On the integral cohomology ring of toric orbifolds and singular toric varieties

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We examine the integral cohomology rings of certain families of $2n$–dimensional orbifolds X that are equipped with a well-behaved action of the n–dimensional real torus. These orbifolds arise from two distinct but closely related combinatorial sources, namely from characteristic pairs (Q, λ), where Q is a simple convex n–polytope and λ a labeling of its facets, and from n–dimensional fans Σ. In the literature, they are referred as toric orbifolds and singular toric varieties, respectively. Our first main result provides combinatorial conditions on (Q, λ) or on Σ which ensure that the integral cohomology groups $H^*(X)$ of the associated orbifolds are concentrated in even degrees. Our second main result assumes these conditions to be true, and expresses the graded ring $H^*(X)$ as a quotient of an algebra of polynomials that satisfy an integrality condition arising from the underlying combinatorial data. Also, we compute several examples.

1 Introduction

There are several advantages to studying topological spaces whose integral cohomology groups $H^*(X)$ are torsion-free and concentrated in even degrees; for example, their complex K–theory and complex cobordism groups may be deduced immediately, because the appropriate Atiyah–Hirzebruch spectral sequences collapse for dimensional

reasons. For convenience, we call such spaces *even*, where integral coefficients are understood unless otherwise stated. Our fundamental aim is to identify certain families of even spaces within the realms of toric topology, and to explain how their evenness leads to a description of the Borel equivariant cohomology rings $H_T^*(X)$, and thence to the multiplicative structure of $H^*(X)$.

Many even spaces arise from complex geometry, and have been of major importance since the early 20$^{\text{th}}$ century. They range from complex projective spaces and Grassmannian manifolds to Thom spaces of complex vector bundles over other even spaces. Examples of the latter include stunted projective spaces, which play an influential and enduring role in homotopy theory, and certain restricted families of weighted projective spaces. In fact *every* weighted projective space is even, thanks to a beautiful and somewhat surprising result of Kawasaki [18], whose calculations lie behind one of our main works in Section 4. In the literature, weighted projective spaces have been viewed as singular toric varieties or as toric orbifolds, which we shall define in Section 3, and our results may be interpreted as an investigation of their generalizations within either context.

We begin in Section 2 by introducing a sequence $\{B_k\}$ of polytopal complexes whose initial term is a simple polytope Q and final term is a vertex of Q. We define the sequence inductively by the rule stated as 2 in Section 2, which is motivated by several spaces called *invariant subspaces*, and *orbifold lens spaces* sitting inside the given toric orbifold.

In Section 3, we summarize the theory of *toric orbifolds* $X = X(Q, \lambda)$,[1] as constructed from an n–dimensional simple convex polytope Q and an $\mathcal{R}$–*characteristic function* λ from its facets to $\mathbb{Z}^n$. The combinatorial data (Q, λ) is called an $\mathcal{R}$–*characteristic pair* associated to the given toric orbifold. The notion of *invariant subspaces* and *orbifold lens spaces* follow from (Q, λ), which we shall explain in the following subsections. Moreover, for each polytopal complex B which appears in a retraction sequence, the $\mathcal{R}$–characteristic function λ may be used to associate a finite group $G_B(v)$ — see (4-8) — to certain vertices v, called free vertices in B, and to define the collection

(1-1) $\{|G_B(v)| : v \text{ is a } free\ vertex \text{ in } B\}.$

Interest in toric orbifolds was stimulated by Davis and Januszkiewicz [9], who saw them as natural extensions to their own smooth toric manifolds.[2] They proved that toric manifolds are always even; however, the best comparable statement for toric *orbifolds* is due to Poddar and the second author [20], who showed that, in general, they are only

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[1]In the literature, these orbifolds are sometimes called *quasitoric* orbifolds.

[2]They are renamed in Buchstaber and Panov [4] as *quasitoric* manifolds.

even over the rationals. We introduce our main result of the first part of this paper in Section 4, as follows.

Theorem 1.1 *Given any toric orbifold $X(Q, \lambda)$, assume that the gcd of the collection (1-1) is 1 for each B which appears in a retraction sequence with $\dim B > 1$; then X is even.*

The proof employs a cofiber sequence involving *orbifold lens spaces*, which are a generalization of *lens complexes*, introduced by Kawasaki [18]. Furthermore, Theorem 1.1 automatically applies to weighted projective spaces.

In Section 5, we restrict our emphasis to projective toric orbifolds, which are realized as toric varieties whose details are admirably presented by Cox, Little and Schenck in their encyclopedic book [6]. Every such variety X_Σ is encoded by a fan Σ in $\mathbb{R}^n$, and admits a canonical action by the n–dimensional real torus T^n. If Σ is *smooth*, then the underlying geometry guarantees that X_Σ is always even. Moreover, it is true that the Borel equivariant cohomology ring $H^*_T(X_\Sigma)$ is isomorphic to the Stanley–Reisner ring $\mathcal{SR}[\Sigma]$, which is also concentrated in even degrees, and $H^*(X_\Sigma)$ is its quotient by a linear ideal determined by (5-2). It is important to note that $\mathcal{SR}[\Sigma]$ is isomorphic to the ring $\mathcal{PP}[\Sigma]$ of *integral piecewise polynomials* on Σ for any *smooth* fan.

For a particular class of singular examples, a comparable description of the ring $H^*(X_\Sigma)$ was given in Bahri, Franz and Ray [1], as follows. If Σ is polytopal and X_Σ is even, then $H^*(X_\Sigma)$ is the quotient of $\mathcal{PP}[\Sigma]$ by the ideal generated by all *global* polynomials. It is no longer possible to use the Stanley–Reisner ring, which only agrees with $\mathcal{PP}[\Sigma]$ over the rationals. In these circumstances, when X_Σ is a toric variety over a polytopal fan, we have a major incentive to develop criteria which test whether or not it is even. There also remains the significant problem of presenting $\mathcal{PP}[\Sigma]$ by generators and relations, as exemplified by the calculation for the weighted projective space $\mathbb{CP}^3_{(1,2,3,4)}$ in [1, Section 4]. So the aim of Section 5 is to find an alternative description for the ring of piecewise polynomials. It is accomplished by defining the *weighted Stanley–Reisner ring* $w\mathcal{SR}[\Sigma]$, which turns out to be a subring of $\mathcal{SR}[\Sigma]$, consisting of polynomials that satisfy an *integrality condition*; see Definition 5.2. The main result of Section 5 combines Theorems 1.1 and 5.3, as follows.

Theorem 1.2 *Given any polytopal fan Σ in $\mathbb{R}^n$, assume that the corresponding $\mathcal{R}$–characteristic pair (Q, λ) satisfies the hypothesis of Theorem 1.1; then X_Σ is even, and there exists an isomorphism*

$$H^*(X_\Sigma) \cong w\mathcal{SR}[\Sigma]/\mathcal{J}$$

of graded rings, where $\mathcal{J}$ is an ideal of linear relations determined by the generators of rays of Σ.

So our combinatorial condition on the fan allows us to give an explicit description of the integral cohomology ring of X_Σ.

Several natural questions present themselves for future discussion. For example, Sections 3 and 5 may be linked more closely by establishing a common framework for toric orbifolds and toric varieties over nonsmooth polytopal fans. The theory of multifans is an obvious candidate, but we have been unable to identify an associated ring of piecewise polynomials with sufficient clarity. However, the third author with Darby and Kuroki [8] has recently proposed a definition of piecewise polynomials on an orbifold torus graph, which does allow those two objects to be dealt with simultaneously.

In view of our opening remarks, another reasonable challenge is to extend our study to the complex K–theory and complex cobordism of toric orbifolds. This program was suggested by work of Harada, Henriques and Holm [13], and begun in Harada, Holm, Ray and Williams [14] by the adoption of a categorical approach to piecewise structures; but overall progress has been limited to a small subfamily of weighted projective spaces, and much further work is required. However, some progress has made by the second author and Uma [22].

2 A retraction of simple polytopes

In this section, we introduce a natural way of retracting a simple polytope Q to a point, which we call a *retraction sequence*. For each polytope, there are finitely many such

retractions, enabling us to develop a sufficient condition for torsion-freeness in the homology of toric orbifolds in the following section. The operation itself is motivated by several spaces which arise in a toric orbifold by decomposing the orbit space. We shall explain this topological interpretation in Section 3. This section is devoted to giving the combinatorial definition and properties of retraction sequences. We begin by introducing the definition of a polytopal complex.

Definition 2.1 [23, Definition 5.1] A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{R}^n$ satisfying:

(1) If E is a face of F and $F \in \mathcal{C}$ then $E \in \mathcal{C}$.

(2) If $E, F \in \mathcal{C}$ then $E \cap F$ is a face of both E and F.

Let $|\mathcal{C}| = \bigcup_{F \in \mathcal{C}} F$ be the underlying set of $\mathcal{C}$.

The elements of $\mathcal{C}$ are called faces and the zero-dimensional faces of $\mathcal{C}$ are called vertices. We denote the set of vertices of $\mathcal{C}$ by $V(|\mathcal{C}|)$. The dimension of $\mathcal{C}$ or $|\mathcal{C}|$ is the maximum of the dimension of its faces. Given a simple polytope Q, let $\mathcal{C}(Q)$ be the collection of all faces of Q and $\mathscr{F}(Q)$ the collection of all facets of Q. Then $\mathcal{C}(Q)$ is a polytopal complex and $|\mathcal{C}(Q)|$ is homeomorphic to Q as manifolds with corners. Throughout this paper, we always write $\ell := |V(Q)|$ for the number of vertices of Q, $m := |\mathscr{F}(Q)|$ for the number of facets of Q and $n := \dim Q$.

Now, given an n–dimensional simple polytope Q, we construct a sequence of triples $\{(B_k, E_k, b_k)\}_{k=1}^{\ell}$, which we call a *retraction sequence* of Q. First, we define $B_1 = Q = E_1$ and $b_1 \in V(B_1)$. The second term (B_2, E_2, b_2) is defined as follows. Consider a subcollection

$$\mathcal{C}_2 = \{E \in \mathcal{C}(Q) \mid b_1 \notin V(E)\}$$

of $\mathcal{C}(Q)$. Then $\mathcal{C}_2$ is an $(n-1)$–dimensional polytopal complex. We define B_2 by the underlying set $|\mathcal{C}_2|$ of $\mathcal{C}_2$. We choose a vertex b_2 of B_2 such that b_2 has a neighborhood diffeomorphic to $\mathbb{R}_{\geq 0}^N$ as manifolds with corners for some $1 \leq N \leq \dim B_2$ and let E_2 be the unique N–dimensional face of B_2 containing b_2. Notice that, in this case, $N = n - 1$ and we have n different choices of b_2 because Q is an n–dimensional simple polytope.

Next we construct the sequence of triples inductively. Given (B_k, E_k, b_k), the next term $(B_{k+1}, E_{k+1}, b_{k+1})$ is defined as follows. First we consider a polytopal complex

$$\mathcal{C}_{k+1} = \{E \in \mathcal{C}_k \mid b_k \notin V(E)\}.$$

Then B_{k+1} is defined by its underlying set $|\mathcal{C}_{k+1}|$. We choose a vertex b_{k+1} in $V(B_{k+1})$ satisfying the condition

($\diamondsuit$) b_{k+1} has a neighborhood homeomorphic to $\mathbb{R}^N_{\geq 0}$ as manifolds with corners for some $N \in \{1, \dots, \dim B_{k+1}\}$,

and E_{k+1} defined to be a unique face of B_{k+1} containing b_{k+1} with $\dim E_{k+1} = N$.

Definition 2.2 We call a vertex v in B_k a *free vertex* if it has a neighborhood in B_k that is diffeomorphic to $\mathbb{R}^N_{\geq 0}$ as manifolds with corners for some $N \in \{1, \dots, \dim B_k\}$. We denote the set of free vertices in B_k by $\mathrm{FV}(B_k)$.

Finally, the sequence stops if the sequence reaches a vertex, ie $B_\ell = E_\ell = b_\ell \in V(Q)$. Essentially, we can think of a retraction sequence as an iterated choice of free vertices at each step. Figure 1 shows an example of retraction sequence for the vertex cut of a cube, where the colored face of each B_k indicates E_k for $k = 1, \dots, 10$.

Proposition 2.3 *Every simple polytope has at least one retraction sequence.*

Proof We begin by following the argument of [9, Proposition 3.1]. First, we realize Q as a convex polytope in $\mathbb{R}^n$ and choose a vector $u \in \mathbb{R}^n$ such that

$$\langle u, v \rangle \neq \langle u, w \rangle \quad \text{whenever } v \neq w \in V(Q) \subset \mathbb{R}^n,$$

with respect to the Euclidean inner product $\langle\,,\,\rangle$. Let $e := e(vw)$ be the oriented edge with the initial vertex $i(e) = v$ and the terminal vertex $t(e) = w$. Here the direction of $e(vw)$ is given by the rule

$$i(e) = v \quad \text{and} \quad t(e) = w \qquad \text{if and only if} \qquad \langle u, v \rangle < \langle u, w \rangle,$$

which makes the one-skeleton of Q into a directed graph.

Let $\mathrm{ind}(v)$ be the number of inward edges at v and we call $\mathrm{ind}(v)$ the index of v (with respect to the choice of generic vector u). Then, for each face $E \subset Q$, there exists a unique vertex v of E having the maximal index among the vertices in E. Moreover, E is locally diffeomorphic to $\mathbb{R}^{\mathrm{ind}(v)}_{\geq 0}$ around v. Conversely, given a vertex $v \in V(Q)$, there exists a unique face E_v such that $\dim E_v = \mathrm{ind}(v)$.
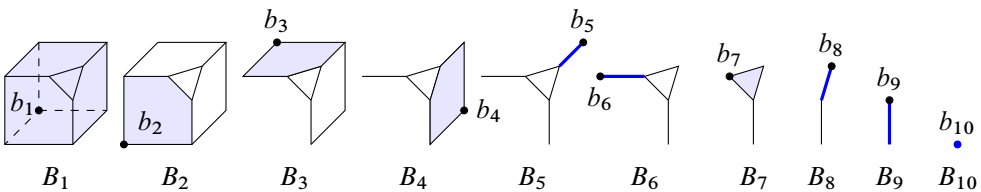

Figure 1: A retraction sequence of a vertex cut of the cube

Let $\{b_k\}_{k=1}^{\ell}$ be a sequence of vertices in Q determined by

$$\langle u, b_1 \rangle > \langle u, b_2 \rangle > \cdots > \langle u, b_\ell \rangle.$$

Notice that $\mathrm{ind}(b_1) = n = \dim Q$, and $\mathrm{ind}(b_\ell) = 0$. Now we claim that the sequence

$$\left\{ \left(B_k := \bigcup_{j \geq k} E_{b_j}, E_{b_k}, b_k \right) \right\}_{k=1,\ldots,\ell},$$

where E_{b_k} is a unique face containing b_k with $\dim E_{b_k} = \mathrm{ind}(b_k)$, is a retraction sequence of Q. Indeed, for each $k \in \{1, \ldots, \ell - 1\}$, we have $\langle u, b_k \rangle > \langle u, v \rangle$ for all $v \in V(B_k) \setminus \{b_k\}$. Hence, there are no outgoing edges from b_k in B_k, which implies that b_k has a neighborhood in $E_{b_k} \subseteq B_k$ homeomorphic to $\mathbb{R}_{\geq 0}^{\mathrm{ind}(b_k)}$ as manifolds with corners. $\qquad\square$

We denote by $\mathfrak{R}(Q)$ the set of all retraction sequences of Q and by $\mathfrak{B}(Q)$ the set of all possible B_i which appear in $\mathfrak{R}(Q)$. Evidently, both $\mathfrak{R}(Q)$ and $\mathfrak{B}(Q)$ are finite sets, because we have finitely many choices of free vertices at each step.

Remark The retraction sequence has a strong relation with *shelling* of a simplicial complex. We are preparing an independent article [2] about the exact correspondence and some other interesting properties.

3 Toric orbifolds and orbifold lens spaces

In this section we recall the *characteristic pairs* (Q, λ) of [9; 20], and explain the way in which they are used to construct *toric orbifolds* $X = X(Q, \lambda)$. If λ obeys Davis and Januszkiewicz's condition $(*)$ (see [9, page 423]), then X is smooth and even; so one of the main goals of this paper is to establish Theorem 1.1, which focuses on singular cases, and states a sufficient condition for the orbifold X to be even. In this section, to complete the proof of Theorem 1.1, we commandeer two additional types of spaces, namely the *invariant subspaces* of X which arise as the preimage of faces via the orbit map, and the *orbifold lens spaces* that arise as quotients of odd-dimensional spheres by the actions of certain finite groups associated to λ.

3.1 Toric orbifolds

In this subsection, we discuss a combinatorial definition of toric orbifolds. Let Q be an n–dimensional simple convex polytope in $\mathbb{R}^n$ and $\mathscr{F}(Q) = \{F_1, \ldots, F_m\}$ the codimension-one faces of Q, which are called *facets*.

Definition 3.1 A function $\lambda\colon \mathscr{F}(Q) \to \mathbb{Z}^n$ is called a *rational characteristic function* (or *R–characteristic function*) for Q if it satisfies the following condition:

$$(3\text{-}1) \qquad \{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\} \text{ is linearly independent whenever } \bigcap_{j=1}^{k} F_{i_j} \neq \varnothing.$$

We write $\lambda_i = \lambda(F_i)$ and call it an *R–characteristic vector* assigned to the facet F_i. The pair (Q, λ) is called an *R–characteristic pair*.

Remark   (1) In the literature about toric manifolds, the pair (Q, λ) satisfying the condition $(*)$ in [9, page 423] is called a *characteristic pair*.

  (2) For convenience, we usually express an *R–characteristic function* λ as an $n \times m$ matrix Λ by listing the λ_i as column vectors. We call Λ an *R–characteristic matrix* associated to λ.

  (3) It is easy to check that it suffices to satisfy the linearly independence at each vertex which is an intersection of n facets.

One canonical example of such functions can be given by a *simple lattice polytope*, which is a convex hull of finitely many points in the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and simple. Namely, we can naturally assign as an *R–characteristic vector* the primitive normal vector on each facet of a simple lattice polytope. In Section 5, we shall see this again as primitive vectors of 1–dimensional cones in a normal fan associated to a simple lattice polytope.

For $x \in Q$, we denote by $E(x)$ the face of Q which contains x in its interior. If $E(x)$ is a face of codimension k, then it is a unique intersection of k facets $F_{i_1}, \ldots, F_{i_k}$. We also denote by $T_{E(x)}$ the subtorus of the standard n–dimensional torus T^n determined by $\lambda_{i_1}, \ldots, \lambda_{i_k}$. To be more precise, we may regard the target space $\mathbb{Z}^n$ of λ as the $\mathbb{Z}$–submodule of the Lie algebra of T^n, and $T_{E(x)}$ is the torus generated by the exponential image of the lines determined by the *R–characteristic vectors* $\lambda_{i_1}, \ldots, \lambda_{i_k}$.

Now we define an equivalence relation $\sim_\lambda$ on the product $T^n \times Q$ by

$$(3\text{-}2) \qquad (t, x) \sim_\lambda (s, y) \qquad \text{if and only if} \qquad x = y \quad \text{and} \quad t^{-1}s \in T_{E(x)}.$$

The quotient space

$$X(Q, \lambda) = (T^n \times Q)/\!\sim_\lambda$$

has an orbifold structure with a natural T^n–action induced by the group operation; see Section 2 in [20]. Clearly, the orbit space of the T^n–action on $X(Q, \lambda)$ is Q. Let

$$(3\text{-}3) \qquad \pi\colon X(Q, \lambda) \to Q, \quad \pi([t, x]_{\sim_\lambda}) = x,$$

be the orbit map, where $[t, x]_{\sim_\lambda}$ is the equivalence class of (t, x) with respect to $\sim_\lambda$. The space $X(Q, \lambda)$ is called the toric orbifold associated to the combinatorial pair (Q, λ).

In analyzing the orbifold structure of $X(Q, \lambda)$, Poddar and Sarkar [20, Section 2.2], gave an axiomatic definition of toric orbifolds, which generalizes the axiomatic definition of toric manifolds of [9].

3.2 Invariant subspaces

In this subsection, we study the $\mathcal{R}$–characteristic pair of some invariant subspaces of $X(Q, \lambda)$. Let $E = F_{i_1} \cap \cdots \cap F_{i_k}$ be a face of Q, where $F_{i_1}, \ldots, F_{i_k}$ are facets. We can define a natural projection

$$(3\text{-}4) \qquad \rho_E \colon \mathbb{Z}^n \to \mathbb{Z}^n / \big((\mathrm{span}\{\lambda_{i_1}, \ldots, \lambda_{i_k}\} \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n\big),$$

where the target space is isomorphic to $\mathbb{Z}^{n-k}$, because $(\mathrm{span}\{\lambda_{i_1}, \ldots, \lambda_{i_k}\} \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^n$ is a rank-k direct summand of $\mathbb{Z}^n$. Notice that the rank of the target space of ρ_E is the same as the dimension of E. We consider E as an independent simple polytope, and denote the set of facets of E by

$$\mathscr{F}(E) = \{E \cap F_j \mid F_j \in \mathscr{F}(Q) \text{ and } j \neq i_1, \ldots, i_k \text{ and } E \cap F_j \neq \varnothing\}.$$

Now the map ρ_E, together with λ, yields an $\mathcal{R}$–characteristic function

$$(3\text{-}5) \qquad\qquad\qquad \lambda_E \colon \mathscr{F}(E) \to \mathbb{Z}^{n-k}$$

on E, defined so that $\lambda_E(E \cap F_j)$ is the primitive vector of $(\rho_E \circ \lambda)(F_j)$. Indeed, the condition (3-1) naturally follows from λ.

Hence, we get an $\mathcal{R}$–characteristic pair (E, λ_E) from (Q, λ), which yields another toric orbifold

$$X(E, \lambda_E) := (T^{n-k} \times E)/\!\sim_{\lambda_E},$$

where the equivalence relation $\sim_{\lambda_E}$ is defined in a manner similar to (3-2).

Proposition 3.2 [20, Section 2.3] *Let $\pi \colon X(Q, \lambda) \to Q$ and (E, λ_E) be as above. Then $\pi^{-1}(E)$ is a T^n–invariant suborbifold. Moreover, it is a toric orbifold homeomorphic to $X(E, \lambda_E)$ as a topological space.*

The second assertion of the above proposition follows from the fact that the circle subgroups determined by $\lambda_E(E \cap F_j)$ and $(\rho_E \circ \lambda)(F_j)$, respectively, are identical.
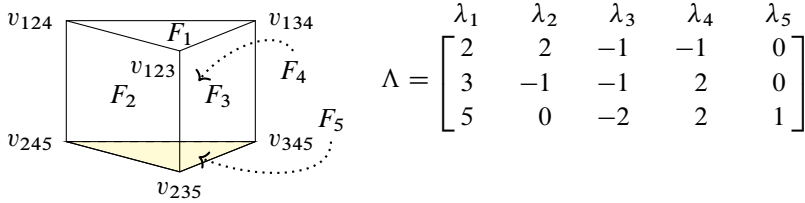
The matrix shown:

$$\Lambda = \begin{array}{c} \begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array} \\ \begin{bmatrix} 2 & 2 & -1 & -1 & 0 \\ 3 & -1 & -1 & 2 & 0 \\ 5 & 0 & -2 & 2 & 1 \end{bmatrix} \end{array}$$

Figure 2

We also remark that the torus T^{n-k} acting on $X(E, \lambda_E)$ can be identified with the image of the map

$$(3\text{-}6) \qquad\qquad \bar{\rho}_E \colon T^n \to T^{n-k},$$

which is induced from the map ρ_E.

Example 3.3 Suppose we have an $\mathcal{R}$–characteristic pair (Q, λ) as described in Figure 2. Notice that Q is a 3–dimensional polytope with 5 facets, say $\mathscr{F}(Q) = \{F_1, \ldots, F_5\}$. Here we assume that the target space $\mathbb{Z}^3$ of λ is generated by the standard basis $\{e_1, e_2, e_3\}$. We choose E to be the facet F_5. So $k = 1$ and $n - k = 2$. Then the projection

$$\rho_E \colon \mathbb{Z}^3 \to \mathbb{Z}^3/\langle e_3 \rangle = \langle e_1, e_2, e_3 \rangle / \langle e_3 \rangle \cong \mathbb{Z}^2$$

is onto the first two coordinates. The facets of E are $F_2 \cap E$, $F_3 \cap E$ and $F_4 \cap E$. Hence, the map

$$\lambda_E \colon \{F_2 \cap E, \ F_3 \cap E, \ F_4 \cap E\} \to \mathbb{Z}^2$$

is defined by

$$\lambda_E(F_2 \cap E) = \rho_E(\lambda(F_2)) = (2, -1) = 2e_1 - e_2,$$
$$\lambda_E(F_3 \cap E) = \rho_E(\lambda(F_3)) = (-1, -1) = -e_1 - e_2,$$
$$\lambda_E(F_4 \cap E) = \rho_E(\lambda(F_4)) = (-1, 2) = -e_1 + 2e_2.$$

The orbifold corresponding to (E, λ_E) is known to be a fake weighted projective space with weight $(1, 1, 1)$. We refer to [5; 17] for the details of fake weighted projective space.

3.3 Orbifold lens spaces

Here we introduce a generalization of *lens complexes* and study their homology groups. Let Δ^{n-1} be the $(n-1)$–dimensional simplex and $\mathscr{F}(\Delta^{n-1}) = \{F_1, \ldots, F_n\}$ the facets of Δ^{n-1}. We begin by introducing the following definition.

Definition 3.4 A function $\xi\colon \mathscr{F}(\Delta^{n-1}) \to \mathbb{Z}^n$ is called an $\mathcal{L}$–*characteristic function* on Δ^{n-1} if $\{\xi(F_1),\ldots,\xi(F_n)\}$ is linearly independent. We set $\xi_i := \xi(F_i)$ for $i = 1,\ldots, n$.

Now we define an equivalence relation $\sim_\xi$ on $T^n \times \Delta^{n-1}$ as follows:

(3-7) $(t, x) \sim_\xi (s, y)$ if and only if $x = y$ and $t^{-1}s \in T_{F(x)}$,

where $F(x)$ is the face containing x in its interior and $T_{F(x)}$ denotes the subtorus of T^n determined by $\xi_{i_1},\ldots,\xi_{i_k}$ if $F(x) = F_{i_1} \cap \cdots \cap F_{i_k}$. The pair (Δ^{n-1}, ξ), together with the equivalence relation $\sim_\xi$, yields the quotient space

$$L(\Delta^{n-1}, \xi) := T^n \times \Delta^{n-1}/\!\sim_\xi,$$

which we call the *orbifold lens space* associated to (Δ^{n-1}, ξ).

Proposition 3.5 *The orbifold lens space $L(\Delta^{n-1}, \xi)$ is homeomorphic to the quotient space of the $(2n-1)$–dimensional sphere S^{2n-1} by the action of a finite group $G_\xi :=$ $\mathbb{Z}^n/\mathrm{span}\{\xi_1,\ldots,\xi_n\}$.*

Proof The proof is essentially same as the proof of [21, Proposition 2.3].   □

Remark   (1) In [21], the function ξ is called a *hypercharacteristic function* if the submodule generated by $\{\xi(F_{i_1}),\ldots,\xi(F_{i_k})\}$ is a direct summand of $\mathbb{Z}^{n+1}$ of rank k whenever $F_{i_1}\cap\cdots\cap F_{i_k}$ is nonempty. In particular, if $\{\xi(F_{i_1}),\ldots,\xi(F_{i_n})\}$ is a linearly independent set, then it becomes an $\mathcal{L}$–characteristic function.

 (2) The action of G_ξ is induced from the standard T^n–action on $S^{2n-1} \subset \mathbb{C}^n$.

 (3) The order $|G_\xi|$ of G_ξ is exactly same as the determinant of the $n \times n$ matrix $[\xi_1 \mid \cdots \mid \xi_n]$.

Proposition 3.5 leads us to the following lemma.

Lemma 3.6 *Let $p_1,\ldots, p_r$ be the prime factors of $|G_\xi|$. Then*

$$H_j(L(\Delta^{n-1}, \xi)) = \begin{cases} \mathbb{Z} & \text{if } j = 0, 2n-1, \\ G_j & \text{if } 1 \le j \le 2n-2, \end{cases}$$

where $G_j = (\mathbb{Z}/p_1^{a_{j_1}}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/p_r^{a_{jr}}\mathbb{Z})$ for some nonnegative integers $a_1,\ldots, a_r$.

Proof We see $H_0(L(\Delta^{n-1}, \xi)) \cong \mathbb{Z}$ trivially. The isomorphism $H_{2n-1}(L(\Delta^n, \xi)) \cong \mathbb{Z}$ follows because the G_ξ–action on S^{2n-1} is induced from the standard action of T^n on $S^{2n-1} \subset \mathbb{C}^n$, which is orientation-preserving. For $j \in \{1,\ldots, 2n-2\}$, recall the

following isomorphism, which can be obtained from the classical result for an action of a finite group G on a locally compact Hausdorff space X:

$$(3\text{-}8) \qquad H^*(X/G;\boldsymbol{k}) \cong H^*(X;\boldsymbol{k})^G,$$

where $\boldsymbol{k}$ is a field of characteristic zero or prime to $|G|$; see [3, III.2].

We apply the isomorphism (3-8) to the orbifold lens space $L(\Delta^{n-1}, \xi) \cong S^{2n-1}/G_\xi$. Since $H^j(S^{2n-1};\boldsymbol{k})^{G_\xi} = 0$ for $j = 1, \dots 2n-2$, the claim is proved by the universal coefficient theorem. $\qquad \square$

Toric orbifolds, invariant subspaces and orbifold lens spaces motivate the definition of retraction sequences which we introduced in the previous section. For a vertex $v \in V(Q)$, let B_2 be the union of all faces in Q which do not contain v. Next we consider a hyperplane

$$(3\text{-}9) \qquad H(v) := \{x \in \mathbb{R}^n \mid \langle x, p_v \rangle = q_v\},$$

where $\langle \, , \rangle$ denotes the Euclidean inner product and $p_v \in \mathbb{R}^n$ and $q_v \in \mathbb{R}$ are chosen in such a way that

- $\{x \in \mathbb{R}^n \mid \langle x, p_v \rangle + q_v \geq 0\} \cap V(Q) = \{v\}$,
- $\{x \in \mathbb{R}^n \mid \langle x, p_v \rangle + q_v \leq 0\} \cap V(Q) = V(Q) \setminus \{v\}$.

Then $\Delta_Q(v) := Q \cap H(v)$ is an $(n-1)$–dimensional simplex, because Q is a simple polytope of dimension n; see Figure 3.

An $\mathcal{L}$–characteristic pair arises naturally from an $\mathcal{R}$–characteristic pair (Q, λ) for each vertex v of Q. Indeed, if $v = F_{j_1} \cap \dots \cap F_{j_n}$, we denote the set of facets of $\Delta_Q(v)$ by

$$\mathscr{F}(\Delta_Q(v)) = \{\Delta_Q(v) \cap F_{j_1}, \dots, \Delta_Q(v) \cap F_{j_n}\}.$$

Now we define a function

$$(3\text{-}10) \qquad \xi_{Q,v} \colon \mathscr{F}(\Delta_Q(v)) \to \mathbb{Z}^n$$

by $\xi_{Q,v}(\Delta_Q(v) \cap F_{j_r}) = \lambda(F_{j_r})$ for $r = 1, \dots, n$. Notice that $\dim \Delta_Q(v) = n-1$, but the rank of target space is n. Since $\{\lambda(F_{i_1}) \dots, \lambda(F_{i_n})\}$ is a linearly independent set, the function $\xi_{Q,v}$ is an $\mathcal{L}$–characteristic function on $\Delta_Q(v)$.

4 Vanishing odd degree homology and torsion-freeness

Now we combine the ingredients which we introduced in the previous sections to derive a sufficient condition for vanishing odd degree cohomology of toric orbifolds.
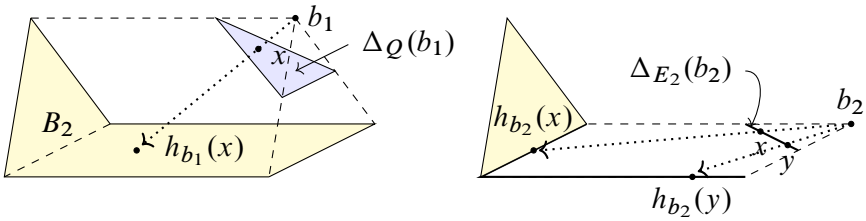
Figure 3: The geometric interpretation of a retraction sequence

In particular, let $X(Q, \lambda)$ be a toric orbifold and the triple $\{(B_k, E_k, b_k)\}_{k=1}^{\ell}$ be a retraction sequence of Q. Given an n–dimensional polytope Q, we begin by defining the map

$$(4\text{-}1) \qquad h_{b_1} \colon \Delta_Q(b_1) \to B_2 = \bigcup \{E \mid E \text{ is a face of } Q, \, b_1 \notin V(E)\}$$

by $h_{b_1}(x) = B_2 \cap (\text{line passing through } x \text{ and } b_1)$, where $\Delta_Q(b_1)$ is an $(n-1)$–dimensional simplex. The map h_{b_1} is well-defined, because Q is convex. The left picture of Figure 3 shows the map h_{b_1} when Q is a prism.

Define a map

$$(4\text{-}2) \qquad f_{b_1} \colon T^n \times \Delta_Q(b_1) \to \bigcup_{E \text{ a face of } B_2} T^{\dim E} \times E$$

by $f_{b_1}(t, x) = (\bar{\rho}_E(t), h_{b_1}(x))$, where $\bar{\rho}_E$ is as defined in (3-6). This induces the map

$$(4\text{-}3) \qquad \bar{f}_{b_1} \colon L(\Delta_Q(b_1), \xi_{Q,b_1}) \to \bigcup_{E \text{ a face of } B_2} X(E, \lambda_E),$$

where ξ_{Q,b_1} is the $\mathcal{L}$–characteristic function defined in (3-10). This map is well-defined by the proof of the following proposition.

Proposition 4.1 *The following diagram commutes:*

$$(4\text{-}4) \qquad
\begin{array}{ccc}
T^n \times \Delta_Q(b_1) & \xrightarrow{\ f_{b_1}\ } & \bigcup_{E \text{ a face of } B_2} (T^{\dim E} \times E) \\
\Big\downarrow{\scriptstyle /\sim_{\xi_{Q,b_1}}} & & \Big\downarrow{\scriptstyle /\sim_{\lambda_E}} \\
L(\Delta_Q(b_1), \xi_{Q,b_1}) & \xrightarrow{\ \bar{f}_{b_1}\ } & \bigcup_{E \text{ a face of } B_2} X(E, \lambda_E) \hookrightarrow X(Q, \lambda)
\end{array}$$

where the equivalence relations $\sim_{\xi_{Q,b_1}}$ *and* $\sim_{\lambda_E}$ *are defined similarly as in (3-7) and (3-2), respectively. Moreover, the bottom row is a cofiber sequence, ie $X(Q, \lambda)$ is homotopy equivalent to the mapping cone* $c(\bar{f}_{b_1})$ *of the map* $\bar{f}_{b_1}$.

Proof We first show that the map $\bar{f}_{b_1}$ is well-defined. Suppose we choose two different representatives, say $[t, x]_{\sim_{\xi_{Q,b_1}}}$ and $[s, y]_{\sim_{\xi_{Q,b_1}}}$ in $L(\Delta_Q(b_1), \xi_{Q,b_1})$. Then $x = y$, so $h_{b_1}(x) = h_{b_1}(y)$. Moreover, if $x \in \Delta_Q(b_1) \cap F$ for some face F of Q, then $h_{b_1}(x) \in F \cap E$ for some face E of B_2. Hence the map $\bar{\rho}_E$ sends the subtorus $T_{F(x)}$ of T^n to $T_{E(h_{b_1}(x))}$ the subtorus of $T^{\dim E}$. Since the map $\bar{\rho}_E$ is a homomorphism, if $t^{-1}s \in T_{F(x)}$, then

$$\bar{\rho}_E(t)^{-1}\bar{\rho}_E(s) = \bar{\rho}_E(t^{-1}s) \in T_{E(h_{b_1}(x))}.$$

Let $C\Delta_Q(b_1)$ be the cone on $\Delta_Q(b_1)$ in Q with the cone point b_1. Then we can decompose Q into two parts as follows:

$$(4\text{-}5) \qquad Q = C\Delta_Q(b_1) \cup_{\Delta_Q(b_1)} \overline{Q \setminus C\Delta_Q(b_1)}.$$

Now we define a continuous surjective map

$$g_{b_1} \colon \overline{Q \setminus C\Delta_Q(b_1)} \to B_2$$

in a manner similar to (4-1). We use it to define a straight line homotopy by

$$\phi \colon \overline{Q \setminus C\Delta_Q(b_1)} \times I \to \overline{Q \setminus C\Delta_Q(b_1)}, \quad (x, u) \mapsto (1-u)x + u \cdot g_{b_1}(x),$$

which preserves the face structure. Thus, ϕ induces a homotopy

$$\hat{\phi} \colon (T^n \times \overline{Q \setminus C\Delta_Q(b_1)})/\sim_\lambda \times I \to (T^n \times \overline{Q \setminus C\Delta_Q(b_1)})/\sim_\lambda,$$

defined by

$$([t, x]_{\sim_\lambda}, u) \mapsto [t, \phi(x, u)]_{\sim_\lambda}.$$

Note that at $u = 0$ the map $\hat{\phi}$ is the identity and at $u = 1$ the image of $\hat{\phi}$ is $\pi^{-1}(B_2)$. Then

$$\begin{aligned}
X(Q, \lambda) &= \pi^{-1}(C\Delta_Q(b_1)) \cup_{L(\Delta_Q(b_1), \xi_{Q,b_1})} \pi^{-1}(\overline{Q \setminus C\Delta_Q(b_1)}) \\
&\simeq C\big(L(\Delta_Q(b_1), \xi_{Q,b_1})\big) \cup_{L(\Delta_Q(b_1), \xi_{Q,b_1})} \pi^{-1}(B_2) \\
&\simeq c(\bar{f}_{b_1}).
\end{aligned}$$

Hence, the result follows. $\qquad\square$

Now the following isomorphisms are straightforward from the cofiber sequence

$$\begin{aligned}
H_*(X(Q, \lambda), \pi^{-1}(B_2)) &\cong H_*\big(C\big(L(\Delta_Q(b_1), \xi_{Q,b_1})\big), \pi^{-1}(B_2)\big) \\
&\cong \tilde{H}_{*-1}\big(L(\Delta_Q(b_1), \xi_{Q,b_1})\big).
\end{aligned}$$

Those two isomorphisms come from the excision and the long exact sequence of the pair, respectively.

So far, we have considered $B_1 = Q$ and B_2, which is the second term of a retraction sequence starting by choosing $b_1 \in \mathrm{FV}(Q) = V(Q)$. However, we can apply similar arguments to each pair B_i and B_{i+1} in a retraction sequence. This leads us to the following lemma, whose proof is essentially same as that of Proposition 4.1. Before we state the lemma, we first set up the notations: Given a retraction sequence $\{(B_k, E_k, b_k)\}_{k=1}^{\ell}$ of Q:

- $\Delta_{E_k}(b_k) := E_k \cap H(b_k) = B_k \cap H(b_k)$ is the simplex obtained by cutting the vertex b_k from B_k.

- ξ_{E_k, b_k} is an $\mathcal{L}$–characteristic function on $\Delta_{E_k}(b_k)$ defined in a similar manner to (3-10) induced from λ_{E_k}.

- The map
$$\bar{f}_{b_k} \colon L(\Delta_{E_k}(b_k), \xi_{E_k, b_k}) \to \bigcup_{E \text{ a face of } B_{k+1}} X(E, \lambda_E) = \pi^{-1}(B_{k+1})$$
   is defined similarly to (4-3) by regarding E_k as a simple polytope.

The right-hand side of Figure 3 illustrates the case of the 3–dimensional prism. The argument above extends to prove the following lemma.

Lemma 4.2 *The sequence*

$$(4\text{-}6) \qquad L(\Delta_{E_k}(b_k), \xi_{E_k, b_k}) \xrightarrow{\bar{f}_{b_k}} \pi^{-1}(B_{k+1}) \hookrightarrow \pi^{-1}(B_k)$$

is a cofiber sequence. Moreover,

$$H_*(\pi^{-1}(B_k), \pi^{-1}(B_{k+1})) \cong \tilde{H}_{*-1}\big(L(\Delta_{E_k}(b_k), \xi_{b_k})\big).$$

Recall from Proposition 3.5 that an $\mathcal{L}$–characteristic function

$$\xi \colon \mathscr{F}(\Delta^{n-1}) \to \mathbb{Z}^n$$

defines a finite abelian group $\mathbb{Z}^n / \mathrm{im}(\xi)$. An $\mathcal{R}$–characteristic pair (Q, λ) induces an $\mathcal{R}$–characteristic pair (E, λ_E) as in (3-5) for any face E of Q. Let E be a k–dimensional face of Q for some $k \leq n$ and $v \in V(E)$. Then $\Delta_E(v) := E \cap H(v)$ is a $(k-1)$–simplex. These give us an $\mathcal{L}$–characteristic function

$$\xi_{E,v} \colon \mathscr{F}(\Delta_E(v)) \to \mathbb{Z}^k,$$

which is defined in a similar manner to (3-10) associated to $\lambda_E \colon \mathscr{F}(E) \to \mathbb{Z}^k$ and $v \in V(E)$. This $\mathcal{L}$–characteristic function defines the finite group

$$(4\text{-}7) \qquad\qquad G_E(v) := \mathbb{Z}^k / \mathrm{im}(\xi_{E,v}).$$

If $G_E(v)$ is trivial, we call a point $\pi^{-1}(v)$ in $\pi^{-1}(E) \cong X(E, \lambda_E)$ a *smooth* point, and otherwise a *singular* point, where $\pi \colon X(Q, \lambda) \to Q$ is the orbit map defined in (3-3).

Furthermore, for each $B \in \mathfrak{B}(Q)$ and a free vertex $v \in \mathrm{FV}(B)$, there exists a unique maximal face, say E_v, of B containing v. Hence, for each $B \in \mathfrak{B}(Q)$, we write

$$(4\text{-}8) \qquad\qquad G_B(v) := G_{E_v}(v)$$

whenever v is a free vertex in B.

Proposition 4.3 *Given a vertex $v \in V(Q)$, let E and E' be two faces containing v such that E is a face of E'. Then $|G_E(v)|$ divides $|G_{E'}(v)|$.*

Proof From Proposition 3.2, we may assume that $E' = Q$ without loss of generality. Suppose that E is a face of Q with codimension k. For convenience, we further assume that $E = F_1 \cap \cdots \cap F_k$ and $v = F_1 \cap \cdots \cap F_k \cap F_{k+1} \cap \cdots \cap F_n$, where the F_i are facets of Q.

From (3-10) and (4-7), we have $G_Q(v) = \mathbb{Z}^n / \langle \lambda(F_1), \ldots, \lambda(F_n) \rangle$ and $G_E(v) = \mathbb{Z}^k / \langle \lambda_E(E \cap F_{k+1}), \ldots, \lambda_E(E \cap F_n) \rangle$. Now we consider the composition

$$\mathbb{Z}^n \xrightarrow{\rho_E} \mathbb{Z}^k \twoheadrightarrow \mathbb{Z}^k / \langle \lambda_E(E \cap F_{k+1}), \ldots, \lambda_E(E \cap F_n) \rangle,$$

where the map ρ_E is defined in (3-4) and the second map is the natural surjection determined by (3-5). Observe that the kernel of the previous composition contains $\langle \lambda(F_1), \ldots, \lambda(F_n) \rangle$. Hence, we get a surjective group homomorphism from $G_Q(v)$ to $G_E(v)$. The result follows from Lagrange's theorem in group theory. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 We prove the claim by the induction on the number of vertices of $B \in \mathfrak{B}(Q)$. First, notice that when the retraction sequence reaches an edge or a union of edges, say B_s, then $\pi^{-1}(B_s)$ is $\mathbb{CP}^1$ or homotopic to a finite wedge of $\mathbb{CP}^1$, which implies that $H_*(\pi^{-1}(B_s))$ is torsion-free and concentrated in even degrees. Therefore, if $|V(B)| \leq 2$ for $B \in \mathfrak{B}(Q)$, then the claim is true.

Now we assume that $\pi^{-1}(B)$ is even for $B \in \mathfrak{B}(Q)$ with $|V(B)| \leq i - 1$. To complete the induction, we shall prove that the same holds for $B' \in \mathfrak{B}(Q)$ with $|V(B')| = i$. Given such B', there exists $B \in \mathfrak{B}(Q)$ such that B is obtained from B' by deleting all faces containing a free vertex of B'. To be more precise, let $\mathrm{FV}(B') = \{v_{i_1}, \ldots, v_{i_r}\}$ be the set of free vertices in B'. Notice that, regarding B' as a generic step of a retraction sequence in $\mathfrak{R}(Q)$, we can produce r many different $B \in \mathfrak{B}(Q)$ with $|V(B)| = i - 1$ from B'. According to the induction hypothesis, we assume that for each $t = 1, \ldots, r$, the group $H_*(\pi^{-1}(B(v_{i_t})))$ is concentrated in even degrees and torsion-free, where $B(v_{i_t}) \in \mathfrak{B}(Q)$ is obtained from B' by deleting faces containing v_{i_t}. This assumption makes sense, because any retraction sequence reaches a union of edges.

For simplicity, we fix the following notation: For each free vertex $v_{i_t} \in \mathrm{FV}(B')$:

- $X' := \pi^{-1}(B')$ and $\dim B' = d' = \frac{1}{2} \dim_{\mathbb{R}} X'$.
- $X(v_{i_t}) := \pi^{-1}(B(v_{i_t}))$ and $\dim B(v_{i_t}) = d = \frac{1}{2} \dim_{\mathbb{R}} X(v_{i_t})$.
- $L(v_{i_t}) := L(\Delta_{E_{i_t}}(v_{i_t}), \xi_{E_{i_t}, v_{i_t}})$, where E_{i_t} denotes the maximal face of B' containing v_{i_t}.

Notice that $\dim L(v_{i_t}) \leq 2d' - 1$ and $d \leq d'$.

Now we consider the long exact sequence of the homology for the pair $(X', X(v_{i_t})) = (\pi^{-1}(B'), \pi^{-1}(B(v_{i_t})))$

$$(4\text{-}9) \quad \cdots \to H_{j+1}(X') \to H_{j+1}(X', X(v_{i_t})) \to H_j(X(v_{i_t}))$$
$$\to H_j(X') \to H_j(X', X(v_{i_t})) \to \cdots .$$

Suppose that j is odd. By the induction hypothesis and Lemma 4.2, the sequence (4-9) becomes

$$(4\text{-}10) \qquad 0 \to H_j(X') \to \widetilde{H}_{j-1}(L(v_{i_t})) \xrightarrow{0} H_{j-1}(X(v_{i_t})).$$

The rightmost map is the zero map because the domain is a torsion group but the target space is free by assumption. Hence, $H_j(X')$ is isomorphic to $\widetilde{H}_{j-1}(L(v_{i_t}))$, and the latter is zero if $j - 1 > \dim L(v_{i_t})$ or a torsion group determined by the prime factors of $|G_{B'}(v_{i_t})|$ if $j - 1 \leq \dim L(v_{i_t})$ by Lemma 3.6. This argument holds for each free vertex $v_{i_1}, \ldots, v_{i_r}$. Hence we have r many different exact sequences like (4-10). Now the assumption of Theorem 1.1 tells us that

$$\gcd\{|\widetilde{H}_{j-1}(L(v_{i_1}))|, \ldots, |\widetilde{H}_{j-1}(L(v_{i_r}))|\} = 1,$$

but $H_j(X')$ stays same. Hence, we conclude that $H_j(X') = 0$ if j is odd. Moreover, $\widetilde{H}_{j-1}(L(v_{i_t})) = 0$ for all $t = 1, \ldots, r$ because of the exactness of (4-10).

Next we assume that j is even. Then the exact sequence (4-9) gives us

$$(4\text{-}11) \qquad \widetilde{H}_j(L(v_{i_t})) \xrightarrow{0} H_j(X(v_{i_t})) \to H_j(X') \to \widetilde{H}_{j-1}(L(v_{i_t})) \to 0.$$

Then we have the following three cases:

$$\cdots \xrightarrow{0} H_j(X(v_{i_t})) \to H_j(X') \to 0 \qquad \text{if } j - 1 > \dim L(v_{i_t}),$$
$$\cdots \xrightarrow{0} H_j(X(v_{i_t})) \to H_j(X') \to \mathbb{Z} \to 0 \qquad \text{if } j - 1 = \dim L(v_{i_t}),$$
$$\cdots \xrightarrow{0} H_j(X(v_{i_t})) \to H_j(X') \to G_{j-1} \to 0 \quad \text{if } j - 1 < \dim L(v_{i_t}),$$

where G_{j-1} is as defined in Lemma 3.6 and $H_j(X(v_{i_t}))$ is free by the induction hypothesis. The free vertices $v_{i_1}, \ldots, v_{i_r}$ in B' gives us r many exact sequences, and each of them is one of the above three cases. If one of the free vertices gives the first or the second type of exact sequence, then $H_j(X')$ cannot have a torsion subgroup

because of the exactness. If all of the sequences are of the third type, then $H_j(X')$ has no torsion because of the assumption of the theorem and arguments similar to those used in the case when j is odd. This completes the induction. □

Notice that Kawasaki [18] has shown that the cohomology ring of weighted projective space $\mathbb{CP}^n_\chi$ with weight $\chi = (\chi_0, \ldots, \chi_n)$ is concentrated in even degrees and torsion-free, if $\gcd(\chi_0, \ldots, \chi_n) = 1$. Theorem 1.1 extends Kawasaki's theorem to the category of toric orbifolds which contains the weighted projective spaces. The following Example 4.4 shows how we can apply this result to a polygon, and Example 4.5 is a practical computation on a higher-dimensional weighed projective space.

Example 4.4 Consider the 4–dimensional toric orbifold X over Q whose $\mathcal{R}$–characteristic pair is described in Figure 4. Let $H(v)$ be an affine hyperplane as defined in (3-9). Then $H(v) \cap Q$ is an 1–simplex. The induced $\mathcal{L}$–characteristic function

$$\xi_{Q,v} \colon \{H(v) \cap F_1,\ H(v) \cap F_m\} \to \mathbb{Z}^2$$

is defined by $\xi_{Q,v}(H(v) \cap F_1) = \lambda(F_1) = (a_1, b_1)$ and $\xi_{Q,v}(H(v) \cap F_m) = \lambda(F_m) = (a_m, b_m)$. Therefore, the orbifold lens space $L(\Delta_Q(v), \xi_{Q,v})$ is homeomorphic to $S^3/G_Q(v)$, where $G_Q(v)$ is a finite abelian group of order $|a_1 b_m - b_1 a_m|$; see Proposition 3.5. Moreover, the prime factors of the order of a torsion element in $H_*(L(\Delta_Q(v), \xi_{Q,v}))$ is a subset of the prime factors of $|a_1 b_m - b_1 a_m|$ by Lemma 3.6.

Now we consider a retraction sequence $\{B_k, E_k, b_k\}_{k=1}^\ell$ starting at v. The second space B_2 is the union $F_2 \cup \cdots \cup F_{m-1}$ of edges whose preimage $\pi^{-1}(B_2)$ is homotopic to the wedge of $m - 2$ copies of $\mathbb{CP}^1$. Hence, $H_*(\pi^{-1}(B_2))$ is torsion-free and $H_{\mathrm{odd}}(\pi^{-1}(B_2))$ vanishes. A cofibration

$$L(\Delta_Q(v), \xi_{Q,v}) \to \pi^{-1}(B_2) \to X$$

gives an isomorphism $H_j(X, \pi^{-1}(B_2)) \cong \tilde{H}_{j-1}(L(\Delta_Q(v), \xi_{Q,v}))$. Hence, the long exact sequence of pair $(X, \pi^{-1}(B_2))$ yields

$$\cdots \to \tilde{H}_j(L(\Delta_Q(v), \xi_{Q,v})) \to H_j(\pi^{-1}(B_2))$$
$$\to H_j(X) \to \tilde{H}_{j-1}(L(\Delta_Q(v), \xi_{Q,v})) \to \cdots,$$

and this shows that, if $H_j(X)$ has a torsion part, then its prime factors must divide $|a_1 b_m - b_1 a_m|$. But the same argument can be applied to all the other vertices in Q. Finally, we may conclude that $H_*(X)$ is torsion-free and concentrated in even degrees if

(4-12) $$\gcd\{|a_1 b_2 - b_1 a_2|, \ldots, |a_{m-1} b_m - b_{m-1} a_m|, |a_1 b_m - b_1 a_m|\} = 1,$$

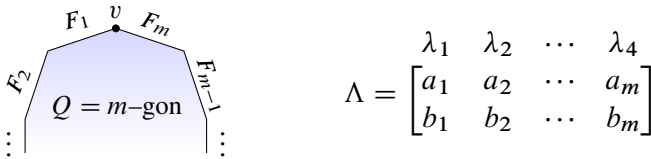which is the assumption of Theorem 1.1.

Figure 4: An $\mathcal{R}$–characteristic function on a polygon

Example 4.5 We consider an $\mathcal{R}$–characteristic pair (Δ^4, λ), where $\lambda\colon \mathscr{F}(\Delta^4) \to \mathbb{Z}^4$ is defined by

$$
\begin{array}{ccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\end{array}
$$
$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 1 & 0 \\
-2 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

The column vectors satisfies the relation $\lambda_1 + \lambda_2 + 2\lambda_3 + 2\lambda_4 + 2\lambda_5 = \mathbf{0}$. Then the resulting toric orbifold is a weighted projective space $\mathbb{CP}^4_{(1,1,2,2,2)}$. We refer to [6, Example 3.1.17] or [12, Section 2.2] for more details.

To check the assumption in Theorem 1.1, it suffices to consider all faces of Δ^4 of dimension greater than 1, because the set $\mathfrak{B}(\Delta^4)$ coincides with the set of all faces of Δ^4. First of all, for Δ^4 itself, it is easy to see that

$$
\gcd\{|G_{\Delta^4}(v)| : v \in V(\Delta^4)\} = \gcd\{2, 2, 2, 1, 1\} = 1.
$$

Since the process is essentially the same, we choose $E = F_1 \cap F_2 = \Delta^2$ as a sample. Observe that

$$
(\langle \lambda_1, \lambda_2 \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^4 = ((\langle -e_1 - 2e_2 - 2e_3 - 2e_4, e_1 \rangle \otimes_{\mathbb{Z}} \mathbb{R}) \cap \mathbb{Z}^4 \cong \langle e_2 + e_3 + e_4, e_1 \rangle.
$$

Hence, we may decompose the target space $\mathbb{Z}^4 \cong \langle e_2 + e_3 + e_4 \rangle \oplus \langle e_1 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4 \rangle$. This derives an $\mathcal{R}$–characteristic function

$$
\lambda_E\colon \{E \cap F_3,\ E \cap F_4,\ E \cap F_5\} \to \mathbb{Z}^2 \cong \langle e_3 \rangle \oplus \langle e_4 \rangle,
$$

defined by $\lambda_E(E \cap F_3) = (-1, -1)$, $\lambda_E(E \cap F_4) = (1, 0)$ and $\lambda_E(E \cap F_5) = (0, 1)$. Hence, $\pi^{-1}(E) = X(\Delta^2, \lambda_E) \cong \mathbb{CP}^2_{(1,1,1)}$. Hence, we have

$$
\gcd\{|G_E(v)| : v \in V(E)\} = \gcd\{1, 1, 1\} = 1.
$$

Sometimes, if the polytope has sufficiently many symmetries, we can analyze all possible retraction sequences efficiently. Proposition 4.3 can then be used to ensure the
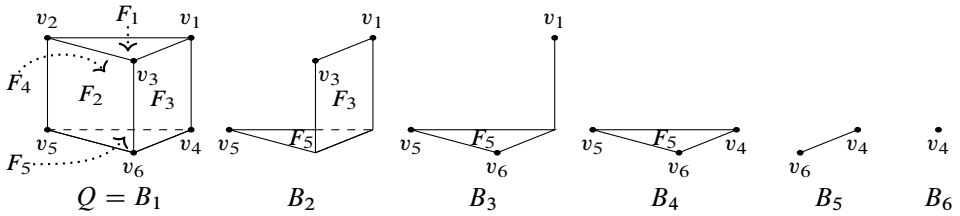
Figure 5: A retraction sequence of a prism

gcd assumption of Theorem 1.1 holds. The main features of the following example are that the polytope has at least two free vertices at each $B \in \mathfrak{B}(Q)$, and that the collection $\{|G_Q(v)| : v \in V(Q)\}$ consists of mutually different prime numbers; in particular, they are pairwise relatively prime.

Example 4.6 Let Q be the 3–dimensional cube whose facets and vertices are illustrated in Figure 5. We assign an $\mathcal{R}$–characteristic function $\lambda: \mathscr{F}(Q) \to \mathbb{Z}^3$ as follows:

$$\lambda(F_1) = (p_1, p_2, p_3), \quad \lambda(F_5) = (p_4, p_5, p_6),$$
$$\lambda(F_2) = e_1, \quad \lambda(F_3) = e_2, \quad \lambda(F_4) = e_3,$$

where the p_i are all prime numbers with $p_i \neq p_j$ whenever $i \neq j$, and e_i is the ith standard unit vector in $\mathbb{Z}^3$. Then it is easy to see that $|G_Q(v_i)| = p_i$ for $i = 1, \ldots, 6$. Hence, we have

$$\gcd\{|G_Q(v)| : v \in V(Q)\} = \gcd\{p_1, \ldots, p_6\} = 1.$$

The same property holds for other polytopal complex $B \in \mathfrak{B}(Q)$ from Proposition 4.3. Indeed, for instance,

$$\gcd\{|G_{B_2}(v)| : v \in \mathrm{FV}(B_2)\} = \gcd\{|G_{B_2}(v_1)|, |G_{B_2}(v_3)|, |G_{B_2}(v_5)|\} = 1$$

because $\gcd\{p_1, p_3, p_5\} = 1$.

5 Cohomology ring of toric orbifolds

The integral equivariant cohomology ring of certain projective toric varieties is given by a ring determined by the fan data. This ring is called the ring of *piecewise polynomials*, which we denote by $\mathcal{PP}[\Sigma]$. For a smooth fan, it uses the fan's combinatorial data only and coincides with the *Stanley–Reisner ring* $\mathcal{SR}[\Sigma]$ of the fan Σ. In general, however, the ring of piecewise polynomials uses all the geometric data in a fan.

To be more precise, let Σ be a fan in $\mathbb{R}^n$ and $\{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{Z}^n$ the set of primitive vectors generating 1–dimensional rays in Σ. Then the Stanley–Reisner ring $\mathcal{SR}[\Sigma]$ is defined by the quotient $\mathbb{Z}[x_1, \ldots, x_m]/\mathcal{I}$ of polynomial ring with m variables by the following ideal generated by squarefree monomials:

$$(5\text{-}1) \qquad \mathcal{I} = \langle x_{i_1} \cdots x_{i_k} \mid \text{cone}\{\lambda_{i_1}, \ldots, \lambda_{i_k}\} \notin \Sigma \rangle,$$

where $\text{cone}\{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$ denotes the cone generated by $\{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$. For the case of smooth toric varieties, their odd-degree cohomology always vanishes, which leads us to the following description of the cohomology ring.

Theorem 5.1 [7; 16] *Let X_Σ be a smooth toric variety. Then there exists a ring isomorphism $H^*(X_\Sigma) \cong \mathcal{SR}[\Sigma]/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by the linear relations*

$$(5\text{-}2) \qquad \sum_{i=1}^{m} \langle \lambda_i, e_j \rangle x_i = 0, \quad j = 1, \ldots, n,$$

where e_j denotes the jth standard unit vector in $\mathbb{Z}^n$.

Notice that, for toric orbifolds, the theorem holds only for $\mathbb{Q}$–coefficients; see for instance [6, Section 12.4]. In order to make the singular theory better resemble the smooth case, we introduce an intermediate ring, which models the Stanley–Reisner ring but is based on a fan $\widehat{\Sigma}$ in $\mathbb{R}^m$ defined from the combinatorial data of Σ, which has m one-dimensional rays. The ring of piecewise polynomials on the original fan Σ is recovered by imposing an *integrality condition*, which leads us to the notion of the *weighted Stanley–Reisner ring* $w\mathcal{SR}[\Sigma]$ of Σ.

5.1 Weighted Stanley–Reisner ring

Let Σ be a simplicial fan in $\mathbb{R}^n$, ie each top-dimensional cone of Σ is generated by n linearly independent primitive vectors in the lattice $\mathbb{Z}^n$. In particular, a simplicial fan Σ is called a *polytopal fan* if it is the normal fan of a simple lattice polytope in $\mathbb{R}^n$; see [6, Chapter 2] or [12, Section 1.5] for more details. Hence, the determinant of generators of each top-dimensional cone is nonzero but not necessarily ± 1, so the corresponding fixed point might be singular. Let $\Sigma^{(j)}$ denotes the set of j–dimensional cones in Σ. To record the singularity of each fixed point in an efficient way, we assign a vector

$$z^\sigma := (z_1^\sigma, \ldots, z_m^\sigma) \in \bigoplus_m \mathbb{Q}[u_1, \ldots, u_n]$$

to each top-dimensional cone $\sigma = \text{cone}\{\lambda_{i_1}, \ldots, \lambda_{i_n}\} \in \Sigma^{(n)}$ by the following rule:

(C1) $z_j^\sigma = 0$ if $j \notin \{i_1, \ldots, i_n\}$.

(C2) $\begin{bmatrix} z_{i_1}^\sigma \\ \vdots \\ z_{i_n}^\sigma \end{bmatrix} = \begin{bmatrix} \lambda_{i_1} \bigg| \cdots \bigg| \lambda_{i_n} \end{bmatrix}^{-1} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$

The inverse matrix in the condition (C2) may have rational entries. The following definition is motivated by this observation.

Definition 5.2 Given a fan Σ in $\mathbb{R}^n$ with m one-dimensional rays, we say a polynomial $h(x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots, x_m]$ satisfies the *integrality condition* with respect to Σ if $h(z^\sigma) \in \mathbb{Z}[u_1, \ldots, u_n]$ for all $\sigma \in \Sigma^{(n)}$.

Notice that the collection of polynomials satisfying the integrality condition is closed under addition and multiplication, which induces the natural ring structure on it inherited from that of $\mathbb{Z}[x_1, \ldots, x_m]$. Moreover, the polynomials in $\mathcal{I}$ defined in (5-1) satisfy the integrality condition, obviously. Indeed, the condition (C1) leads $h(z^\sigma)$ to be the zero polynomial for all $\sigma \in \Sigma^{(n)}$ whenever $h(x_1, \ldots, x_m) \in \mathcal{I}$.

Finally, we define the weighted Stanley–Reisner ring $w\mathcal{SR}[\Sigma]$ as follows:

(5-3) $w\mathcal{SR}[\Sigma] := \{h \in \mathbb{Z}[x_1, \ldots, x_m] \mid h$ satisfies the *integrality condition*$\}/\mathcal{I}.$

Remark When the fan Σ is smooth, $w\mathcal{SR}[\Sigma] = \mathcal{SR}[\Sigma]$. Indeed, the determinant of a smooth top-dimensional cone is ± 1, which implies that its inverse has integer entries.

Now we introduce the second main theorem of this paper. The proof will be given in the next subsection.

Theorem 5.3 Let X_Σ be a projective toric orbifold over a polytopal fan Σ with $H^{\mathrm{odd}}(X) = 0$. Then there is a ring isomorphism

$$H^*(X_\Sigma) \cong w\mathcal{SR}[\Sigma]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by the linear relations (5-2).

Consider a simple lattice polytope Q in $\mathbb{R}^n$ whose normal fan is Σ. Then the normal vectors of each facet define an $\mathcal{R}$–characteristic function $\lambda \colon \mathcal{F}(Q) \to \mathbb{Z}^n$. Now we have a natural $\mathcal{R}$–characteristic pair (Q, λ) from Σ, which allows us to apply the results of Sections 2 and 4. Hence, we have a concrete statement, which is Theorem 1.2, with a sufficient condition for $H^{\mathrm{odd}}(X_\Sigma) = 0$.

We complete this subsection by applying Theorem 1.2 to a weighted projective space $\mathbb{CP}^2_{(1,a,b)}$. We shall recover Kawasaki's result [18, Theorem 1].

Example 5.4 Let Σ be a fan in $\mathbb{R}^2$ generated by

$$(5\text{-}4) \qquad \lambda_1 = (a, b), \quad \lambda_2 = (-1, 0), \quad \lambda_3 = (0, -1) \in \mathbb{Z}^2,$$

where a and b are relatively prime. The 2–dimensional cones are σ_{12}, σ_{13} and σ_{23}, where $\sigma_{ij} = \text{cone}\{\lambda_i, \lambda_j\}$. Since $\{\lambda_1, \lambda_2, \lambda_3\}$ generates the lattice $\mathbb{Z}^2$ and satisfies $\lambda_1 + a\lambda_2 + b\lambda_3 = (0, 0)$, the toric variety X_Σ is isomorphic to the weighed projective space $\mathbb{CP}^2_{(1,a,b)}$. We refer to [6, Example 3.1.17] or [12, Section 2.2] for the characterization of a fan corresponding to weighted projective spaces.

The direct computation of inverse matrices for $[\lambda_i \mid \lambda_j]$ gives us the following list of vectors:

$$z^{\sigma_{12}} = \left(\frac{1}{b}u_2, \, -u_1 + \frac{a}{b}u_2, \, 0\right),$$

$$z^{\sigma_{13}} = \left(\frac{1}{a}u_1, \, 0, \, \frac{b}{a}u_1 - u_2\right),$$

$$z^{\sigma_{23}} = (0, \, -u_1, \, -u_2).$$

Hence, we have

$$(5\text{-}5) \quad w\mathcal{SR}[\Sigma] =$$
$$\{h(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3] \mid h(z^{\sigma_{ij}}) \in \mathbb{Z}[u_1, u_2] \text{ for } 1 \le i < j \le 3\}/\mathcal{I}.$$

Finding elements at each degree is straightforward. For instance, for a degree-2 polynomial, $k_1 x_1 + k_2 x_2 + k_3 x_3 \in w\mathcal{SR}[\Sigma]$ if and only if the following three polynomials have integer coefficients:

$$-k_2 u_1 + \left(\frac{1}{b}k_1 + \frac{a}{b}k_2\right)u_2, \quad \left(\frac{1}{a}k_1 + \frac{b}{a}k_3\right)u_1 - k_3 u_2, \quad -k_2 u_1 - k_2 u_2,$$

which is exactly the case when $k_1 + ak_2 \in b\mathbb{Z}$ and $k_1 + bk_3 \in a\mathbb{Z}$. Hence, one can show that the integers (k_1, k_2, k_3) are

$$(a, -1, 0), \quad (b, 0, -1), \quad (ab, 0, 0), \quad (0, b, 0), \quad (0, 0, a),$$

and $\mathbb{Z}$–linear combinations of them. They give us the following degree-2 elements in $w\mathcal{SR}[\Sigma]$:

$$(5\text{-}6) \qquad ax_1 - x_2, \quad bx_1 - x_3, \quad abx_1, \quad bx_2, \quad ax_3,$$

and $\mathbb{Z}$–linear combinations of them. Similarly, we can find the degree-4 elements:

$$(5\text{-}7) \qquad a^2b^2x_1^2, \quad b^2x_2^2, \quad a^2x_3^2, \quad abx_1x_2, \quad a^2x_1x_3, \quad x_2x_3,$$

and $\mathbb{Z}$–linear combinations of them.

We continue to calculate the ring structure of $H^*(\mathbb{CP}^2_{(1,a,b)})$ using Theorem 1.2. Indeed, the $\mathcal{R}$–characteristic pair (Δ^2, λ) induced from Σ satisfies the assumption

of Theorem 1.1; see Example 4.4. Hence, we conclude that $\mathbb{CP}^2_{(1,a,b)}$ is *even*, which implies that the rank of the integral cohomology group is 1 in each even degree and 0 otherwise.

Remark In general, the integral Betti numbers of a toric manifold or the rational Betti numbers of a toric orbifold are given by the h–vector of its underlying polytope; see [9, Section 3] or [20, Section 4]. Hence, if a toric orbifold is even, then its integral Betti numbers are obtained by the h–vector of the underlying polytope.

Now the characteristic vectors (5-4) and the relation (5-2) determine the ideal $\mathcal{J} = \langle ax_1 - x_2, bx_1 - x_3 \rangle$ whose generators are first two items in (5-6). Hence, the elements in (5-6) except the first two are are the same modulo $\mathcal{J}$. Hence, they represent the same element in $H^*(\mathbb{CP}^2_{(1,a,b)})$. We put

$$w_1 := abx_1 = bx_2 = ax_3.$$

Since rank $H^4(\mathbb{CP}^2_{(1,a,b)}) = 1$, we choose an element in (5-7) which has the minimal divisibility. In this case, we pick

$$w_2 := x_2 x_3.$$

Then we have the multiplicative structure $w_1^2 = abw_2$. Finally, we have the presentation

$$H^*(\mathbb{CP}^2_{(1,a,b)}) \cong \mathbb{Z}[w_1, w_2]/\langle w_1^2 - abw_2, w_1 w_2 \rangle,$$

where deg $w_1 = 2$ and deg $w_2 = 4$. Notice that the monomial $w_1 w_2$ comes from the Stanley–Reisner ideal $x_1 x_2 x_3$.

Remark Even if we can find elements in $w\mathcal{SR}[\Sigma]$ by the direct computation of the integrality condition, finding the minimal set of generators in $w\mathcal{SR}[\Sigma]$ for an arbitrary simplicial fan is not obvious, in general. However, when X_Σ is a weighted projective space, a result of [1] allows us to find generators of the ring of piecewise polynomials $\mathcal{PP}[\Sigma]$ and, hence, generators in $w\mathcal{SR}[\Sigma]$, by a method in the next subsection. Moreover, the identification result, Corollary 5.8, tells us how to interpret those generators in terms of elements in $w\mathcal{SR}[\Sigma]$.

5.2 Piecewise algebra and cohomology ring

We introduce now the ring of piecewise polynomials, which is determined by a fan and describes the equivariant cohomology of a large class of toric orbifolds. As mentioned above, unlike the Stanley–Reisner ring, which encodes combinatorial data only, the ring of piecewise polynomials depends on the full geometric information in a fan.

We begin by introducing piecewise polynomials. Let Σ be a fan in $\mathbb{R}^n$. A function $f \colon \mathbb{Z}^n \to \mathbb{Z}$ is called a *piecewise polynomial* on Σ if, for each cone $\sigma \in \Sigma$, the

restriction $f|_\sigma$ is a polynomial function on $\sigma \cap \mathbb{Z}^n$. Such a function can be interpreted as a collection $\{f_\sigma\}_{\sigma \in \Sigma^{(n)}}$, which we denote by $\{f_\sigma\}$ for simplicity, such that

$$(5\text{-}8) \qquad\qquad f_\sigma|_{\sigma \cap \sigma'} = f_{\sigma'}|_{\sigma \cap \sigma'}.$$

In other words, it is enough to consider the polynomials on each top-dimensional cone. The polynomials on lower-dimensional cones are determined by (5-8).

The set $\mathcal{PP}[\Sigma]$ of piecewise polynomial functions on Σ with integer coefficients on Σ has a ring structure under pointwise addition and multiplication. Moreover, the natural inclusion of global polynomials $\mathbb{Z}[u_1, \dots, u_n]$ into $\mathcal{PP}[\Sigma]$ induces a $\mathbb{Z}[u_1, \dots, u_n]$–algebra structure on $\mathcal{PP}[\Sigma]$. Furthermore, by considering $\mathbb{Q}^n$ instead of $\mathbb{Z}^n$, we can define piecewise polynomial functions with rational coefficients $f: \mathbb{Q}^n \to \mathbb{Q}$, and we denote the ring of piecewise polynomial functions with rational coefficients by $\mathcal{PP}[\Sigma; \mathbb{Q}]$.

It is well known that the equivariant cohomology ring with rational coefficients of a toric variety over a simplicial fan is isomorphic to $\mathcal{PP}[\Sigma; \mathbb{Q}]$; see [6]. On the other hand, for the case of polytopal fans, Bahri, Franz and Ray [1] proved the following proposition over $\mathbb{Z}$.

Proposition 5.5 [1, Proposition 2.2] *Let Σ be a polytopal fan in $\mathbb{R}^n$, X_Σ the associated compact projective toric variety with $H^{\mathrm{odd}}(X_\Sigma) = 0$, and $T = T^n$ the n–dimensional torus acting on X_Σ. Then $H_T^*(X_\Sigma)$ is isomorphic to $\mathcal{PP}[\Sigma]$ as an $H^*(BT)$–algebra.*

Here $H^*(BT)$–algebra structure on $\mathcal{PP}[\Sigma]$ is obtained by identifying $H^*(BT)$ with the global polynomials $\mathbb{Z}[u_1, \dots, u_n]$, where u_i is the first Chern class of the canonical line bundle given by the ith projection $T \to S^1$.

On the other hand, the combinatorial structure of Σ determines a canonical fan in a higher-dimensional lattice as follows: Let $\Sigma^{(1)} = \{\lambda_1, \dots, \lambda_m\}$ be the set of primitive vectors generating 1–dimensional rays in Σ. We define a linear map $\Lambda: \mathbb{Z}^m \to \mathbb{Z}^n$ by $\Lambda(e_i) = \lambda_i$, where $e_1, \dots, e_m$ denote the standard unit vectors in $\mathbb{Z}^m$. By the pull-back of Σ through Λ, we can define a fan

$$\widehat{\Sigma} = \{\widehat{\sigma} := \Lambda^{-1}(\sigma) \mid \sigma \in \Sigma\}$$

in $\mathbb{R}^m$. To be more precise, if σ is the cone generated by $\lambda_{i_1}, \dots, \lambda_{i_k}$, then $\widehat{\sigma}$ is the cone generated by $e_{i_1}, \dots, e_{i_k}$. Moreover, for a commutative ring k, a linear map Λ induces a ring homomorphism

$$(5\text{-}9) \qquad\qquad \Lambda^*: \mathcal{PP}[\Sigma; k] \to \mathcal{PP}[\widehat{\Sigma}; k]$$

of piecewise polynomial rings, where the map is defined by

$$\Lambda^*(\{f_\sigma\}) = \{g_{\widehat{\sigma}}(x_{i_1},\ldots,x_{i_n}) := f_\sigma(\Lambda_\sigma \cdot [x_{i_1},\ldots,x_{i_n}]^T)\}_{\widehat{\sigma} \in \widehat{\Sigma}^{(n)}},$$

where $\Lambda_\sigma = [\lambda_{i_1} \mid \cdots \mid \lambda_{i_n}]$ is a square matrix and $\Lambda_\sigma \cdot [x_{i_1},\ldots,x_{i_n}]^T$ is the usual matrix multiplication of $n \times n$ and $n \times 1$ matrices.

Indeed, the map Λ^* is well-defined, since $g_{\widehat{\sigma}}|_{\widehat{\sigma} \cap \widehat{\sigma}'} = g_{\widehat{\sigma}'}|_{\widehat{\sigma} \cap \widehat{\sigma}'}$.

Lemma 5.6 *Given a polytopal fan Σ, as $H^*(BT;\boldsymbol{k})$–algebras:*

(1) *When $\boldsymbol{k} = \mathbb{Q}$, $\mathcal{PP}[\Sigma;\mathbb{Q}]$ is isomorphic to $\mathcal{PP}[\widehat{\Sigma};\mathbb{Q}]$.*

(2) *When $\boldsymbol{k} = \mathbb{Z}$, there is a monomorphism from $\mathcal{PP}[\Sigma]$ to $\mathcal{PP}[\widehat{\Sigma}]$.*

Proof For each top-dimensional cone $\sigma = \mathrm{cone}\{\lambda_{i_1},\ldots,\lambda_{i_n}\} \in \Sigma^{(n)}$, we set the following notation:

- $f_\sigma(u_1,\ldots,u_n)$, $g_{\widehat{\sigma}}(x_{i_1},\ldots,x_{i_n})$ are polynomial functions defined on $\sigma \in \Sigma$ and $\widehat{\sigma} \in \widehat{\Sigma}$, respectively.
- $\{f_\sigma\} := \{f_\sigma(u_1,\ldots,u_n) \mid \sigma \in \Sigma^{(n)}\} \in \mathcal{PP}[\Sigma]$.
- $\{g_{\widehat{\sigma}}\} := \{g_{\widehat{\sigma}}(x_{i_1},\ldots,x_{i_n}) \mid \widehat{\sigma} \in \widehat{\Sigma}^{(n)}\} \in \mathcal{PP}[\widehat{\Sigma}]$.
- $\Lambda_\sigma := [\lambda_{i_1} \mid \cdots \mid \lambda_{i_n}]$ is an $n \times n$ matrix with column vectors $\lambda_{i_1},\ldots,\lambda_{i_n}$.

Recall the ring homomorphism Λ^* introduced in (5-9). If we restrict $\boldsymbol{k}$ to $\mathbb{Q}$, the map Λ^* has the natural inverse

$$(5\text{-}10) \qquad\qquad \Theta\colon \mathcal{PP}[\widehat{\Sigma};\boldsymbol{k}] \to \mathcal{PP}[\Sigma;\boldsymbol{k}],$$

defined by

$$\Theta(\{g_{\widehat{\sigma}}\}) = \{f_\sigma(u_1,\ldots,u_n) := g_{\widehat{\sigma}}(\Lambda_\sigma^{-1} \cdot [u_1,\ldots,u_n]^T) \mid \sigma \in \Sigma^{(n)}\},$$

where Λ_σ^{-1} is regarded as a linear automorphism of $\mathbb{Q}^n$. Indeed,

$$(\Theta \circ \Lambda^*)(\{f_\sigma\}) = \{f_\sigma(\Lambda_\sigma \cdot \Lambda_\sigma^{-1} \cdot [u_1,\ldots,u_n]^T) \mid \sigma \in \Sigma^{(n)}\}$$
$$= \{f_\sigma(u_1,\ldots,u_n) \mid \sigma \in \Sigma^{(n)}\} = \{f_\sigma\}.$$

In particular, Λ^* is a monomorphism in $\mathbb{Z}$–coefficients. Finally, the $H^*(BT;\boldsymbol{k})$–algebra structure on $\mathcal{PP}[\widehat{\Sigma};\boldsymbol{k}]$ is naturally inherited from that of $\mathcal{PP}[\Sigma;\boldsymbol{k}]$ via the map Λ^*. $\qquad\square$

Recall that the Stanley–Reisner ring $\mathcal{SR}[\Sigma; \boldsymbol{k}]$ has combinatorial data only, while $\mathcal{PP}[\Sigma; \boldsymbol{k}]$ contains both combinatorial and geometric data. However, $\mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$ has only combinatorics, but looks like $\mathcal{PP}[\Sigma; \boldsymbol{k}]$. In this point of view, $\mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$ is an intermediate object between $\mathcal{SR}[\Sigma; \boldsymbol{k}]$ and $\mathcal{PP}[\Sigma; \boldsymbol{k}]$. The following lemma, together with Lemma 5.6, concludes the relations among those three objects.

Lemma 5.7 As an $H^*(BT; \boldsymbol{k})$–algebra, $\mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$ is isomorphic to $\mathcal{SR}[\Sigma; \boldsymbol{k}]$ for $\boldsymbol{k} = \mathbb{Z}$ or $\mathbb{Q}$.

Proof We construct an isomorphism between $\mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$ and $\mathcal{SR}[\Sigma; \boldsymbol{k}]$, where $\boldsymbol{k} = \mathbb{Z}$ or $\mathbb{Q}$. Assume that $|\Sigma^{(1)}| = m$. Define a map

$$(5\text{-}11) \qquad \phi \colon \boldsymbol{k}[x_1, \ldots, x_m] \to \mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$$

by restriction to each cone of $\widehat{\Sigma}$. Then this map ϕ is a surjective ring homomorphism. Indeed, given $\{g_{\widehat{\sigma}}\} \in \mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$, we can apply the *inclusion–exclusion principle* to obtain

$$(5\text{-}12) \qquad h(x_1, \ldots, x_m) = \sum_{j=0}^{n-1} \left((-1)^j \sum_{\substack{\widehat{\tau} \in \widehat{\Sigma} \\ \dim \widehat{\tau} = n-j}} g_{\widehat{\tau}}(x_{i_1}, \ldots, x_{i_{n-j}}) \right)$$

which is the desired global function h satisfying $\phi(h) = \{g_{\widehat{\sigma}}\}$, where $\widehat{\sigma} \in \widehat{\Sigma}^{(n)}$.

Moreover, since the zero element in $\mathcal{PP}[\widehat{\Sigma}; \boldsymbol{k}]$ is $\{g_{\widehat{\sigma}} = 0 \mid \widehat{\sigma} \in \widehat{\Sigma}^{(n)}\}$, the kernel is

$$\ker \phi = \mathrm{span}\left\{ \prod_{j=1}^{k} x_{i_j} \;\middle|\; \mathrm{cone}\{e_{i_1}, \ldots, e_{i_k}\} \notin \widehat{\Sigma} \right\},$$

which is exactly the Stanley–Reisner ideal $\mathcal{I}$ of Σ. Hence, the result follows. ☐

Corollary 5.8 There is an isomorphism $\mathcal{PP}[\Sigma] \cong w\mathcal{SR}[\Sigma]$ (see (5-3)) as $H^*(BT)$–algebras.

Proof Consider the composition of ring homomorphisms

$$\mathcal{PP}[\Sigma] \xrightarrow{\Lambda^*} \mathcal{PP}[\widehat{\Sigma}] \xrightarrow{\Phi^{-1}} \mathcal{SR}[\Sigma],$$

where $\Phi \colon \mathcal{SR}[\Sigma] \to \mathcal{PP}[\widehat{\Sigma}]$ is the isomorphism induced by ϕ. With $\mathbb{Z}$–coefficients, the map Λ^* is injective by Lemma 5.6. Hence, $\mathcal{PP}[\Sigma]$ is isomorphic to its image in $\mathcal{SR}[\Sigma]$ via the composition $\Phi^{-1} \circ \Lambda^*$.

Recall that the composition $\Phi^{-1} \circ \Lambda^*$ is an isomorphism over $\mathbb{Q}$, whose inverse $\Theta \circ \Phi^{-1}$ maps an element $[h] \in \mathcal{SR}[\Sigma; \mathbb{Q}]$ to $\{h(z^\sigma)\}_{\sigma \in \Sigma^{(n)}} \in \mathcal{PP}[\Sigma; \mathbb{Q}]$. Therefore,

over integer coefficients, $[h] \in \operatorname{im}(\Phi^{-1} \circ \Lambda^*)$ if and only if the polynomial h satisfies the integrality condition. Hence, the result follows. □

Finally, we conclude this subsection with a proof of Theorem 5.3.

Proof of Theorem 5.3 Since $H^*(X_\Sigma; \mathbb{Z})$ concentrated in even degrees, the Serre spectral sequence for the fibration

$$X_\Sigma \to ET \times_T X_\Sigma \xrightarrow{\pi} BT$$

degenerates at the E_2 level. By the result from Franz and Puppe [11, Theorem 1.1], we get isomorphisms of $H^*(BT)$–algebras,

$$H^*(X_\Sigma) \cong H_T^*(X_\Sigma) \otimes_{H^*(BT)} \mathbb{Z} \cong H_T^*(X_\Sigma)/\operatorname{Im}(\pi^*\colon H^*(BT) \to H_T^*(X_\Sigma)).$$

By Proposition 5.5 and Corollary 5.8, we have $H_T^*(X_\Sigma) \cong w\mathcal{SR}[\Sigma]$. Moreover, for each $u_j \in \mathbb{Z}[u_1, \ldots, u_n] \cong H^*(BT)$,

$$(\Phi \circ \Lambda^*)(u_j) = \sum_{i=1}^{m} \langle \lambda_i, e_j \rangle x_i.$$

Hence, we conclude that $\operatorname{im}(\pi^*\colon H^*(BT) \to H_T^*(X_\Sigma)) = \mathcal{J}$. □

6 Example: orbifold Hirzebruch varieties

We finish this paper by illustrating the results of the previous sections with a concrete example which is not a weighted projective space. Consider a primitive vector $(a, b) \in \mathbb{Z}^2$ with $a > 0$. Together with $(-1, 0)$, $(0, 1)$ and $(0, -1)$, we can make a complete fan Σ in $\mathbb{R}^2$ which gives us a compact toric variety with two singular points. We denote this toric variety by $\mathscr{H}_{(a,b)}$. See Figure 6 for the fan and corresponding $\mathcal{R}$–characteristic pair (Q, λ). When $a = 1$, the toric variety is known as a Hirzebruch surface, say $\mathscr{H}_b$. In this point of view, let us call $\mathscr{H}_{(a,b)}$ an *orbifold Hirzebruch variety*.
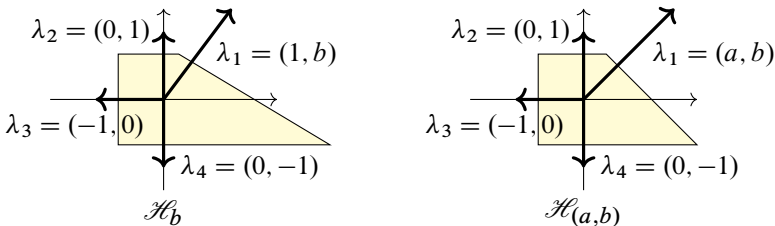

Figure 6: A Hirzebruch surface and an orbifold Hirzebruch variety

Since the collection in (1-1) becomes $\{|G_Q(v)| : v \in V(Q)\} = \{1, 1, a, a\}$ when $B_1 = Q$, its gcd is 1. Moreover, in any retraction sequence, B_2 is given by a union of edges, which guarantees that (Q, λ) satisfies the assumption of Theorem 1.1; see Example 4.4. Moreover, since the underlying polytope is a square, the integral Betti numbers are given by $\beta^0 = \beta^4 = 1$ and $\beta^2 = 2$ by the remark on page 3802.

Remark We may compute the (co)homology groups of low-dimensional toric orbifolds by the spectral sequence whose E_1 page is described by the fan data; see [10; 15]. More generally, the low-dimensional calculations of Kuwata, Masuda and Zeng [19] apply to the category of torus orbifolds.

Let $\sigma_{ij} = \text{cone}\{\lambda_i, \lambda_j\}$, where $\lambda_1, \ldots, \lambda_4$ are described in the right-hand side of Figure 6. Then the integrality condition of Definition 5.2 is given by the vectors

$$z^{\sigma_{12}} = \left(\frac{1}{a}u_1, -\frac{b}{a}u_1 + u_2, 0, 0\right),$$
$$z^{\sigma_{14}} = \left(\frac{1}{a}u_1, 0, 0, \frac{b}{a}u_1 - u_2\right),$$
$$z^{\sigma_{23}} = (0, u_2, -u_1, 0),$$
$$z^{\sigma_{34}} = (0, 0, -u_1, -u_2).$$

Notice that the last two vectors, $z^{\sigma_{23}}$ and $z^{\sigma_{34}}$, don't contribute to the integrality condition, because their entries have integral coefficients.

A similar computation to Example 5.4 shows that the following polynomials are elements of degree 2 in $w\mathcal{SR}[\Sigma]$:

(6-1) $\qquad ax_1 - x_3, \quad bx_1 + x_2 - x_4, \quad ax_1, \quad ax_2, \quad x_3, \quad ax_4,$

as are $\mathbb{Z}$–linear combinations of them. The first two elements are actually the linear relations in $\mathcal{J}$, which means that they come from the global polynomials in $\mathcal{PP}[\Sigma]$. Since rank $H^2(\mathcal{H}_{(a,b)}) = 2$, we choose two linearly independent elements as follows:

$$w_1 := ax_1 \quad \text{and} \quad w_2 := ax_4.$$

Next, degree-4 elements in $w\mathcal{SR}[\Sigma]$ are

(6-2) $\quad a^2x_1^2, \quad a^2x_2^2, \quad x_3^2, \quad a^2x_4^2, \quad a^2x_1x_2, \quad a^2x_1x_4, \quad x_2x_3 \quad \text{and} \quad x_3x_4,$

and their $\mathbb{Z}$–linear combination. The first four of (6-2) are just the square of degree-2 elements. The remaining four monomials are

- $a^2x_1x_2 = ax_1ax_2 = ax_1a(-bx_1 + x_4) = w_1(-bw_1 + w_2),$
- $a^2x_1x_4 = ax_1ax_4 = w_1^2,$

Notice that the final two monomials x_2x_3 and x_3x_4 cannot come from degree-2 elements. Hence, we put

$$w_3 := x_3x_4.$$

Then

$$x_2x_3 = (-bx_1 + x_4)x_3 = x_3x_4 = ax_1x_4 = w_3.$$

The second equality holds because of the Stanley–Reisner ideal $\mathcal{I} = \langle x_1x_3, x_2x_4 \rangle$. Finally, the ideal $\mathcal{I}$ and $\mathcal{J}$ determine the multiplicative structures as follows:

$$w_1^2 = (ax_1)^2 = (ax_1)(x_3) = 0,$$
$$w_1w_2 = (ax_1)(ax_4) = x_3(ax_4) = aw_3,$$
$$w_2^2 = (ax_4)(ax_4) = a(bx_1 + x_2)(ax_4) = abx_3x_4 = abw_3,$$
$$w_1w_3 = (ax_1)(x_3x_4) = 0,$$
$$w_2w_3 = (ax_4)(x_3x_4) = ax_4x_3(bx_1 + x_2) = 0,$$
$$w_3^2 = (x_3x_4)^2 = x_3^2x_4^2 = (ax_1x_3)(x_4^2) = 0.$$

Therefore, we get the following presentation for the cohomology ring of orbifold Hirzebruch varieties:

$$(6\text{-}3) \quad H^*(\mathcal{H}_{(a,b)})$$
$$\cong \mathbb{Z}[w_1, w_2, w_3]/\langle w_1^2, w_1w_2 - aw_3, w_2^2 - abw_3, w_1w_3, w_2w_3, w_3^2 \rangle,$$

where $\deg w_1 = \deg w_2 = 2$ and $\deg w_3 = 4$.

Remark The cohomology ring of Hirzebruch surfaces, by way of comparison, can be computed from the results of [7], [9] or [16]. Indeed it has the presentation

$$H^*(\mathcal{H}_b) \cong \mathbb{Z}[w_1, w_2]/(w_1^2, w_2^2 - bw_1w_2),$$

where $\deg w_1 = \deg w_2 = 2$, which means that it is generated by degree-2 elements. However, $H^*(\mathcal{H}_{(a,b)})$ has the degree-4 generator w_3 which cannot be generated by degree-2 elements, ie $w_1w_2 = aw_3$. Notice that we can recover the presentation of $H^*(\mathcal{H}_b)$ by replacing a by 1 in (6-3).

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