup K such that for any $\chi \in \text{Hom}(\text{PLB}_n, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(\text{PLB}_n)^c$, which means $[\chi] \in \Sigma^1(\text{PLB}_n)^c$. By Proposition 1.4, there exists $\chi \in \text{Hom}(\text{PLB}_n, \mathbb{R})$ with $\ker(\chi) = \sqrt{K[\text{PLB}_n, \text{PLB}_n]}$. Since $\chi(K) = 0$ we know $[\chi] \in \Sigma^1(\text{PLB}_n)^c$, which implies that either $n = 3$ or else χ is induced from some standard projection onto PLB_2 or PLB_3 .

In particular, if $n \geq 4$ then there exists i such that $\chi(\alpha_{i,j}) = \chi(\alpha_{j,i}) = 0$ for all $i \neq j$ (just choose i such that x_i is sent to 1 in the projection of F_n inducing the standard

projection of PLB_n), which implies that $\alpha_{i,j}, \alpha_{j,i} \in \sqrt{K[\text{PLB}_n, \text{PLB}_n]}$ for all $i \neq j$. Up to automorphisms (note that the BNS invariant is invariant under automorphisms) we can assume $i = 1$, so in particular $\alpha_{1,2}, \alpha_{2,1} \in \sqrt{K[\text{PLB}_n, \text{PLB}_n]}$. Choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\text{PLB}_n, \text{PLB}_n]$ such that $\alpha_{1,2}^q c, \alpha_{2,1}^r d \in K$, which, since K is NF, implies that $\alpha_{1,2}^q c$ and $\alpha_{2,1}^r d$ do not generate a copy of F_2 . Now consider the standard projection $\pi: \text{PLB}_n \rightarrow \text{PLB}_2$ given by sending all but the first two generators of F_n to 1 and the first two to the generators of F_2 (in order). Then $\alpha_{1,2}^q \pi(c)$ and $\alpha_{2,1}^r \pi(d)$ do not generate a copy of F_2 in PLB_2 . Since $\text{PLB}_2 \cong F_2$, this means $\alpha_{1,2}^q \pi(c)$ and $\alpha_{2,1}^r \pi(d)$ have a common power, say $(\alpha_{1,2}^q \pi(c))^a = (\alpha_{2,1}^r \pi(d))^b$ for $a, b \in \mathbb{Z}$. But abelianizing F_2 to $\mathbb{Z}^2 = \langle \bar{\alpha}_{1,2}, \bar{\alpha}_{2,1} \rangle$, this implies that $qa\bar{\alpha}_{1,2} = rb\bar{\alpha}_{2,1}$, which is absurd.

Now suppose $n = 3$. If χ is induced from a standard projection $\text{PLB}_3 \rightarrow \text{PLB}_2$ then we can use the above argument to get our contradiction, so suppose it is not. Hence, we have $\chi(\alpha_{2,1}) + \chi(\alpha_{3,1}) = 0$, $\chi(\alpha_{1,2}) + \chi(\alpha_{3,2}) = 0$ and $\chi(\alpha_{1,3}) + \chi(\alpha_{2,3}) = 0$. In particular, $\alpha_{1,2}\alpha_{3,2}, \alpha_{2,1}\alpha_{3,1} \in \ker(\chi) = \sqrt{K[\text{PLB}_3, \text{PLB}_3]}$, so we can choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\text{PLB}_3, \text{PLB}_3]$ such that $(\alpha_{1,2}\alpha_{3,2})^q c$ and $(\alpha_{2,1}\alpha_{3,1})^r d$ lie in K , and hence do not generate a copy of F_2 . Their images under the standard projection $\pi: \text{PLB}_3 \rightarrow \text{PLB}_2$ induced by the projection $F_3 \rightarrow F_2$ sending x_1 to x_1 , x_2 to x_2 and x_3 to 1 also do not generate a copy of F_2 , so $\alpha_{1,2}^q \pi(c)$ and $\alpha_{2,1}^r \pi(d)$ do not generate a copy of F_2 in PLB_2 . We are now in the same situation as in the proof of the $n \geq 4$ case, and as in that proof we reach a contradiction. \square

Much like in the braid group case, it would not have worked to try and apply this technique to LB_n itself, so working with PLB_n really was necessary. In fact LB_n has finite abelianization, so it is impossible to find nontrivial characters killing arbitrary NF subgroups simply because there no nontrivial characters at all.

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Received: 28 August 2017 Revised: 16 May 2018