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Nine generators of the skein space of the 3–torus

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We show that the skein vector space of the 3–torus is finitely generated. We show that it is generated by nine elements: the empty set, some simple closed curves representing the nonzero elements of the first homology group with coefficients in $\mathbb{Z}_2$, and a link consisting of two parallel copies of one of the previous nonempty knots.

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1 Introduction

An alternative approach to representation theory for *quantum invariants* is provided by *skein theory*. The word "skein" and the notion were introduced by Conway in 1970 for his model of the *Alexander polynomial*. This idea became quite useful after the work of Kauffman [10] which redefined the *Jones polynomial* in a very simple and combinatorial way passing through the *Kauffman bracket*. These combinatorial techniques allow us to reproduce all quantum invariants arising from the representations of $U_q(\mathfrak{sl}_2)$ without any reference to representation theory. This also leads to many interesting and quite easy computations. This skein method was used by Blanchet, Habegger, Masbaum and Vogel [1], Kauffman and Lins [11] and Lickorish [12; 13; 15; 14] to reinterpret and extend some of the methods of representation theory.

The first notion in skein theory is that of a "*skein vector space*" (or *skein module*). These are vector spaces (R–modules) associated to oriented 3–manifolds, where the base field is equipped with a fixed invertible element A. These were introduced independently in 1988 by Turaev [24] and in 1991 by Przytycki [20]. We can think of them as an attempt to get an algebraic topology for knots: they can be seen as homology spaces obtained using isotopy classes instead of homotopy or homology classes. In fact, they are defined taking a vector space generated by subobjects (*framed links*) and then quotienting them by some relations. In this framework, the following questions arise naturally and are still open in general:

Question 1.1
- Are skein spaces (modules) computable?
- How powerful are they to distinguish 3–manifolds and links?

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- Do the vector spaces (modules) reflect the topology/geometry of the 3–manifolds (eg surfaces, geometric decomposition)?
- Does this theory have a functorial aspect? Can it be extended to a functor from a category of cobordisms to the category of vector spaces (modules) and linear maps?

Skein spaces (modules) can also be seen as deformations of the ring of the $\mathrm{SL}_2(\mathbb{C})$–*character variety* of the 3–manifold; see Bullock [3]. Moreover, they are useful to generalize the Kauffman bracket, hence the Jones polynomial, to manifolds other than S^3. Thanks to Hoste and Przytycki [9], Przytycki [22] and (with different techniques) Costantino [4], now we can define the Kauffman bracket also in the connected sum $\#_g(S^1 \times S^2)$ of $g \geq 0$ copies of $S^1 \times S^2$.

Currently, there are only few 3–manifolds whose skein space (module) is known; see for instance Bullock [2], Hoste and Przytycki [7; 8; 9], Marché [16], Mroczkowski [18; 17], Mroczkowski and Dabkowski [19] and Przytycki [21; 22; 23]. Another natural question is:

Question 1.2 Is the skein vector space of a closed oriented 3–manifold always finitely generated?

In this paper, we take as base field the set $\mathbb{Q}(A)$ of all rational functions with rational coefficients and abstract variable A, and we note that every result in this work holds also for the field $\mathbb{C}$ of complex numbers with $A \in \mathbb{C}$ a nonzero number such that $A^{2n} \neq 1$ for every $n > 0$.

Theorem 1.3 *The skein space $K(T^3)$ of the 3–torus $T^3 = S^1 \times S^1 \times S^1$ is finitely generated.*

A set of nine generators is given by the empty set $\varnothing$, some simple closed curves representing the nonzero elements of the first homology group $H_1(T^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^3$ with coefficients in $\mathbb{Z}_2$, and a skein element α that is equal to the link consisting of two parallel copies of any previous nonempty knots.

Our main tool is the algebraic work of Frohman and Gelca [5]. The skein space (module) of a (thickened) surface has a natural algebra structure obtained by overlap of framed links. In their work, Frohman and Gelca gave a nice formula that describes the product in the skein space (algebra) $K(T^2)$ of the 2–torus $T^2 = S^1 \times S^1$. A standard embedding of T^2 in T^3 makes this product commutative; hence we can get further relations from the formula of Frohman and Gelca.

A natural question is the following:

Question 1.4 Is 9 the dimension of the skein vector space $K(T^3)$ of the 3–torus?

After this paper was submitted, P Gilmer [6] answered this question positively.

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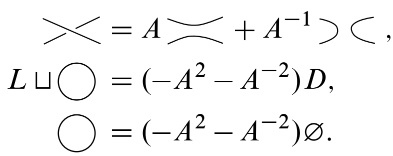
2 The result

2.1 Definition of skein module

Let M be an oriented 3–manifold, R a commutative ring with unit and $A \in R$ an invertible element of R. Let V be the abstract free R–module generated by all framed links in M (considered up to isotopies) including the empty set $\varnothing$.

Definition 2.1 The (R, A)*–Kauffman bracket skein module* of M, or the R*–skein module*, or simply the *KBSM*, sometimes indicated with $\mathrm{KM}(M; R, A)$, is the quotient of V by all the possible *skein relations*:



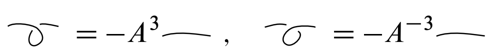
These are local relations where the framed links in an equation differ just in the pictured 3–ball that is equipped with a positive trivialization. An element of $\mathrm{KM}(M; R, A)$ is called a *skein* or a *skein element*. If M is the oriented I–bundle over a surface S (that is, $M = S \times [-1, 1]$ if S is oriented), we simply write $\mathrm{KM}(S; R, A)$ and call it the *skein module* of S.

Let $\mathbb{Q}(A)$ be field of all rational function with rational coefficients and abstract variable A. We set

$$K(M) := \mathrm{KM}(M; \mathbb{Q}(A), A),$$

and we call it the *skein vector space*, or simply the *skein space*, of M.

Remark 2.2 It is easy to verify that if we modify the framing of a component of a framed link, the skein changes by the multiplication of an integer power of $-A^3$:



2.2 The skein algebra of the 2–torus

Definition 2.3 Let S be a surface; the skein module $\mathrm{KM}(S; R, A)$ has a natural structure of an R–algebra that is given by the linear extension of the multiplication defined on framed links. Given two framed links $L_1, L_2 \subset S \times [-1,1]$, the product $L_1 \cdot L_2 \subset S \times [-1,1]$ is obtained by putting L_1 above L_2, so $L_1 \cdot L_2 \cap S \times [0,1] = L_1$ and $L_1 \cdot L_2 \cap S \times [-1,0] = L_2$.

Consider the 2–torus T^2 as the quotient of $\mathbb{R}^2$ modulo the standard lattice of translations generated by $(1,0)$ and $(0,1)$; hence for any nonzero pair (p,q) of integers, we have the notion of *(p,q)–curve*: the simple closed curve in the 2–torus that is the quotient of the line passing trough $(0,0)$ and (p,q).

Definition 2.4 Let p and q be two coprime integers; hence $(p,q) \neq (0,0)$. We denote by $(p,q)_T$ the (p,q)–curve in the 2–torus T^2 equipped with the blackboard framing. Given a framed knot γ in an oriented 3–manifold M and an integer $n \geq 0$, we denote by $T_n(\gamma)$ the skein element defined by induction as follows:

$$\begin{aligned} T_0(\gamma) &:= 2 \cdot \varnothing, \\ T_1(\gamma) &:= \gamma, \\ T_{n+1}(\gamma) &:= \gamma \cdot T_n(\gamma) - T_{n-1}(\gamma), \end{aligned}$$

where $\gamma \cdot T_n(\gamma)$ is the skein element obtained adding a copy of γ to all the framed links that compose the skein $T_n(\gamma)$. For $p, q \in \mathbb{Z}$ such that $(p,q) \neq (0,0)$, we denote by $(p,q)_T$ the skein element

$$(p,q)_T := T_{\mathrm{MCD}(p,q)}\left(\left(\frac{p}{\mathrm{MCD}(p,q)}, \frac{q}{\mathrm{MCD}(p,q)}\right)_T\right),$$

where $\mathrm{MCD}(p,q)$ is the maximum common divisor of p and q. Finally, we set

$$(0,0)_T := 2 \cdot \varnothing.$$

It is easy to show that the set of all the skein elements $(p,q)_T$ with $p, q \in \mathbb{Z}$ generates $\mathrm{KM}(T^2; R, A)$ as R–module.

This is not the standard way to color framed links in a skein module. The colorings $\mathrm{JW}_n(\gamma)$, $n \geq 0$, with the Jones–Wenzl projectors are defined in the same way as $T_n(\gamma)$, but at the 0–level we have $\mathrm{JW}_0(\gamma) = \varnothing$.

Theorem 2.5 (Frohman and Gelca [5]) *For any $p, q, r, s \in \mathbb{Z}$, the following holds in the skein module $\mathrm{KM}(T^2; R, A)$ of the 2–torus T^2:*

$$(p,q)_T \cdot (r,s)_T = A^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p+r, q+s)_T + A^{-\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|}(p-r, q-s)_T,$$

where $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|$ is the determinant $ps - qr$. □

2.3 The abelianization

Definition 2.6 Let B be a R–algebra for a commutative ring with unity R. We denote by $C(B)$ the R–module defined as the quotient

$$C(B) := \frac{B}{[B,B]},$$

where $[B,B]$ is the submodule of B generated by all the elements of the form $ab-ba$ for $a,b \in B$. We call $C(B)$ the *abelianization* of B.

Remark 2.7 Usually in noncommutative algebra, the *abelianization* is the R–algebra defined as the quotient of B modulo the subalgebra (submodule and ideal) generated by all the elements of the form $ab-ba$. In our definition, the denominator is just a submodule and we only get an R–module. We use the word "abelianization" anyway.

Now we work with $C(K(T^2))$, and we still use $(p,q)_T$ and $(p,q)_T \cdot (r,s)_T$ to denote the class of $(p,q)_T \in K(T^2)$ and $(p,q)_T \cdot (r,s)_T \in K(T^2)$ in $C(K(T^2))$.

Lemma 2.8 *Let (p,q) be a pair of integers different from $(0,0)$. Then in the abelianization $C(K(T^2))$ of the skein algebra $K(T^2)$ of the 2–torus T^2, we have*

$$(p,q)_T = \begin{cases} (1,0)_T & \text{if } p \in 2\mathbb{Z}+1 \text{ and } q \in 2\mathbb{Z}, \\ (0,1)_T & \text{if } p \in 2\mathbb{Z} \text{ and } q \in 2\mathbb{Z}+1, \\ (1,1)_T & \text{if } p,q \in 2\mathbb{Z}+1, \\ (2,0)_T & \text{if } p,q \in 2\mathbb{Z}. \end{cases}$$

Hence $C(K(T^2))$ is generated as a $\mathbb{Q}(A)$–vector space by the empty set $\varnothing$, the framed knots $(1,0)_T$, $(0,1)_T$, $(1,1)_T$, and a framed link consisting of two parallel copies of $(1,0)_T$.

Proof By Theorem 2.5, for every $p,q \in \mathbb{Z}$, we have

$$\begin{aligned} A^{-q}(p+2,q)_T + A^{q}(p,q)_T &= (p+1,q)_T \cdot (1,0)_T \\ &= (1,0)_T \cdot (p+1,q)_T \\ &= A^{q}(p+2,q)_T + A^{-q}(-p,-q)_T. \end{aligned}$$

Since $(p,q)_T = (-p,-q)_T$, we have $(A^q - A^{-q})(p,q)_T = (A^q - A^{-q})(p+2,q)_T$. Hence if $q \neq 0$, we get $(p,q)_T = (p+2,q)_T$ (here we use the fact that the base ring is a field and that $A^{2n} \neq 1$ for every $n > 0$). Thus

$$(p,q)_T = \begin{cases} (0,q)_T & \text{if } p \in 2\mathbb{Z} \text{ and } q \neq 0, \\ (1,q)_T & \text{if } p \in 2\mathbb{Z}+1 \text{ and } q \neq 0. \end{cases}$$

Analogously, by using $(0,1)_T$ instead of $(1,0)_T$ for $p \neq 0$, we get

$$(p,q)_T = \begin{cases} (p,0)_T & \text{if } q \in 2\mathbb{Z} \text{ and } q \neq 0, \\ (p,1)_T & \text{if } q \in 2\mathbb{Z}+1 \text{ and } q \neq 0. \end{cases}$$

Therefore, if $p,q \in 2\mathbb{Z}+1$, we have $(p,q)_T = (1,1)_T$. If $p \neq 0$, we get

$$(p,0)_T = (p,2)_T = \begin{cases} (0,2)_T & \text{if } p \in 2\mathbb{Z}, \\ (1,2)_T = (1,0)_T & \text{if } p \in 2\mathbb{Z}+1. \end{cases}$$

In the same way for $q \neq 0$, we get

$$(0,q)_T = (2,q)_T = \begin{cases} (2,0)_T & \text{if } p \in 2\mathbb{Z}, \\ (2,1)_T = (0,1)_T & \text{if } p \in 2\mathbb{Z}+1. \end{cases}$$

In particular, we have

$$(2,0)_T = (2,2)_T = (2,-2)_T = (0,2)_T = (p,q)_T \quad \text{for } (p,q) \neq (0,0),\ p,q \in 2\mathbb{Z}. \qquad \square$$

2.4 The (p,q,r)–type curves

As for the 2–torus T^2, we look at the 3–torus T^3 as the quotient of $\mathbb{R}^3$ modulo the standard lattice of translations generated by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Definition 2.9 Let (p,q,r) be a triple of coprime integers; that means we have $\mathrm{MCD}(p,q,r) = 1$, where $\mathrm{MCD}(p,q,r)$ is the maximum common divisor of p, q and r, and in particular, we have $(p,q,r) \neq (0,0,0)$. The (p,q,r)*–curve* is the simple closed curve in the 3–torus that is the quotient (under the standard lattice) of the line passing through $(0,0,0)$ and (p,q,r). We denote by $[p,q,r]$ the (p,q,r)–curve equipped with the framing that is the collar of the curve in the quotient of any plane containing $(0,0,0)$ and (p,q,r). The framing does not depend on the choice of the plane.

Definition 2.10 An embedding $e\colon T^2 \to T^3$ of the 2–torus in the 3–torus is *standard* if it is the quotient (under the standard lattice) of a plane in $\mathbb{R}^3$ that is the image of the plane generated by $(1,0,0)$ and $(0,1,0)$ under a linear map defined by a matrix of $\mathrm{SL}_3(\mathbb{Z})$ (a 3×3 matrix with integer entries and determinant 1).

Remark 2.11 There are infinitely many standard embeddings, even up to isotopies. A standard embedding of T^2 in T^3 is the quotient under the standard lattice of the plane generated by two columns of a matrix of $\mathrm{SL}_3(\mathbb{Z})$.

Lemma 2.12 *Let (p,q,r) be a triple of coprime integers. Then the skein element $[p,q,r] \in K(T^3)$ is equal to $[x,y,z]$, where $x,y,z \in \{0,1\}$ and they have respectively the same parities as p, q and r.*

Proof Every embedding $e\colon T^2 \to T^3$ of the 2–torus defines a linear map between the skein spaces

$$e_*\colon K(T^2) \to K(T^3).$$

The map e_* factorizes with the quotient map $K(T^2) \to C(K(T^2))$. In fact, we can slide the framed links in $e(T^2 \times [-1,1])$ from above to below, getting $e_*(L_1 \cdot L_2) = e_*(L_2 \cdot L_1)$ for every two framed links, L_1 and L_2, in $T^2 \times [-1,1]$. As said in Remark 2.11, a standard embedding $e\colon T^2 \to T^3$ corresponds to the plane generated by two columns $(p_1,q_1,r_1),(p_2,q_2,r_2) \in \mathbb{Z}^3$ of a matrix in $\mathrm{SL}_3(\mathbb{Z})$. In this correspondence, $e_*((a,b)_T) = [ap_1+bp_2, aq_1+bq_2, ar_1+br_2]$ for every coprime $a,b \in \mathbb{Z}$. Therefore, by Lemma 2.8, we get

$$\begin{aligned}[][a'p_1+b'p_2, a'q_1+b'q_2, a'r_1+b'r_2] &= e_*((a',b')_T)\\ &= e_*((a,b)_T)\\ &= [ap_1+bp_2, aq_1+bq_2, ar_1+br_2]\end{aligned}$$

for every two pairs $(a,b),(a',b') \in \mathbb{Z}^2$ of coprime integers such that $a+a', b+b' \in 2\mathbb{Z}$.

Let (p,q,r) be a triple of coprime integers. By permuting p, q and r, we get either $(p,q,r) = (1,0,0)$ or $p,q \neq 0$. Consider the case where $p,q \neq 0$. Let d be the maximum common divisor of p and q, and let $\lambda,\mu \in \mathbb{Z}$ such that $\lambda p + \mu q = d$. The following matrix belongs in $\mathrm{SL}_3(\mathbb{Z})$:

$$M_1 := \begin{pmatrix} p/d & -\mu & 0\\ q/d & \lambda & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Let $v_1^{(1)}$ and $v_3^{(1)}$ be the first and the third columns of M_1. We have $(p,q,r) = dv_1^{(1)} + rv_3^{(1)}$. Hence

$$[p,q,r] = \begin{cases} \left[\frac{p}{d},\frac{q}{d},0\right] & \text{if } d \in 2\mathbb{Z}+1 \text{ and } r \in 2\mathbb{Z},\\ [0,0,1] & \text{if } d \in 2\mathbb{Z} \text{ and } r \in 2\mathbb{Z}+1,\\ \left[\frac{p}{d},\frac{q}{d},1\right] & \text{if } d,r \in 2\mathbb{Z}+1.\end{cases}$$

The integers p,q,r cannot be all even because they are coprime; hence d and r cannot be both even. Therefore, we just need to study the cases where $r \in \{0,1\}$.

If $r=0$, we consider the trivial embedding of T^2 in T^3. The corresponding matrix of $\mathrm{SL}_3(\mathbb{Z})$ is the identity. We have $\left(\frac{p}{d},\frac{q}{d},0\right) = \frac{p}{d}(1,0,0) + \frac{q}{d}(0,1,0)$; hence

$$[p,q,0] = \left[\frac{p}{d},\frac{q}{d},0\right] = \begin{cases} [1,0,0] & \text{if } \frac{p}{d} \in 2\mathbb{Z}+1 \text{ and } \frac{q}{d} \in 2\mathbb{Z},\\ [0,1,0] & \text{if } \frac{p}{d} \in 2\mathbb{Z} \text{ and } \frac{q}{d} \in 2\mathbb{Z}+1,\\ [1,1,0] & \text{if } \frac{p}{d},\frac{q}{d} \in 2\mathbb{Z}+1.\end{cases}$$

If $r = 1$, we take the matrix of $\mathrm{SL}_3(\mathbb{Z})$

$$M_2 := \begin{pmatrix} 0 & 0 & 1 \\ q & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $v_1^{(2)}$ and $v_3^{(2)}$ be the first and the third columns of M_2. We have $(p,q,1) = pv_3^{(2)} + v_1^{(2)}$; hence

$$[p,q,1] = \begin{cases} [1,q,1] & \text{if } p \in 2\mathbb{Z}+1, \\ [0,q,1] & \text{if } p \in 2\mathbb{Z}. \end{cases}$$

By permuting p, q and r, we reduce the case $(p,q,r) = (0,q,1)$ to the case $p,q \neq 0$, $r = 0$ that we studied before.

It remains to consider the case $p = r = 1$. We consider the matrix of $\mathrm{SL}_3(\mathbb{Z})$

$$M_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let $v_1^{(3)}$ and $v_2^{(3)}$ be the first and the second columns of M_3. We have $(1,q,1) = v_1^{(3)} + qv_2^{(3)}$. Hence

$$[1,q,1] = \begin{cases} [1,0,1] & \text{if } q \in 2\mathbb{Z}, \\ [1,1,1] & \text{if } q \in 2\mathbb{Z}+1. \end{cases}$$

□

Lemma 2.13 *The intersection of any two different standardly embedded 2–tori in T^3 contains a (p,q,r)–type curve.*

Proof Let T_1 and T_2 be two standardly embedded tori in the 3–torus, and let π_1 and π_2 be two planes in $\mathbb{R}^3$ whose projections under the standard lattice are respectively T_1 and T_2. The intersection $T_1 \cap T_2$ contains the projection of $\pi_1 \cap \pi_2$. We just need to prove that in $\pi_1 \cap \pi_2$, there is a point $(p,q,r) \neq (0,0,0)$ with integer coordinates $p,q,r \in \mathbb{Z}$. Every plane defining a standardly embedded torus is generated by two vectors with integer coordinates, and hence it is described by an equation $ax + by + cz = 0$ with integer coefficients $a,b,c \in \mathbb{Z}$. Applying a linear map described by a matrix of $\mathrm{SL}_3(\mathbb{Z})$, we can suppose that π_1 is the trivial plane $\{z = 0\}$. Let $a,b,c \in \mathbb{Z}$ such that $\pi_2 = \{ax + by + cz = 0\}$. The vector $(-b,a,0)$ is nonzero and lies on $\pi_1 \cap \pi_2$. □

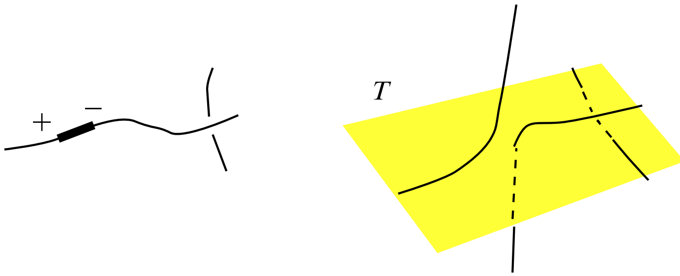


Figure 1: Diagrams of framed links in T^3. The plane is a part of the standardly embedded torus $T \subset T^3$ where the links project. If we look at the framed links in T^3 as framed tangles in $T \times [-1,1]$, the two strands that get out vertically from the plane end in the boundary points $(x,1)$ and $(x,-1)$ for some $x \in T$.

2.5 Diagrams

Framed links in T^3 can be represented by diagrams in the 2–torus T^2. These diagrams are like the usual link diagrams but with further oriented signs on the edges; see Figure 1 (left). Fix a standardly embedded 2–torus T in T^3. After a cut along a parallel copy T' of T, the 3–torus becomes diffeomorphic to $T \times [-1,1]$, and framed links in T^3 correspond to framed tangles of $T \times [-1,1]$. These diagrams are generic projections on T of the framed tangles in $T \times [-1,1]$ via the natural projection $(x,t) \mapsto x$. The further signs on the diagrams represent the intersection of the framed links with the boundary $T \times \{-1,1\}$. In other words, they represent the passages of the links along the (p,q,r)–type curve that, in the Euclidean metric, is orthogonal to T; see Figure 1 (right). If T is the trivial torus $S^1 \times S^1 \times \{x\}$, the further signs represent the passages through the third S^1–factor. We use the proper notion of blackboard framing.

2.6 Generators for the 3–torus

The following is the main theorem proved in this paper. We use all the previous lemmas to get a set of nine generators of $K(T^3)$.

Theorem 2.14 *The skein space $K(T^3)$ of the 3–torus T^3 is generated by the empty set $\varnothing$, $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, $[1,1,1]$ and a skein α that is equal to the framed link consisting of two parallel copies of any (p,q,r)–type curve.*

Proof Let T be the trivial embedded 2–torus: the one containing the (p,q,r)–type curves with $r=0$. Use T to project the framed links and make diagrams. By using the first skein relation on these diagrams, we can see that $K(T^3)$ is generated by the framed links described by diagrams on T without crossings. These diagrams are unions of simple closed curves on T equipped with some signs as the one with +

and $-$ in Figure 1. These simple closed curves are either parallel to a (p,q)–curve or homotopically trivial. The framed links described by these diagrams lie in the standardly embedded tori that are the projections (under the standard lattice) of the planes generated by $(0,0,1)$ and $(p,q,0)$ for some p and q. Therefore, $K(T^3)$ is generated by the images of $K(T^2)$ under the linear maps induced by the standard embeddings of T^2 in T^3.

As said in the proof of Lemma 2.12, the linear map e_* induced by any standard embedding $e\colon T^2 \to T^3$ factorizes with the quotient map $K(T^2) \to C(K(T^2))$. Lemma 2.8 applied to the standard embedding e shows that the image $e_*(K(T^2))$ is generated by $\varnothing$, three (p,q,r)–type curves lying on $e(T^2)$, and the skein α_e that is equal to the framed link consisting of two parallel copies of any (p,q,r)–type curve lying on $e(T^2)$.

Let $e_1, e_2\colon T^2 \to T^3$ be two standard embeddings. By Lemma 2.13, $e_1(T^2) \cap e_2(T^2)$ contains a (p,q,r)–type curve γ; hence α_{e_1} and α_{e_2} coincide with the framed link that is two parallel copies of γ. Therefore, the skein element α_e does not depend on the embedding e.

We conclude by using Lemma 2.12, which says that the skein of any (p,q,r)–type curve is equal to the one of a standard representative of a nonzero element of the first homology group $H_1(T^3;\mathbb{Z}_2)$ with coefficient in $\mathbb{Z}_2$, namely a (p,q,r)–type curve with $p,q,r \in \{0,1\}$. □

Remark 2.15 Theorem 2.14, Lemma 2.8 and Lemma 2.12 work for every base pair (R,A) such that $A^{2n}-1$ is an invertible element of R for any $n>0$. In particular, they work for $(\mathbb{C},A)$, where $A^{2n} \neq 1$ for any $n>0$. Unfortunately, we do not know what happens with the base pair $(\mathbb{C},\pm 1)$, which is the one used for the connection with the $\mathrm{SL}_2(\mathbb{C})$–character variety [3]. In fact, in Lemma 2.8, we would get just trivial equalities if $A=\pm 1$.

2.7 Linear independence

Here we talk about the linear independence of our generators of $K(T^2)$. The following proposition shows a direct sum decomposition of $K(T^3)$.

Proposition 2.16 *The skein space $K(T^3)$ is the direct sum of eight subspaces,*

$$K(T^3) = V_0 \oplus V_1 \oplus \cdots \oplus V_7,$$

such that

(1) *V_0 is generated by the empty set $\varnothing$ and the skein α (see Theorem 2.14);*

(2) *every (p,q,r)–type curve generates a V_j with $j>0$, and every V_j with $j>0$ is generated by one such curve.*

Proof The skein relations relate framed links in the same $\mathbb{Z}_2$–homology class. Hence for every oriented 3–manifold M, we have a direct sum decomposition

$$\mathrm{KM}(M; R, A) = \bigoplus_{h \in H_1(M;\mathbb{Z}_2)} V_h,$$

where V_h is generated by the framed links whose $\mathbb{Z}_2$–homology class is h. The statement follows by this observation and the fact that if $[p,q,r]$ and $[p',q',r']$ represent the same $\mathbb{Z}_2$–homology class, then $[p,q,r] = [p',q',r'] \in K(T^3)$. □

Remark 2.17 Given a triple of integers $(x, y, z) \neq (0, 0, 0)$ such that $x, y, z \in \{0, 1\}$, we can easily find an orientation-preserving diffeomorphism of the 3–torus T^3 sending $[x, y, z]$ to $[1, 0, 0]$. Hence if the skein of one such curve $[x, y, z]$ is null, then also all the other skein elements of such curves are null. Therefore, by Proposition 2.16, the possible dimensions of the skein space $K(T^3)$ are 0, 1, 2, 7, 8 and 9.

After the submission of this paper, P Gilmer [6] showed that the skein of the $(1,0,0)$–curve is not null and that the empty set and the skein α are linear independent. This answers Question 1.4 in the affirmative by proving that the set of nine generators is actually a basis for the skein space.

References

[1] **C Blanchet**, **N Habegger**, **G Masbaum**, **P Vogel**, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (1995) 883–927 MR

[2] **D Bullock**, *The $(2,\infty)$–skein module of the complement of a $(2, 2p+1)$ torus knot*, J. Knot Theory Ramifications 4 (1995) 619–632 MR

[3] **D Bullock**, *Rings of $\mathrm{SL}_2(\mathbf{C})$–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997) 521–542 MR

[4] **F Costantino**, *Coloured Jones invariants of links and the volume conjecture*, J. Lond. Math. Soc. 76 (2007) 1–15 MR

[5] **C Frohman**, **R Gelca**, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. 352 (2000) 4877–4888 MR

[6] **P M Gilmer**, *On the Kauffman bracket skein module of the 3–torus*, preprint (2016) arXiv To appear in Indiana J. of Math.

[7] **J Hoste**, **J H Przytycki**, *The $(2,\infty)$–skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications 2 (1993) 321–333 MR

[8] **J Hoste**, **J H Przytycki**, *The $(2,\infty)$–skein module of Whitehead manifolds*, J. Knot Theory Ramifications 4 (1995) 411–427 MR

[9] **J Hoste**, **J H Przytycki**, *The Kauffman bracket skein module of $S^1 \times S^2$*, Math. Z. 220 (1995) 65–73 MR

[10] **L H Kauffman**, *New invariants in the theory of knots*, Amer. Math. Monthly 95 (1988) 195–242 MR

[11] **L H Kauffman**, **S L Lins**, *Temperley–Lieb recoupling theory and invariants of 3–manifolds*, Annals of Mathematics Studies 134, Princeton Univ. Press (1994) MR

[12] **W B R Lickorish**, *Three-manifolds and the Temperley–Lieb algebra*, Math. Ann. 290 (1991) 657–670 MR

[13] **W B R Lickorish**, *Calculations with the Temperley–Lieb algebra*, Comment. Math. Helv. 67 (1992) 571–591 MR

[14] **W B R Lickorish**, *The skein method for three-manifold invariants*, J. Knot Theory Ramifications 2 (1993) 171–194 MR

[15] **W B R Lickorish**, *Skeins and handlebodies*, Pacific J. Math. 159 (1993) 337–349 MR

[16] **J Marché**, *The skein module of torus knots*, Quantum Topol. 1 (2010) 413–421 MR

[17] **M Mroczkowski**, *Kauffman bracket skein module of a family of prism manifolds*, J. Knot Theory Ramifications 20 (2011) 159–170 MR

[18] **M Mroczkowski**, *Kauffman bracket skein module of the connected sum of two projective spaces*, J. Knot Theory Ramifications 20 (2011) 651–675 MR

[19] **M Mroczkowski**, **M K Dabkowski**, *KBSM of the product of a disk with two holes and S^1*, Topology Appl. 156 (2009) 1831–1849 MR

[20] **J H Przytycki**, *Skein modules of 3–manifolds*, Bull. Polish Acad. Sci. Math. 39 (1991) 91–100 MR

[21] **J H Przytycki**, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. 16 (1999) 45–66 MR

[22] **J H Przytycki**, *Kauffman bracket skein module of a connected sum of 3–manifolds*, Manuscripta Math. 101 (2000) 199–207 MR

[23] **J H Przytycki**, *Skein modules*, preprint (2006) arXiv

[24] **V G Turaev**, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988) 79–89 MR In Russian; translated in J. Soviet Math. 52 (1990) 2799–2805

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